On the Generalized Numerical Range

by

M.K.H. Fan and A.L. Tits
ON THE GENERALIZED NUMERICAL RANGE

by

M.K.H. Fan & A.L. Tits
On the Generalized Numerical Range *

M.K.H. Fan
A.L. Tits †

Electrical Engineering Department and Systems Research Center
University of Maryland, College Park, MD 20742

June 9, 1987

Abstract

Let $A_k, k = 1, \ldots, m$ be $n \times n$ Hermitian matrices and let $f : \mathbb{C}^n \to \mathbb{R}^m$ have components $f^k(x) = x^H A_k x, k = 1, \ldots, m$. When $n \geq 3$ and $m = 3$, the set $W(A_1, \ldots, A_m) = \{ f(x) : \| x \| = 1 \}$ is convex. This property does not hold in general when $m > 3$. These particular cases of known results are proven here using a direct, geometric approach. A geometric characterization of the contact surfaces is obtained for any $n$ and $m$. Necessary conditions are given for $f(x)$ to be on boundary of $W(A_1, \ldots, A_m)$ or on certain subsets of this boundary. These results are of interest in the context of the computation of the structured singular value, a recently introduced tool for the analysis and synthesis of control systems.

---

*This research was supported by the National Science Foundation under grants No. DMC-84-51515 and OIR-85-00108.
†Please address all correspondence to the second author.
1 Introduction

Let $A_k, k = 1,\ldots,m$, be $n \times n$ Hermitian matrices and let $f : \mathbb{C}^n \to \mathbb{R}^m$ have components $f^k(x) = x^H A_k x$, $k = 1,\ldots,m$. The \textit{generalized numerical range of matrices} $A_1,\ldots,A_m$ is the set $W(A_1,\ldots,A_m) = \{f(x) : ||x|| = 1\}$, a subset of $\mathbb{R}^m$ (e.g., [1–3]). It has been long known that, when $m = 2$, this set is always convex [1] and that, when $m = 3$, it still has a convex boundary [1,4]. Here a set is said to have a convex boundary if its intersection with each of its support hyperplane is convex [1,2,4]. More recently, it was shown [5–7], as a particular case of a more general result, that the generalized numerical range is still convex when $m = 3$ and $n > 2$, but that this property fails to hold in general if $m > 3$ or $n = 2$. In this note, a direct, geometric proof of convexity is given for the case $m = 3, n > 2$. For $m > 3$ or $n = 2$, a canonical family of examples is exhibited where $W(A_1,\ldots,A_m)$ is not convex. For any $m$ and $n$, a geometric characterization of the intersections of $W(A_1,\ldots,A_m)$ with its supporting hyperplanes is derived. Necessary conditions on $x$ are given for $f(x)$ to be (i) on the boundary of $W(A_1,\ldots,A_m)$, (ii) on the intersection of this boundary with the boundary of the cone $\hat{W}(A_1,\ldots,A_m)$ it generates, and (iii) on a certain type of ‘corner’ of $W(A_1,\ldots,A_m)$. These results are of interest in the context of the computation of the structured singular value, a quantity recently introduced by Doyle [8] as a tool in control system analysis and synthesis (see [9]).

We will make repeated use of the concept of \textit{3D-ellipsoid}, defined as follows.

\textbf{Definition 1.} We call \textit{3D-ellipsoid} the image in $\mathbb{R}^m$ of the unit sphere in $\mathbb{R}^3$ under an affine map. A 3D-ellipsoid is degenerate if it is entirely contained.
in a two-dimensional affine set. □

With this definition, a 3D-ellipsoid is a compact set entirely contained in a subspace of \( \mathbb{R}^m \) of dimension three (the range of the affine map). It can consist in either the boundary of a nondegenerate ellipsoid, a solid ellipse, a line segment, or a point.

In the sequel, \( \partial B \) is the unit sphere in \( \mathbb{C}^n \), \( \Re \) and \( \Im \) indicate the real and imaginary parts and, for any set \( S \), \( \text{co}S \) denotes its convex hull.

2 Main Results

The following two propositions hold for any \( m \). The first one is a straightforward extension of a result in [8].

**Proposition 1.** If \( n = 2 \), \( W(A_1, \ldots, A_m) \) is a 3D-ellipsoid. The \( k \)th coordinate of its center is \( \text{trace}(A_k)/2 \).

**Proof.** For \( k = 1, \ldots, m \), let

\[
A_k = \begin{bmatrix}
  a_k & b_k \\
  \overline{b_k} & c_k
\end{bmatrix},
\]

where \( a_k, c_k \in \Re \), \( b_k \in \mathbb{C} \), and \( \overline{b_k} \) is the complex conjugate of \( b_k \). The unit sphere in \( \mathbb{C}^2 \) can be expressed as

\[
\left\{ e = \exp(i\phi) \begin{bmatrix}
  \cos \theta \\
  \sin \theta \exp(i\psi)
\end{bmatrix} : \theta, \phi, \psi \in \Re \right\}
\]

where \( i \) is the square root of -1. For \( e \) as in (1), elementary manipulations give

\[
e^H A_k e = \frac{\text{trace}(A_k)}{2} + \frac{a_k - c_k}{2} \Re b_k - \Im b_k \begin{bmatrix}
  \cos(2\theta) \\
  \sin(2\theta) \cos \psi \\
  \sin(2\theta) \sin \psi
\end{bmatrix}.
\]
Since \( \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \cos \psi \\ \sin(2\theta) \sin \psi \end{pmatrix} : \theta, \psi \in \mathbb{R} \) is the unit sphere in \( \mathbb{R}^3 \), the claim is proven. \( \square \)

**Proposition 2.** If \( n \geq 3 \), \( W(A_1, \ldots, A_m) \) is not a nondegenerate 3D-ellipsoid.

**Proof.** If \( W(A_1, \ldots, A_m) \) is a singleton, the claim holds. Thus suppose it is not, i.e., suppose there exist \( y, z' \in \partial B \) and \( k_0 \in \{1, \ldots, m\} \) such that

\[
y^H A_{k_0} y \not\equiv z'^H A_{k_0} z'.
\]  

(2)

Since \( n \geq 3 \), there exists an \( x \in \partial B \) such that

\[
x^H y = x^H z' = 0
\]

and, without loss of generality (in view of (2)),

\[
x^H A_{k_0} x \not\equiv y^H A_{k_0} y.
\]  

(3)

In view of (2), continuity implies that there exists a \( z \in \partial B \) in the subspace of \( \mathbb{C}^n \) generated by \( y \) and \( z' \) such that

\[
z^H A_{k_0} z \not\equiv x^H A_{k_0} x
\]

and

\[
z^H A_{k_0} z \not\equiv y^H A_{k_0} y.
\]  

(4)

Now consider the sets

\[
W_y = W([x^|y]^H A_1[x^|y], \ldots, [x^|y]^H A_m[x^|y])
\]

and

\[
W_z = W([x^|z]^H A_1[x^|z], \ldots, [x^|z]^H A_m[x^|z]).
\]
Since both \( y \) and \( z \) are orthogonal to \( x \), both \( W_y \) and \( W_z \) are subsets of \( W(A_1, \ldots, A_m) \). By Proposition 1, both are 3D-ellipsoids and their centers have as \( k_0 \)th coordinate respectively \( (x^H A_{k_0} x + y^H A_{k_0} y)/2 \) and \( (x^H A_{k_0} x + z^H A_{k_0} z)/2 \), so that, in view of (4), the two sets are distinct. Thus at least one of them is a proper subset of \( W(A_1, \ldots, A_m) \). Since the known properties of \( y \) and \( z \) are identical, there is no loss of generality in assuming that this set is \( W_y \). Also, clearly, \( W_y \) passes through the two points in \( \mathbb{R}^m \) whose \( k \)th coordinates are \( x^H A_k x \) and \( y^H A_k y \). Thus, in view of (3), \( W_y \) is not a singleton. Since clearly a nondegenerate 3D-ellipsoid cannot have any 3D-ellipsoid but singletons as proper subsets, the proof is complete. \( \square \)

In proving the next proposition, we will make use of the following lemma, which extends a result in [8]. It holds for any \( n \) and \( m \).

**Lemma 1.** Given any \( u, v_0, v_1 \in W(A_1, \ldots, A_m) \), there exists a point-to-set map \( E_{uv_0v_1} : [0, 1] \to 2^{\mathbb{R}^m} \), continuous in the Hausdorff topology, such that \( u, v_0 \in E_{uv_0v_1}(0) \) and \( u, v_1 \in E_{uv_0v_1}(1) \) and such that for all \( t \in [0, 1] \), \( E_{uv_0v_1}(t) \) is a 3D-ellipsoid contained in \( W(A_1, \ldots, A_m) \).

**Proof.** First, suppose that \( v_0 \neq u \neq v_1 \), and let \( x, y_0, y_1 \in \partial B \) be unit vectors such that, for \( k = 1, \ldots, m \), \( u^k = x^H A_k x \), \( v_0^k = y_0^H A_k y_0 \), \( v_1^k = y_1^H A_k y_1 \). Clearly, \( \{x, y_0\} \) and \( \{x, y_1\} \) are both linearly independent over \( \mathbb{C} \) and the vectors \( y_0' \) and \( y_1' \), given by

\[
y_0' = \frac{1}{\|y_0 - (x^H y_0)x\|}(y_0 - (x^H y_0)x)
\]

and

\[
y_1' = \frac{1}{\|y_1 - (x^H y_1)x\|}(y_1 - (x^H y_1)x)
\]

are both orthogonal to \( x \) and have unit length. Let \( y : [0, 1] \to \partial B \) be any continuous map such that \( y(0) = y_0' \) and \( y(1) = y_1' \) and such that, for all
\( t \in [0, 1], y(t) \) belongs to the subspace of \( \mathbb{C}^n \) generated by \( y_0 \) and \( y_1 \). Next, for \( k = 1, \ldots, m \), let \( B_k : [0, 1] \to \mathbb{C}^{2 \times 2} \) be the continuous map defined by

\[
B_k(t) = \begin{bmatrix} x & y(t) \end{bmatrix}^H A_k \begin{bmatrix} x & y(t) \end{bmatrix} \quad \forall t \in [0, 1].
\]

Proposition 1 implies that, for each \( t \in [0, 1] \), \( W(B_1(t), \ldots, B_m(t)) \) is a 3D-ellipsoid, say \( E_{w_0v_1}(t) \). It is easily checked that \( E_{w_0v_1} \) satisfies the required conditions. Finally, if \( u = v_0 \) (resp. \( u = v_1 \)), pick \( E_{w_0v_1} \) to be the constant map whose value is any 3D-ellipsoid contained in \( W(A_1, \ldots, A_m) \) and passing through \( u \) and \( v_1 \) (resp. \( u \) and \( v_0 \)). □

**Proposition 3.** If \( n \geq 3 \), \( W(A_1, A_2, A_3) \) is convex.

**Proof.** Let \( u, v \in W(A_1, A_2, A_3) \) and let \( E \subset W(A_1, A_2, A_3) \) be a 3D-ellipsoid passing through \( u \) and \( v \) (see Lemma 1). We will show that the convex hull of \( E \), denoted by \( \text{co}E \), is contained in \( W(A_1, A_2, A_3) \), thus proving convexity. If \( E \) is degenerate, the result is clear. Thus assume \( E \) is nondegenerate. In view of Proposition 2, \( E \) must be a proper subset of \( W(A_1, A_2, A_3) \). Thus let \( \hat{w} \in W(A_1, A_2, A_3) \), \( \hat{w} \not\in E \), and let \( w \) be any point in \( \text{co}E \). We prove that \( w \in W(A_1, A_2, A_3) \). If \( w = \hat{w} \), the claims holds. Thus suppose that \( w \neq \hat{w} \). Let \( w_0 \) and \( w_1 \) be the intersection points with \( E \) of the straight line through \( w \) and \( \hat{w} \) and without loss of generality suppose that \( w \) lies between \( \hat{w} \) and \( w_0 \). Let \( E_{\hat{w}w_0w_1} : [0, 1] \to \mathbb{R}^m \) be as specified by Lemma 1. Clearly \( w \in \text{co}E_{\hat{w}w_0w_1}(0) \) and \( w \not\in \text{co}E_{\hat{w}w_0w_1}(1) \). Since \( E_{\hat{w}w_0w_1} \) is a continuous map, there must exist \( t \in [0, 1] \) such that \( w \in E_{\hat{w}w_0w_1}(t) \). Thus \( w \in W(A_1, A_2, A_3) \). □

A canonical family of examples is easily constructed, showing that Proposition 3 cannot be extended to the case of more than three matrices. More precisely, for any \( m \geq 4 \), \( n \geq 2 \), one can find matrices \( A_1, \ldots, A_m \) such
that $W(A_1, \ldots, A_m)$ does not have a convex boundary (and thus is not convex). The construction is as follows. For $k = 1, \ldots, m - 1$, let $B_k \in \mathbb{C}^{2 \times 2}$ be Hermitian matrices such that $W(B_1, \ldots, B_{m-1})$ is a nondegenerate 3D-ellipsoid (see Proposition 1). Then, for $k = 1, \ldots, m - 1$, let $A_k \in \mathbb{C}^{n \times n}$ be Hermitian matrices such that $A_k$ has $B_k$ as its top left corner and let $A_m = \text{diag}(\sigma_1, \ldots, \sigma_m)$ with $\sigma_1 = \sigma_2 > \sigma_3 \geq \ldots \geq \sigma_m$. It is easily checked that the intersection of $W(A_1, \ldots, A_m)$ with its supporting hyperplane $\{u \in \mathbb{R}^m : u^m = \sigma_1\}$ is an $\mathbb{R}^m$-imbedding of $W(B_1, \ldots, B_{m-1})$, which is not convex.

Using the construction just described, the following proposition can be easily proved.

**Proposition 4.** The intersection of $W(A_1, \ldots, A_m)$ with any of its supporting hyperplanes is an $\mathbb{R}^m$-imbedding of the generalized numerical range of some matrices. □

It is easy to see that, for any $m$ and $n$, points $f(x)$ on the intersection of $W(A_1, \ldots, A_m)$ with any supporting hyperplane are characterized by the fact that the corresponding $x$ is an eigenvector to the smallest eigenvalue of $\sum_{k=1}^{m} w^k A_k$ where the $w^k$'s are the components of a vector orthogonal to the hyperplane, pointing toward $W(A_1, \ldots, A_m)$. This fact is used by Doyle to construct the projection of the origin on $W(A_1, \ldots, A_m)$ when $W(A_1, \ldots, A_m)$ is convex ([8], see also [10]). Below, we derive properties of any point on the boundary of $W(A_1, \ldots, A_m)$ as well as properties of points on certain subsets of this boundary.

**Proposition 5.** If $x \in \partial B$ is such that $f(x)$ is on the boundary of $W(A_1, \ldots, A_m)$ then there exists a direction $w \in \mathbb{R}^m$ such that $x$ is an eigenvector of $\sum_{k=1}^{m} w^k A_k$. Moreover (i) if $x$ is any supporting hyperplane to
$W(A_1, \ldots, A_m)$ at $f(x)$, then the direction orthogonal to $N$ is a valid choice for $w$. (ii) If $f(x)$ is on the boundary of any cone containing $W(A_1, \ldots, A_m)$ (or, equivalently, of the cone generated by $W(A_1, \ldots, A_m)$), then $w$ can be chosen in such a way that

$$\sum_{k=1}^{m} w^k A_k x = 0.$$  

(iii) If there exists no subset of $W(A_1, \ldots, A_m)$ containing $f(x)$ that is locally homeomorphic to $\mathbb{R}^{m-(q-1)}$, $1 \leq q \leq m$, around $f(x)$, then there is a $q$-dimensional subspace $S$ of $\mathcal{V} = \{ A \in \mathcal{C}^{m \times n} : A = \sum_{k=1}^{m} w^k A_k, w^t \in \mathbb{R} \}$ such that all matrices in $S$ admit $x$ as an eigenvector.

**Proof.** Suppose that $x \in \partial B$ is such that $f(x)$ is on the boundary of $W(A_1, \ldots, A_m)$. Let

$$\partial B_x = \{ z \in \partial B \mid x^H z = 0 \}$$

and, for $k = 1, \ldots, m$, let $y_k$ be given by

$$y_k = A_k x - (x^H A_k x) x. \tag{5}$$

Clearly, for any $z \in \partial B_x$,

$$y_k^H z = x^H A_k z. \tag{6}$$

Next, for any $\theta \in \mathbb{R}$, $z \in \partial B_x$ define

$$f_x(\theta, z) = f(\cos \theta x + \sin \theta z)$$

$$= \cos^2 \theta f(x) + \sin^2 \theta f(z) + 2 \cos \theta \sin \theta R \left[ \begin{array}{c} x^H A_1 z \\ \vdots \\ x^H A_m z \end{array} \right].$$
In view of (6), we can write
\[
\frac{\partial f_z(0, z)}{\partial \theta} = 2\Re \begin{bmatrix} x^H A_1 z \\ \vdots \\ x^H A_m z \end{bmatrix} = 2M \begin{bmatrix} \Re z \\ -\Im z \end{bmatrix}
\]
where
\[
M = \begin{bmatrix} \Re y_1^T & \Im y_1^T \\ \vdots & \vdots \\ \Re y_m^T & \Im y_m^T \end{bmatrix}.
\]

Let
\[
F = \left\{ \frac{\partial f_z(0, z)}{\partial \theta} \mid z \in \partial B_z \right\}.
\]

Since for all \(k, y_k \in \partial B_z\), the ellipsoid \(G\) given by
\[
G = \left\{ 2M \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} M^T b \mid \|M^T b\| = 1, \ b \in \mathbb{R}^m \right\}
\]
is a subset of \(F\). Clearly, since \(f(z)\) is on the boundary of \(W(A_1, \ldots, A_m)\), \(F\) cannot contain any neighborhood of the origin, so that \(G\) must be contained in an \(m - 1\) dimensional subspace of \(\mathbb{R}^m\). This implies that \(M\) is singular, i.e., \(\sum_{k=1}^m w^k y_k = 0\) for some nonzero \(w \in \mathbb{R}^m\). Thus it follows from (5) that \(z\) is an eigenvector of \(\sum_{k=1}^m w^k A_k\) as claimed. The corresponding eigenvalue is \(z^H (\sum_{k=1}^m w^k A_k) z\). If \(\mathcal{H}\) is any hyperplane supporting \(W(A_1, \ldots, A_m)\) at \(f(z)\), then \(G\) must be contained in \(\mathcal{H}\) and (i) easily follows. Consider now, the cone \(C\) generated by the ellipsoid \(f(z) + G\) and suppose that \(f(z)\) is on the boundary of a cone containing \(W(A_1, \ldots, A_m)\). Clearly, since \(G \subset F\), the ray \(r = \{ \alpha f(z) : \alpha > 0 \}\) cannot be an interior ray of \(\text{co}C\). Since \(r\) passes through the center of every section of \(C\), it implies that the interior of \(\text{co}C\)
is empty. Thus, $C$ must be entirely contained in a hyperplane $\mathcal{H}$ passing through the origin. Since $r$ belongs to $\mathcal{H}$, it follows that
\[ \sum_{k=1}^{m} w^k f^k(x) = 0, \]
i.e.,
\[ x^H (\sum_{k=1}^{m} w^k A_k) x = 0 \]
for any $w$ normal to $\mathcal{H}$. Claim (ii) follows. Finally, if no subset of $W(A_1, \ldots, A_m)$ containing $f(x)$ is homeomorphic to $\mathbb{R}^{m-(q-1)}$, $1 \leq q \leq m$, around $f(x)$, $G$ must be contained in subspace $\mathcal{T}$ of dimension $m - q$. The subspace $S = \{ A \in \mathcal{V} : A = \sum_{k=1}^{m} w^k A_k, \ w \perp \mathcal{T} \}$ satisfies claim (iii). □

**Corollary.** If $W(A_1, A_2)$ is nonsmooth at a boundary point $f(x)$, then $x$ is an eigenvector of both $A_1$ and $A_2$. □

The well-known fact that such $x$ is an eigenvector of $A_1 + jA_2$ [3,11] is a direct consequence of this corollary.

**Acknowledgment.** The authors are grateful to Dr. A. Bhaya and to J.C. Wang for helpful discussions and to Drs. A. Laub and M. Marcus for pointing out the result in [7] and related references.
References


