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An Enumeration Problem in Digital Geometry

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Abstract:

We prove that the number L(N) of digital line segments of length N (corresponding to lines y= ax+b, $0 \leq a < 1$, $0 \leq b <$) has the asymptotic expansion

$$L(N) = N^3/\pi^2 + O(N^2 \log N)$$

This expression has application in image registration problems and originated in a question posed by NASA.

An Enumeration Problem in Digital Geometry

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I. Introduction

Image processing problems in high accuracy matching, edge detection, and measurement can be approached using probabilistic methods in digital geometry. We have considered this approach in [1, 2, 3, 4] in the problem of subpixel accuracy in feature based image registration. Related techniques have been developed by Dorst and Smeulders [5]. The digital natrual of these problems gives rise to a need for the study of the properties of digital line segments, i.e. the digitization of real line segments. Work in this area characterizing which collections of pixel are digital lines was done by Rosenfeld and others ([6], [7]). Dorst and Smeulders [5] introduced a useful methods of representing digital line segments in terms of four parameters, which we used in [1-4]. The above applications lead to problems related to integral geometry and hence the first question to consider is that of putting a measure on the set of digital line segments. Dorst and Smeulders [5] suggested using the unique invariant measure [8] on the space of real lines to induce a measure on the space of digital lines. On the other hand, the counting measure on digital lines is often more convenient for computational purposes. The use of this counting measures necessitates the development of an expression for the number of digital lines of a given length, which was obtained in [4] (see formulas (5) and (6) below). Since this exact formula is very involved, it is convenient to have an asymptotic expression for it, when the length of the digital lines is large. The purpose of this paper is to prove such an expression.

II. Background

This section describes the parametrization [5] of digital lines mentioned in the introduction. We also recall the results obtained in [1] describing the set of all digital lines in terms of these parameters. We will be concerned with lines with slope in the range $[0,1)$ and crossing the y-axis in the interval

[0,1). (The family of all lines can be reduced to this one modulo translation and relabeling of the axes.) To each such line, we can associate a digital line by the following procedure. For each nonnegative integer a, if the line crosses the vertical line $x = a$ at the point (a,b), then we mark the square pixel whose lower left hand corner is $(a,[b])$, where $[b]$ denotes the integral part of b. The set of marked pixels obtained in this way is called the digital line associated to the original line. In practice, we are only interested in line segments, hence the integer will be in the interval $[0,N]$. N will be called the length of the digital line segment. For this digital segment, one can assign a sequence $\{c_j\}$ of zeros and ones as follows. Let $b_0, b_1, \ldots, b_N$ be the ordinates of the lower left hand corners of the pixels of that segment. Define

$$c_j = \begin{cases} 0 & \text{if } b_j = b_{j-1} \\ 1 & \text{otherwise} \end{cases}$$

This sequence has N elements.

The period, q, of this sequence is defined to be the smallest integer such that there exists an infinite periodic extension $c_{N+1}, c_{N+2}, \ldots,$ with period q. It is clear that $1 \le q \le N$ and the case $q = 1$ corresponds to a horizontal digital segment, i.e. $c_j = 0$ for all j. Define p to be the number of ones in a period. If p is different from zero, then p and q are relatively prime.

The fourth parameter, called the shift s, can be defined by the property that

$$(1) \qquad c_j = [(j-s)(p/q)] - [(j-s-1)(p/q)], \quad j = 1, \ldots, N$$

We impose the constraint that $0 \le s \le q-1$

It can be shown that a quadruple (N,q,p,s) subject to the above mentioned restrictions determines a digital line segment by (1). On the other hand, this correspondence between digital line segments and quadruples is not 1-1. In [4], we gave a 1-1 correspondence between the family of all digital segments and a subset of the quadruples (N,q,p,s). This subset if determined by the single condition given in Proposition 1 below. In order to state this result, we must introduce an auxiliary parameter, ℓ, given by

(2) $$\ell p = -1 \pmod q \quad \text{and} \quad 0 < \ell < q \ .$$

Proposition 1. (see [4]). The family of digital line segments is in a 1-1 correspondence with the set of quadruples (N,q,p,s) such that the quantity

(3) $$[(N-s)/q]\, q + [(s+\ell)/q]\, q - \ell$$

is positive.

We want to compute the number $L(N,q)$ of digital lines of length N and period q. Clearly $L(N,1) = 1$, so we can consider $q > 1$. Proposition 1 reduced the problem of counting the number of lines to determining the number of values of s for which the expression (3) is positive. It is clear that if $N - s \geq q$, then (3) is positive. The only time we must be careful is when $N - s < q$. This can only arise if $N \leq q + s - 1 \leq 2q - 2$, that is, $(N+2)/2 \leq q$. Hence, if $q < (N+2)/2$, s can take arbitrary values and it follows that

(4) $$L(N,q) = q\phi(q) \quad \text{for} \quad 2 \leq q < (N+2)/2$$

where $\phi(q)$ is the Euler function that counts the number of values p, $1 \leq p \leq q$, with p and q relatively prime. This formula is clearly valid for $q = 1$ since $\phi(1) = 1$. In the remaining range of q, one has to be more careful but a relatively simple argument which can be found in [1] leads to an exact formula for the number $L(N,q)$ of digital line segments in this range of q:

(5) $$L(N,q) = (N-q+2)\,\phi(q) + \sum_{\ell} \min(2q-N-2,\ q-\ell-1,\ \ell-1,\ N-q)\ ,$$

where the sum takes place over all values ℓ, $0 < \ell < q$, with ℓ and q relatively prime.

We summarize the above as:

<u>Proposition 2</u>. Let $L(N)$ be the number of digital lines of length N with both slope and y-intercept between 0 and 1. Then

(6) $$L(N) = \sum_{q=1}^{N} L(N,q)\ ,$$

where $L(N,q)$ is given by (4) if $0 < q < (N+2)/2$ and by (5) otherwise.

III. Asymptotic formula for the number of lines

The exact formula given in Proposition 2 is difficult to evaluate for large N and so an asymptotic formula becomes desirable. The derivation of such a formula is based on the heuristic fact that the numbers relatively prime to a given number q appear to be uniformly distributed in the interval $[1,q]$. In fact this heuristic can be made precise by the following auxiliary proposition (a particular case of the theorem of Bombieri-Vinogradov):

<u>Proposition 3</u>. For fixed n, let the function $F(x)$ be defined as the cardinality of the set of positive integers, p, such that $p \leq x$ and $(p,n) = 1$. Then

(7) $$F(x) = \frac{\phi(n)}{n}\,x + O(\log\log n)\ .$$

It is understood that $O(\log\log n)$ represents a quantity bounded in absolute value by a constant, independent of n, times $\log\log n$.

The proof of proposition 3 requires the following standard facts from number theory. We first recall the definition of the Moebius function, μ, defined by

$$\mu(d) = \begin{cases} (-1)^r & \text{where } r \text{ is the number of distinct primes dividing } d \text{ if there exists no prime whose square divides } d \\ 0 & \text{otherwise.} \end{cases}$$

Hence $|\mu(d)| = 1$ or 0 according to whether d is square-free or not.

Lemma 4. The following two estimates hold:

(8) $$\sum_{d|n} \frac{|\mu(d)|}{d} = O(\log\log n)$$

(9) $$\sum_{d|n} \frac{1}{d} = O(\log\log n)$$

Proof. The first sum is over the divisors of n that are square-free. It coincides with $\prod_{p|n} (1+\frac{1}{p})$. (The product runs over the prime divisors of n). This is clearly as large as possible if n itself is the product of the first r primes, $n = p_1 \cdots p_r$. We now estimate r and p_r. Using Theorem 414 from [9] we have for some constant $C > 0$,

$$\log n = \Sigma \log p_i \geq Cp_i .$$

Also, by the Prime Number Theorem, we have $p_r \approx r \log r$. Hence we obtain

$$\log n \geq C\, r \log r$$

where C denotes a different positive constant than above. Therefore

$$r \leq C \log n/\log\log n .$$

Now, by Theorem 427 in [9] we have

$$\log \prod_{p|n} (1+\frac{1}{p}) \leq \sum_{p \leq p_r} \frac{1}{p} \approx \log\log p_r$$

Using the previous inequality this leads to

$$\prod_{p|n} (1+\frac{1}{p}) = O(\log\log n) \ ,$$

which proves the estimate (8).

To prove the second estimate one needs to show that

$$(\sum_{d|n} \frac{1}{d}) / (\sum_{d|n} \frac{|\mu(d)|}{d}) \leq C \ .$$

We have

$$(\sum_{d|n} \frac{1}{d}) / (\sum_{d|n} \frac{|\mu(d)|}{d}) \leq$$

$$\prod_{p|n} \frac{(1+\frac{1}{p}+\frac{1}{p^2}+\ldots)}{(1+\frac{1}{p})} \leq \prod_{p} \frac{(1+\frac{1}{p}+\frac{1}{p^2}+\ldots)}{(1+\frac{1}{p})} =$$

$$= \prod_{p} \frac{(1-\frac{1}{p})^{-1}}{(1+\frac{1}{p})} = \prod_{p} (1-\frac{1}{p^2})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ . \qquad \square$$

<u>Proof of Proposition 3</u>:

$$\text{Let } F(x) = \sum_{\substack{(\ell,n)=1 \\ \ell\leq x}} 1 = x - \sum_{\substack{p|n \\ (p\leq x)}} [\frac{x}{p}] + \sum_{\substack{p|n \\ q|n \\ p\neq q \\ pq\leq x}} [\frac{x}{pq}] - \sum [\frac{x}{pqr}] + \ldots$$

$$= x(1 - \sum_{p|n} \frac{1}{p} + \sum_{\substack{p|n \\ q|n}} \frac{1}{pq} \ldots) + \text{error} = x\frac{\phi(n)}{n} + \text{error}$$

Using Lemma 4 we can now estimate the error term:

$$\text{Error} \leq \sum_{\substack{p|n \\ p\leq x}} \frac{1}{p} + \sum_{\substack{p|n \\ q|n \\ p\neq q \\ pq\leq x}} \frac{1}{pq} + \ldots \leq \sum_{\substack{d|n \\ d\leq x}} \frac{|\mu(d)|}{d} \leq \sum_{d|n} \frac{|\mu(d)|}{d} = O(\log\log n)$$

Therefore the distribution function $F(x)$ of the number of ℓ relatively prime to n is given by:

$$F(x) = \frac{\phi(n)}{n} x + O(\log\log n) \quad (0 \leq x \leq n) .$$

We obtain the following corollaries of Proposition 3.

$$(10) \qquad \sum_{\substack{\ell \leq n \\ a \leq \ell \leq b \leq n}} 1 = F(b) - F(a) = (b-a) \frac{\phi(n)}{n} + O(\log\log n)$$

$$(11) \qquad \sum_{\substack{1|n \\ a \leq \ell \leq b}} \ell = \int_a^b x dF = xF(x) \Big|_a^b - \int_a^b F(x) dx$$

$$= x^2 \frac{\phi(n)}{n} \Big|_a^b + O(\log\log n)\ (b-a) - \frac{\phi(n)}{n} \int_a^b x dx + (b-a)\ O(\log\log n$$

$$= \frac{1}{2} x^2 \frac{\phi(n)}{n} \Big|_a^b + (b-a)\ O(\log\log n)$$

$$= \frac{b^2 - a^2}{2} \frac{\phi(n)}{n} + (b-a)\ O(\log\log n) .$$

The following proposition follows now immediately from the explicit formulas (5) and (6) and these corollaries.

Proposition 5. The following asymptotic development holds:

$$(12) \qquad L(N) = N^3/\pi^2 + O(N^2 \log N) .$$

Values of $L(N)$ and the leading term, $L'(N)$, of its asymptotic formula were computed for $N = 100$:

$$L(N) = 104,359 , \quad L'(N) = 104,949 ,$$

resulting in a relative error of 1/2% which shows that the asymptotic formula is quite effective for use in problems of a statistical nature. In particular, this formula proves the image registration accuracy results conjectured in [4]. In effect, the result is the following:

Given a digital line segment, λ, of length N, we gave in [4], a choice of a linear function $\ell^*(x) = a^*x + b^*$ such that the line with equation $y = \ell^*(x)$ has digitization λ and minimizes the quantity

$$\varepsilon(\lambda) = \min_{\ell^*} \max_{\ell} \max_{0 \leq x \leq N} |\ell(x) - \ell^*(x)|$$

where ℓ range also over all linear functions such that the digitization of the line $y = \ell(x)$ is λ. From Proposition 3 one can deduce that the expected value, $E(N)$, of $\varepsilon(\lambda)$ over all λ of length N is given by

$$\left(\frac{3(1-\log 2)}{\pi^2}\right) \frac{1}{N} + O(\log N/N^2) . \tag{13}$$

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