MARYLAND

TECHNICAL RESEARCH REPORT

SRC TR 86-16

Geometric Aspects In The Computation Of The Structured Singular Value

By

M. K. Fan & A. L. Tits

SYSTEMS RESEARCH CENTER

UNIVERSITY OF MARYLAND

COLLEGE PARK, MARYLAND 20742

Proceedings of 1986 ACC Seattle, WA June 1986

GEOMETRIC ASPECTS IN THE COMPUTATION OF THE STRUCTURED SINGULAR VALUE[1]

Michael K.H. Fan and André L. Tits

Electrical Engineering Department and Systems Research Center
University of Maryland, College Park, MD 20742

ABSTRACT

The concept of *structured singular value* was recently introduced by Doyle (Proc. IEE, vol. 129, pp. 242-250, 1982) as a tool for the analysis and synthesis of feedback systems with structured uncertainties. In this paper an equivalent expression for the structured singular value is proposed, leading to an alternative algorithm for its computation. The new approach is based on the geometric properties of a certain family of sets. Similar to previously considered schemes, the algorithm proposed here is proven to give the correct value for block-structures of size up to 3. For larger sizes, insight is gained in the question of the possible 'gap' between the structured singular value and its known upper bound.

INTRODUCTION AND PRELIMINARIES

The concept of *structured singular value* was recently introduced by Doyle [1] as a tool for the analysis and synthesis of feedback systems with structured uncertainties. In this paper an equivalent expression for the structured singular value is proposed, leading to an alternative algorithm for its computation. The new approach is based on the geometric properties of a certain family of sets. Similar to previously considered schemes, the algorithm proposed here is proven to give the correct value for block-structures of size up to 3. For larger sizes, insight is gained in the question of the possible 'gap' between the structured singular value and its known upper bound.

Throughout the paper, given any square complex matrix A, we denote by $\overline{\sigma}(A)$ its largest singular value and by A^H its complex conjugate transpose. If A is Hermitian, $A>0$ (resp. $A\geq 0$) expresses that A is positive definite (resp. semi-positive definite). Given any complex vector x, x^H indicates its complex conjugate transpose and $\|x\|$ its Euclidean norm. The unit sphere in C^n is denoted by ∂B, i.e., $\partial B=\{x \in C^n, \|x\|=1\}$. Given a set S, co$S$ and int$S$ respectively denote its convex hull and interior. A *block-structure* of size m is any m-tuple $\mathcal{k}=(k_1,\cdots,k_m)$ of positive integers.[2] Given a block-structure $\mathcal{k}$ of size m, we make use of the family of diagonal matrices

$$\mathcal{d} = \{\text{block diag}\,(d_1 I_{k_1},\cdots,d_m I_{k_m}) \mid d_i \in (0,\infty)\}$$

and of the projection matrices

$$P_i = \text{block diag}\,(O_{k_1},\cdots,O_{k_{i-1}},I_{k_i},O_{k_{i+1}},\cdots,O_{k_m}),$$

where, for any positive integer k, I_k is the $k\times k$ identity matrix and O_k the $k\times k$ zero matrix.

Definition 1. [3] [2, 3] The *structured singular value* of a complex $n\times n$ matrix M *with respect to the block-structure* $\mathcal{k}=(k_1,\cdots,k_m)$ of size m, where $n=\sum_{i=1}^{m} k_i$, is the nonnegative scalar

$$\mu = \max_{x\in C^n} \{\, \|Mx\| \mid \|P_i x\|\,\|Mx\| = \|P_i Mx\|, \; i=1,\cdots,m \,\}. \qquad (1)$$

Notice in particular that, if $\mathcal{k}=(n)$, the structured singular value is equal to the largest singular value $\overline{\sigma}(M)$.

The question of how to compute μ has been addressed by several authors. Doyle [1] showed that, while in general

$$\mu \leq \inf_{D\in\mathcal{d}} \overline{\sigma}\,(DMD^{-1}), \qquad (2)$$

if $m\leq 3$,

$$\mu = \inf_{D\in\mathcal{d}} \overline{\sigma}\,(DMD^{-1}). \qquad (3)$$

He produced a counterexample to (3) for the case $m=4$ [4]. The minimization problem in (2) has no stationary point other than its global minimizers [5] and algorithms exist to solve it [1, 6]. Another approach for computing μ is to solve directly optimization problem (1). Although local maxima are generally present, global optimality can be checked whenever (3) holds [3]. For such cases, an algorithm proposed in [3] results in a typical speedup of an order of magnitude over algorithms based on (3). However, the question of obtaining an algorithm to compute μ for any block-size remains open. It is hoped that the geometric framework introduced in this paper for the computation of μ will contribute to the progress towards answering this question.

In the next section, we propose a new algorithm for computing the structured singular value, based on the distance

[1] This research was supported by the National Science Foundation under grants No DMC-84-51515 and OIR-85-00108. During the time the research was performed, the first author was a Fellow of the Minta Martin Foundation, College of Engineering, University of Maryland.

[2] This corresponds, in the terminology of [1], to structures with no repeated blocks

[3] This definition of the structured singular value, while computationally more tractable, is equivalent to that originally proposed by Doyle [1].

between the origin and members of a certain family of subsets $V(\alpha)$ of $\mathbb{R}^m$. This distance can be efficiently computed when these subsets are convex, which is shown to be the case for block-structures of size no larger than 3. In the final section, a new upper bound for μ is derived for the case $m \geq 4$ and upper bound gaps are related to a specific type of non-convexity of $V(\mu^2)$. An algorithm for plotting the boundary of a set $V(\alpha)$ when $m = 2$ is given in the appendix.

CONCEPTS AND ALGORITHM

For $i = 1, \cdots, m$, and any real number α, let $A_i(\alpha)$ be defined as

$$A_i(\alpha) = \alpha P_i - M^H P_i M ,$$

and consider the nonnegative scalar function $c(\cdot)$ defined as

$$c(\alpha) = \min \{ \|v\| \mid v \in V(\alpha) \}$$

where

$$V(\alpha) = \{ v \in \mathbb{R}^m \mid v_i = x^H A_i(\alpha) x , \; x \in \partial B \} .$$

Our first theorem gives an equivalent formula for μ.

Theorem 1. $c(\mu^2) = 0$, and $c(\alpha) > 0$ for all $\alpha > \mu^2$, so that

$$\mu = (\inf \{ \alpha \mid c(\beta) > 0 \text{ for all } \beta > \alpha \})^{1/2} .$$

Proof. First, let $\bar{x} \in \partial B$ be a global solution of (1), i.e., $\mu = \|M\bar{x}\|$ and $\mu \|P_i \bar{x}\| = \|P_i M\bar{x}\|$ for $i = 1, \cdots, m$. This implies

$$c(\mu^2) = \min_{x \in \partial B} \{ (\sum_{i=1}^m (x^H A_i(\mu^2) x)^2)^{1/2} \}$$

$$\leq (\sum_{i=1}^m (\bar{x}^H A_i(\mu^2) \bar{x})^2)^{1/2}$$

$$= (\sum_{i=1}^m (\mu^2 \|P_i \bar{x}\|^2 - \|P_i M\bar{x}\|^2))^{1/2} = 0 .$$

Thus

$$c(\mu^2) = 0 .$$

Second, let α be such that $c(\alpha) = 0$. There must exist some $x \in \partial B$ such that, for $i = 1, \cdots, m$,

$$\alpha \|P_i x\|^2 - \|P_i Mx\|^2 = 0 . \qquad (4)$$

For any such x, one then has

$$\alpha = \sum_{i=1}^m \alpha \|P_i x\|^2 = \|Mx\|^2 . \qquad (5)$$

Relations (4) and (5) imply that x is a feasible point for (1). Hence, by (1),

$$\mu^2 \geq \|Mx\|^2 = \alpha .$$

[]

Proposition 1. For any $s \geq 0$, and for any real α

$$c(\alpha + s) \leq c(\alpha) + s$$

Proof.

$$c(\alpha + s) = \min_{x \in \partial B} \{ (\sum_{i=1}^m (x^H A_i(\alpha + s) x)^2)^{1/2} \}$$

$$= \min_{x \in \partial B} \{ (\sum_{i=1}^m (x^H A_i(\alpha) x + s \|P_i x\|^2)^2)^{1/2} \}$$

Using the triangular inequality in $\mathbb{R}^m$, we obtain

$$c(\alpha + s) \leq \min_{x \in \partial B} \{ (\sum_{i=1}^m (x^H A_i(\alpha) x)^2)^{1/2} + s (\sum_{i=1}^m (\|P_i x\|^2)^2)^{1/2} \}$$

Replacing the second term in the 'min' by its constrained maximum, since $s \geq 0$, we can write

$$c(\alpha + s) \leq \min_{x \in \partial B} \{ (\sum_{i=1}^m (x^H A_i(\alpha) x)^2)^{1/2} + s \} = c(\alpha) + s$$

[]

Based on these facts, provided one has an algorithm to compute $c(\alpha)$, μ can be obtained as follows.

Algorithm 1. Computation of μ

Step 0. Set $\alpha_0 = \bar{\sigma}^2(M)$ and $k = 0$.
Step 1. Set $\alpha_{k+1} = \alpha_k - c(\alpha_k)$.
Step 2. Set $k = k+1$ and go to Step 1.

[]

Theorem 2. The sequence $\{\alpha_k\}$ generated by Algorithm 1 is monotone decreasing and

$$\lim_{k \to \infty} \alpha_k = \mu^2 .$$

Proof. We first show by induction that $\alpha_k \geq \mu^2$ for all k. Clearly, $\alpha_0 = \bar{\sigma}^2(M) \geq \mu^2$. Assuming the claim is true for k and using Proposition 1 and Theorem 1, we can write

$$c(\alpha_k) = c(\mu^2 + (\alpha_k - \mu^2)) \leq c(\mu^2) + \alpha_k - \mu^2 = \alpha_k - \mu^2 .$$

Thus, in view of the construction in Step 1 of Algorithm 1, the claim is true for $k+1$. Now, since the sequence $\{\alpha_k\}$ is monotone nonincreasing, it follows that it converges to a limit α^* satisfying

$$\alpha^* \geq \mu^2 .$$

Since $c(\cdot)$ is clearly continuous, the construction in Step 1 now implies, letting $k \to \infty$,

$$\alpha^* = \alpha^* - c(\alpha^*)$$

and thus $c(\alpha^*) = 0$. The result now follows from Proposition 1.

[]

Existing algorithms [1, 6], proposed in a slightly different context, can be used to compute the distance between the origin and $\mathrm{co}V(\alpha)$, thus yielding $c(\alpha)$, among other instances, when $V(\alpha)$ is convex. The next proposition shows that the latter case is of definite interest.

Proposition 2. For $m \leq 3$ and for any real number α, $V(\alpha)$ is convex.

[]

In proving this proposition we will make use of the following three lemmas.

Lemma 1. Let $m = 3$. Then for any real number α and for any $u, v \in V(\alpha)$, there exists an ellipsoid[4] $E(u, v)$, *possibly*

[4] Following the standard usage we call ellipsoid a surface, not a volume, to be compared to a sphere rather than to a ball. Lemma 1 states that such a surface 'passes through' u and v and is entirely in $V(\alpha)$.

degenerate, containing u and v and contained in $V(\alpha)$.

Proof. For $i=1, 2, 3$, let

$$H_i = \begin{bmatrix} a_i & b_i \\ \overline{b}_i & c_i \end{bmatrix},$$

where $a_i, c_i \in \mathbb{R}$, $b_i \in C$, and $\overline{b}_i$ is the complex conjugate of b_i. Any unit vector e in C^2 can be written as

$$e = \exp(j\,\phi) \begin{bmatrix} \cos(\theta) \\ \sin(\theta)(\cos(\psi)+j\sin(\psi)) \end{bmatrix}$$

for some $\phi, \theta, \psi \in \mathbb{R}$, where j is the square root of -1. Elementary manipulations yield

$$e^H H_i e = \frac{a_i+c_i}{2} + \begin{bmatrix} \frac{a_i-c_i}{2} & \mathrm{Re}(b_i) & \mathrm{Im}(b_i) \end{bmatrix} \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta)\cos(\psi) \\ \sin(2\theta)\sin(\psi) \end{bmatrix} \tag{6}$$

so that the set E defined by

$$E = \left\{ \begin{bmatrix} e^H H_1 e \\ e^H H_2 e \\ e^H H_3 e \end{bmatrix} \mid e \in C^2, \|e\|=1 \right\} \tag{7}$$

is a possibly degenerate ellipsoid centered at

$$\frac{1}{2}\begin{bmatrix} a_1+c_1 \\ a_2+c_2 \\ a_3+c_3 \end{bmatrix}.$$

Now let $\alpha \in \mathbb{R}$ and $u, v \in V(\alpha)$ and let $x, y \in C^n$ be the unit vectors such that, for $i=1, 2, 3$, $u_i = x^H A_i(\alpha) x$ and $v_i = y^H A_i(\alpha) y$. Without loss of generality, assume that x and y are linearly independent. Pick an orthogonal basis $\{\overline{x}, \overline{y}\}$ for the subspace of $\mathbb{R}^n$ spanned by $\{x, y\}$. For $i=1, 2, 3$, let

$$H_i = [\overline{x}\ \overline{y}]^H A_i(\alpha) [\overline{x}\ \overline{y}] \tag{8}$$

and denote by $E(u,v)$ the corresponding ellipsoid (7). It is easily checked that $E(u,v)$ satisfies the required conditions.

□

Lemma 2. $V(\alpha)$ is not a *non-degenerate* ellipsoid.

Proof. By contradiction. Suppose $V(\alpha)$ is a non-degenerate ellipsoid. Clearly such a set does not have any ellipsoid as a proper subset (except for single points). Thus, in particular, if x, y and z are any points in ∂B such that[5] $<x,y> = <x,z> = 0$ and if u, v and w in $\mathbb{R}^3$ have components $u_i = x^H A_i(\alpha) x$, $v_i = y^H A_i(\alpha) y$, $w_i = z^H A_i(\alpha) z$,

$$E(u,v) = E(u,w)$$

where these sets are ellipsoids as in Lemma 1. The centers of these two ellipsoids must coincide, i.e., from (6) and (8),

$$x^H A_i(\alpha) x + y^H A_i(\alpha) y = x^H A_i(\alpha) x + z^H A_i(\alpha) z \quad i=1, 2, 3$$

so that

$$y^H A_i(\alpha) y = z^H A_i(\alpha) z \quad i=1, 2, 3.$$

Since y and z are arbitrary, this implies that, for some $\gamma_i \in C$

$$A_i(\alpha) = \gamma_i I \quad i=1, 2, 3$$

so that $V(\alpha)$ is a singleton. This is a contradiction.

□

Lemma 3. For any $u, v \in V(\alpha)$, $\mathrm{co}E(u,v) \subset V(\alpha)$, where $E(u,v)$ is as in Lemma 1.

Proof. Let $u, v \in V(\alpha)$. If $E(u,v)$ is a degenerate ellipsoid, then $E(u,v) = \mathrm{co}E(u,v)$ and the result follows from Lemma 1. Suppose now that $E(u,v)$ is non-degenerate. Lemma 2 implies that there exists $w^* \in V(\alpha)$ with $w^* \notin E(u,v)$. Suppose first that $w^* \in \mathrm{int\,co}E(u,v)$. For any $w \in \mathrm{int\,co}E(u,v)$, $w \neq w^*$, denote by w_1 and w_2 the intersections of the line through w^* and w with $E(u,v)$, with w_1 on the side of w^* and w_2 on the side of w. Clearly

$$w \in \mathrm{co}E(w^*, w_2)$$

but

$$w \notin \mathrm{co}E(w^*, w_1).$$

Continuity considerations then show that there is some $\overline{w} \in E(u,v)$ such that a corresponding $E(w^*, \overline{w})$ passes through w. This implies that $w \in V(\alpha)$, so that $\mathrm{co}E(u,v) \subset V(\alpha)$. Suppose now that $w^* \notin \mathrm{int\ co}E(u,v)$. Again, for any $w \in \mathrm{int\ co}E(u,v)$ denote by w_1 and w_2 the intersections of the line through w^* and w with $E(u,v)$, this time with w_1 on the far side from w^*. Since, if $E(w^*, w_1)$ is nondegenerate, $w_2 \in V(\alpha) \cap \mathrm{int\ co}E(w^*, w_1)$, it follows from the discussion of the previous case that $\mathrm{co}E(w^*, w_1) \subset V(\alpha)$. Since, clearly $w \in \mathrm{co}E(w^*, w_1)$, it follows that $\mathrm{co}E(u,v) \subset V(\alpha)$.

□

Proof of Proposition 2. If $m=1$, $V(\alpha)$ is an interval, thus it is convex. The proof for the case $m=2$ can be found in another context in [7] and [1]. For $m=3$, the result follows directly from Lemmas 1 and 3.

□

Figure 1 shows the boundaries of sets $V(\alpha)$ for the matrix

$$M = \begin{bmatrix} (1,-0.2) & (-0.1,5) & (0.2,3) \\ (0,1.2) & (0.1,0.1) & (0.1,0) \\ (1,2) & (0.3,-0.2) & (0.1,-2) \end{bmatrix}$$

with block structure $\mathcal{K} = [1,2]$, for α equal to the successive values 0, 15, $\mu^2 = 21.1$, 29, and $\overline{\sigma}^2(M) = 36.6$. The algorithm used for plotting these boundaries is given in the appendix.

In view of Proposition 2, Algorithm 1 provides an alternative way to compute the structured singular value for block-structures of size no larger than 3. It is not clear whether the proposed algorithm has any computational advantage over existing methods.

PROPERTIES OF UPPER BOUNDS FOR μ

For block-structures of size larger than 3, $V(\alpha)$ may not be convex. Algorithms from [1] and [6] would then yield, instead of $c(\alpha)$, the value

[5] Such a choice is possible because $n \geq m = 3$. This is in fact the only use we make of the properties of the A_i's. If matrices of size 2×2 were considered, Proposition 2 would have to be replaced by the weaker result that the corresponding sets

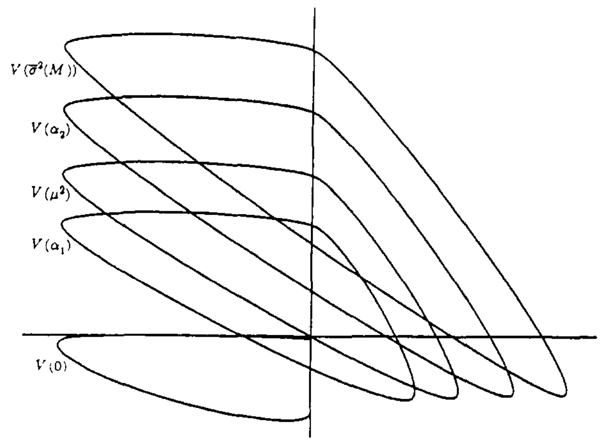

Figure 1. $V(\alpha)$ Sets, with $0<\alpha_1<\mu^2<\alpha_2<\bar{\sigma}^2(M)$

$$c'(\alpha) = \min \{\|v\| \mid v \in \mathrm{co}V(\alpha)\}$$

which is a lower bound for $c(\alpha)$. Using this value instead of $c(\alpha)$ in Algorithm 1 would then yield, as the limit of the sequence $\{\alpha_k^{1/2}\}$, an upper bound μ' for μ, with

$$\mu' = (\inf \{\alpha \mid c'(\beta)>0 \text{ for all } \beta>\alpha\})^{1/2} . \tag{9}$$

While it is not clear that this upper bound can be better than $\inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$, the following proposition shows that it is never worse.

Proposition 3. Let μ' be defined by (9). Then

$$\mu \leq \mu' \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) . \tag{10}$$

Proof. Let $\alpha > \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1})$. There exists $D =$block diag $(d_i I_{k_i}) \in \mathcal{D}$ such that, for any $\beta>\alpha$,

$$\beta I - (DMD^{-1})^H (DMD^{-1}) > 0 ,$$

which implies

$$\beta D^2 - M^H D^2 M > 0 ,$$

i.e.,

$$\sum_{i=1}^{m} d_i^2(\beta P_i - M^H P_i M) > 0$$

so that

$$\sum_{i=1}^{m} d_i^2 x^H (\beta P_i - M^H P_i M) x > 0 \quad \text{for all } x \in \partial B .$$

Hence,

$$<v,\lambda> > 0 \quad \text{for all } v \in V(\beta)$$

where $\lambda = [d_1^2 \cdots d_m^2]^T$. Therefore, for any $\beta>\alpha$, $\mathrm{co}V(\beta)$ does not contain the origin, so that $c'(\beta)>0$. In view of the definition of μ', we conclude that $\alpha \geq \mu'^2$.

[]

The next proposition shows that the absence of gap between the three quantities in (10) is equivalent to a separation condition for $V(\mu^2)$.

Proposition 4.

$$\mu = \mu' = \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) \tag{11}$$

if, and only if there exists a vector $\lambda \in \mathbb{R}^m$, with $\lambda_i \geq 0$ for all i, such that $V(\mu^2)$ is contained in the closed half space

$$H(\lambda) = \{v \in \mathbb{R}^m \mid <v,\lambda> \geq 0\} .$$

Furthermore, the infimum in (11) is achieved if, and only if all the λ_i's can be chosen strictly positive. In this case,

$$D^* = \text{block diag } (\lambda_i^{1/2} I_{k_i})$$

is a minimizer.

Proof. We first prove the second statement. Let $\lambda_i > 0$ and $D^* =$ block diag $(\lambda_i^{1/2} I_{k_i})$. Using the definition of $V(\mu^2)$, it is easily checked that $V(\mu^2) \subset H(\lambda)$ if, and only if

$$\sum_{i=1}^{m} \lambda_i x^H (\mu^2 P_i - M^H P_i M) x \geq 0 \quad \text{for all } x \in \partial B$$

or, equivalently

$$\mu^2 D^{*2} - M^H D^{*2} M \geq 0 .$$

Since D^* is invertible, this can occur if, and only if

$$\mu^2 I - (D^* M D^{*-1})^H (D^* M D^{*-1}) \geq 0$$

i.e., if, and only if

$$\bar{\sigma}(D^* M D^{*-1}) \leq \mu .$$

Clearly, in view of (2), this happens if, and only if

$$\bar{\sigma}(D^* M D^{*-1}) = \mu = \min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) .$$

If the infimum in (11) is not achieved, we can always find a matrix N arbitrarily small such that $\inf_{D \in \mathcal{D}} \bar{\sigma}(D(M+N)D^{-1})$ is achieved. The result then follows by continuity.

[]

Referring back to Fig. 1, one can check that there does indeed exist a half space $H(\lambda)$ as specified in the proposition.

Finally, the following proposition gives a condition verifiable *a priori*, under which the infimum in (10) is achieved. In this proposition, we denote by M_{ij} the ijth block of M for the given structure $\mathcal{K}$, i.e., the $k_i \times k_j$ matrix whose (p,q) entry is the $(\sum_{l=1}^{i-1} k_l + p, \sum_{l=1}^{j-1} k_l + q)$ entry in M.

Proposition 5. Suppose that for any nonempty proper subset I of $\{1, \cdots, m\}$ there exist $i \in I$, $j \notin I$ such that M_{ij} is not identically zero. Then the infimum in (10) is achieved.

Proof. By contraposition. Suppose the infimum in (10) is not achieved and let $D^* =$ block diag $(d_i^* I_{k_i})$, where some of the d_i^* may be infinite, be such that, for some sequence $\{D_k\}$ converging to D^*

$$\lim_{k \to \infty} \bar{\sigma}(D_k M D_k^{-1}) = \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) .$$

Without loss of generality, assume that there exist integers $i, j \in \{1, \cdots, m\}$ such that $d_i^*=0$ and $d_j^* \neq 0$. Let

$$I = \{ i \in \{1, \cdots, m\} \mid d_i^* \neq 0 \} .$$

Since the ijth block of DMD^{-1} is $(d_i/d_j)M_{ij}$, in order to ensure that $\bar{\sigma}(DMD^{-1})$ is finite when $D \to D^*$, it is necessary that, for all $i \in I$, $j \notin I$,

$$M_{ij} = 0 .$$

APPENDIX

We describe an algorithm to plot the boundary of $V(\alpha)$ when this set is in $\mathbb{R}^2$ $(m=2)$. Such an algorithm was used to generate the plots of Fig. 1.

Suppose $V(\alpha)$ is strictly convex, i.e., for any $u, v \in V(\alpha)$,

$$\lambda u + (1-\lambda)v \in \text{int}V(\alpha) \quad \text{for all } \lambda \in (0,1) .$$

Then clearly there is a one to one correspondence between the points of the boundary of $V(\alpha)$ and the support hyperplanes to $V(\alpha)$, namely, for any $u \in \text{bd}V(\alpha)$ there exists a unit vector $h = [\cos\theta \;\; \sin\theta]^T$ such that u achieves the minimum in

$$\min\{ < v, h > \mid v \in V(\alpha)\} .$$

In view of the definition of $V(\alpha)$, $u_i = x^H A_i(\alpha)x$, for $i=1, 2$, where x achieves the minimum in

$$\min_{x \in \partial B} \{ x^H (\cos\theta A_1(\alpha)+\sin\theta A_2(\alpha))x \}$$

i.e., x is a unit eigenvector corresponding to the smallest eigenvalue of $\cos\theta A_1(\alpha)+\sin\theta A_2(\alpha)$. This leads to the following algorithm.

Algorithm A.

Step 0. Set $\theta=0$ and $N=$ a large integer.

Step 1. Let x be any unit eigenvector corresponding to the smallest eigenvalue of $\cos\theta A_1(\alpha)+\sin\theta A_2(\alpha)$. Set

$$y_2 = \begin{bmatrix} x^H A_1(\alpha)x \\ x^H A_2(\alpha)x \end{bmatrix} .$$

If $\theta \neq 0$, draw the line segment $\overline{y_1 y_2}$. If $\theta \geq 2\pi$, stop

Step 2. Set $y_1 = y_2$, $\theta=\theta+2\pi/N$ and go to Step 1.

REFERENCES

[1] J.C. Doyle, "Analysis of Feedback Systems with Structured Uncertainties," *Proceedings of IEE* vol. 129, no. 6, pp. 242-250 (November 1982).

[2] M.K.H. Fan and A.L. Tits, "A New Formula for the Structured Singular Value," *Proceedings of the 24th IEEE Conference on Decision and Control*, pp. 595-596 (December 1985).

[3] M.K.H. Fan and A.L. Tits, "Characterization and Efficient Computation of the Structured Singular Value," to appear in IEEE Trans. on Automatic Control (1986).

[4] J.C. Doyle, *Private communication.* 1984.

[5] M.G. Safonov and J.C. Doyle, "Minimizing Conservativeness of Robustness Singular Values," pp. 197-207 in *Multivariable Control*, ed. S.G. Tzafestas, D. Reidel Publishing Company (1984).

[6] M.K.H. Fan, "An Algorithm to Compute the Structured Singular Value," Technical Report TR-86-8, Systems Research Center, University of Maryland, College Park (1986).

[7] A.S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdel, New York (1964).