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ON OBSERVER PROBLEMS FOR SYSTEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS

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Contents

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1 - INTRODUCTION

The problem of observers has been introduced in the control litterature by D. LUENBERGER [15]. Let us consider a dynamic system which is deterministic, but whose initial state is unknown. An observer is a model which mimics the behaviour of the physical system, and in particular its state becomes closer and closer as time evolves to the state of the physical system. There is a great deal of freedom in such a design and it is important to investigate various kinds of observers.

Since, after all, the observer problem presents analogies with the filtering problem (estimating the state of a stochastic dynamic system), although there are no stochastic disturbances, it is natural to exploit the analogy. This idea has been used by J.S. BARAS. P.S. KRISHNAPRASAD [4] and leads to an observer which is different from Luenberger's observer. It presents several advantages. In particular, it is obtained in a constructive way and it has robustness properties.

From the very definition it applies identically when there are disturbances, whereas the Luenberger observer is strictly limited to the deterministic case and is not obtained in a constructive way. Also it may apply to more general cases, in the sense that when the Luenberger observer exists, the observer arising from the Kalman filter theory exists as well.

In this article we consider dynamic systems whose evolution is governed by a parabolic partial differential equation, or more generally a differential operational equation in the sense of J.L. LIONS [12].

The Luenberger theory has been extended to infinite dimensional systems (see in particular M.J. CHAPMAN - A.J. PRITCHARD [6], A. ICHIKAWA - A.J. PRITCHARD [10]). We explore here the observer based upon Kalman filter theory.

Let us discuss here an example to present the main results of the paper. Let Ω be a smooth domain of R^n. Consider the P.D.E.

$$\frac{\partial y}{\partial t} - \Delta y = 0 \quad \text{in } \Omega$$

(1) $$\left.\frac{\partial y}{\partial \nu}\right|_{\Gamma} = 0$$

$$y(x,o) = y_0(x) \in H = L^2(\Omega)$$

where $\Gamma = \partial\Omega$ denotes the boundary of Ω. The state y_0 is unknown. If instead of (1) we would have

(2)
$$\frac{\partial y}{\partial t} - \Delta y + \lambda y = 0 \qquad \lambda > 0$$
$$\frac{\partial y}{\partial \nu}\Big|_\Gamma = 0$$
$$y(x,o) = y_0$$

then the system is stable and $y(x,T) \to 0$ as $T \to \infty$. In other words, whatever be y_0, the state becomes closer and closer to 0 as T evolves and thus becomes "more and more known". An observer could be simply the model (2) itself with an arbitrary value of y_0, independently of the available observation.

Such a stability property is not present in the model (1). Let us assume that we observe

(1)' $\quad z = y\Big|_\Gamma$

i.e. the value of the state on the boundary.

An observer in the spirit of Luenberger would be the following model

(3)
$$\frac{\partial m}{\partial t} - \Delta m = 0 \quad \text{in } \Omega$$
$$\frac{\partial m}{\partial \nu} = (z-m)_\Gamma$$
$$m(x,o) = m_0(x)$$

where m_0 is arbitrary. The error $\eta = y - m$ appears as the solution of

(4)
$$\frac{\partial \eta}{\partial t} - \Delta \eta = 0 \quad \text{in } \Omega$$
$$\frac{\partial \eta}{\partial \nu} + \eta\Big|_\Gamma = 0$$
$$\eta(x,o) = \eta_0(x) = y_0 - m_0 .$$

Multiplying (4) by η integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \eta^2 dx + \int_\Omega |D\eta|^2 dx + \int_\Gamma \eta^2 d\Gamma = 0$$

But (see SORINE [16])

$$\int_\Omega |D\eta|^2 dx + \int_\Gamma \eta^2 d\Gamma \geq \beta \int_\Omega \eta^2 dx \quad , \qquad \beta > 0$$

and thus

$$\int_\Omega \eta^2(x,t)dx \leq C\, e^{-2\beta t}$$

which proves the exponential decay of the error.

The theory developed in this paper leads to the following observer

$$\frac{\partial m}{dt} - \Delta m = \int_\Gamma P(x,\xi,t)\, \big(z(\xi,t) - m(\xi,t)\big)\, d\xi$$

$$(5) \qquad \frac{\partial m}{\partial \nu}\Big|_\Gamma = 0$$

$$m(x,o) = m_o(x)$$

where $P(x,\xi,t)$ appears as the solution of a Riccati equation, connected to a filtering problem, or by duality to a control problem. We study the type of control problems which may be introduced in order to derive exponential decay for the error.

2 - SETTING OF THE PROBLEM

2.1. Notation - Assumptions

Let V, H be two separable real Hilbert spaces, such that identifying H and its dual H', one has

$$(2.1) \qquad V \subset H = H' \subset V'$$

each space being dense in the next one with continuous injection

We denote by $((\ ,\ ))$ $\|\ \|$ and $(\ ,\ )$ $|\ |$ the scalar product and norm in V, H respectively. For another Hilbert space X we shall use the notation $(\ ,\ )_X$ and $|\ |_X$. We denote by $<\ ,\ >$ the duality V, V' (*).

Let A(t) be a family of operators such that

(2.2)
$$A(.) \in L^{\infty}\left(o,\infty;\mathscr{L}(V;V')\right)$$
$$< A(t)\phi,\phi > + \lambda|\phi|^2 \geq \alpha\|\phi\|^2 , \quad \forall\ \phi \in V ,$$
$$\alpha > 0 , \ \lambda \geq 0 , \ \forall t$$

We consider a dynamic system whose evolution is governed by the differential operational equation (see J.L. LIONS [12])

(2.3)
$$\frac{dy}{dt} + A(t)y = f$$
$$y(o) = y_o$$

where

(2.4) $f \in L^2(o,\infty;V')$ given

(2.5) $y_o \in H$, y_o unknown

It is well known that for any y_o, (2.3) defines the state y(.) in the sense

(2.6) $y \in L^2_{loc}(o,\infty;V)$, $\frac{dy}{dt} \in L^2_{loc}(o,\infty;V')$.

We perform an observation on the state y(.) as follows

(2.7) $z(t) = C(t)\ y(t)$

where $C(.) \in L^{\infty}\left(o,\infty;\mathscr{L}(V;F)\right)$

where F is a given Hilbert space.

(*) More generally $<\ ,\ >_X$ will represent the duality between a Hilbert space X and its dual X' and Λ_X the canonical isomorphism between X and X'.

2.2. Definition of an observer

An observer is a function m(t) measurable with values in H, whose value can be computed at each time t in terms of the known data (in particular the observation z(.)) and such that the error

$$e(t) = y(t) - m(t)$$

satisfies $|e(t)| \to 0$ as $t \to \infty$.

Therefore m(t) will reasonably estimate the state of the system at time t (at t=0 this estimate may be very bad, but it improves more and more as $t \to \infty$; it is of course nice to get an exponential decay for $|e(t)|$).

This problem has been extensively studied in the finite dimensional case, starting with the seminal work of D. LUENBERGER [15]. For infinite dimensional systems, the Luenberger observer has been extended in a natural fashion, however the research has mostly concentrated in the design of compensation (i.e. use the possibility of controlling the system, so that the global system made of the system itself and of the observer be stable) (*) While the control is generally present in such contexts, it is worthwile to separate the observer problem from the compensation problem, and consider cases as (2.3), (2.7) when there is no control. In particular nothing can be done to stabilize the system itself.

We shall thus prefer to introduce observers constructed in a different way, following ideas introduced by J.S. BARAS, P.S. KRISHNAPRASAD [4], and which consist in artificially randomizing the problem and use Kalman filter theory for infinite dimensional systems.

2.3. A randomizing system

The theory of linear stochastic infinite dimensional systems has attracted a considerable interest in the litterature. One of the main objectives has been to develop a

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(*) for details in this direction see M.J. CHAPMAN - A.J. PRITCHARD [6], A. ICHIKAWA - A.J. PRITCHARD [10].

rigorous theory of the Kalman filter applicable to distributed parameter systems. Among the main contributions one can refer to A.V. BALAKRISHNAN [1], [2], A. BENSOUSSAN [5], R.F. CURTAIN [8], R.F. CURTAIN, A.J. PRITCHARD [9].

Let us describe briefly what we need in the present context. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\mathcal{F}^t$, satisfying the usual conditions.

Let E be a Hilbert space, a generalized E Wiener space is a stochastic process indexed by an element $e_*(.) \in L^2_{loc}(0,\infty;E')$, denoted by $\mu_t^{e_*}(\omega)$ satisfying

(2.8) $\mu_t^{e_*}$ is a $\mathcal{F}^t$ Wiener, $\forall\, e_*$

(2.9) $$E\, \mu_t^{e_*}\, \mu_s^{e_*} = \int_0^{t \wedge s} < e_*(\lambda)\, ,\ \Lambda_E^{-1}\ e_*(\lambda) > d\lambda\, ,$$

(2.10) the map $e_* \to \mu_t^{e_*}$ is linear.

We assume that such a generalized E Wiener process exists. Similarly we shall assume the existence of a generalized F Wiener process $\nu_t^{h_*}$ and we suppose that

(2.11) $\mu_t^{e_*}$ and $\nu_t^{f_*}$ are independent.

Let $G(.) \in L^\infty\left(0,\infty;\mathcal{L}(E;V')\right)$ and $M(.) \in L^\infty\left(0,\infty;\mathcal{L}(F;F)\right)$, we shall consider the stochastic processes $\mu_t^{G^* v}$ and $\nu_t^{M^* f_*}$ indexed respectively by $v(.) \in L^2_{loc}(0,\infty;V)$ and $f_*(.) \in L^2_{loc}(0,\infty;F')$. They are clearly independent $\mathcal{F}^t$ processes and one has

(2.12) $$E\, \mu_t^{G^* v_1}\, \mu_s^{G^* v_2} = \int_0^{t \wedge s} < G\, \Lambda_E^{-1}\, G^* v_1(\lambda), v_2(\lambda) > d\lambda$$

Let finally be a family of random variables indexed by $h \in H$ denoted by ξ^h such that

(2.13) $\forall\, h$, ξ^h is a gaussian with mean O, and

$$E\, \xi^h\, \xi^{\tilde{h}} = (P_0 h, \tilde{h})\, , \quad P_0 \in \mathcal{L}(H;H)$$

symmetric, semi positive definite.

(2.14) ξ^h is independent from $\mu_t^{e_*}$ and $\nu_t^{f_*}$.

Let $\Gamma(t,s)$, $t \geq s$ be the Green operator corresponding to $A(t)$, i.e.

(2.15) $\Gamma(t,s) \in \mathscr{L}(H;H)$ and $\alpha(t) = \Gamma(t,s)h$

is the solution of

$$\frac{d\alpha}{dt} + A\alpha = 0 \qquad t > s, \quad \alpha(s) = h .$$

For simplicity we write $\Gamma(t) = \Gamma(t,o)$.

A <u>stochastic linear system</u> is a process y_t^h indexed by $h \in H$, defined by the formula

(2.16) $$y_t^h = (\bar{y}(t),h) + \xi^{\Gamma^*(t)h} + \mu_t^{G^*\Gamma^*(t,.)h}$$

where $\bar{y}(t)$ is the solution of

(2.17) $$\frac{d\bar{y}}{dt} + A(t)\bar{y} = f(t) \qquad \bar{y}(o) = y_o$$

y_o given in H.

We are to be careful in the interpretation of $\mu_t^{G^*\Gamma^*(t,o)h}$. In fact consider $s \to G^*(s)\Gamma^*(t,s)h$ for $s < t$; it coïncides with $G^*(s)\beta.(s)$ where β is the solution of

$$\frac{\partial\beta}{ds} + A^*(s)\beta = 0 \qquad \beta(t) = h$$

and $\beta(s) \in L^2(o,t,V)$, so $s \to G^*(s)\Gamma^*(t,s)h = e_*^t(s)$ belongs to $L^2(o,t;E')$. Since $\mu_t^{e_*}$ depends only on the restriction of e_* on (o,t), the quantity $\mu_t^{e_*t}$ is well defined for any t.

We now define the observation at time T as follows. Let $f_* \in L^2(op,T,F')$ and η be the solution of

(2.18) $$-\frac{d\eta}{dt} + A^*\eta = C^*f_*$$

$$\eta(T) = 0$$

then one sets

(2.19) $$z_T^{f_*} = \int_o^T < f_*(t), C(t)\bar{y}(t) > dt + \xi^{\eta(o)} + \mu_T^{G^*\eta} + \nu_T^{M^*f_*}$$

2.4. Kalman filter

This problem can be stated as follows. Let

$$Z^T = \sigma(z^{f^*}, f_* \in L^2(o,T;F'))$$

find

$$(2.20) \qquad \hat{y}_T^h = E\,[y_T^h \mid Z^T]$$

Without redoing the theory, for which we refer to A. BENSOUSSAN [5], we shall only recall that it can be obtained by solving a deterministic control problem, related to maximum likelihood. We assume that M is invertible as well as P_o and set

$$(2.21) \qquad R = M^{*-1}\,\Lambda_F\,M^{-1} \in \mathscr{L}(F,F').$$

We introduce the control problem

$$(2.22) \qquad \begin{cases} \dfrac{dy}{dt} + Ay = f + G\,e & e(.) \in L^2(o,T;E) \\ y(o) = y_o + \xi & \end{cases}$$

in which ξ and e(.) are the decision variables.

We are interested in minimizing the following cost

$$(2.23) \qquad J(\xi,e(.)) = \int_o^T \left[< \Lambda_E\, e,e > + < R\,(\zeta - Cy), \zeta - Cy >\right] dt + (P_o^{-1}\,\xi,\xi)$$

where in the right hand side of (2.23) the function $\zeta(.)$ is given in $L^2(o,T;F)$.

Define

$$(2.24) \qquad \begin{cases} D_1 = G\,\Lambda_E^{-1}\,G^* \\ D_2 = C^*R\,C \end{cases}$$

and consider the pair y(.), p(.) defined by the system of coupled equations

$$\frac{dy}{dt} + A(t)y + D_1(t)p = f$$

$$-\frac{dp}{dt} + A^*(t)p - D_2(t)y = -C^*R\zeta \tag{2.25}$$

$$y(o) = y_o - P_o\, p(o)$$

$$p(T) = 0$$

then the optimal control is given by

$$\hat{e}(t) = -\Lambda_E^{-1} G^*(t)\, p(t) \tag{2.26}$$

$$\xi = -P_o\, p(o)$$

The decoupling theory leads to the Riccati equation (written formally)

$$\frac{dP}{dt} + AP + PA^* + PD_2P = D_1 \tag{2.27}$$

$$P(o) = P_o$$

and the linear equation

$$\frac{dr}{dt} + A\, r = f + P\, C^*R\, (\zeta - Cr) \tag{2.28}$$

$$r(o) = y_o$$

It can be proven that if $\phi_*^h \in L^2(o,T;F')$ is defined by

$$\int_o^T < \phi_*^h, \zeta > dt = (r(T) - \bar{y}(T), h)\ , \quad \forall\, \zeta$$

then the quantity (2.20) is given by

$$\hat{y}_T^h = z^{\phi_*^h} + (\bar{y}(T), h)$$

3 - STUDY OF THE OPERATOR P(T)

3.1. Definition of P(T)

Consider the coupled system

$$(3.1)\qquad \begin{aligned} &\frac{d\hat{\alpha}}{dt} + A\hat{\alpha} + D_1\hat{\beta} = 0 \\ &-\frac{d\hat{\beta}}{dt} + A^*\hat{\beta} - D_2\,\hat{\alpha} = 0 \\ &\hat{\alpha}(o) = -\,P_o\,\hat{\beta}(o) \\ &\hat{\beta}(T) = h \end{aligned}$$

then we set

$$(3.2)\qquad P(T)h = -\hat{\alpha}(T)\ .$$

The system (3.1) is related to the following control problem

$$\frac{d\alpha}{dt} + A(t)\alpha = G\ e$$

$$(3.3)\qquad \alpha(o) = \xi$$

$$J_T^h(\xi,e(.)) = \frac{1}{2}\ \left\{(P_o^{-1}\ \xi,\ \xi) + \int_o^T\ [\,|e|_E^2\ + < D_2\alpha,\alpha >\,]\ dt\right\}\ +\ \left(h,\alpha(T)\right)\ .$$

We deduce easily that

$$(3.4)\qquad \frac{1}{2}\ (P(T)h,h) = -\inf_{\xi,e(.)}\ J_T^h\ (\xi,e(.)).$$

We can also characterize this quantity in a different way. Consider the control problem

$$(3.5)\qquad \begin{aligned} &-\frac{d\beta}{dt} + A^*\beta = C^*\phi_* \qquad \phi_* \in L^2(o,T;F') \\ &\beta(T) = h \end{aligned}$$

and the cost

$$\tilde{J}_T^h\ (\phi_*(.)) = (P_o\beta(o),\beta(o)) + \int_o^T\ < R^{-1}\ \phi_*,\ \phi_* > dt\ +\ \int_o^T < D_1\beta,\beta > dt$$

then one has also

(3.6) $\quad (P(T)h,h) = \inf \tilde{J}_T^h (\phi_*(.))$

3.2. Detectability and stabilisability

Definition 3.1. We shall say that the pair A(.), C(.) is detectable if $\forall$ h , $\exists$ $\phi_*^{T,h}$ such that

$$\int_0^T |\phi_*^{T,h}|_{F'}^2 \, dt + \int_0^T |\beta^{T,h}|_H^2 \, dt < K_h \tag{3.7}$$

independent of T, where $\beta^{T,h}$ represents the solution of (3.5) corresponding to $\phi_*^{T,h}$

□

In the stationary case, i.e. A, C independent of time, it is sufficient to assume the following.

Definition 3.2. We say that A^*, C^* is stabilisable if $\forall$ h, $\exists$ $\phi_*^h \in L^2(o,\infty;F')$ such that the solution γ^h of

$$\frac{d\gamma^h}{dt} + A^* \gamma^h = C^* \phi_*^h$$

$$\gamma^h(o) = h$$

satisfies $\gamma^h \in L^2(o,\infty;H)$.

□

Proposition 3.1. In the stationary case if A^*, C^* is stabilisable, then A, C is detectable.

Proof.

Define

$$\phi_*^{T,h}(t) = \phi_*^h(T-t) \ , \qquad \beta^{T,h}(t) = \gamma^h(T,t)$$

then clearly $\beta^{T,h}(t)$ is the solution of (2.5) corresponding to $\phi_*^{T,h}(t)$, and

$$\int_0^T |\phi_*^{T,h}|_F^2 dt = \int_0^T |\phi_*^h|^2 dt < \int_0^\infty |\phi_*^h|^2 dt$$

$$\int_0^T |\beta^{T,h}|^2 dt = \int_0^T |\gamma^h|^2 dt < \int_0^\infty |\gamma^h|^2 dt$$

and the desired property follows.

□

Theorem 3.1. Assume that A(.), C(.) is detectable, then

(3.8) $\quad |P(T)|_{\mathscr{L}(H;H)} \leq p$

Proof.

From (3.5) it follows that

$$\frac{1}{2}|\beta(o)|^2 + \int_0^T \langle A^*\beta,\beta \rangle dt = \frac{1}{2}|h|^2 + \int_0^T \langle \beta, C^*\phi_* \rangle dt$$

hence

$$\frac{1}{2}|\beta(o)|^2 + \alpha\int_0^T \|\beta\|^2 dt \leq \frac{1}{2}|h|^2 + \int_0^T \langle \beta, C^*\phi_* \rangle dt + \lambda\int_0^T |\beta|^2 dt$$

and the detectability condition implies immediately

$$|\beta^{T,h}(o)| \leq K'_h \qquad \int_0^T \|\beta^{T,h}\|^2 dt \leq K'_h$$

therefore from the definition (3.6)

$$(P(T)h,h) \leq \tilde{J}_T^h(\phi_*^{T,h}) \leq K''_h \ .$$

This and the fact that P(T) is symmetric positive semi definite implies the desired result.

□

Theorem 3.2. If there exists a family $\Gamma(.) \in L^\infty(o,\infty;\mathscr{L}(F;V'))$ such that

(3.9) $\quad \langle (A(t) + \Gamma(t)\,C(t))\,\xi, \xi \rangle \geq \alpha_o \|\xi\|^2$

then the pair A(.), C(.) is detectable.

Proof.

Indeed pick the feedback

$$\phi_* = - \Gamma^* \beta$$

the corresponding trajectory is given by

$$- \frac{d\beta}{dt} + (A^* + C^* \Gamma^*) \beta = 0 \qquad \beta(T) = h$$

and thus $\int_0^T \|\beta\|^2 dt$ is bounded by a constant independent of T. The desired result follows.

□

3.3. Invertibility

We turn now to the question of the invertibility of the operator P(T). We shall need another property. Consider the dynamic system

$$(3.10) \quad \begin{aligned} &\frac{d\alpha}{dt} + A\alpha = Ge \\ &\alpha(o) = \xi \end{aligned}$$

Definition 3.3. We shall say that the pair A(.), G(.) is controllable if $\forall$ h, $\exists$ $e^{T,h}$ and $\xi^{T,h}$ such that

$$(3.11) \quad |\xi^{T,h}|^2 + \int_o^T |e^{T,h}|^2 dt + \int_o^T |\alpha^{T,h}|^2 dt \le L_h$$

and

$$(3.12) \quad \alpha^{T,h}(T) = h$$

whose $\alpha^{T,h}$ designates the solution of (3.10) corresponding to $\xi^{T,h}$ and $e^{T,h}$. The constant L_h is independent of T.

□

We can give an example of controllability (probably the only one really applicable in practice).

Proposition 3.2. Assume that

(3.13) D_1 is invertible

then the pair $A(.), G(.)$ is controllable.

Proof.

Let $\phi_h \in L^2(o,\infty,V)$, $\frac{d\phi_h}{dt} \in L^2(o,\infty;V')$ and $\phi_h(o) = h$. We set

$$\alpha^{T,h}(t) = \phi_h(T-t) \quad , \quad \xi^{T,h} = \alpha^{T,h}(o) = \phi_h(T)$$

and

$$(3.14) \quad e^{T,h}(t) = \Lambda_E^{-1} G^*(t) D_1(t)^{-1} \left(\frac{d\alpha^{T,h}}{dt} + A(t)\,\alpha^{T,h}(t)\right)$$

then

$$|\xi^{T,h}|^2 = |\phi_h(T)|^2 = 2\int_o^T < \phi_h, \frac{d\phi_h}{dt} > dt + |h|^2$$

$$< |h|^2 + 2\int_o^\infty < \phi_h, \frac{d\phi_h}{dt} > dt .$$

$$\int_o^T |\alpha^{T,h}|^2\, dt = \int_o^T |\phi_h(T-t)|^2\, dt = \int_o^T |\phi_h(t)|^2\, dt < \int_o^\infty |\phi_h|^2\, dt$$

and

$$\int_o^T |e^{T,h}(t)|^2\, dt \leq C \left[\int_o^T \left\|\frac{d\alpha^{T,h}}{dt}\right\|_{V'}^2 dt + \int_o^T \|\alpha^{T,h}(t)\|^2\, dt\right]$$

$$\leq C \left[\int_o^\infty \left\|\frac{d\phi_h}{dt}\right\|_{V'}^2 dt + \int_o^\infty \|\phi_h\|^2\, dt\right] .$$

Finally from (3.4) follows

$$G(t)\, e^{T,h}(t) = \frac{d\alpha^{T,h}}{dt} + A(t)\,\alpha^{T,h}(t)$$

and

$$\alpha^{T,h}(T) = h .$$

The desired result has been established.

□

Remark 3.1. As it is well known (cf. R. LATTES - J.L. LIONS [11]), we cannot solve a priori the backward problem

$$(3.15)\qquad \frac{d\alpha}{dt} + A\alpha = Ge \qquad \alpha(T) = \dot{h} .$$

The situation is different from ordinary differential equation. The problem (3.15) (for given e) is a priori ill posed. We refer to J.L. LIONS [14] for a detailed study of these ill posed problems in the context of control theory.

□

We shall now consider control problems similar to those described in §. 3.1. The response is described by

$$(3.16)\qquad \begin{aligned} &\frac{d\alpha}{dt} + A\alpha = Ge \\ &\alpha(o) = \xi \end{aligned}$$

We impose the constraint

$$(3.17)\qquad \alpha(T) = h$$

and minimize the cost (recall that P_o is invertible (cf. §. 2.4)

$$(3.18)\qquad K_T^h(\xi, e(.)) = (P_o^{-1}\xi,\xi) + \int_o^T [\,|e|_E^2 + < D_2\alpha,\alpha >]\, dt.$$

Note that (3.17) must be considered as a constraint and not an initial condition. We can assert

Theorem 3.3. Assume that the pair A(.), G(.) is controlable. Then the control problem (3.16), (3.17), (3.18) has a unique solution. There exists a unique pair $\bar{\alpha}$, $\bar{\beta}$ such that

$$(3.19)\qquad \begin{aligned} &\frac{d\bar{\alpha}}{dt} + A\bar{\alpha} + D_1\bar{\beta} = 0 \\ &-\frac{d\bar{\beta}}{dt} + A^*\bar{\beta} - D_2\bar{\alpha} = 0 \\ &\bar{\alpha}(o) = -P_o\bar{\beta}(o) \\ &\bar{\alpha}(T) = h \end{aligned}$$

and the optimal control is

$$\bar{e}(t) = - \Lambda_E^{-1} G^*(t)\, \bar{\beta}(t)$$

(3.20)

$$\bar{\xi} = - P_0 \bar{\beta}(o)$$

Proof.

We follow the technique introduced by J.L. LIONS [13] to deal with this type of ill posed problems. It relies on the use of the penalty technique. We shall penalize the constraint (3.17).

a. Existence and uniqueness of the optimal control

Let $\Gamma_h = \{e(.), \xi \mid \alpha(T) = h\}$. By the controllability assumption Γ_h is not empty. It is a convex closed subset of the Hilbert space $L^2(o,T;E) \times H$ and the functional (3.18) is a coercive quadratic functional. Hence the existence and uniqueness of the optimal control $\bar{e}(.), \bar{\xi}$.

b. Necessary and sufficient conditions of optimality

Consider Γ_0 (i.e. Γ_h with h=0). It is obviously a closed subvector space of $L^2(o,T;E) \times H$. Noting that $\bar{e}(.) + \lambda\, e(.)$, $\bar{\xi} + \lambda\xi$ $\forall$ $e(.),\xi$ belonging to Γ_0, we deduce easily from the relation

$$K_T^h(\bar{\xi} + \lambda\xi\, , \bar{e}(.) + \lambda e(.)) \geq K_T^h(\bar{\xi}, \bar{e}(.))$$

that the following condition must hold

$$(3.21) \qquad (P_0^{-1}, \bar{\xi}, \xi) + \int_0^T < \Lambda_E\, \bar{e}, e > dt + \int_0^T < D_2\, \bar{\alpha}, \alpha > dt = 0$$

$\forall$ ξ, e(.) in E_0, α being the corresponding solution of (3.16), and $\bar{\alpha}$ designates the optimal trajectory

This condition is also sufficient, since $\forall\, \tilde{\xi}, \tilde{e}(.) \in E_h$, $\tilde{\xi} - \bar{\xi}$, $\tilde{e}(.) - \bar{e}(.) \in E_0$, hence

$$K_T^h(\bar{\xi}, \bar{e}(.)) = (P^{-1}_0\, \bar{\xi}, \bar{\xi}) + \int_0^T < \Lambda_E\, \bar{e}, \bar{e} > dt + \int_0^T < D_2\bar{\alpha}, \bar{\alpha} > dt$$

such implies

$$K_T^h(\tilde{\xi},\tilde{e}(.)) > K_T^h(\bar{\xi},\bar{e}(.)) .$$

c. Adjoint system. Uniqueness

Assume that we have a solution of (3.19). Then define $\bar{e}(.)$ and $\bar{\xi}$ by (3.20). Let us show that it is optimal. Clearly $\bar{\alpha}$ is the trajectory corresponding to the control $\bar{e}(.)$, $\bar{\xi}$ and it is admissible (belongs to Γ_h). It is easy to check that it satisfies (3.21). It is thus the optimal control and therefore $\bar{e}(.)$, $\bar{\xi}$ and $\bar{\alpha}(.)$ are uniquely defined. It is also the case of $\bar{\beta}(o)$ and $-\frac{d\bar{\beta}}{dt} + A^*\bar{\beta}$. This implies that $\bar{\beta}$ is unique as well.

d. Adjoint system. Existence

It remains to prove the existence of the pair $\bar{\alpha}$, $\bar{\beta}$ solution of (3.19). The penalty technique is now used. Consider the functional

$$K_T^{h,\epsilon}(\epsilon,e(.)) = K_T^h(\xi,e(.)) + \frac{1}{\epsilon}\,|\alpha(T) - h|^2 . \tag{3.22}$$

Then the control problem (3.16), (3.22) becomes classical, and there exists a pair α^ϵ, β^ϵ solution of

$$\begin{aligned}
&\frac{d\alpha_\epsilon}{dt} + A\alpha_\epsilon + D_1\beta_\epsilon = 0\\
&-\frac{d\beta_\epsilon}{dt} + A^*\beta_\epsilon - D_2\alpha_\epsilon = 0\\
&\alpha_\epsilon(o) = -P_0\beta_\epsilon(o)\\
&\beta_\epsilon(T) = \frac{1}{\epsilon}(\alpha_\epsilon(T) - h)
\end{aligned} \tag{3.23}$$

and the optimal control is

$$e_\epsilon = -\Lambda_E^{-1} G^* \beta_\epsilon \qquad\qquad \xi_\epsilon = -P_0\,\beta_\epsilon(o) .$$

Now from

$$K_T^{h,\epsilon}(\xi_\epsilon, e_\epsilon(.)) \le K_T^h(\xi^{T,h}, e^{T,h}) \le C_h$$

we deduce

$$(3.24)\quad \begin{array}{l} |\xi_\epsilon|_H , \int_0^T |e_\epsilon|_E^2\, dt \quad \int_0^T < D_2\alpha_\epsilon, \alpha_\epsilon > dt < C_h \\ \frac{1}{\epsilon}|\alpha_\epsilon(T) - h|^2 < C_h \end{array}$$

and since α_ϵ is the trajectory corresponding to e_ϵ and ξ_ϵ

$$(3.25)\quad \int_0^T \|\frac{d\alpha_\epsilon}{dt}\|^2 dt + \int_0^T \|\alpha_\epsilon\|^2 dt < C_{h,T} \quad \text{(this last constant may depend on T).}$$

Now for any ξ, e(.) we have (necessary condition of optimality for the problem (3.16), (3.22)),

$$(3.26)\quad (P_0^{-1}\xi_\epsilon, \xi) + \int_0^T < \Lambda_E\, e_\epsilon, e > dt + \int_0^T < D_2\, \alpha_\epsilon, \alpha > dt + \frac{1}{\epsilon}(\alpha_\epsilon(T) - h, \alpha(T)) = 0$$

Let us pick $\xi = \xi^{T,k}$, $e = e^{T,k}$ where k is arbitrary in H. We deduce from (3.26) and the estimates (3.24), (3.25), that

$$|(\frac{\alpha_\epsilon(T)-h}{\epsilon}, k)| \le C_{h,T,k} = C_k$$

since in this context h,T are fixed. From Banach Steinhaus theorem it follows that $|\frac{\alpha_\epsilon(T)-h}{\epsilon}| \le C_{h,T}$. Therefore also using (3.23)

$$(3.27)\quad \int_0^T \|\frac{d\beta_\epsilon}{dt}\|^2 dt + \int_0^T \|\beta_\epsilon\|^2 dt < C_{h,T}$$

With the estimates (3.24), (3.25), (3.27) we can extract a subsequence $\alpha_{\epsilon'}$, $\beta_{\epsilon'}$ converging weakly to $\frac{d\bar\alpha}{dt}$, $\frac{d\bar\beta}{dt}$ in $L^2(o,T;V')$. We obtain easily that $\bar\alpha$, $\bar\beta$ is a solution of (3.19), the proof has been completed.

□

Theorem 3.4. Under the assumptions of Theorem 3.3 one has

$$(3.28)\quad \bar\beta(T) = - Q(T)\, h$$

where $Q(T) \in \mathscr{L}(H;H)$ symmetric positive semi definite.
Moreover,

$$(3.29)\quad |Q(T)|_{\mathscr{L}(H;H)} \le C$$

Proof.

Consider (3.23). The quantities α_ϵ , β_ϵ are linear continuous functions of h, hence we can write

$$\beta_\epsilon(T) = - Q_\epsilon(T)\, h \tag{3.30}$$

where it is easily seen that $Q_\epsilon(T) \in \mathscr{L}(H;H)$ symmetric, positive semi definite. Moreover one easily computes

$$\inf_{\xi,e(.)} K_T^{h,\epsilon}(\xi,e(.)) = (Q_\epsilon(T)h,h) . \tag{3.31}$$

Since the left hand side increases as ϵ decreases, it follows that $Q_\epsilon(T)$ is an increasing family of symmetric positive semi definite operators in $\mathscr{L}(H;H)$.

But

$$K_T^{h,\epsilon}(\xi,e(.)) = K_T^h(\xi,e(.)) \quad \forall\ \xi,e(.) \text{ in } \Gamma_h .$$

Therefore

$$\begin{aligned} (Q_\epsilon(T)h,h) &\leq \inf_{\xi,e(.)\in\Gamma_h} K_T^h(\xi,e(.)) \\ &\leq K_T^h(\xi^{T,h}, e^{T,h}) \leq C_h \end{aligned} \tag{3.32}$$

independent of T and ϵ.

Necessarily $Q_\epsilon(T) \uparrow Q(T) \in \mathscr{L}(H;H)$ positive semi definite and

$$|Q(T)|_{\mathscr{L}(H;H)} \leq C \quad \text{independent of T.} \tag{3.33}$$

But

$$\inf_{\xi,e(.)} K_T^{h,\epsilon}(\xi,e(.)) \to \inf_{\xi,e(.)\in\Gamma_h} K_T^h(\xi,e(.)) . \tag{3.34}$$

Indeed

$$\begin{aligned} (P_o^{-1}\xi_\epsilon, \xi_\epsilon) + \int_o^T |e_\epsilon|^2\, dt + \int_o^T < D_2\alpha_\epsilon, \alpha_\epsilon > dt + \frac{1}{\epsilon}\, |\alpha_\epsilon(T) - h|^2 \\ \leq (P_o^{-1}\, \bar{\xi}, \bar{\xi}) + \int_o^T |\bar{e}|^2\, dt + \int_o^T < D_2,\bar{\alpha}, \bar{\alpha} > dt \end{aligned}$$

and from the weak convergence

$$\leq \underline{\lim} \{(P_0^{-1}\xi_\epsilon, \xi_\epsilon) + \int_0^T |e_\epsilon|^2 \, dt + \int_0^T < D_2 \alpha_\epsilon, \alpha_\epsilon > dt\}$$

This implies the strong convergence of ξ_ϵ , e_ϵ , α_ϵ to $\bar{\xi}$, $\bar{e}$, $\bar{\alpha}$ and also that $\frac{1}{\epsilon}|\alpha_\epsilon(T)-h|^2 \to 0$. (*) Therefore (3.34) is demonstrated and thus

(3.35) $$\inf_{\xi, e(.) \in \Gamma_h} K_T^h(\xi, e(.)) = (Q(T)h,h) .$$

Moreover since $\beta_\epsilon(T) \to \bar{\beta}(T)$ in H weakly we deduce also the property (3.28).

□

Remark 3.2. Before being used as a useful technique to treat ill posed problems, the penalty technique which goes back to COURANT [7], has been widely used as an approximation technique for the control of systems governed by partial differential equations, see A.V. BALAKRISHNAN [3], J.L. LIONS [13].

□

We can now compare Q(T) and P(T).

Theorem 3.5. Assume that A(.), C(.) is detectable and A(.), G(.) is controlable. Then one has

(3.36) $$Q(T)\, P(T) = P(T)\, Q(T) = I$$

(3.37) $$|Q(T)|_{\mathscr{L}(H;H)} \leq q \, , \quad |P(T)|_{\mathscr{L}(H;H)} \leq p \, ,$$

where p, q are constants independent of T.

Proof.

The property (3.37) has already been proved. The property (3.36) follows by comparing (3.1), (3.2) to (3.19), (3.28). Indeed set in (3.19) $h = - P(T)\, h$, then we have $\bar{\alpha} = \hat{\alpha}$ and $\bar{\beta} = \hat{\beta}$ (by uniqueness) hence

$$\bar{\beta}(T) = h = - Q(T)(-P(T)h) = Q(T)P(T)h$$

A similar proof is made to prove that PQ = I .

□

(*) this follows also directly form the fact that $|\frac{\alpha_\epsilon(T)-h}{\epsilon}|$ is bounded in ϵ.

4 - OBSERVER BASED UPON KALMAN FILTER

4.1. The model

Motivated by the form of the Kalman filter (see (3.28)), we shall define the observer by the solution of the equation

$$(4.1) \qquad \frac{dm}{dt} + Am = f + PC^{*}R(z-Cm)$$

$$m(o) = m_{o}$$

where m_o is arbitrary and z is the observation corresponding to the state (2.3).

The writing (4.1) is somewhat formal, since P is not defined on V'. In fact we shall give a meaning to

$$(4.2) \qquad y - m = \eta$$

where η appears as the solution of

$$(4.3) \qquad \begin{cases} \frac{d\eta}{dt} + (A + PD_2)\eta = 0 \\ \eta(o) = y_o - m_o \end{cases}$$

The solution $\eta(t)$ of (4.3) is defined by duality.
Indeed considering (3.1) we see that the equation

$$(4.4) \qquad \begin{cases} -\frac{d\beta^T}{dt} + (A^{*} + D_2P)\beta^T = 0 \\ \beta^T(T) = h \end{cases}$$

has a solution in the functional space

$$(4.5) \qquad W_P(o,T) = \{\phi \in L^2(o,T;V) \, , \ \frac{d\phi}{dt} \in L^2(o,T;V') \, , \ P\phi \in L^2(o,T;V)\}$$

Then the value $\eta(T)$ is defined by

$$(4.6) \qquad (\eta(T),h) = (\beta^T(o), y_o - m_o) \, , \quad \forall \, h$$

which defines $\eta(T)$ uniquely in H.

Our problem amounts to studying the behaviour of $\eta(T)$ as $T \to 0$.

4.2. Estimates

We shall prove the following (main) result.

Theorem 4.1. Assume A(.), C(.) detectable, and

(4.7) $\quad < D_1(t)\, v,v > \ \geq \ k\,|v|_H^2 \quad \forall\, v \in V$ (*)

then one has

(4.8) $\quad |y(t) - m(t)| \leq C\,|y_o - m_o|\, e^{-\gamma t} \quad , \quad \gamma > 0$

Proof.

Considering the system (3.1) and the relation (4.6) we know that

$$(P(T)h,h) = (P_o\beta^T(o), \beta^T(o)) + \int_o^T < D_1\beta^T(s), \beta^T(s) > ds + \int_o^T < D_2P\beta^T(s), P\beta^T(s) > ds$$

where we have written $\beta^T(t)$ instead of β to emphasize the dependence on T. Recall that $P\beta^T = -\bar{\alpha}$.

In a similar manner we can write the relation

(4.9) $$(P(T)h,h) = (P(t)\beta^T(t),\beta^T(t) + \int_t^T < D_1\beta^T(s), \beta^T(s) > ds + \int_t^T < D_2P\beta^T(s), P\beta^T(s) > ds$$

which holds for any $t \in (o,T)$. Therefore it follows that

$$\frac{d}{dt}(P(t)\beta^T(t),\beta^T(t)) = + < D_1(t)\beta^T(t),\beta^T(t) > + < D_2(t)P(t)\beta^T(t),P(t)\beta^T(t) >$$

(*) of course if $< D_1(t)v,v > \ \geq \ k\,||v||_V^2$, (4.7) holds as well as the controllability property (see Proposition 3.2). But (4.7) is weaker.

and from (4.7)

$$(4.10) \quad \frac{d}{dt}(P(t)\beta^T(t),\beta^T(t)) \geq k\,|\beta^T(t)|^2$$

Note that, from Theorem 3.1

$$(P(t)\beta^T(t),\beta^T(t)) \leq p\,|\beta^T(t)|^2$$

therefore (4.10) implies

$$\frac{d}{dt}|P^{\frac{1}{2}}(t)\,\beta^T(t)|^2 \;\geq\; \frac{k}{p}|P^{\frac{1}{2}}(t)\,\beta^T(t)|^2$$

hence

$$(4.11) \quad (P_o\beta^T(o),\beta^T(o)) \;\leq\; (P(T)h,h)\,e^{-\frac{k}{p}T} \;\leq\; p\,e^{-\frac{k}{p}T}\,|h|^2$$

Therefore if $(P_o\,\xi,\xi) \geq \nu_o|\xi|^2$,

we have

$$|\beta^T(o)|^2 \;\leq\; \frac{p}{\nu_o}\,e^{-\frac{k}{p}T}\,|h|^2 .$$

From (4.6) it follows that

$$|\eta(T)| \;\leq\; |y_o - m_o|\,\sqrt{\frac{p}{\nu_o}}\,e^{-\frac{k}{2p}T}$$

□

4.3. Example

Let us turn to the example considered in the introduction. We shall take

$H = L^2(\Omega)$, $V = H^1(\Omega)$

$E = H$, $G = I$, hence $D_1 = I$

$F = L^2(\Gamma)$ $C = \gamma$ = trace operator

The system of optimality (3.1) looks as follows

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ON OBSERVER PROBLEMS FOR SYSTEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS

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Paper prepared in honor to Prof. A.V. BALAKRISHNAN, for his 60th anniversary

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