SRC TR 86-19

Dynamic Observers as Asymptotic Limits of Recursive Filters: Linear Case

by

J. S. Baras and M. R. James

Dynamic Observers as Asymptotic Limits of Recursive Filters : Linear Case

J. S. Baras   M. R. James

Systems Research Centre,
University of Maryland,
College Park, MD, 20742.

8 April 1986

1 Introduction

Our objective is to construct an observer for the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

as the asymptotic limit of (Kalman) filters for a family of associated filtering problems

$$\begin{aligned} dx^\epsilon(t) &= Ax^\epsilon(t)dt + Bu(t)dt + \sqrt{\epsilon}N dw(t), \quad x^\epsilon(0) = x_0^\epsilon, \\ d\xi^\epsilon(t) &= Cx^\epsilon(t)dt + \sqrt{\epsilon}R dv(t), \quad \xi^\epsilon(0) = 0. \end{aligned} \tag{2}$$

Such a construction is suggested by the fact that for certain choices of $P_0^\epsilon = \text{cov}(x_0^\epsilon)$, the filters are independent of ϵ, as discussed in Baras and Krishnaprasad [1]. Also, the theory of large deviations says that the solutions of (2) "converge" as $\epsilon \to 0$ to the solution of (1).

The motivation for such an approach is the hope that it might lead to the construction of observers for certain nonlinear systems; for example, the Beneš case [4].

The work of Hijab [2], [3] is indispensible here in deriving a large deviation principle for the conditional measures $P_{x|\xi}^\epsilon$ (see Section 4), and identifying the limit of the filters for (2) as an associated deterministic estimator.

2 Observers and Filters

We assume as usual that $x(t) \in I\!R^n$, $u(t) \in I\!R^m$, $y(t) \in I\!R^p$, and $t \mapsto u(t)$ is piecewise continuous.

Recall that the *observer* problem consists of constructing a dynamical system

$$\dot{z}(t) = Ez(t) + Fu(t) + Gy(t), \quad z(0) = z_0, \tag{3}$$

so that the error

$$e(t) = x(t) - z(t) \tag{4}$$

decays exponentially fast to zero, at a rate controlled by the designer, independent of choice of z_0 and x_0. This reflects the fact that the initial condition x_0 is unknown, and the best that can be done is to approximately estimate $x(t)$ by $z(t)$ in this way.

Solutions to this problem are well known, first given by Luenberger [5]. In particular, if the pair (A,C) is detectable, then there exists a matrix Γ such that the matrix $A + \Gamma C$ has eigenvalues in the open left half plane. Then set

$$E = A + \Gamma C, \quad F = B, \quad G = -\Gamma.$$

In this case the error (4) satisfies

$$\dot{e}(t) = (A + \Gamma C)e(t), \quad e(0) = x_0 - z_0,$$

and the eigenvalues of $A + \Gamma C$ can be arbitrarily assigned by the designer if and only if (A,C) is observable.

Consider the system (1). Define $\xi(t) = \int_0^t y(s)ds$, so that (1) becomes

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ \dot{\xi}(t) &= Cx(t), \quad \xi(0) = 0. \end{aligned} \tag{5}$$

Then associate with (5) the family of filtering problems (2), where w, v are independent standard n-dimensional, respectively p-dimensional Brownian motions. The initial condition x_0 is Gaussian, independent from v, w with $E(x_0^\epsilon) = \bar{x}_0^\epsilon$, $\text{cov}(x_0^\epsilon) = P_0^\epsilon$, where P_0^ϵ is positive definite. Note that the (small) parameter ϵ controls the intensity of the noise.

As is well known, the minimum variance estimate $\hat{x}^\epsilon(t) = E(x(t) \mid \xi^\epsilon(s), 0 \leq s \leq t)$ for the linear Gausssian filtering problem (2) is given by the Kalman filter [6]

$$\begin{aligned} d\hat{x}^\epsilon(t) &= A\hat{x}^\epsilon(t)dt + Bu(t)dt + K^\epsilon(t)C'(RR')^{-1}C(d\xi^\epsilon(t) - C\hat{x}^\epsilon(t)dt), \\ \hat{x}^\epsilon(0) &= \bar{x}_0^\epsilon, \end{aligned} \tag{6}$$

where K^ϵ satisfies the Riccati equation

$$\begin{aligned} \dot{K}^\epsilon(t) &= AK^\epsilon(t) + K^\epsilon(t)A' - K^\epsilon(t)C'(RR')^{-1}CK^\epsilon(t) + NN', \\ K^\epsilon(0) &= P_0^\epsilon/\epsilon. \end{aligned} \tag{7}$$

Note that these filters depend on ϵ only via the matrix P_0^ϵ/ϵ. In fact, if we choose $P_0^\epsilon = \epsilon P_0$, then all the filters are independent of ϵ and identical with the filter for $\epsilon = 1$.

Following Hijab [2], it is convenient to consider the filter (6), (7) as a map

$$\begin{aligned} F^{\epsilon}: C\left([0,t], \mathbb{R}^{p}\right) &\longrightarrow \mathbb{R}, \\ \xi(s), 0 \leq s \leq t &\longmapsto \hat{x}^{\epsilon}(t). \end{aligned}$$

3 Deterministic Estimation

Following Mortensen [7] and Hijab [2], we associate with (5) the deterministic (noisy) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Nw(t), \quad x(0) = \bar{x}, \\ \dot{\xi}(t) &= Cx(t) + Rv(t), \quad \xi(0) = 0, \end{aligned} \tag{8}$$

and energy cost functional

$$J_t(\bar{x}, w, v) = (\bar{x} - \bar{x}_0)\, P_0^{-1}\, (\bar{x} - \bar{x}_0) + \frac{1}{2}\int_0^t \left(w(s)'w(s) + v(s)'(RR')^{-1}v(s)\right) ds, \tag{9}$$

where $t \mapsto w(t)$, $t \mapsto v(t)$ are piecewise contiuous.

A minimum energy input triple $(\bar{x}^*, w^*, v^*)$ given $\xi(s)$, $0 \leq s \leq t$, is a triple that minimises J_t subject to (8) and produces the given output record $\xi(s)$, $0 \leq s \leq t$. The *deterministic* or minimum energy *estimate* of $x(t)$ given $\xi(s)$, $0 \leq s \leq t$, is the endpoint $\hat{x}(t)$ of the trajectory of (8) corresponding to a minimum energy input triple.

According to Krener [8], $\hat{x}$ is the solution of the Kalman filter equations

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + P(t)C'(RR')^{-1}(\dot{\xi}(t) - C\hat{x}(t)), \\ \hat{x}(0) &= 0, \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{P}(t) &= AP(t) + P(t)A' - P(t)C'(RR')^{-1}CP(t) + NN', \\ P(0) &= P_0. \end{aligned} \tag{11}$$

As in the stochastic case (Section 2), it is convenient to consider the deterministic filter (10), (11) as a map

$$\begin{aligned} F: C([0,t], \mathbb{R}^{p}) &\longrightarrow \mathbb{R}, \\ \xi(s),\ 0 \leq s \leq t &\longmapsto \hat{x}(t). \end{aligned}$$

Note that this corresponds to the stochastic filter for $\epsilon = 1$, that is, F^1. Also, $\hat{x}(t)$ is obtained from an optimal control problem, and is determined by a Hamilton–Jacobi–Belman equation [2], [7].

4 Large Deviations

Consider the stochastic differential equation (2). Let P_x^ϵ be the probability measure induced on $\Omega^n = C([0,t], \mathbb{R}^n)$ by the diffusion x^ϵ. Assuming that $x_0^\epsilon \to x_0$ as $\epsilon \to 0$, the large deviation theory of Ventcel–Friedlin (see Varadhan [9]) suggests that as $\epsilon \to 0$, P_x^ϵ converges weakly to the degenerate measure concentrated on the unique solution x of (1).

There are several formulations of this large deviation principle available in the literature, for example Azencott [11], Varadhan [9] and Hijab [3]. It is convenient for us to follow Hijab's approach. In order to do so directly, we shall assume in this section that $x_0^\epsilon = x_0$ a.s., and $u \equiv 0$. For $\omega \in \Omega^n$, let x_ω denote the unique solution to (8) with initial condition $x_\omega(0) = x_0$. The following result is proven in Hijab [3].

Theorem 4.1 *For any open subset $\mathcal{O}$ and any closed subset $\mathcal{C}$ of Ω^n,*

$$\begin{aligned} \limsup_{\epsilon \to 0} \epsilon \log P_x^\epsilon(\mathcal{O}) &\geq -\inf \left\{ \frac{1}{2} \int_0^T \omega(s)'\omega(s) ds \mid x_\omega \in \mathcal{O} \right\}, \\ \liminf_{\epsilon \to 0} \epsilon \log P_x^\epsilon(\mathcal{C}) &\leq -\inf \left\{ \frac{1}{2} \int_0^T \omega(s)'\omega(s) ds \mid x_\omega \in \mathcal{C} \right\}. \end{aligned}$$

Remark 4.2 Azencott [11], Proposition 2.10, shows that the action functional or rate function is given by

$$\begin{aligned} I(\phi) &= \frac{1}{2} \int_0^T \langle \dot{\phi}(t) - A\phi(t), (NN')^{-1} \left(\dot{\phi}(t) - A\phi(t) \right) \rangle dt \\ &= \inf \left\{ \frac{1}{2} \int_0^T \omega(s)'\omega(s) ds \mid \phi = x_\omega \right\}. \end{aligned}$$

We now consider the observation equation in (2). Let $P^\epsilon_{x|\xi}$ be the conditional probability measure on Ω^n of x^ϵ given $\xi^\epsilon \in \Omega^p = C([0,T], \mathbb{R}^p)$. Hijab proves the following large deviation result for $P^\epsilon_{x|\xi}$.

Theorem 4.3 *For any open subset $\mathcal{O}$ and any closed subset $\mathcal{C}$ of Ω^n,*

$$\begin{aligned}
\liminf_{\epsilon\to 0} \epsilon \log P^\epsilon_{x|\xi}(\mathcal{O}) &\geq -\inf\{I(\omega,\xi) \mid x_\omega \in \mathcal{O}\} \\
\limsup_{\epsilon\to 0} \epsilon \log P^\epsilon_{x|\xi}(\mathcal{C}) &\leq -\inf\{I(\omega,\xi) \mid x_\omega \in \mathcal{C}\}
\end{aligned}$$

where

$$\begin{aligned}
I(\omega,\xi) &= J(\omega,\xi) - \inf\{J(\omega,\xi) \mid \omega \in \Omega^n\} \\
J(\omega,\xi) &= \int_0^T \left(\frac{1}{2}\omega(s)'\omega(s) + \frac{1}{2}x_\omega(s)'C'Cx_\omega(s)\right) ds - \int_0^T Cx_\omega(s)d\xi(s).
\end{aligned}$$

Proof Define, for each $\xi \in \Omega^p, \omega \in \Omega^n$,

$$\begin{aligned}
\phi(\omega,\xi) &= -\xi(T)'C\omega(T) + \\
&\int_0^T \left(\xi(t)'CA\omega(t) + \frac{1}{2}\omega(t)'C'C\omega(t) - \frac{1}{2}\xi(t)'CNN'C'\xi(t)\right) dt.
\end{aligned}$$

There exist constants A, B depending only on ξ, such that

$$-\phi(\omega,\xi) \leq A + B\|\omega\|.$$

Then arguing as in Varadhan [10],

$$\lim_{R\to\infty} \limsup_{\epsilon\to 0} \epsilon \log \int_{\{\omega:\ -\phi(\omega,\xi)\geq R\}} \exp\left(-\frac{1}{\epsilon}\phi(\omega,\xi)\right) dP^\epsilon_x = -\infty.$$

But this estimate is enough to prove the theorem. See Hijab [3] and Varadhan [10] for details. □

This theorem implies that if ξ is an actual output record of the system (5), then as $\epsilon \to 0$, $P^\epsilon_{x|\xi}$ converges weakly to a degenerate measure concentrated on the corresponding solution of (5).

Note 4.4 The minimisation of the action functionals I, J is related to the deterministic estimator of Section 3. It is suspected that more general versions of Theorems 4.1 and 4.3 are true, taking initial conditions into acount. This would make the relationship clearer.

5 Observer Construction

We now prove that as $\epsilon \to 0$ the stochastic filter F^ϵ converges to the deterministic filter F.

Theorem 5.1 *Suppose that (2) has initial conditions x_0^ϵ Gaussian with mean $\bar{x}_0$ and covariance P_0^ϵ satisfying*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} P_0^\epsilon = P_0,$$

where P_0 is positive definite. Let $\xi \in \Omega^p$ and $\hat{x}^\epsilon(t), \hat{x}(t)$ be the corresponding estimates. Then

$$\lim_{\epsilon \to 0} \hat{x}^\epsilon(t) = \hat{x}(t).$$

Proof Let $\phi^\epsilon(t,s), \phi(t,s)$ be the transition matrices for $A - K^\epsilon(t)C'(RR')^{-1}C$, $A - P(t)C'(RR')^{-1}C$ respectively. Since linear ordinary differential equations depend continuously on their initial conditions, then as $\epsilon \to 0$,

$$\begin{aligned} K^\epsilon(s) &\longrightarrow P(s) \text{ uniformly in } s \in [0,t], \\ \phi^\epsilon(\tau,s) &\longrightarrow \phi(\tau,s) \text{ uniformly in } \tau, s \in [0,t]. \end{aligned}$$

From Sections 2 and 3, the estimates are given by

$$\begin{aligned} \hat{x}^\epsilon(t) &= \phi^\epsilon(t,0)\bar{x}_0 + \int_0^t \phi^\epsilon(t,s)\left(Bu(s) + K^\epsilon(s)C'(RR')^{-1}\dot{\xi}(s)\right) ds, \\ \hat{x}(t) &= \phi(t,0)\bar{x}_0 + \int_0^t \phi(t,s)\left(Bu(s) + P(s)C'(RR')^{-1}\dot{\xi}(s)\right) ds. \end{aligned}$$

The theorem then follows using the triangle inequality. □

Next we construct an observer for the system (1). Assume that (C, A) is observable and (A, N) is controllable. Then as $t \to \infty$, $P(t) \to \bar{P}$, where $\bar{P}$ is the unique positive definite solution of the algebraic equation [6]

$$A\bar{P} + \bar{P}A' - \bar{P}C'(RR')^{-1}C\bar{P} + NN' = 0. \tag{12}$$

Furthermore, the matrix

$$\bar{A} = A - \bar{P}C'(RR')^{-1}C \tag{13}$$

is exponentially stable.

Choose the initial covariances P_0^ϵ in such a way that $P_0 = \bar{P}$ (for example, $P_0^\epsilon = \epsilon\bar{P}$). Then the deterministic filter (10), (11) is time invariant. We use this filter to specify the matrices $E = \bar{A},\ F = B,\ G = -\bar{P}C'(RR')^{-1}$, giving the observer

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) + \bar{P}C'(RR')^{-1}\left(y(t) - Cz(t)\right), \\ z(0) &= z_0, \end{aligned} \tag{14}$$

where $\bar{P}$ satisfies (12).

When $z_0 = x_0,\ z(t) = \hat{x}(t)$ and by Theorem 5.1, $\hat{x}^\epsilon(t) \to z(t)$ as $\epsilon \to 0$. The observer error satisfies

$$\dot{e}(t) = \bar{A}e(t), \quad e(0) = x_0 - z_0, \tag{15}$$

which is exponentially stable. We have proved the following theorem.

Theorem 5.2 *Given the linear system (1), where (C, A) is observable, an $n \times n$ matrix N such that (A, N) is controllable, and a $p \times n$ matrix R such that RR' is positive definite, then there exists a unique positive definite solution $\bar{P}$ to the algebraic Riccati equation (12) , the matrix $\bar{A}$ is exponentially stable, and the system (14) is an observer for the given system (1).*

Note 5.3 If we make the weaker assumptions that (C, A) is detectable, (A, N) is stabilisable, then $P_0 = \bar{P}$ is positive semi–definite and the energy functional (9) may not be defined. However, (14) is still an observer since $\bar{A}$ will still be exponentially stable.

Remark 5.4 The exponential decay of the observer error equation (15) is controlled by the design matrices R, N via the Riccati equation (12). An interesting algebraic problem is the following:

> *Given an observable (detectable) pair (C, A), analyse the dependence of the spectrum of $\bar{A}$, given by (12), (13), on matrices R, N such that RR' is positive definite and the pair (A, N) is controllable (stabilisable).*

6 Conclusion

We have shown how an observer for a linear system can be obtained as a limit of Kalman filters for an associated family of filtering problems. The limit was identified as a deterministic estimator for an associated problem. The theory of large deviations suggests deeper connections between these ideas. Finally, an interesting algebraic problem was posed.

Much of what we have done can be extended to certain nonlinear systems. Following Hijab [2], the above suggests that the limiting filters correspond to deterministic estimators, which depend on the Hamilton–Jacobi–Bellman equation. In general, this is infinite dimensional. In cases where finite dimensional filters exist, observability and controllability conditions need to be examined to determine whether the corresponding error equation is asymptotically stable, and so giving an observer. The Beneš case [4], with appropriate observations, is a candidate for such an investigation. This will be taken up in a latter paper.

References

[1] J. S. Baras and P. S. Krishnaprasad, *Dynamic Observers as Asymptotic Limits of Recursive Filters*, IEEE Proc. 21st CDC, Orlando, Florida, Dec., 1982.

[2] O. Hijab, *Minimum Energy Estimation*, PhD Dissertation, University of California, Berkely, December, 1980.

[3] O. Hijab, *Asymptotic Estimation of a First Order Equation with Small Diffusion*, May 2, 1983, Submitted.

[4] V. E. Beneš, *Exact Finite-Dimensional Filters for Certain Diffusions witn Nonlinear Drift*, Stochastics, Vol. 5, pp65–92, 1981.

[5] D. G. Luenberger, *Observers for Multivariable Systems*, IEEE Trans. Aut. Control, Vol. AC–11, pp190–199, 1966.

[6] M. H. A. Davis, *Linear Estimation and Stochastic Control*, London, Chapman and Hall, 1977.

[7] R. E. Mortensen, *Maximum–Likelihood Recursive Nonlinear Filtering*, J. Opt. Theory and Appl., Vol. 2, No. 6, pp386–394, 1968.

[8] A. J. Krener, *Minimum Covariance, Minimax and Minimum Energy Estimators*, in *Stochastic Control Theory and Stochastic Differential systems*, M. Kohlmann and W. Vogel (eds.), Berlin, Springer–Verlag, 1979.

[9] S. R. S. Varadhan, *Large Deviations and Applications*, CBMS–NSF Regional Conf. Series in Appl. Math., Philadelphia, SIAM, 1984.

[10] S. R. S. Varadhan, *Asymptotic Probabilities and Differential Equations*, Comm. Pure and Appl. Math, Vol. 19, pp261–286, 1966.

[11] R. Azencott, *Grandes Deviations et Applications*, Lecture Notes in Math., No. 774, Springer–Verlag, 1978.