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The Dynamics of Coupled Planar Rigid Bodies
Part I: Reduction, Equilibria and Stability

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The dynamics of coupled planar rigid bodies. Part I: Reduction, equilibria and stability

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Abstract

This paper studies the dynamics of coupled planar rigid bodies, concentrating on the case of two or three bodies coupled with a hinge joint. The Hamiltonian structure is non-canonical and is obtained using the methods of reduction, starting from canonical brackets on the cotangent bundle of the configuration space in material representation. The dynamics on the reduced space for two bodies occurs on cylinders in $\mathbb{R}^3$; stability of the equilibria is studied using the energy-Casimir method and is confirmed numerically. The phase space of the two bodies contains a homoclinic orbit which produces chaotic solutions when the system is perturbed by a third body. This and a study of periodic orbits are discussed in part II. The number and stability of equilibria and their bifurcations for three bodies as system parameters are varied are studied here; in particular, it is found that there are always four or six equilibria.

1. Introduction

The techniques of reduction of Hamiltonian systems with symmetry and the attendant energy-Casimir method have proved to be useful in a wide variety of problems, including fluid and plasma stability (Holm, Marsden, Ratiu and Weinstein, 1985), rigid-body dynamics with attachments and internal rotors (Holmes and Marsden, 1983; Koiller, 1985; Krishnaprasad, 1985; Krishnaprasad and Marsden, 1987), and bifurcations of liquid drops (Lewis, Marsden and Ratiu, 1986a,b). In this paper we shall apply these techniques to the case of planar rigid bodies coupled by a hinge joint. Many of the results for the two and three bodies generalize to multibody structures and other modifications, such as the inclusion of hinge torques. In subsequent papers we shall be studying this as well as the problem of coupled three-dimensional rigid bodies (for example, with a ball-in-socket or hinge joint). We also expect that the non-canonical Hamiltonian methods that are useful here will be useful in related problems of control (see (Van der Schaft, 1984; Sanchez de Alvarez, 1986).

The reduction technique used here goes back to Arnold (1966), Meyer (1973), and Marsden and Weinstein (1974), amongst others. It involves starting with a Poisson manifold P and a Lie group G acting on P by canonical transformations. The reduced phase space P/G (assume it has no singularities) has a natural Poisson structure whose symplectic leaves are the Marsden–Weinstein–Meyer spaces $J^{-1}(\mu)/G_\mu \approx J^{-1}(\mathcal{O})/G$, where $\mu \in g^*$, the dual of the Lie algebra of G, $J: P \to g^*$ is an equivariant momentum map for the action of G on P, G_μ is the isotropy group of μ (relative to the coadjoint action) and $\mathcal{O}$ is the coadjoint orbit through μ. If $P = T^*G$ and G acts by *left* translations, then P/G is identifiable with g^* equipped with the $(-)$ Lie–Poisson bracket:

$$\{F, H\}(\mu) = -\left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right\rangle. \tag{1.1}$$

The symplectic leaves in this case are just the coadjoint orbits. For $G = \mathrm{SO}(3)$ we get the (Pauli–Martin) bracket for rigid body dynamics:

$$\{F, H\}(l) = -l \cdot (\nabla F \times \nabla H). \tag{1.2}$$

Here $l \in \mathrm{SO}(3)^*$ is identified with a vector in $\mathbb{R}^3$ and represents the angular momentum of the rigid body in a body-fixed frame. If $\mathbf{I}$ is the moment-of-inertia tensor so that $l = \mathbf{I}\omega$; where ω is the body angular velocity, then Euler's equations

$$\frac{dl}{dt} = l \times \omega \tag{1.3}$$

are equivalent to Hamilton's equations

$$\dot{F} = \{F, H\}, \tag{1.4}$$

where $H(l) = \frac{1}{2}\langle l, \omega\rangle = \frac{1}{2}\langle \mathbf{I}^{-1}l, l\rangle$.

Notice that (1.2) is a non-canonical bracket; that is, the usual (q, p) Poisson-bracket formalism has disappeared through the reduction process. One of our first goals in the paper will be to develop a similar bracket for the dynamics of two coupled planar rigid bodies. We start with the canonical bracket on the cotangent bundle of configuration space just as one starts with $T^*\mathrm{SO}(3)$ (parametrized by Euler angles (θ, φ, ψ) and their conjugate momenta $(p_\theta, p_\varphi, p_\psi)$) in rigid-body dynamics.

When these procedures are carried out for coupled rigid-body dynamics (§§2 to 4) we find that concepts akin to the 'augmented body' (cf. (Wittenburg, 1977)) come out in a natural way. The reduced Poisson structure obtained is a Poisson structure in $\mathbb{R}^3$ (not of Lie–Poisson type, however) whose symplectic leaves are cylinders. The reduced dynamics on one of these cylinders for specific rigid-body

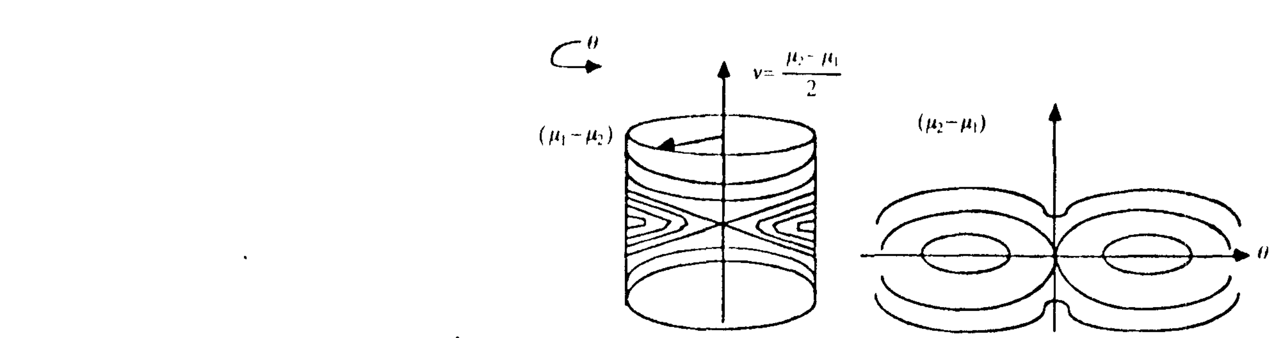


Fig. 1. Phase portrait of a planar two-body system

parameters† is shown in Fig. 1. Here, μ_1, μ_2 are closely related to the angular momenta of the two bodies and θ is the joint angle.

Being two-dimensional and Hamiltonian, the flow on the cylinder is completely integrable. Notice that there are two equilibria, one a saddle and one a stable point. This is confirmed by a linearized analysis for the saddle point and an energy-Casimir analysis for the stable point (see (Holm *et al.*, 1985)). The stable point corresponds to the two bodies uniformly rotating in an extended position, while the saddle point corresponds to uniform rotation in a folded position (Fig. 2). There are, of course, corresponding equilibria for oppositely oriented rotational motions.

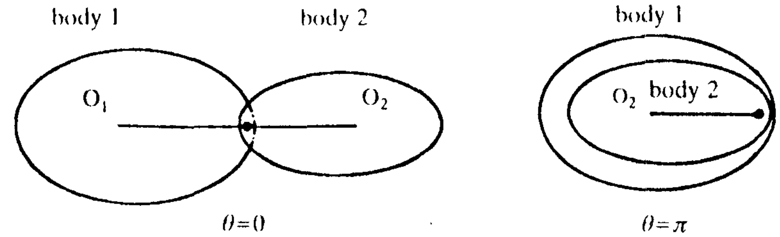
Notice from Fig. 1 that there are two homoclinic orbits from the unstable equilibrium back to itself. Thus one can expect that when, for example, an additional third body is attached nearly at the centre of mass of body 2 or the system is forced (for instance by joint torques), there will be a splitting of these homoclinic orbits resulting in chaotic dynamics. One way to proceed with an analysis of this sort is via the Melnikov method (see (Holmes and Marsden, 1982, 1983; Guckenheimer and Holmes, 1983)). This analysis, together with more information on instability and periodic orbits will be given in part II of this paper.

Another benefit of doing the analysis systematically using the reduction procedure is that the generalization to multibody problems and three-dimensional motion can be made using similar ideas. We discuss the planar multibody case in §6 and the three-dimensional case in another publication.

We now summarize one of the results of the present work; namely we display the Hamiltonian form for the dynamic equations. The details of the derivation of this structure are given in §§2 to 4. Refer to Fig. 3 and define the following quantities.

d_i	distance from the hinge to the centre of mass of body $i = 1, 2$
ω_i	angular velocity of body $i = 1, 2$
θ	joint angle from body 1 to body 2
$\lambda(\theta)$	$d_1 d_2 \cos\theta$
m_i	mass of body $i = 1, 2$
ε	$m_1 m_2/(m_1 + m_2)$ = reduced mass

† The parameters chosen, in the notation of §§2 to 4 are $\bar{I}_1 = 105.55$, $\bar{I}_2 = 70$, $\varepsilon = 55.55$, and $\mu_1 + \mu_2 = 50$.



(a) Stable equilibrium (b) Unstable equilibrium

Fig. 2. Equilibria of a planar two-body system

I_i moment of inertia of body i about its centre of mass
$\bar{I}_i$ $I_i + \varepsilon d_i^2$; $i = 1, 2$ (augmented moments of inertia)
γ $\varepsilon\lambda'/(\bar{I}_1\bar{I}_2 - \varepsilon^2\lambda^2)$

The dynamics of the system is described by the following Euler–Lagrange equations for θ, ω_1, ω_2:

$$\left.\begin{aligned}\dot\theta &= \omega_2 - \omega_1,\\ \dot\omega_1 &= -\gamma(\bar{I}_2\omega_2^2 + \varepsilon\lambda\omega_1^2),\\ \dot\omega_2 &= \gamma(\bar{I}_1\omega_1^2 + \varepsilon\lambda\omega_2^2).\end{aligned}\right\} \tag{1.5}$$

For the Hamiltonian structure it is convenient to introduce the momenta

$$\mu_1 = \bar{I}_1\omega_1 + \varepsilon\lambda\omega_2, \qquad \mu_2 = \bar{I}_2\omega_2 + \varepsilon\lambda\omega_1, \tag{1.6a}$$

that is,

$$\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \mathbf{J}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix}\bar{I}_1 & \varepsilon\lambda\\ \varepsilon\lambda & \bar{I}_2\end{pmatrix} \tag{1.6b}$$

(this is done via the Legendre transform in §4). The evolution equations for μ_i are obtained by solving (1.6) for ω_1, ω_2 and substituting into (1.5). The Hamiltonian is

$$H = \tfrac{1}{2}(\omega_1, \omega_2)\mathbf{J}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, \tag{1.7a}$$

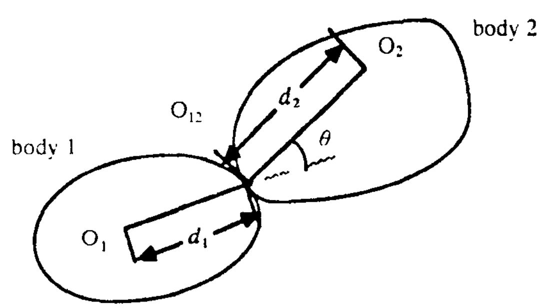


Fig. 3. Planar two-body system

that is,

$$H = \tfrac{1}{2}(\mu_1, \mu_2)\mathbf{J}^{-1}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \tag{1.7b}$$

which is the total kinetic energy for the two bodies. The Poisson structure on the (θ, μ_1, μ_2)-space (called P in §3) is

$$\{F, H\} = \{F, H\}_2 - \{F, H\}_1, \tag{1.8}$$

where

$$\{F, H\}_i = \frac{\partial F}{\partial \theta}\frac{\partial H}{\partial \mu_i} - \frac{\partial H}{\partial \theta}\frac{\partial F}{\partial \mu_i}.$$

The evolution equations (1.5) are then equivalent to Hamilton's equations $\dot{F} = \{F, H\}$. Casimirs for the bracket (1.8) are readily checked to be

$$C = \Phi(\mu_1 + \mu_2) \tag{1.9}$$

for Φ any function of one variable; that is, $\{F, C\} = 0$ for any F. One can also verify directly from (1.5) that, correspondingly, $d\mu/dt = 0$, where $\mu = \mu_1 + \mu_2$ is the system angular momentum.

The symplectic leaves of (1.8) are described by the variables $\nu = (\mu_2 - \mu_1)/2$, θ which parametrize the cylinder shown in Fig. 1. The bracket in terms of (θ, ν) is the canonical one on T^*S^1:

$$\{F, H\} = \frac{\partial F}{\partial \theta}\frac{\partial H}{\partial \nu} - \frac{\partial H}{\partial \theta}\frac{\partial F}{\partial \nu}. \tag{1.10}$$

As we shall see, this canonical structure on T^*S^1 is consistent with the Satzer–Marsden–Kummer cotangent bundle reduction theorem (Abraham and Marsden, 1978; Kummer, 1981).

2. Kinematical preliminaries (for two coupled planar rigid bodies)

In this section we set up the phase space for the dynamics of our problem. Refer to Fig. 4 and define the following quantities.

$\mathbf{d}_{12}$	the vector from the centre of mass of body 1 to the hinge point in a reference configuration (fixed)
$\mathbf{d}_{21}$	the vector from the centre of mass of body 2 to the hinge point in a reference configuration (fixed)
$R(\theta_i)$	$\begin{pmatrix}\cos\theta_i & -\sin\theta_i\\ \sin\theta_i & \cos\theta_i\end{pmatrix}$ the rotation through angle θ_i giving the current orientation of body i (written as a matrix relative to the fixed standard inertial frame)
$\mathbf{r}_i$	current position of the centre of mass of body i
$\mathbf{r}$	current position of the system centre of mass
$\mathbf{r}_i^0$	the vector from the system centre of mass to the centre of mass of body i
θ	$\theta_2 - \theta_1$ joint angle
$R(\theta)$	joint rotation, $R(\theta_2) \cdot R(-\theta_1)$

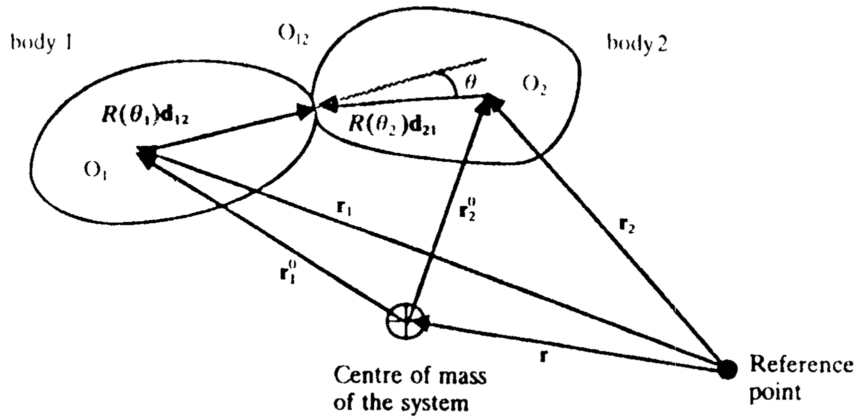


Fig. 4. Planar two-body system in the reference frame

The basic configuration space we start with is Q, the subset of $SE(2) \times SE(2)$ (two copies of the special Euclidean group of the plane) consisting of pairs $((R(\theta_1), \mathbf{r}_1), (R(\theta_2), \mathbf{r}_2))$ satisfying the *hinge constraint*

$$\mathbf{r}_2 = \mathbf{r}_1 + R(\theta_1)\mathbf{d}_{12} - R(\theta_2)\mathbf{d}_{21}. \tag{2.1}$$

Notice that Q is of dimension 4 and is parametrized by θ_1, θ_2 and, say $\mathbf{r}_1$; that is, $Q \approx S^1 \times S^1 \times \mathbb{R}^2$. We form the velocity phase space TQ and momentum phase space T^*Q.

The Lagrangian on TQ is just the kinetic energy (relative to the inertial frame) given by summing the kinetic energies of each body. For convenience, we recall how this proceeds: let $\mathbf{X}_1$ denote a position vector in body 1 relative to the centre of mass of body 1, and let $\rho_1(\mathbf{X}_1)$ denote the mass density of body 1. Then the current position of the point with material label $\mathbf{X}_1$ is

$$\mathbf{x}_1 = R(\theta_1)\mathbf{X}_1 + \mathbf{r}_1. \tag{2.2}$$

Thus

$$\dot{\mathbf{x}}_1 = \dot{R}(\theta_1)\mathbf{X}_1 + \dot{\mathbf{r}}_1,$$

and so the kinetic energy of body 1 is

$$\begin{aligned} K_1 &= \frac{1}{2}\int \rho_1(\mathbf{X}_1)\, \|\dot{\mathbf{x}}_1\|^2\, d^2\mathbf{X}_1 \\ &= \frac{1}{2}\int \rho_1(\mathbf{X}_1)\langle \dot{R}\mathbf{X}_1 + \dot{\mathbf{r}}_1, \dot{R}\mathbf{X}_1 + \dot{\mathbf{r}}_1\rangle\, d^2\mathbf{X}_1 \\ &= \frac{1}{2}\int \rho_1(\mathbf{X}_1)[\langle \dot{R}\mathbf{X}_1, \dot{R}\mathbf{X}_1\rangle + 2\langle \dot{R}\mathbf{X}_1, \dot{\mathbf{r}}_1\rangle + \|\dot{\mathbf{r}}_1\|^2]\, d^2\mathbf{X}_1. \end{aligned} \tag{2.3}$$

But

$$\langle \dot{R}\mathbf{X}_1, \dot{R}\mathbf{X}_1\rangle = \operatorname{tr}(\dot{R}\mathbf{X}_1(\dot{R}\mathbf{X}_1)^{\mathrm{T}}) = \operatorname{tr}(\dot{R}\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_1\dot{R}^{\mathrm{T}}) \tag{2.4}$$

and

$$\int \rho_1(\mathbf{X}_1)\langle \dot{R}\mathbf{X}_1, \dot{\mathbf{r}}_1\rangle\, d^2\mathbf{X}_1 = \left\langle \dot{R}\int \rho_1(\mathbf{X}_1)\mathbf{X}_1\, d^2\mathbf{X}_1, \dot{\mathbf{r}}_1\right\rangle = 0 \tag{2.5}$$

since $\mathbf{X}_1$ is the vector relative to the center of mass of body 1. Substituting (2.4) and (2.5) into (2.3) and defining the matrix

$$\mathbf{I}^1 = \int \rho(\mathbf{X}_1)\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\, d^2\mathbf{X}_1 \tag{2.6}$$

we get

$$K_1 = \tfrac{1}{2}\operatorname{tr}(\dot{R}(\theta_1)\mathbf{I}^1\dot{R}(\theta_1)^{\mathrm{T}}) + \tfrac{1}{2}m_1\|\dot{\mathbf{r}}_1\|^2; \tag{2.7}$$

with a similar expression for K_2 we let

$$L: TQ \to \mathbb{R} \quad \text{be} \quad L = K_1 + K_2. \tag{2.8}$$

The equations of motion then are the Euler–Lagrange equations for this L on TQ. Equivalently, they are Hamilton's equations for the corresponding Hamiltonian.

For later convenience, we shall rewrite the energy (2.8) in terms of $\omega_1 = \dot{\theta}_1$, $\omega_2 = \dot{\theta}_2$, $\mathbf{r}_1^0$ and $\mathbf{r}_2^0$. To do this note that, by definition,

$$m\mathbf{r} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2, \tag{2.9}$$

where $m = m_1 + m_2$, and so, as $\mathbf{r}_1 = \mathbf{r} + \mathbf{r}_1^0$,

$$0 = m_1\mathbf{r}_1^0 + m_2\mathbf{r}_2^0 \tag{2.10}$$

and, subtracting $\mathbf{r}$ from both sides of (2.1),

$$\mathbf{r}_2^0 = \mathbf{r}_1^0 + R(\theta_1)\mathbf{d}_{12} - R(\theta_2)\mathbf{d}_{21}. \tag{2.11}$$

From (2.10) and (2.11) we find that

$$\mathbf{r}_2^0 = \frac{m_1}{m}(R(\theta_1)\mathbf{d}_{12} - R(\theta_2)\mathbf{d}_{21}) \tag{2.12a}$$

and

$$\mathbf{r}_1^0 = -\frac{m_2}{m}(R(\theta_1)\mathbf{d}_{12} - R(\theta_2)\mathbf{d}_{21}). \tag{2.12b}$$

Now we substitute

$$\mathbf{r}_1 = \mathbf{r} + \mathbf{r}_1^0 \quad \text{so} \quad \dot{\mathbf{r}}_1 = \dot{\mathbf{r}} + \dot{\mathbf{r}}_1^0 \tag{2.13a}$$

and

$$\mathbf{r}_2 = \mathbf{r} + \mathbf{r}_2^0 \quad \text{so} \quad \dot{\mathbf{r}}_2 = \dot{\mathbf{r}} + \dot{\mathbf{r}}_2^0 \tag{2.13b}$$

into (2.8) to give

$$L = \tfrac{1}{2}\operatorname{tr}(\dot{R}(\theta_1)\mathbf{I}^1\dot{R}(\theta_1)^{\mathrm{T}} + \dot{R}(\theta_2)\mathbf{I}^2\dot{R}(\theta_2)^{\mathrm{T}}) + \tfrac{1}{2}[m_1(\|\dot{\mathbf{r}} + \dot{\mathbf{r}}_1^0\|^2) + m_2\|\dot{\mathbf{r}} + \dot{\mathbf{r}}_2^0\|^2]. \tag{2.14}$$

But $m_1\langle\dot{\mathbf{r}}, \dot{\mathbf{r}}_1^0\rangle + m_2\langle\dot{\mathbf{r}}, \dot{\mathbf{r}}_2^0\rangle = 0$ since $m_1\dot{\mathbf{r}}_1^0 + m_2\dot{\mathbf{r}}_2^0 = 0$ from (2.10). Thus (2.14) simplifies to

$$\begin{aligned} L = {} & \tfrac{1}{2}\operatorname{tr}(\dot{R}(\theta_1)\mathbf{I}^1\dot{R}(\theta_1)^{\mathrm{T}} + \dot{R}(\theta_2)\mathbf{I}^2\dot{R}(\theta_2)^{\mathrm{T}}) \\ & + (p^2/2m) + \tfrac{1}{2}m_1\|\dot{\mathbf{r}}_1^0\|^2 + \tfrac{1}{2}m_2\|\dot{\mathbf{r}}_2^0\|^2, \end{aligned} \tag{2.15}$$

where $p = m\|\dot{\mathbf{r}}\|$ is the magnitude of the system momentum.

Now write

$$\dot R(\theta_1)=\frac{d}{dt}\begin{pmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}$$

$$=\begin{pmatrix}-\sin\theta_1 & -\cos\theta_1\\ \cos\theta_1 & -\sin\theta_1\end{pmatrix}\omega_1:=R(\theta_1)\begin{pmatrix}0 & -\omega_1\\ \omega_1 & 0\end{pmatrix}=R(\theta_1)\hat\omega_1, \qquad (2.16)$$

so that (2.12) gives

$$\dot{\mathbf r}_2^0=\frac{m_1}{m}(R(\theta_1)\hat\omega_1\mathbf d_{12}-R(\theta_2)\hat\omega_2\mathbf d_{21}), \qquad \dot{\mathbf r}_1^0=-\frac{m_2}{m}(R(\theta_1)\hat\omega_1\mathbf d_{12}-R(\theta_2)\hat\omega_2\mathbf d_{21}). \qquad (2.17)$$

Substituting (2.17) and (2.16) into (2.15) gives

$$L=\tfrac12\operatorname{tr}(\hat\omega_1\mathbf I^1\hat\omega_1^{\mathrm T})+\hat\omega_2\mathbf I^2\hat\omega_2^{\mathrm T}))+\frac{p^2}{2m}+\frac{m_1m_2}{m}\|\hat\omega_1\mathbf d_{12}-R(\theta_2-\theta_1)\hat\omega_2\mathbf d_{21}\|^2. \qquad (2.18)$$

Finally we note that

$$\tfrac12\operatorname{tr}(\hat\omega_1\mathbf I^1\hat\omega_1^{\mathrm T})=\tfrac12\operatorname{tr}(\hat\omega_1^{\mathrm T}\hat\omega_1\mathbf I^1)=\tfrac12\operatorname{tr}\left(\begin{pmatrix}\omega_1^2 & 0\\ 0 & \omega_1^2\end{pmatrix}\mathbf I^1\right)=\omega_1^2\operatorname{tr}\mathbf I^1:=\omega_1^2 I_1, \qquad (2.19.1)$$

where

$$I_1=\int\rho(X_1,Y_1)(X_1^2+Y_1^2)\,dX_1\,dY_1$$

is the usual moment of inertia of body 1 about its centre of mass. One similarly derives (2.19.2) where 1 is replaced by 2 throughout. The final term in (2.18) is manipulated as follows:

$$\begin{aligned}\|\hat\omega_1\mathbf d_{12}-R(\theta)\hat\omega_2\mathbf d_{21}\|^2&=\|\hat\omega_1\mathbf d_{12}\|^2-2\langle\hat\omega_1\mathbf d_{12},R(\theta)\hat\omega_2\mathbf d_{21}\rangle+\|\hat\omega_2\mathbf d_{21}\|^2\\&=\omega_1^2d_1^2+\omega_2^2d_2^2-2\langle\hat\omega_1\mathbf d_{12},\hat\omega_2R(\theta)\mathbf d_{21}\rangle\\&=\omega_1^2d_1^2+\omega_2^2d_2^2-2\omega_1\omega_2\langle\mathbf d_{12},R(\theta)\mathbf d_{21}\rangle.\end{aligned} \qquad (2.20)$$

Substituting (2.19.1), (2.19.2) and (2.20) into (2.18) gives

$$L=\tfrac12[(\omega_1^2\bar I_1+\omega_2^2\bar I_2)+2\omega_1\omega_2\varepsilon\lambda(\theta)]+\frac{p^2}{2m}, \qquad (2.21)$$

where

$$\lambda(\theta)=-\langle\mathbf d_{12},R(\theta)\mathbf d_{21}\rangle=-[\mathbf d_{12}\cdot\mathbf d_{21}\cos\theta-(\mathbf d_{12}\times\mathbf d_{21})\cdot\hat Z\sin\theta]. \qquad (2.22)$$

Remarks 1. If $\mathbf d_{12}$ and $\mathbf d_{21}$ are parallel (that is, the reference configuration is chosen with $\mathbf d_{12}$ and $\mathbf d_{21}$ aligned), then (2.22) gives $\lambda(\theta)=d_1d_2\cos\theta$, as in §1.

2. The quantities $\bar I_1$, $\bar I_2$ are the moments of inertia of 'augmented' bodies as defined in §1; for example $\bar I_1$ is the moment of inertia of body 1 augmented by putting a mass ε at the hinge point.

3. Reduction to the centre of mass frame

In this section we reduce the dynamics by the action of the translation group $\mathbb R^2$. This group acts on the original configuration space Q by

$$\mathbf v\cdot((R(\theta_1),\mathbf r_1),(R(\theta_2),\mathbf r_2))=((R(\theta_1),\mathbf r_1+\mathbf v),(R(\theta_2),\mathbf r_2+\mathbf v)). \qquad (3.1)$$

This is well defined since the hinge constraint (2.1) is preserved by this action. The induced momentum map on TQ is calculated by the standard formula

$$J_\xi = \frac{\partial L}{\partial \dot{q}_i} \xi^i_Q(q), \tag{3.2a}$$

or on T^*Q by

$$J_\xi = p_i \xi^i_Q(q), \tag{3.2b}$$

where ξ^i_Q is the infinitesimal generator of the action on Q (see (Abraham and Marsden, 1978)). To implement (3.2) we parametrize Q by θ_1, θ_2 and $\mathbf{r}$ with $\mathbf{r}_1$ and $\mathbf{r}_2$ determined by (2.12) and (2.13). From (2.15) we see that the momentum conjugate to $\mathbf{r}$ is

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} \tag{3.3}$$

and so (3.2) gives

$$J_\xi = \langle \mathbf{p}, \xi \rangle, \qquad \xi \in \mathbb{R}^2. \tag{3.4}$$

Thus $J = \mathbf{p}$ is conserved since H is cyclic in $\mathbf{r}$ and so H is translation invariant. The corresponding reduced space is obtained by fixing $\mathbf{p} = \mathbf{p}_0$ and letting

$$P_{p_0} = \mathbf{J}^{-1}(\mathbf{p}_0)/\mathbb{R}^2$$

(see (Abraham and Marsden, 1978, Chapter 4)). But P_{p_0} is clearly isomorphic to $T^*(S^1 \times S^1)$, that is, to the space of θ_1, θ_2 and their conjugate momenta. The reduced Hamiltonian is simply the Hamiltonian corresponding to (2.21) with p regarded as a constant.

Note that in this case the reduced symplectic manifold is a cotangent bundle, in agreement with the cotangent-bundle reduction theorem (Abraham and Marsden, 1978; Kummer, 1981). The reduced phase space has the *canonical* symplectic form; one can also check this directly here.

In (2.21) we can adjust L by a constant and thus assume that $p = 0$; this obviously does not affect the equations of motion.

Let us observe that the reduced system is given by geodesic flow on $S^1 \times S^1$ since (2.21) is quadratic in the velocities. Indeed the metric tensor is just the matrix $\mathbf{J}$ given by (1.6), so the conjugate momenta are μ_1, μ_2 given by (1.6).

We remark, finally, that the reduction to centre-of-mass coordinates here is somewhat simpler and more symmetric than the Jacobi–Haretu reduction to centre-of-mass coordinates for n point masses. (Just taking the positions relative to the centre of mass does not achieve this since this does not reduce the dimension at all!) What is different here is that the two bodies are hinged, and so by (2.12), $\mathbf{r}_1^0$ and $\mathbf{r}_2^0$ are determined by the other data.

4. Reduction by rotations

To complete the reduction, we reduce by the diagonal action of S^1 on the configuration space $S^1 \times S^1$ that was obtained in §3. The momentum map for this action is obviously given by

$$J((\theta_1, \mu_1), (\theta_2, \mu_2)) = \mu_1 + \mu_2. \tag{4.1}$$

For purposes of later stability calculations, we shall find it convenient to form the Poisson reduced space

$$P := T^*(S^1 \times S^1)/S^1 \tag{4.2}$$

whose symplectic leaves are the reduced symplectic manifolds

$$P_\mu = J^{-1}(\mu)/S^1 \subset P.$$

We coordinatize P by $\theta = \theta_2 - \theta_1$, μ_1 and μ_2; topologically, $P = S^1 \times \mathbb{R}^2$. The Poisson structure on P is computed in the standard way: take two functions $F(\theta, \mu_1, \mu_2)$ and $H(\theta, \mu_1, \mu_2)$. Regard them as functions of θ_1, θ_2, μ_1, μ_2 by substituting $\theta = \theta_2 - \theta_1$ and compute the canonical bracket. It is clear that the asserted bracket (1.8) is what results. The Casimirs on P are obtained by composing J with Casimirs on the dual of the Lie algebra of S^1; that is, with arbitrary functions of one variable; thus (1.9) results. This can of course be checked directly.

If we parametrize P_μ by θ and $\nu = \frac{1}{2}(\mu_2 - \mu_1)$, then the Poisson bracket on P_μ becomes the canonical one. This, again, is consistent with the cotangent-bundle reduction theorem which asserts in this case that the reduction of $T^*(S^1 \times S^1)$ by S^1 is symplectically diffeomorphic to $T^*((S^1 \times S^1)/S^1) \cong T^*S^1$. There are no 'magnetic' terms since the reduced configuration space S^1 is one-dimensional, and hence has no non-zero two-forms.

The realization of P_μ as T^*S^1 is not unique. For example we can parametrize P_μ by (θ_2, μ_2) or by (θ_1, μ_1), each of which also gives the canonical bracket. (In the general theory there can be more than one one-form 'α_μ' by which one embeds P_μ into T^*S^1, as well as more than one way to identify $(S^1 \times S^1)/S^1 \cong S^1$. The three listed above correspond to three such choices of α_μ.)

Remark The reduced bracket on $T^*(S^1 \times S^1)/S^1$ can also be obtained from the general formula for the bracket on $(P \times T^*G)/G \cong P \times g^*$ found in (Krishnaprasad and Marsden, 1987); it produces one of the variants above, depending on whether we take G to be parametrized by θ_1 or θ_2, or $\theta_2 - \theta_1$.

The reduced Hamiltonian on P is just (1.7b) regarded as a function of μ_1, μ_2 and θ. We therefore know that the Euler–Lagrange equations (1.5) are equivalent to $\dot{F} = \{F, H\}$ for the reduced bracket (1.8).

We can also obtain a Hamiltonian system on the leaves, parametrized by say (θ, ν). We simply take (1.7b), namely

$$H = \frac{1}{2\Delta}(\mu_1, \mu_2)\begin{pmatrix} \bar{I}_2 & -\varepsilon\lambda \\ -\varepsilon\lambda & \bar{I}_1 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \tag{4.3}$$

where $\Delta = \bar{I}_1\bar{I}_2 - \varepsilon^2\lambda^2$, and substitute $\mu_1 = \frac{1}{2}\mu - \nu$, $\mu_2 = \nu + \frac{1}{2}\mu$ producing

$$H = \frac{1}{2\Delta}(\bar{I}_1 + \bar{I}_2 + 2\varepsilon\lambda)\nu^2 + \frac{1}{2\Delta}[(\bar{I}_1 - \bar{I}_2)\mu]\nu + \frac{1}{2\Delta}(\tfrac{1}{4}\mu^2(\bar{I}_1 + \bar{I}_2 - 2\varepsilon\lambda)). \tag{4.4}$$

The presence of the linear term in ν can be eliminated by completion of squares: it is not there in the general theory (Abraham and Marsden, 1978; Smale, 1970) because reduced coordinates adapted to the metric of the kinetic energy are used; these are produced by the completion of squares. Notice that the Hamiltonian now is the form of kinetic plus potential energy but that the metric now on S^1

is θ-dependent and, unless d_1 or d_2 vanishes, it is a non-trivial dependence. The potential piece is usually referred to as the *amended potential.*

We summarize as follows.

Theorem 1 *The reduced phase space for two coupled planar rigid bodies is the three-dimensional Poisson manifold $P = S^1 \times \mathbb{R}$ with the bracket* (1.8)*; its symplectic leaves are the cylinders with canonical variables* (θ, v). *Casimirs are given by* (1.9).

The reduced dynamics are given by $\dot{F} = \{F, H\}$ or, equivalently,

$$\dot{\theta} = \frac{\partial H}{\partial \mu_2} - \frac{\partial H}{\partial \mu_1}, \qquad \dot{\mu}_1 = \frac{\partial H}{\partial \theta}, \qquad \dot{\mu}_2 = -\frac{\partial H}{\partial \theta}, \tag{4.5}$$

where H is given by (1.7b). *The equivalent dynamics on the leaves is given by*

$$\frac{\partial \theta}{\partial t} = \frac{\partial H}{\partial v}, \qquad \frac{\partial v}{\partial t} = -\frac{\partial H}{\partial \theta}, \tag{4.6}$$

where H is given by (4.4).

5. Equilibria and stability by the energy-Casimir method

We now use Arnold's energy-Casimir method, as summarized in (Holm *et al.*, 1985; Krishnaprasad and Marsden, 1987) to determine the equilibrium points and their stability. An equivalent alternative to this method is to look for critical points of H given by (4.4) in (θ, v)-space and test d^2H for definiteness at these equilibria.

To search for equilibria we look directly at Hamilton's equations on P. Using the bracket (1.8) and $\dot{F} = \{F, H\}$, we obtain equations (4.5), where H is given by (1.7b). The conditions $\dot{\mu}_1 = \dot{\mu}_2 = 0$ become

$$\partial H / \partial \theta = 0; \tag{5.1a}$$

that is,

$$-\tfrac{1}{2}(\mu_1, \mu_2)\mathbf{J}^{-1}\frac{d\mathbf{J}}{d\theta}\mathbf{J}^{-1}\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = 0. \tag{5.1b}$$

Clearly

$$\frac{d\mathbf{J}}{d\theta} = \begin{pmatrix} 0 & \varepsilon\lambda' \\ \varepsilon\lambda' & 0 \end{pmatrix} \tag{5.2}$$

from (1.6), so (5.2) becomes

$$-\tfrac{1}{2}(\omega_1, \omega_2)\begin{pmatrix} 0 & \varepsilon\lambda' \\ \varepsilon\lambda' & 0 \end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = 0; \tag{5.3}$$

that is,

$$-\varepsilon\lambda'\omega_1\omega_2 = 0. \tag{5.4}$$

The equilibrium condition $\dot{\theta} = 0$ becomes $\bar{I}_1\mu_1 - \varepsilon\lambda\mu_2 = \bar{I}_2\mu_2 - \varepsilon\lambda\mu_1$ or, equivalently, $\omega_1 = \omega_2$.

Thus, the equilibria are given by

(i) $\omega_1 = \omega_2 = 0$, or

(ii) $\omega_1 = \omega_2 \neq 0$, $\lambda' = 0$.

Let us, for simplicity, choose our reference configuration so that $\mathbf{d}_{12}$ and $\mathbf{d}_{21}$ are parallel. Then

$$\lambda'(\theta) = \mathbf{d}_{12} \cdot \mathbf{d}_{21} \sin \theta$$

so the equilibria in case (ii) occur when

(ii)′ either (a) $\mathbf{d}_{12} = 0$ or $\mathbf{d}_{21} = 0$, or (b) $\theta = 0$ or π.

The case in which $\theta = \pi$ corresponds to the case of folded bodies, while $\theta = 0$ corresponds to extended bodies.

The first step in the energy-Casimir method is to realize the equilibria as critical points of $H + C$, where H is given by (1.7b) and $C = \Phi(\mu_1 + \mu_2)$.

One calculates from (5.2) and (1.7) that

$$\left.\begin{aligned} &\frac{\partial H}{\partial \theta} = \varepsilon \lambda' \omega_1 \omega_2, \\ &\frac{\partial H}{\partial \mu_1} = \omega_1; \qquad \frac{\partial H}{\partial \mu_2} = \omega_2, \end{aligned}\right\} \tag{5.5}$$

where

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \tilde{I}_2 \mu_1 - \varepsilon \lambda \mu_2 \\ \tilde{I}_1 \mu_2 - \varepsilon \lambda \mu_1 \end{pmatrix}.$$

The first variation is

$$d(H + C) = \frac{\partial H}{\partial \theta} d\theta + \left(\frac{\partial H}{\partial \mu_1} + \Phi' \right) d\mu_1 + \left(\frac{\partial H}{\partial \mu_2} + \Phi' \right) d\mu_2, \tag{5.6}$$

from which it is clear that critical points of $H + C$ correspond to equilibria of (4.5) with

$$\Phi'(\mu_e) = -\left(\frac{\partial H}{\partial \mu_1} \right)_e = -\left(\frac{\partial H}{\partial \mu_2} \right)_e, \tag{5.7}$$

where the subscript e means evaluation at the equilibrium. As in other examples (the rigid body and heavy top in (Holm *et al.*, 1985)), $\Phi''(\mu_e)$ is arbitrary.

The matrix of the second variation is

$$\delta^2(H + C) = \begin{pmatrix} \dfrac{\partial^2 H}{\partial \theta^2} & \dfrac{\partial^2 H}{\partial \theta\, \partial \mu_1} & \dfrac{\partial^2 H}{\partial \theta\, \partial \mu_2} \\ \dfrac{\partial^2 H}{\partial \theta\, \partial \mu_1} & \dfrac{\partial^2 H}{\partial \mu_1^2} + \Phi'' & \dfrac{\partial^2 H}{\partial \mu_1\, \partial \mu_2} + \Phi'' \\ \dfrac{\partial^2 H}{\partial \theta\, \partial \mu_2} & \dfrac{\partial^2 H}{\partial \mu_1\, \partial \mu_2} + \Phi'' & \dfrac{\partial^2 H}{\partial \mu_2^2} + \Phi'' \end{pmatrix}, \tag{5.8}$$

where

$$\begin{pmatrix} \dfrac{\partial^2 H}{\mu_1^2} & \dfrac{\partial^2 H}{\partial\mu_1\,\partial\mu_2} \\ \dfrac{\partial^2 H}{\partial\mu_1\,\partial\mu_2} & \dfrac{\partial^2 H}{\partial\mu_2^2} \end{pmatrix} = \mathbf{J}^{-1} = \frac{1}{\Delta}\begin{pmatrix} \bar{I}_2 & -\varepsilon\lambda \\ -\varepsilon\lambda & \bar{I}_1 \end{pmatrix},$$

$$\frac{\partial^2 H}{\partial\theta\,\partial\mu_1} = -\frac{\varepsilon\lambda'}{\Delta^2}(\bar{I}_2\omega_2 - \varepsilon\lambda\omega_1), \qquad \frac{\partial^2 H}{\partial\theta\,\partial\mu_2} = -\frac{\varepsilon\lambda'}{\Delta^2}(-\varepsilon\lambda\omega_2 + \bar{I}_1\omega_1),$$

and

$$\frac{\partial^2 H}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left[-\varepsilon\lambda'\frac{\partial H}{\partial\mu_1}\frac{\partial H}{\partial\mu_2}\right] = -\varepsilon\lambda''\omega_1\omega_2 - \varepsilon\lambda'\frac{\partial^2 H}{\partial\theta\,\partial\mu_1}\omega_2 - \varepsilon\lambda'\omega_1\frac{\partial^2 H}{\partial\theta\,\mu_2}.$$

At equilibrium, $\lambda = \pm d_1 d_2$ ($+$ if $\theta = 0$, $-$ if $\theta = \pi$) so

$$\mathbf{J}^{-1} = \frac{1}{(\bar{I}_1\bar{I}_2 - \varepsilon^2 d_1^2 d_2^2)}\begin{pmatrix} \bar{I}_2 & \mp\varepsilon d_1 d_2 \\ \mp\varepsilon d_1 d_2 & \bar{I}_1 \end{pmatrix},$$

$$\frac{\partial^2 H}{\partial\theta\,\partial\mu_1} = 0 = \frac{\partial^2 H}{\partial\theta\mu_2},$$

and

$$\frac{\partial^2 H}{\partial\theta^2} = -\varepsilon\lambda''\omega_e^2 = \pm\varepsilon d_1 d_2\omega_e^2,$$

where $\omega_e = \omega_1 = \omega_2 \neq 0$ at equilibrium. Thus (5.8) becomes

$$\delta^2(H + C) = \begin{pmatrix} \pm\varepsilon\lambda d_1 d_2\omega_e^2 & 0 \\ 0 & \mathbf{J}^{-1} + \Phi''\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{pmatrix}. \tag{5.9}$$

This matrix is clearly positive definite if $d_1 \neq 0$, $d_2 \neq 0$ if $\theta = 0$ ($+$sign) and $\Phi''(\mu_e) \geq 0$ and is indefinite for any choice of $\Phi''(\mu_e)$ if $\theta = \pi$.

Another way to do the stability analysis is to use the reduced Hamiltonian on T^*S^1 given by equation (4.4). After completing squares, H will have the form of kinetic plus potential energy with effective potential given by

$$V(\theta) = \frac{1}{2\Delta}\left[\tfrac{1}{4}\mu^2(\bar{I}_1 + \bar{I}_2 - 2\varepsilon\lambda) + \frac{(\bar{I}_1 - \bar{I}_2)^2\mu^2}{4(\bar{I}_1 + \bar{I}_2 + 2\varepsilon\lambda)}\right]. \tag{5.10}$$

Minima of V are then the stable equilibria while maxima are unstable.

For three or more bodies, this method of looking for minima of the potential will not work because the symplectic structures on the symplectic leaves will have magnetic terms.

Theorem 2 *The dynamics of the 2-body problem is completely integrable and contains one stable relative equilibrium solution ($\theta = 0$—the stretched-out case) and one unstable relative equilibrium solution ($\theta = \pi$—the folded-over case). The dynamics contain a homoclinic orbit, as in Fig. 1.*

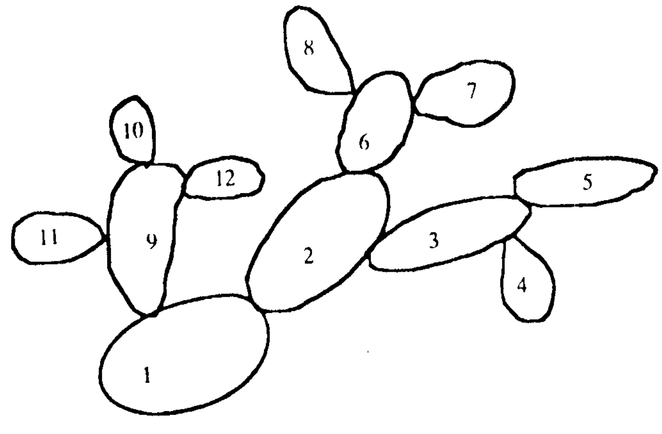


Fig. 5. Planar multi-body system—tree case

6. Multibody problems

We have proved that the Hamiltonian formulation of the previous sections extends in a natural way to systems of N planar rigid bodies connected to form a *tree structure* (Fig. 5). Since the general statement of this result requires significant additional notation and the explicit introduction of the notion of *nested bodies*, we limit ourselves to the special case of a chain of N bodies (Fig. 6).

Theorem 3 *The total kinetic energy (Hamiltonian) for an open chain of N planar rigid bodies connected together by hinge joints takes the form*

$$H = \boldsymbol{\mu}^{\mathrm{T}} \cdot \mathbf{J}^{-1} \cdot \boldsymbol{\mu} \tag{6.1}$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_N)^{\mathrm{T}}$ *is the momentum vector and* $\mathbf{J}$ *is the corresponding* $N \times N$ *pseudo-inertia matrix which is a function of the set of relative (or joint)*

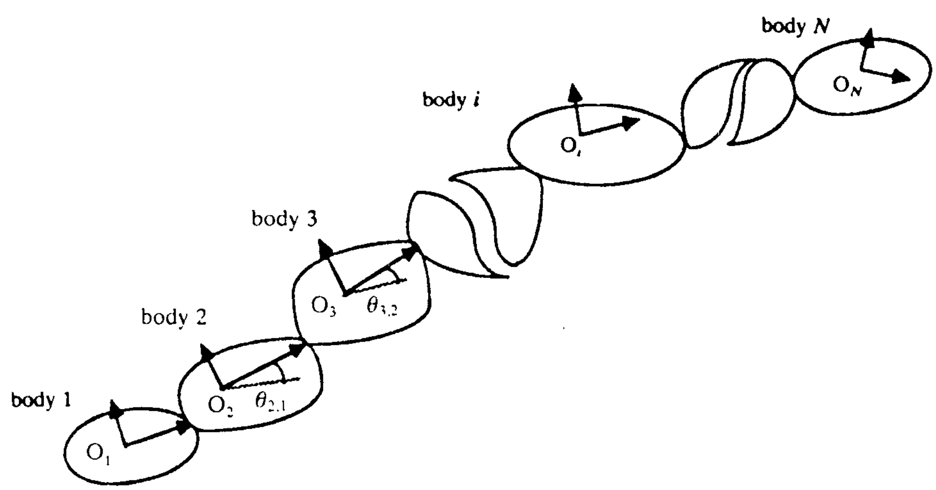


Fig. 6. Planar multi-body system—chain case

angles between adjacent bodies. The reduced dynamics takes the form

$$\left.\begin{aligned}
\dot{\mu}_1 &= \frac{\partial H}{\partial \theta_{2,1}}, \\
\dot{\mu}_2 &= \frac{\partial H}{\partial \theta_{3,2}} - \frac{\partial H}{\partial \theta_{2,1}}, \\
&\dots, \\
\dot{\mu}_i &= \frac{\partial H}{\partial \theta_{i+1,i}} - \frac{\partial H}{\partial \theta_{i,i-1}}, \\
&\dots, \\
\dot{\mu}_N &= -\frac{\partial H}{\partial \theta_{N,N-1}}, \\
\dot{\theta}_{i+1,i} &= \frac{\partial H}{\partial \mu_{i+1}} - \frac{\partial H}{\partial \mu_i} \qquad \text{for } i = 1, \dots, N-1,
\end{aligned}\right\} \tag{6.2}$$

where $\theta_{i+1,i}$ is the joint angle between body $i+1$ and body i.

The associated Poisson structure is given by

$$\{f, g\} = \sum_{i=1}^{N-1} \left(\frac{\partial f}{\partial \mu_i} - \frac{\partial f}{\partial \mu_{i+1}}\right) \frac{\partial g}{\partial \theta_{i+1,i}} - \frac{\partial f}{\partial \theta_{i+1,i}} \left(\frac{\partial g}{\partial \mu_i} - \frac{\partial g}{\partial \mu_{i+1}}\right). \tag{6.3}$$

This is proved in a way similar to the two-body case (see Sreenath, Krishnaprasad and Marsden, 1987)).

The structure of equilibria and the associated stability analysis become quite complex and interesting as the number of interconnected bodies increases. A mixture of topological and geometric methods may be necessary to extract useful information on the phase portraits.

In the remainder of this section, we illustrate some of the complexities of multibody problems by giving an analysis of the equilibria and stability for a system of three planar rigid bodies connected by hinge joints (see Fig. 7).

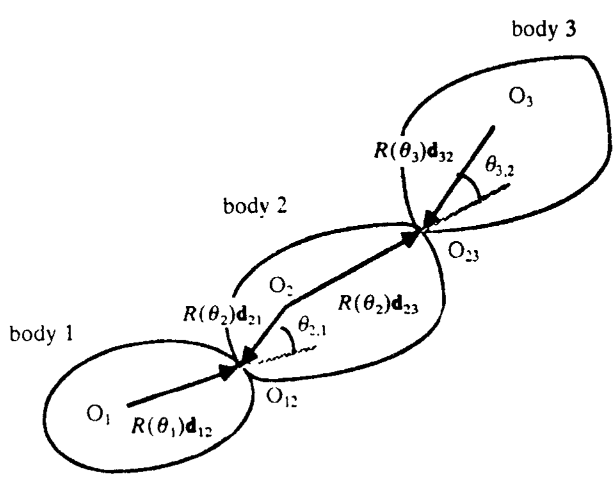


Fig. 7. Planar three-body system

6.1 Three-body problem

The Hamiltonian of the planar three-body problem is given by equation (6.1) with the momentum vector $\boldsymbol{\mu}$ and the coefficient of inertia matrix $\mathbf{J}$ being defined as below:

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T,$$
$$\mathbf{J} = \begin{pmatrix} \bar{I}_1 & \bar{\lambda}_{12}(\theta_{2,1}) & \bar{\lambda}_{31}(\theta_{2,1} + \theta_{3,2}) \\ \bar{\lambda}_{12}(\theta_{2,1}) & \bar{I}_2 & \bar{\lambda}_{23}(\theta_{3,2}) \\ \bar{\lambda}_{31}(\theta_{2,1} + \theta_{3,2}) & \bar{\lambda}_{23}(\theta_{3,2}) & \bar{I}_3 \end{pmatrix}, \tag{6.4}$$

where the $\bar{I}$ and $\bar{\lambda}$ are defined later. Here $\theta_{2,1}$ and $\theta_{3,2}$ are the relative angles between bodies 2 and 1, and bodies 3 and 2, respectively.

The dynamics of a three-body system of planar, rigid bodies in the Hamiltonian setting is given by:

$$\left.\begin{aligned} \dot{\mu}_1 &= \frac{\partial H}{\partial \theta_{2,1}}, \\ \dot{\mu}_2 &= -\frac{\partial H}{\partial \theta_{2,1}} + \frac{\partial H}{\partial \theta_{3,2}}, \\ \dot{\mu}_3 &= -\frac{\partial H}{\partial \theta_{3,2}}, \\ \dot{\theta}_{2,1} &= \frac{\partial H}{\partial \mu_2} - \frac{\partial H}{\partial \mu_1}, \\ \dot{\theta}_{3,2} &= \frac{\partial H}{\partial \mu_3} - \frac{\partial H}{\partial \mu_2}. \end{aligned}\right\} \tag{6.5}$$

Remark The sum $(\mu_1 + \mu_2 + \mu_3)$ of momentum variables is a constant.

Remark The coefficients of inertia $\bar{I}_i$ and $\bar{\lambda}_{ij}$ are given by

$$\begin{aligned} \bar{I}_1 &= [I_1 + (\varepsilon_{12} + \varepsilon_{31})\langle \mathbf{d}_{12}, \mathbf{d}_{12}\rangle], \\ \bar{I}_2 &= [I_2 + \varepsilon_{12}\langle \mathbf{d}_{21}, \mathbf{d}_{21}\rangle + \varepsilon_{23}\langle \mathbf{d}_{23}, \mathbf{d}_{23}\rangle \\ &\quad + \varepsilon_{31}\langle(\mathbf{d}_{23} - \mathbf{d}_{21}), (\mathbf{d}_{23} - \mathbf{d}_{21})\rangle] \\ \bar{I}_3 &= [I_3 + (\varepsilon_{23} + \varepsilon_{31})\langle \mathbf{d}_{32}, \mathbf{d}_{32}\rangle], \\ \bar{\lambda}_{12}(\theta_{2,1}) &= [\varepsilon_{12}\lambda_{(-\mathbf{d}_{21}, \mathbf{d}_{12})}(\theta_{2,1}) + \varepsilon_{31}\lambda_{(\mathbf{d}_{23}-\mathbf{d}_{21}, \mathbf{d}_{12})}(\theta_{2,1}), \\ \bar{\lambda}_{23}(\theta_{3,2}) &= [\varepsilon_{23}\lambda_{(-\mathbf{d}_{32}, \mathbf{d}_{23})}(\theta_{3,2}) + \varepsilon_{31}\lambda_{(-\mathbf{d}_{32}, \mathbf{d}_{23}-\mathbf{d}_{21})}(\theta_{3,2})], \\ \bar{\lambda}_{31}(\theta_{2,1} + \theta_{3,2}) &= \varepsilon_{31}\lambda_{(-\mathbf{d}_{32}, \mathbf{d}_{12})}(\theta_{2,1} + \theta_{3,2}), \\ \varepsilon_{ij} &= \frac{m_i m_j}{m_1 + m_2 + m_3}, \qquad i \neq j \quad \text{and} \quad i, j = 1, 2, 3, \\ \lambda_{(\mathbf{x}, \mathbf{y})}(\alpha) &= \mathbf{x} \cdot \mathbf{y} \cos(\alpha) + [\mathbf{x} \times \mathbf{y}] \sin(\alpha), \end{aligned}$$

where the m_i and I_i are the mass and inertia respectively of the body i, and the $\mathbf{d}_{ij}$ are defined as in Fig. 7.

6.2 Three-body problem: equilibria

Refer to Fig. 7. Let the centres of mass of the bodies O_1, O_2 and O_3 respectively be the origins of the local frames of references also. Let O_{12} be the joint between body 1 and body 2, and let O_{23} be the joint between body 2 and body 3. The local coordinate system for body 1 is chosen such that the x-axis is parallel to the line joining O_1 and O_{12}. Similarly, the coordinate systems for body 2 and body 3 are chosen to be parallel to (a) the line joining O_2 and O_{12}, and (b) the line joining O_3 and O_{23}, respectively. Define the vectors $\mathbf{d}_{12}$, $\mathbf{d}_{21}$, $\mathbf{d}_{23}$, $\mathbf{d}_{32}$, in their respective local coordinate systems to be

$$\mathbf{d}_{12}=[c_1,0],\qquad \mathbf{d}_{21}=[-b_1,0],\qquad \mathbf{d}_{23}=[e_1,e_2],\qquad \mathbf{d}_{32}=[-d_1,0].$$

The equilibria for the three-body system can be found by setting the dynamical equations in (6.5) to be zero. This results in the following equations:

$$\frac{\partial H}{\partial\theta_{2,1}}=\frac{\partial H}{\partial\theta_{3,2}}=0,\qquad \dot\theta_{2,1}=\omega_2-\omega_1=0,\qquad \dot\theta_{3,2}=\omega_3-\omega_2=0. \tag{6.6}$$

From the above equations it can be seen that

$$\omega_1=\omega_2=\omega_3=\omega_0\ \text{(constant)}. \tag{6.7}$$

The system angular momentum μ_s and the Hamiltonian H are given by

$$\mu_s=\omega_0\left[\sum_{i=1}^{3}\tilde I_i+2(\tilde\lambda_{12}(\theta_{2,1})+\tilde\lambda_{23}+\tilde\lambda_{31(\theta_{3,2})}(\theta_{2,1}+\theta_{3,2}))\right], \tag{6.8}$$

$$H=\tfrac12\omega_0^2\left[\sum_{i=1}^{3}\tilde I_i+2(\tilde\lambda_{12}(\theta_{2,1})+\tilde\lambda_{23}(\theta_{3,2})+\tilde\lambda_{31}(\theta_{2,1}+\theta_{3,2}))\right]=\tfrac12\omega_0\mu_s, \tag{6.9}$$

or

$$\omega_0=2H/\mu_s. \tag{6.10}$$

It is a consequence of Theorem 3 and (6.6) that,

$$\begin{aligned}\left[\frac{\partial H}{\partial\theta_{2,1}}\right]_e&=\frac12\frac{\partial}{\partial\theta_{2,1}}\langle\boldsymbol\mu,\mathbf{J}^{-1}\boldsymbol\mu\rangle_e=-\frac12\left\langle\mathbf{J}^{-1}\boldsymbol\mu,\frac{\partial\mathbf{J}}{\partial\theta_{2,1}}\mathbf{J}^{-1}\boldsymbol\mu\right\rangle_e=-\frac12\left\langle\boldsymbol\omega,\frac{\partial\mathbf{J}}{\partial\theta_{2,1}}\boldsymbol\omega\right\rangle_e\\&=-\frac{\omega_0^2}{2(m_1+m_2+m_3)}[A_1\sin(\theta_{2,1}+\theta_{3,2})+B_1\sin(\theta_{2,1})+C_1\cos(\theta_{2,1})]\\&=0\end{aligned}$$

or, for the non-degenerate case ($\omega_0\neq0$),

$$A_1\sin(\theta_{2,1}+\theta_{3,2})+B_1\sin(\partial_{2,1})+C_1\cos(\theta_{2,1})=0, \tag{6.11}$$

where

$$A_1=m_1m_3c_1d_1, \tag{6.12}$$

$$B_1=[m_3(b_1+e_1)+m_2b_1]m_1c_1, \tag{6.13}$$

$$C_1=m_1m_3c_1e_2. \tag{6.14}$$

Similarly, for $\partial H/\partial\theta_{3,2}$ we get

$$\frac{\partial H}{\partial\theta_{3,2}}=\frac{\omega_0^2}{2(m_1+m_2+m_3)}[A_1\sin(\theta_{2,1}+\theta_{3,2})+B_2\sin(\theta_{3,2})+C_2\cos(\theta_{3,2})]=0, \tag{6.15}$$

where

$$B_2 = [m_1(b_1 + e_1) + m_2 e_1]m_3 d_1, \tag{6.16}$$

$$C_2 = (m_1 + m_2)m_3 d_1 e_2. \tag{6.17}$$

We assemble the final equilibrium equations from equations (6.11) and (6.15):

$$\left.\begin{aligned} A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_1 \sin(\theta_{2,1}) + C_1 \cos(\theta_{2,1}) = 0, \\ A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_2 \sin(\theta_{3,2}) + C_2 \cos(\theta_{3,2}) = 0. \end{aligned}\right\} \tag{6.18}$$

6.3 Three-body system: special kinematic case

We consider here a case of the three-body system with a special kinematic structure where the centres of mass of the bodies are aligned with the joints in a straight line when the bodies are in a stretched-out position. In this case we shall prove that equations (6.18) have four or six solutions. For this situation $\mathbf{e} = [e_1, e_2]^T = [e_1, 0]^T$, and so from (6.14) and (6.17)

$$e_2 = 0 \quad \text{implies that} \quad C_1 = C_2 = 0.$$

Thus (6.18) reduces to

$$A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_1 \sin(\theta_{2,1}) = 0, \tag{6.19}$$

$$A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_2 \sin(\theta_{3,2}) = 0, \tag{6.20}$$

with

$$A_1 = c_1 d_1 m_1 m_3, \tag{6.21}$$

$$B_1 = [(b_1 + e_1)m_3 + b_1 m_2]c_1 m_1, \tag{6.22}$$

$$B_2 = [(b_1 + e_1)m_1 + e_1 m_2]d_1 m_3. \tag{6.23}$$

Subtracting (6.19) from (6.20) we get

$$\sin(\theta_{3,2}) = \kappa \sin(\theta_{2,1}), \tag{6.24}$$

where

$$\kappa = B_1/B_2. \tag{6.25}$$

Expanding (6.19) and substituting (6.24), we get

$$A_1 \sin(\theta_{2,1})[\cos(\theta_{3,2}) + \kappa \cos(\theta_{2,1}) + \tau] = 0, \tag{6.26}$$

where

$$\tau = B_1/A_1. \tag{6.27}$$

Consequently from (6.24) and (6.26) we have

$$\sin(\theta_{2,1}) = 0 \quad \text{and} \quad \sin(\theta_{3,2}) = 0 \tag{6.28}$$

or

$$\sin(\theta_{3,2}) = \kappa \sin(\theta_{2,1}), \tag{6.29}$$

$$\cos(\theta_{3,2}) + \kappa \cos(\theta_{2,1}) + \tau = 0. \tag{6.30}$$

It is obvious from considering (6.28) that the following four roots of the $\{\theta_{2,1}, \theta_{3,2}\}$ pair can be readily identified:

$$\{0, 0\}, \qquad \{0, \pi\}, \qquad \{\pi, 0\}, \qquad \{\pi, \pi\}. \tag{6.31}$$

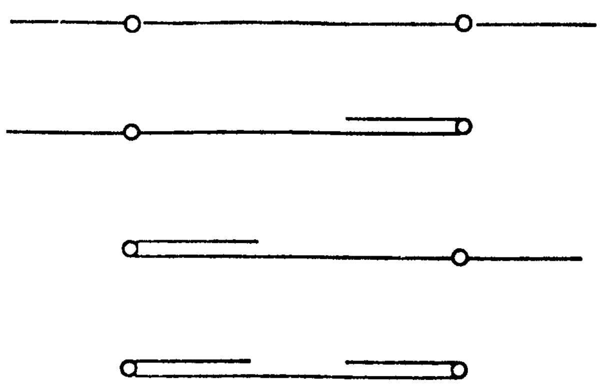


Fig. 8. Fundamental equilibria

We label these equilibria as the *fundamental equilibria.* A stick-figure representation (Fig. 8) helps in bringing out the symmetrical way in which these equilibria occur.

The remaining equilibria for this system are computed as the solutions to (6.29) and (6.30). Since the equilibrium equations are nonlinear and parameter dependent, one needs to exercise care while solving them. The parameter dependence of the equilibrium solutions can be summarized by two sets of constraints—*parameter-sign* and *parameter-value* constraints respectively. It was found that two *extra equilibria* (other than the *fundamental equilibria*) can exist at a time, subject to the existence of suitable values of κ and τ satisfying these constraints. The maximum number of equilibria for a general three-body system (special kinematic case) is thus six. For some values of κ and τ not satisfying these constraints and for the cases with κ and/or τ being zero these extra equilibria merge with the fundamental equilibria to give a total of four equilibria.

6.3.1 *Parameter-sign constraints*

This constraint set restricts the existence of values of the pair $\{\theta_{2,1}, \theta_{3,2}\}$ depending on the signs of κ and τ. Using (6.27) in (6.19) we get

$$\sin(\theta_{2,1} + \theta_{3,2}) = -\tau \sin(\theta_{2,1}). \tag{6.32}$$

Taking into account the signs of κ and τ, from (6.29) and (6.32) we get Fig. 9, which illustrates the feasible regions of the solution pair $\{\theta_{2,1}, \theta_{3,2}\}$ to form the *parameter-sign* constraints.

6.3.2 *Parameter-value constraints*

The existence of solutions of (6.29) and (6.30) is also dependent on the actual values of κ and τ (which are constants for a given three-body system). The *parameter-value* dependence of the solutions can be formulated by squaring and adding (6.29) and (6.30), and simplifying to get

$$\cos(\theta_{2,1}) = \frac{1 - \kappa^2 - \tau^2}{2\kappa\tau}, \tag{6.33}$$

$$\cos(\theta_{3,2}) = \frac{\kappa^2 - \tau^2 - 1}{2\tau}, \tag{6.34}$$

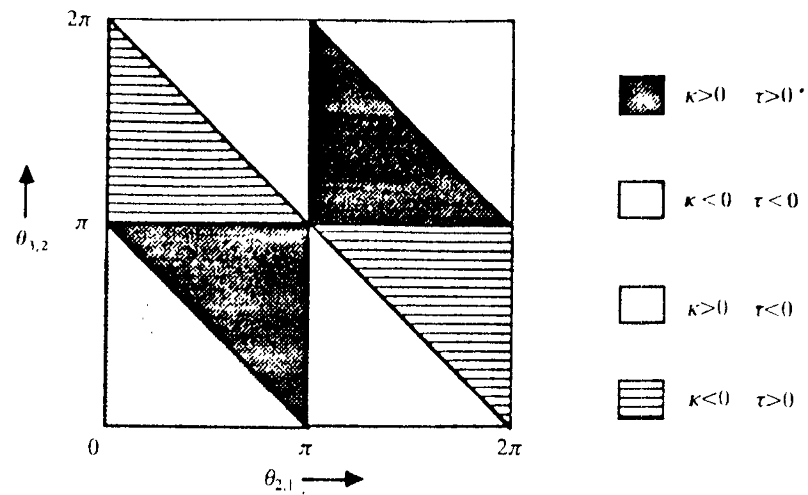


Fig. 9. Parameter-sign constraints

so that

$$-1<\frac{1-\kappa^2-\tau^2}{2\kappa\tau}<1, \tag{6.35}$$

$$-1<\frac{\kappa^2-\tau^2-1}{2\tau}<1. \tag{6.36}$$

These equations could be represented in the form of a graph as in Fig. 10. The graph has been drawn for $\kappa'>0$ and $\tau'>0$, where

$$\kappa'=|\kappa|, \qquad \tau'=|\tau|.$$

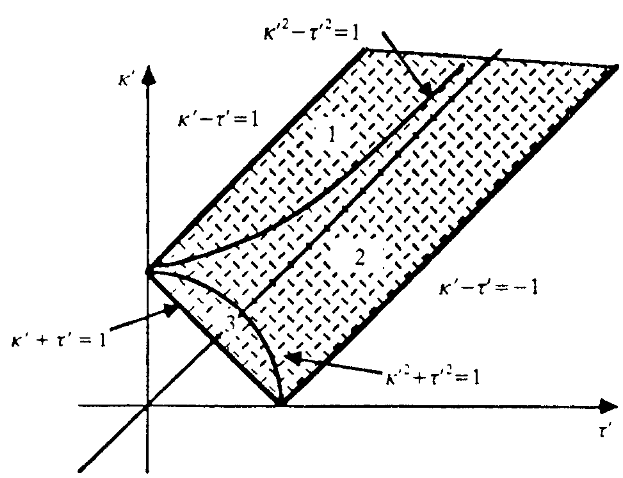


Fig. 10. Parameter-value constraints

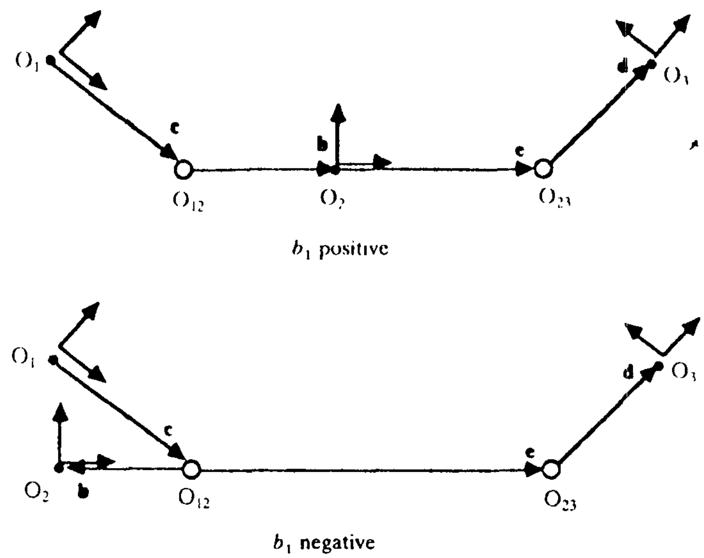


Fig. 11. Reference configuration

6.3.3 *Local frames of reference*

It is necessary to choose a local frame of reference for each of the bodies in order to parametrize the system and study the system equilibria; refer to Fig. 11. Proper choice of the local frames of reference for bodies 1 and 3 results in the vectors $\mathbf{c} = [c_1, 0]^T$ and $\mathbf{d} = [d_1, 0]^T$, where both c_1 and d_1 are positive. In general, the local frame of reference of body 2 could be chosen in such a way that $\mathbf{e} = [e_1, e_2]^T = [e_1, 0]^T$, where e_1 is positive. Note that if $\mathbf{b} = [b_1, 0]^T$, the kinematic parameter b_1 could be either negative or positive. The two signs of b_1 represent the cases when the centre of mass of body 2 is (a) inside the line segment joining the hinges O_{12} and O_{23}, and (b) outside it. If any of the kinematic parameters c_1 or d_1 is equal to zero then the three-body problem decomposes into a two-body problem and a one-body problem. It is also important to observe that with this choice of local frames of reference, A_1 is positive (see (6.21)).

6.3.4 *Parameter-dependent equilibria*

We now delve into particular cases of the signs of parameters κ and τ and establish the solutions to the equilibrium equations. We constantly refer to (6.21) to (6.27) while formulating the necessary conditions.

In all the cases we consider, we first ascertain that there exist physically realizable values of the kinematic parameters c_1, b_1, e_1 and d_1 before finding the actual solutions. The equilibria are evaluated based on the signs of $\cos(\theta_{2,1})$ and $\cos(\theta_{3,2})$ (see (6.33) and (6.34)), and according to the parameter-sign and parameter-value constraints. The results are presented in the form of a table for each case. The graphs under the column parameter-sign constraints have to be read with $\theta_{2,1}$ as the **X**-axis and $\theta_{3,2}$ as the **Y**-axis. The shaded regions represent

Table 2. $\kappa < 0$, $\tau < 0$

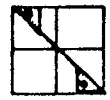
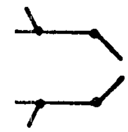
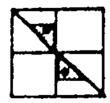
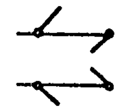
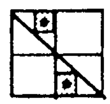
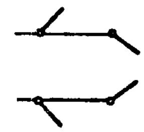
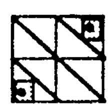
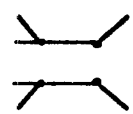
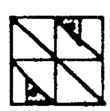
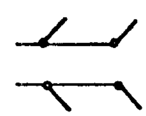
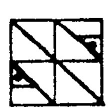
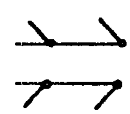
Case	cos (θ_{12})	cos (θ_{23})	Parameter-sign constraints	Parameter-value constraints	Equilibria
2.1	>0	>0		region 3	
2.2	<0	<0		region 1	
2.3	<0	>0		region 2	
2.4	>0	<0		not satisfied	

Table 3. $\kappa > 0$, $\tau < 0$

Case	cos (θ_{12})	cos (θ_{23})	Parameter-sign constraints	Parameter-value constraints	Equilibria
3.1	>0	>0		region 2	
3.2	<0	<0	not satisfied		
3.3	<0	>0		region 3	
3.4	>0	<0		region 1	

Naturally, equation (6.38) indicates that this case is possible only if b_1 is negative (since $e_1 > 0$).

Table 2 gives the equilibria associated with this case if (6.38) is satisfied.

Case 3, *in which* $\kappa > 0$, $\tau < 0$. For this case since $A_1 > 0$ we have to have B_1, $B_2 < 0$, that is,

$$e_1 < -\left(1 + \frac{m_2}{m_3}\right) b_1 < -\left(\frac{m_1}{m_1 + m_2}\right) b_1.$$

With the choice of local frames of reference, $e_1 > 0$ and so this case is possible only if b_1 is negative and

$$e_1 < -\left(\frac{m_1}{m_1 + m_2}\right) b_1. \tag{6.39}$$

The equilibria are as given in Table 3.

Case 4, *in which* $\kappa < 0$, $\tau > 0$. The necessary condition for this case is

$$-b_1\left(1 + \frac{m_2}{m_3}\right) < e_1 < -b_1\left(\frac{m_1}{m_1 + m_2}\right). \tag{6.40}$$

But $e_1 > 0$, and so b_1 has to be negative. Then from (6.40) $e_1/|b_1|$ is greater than 1 but less than a fraction—which is impossible.

So kinematic parameters satisfying $\kappa < 0$ and $\tau > 0$ can never exist.

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