

An Inverse Neumann Problem

by

Carlos A. Berenstein
Department of Mathematics
University of Maryland
College Park, MD 20742

P.C. Yang
Dept of Mathematics
University of Southern California
Los Angeles, CA 90089-1113

Abstract: We consider the problem of deciding whether the over determined Neumann eigenvalue boundary value problem: $\Delta u + \alpha u = 0$ in D ; $u=1$, $\partial u/\partial n = 0$ on ∂D has a solution. This problem arises in thermodynamics and in harmonic analysis. We show that the existence of infinitely many solutions is equivalent to D being a ball.

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50 Introduction

The harmonic analysis problem in symmetric space known as the Pompeiu problem leads to very classical inverse problems in partial differential equations ([4]). For the Euclidean case, we have: given a bounded domain Ω in \mathbb{R}^{n+1} with a Lipschitz connected boundary does there exist a non-trivial function $f \in C^\infty(\mathbb{R}^{n+1})$ satisfying $\int_{\sigma(\Omega)} f(x)dx = 0$ for all rigid motions σ of \mathbb{R}^{n+1} ? It is known that ([3]) the existence of such a function f would lead to the existence of an over-determined eigenfunction u solving what we call the over-determined Neumann problem:

$$(N) \begin{cases} \Delta u + \alpha u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ and } u \equiv c & \text{on } \partial\Omega \end{cases}, \quad \alpha > 0.$$

The overdetermined Neumann problem can be transformed by the linear substitution $v = u/\alpha c - 1/\alpha$ to

$$(1) \begin{cases} \Delta v + \alpha v = -1 & \text{in } \Omega \\ v = 0, \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

For $\Omega \subset \mathbb{R}^2$ in problem (1) if $\alpha = 0$, the existence of a positive solution was shown by Serrin [12] and Weinberger [13] to imply that Ω is a disk. In case $\alpha = \lambda_2$ the second eigenvalue of the Dirichlet eigenvalue problem in Ω , it is a consequence of the isoperimetric inequality of Payne and Weinberger ([11]) that Ω is a disk. More recently Aviles ([2]) has shown that if the problem (1) has a solution with $\alpha < \lambda_{n+1}$ where λ_j is the j^{th} eigenvalue of the Dirichlet eigenvalue problem and $\partial\Omega$ has nonnegative mean curvature with

respect to the outward normal, then Ω is a ball; in case $\Omega \subset \mathbb{R}^2$, a slightly weaker assumption namely that $\alpha < \lambda_7$ will suffice.

Analogously we may consider the overdetermined Dirichlet eigenvalue problem:

$$(D) \quad \begin{cases} \Delta u + \alpha u = 0 \\ u|_{\partial\Omega} = 0 \text{ and } \frac{\partial u}{\partial n} \equiv c. \end{cases}$$

In either of the problem (N) or (D), if there is a solution and $\partial\Omega$ is $C^{2+\epsilon}$, the regularity results of Kinderlehrer-Nirenberg ([9]) show that $\partial\Omega$ is in fact real analytic. In fact, Williams shows in ([14]) that the same conclusion holds if $\partial\Omega$ is only locally Lipschitz.

In case Ω is the ball, the radial eigenfunctions to either the Neumann or Dirichlet problem satisfy the additional overdetermined condition at $\partial\Omega$, showing there are infinitely many such eigenfunctions. In [3], it was shown that for a simply connected plane domain Ω with a sequence of solutions u_α to either the problems (N) or (D), Ω is necessarily the disk. For Ω a convex domain in the plane, Brown and Kahane [6] gave a quantitative version of [3], where they consider the width of Ω in the direction v (v a unit vector in \mathbb{R}^2)

$$w(v) = [\max_{x \in \Omega} v \cdot x - \min_{x \in \Omega} v \cdot x], \text{ and proved that for}$$

$$m(\Omega) = \inf_{|v|=1} w(v), \quad M(\Omega) = \sup_{|v|=1} w(v)$$

if $m(\Omega) < \frac{1}{2} M(\Omega)$, then the problem (N) has no solution in Ω . However there are many real analytic Ω with $m(\Omega) = M(\Omega)$ which are not disks.

The Berenstein result for \mathbb{R}^2 given in [3] was extended in [5] to the Poincare disk with the hyperbolic metric.

In this paper we extend these results to all dimensions for $\Omega \subset \mathbb{R}^{n+1}$ and as well as for $\Omega \subset \mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1}, |x|^2 < 1\}$ with the complete metric of constant negative sectional curvature.

Theorem 1 Let Ω be a bounded Lipschitz domain in \mathbb{R}^{n+1} with connected boundary with a sequence of over-determined eigenfunctions u_j to problem (N) or (D), then Ω is a ball.

Theorem 2 Let Ω be bounded Lipschitz domain in hyperbolic space \mathbb{H}^{n+1} with connected boundary with a sequence of over-determined eigenfunctions u_j to problem (N) or (D), then Ω is a geodesic ball.

Outline The proof for both of these results depends strongly on the existence of plane waves in the spaces of constant curvature. For each eigenvalue α_j , we find explicit eigenfunctions φ_j in \mathbb{R}^{n+1} or \mathbb{H}^{n+1} which are of the form $f_j e^{i\lambda_j \psi_j}$, f_j, ψ_j real with $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$. We then evaluate the right hand side of the Green's identity:

$$0 = \int_{\Omega} u_j (\Delta \varphi_j) - (\Delta u_j) \varphi_j = \int_{\partial\Omega} u_j \frac{\partial \varphi_j}{\partial n} - \frac{\partial u_j}{\partial n} \varphi_j;$$

which reduces to integrals of the form

$$(0.1) \quad \int_{\partial\Omega} f_j e^{i\lambda_j \psi_j}.$$

An asymptotic evaluation of the integral via the method of the stationary phase then yields the following geometric information about Ω .

I. Ω is a domain of constant breadth. By this we mean $\partial\Omega$ has the following property. At each $x \in \partial\Omega$ let $n(x)$ denote the interior unit normal. The oriented geodesic line through x in the direction $n(x)$ will meet $\partial\Omega$ orthogonally at a constant distance L in a point x^* in opposite direction to $n(x^*)$. This property defines an involution $*$ on $\partial\Omega$.

II. For each such pair of points, let $\lambda_1, \dots, \lambda_n$ and $\lambda_1^*, \dots, \lambda_n^*$ denote the principal curvatures of the hypersurfaces, w.r.t. to interior normal the following identity holds:

a) in case of Euclidean space:

$$\prod_{i=1}^n \lambda_i = \pm \prod_{i=1}^n \lambda_i^*$$

b) in case of the hyperbolic space:

$$\pm \prod_{i=1}^n (\lambda_i - 1) = \left(\prod_{i=1}^n (-\lambda_i^* - 1) \right) e^{nL}.$$

Geometric considerations show that hypersurfaces possessing properties I and II satisfy the following simple identities involving the principal curvatures λ_i :

a) In the Euclidean case:

$$(0.2) \quad \prod_{i=1}^n (L\lambda_i - 1) = \pm 1$$

b) In the hyperbolic case

$$(0.3) \quad \prod_{i=1}^n (\cosh L - \sinh L \lambda_i) = \pm 1$$

These hypersurfaces have been studied by A.D. Alexandrov ([1]), where he proved:

Theorem Embedded closed hypersurfaces M of constant curvature spaces (in case of spheres, M are required to lie in a hemisphere) satisfying an identity of the form $F(\lambda_1, \dots, \lambda_n) = 0$ where $\frac{\partial F}{\partial \lambda_i} > 0, i = 1, \dots, n$; are geodesic spheres.

In the case all factors in (0.2) or (0.3) have the same sign (this amounts to requiring the domain to be convex, i.e. they are intersections of half spaces in the sense of Busemann) then the conditions $\frac{\partial F}{\partial \lambda_i} > 0$ hold, hence Alexandrov's theorem applies.

The general case requires further argument. We choose to proceed by looking at the next term in the asymptotic expansion of (0.1). It turns out that all principal curvatures λ_i can be determined from L alone, and as a direct consequence, Ω is a ball.

In §1 we recall the asymptotic expansion for oscillatory integrals developed in Hormander ([8]). In §2, we set up the Euclidean eigenfunctions φ_j consisting of plane waves and apply the expansion of §1. We then deduce properties I and II using the highest order terms in the expansion and then verify the identity (0.2). In §3 we examine the consequences of I and II, showing that some additional assumptions lead to the conclusion that Ω is a ball. We also give a general argument using the second highest order terms in the expansion to conclude that all λ_i are constants, thus proving Theorem 1. In §4 we carry out the parallel arguments for the hyperbolic space.

Remark 1. There remains the case where the ambient space is the Euclidean spheres, there we have more interesting examples. For example in $S^3 = \{x \in \mathbb{R}^4, |x|^2 = 1\}$, the Clifford torus $\Sigma = \{x \in S^3, x_1x_2 - x_3x_4 = 0\}$ bounds on each side two domains Ω_+ and Ω_- . It is easily checked that the function $u = x_1x_2 - x_3x_4$ is a spherical harmonic of order 2 hence is an eigenfunction of S^3 for the Laplace Beltrami operator on S^3 . In addition it satisfies the identity $|\nabla u|^2 = 1 - 4u^2$ making it possible to look for eigenfunctions v of the form $v = f \circ u$ by solving the ordinary differential equation $\Delta v + \alpha v = f''(u)(1 - 4u^2) + f'(u)(-8u) + \alpha f(u) = 0$ with the boundary condition $f(\frac{1}{2}) = 1, f(0) = 0$ or $f'(0) = 0$ on the interval $[0, 1.2]$. The point $u = \frac{1}{2}$ being a regular singular point, there are thus infinitely many eigenfunctions to both problem (N) and (D). Observe also that the domains Ω_{\pm} are again domains satisfying the geometric properties I and II.

2. There remains also the open question which embedded hypersurfaces in addition to the geodesic spheres satisfy curvature identities of the form (0.2) in \mathbb{R}^{n+1} or (0.3) in \mathbb{H}^{n+1} .

We hope to return to these problem on a later occasion.

§1 Asymptotic expansion for an oscillatory integral.

Let f be a real valued C^∞ function defined in a compact neighborhood K of the origin in \mathbb{R}^n , assume that in K f has only a non-degenerate critical point at the origin. After making an orthogonal transformation the Taylor's expansion for f has the form:

$$f(x) = f(0) + \sum_i \frac{1}{2} k_i x_i^2 + \sum A_{ijk} x_i x_j x_k + O(|x|^4).$$

where A_{ijk} is symmetric in all three indices. Set

$$g(x) = f(x) - f(0) - \sum_{i=1}^n \frac{k_i}{2} x_i^2 = \sum A_{ijk} x_i x_j x_k + O(|x|^4).$$

$$\square = \sum \frac{1}{k_i} \frac{\partial^2}{\partial x_i^2}.$$

Let $u \in C_0^\infty(K)$, then we have (Hormander [8], 7.7)

$$(1.1) \quad \left| \int u(x) e^{i\lambda f(x)} dx - e^{i\lambda f(0)} \left(\prod_{j=1}^n \lambda k_j / 2\pi i \right)^{-1/2} (u(0) + L(u)/\lambda) \right|_{u=0} \\ < c \lambda^{-\frac{n}{2}-2} \sum_{|\alpha| \leq 4} \sup |D^\alpha u| \quad \text{for } \lambda > 0,$$

where c stays bounded when f stays in a bounded set in $C^7(K)$ and

$\frac{|x|}{|f'(x)|}$ stays bounded, and

$$L(u) = -i \left[\frac{1}{2} \square u(0) + \frac{1}{2^2 2!} \square^2 (gu)(0) + \frac{1}{2^3} \frac{1}{2!} \frac{1}{3!} \square^3 (g^2 u)(0) \right].$$

Remark: In applying this expansion for an integral over a compact hypersurface for a phase function f having only a finite number of

nondegenerate critical points, a suitable partition of unity may be introduced to reduce the integral to a finite sum of the same type of integrals considered above. Outside the critical points, the integral decreases faster than λ^{-N} for any N .

§2. Eigenfunctions in Euclidean space.

In \mathbb{R}^{n+1} , fix a pair of mutually orthogonal unit vectors ξ and η and consider

$$\varphi_{\lambda,t}(x) = e^{i\langle x, \lambda\xi - t\eta \rangle} = e^{t\langle x, \eta \rangle} e^{i\lambda\langle x, \xi \rangle}.$$

It is easily seen that (using orthogonality) $\Delta\varphi_{\lambda,t}(x) + (\lambda^2 - t^2)\varphi_{\lambda,t}(x) = 0$. The phase function $x \rightarrow \langle x, \xi \rangle$ considered as a function on a hypersurface $M \subset \mathbb{R}^{n+1}$ has critical points precisely at those points where ξ is normal to M . If M is a smooth hypersurface, the Gauss normal map $x \rightarrow n(x)$ with $n(x)$ the interior normal to M at x is smooth, thus generically, for an open set of vectors ξ in the unit sphere, the critical points of $x \rightarrow \langle x, \xi \rangle$ will be finite in number, and vary locally smoothly in ξ . Given $\alpha > 0$, we will choose $\varphi_{\lambda,t}$ with

$$(i) \quad \lambda^2 - t^2 = \alpha$$

$$(ii) \quad \text{As } \alpha \rightarrow \infty, \quad |t| \rightarrow \infty \text{ and } \frac{t}{\log \lambda} \rightarrow 0.$$

The relevant integrals in case of a (\mathcal{D}) eigenfunction is

$$\int_{\partial\Omega} I_{\mathcal{D}} = \int_{\partial\Omega} e^{+i\langle \lambda\xi - t\eta, x \rangle},$$

in case of an (\mathcal{N}) eigenfunction it is

$$\int_{\partial\Omega} I_{\mathcal{N}} = \int_{\partial\Omega} \frac{\partial}{\partial n} e^{+i\langle \lambda\xi - t\eta, x \rangle} = \int_{\partial\Omega} \langle i\lambda\xi + t\eta, \frac{\partial x}{\partial n} \rangle e^{+i\langle \lambda\xi - t\eta, x \rangle}.$$

For the asymptotics we use a partition of unity on $\partial\Omega$ to cut down the region of integration to neighborhoods U_i of the points p_i where $\pm n(p_i) = \xi$ such that p_i is the unique point in U_i with this property. Over such a neighborhood, it is simplest to use the projection onto the tangent plane at p_i as coordinates, thus W.L.O.G. we may assume

$$\partial\Omega = U = \{x_{n+1} = y(x_1, \dots, x_n)\}, \quad p_i \text{ is the origin and } \xi = (0, \dots, 0, 1).$$

By making an orthogonal transformation of x_1, \dots, x_n space if necessary, we may arrange to have

$$y(x_1, \dots, x_n) = \sum \frac{1}{2} k_i x_i^2 + o(|x|^3).$$

For a general point $X = (x, y(x))$, $v = (-y_1, \dots, -y_n, 1)/(1 + |\nabla y|^2)^{1/2}$ is a unit normal, hence the volume element is given by

$$dv = \left| \det \left(\frac{\partial X}{\partial x_1}, \dots, \frac{\partial X}{\partial x_n} \right) \right| dx_1 \dots dx_n = (1 + |\nabla y|^2)^{1/2} dx_1 \dots dx_n.$$

The second fundamental form is given by

$$II(v, w) = \langle D_v v, w \rangle$$

and in terms of the tangent frame $e_i = (0, \dots, 0, 1, 0, \dots, 0, y_i)$

$$\begin{aligned} II(e_i, e_j) &= \left\langle \frac{\partial}{\partial x_i} \left(\frac{-y_1, \dots, -y_n, 1}{(1 + |\nabla y|^2)^{1/2}} \right), (0, \dots, 0, 1, 0, \dots, 0, y_j) \right\rangle \\ &= (1 + |\nabla y|^2)^{-1/2} (-y_{ij}) = a_{ij} \end{aligned}$$

hence at the origin where (e_i) are orthonormal. Infact,

$$II(e_i, e_j) = -y_{ij} = -k_i \delta_{ij}.$$

i.e. $-k_i$ are the principal curvatures of ∂M at 0.

Writing the integrands

$$(2.2) \quad \dots I_{\mathcal{D}} = \delta e^{+\tau\eta \cdot x} e^{i\lambda\xi \cdot x} \text{ in case of problem } (\mathcal{D}).$$

$$(2.3) \quad I_{\mathcal{N}} = \langle +\tau\eta + i\lambda\xi, \nu \rangle \delta e^{+\tau\eta \cdot x} e^{i\lambda\xi \cdot x} \text{ in case of problem } (\mathcal{N}'),$$

where $\delta = (1 + |\nabla y|^2)^{1/2}$.

$$\text{Set } \square = + \sum \frac{1}{k_i} \frac{\partial^2}{\partial x_i^2}.$$

We find that for a regular value ξ of the Gauss map, there are finite number of points p_1, \dots, p_N where $\nu(p_i) = \pm\xi$, and

$$(2.4) \quad 0 = \int I_{\mathcal{D}} = \sum_{\nu(p_i)=\pm\xi} (2\pi i\lambda)^{-\frac{n}{2}} \left(\prod_{j=1}^n k_j \right)^{-1/2} e^{i\lambda\xi \cdot p_i} e^{+\tau\eta \cdot p_i} + \sum_{\nu(p_i)=\pm\xi} (2\pi i)^{-\frac{n}{2}} \lambda^{-\frac{n}{2}+1} \left(\prod k_j \right)^{-1/2} e^{i\lambda\xi \cdot p_i} L(e^{+\tau\eta \cdot x} \delta(x))_{x=p_i} + O\left(\frac{1}{\lambda}\right)^{\frac{n}{2}+2}.$$

$$(2.5) \quad 0 = \int I_{\mathcal{N}} = \sum_{\nu(p_i)=\pm\xi} (2\pi i\lambda)^{-\frac{n}{2}} \left(\prod k_j \right)^{-1/2} e^{i\lambda\xi \cdot p_i} i\lambda\xi \cdot \nu e^{+\tau\eta \cdot p_i} + \sum_{\nu(p_i)=\pm\xi} (2\pi i)^{-\frac{n}{2}} \lambda^{-\frac{n}{2}+1} \left(\prod k_j \right)^{-1/2} e^{i\lambda\xi \cdot p_i} L(\langle \tau\eta + i\lambda\xi, \nu \rangle e^{+\tau\eta \cdot x} \delta(x))_{x=p_i}$$

$$+ O\left(\frac{1}{\lambda}\right)^{\frac{n}{2}+2}$$

In order for cancellation of the highest order terms to take place, we must have

$$(2.4') \quad \sum (\prod k_j)^{-1/2} e^{i\lambda\xi \cdot p_i} e^{t\eta \cdot p_i} = O(\lambda^{-1}) \quad \text{or}$$

$$(2.5') \quad \sum \pm (\prod k_j)^{-1/2} e^{i\lambda\xi \cdot p_i} e^{t\eta \cdot p_i} = O(\lambda^{-1}).$$

If in addition, we consider the following 1-parameter of variations for the eigenfunctions $\varphi_{\lambda, t, \theta}$ with

$$\begin{cases} \xi(\theta) = \cos\theta, \xi + \sin\theta, \eta \\ \eta(\theta) = -\sin\theta, \xi + \cos\theta, \xi \end{cases}$$

then we have as before,

$$0 = \left(\frac{d}{d\theta}\right)^{\ell} \Big|_{\theta=0} \int_{\partial\Omega} \varphi_{\lambda, t, \theta} = \int_{\partial\Omega} \langle i\lambda\eta - t\xi, x \rangle^{\ell} e^{i\lambda\xi \cdot x} e^{t\eta \cdot x}$$

$$0 = \left(\frac{d}{d\theta}\right)^{\ell} \Big|_{\theta=0} \int_{\partial\Omega} \frac{\partial}{\partial v} \varphi_{\lambda, t, \theta}$$

$$= \int [\langle i\lambda\eta - t\xi, x \rangle^{\ell} + \text{lower powers of } \langle i\lambda\eta - t\xi, x \rangle \langle t\eta + i\lambda\xi, v \rangle] e^{i\lambda\xi \cdot x} e^{t\eta \cdot x} .$$

The leading order asymptotics yield in both cases,

$$\sum_{v(p_i) = \pm\xi} \pm \langle i\lambda\eta - t\xi, p_i \rangle^{\ell} (\prod k_j)^{-1/2} e^{i\lambda\xi \cdot p_i} e^{t\eta \cdot p_i} = O(\lambda^{-1}).$$

Regarded as a system of linear equations with

$$A_{i\ell} = \langle i\lambda\eta - t\xi, p_i \rangle^{\ell}$$

as coefficients, the Van der Monde determinant of the $N \times N$ system must vanish, hence we conclude that there is a pair say p_i and p_j such that

$$\eta \cdot p_i = \eta \cdot p_j .$$

Applying this argument for each η orthogonal to this fixed ξ , we conclude there is a common pair, call it p and p^* , such that p and p^* lie on the same line orthogonal to $\partial\Omega$ at p and p^* , this means that $p^* = p + Lv$ where L is a scalar. Differentiation of this equation yields that $dL = 0$, hence L is constant.

Since $\partial\Omega$ is real analytic, we may assume that the involution is globally defined $p^* = p + Lv$, v is the inward pointing normal, and we must have the normal at p^* is opposite v : $v(p^*) + v(p) = 0$. For if not, then $p + 2Lv$ would be in $\partial\Omega$, hence by continuity $p + 3Lv$ would be in $\partial\Omega$ and so on, this would contradict the boundedness of Ω .

We now claim that for an open dense set of ξ , on the line containing the pair of points p and p^* , there are no other points p_j with the same normal vector ξ , for if not then the above argument still applies and $\{p, p_j\}$ and $\{p^*, p_j\}$ would be constant distance L 'apart, since either $\overrightarrow{pp_j}$ or $\overrightarrow{p^*p}$ has the same direction as normal at p say $\overrightarrow{pp_j}$, then $p + L'v, p + 2L'v, \dots$ would all lie on the surface, contradicting boundedness of Ω .

The foregoing analysis shows that the points $\{p_1, \dots, p_N\}$ are partitioned into pairs p_i, p_i^* given by the involution thus N is even and all pairs are distance L apart. This verifies property I.

To verify property II, we compare for a fixed pair ξ, η the terms in (2.4'), (2.5') with the largest exponents, $\langle \eta, p \rangle$ to conclude that $\prod k_j(p) = \pm \prod k_j(p^*)$. Again, since this is an analytic relation holding locally on the analytic surface, it must hold globally on $\partial\Omega$. This finishes the proof of II.

We proceed to show I and II impose the curvature identity (0.2).

Lemma For a domain of constant breadth L with the given involution $p \rightarrow p^* = p + Lv$ on the boundary the principal curvature directions at p correspond precisely to the principal curvature directions at p^* , and we have the identities:

$$\frac{1}{\lambda_i} + \frac{1}{\lambda_i^*} = L \quad i = 1, \dots, n,$$

where $\{\lambda_i\}$ and $\{\lambda_i^*\}$ are the principal curvatures at p and p^* in the same principal curvature directions.

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{v} & S^n \\ \downarrow * & & \downarrow \tau \\ M & \xrightarrow{v} & S^n \end{array}$$

where τ denotes the antipodal map $\xi \rightarrow -\xi$. Consider the locally defined inverse map to the Gauss normal map say $x = v^{-1}(\xi)$, we have the equation

$$x^* = v^{-1}(-\xi) = x - L\xi$$

hence $dx_{\xi}^* = dx_{\xi} - L \cdot I$. This means that the inverses to the Gauss maps have commuting differentials, hence simultaneously diagonalizable. This proves the first assertion. Evaluating this equation on a principal vector and recalling that the interior normal to M at P^* is opposite to that at p , yield the equation

$$-\frac{1}{\lambda_i^*} = \frac{1}{\lambda_i} - L$$

hence the second assertion.

Proposition: For a domain satisfying the geometric properties I and II we have the following relation for the principal curvatures:

$$\pm \Pi \frac{1}{\lambda_i} = \Pi \frac{1}{\lambda_i^*} = \Pi \left(L - \frac{1}{\lambda_i} \right)$$

or

$$\pm \frac{\Pi \frac{1}{\lambda_i^*}}{\Pi \frac{1}{\lambda_i}} = \frac{\Pi \left(L - \frac{1}{\lambda_i} \right)}{\Pi \frac{1}{\lambda_i}} = \Pi (L \lambda_i - 1) = \pm 1.$$

§3 Proof of Theorem 1

We begin with the remark that in the case Ω is convex, we have immediately from the equations $\frac{1}{\lambda_i} + \frac{1}{\lambda_i^*} = L$ that $L\lambda_i - 1 > 0$ for all i . And conversely if for a pair of points p and p^* all λ_i and $\lambda_i^* > 0$, then it follows from the equation that $L\lambda_i - 1 > 0$; hence by continuity and the equation $\prod_{i=1}^n (L\lambda_i - 1) = 1$ that all $L\lambda_i - 1 > 0$ on $\partial\Omega$, hence Ω is convex. In this case in the curvature relation $F(\lambda_1, \dots, \lambda_n) = \pi(L\lambda_i - 1) - 1 = 0$ we find $\frac{\partial F}{\partial \lambda_i} > 0$; hence Alexandrov's theorem implies that Ω is a ball.

As a further remark, when Ω is topologically a ball and hence $\partial\Omega$ is topologically S^n , if n is even we must have all $L\lambda_i - 1 > 0$. This follows from the fact that even dimensional spheres do not have non-trivial subbundles in their tangent bundle ([8]), while the existence of some $L\lambda_i - 1 < 0$ would force a splitting of the tangent bundle.

The condition I, II allow us to restrict ourselves to values λ, t such that the first term in the asymptotics (2.4) or (2.5) vanishes.

We now proceed to the evaluation of the second term in the asymptotics for the integrals $\int I_M$ and $\int I_D$. We need to evaluate Lu where for I_D

$$Lu = -i \left[\frac{1}{2} \square u(0) + \frac{1}{8} \square^2 (gu)(0) + \frac{1}{2^3} \frac{1}{2!} \frac{1}{3!} \square^3 (g^2 u)(0) \right]$$

$$\square = \sum \frac{1}{k_i} \frac{\partial^2}{\partial x_i^2}$$

$$g = \langle x, \xi \rangle - \langle x_0, \xi \rangle - \sum \frac{k_i}{2} x_i^2$$

$$u = e^{t\eta \cdot x} \delta(x).$$

$$Lu = -\frac{i}{2} \sum \frac{1}{k_i} (t^2 \eta^2 + k_i^2) e^{t\eta \cdot x} + \frac{-i}{8} (\square^2 g) e^{t\eta \cdot x} + \frac{1}{8 \cdot 2 \cdot 6} (\square^3 g^2) e^{t\eta \cdot x}$$

Note that the latter two terms do not contain powers of t , and hence are negligible compared with the first.

Fix $\eta = \eta_0$, since the second term of the expansion (2.4) must now be $O(\lambda^{-\frac{n}{2}-2})$, if we consider the pair p, p^* of points with the largest value of $\eta \cdot p$ we obtain

$$\sum \frac{1}{k} \eta_i^2 = \sum \frac{1}{k_1^*} \eta_i^2 ;$$

where these curvatures are evaluated at p and p^* . For the second term in $\int I_{\mathcal{N}}$, $v = (1+|\nabla y|^2)^{1/2}(y_1, \dots, y_n, -1)$

$$u = \langle t\eta + i\lambda\xi, v \rangle e^{t\eta \cdot x} (1+|\nabla y|^2)^{1/2}$$

and

$$L(u) = -\frac{i}{2} \sum \frac{1}{k_i} (i\lambda t^2 \eta_i^2 + t^2 \eta_i^2 k_i + i\lambda - k_i^2) e^{t\eta \cdot x}$$

hence the leading part of the second term in $\int I_{\mathcal{N}}$ is

$$\sum_{v(p_i) = \pm \xi} (2\pi i)^{-\frac{n}{2}} \lambda^{-\frac{n}{2}-1} (\prod k_j)^{-1/2} e^{i\lambda \xi \cdot p_i} \left(\frac{1}{2} \sum \frac{1}{k_i} (\lambda t^2 \eta_i^2 k_i) \right).$$

Again the cancellation in the pair p, p^* with highest exponent $\eta \cdot p$ require (recall $t = O(\log \lambda)$)

$$\sum \frac{1}{k_i} \eta_i^2 = \sum \frac{1}{k_i^*} \eta_i^2$$

Since the same identity holds for all η lying in a small neighborhood in η_0 ,

we see that

$$k_i(p) = k_i(p^*) \quad \text{for all } i$$

Then by continuity, we have

$$k_i(p) = k_i(p^*) \quad \text{for all } i, p.$$

The equation, $\frac{1}{k_i} + \frac{1}{k_i^*} = L$ then forces $k_i = \frac{2}{L}$, hence B is a ball.

54. Eigenfunctions in the hyperbolic space

In the Poincare model $H^{n+1} = \{x \in R^{n+1}, |x|^2 < 1\}$ with $ds^2 = 4|dx|^2/(1 - |x|^2)^2$, the isometries of H^{n+1} are restrictions of conformal maps of $S^{n+1} = R^{n+1} \cup \{\infty\}$ which preserve the unit sphere ∂H^{n+1} .

The hyperbolic distance between x and $x^* \in H^{n+1}$ is given by

$$d(x, x^*) = \log \frac{1+A(x, x^*)^{1/2}}{1-A(x, x^*)^{1/2}} \quad \text{where } A(x, x^*) = \frac{|x-x^*|^2}{1-2x \cdot x^* + |x|^2 |x^*|^2}.$$

Given a boundary point $b \in \partial H^{n+1}$ we have the Busemann function $x \rightarrow \langle x, b \rangle$ defined by

$$\langle x, b \rangle = \lim_{x^* \rightarrow b} [d(x, x^*) - d(0, x^*)].$$

A short computation shows

$$(4.1) \quad e^{\langle x, b \rangle} = \frac{1-2x \cdot b + |x|^2}{1-|x|^2}$$

and we note that

$$\frac{1-x \cdot b}{1-|x|^2} = \frac{1+e^{\langle x, b \rangle}}{2}.$$

Geometrically the level sets of $\langle \cdot, b \rangle$ are spheres in H tangent to ∂H at b , called horospheres. The superlevel sets of $\langle \cdot, b \rangle$ play the same role as half spaces. The horospheres at b are everywhere orthogonal to the geodesics of H ending at b . As hypersurfaces of R^{n+1} , the horospheres Σ are Euclidean spheres, hence have constant principal curvatures $\tilde{k}_i = \frac{1}{R}$ where R is the Euclidean radius. As hypersurfaces of H^{n+1} they have constant principal curvatures $k = 1$. This is most easily seen by the following considerations.

Let $\tilde{e}_1, \dots, \tilde{e}_n$ be an Euclidean orthonormal tangent frame field along a hypersurface M , let \tilde{e}_{n+1} be a unit normal to M , then $\{e_i = \frac{1-|x|^2}{2} \tilde{e}_i\}$ is a

hyperbolic orthonormal frame field with e_{n+1} normal to M . Let $\tilde{\omega}_1, \dots, \tilde{\omega}_{n+1}$ be the dual one forms to the frame $\{\tilde{e}_i\}$, then $\{\omega_i = \frac{2}{1-|x|^2} \tilde{\omega}_i\}$ form the dual one forms to $\{e_i\}$. Recall that the connection one forms ω_{ij} and $\tilde{\omega}_{ij}$ are uniquely determined by the conditions

$$\begin{cases} d\omega_i = \omega_j \wedge \omega_{ji} \\ \omega_{ij} + \omega_{ji} = 0 \end{cases} \quad \begin{cases} d\tilde{\omega}_i = \tilde{\omega}_j \wedge \tilde{\omega}_{ji} \\ \tilde{\omega}_{ij} + \tilde{\omega}_{ji} = 0 \end{cases}$$

Hence ω_{ij} may be computed from $\tilde{\omega}_{ij}$:

$$d\omega_i = \omega_j \wedge (e_j \cdot x \omega_i + \tilde{\omega}_{ji})$$

thus setting

$$\omega_{ji} = \tilde{\omega}_{ji} + e_j \cdot x \omega_i - e_i \cdot x \omega_j$$

satisfies the equations. In particular

$$\begin{aligned} \omega_{jn+1} &= \tilde{\omega}_{jn+1} + e_j \cdot x \omega_{n+1} - e_{n+1} \cdot x \omega_j \\ &= \tilde{\omega}_{jn+1} - e_{n+1} \cdot x \omega_j \end{aligned}$$

Setting $\omega_{jn+1} = h_{jk} \omega_k$, defines the second fundamental form h_{jk} , which is symmetric in its indices and its eigenvalues are the principal curvatures k_i . Thus we find

$$(4.2) \quad k_i = \tilde{k}_i \frac{(1-|x|^2)}{2} - e_{n+1} \cdot x.$$

Since the isometries of H fixing b acts transitively on the set of horospheres based at b , and the subgroup with Jacobian determinant 1 at b acts transitively on each such horosphere, it suffices to compute k_i for the horosphere passing through the origin, there $\tilde{k}_i = 2$, and $k_i = 1$ follows from (4.2) immediately.

To construct eigenfunctions we shall first compute the Laplacian of the Busemann function. Recall the intrinsic definition of the Laplacian Δ : Take an orthonormal frame field $\{e_i\}$, then

$$\Delta f = \sum_i (e_i e_i f - \nabla_{e_i} e_i f),$$

where $\nabla_{e_i} e_i$ means the covariant derivative of e_i in the direction e_i . To evaluate $\Delta \langle x, b \rangle$ at a point, we will choose frame fields e_1, \dots, e_n tangent to the horosphere passing through p and e_{n+1} tangent to the geodesic ending at b ; we find

$$\nabla_{e_i} e_i \equiv e_{n+1} \pmod{e_1, \dots, e_n}, \quad i = 1, \dots, n$$

$$\nabla_{e_{n+1}} e_{n+1} = 0$$

$$e_i \langle x, b \rangle = e_i e_i \langle x, b \rangle = 0$$

$$e_{n+1} \langle x, b \rangle = -1$$

so that

$$\Delta \langle x, b \rangle = n$$

and

$$\Delta e^{\mu \langle x, b \rangle} = \mu(\mu+n)e^{\mu \langle x, b \rangle}.$$

Setting $\mu = -\frac{n}{2} + i\lambda$ we find

$$\Delta e^{\mu \langle x, b \rangle} + \left[\left(\frac{n}{2}\right)^2 + \lambda^2 \right] e^{\mu \langle x, b \rangle} = 0.$$

For a hypersurface $M = \partial\Omega$ in H^{n+1} , there is a generalized notion of the Gauss normal map. Given $x \in M$, $\nu(x)$ the outward unit normal to M at x , let $b_{\pm}(x)$ be the unique end point on ∂H^{n+1} of the geodesic starting at x in the direction $\pm\nu(x)$. Since the geodesics ending at b are orthogonal to all level sets of the Busemann function, the function $x \mapsto \langle x, b \rangle$ on the hypersurface M , has its critical points precisely at those points x with $b_{\pm}(x) = b$.

We derive the equations describing the geodesics ending at b :

Lemma (4.3) Given $b \in \partial H^{n+1}$ and vectors v_1, \dots, v_n orthonormal (Euclidean) and orthogonal to b , the geodesics in H^{n+1} ending at b are given by the equations:

$$\frac{v_i \cdot x}{1 - 2b \cdot x + |x|^2} = \beta_i \quad i = 1, \dots, n.$$

Proof Consider the half space representation of H^{n+1} :

$$H_+ = \{y \in \mathbb{R}^{n+1}, y_{n+1} > 0\}, \quad ds^2 = \frac{|dy|^2}{y_{n+1}^2}.$$

The geodesics ending at ∞ are clearly given by the vertical lines $y_i = \alpha_i$, $i = 1, \dots, n$. The upper half space H_+ is isometric to H by the following Cayley transformation:

$$(4.4) \quad y_i = \frac{x \cdot v_i}{1 - 2b \cdot x + |x|^2}, \quad y_{n+1} = \frac{2(1 + x_{n+1})}{1 - 2b \cdot x + |x|^2} - 1.$$

The assertion is now obvious.

There is a notion of domains of constant breadth: On $M = \partial\Omega$, to each $p \in \partial M$ the geodesic beginning at p in the direction $-\nu(p)$ meets M again at a point p^* orthogonally at a constant distance L from p . We have the following

Lemma (4.5) Consider two pieces of local hypersurfaces M and M^* . Suppose to each $x \in M$, the normal $\nu(x)$ at x defines a geodesic γ meeting M^* orthogonally at x^* . We assert that

- (1) $d(x, x^*)$ is a constant L
- (2) The principal curvature directions correspond under the map $x \rightarrow x^*$.
- (3) The principal curvatures at x and x^* satisfy

$$(4.6) \quad -k^* = \frac{\sinh L - k \cosh L}{\cosh L - k \sinh L}$$

where k^* is with respect to $\dot{\gamma}(L)$ and k is with respect to $\dot{\gamma}(0)$.

Let $\alpha(s)$ be a curve in M with $\alpha(0) = x$ and $\alpha'(0) = e$. For each $\alpha(s)$ let γ_s be the geodesic passing through $\alpha(s)$ and orthogonal to M . γ_s meets the hypersurface M^* in $\alpha^*(s)$ tracing out a smooth curve in M^* with $\alpha^*(0) = x^*$ and $\alpha^{*\prime}(0) = e^*$. It is easy to see that the length of the geodesic γ_s between $\alpha(s)$ and $\alpha^*(s)$ has the following variational formula.

$$\frac{d}{ds} L(s) = \dot{\gamma}_s(L) \cdot \frac{d\alpha}{ds} - \dot{\gamma}_s(0) \frac{d\alpha^*}{ds} = 0 .$$

Hence the first assertion (1).

Now we can parametrize all geodesics $\gamma_s(t) = \gamma(s,t)$ by arc length over the same t interval $[0,L]$. It is well known ([10]) that the variational vector field $J = \left. \frac{d\gamma}{ds} \right|_{s=0}$ is a Jacobi vector field along γ :

$$\frac{D^2 J}{dt^2} + R(T,J)T = 0 .$$

To relate the principal curvature of M and M^* , choose an orthonormal frame $e_1, \dots, e_{n+1} = \nu = \dot{\gamma}(0)$ at x and then parallel translate along γ to obtain a parallel orthonormal frame field E_i along γ . Write $\alpha'(0) = \sum a_i e_i$ and let $T = \frac{\partial \gamma}{\partial t}$, $J = \frac{\partial \gamma}{\partial s}$ be an associated Jacobi field to the variation γ . Since $\nabla_T J - \nabla_J T = [T,J] = 0$, the second fundamental form at x , $II_x(e, e')$ can be computed as follows: let J, J' be the Jacobi vector fields associated to the variations $\gamma(s,t)$ corresponding to the direction e and e' respectively.

$$-II_x(e, e') = \nabla_e T \cdot e' = \nabla_T J \cdot J'$$

$$-II_x^*(J, J') = \nabla_J T \cdot J' = \nabla_T J \cdot J' .$$

Since orthogonality is preserved for Jacobi vector fields, it follows that principal curvature directions are preserved. Suppose $J(0)$ represents a principal curvature direction with principal curvature λ , then writing $J(t) = \sum b_i(t)E_i$ we find, using Jacobi's equation,

that
$$b_i'' = b_i,$$

$$b_i(t) = c_i e^t + d_i e^{-t}$$

and

$$\frac{DJ}{dt} = \sum (c_i e^t - d_i e^{-t})E_i,$$

If $|J(0)| = 1$ we have

$$\begin{cases} c_i + d_i = 1 & c_i^2 - d_i^2 = -II(J,J) = -k \\ c_j = d_j = 0 & \text{all } j \neq i \end{cases}$$

hence

$$\begin{cases} c_i = \frac{1-k}{2}, & d_i = \frac{1+k}{2} \\ c_j = d_j = 0 \end{cases}$$

It follows that the corresponding curvatures at x^* is given by (with respect to the normal given by $\gamma'(L)$)

$$\begin{aligned}
 -k^* &= \frac{1}{|J|^2} \nabla_T J \cdot J = (c_1^2 e^{2L} - d_1^2 e^{-2L})(c_1 e^L + d_1 e^{-L})^{-2} \\
 &= \frac{\sinh L - k \cosh L}{\cosh L - k \sinh L} \quad \text{as claimed.}
 \end{aligned}$$

Assume now there is a sequence of eigenfunctions u_j satisfying (D) or (N) in the domain Ω . For each $b \in \partial \mathbb{H}^{n+1}$ we consider the functions φ_j with $(\frac{n}{2})^2 + \lambda_j^2 = \alpha_j$

$$\varphi_j = e^{(-\frac{n}{2} + i\lambda_j)\langle x, b \rangle}$$

which satisfy

$$\Delta \varphi_j + ((\frac{n}{2})^2 + \lambda_j^2) \varphi_j = 0.$$

Then Green's identity yields as before

$$0 = \int_{\Omega} [(\Delta u_j) \varphi_j - (u_j \Delta \varphi_j)] = \int_{\partial \Omega} \left(\frac{\partial u_j}{\partial n} \varphi_j - u_j \frac{\partial \varphi_j}{\partial n} \right).$$

The right hand side is

$$c \int_{\partial \Omega} \varphi_j = \int_{\partial \Omega} I_D \quad \text{in case } u_j \text{ solves (D)}$$

$$-c \int_{\partial \Omega} \frac{\partial \varphi_j}{\partial n} = \int_{\partial \Omega} I_N \quad \text{in case } u_j \text{ solves (N)}.$$

To apply the asymptotic expansion formula in §1, we assume b is a regular value of the Gauss map $x \rightarrow b(x)$. We coordinatize the hypersurface by going over to the upper half space \mathbb{H}_+ mapping \mathbb{H}^{n+1} isometrically to \mathbb{H}_+ by (4.4), thus sending b to ∞ . The hypersurfaces will be tangent to the horospheres

$y_{n+1} = \langle p_j, b \rangle$ at a finite number of points p_1, \dots, p_N ; and locally expressible as a graph $y_{n+1} = h(y_1, \dots, y_n)$. In terms of this representation the unit upward normal is given by

$$v = h(1 + |\nabla h|^2)^{-1/2}(-\partial_1 h, \dots, -\partial_n h, 1)$$

and the volume element is

$$dv = h^{-n}(1 + |\nabla h|^2)^{1/2} dy_1 \cdots dy_n$$

Writing the integrands $\int I_D$ and $\int I_N$ in the form $ue^{i\lambda f}$, by (4.1):

$$(4.7) \quad I_D = e^{-\frac{n}{2} \langle x, b \rangle} e^{i\lambda \langle x, b \rangle} dv \\ = h^{-\frac{n}{2}} (1 + |\nabla h|^2)^{1/2} e^{+i\lambda \log h} dy_1 \cdots dy_n$$

$$(4.8) \quad I_N = (-\frac{n}{2} + i\lambda) \frac{\partial}{\partial v} \langle x, b \rangle e^{-\frac{n}{2} \langle x, b \rangle} e^{i\lambda \langle x, b \rangle} dv \\ = \pm(\frac{n}{2} - i\lambda)(1 + |\nabla h|^2)^{1/2} h^{-\frac{n}{2}} e^{+i\lambda \log h} dy_1 \cdots dy_n .$$

We express the Hessian $\frac{\partial^2}{\partial y_i \partial y_j} \log h$ in terms of intrinsic geometric quantities. For this, observe that

$$\frac{\partial^2}{\partial y_i \partial y_j} \log h = \frac{\partial^2 h}{\partial y_i \partial y_j} h^{-1} - \frac{\partial h}{\partial y_i} \frac{\partial h}{\partial y_j} h^{-2}$$

can be evaluated at p_i where $\frac{\partial h}{\partial y_j} = 0$. Hence,

$$\frac{\partial^2 \log h}{\partial y_i \partial y_j} = \frac{\partial^2 h}{\partial y_i \partial y_j} h^{-1},$$

thus in its diagonalized form

$\frac{\partial^2 \log h}{\partial y_i \partial y_j} = \delta_{ij} \tilde{k}_i h^{-1}$ where \tilde{k}_i denotes the Euclidean principal curvature with respect to the upward normal. The Euclidean principal curvature \tilde{k}_i can be related to the hyperbolic principal curvatures by a computation similar to that of (4.2), we omit the details but record the result:

$$\tilde{k}_i = y_{n+1}^{-1} [k_i - \nu \log (y_{n+1})],$$

in particular, when ν is vertical,

$$\tilde{k}_i = y_{n+1}^{-1} [k_i - 1] = \frac{1}{k} (k_i - 1).$$

We can now write down the leading order term in the asymptotic expansion of

$\int I_{\mathcal{D}}$ and $\int I_{\mathcal{N}}$:

$$\int I_{\mathcal{D}} = \sum_{b_{\pm}(p_j) \rightarrow \infty} h^{-\frac{n}{2}(p_j)} (2\pi i\lambda)^{-\frac{n}{2}} \left(\prod_i^n \tilde{k}_i(p_i) \right)^{-1/2} h^{\frac{n}{2}(p_j)} e^{-i\lambda \log h(p_j)} + O\left(\frac{1}{\lambda}\right)^{\frac{n}{2}+1}$$

or intrinsically,

$$\int I_{\mathcal{D}} = \sum_{b_{\pm}(p_j)=b} (2\pi i\lambda)^{-\frac{n}{2}} \left(\prod_i^n (k_i(p_j)-1) \right)^{-1/2} e^{-\frac{n}{2} \langle p_j, b \rangle} e^{i\lambda \langle p_j, b \rangle} + O\left(\frac{1}{\lambda}\right)^{\frac{n}{2}+1}.$$

Similarly

$$\int I_{\mathcal{N}} = \sum_{b_{\pm}(p_j) \rightarrow \infty} \pm \left(\frac{n}{2} - i\lambda\right) (2\pi i\lambda)^{-\frac{n}{2}} \left(\prod_{i=1}^n (k_i(p_j) - 1) \right)^{-\frac{1}{2}} \frac{n}{h^2(p_j)} e^{-i\lambda \log h(p_j)} + O\left(\frac{1}{\lambda}\right)^{\frac{n}{2}}$$

or intrinsically,

$$\int I_{\mathcal{N}} = \sum_{b_{\pm}(p_j) = b} \pm (2\pi i\lambda)^{-\frac{n}{2}} \left(\frac{n}{2} - i\lambda\right) \left(\prod_{i=1}^n (k_i(p_j) - 1) \right)^{-\frac{1}{2}} e^{i\lambda \langle p_j, b \rangle} + O\left(\frac{1}{\lambda}\right)^{\frac{n}{2}}$$

To draw geometric conclusions, we consider a one parameter variations by taking $|v| = 1$, v orthogonal to b , defining

$$b(\theta) = (\cos \theta)b + (\sin \theta)v.$$

Then by differentiating the identities (2.4) and (2.5) with respect to $\frac{d}{d\theta}$, we obtain

$$0 = \int \left(\frac{d}{d\theta}\right)^{\ell} I_{\mathcal{D}} \quad \text{and} \quad 0 = \int \left(\frac{d}{d\theta}\right)^{\ell} I_{\mathcal{N}}.$$

Note that

$$\frac{d}{d\theta} \langle x, b(\theta) \rangle = - \frac{2x \cdot v}{1 - 2x \cdot b + |x|^2}$$

$$\frac{d}{d\theta} e^{(-\frac{n}{2} + i\lambda) \langle x, b \rangle} = \left(\frac{n}{2} - i\lambda\right) \frac{2x \cdot v}{1 - 2x \cdot b + |x|^2} e^{(-\frac{n}{2} + i\lambda) \langle x, b \rangle}$$

$$\begin{aligned} \left(\frac{d}{d\theta}\right)^{\ell} e^{(-\frac{n}{2} + i\lambda) \langle x, b \rangle} &= \left\{ \left[\left(\frac{n}{2} - i\lambda\right) \frac{2x \cdot v}{1 - 2x \cdot b + |x|^2} \right]^{\ell} + \text{lower degree terms in } \left(\frac{n}{2} - i\lambda\right) \right\} \\ &\quad \cdot e^{(-\frac{n}{2} + i\lambda) \langle x, b \rangle} \end{aligned}$$

$$\left(\frac{d}{d\theta}\right)^{\ell} \frac{\partial}{\partial v} e^{(-\frac{n}{2} + i\lambda) \langle x, b \rangle} = \left\{ -\left(\frac{n}{2} - i\lambda\right)^{\ell+1} \left(\frac{2x \cdot v}{1 - 2x \cdot b + |x|^2}\right) \cdot \frac{\partial}{\partial v} \langle x, b \rangle \right.$$

$$\begin{aligned}
 & + \left(\frac{n}{2} - i\lambda\right)^\ell \ell \left(\frac{2x \cdot v}{1 - 2x \cdot b + |x|^2}\right)^{\ell-1} \frac{\partial}{\partial v} \left(\frac{2x \cdot v}{1 - 2x \cdot b + |x|^2}\right) \\
 & + \text{lower degree terms in } \left(\frac{n}{2} - i\lambda\right) \} e^{(-\frac{n}{2} + i\lambda)\langle x, b \rangle}.
 \end{aligned}$$

Going over to the y coordinates and evaluating the leading order asymptotics we find in both cases:

$$(4.9) \quad \sum_{b_{\pm}(p_j)=b} \pm (i\lambda)^\ell \left(\frac{2p_j \cdot v}{1 - 2p_j \cdot b + |p_j|^2}\right)^\ell \Pi(k_j - 1) e^{-\frac{1}{2} - \frac{n}{2} \langle p_j, b \rangle} e^{i\lambda \langle p_j, b \rangle} = O(\lambda^{-1/2}).$$

Regarded as a system of linear equations with coefficients

$$A_{j\ell} = (-i\lambda)^\ell \left(\frac{2p_j \cdot v}{1 - 2p_j \cdot b + |p_j|^2}\right)^\ell \quad 1 \leq j, \ell \leq N,$$

the Van der Monde determinant must vanish. Hence by lemma 4.3, there is a pair of points say $p_i = p$ and $p_j = p^*$ which lie on the same geodesic joining p to b . Furthermore, the geodesic joining p to p^* is orthogonal to $\partial\Omega$ at p and p^* hence Lemma 4.5 shows the distance L is constant as b , and hence p , varies locally. By analytic continuation, we conclude that Ω is a domain of constant breadth. In addition, for each b and each p with $b_{\pm}(p) = b$, there is exactly one other p^* with the same property lying on the geodesic joining p to b . Otherwise, we will find a pair p, p' with the exterior normals pointing in the same direction as the geodesic γ from p_i to p_j , this means that by analytic continuation there will be infinitely many points on the geodesic γ spaced evenly apart, contradicting the boundedness of Ω . Thus all points p_j with $b_{\pm}(p_j) = b$ can be partitioned into pairs $\{p_1, p_1^*\}, \dots, \{p_{N/2}, p_{N/2}^*\}$, such that no two pairs lie on the same geodesics, and the geodesic from p_j to b

enters Ω at p_j and leaves Ω at p_j^* (it may meet $\partial\Omega$ at other points as well but not orthogonally). The equation (4.9) can now be written as

$$(4.10) \quad \sum_{j=1}^{N/2} \left(\frac{p_j \cdot v}{1 - 2p_j \cdot b + |p_j|^2} \right)^2 \left[\frac{e^{i\lambda \langle p_j, b \rangle - \frac{n}{2} \langle p_j, b \rangle}}{(\prod (k_i(p_j) - 1))^{1/2}} + \frac{e^{i\lambda \langle p_j^*, b \rangle - \frac{n}{2} \langle p_j^*, b \rangle}}{(\prod (k_i(p_j^*) - 1))^{1/2}} \right] = 0 \left(\frac{1}{\lambda} \right).$$

The determinant of the system is now non-singular, hence the contribution from each pair must cancel, thus we have the identity:

$$(4.11) \quad \dots \pm \frac{-\prod (k_j^* - 1)}{\prod (k_j - 1)} = e^{nL}$$

Recalling the identity (4.6),

$$-k_j^* = \frac{\sinh L - k_j \cosh L}{\cosh L - k_j \sinh L}$$

we conclude that the curvature identity

$$\pm \prod_{j=1}^n (\cosh L - k_j \sinh L) = (-1)^n$$

holds.

Proof of Theorem 2 It will suffice to show each $k_j > 1$ where now k_j is taken

with respect to the interior normal, for then the curvature relation

$$F(k_1, \dots, k_n) = \prod_i^n (\cosh L - k_i \sinh L) \pm (-1)^n = 0$$

holds.

Recall that in (4.6) the normal was considered to be in the exterior direction hence (4.6) becomes

$$1 < k_j^* = \frac{\sinh L - k_j \cosh L}{\cosh L - k_j \sinh L}.$$

Because $\sinh L - k_j \cosh L < \sinh L - \cosh L < 0$ we find $\cosh L - k_j \sinh L < 0$ always. Hence all partial derivatives $\frac{\partial F}{\partial k_j}$ have the same sign so that Alexandrov's theorem applies.

To prove the inequality $k_j > 1$ we will show that for each b there is exactly a pair of points p and p^* such that $b_+(p) = b = b_-(p^*)$ so that $\langle x, b \rangle = \langle p, b \rangle$ is the smallest horosphere to contain Ω , hence $k_i(p) > 1$. We consider the next highest order term in the asymptotic expansion of $\int (\frac{d}{d\theta})^\ell I$ and $\int (\frac{d}{d\theta})^\ell I$: In the isometry of (4.4), $\frac{x \cdot v}{1 - 2x \cdot b + |x|^2}$ goes over to without loss of generality one of the coordinates y_α , with $1 \leq \alpha \leq n$. Then the integrand for the highest order term in $\int (\frac{d}{d\theta})^\ell I_{\mathcal{D}}$ appears as

$$(-i\lambda)^\ell (2y_\alpha)^\ell h^{-n/2} (1 + |\nabla h|^2)^{1/2} e^{-i\lambda \log h} dy_1 \cdots dy_n,$$

and the integrand for the highest order term in $\int (\frac{d}{d\theta})^\ell I_{\mathcal{N}}$ appears as

$$-(-i\lambda)^{\ell+1} (2y_\alpha)^\ell h^{-n/2} e^{i\lambda \log h} dy_1, \dots, dy_n.$$

The next order asymptotic for these integrals give

$$\sum_{j=1}^{N/2} \left[\frac{h^n(p_j)}{(\Pi(k_i(p_j)-1))^{1/2}} e^{+i\lambda \log h(p_j)} L_j u(p_j) + \frac{h^n(p_j^*)}{(\Pi(k_i(p_j^*)-1))^{1/2}} e^{+i\lambda \log h(p_j^*)} L_j u(p_j^*) \right] = o(1)$$

where

$$u = \begin{cases} y_\alpha^\ell h^{-n/2} (1 + |\nabla h|^2)^{1/2} \\ y_\alpha^\ell h^{-n/2} \end{cases}$$

$$\square_j u = \sum_i \frac{h^2(p_j)}{k_i(p_j)-1} \left(\frac{\partial^2}{\partial y_i^2} \right) u$$

$$g_j = \log h - \log h(p_j) - \sum \frac{k_i(p_j)-1}{h(p_j)} (y_i - y_i(p_j))^2$$

and
$$L_j u = -i \left[\frac{1}{2} \square_j u + \frac{1}{2 \cdot 2!} \square_j^2 (g_j u) + \frac{1}{2 \cdot 3 \cdot 3!} \square_j^3 (g_j^2 u) \right].$$

We display the computations: (recall that the p_j are critical points of h)

$$\begin{aligned} j y_\alpha^\ell h^{-n/2} (1+|\nabla h|^2)^{1/2} &= j (y_\alpha^\ell) \cdot h^{-n/2} (1+|\nabla h|^2)^{1/2} + y_\alpha^\ell j (h^{-n/2} (1+|\nabla h|^2)^{1/2}) \\ &= \ell(\ell-1) y_\alpha^{\ell-2} \frac{h^2(p_j)}{k_\alpha(p_j)-1} h^{-n/2}(p_j) + y_\alpha^\ell h^{-n/2} \left[-\frac{n}{2} + \sum (k_i(p_j)-1) \right] \end{aligned}$$

$$\begin{aligned} j y_\alpha^\ell h^{-n/2} &= j (y_\alpha^\ell) h^{-n/2} + y_\alpha^\ell j h^{-n/2} \\ &= \ell(\ell-1) y_\alpha^{\ell-2} \frac{h^2(p_j)}{k_\alpha(p_j)-1} h^{-n/2} + y_\alpha^\ell \left(-\frac{n}{2} \right) h^{-n/2} \end{aligned}$$

$$\begin{aligned} 2_j (g_j y_\alpha^\ell h^{-n/2} (1+|\nabla h|^2)^{1/2}) &= 2_j (g_j y_\alpha^\ell) h^{-n/2} (1+|\nabla h|^2)^{1/2} \\ &= \sum_{i,\ell} \frac{h^4}{(k_i(p_j)-1)(k_\ell(p_j)-1)} \left[\frac{h_{i\ell\ell\ell}}{h} - \left(\frac{(k_i-1)(k_\ell-1)}{h^4} + \delta_{i\ell} \frac{(k_\ell-1)(k_j-1)}{h^4} \right) \right] y_\alpha^\ell h^{-n/2} \\ &\quad + \sum \frac{h^4}{(k_i(p_j)-1)(k_\alpha(p_j)-1)} \frac{h_{i\ell\alpha}}{h} \ell y_\alpha^{\ell-1} h^{-n/2} \end{aligned}$$

$$2_j (g_j y_\alpha^\ell h^{-n/2}) = 2_j (g_j y_\alpha^\ell) h^{-n/2}$$

$$\begin{aligned} {}_3(g_j^2 u) &= {}_3(g_j^2)u \\ &= \sum_{i,m,n} \frac{h^4 h_{imn}^2}{(k_i-1)(k_m-1)(k_n-1)} h^{-n/2} y_\alpha^\ell(p_j). \end{aligned}$$

Summing up; by grouping into powers of y_α^ℓ , $y_\alpha^{\ell-1}$ and $y_\alpha^{\ell-2}$ we find, setting

$$\Lambda_j = \frac{h^{n/2} (p_j)}{(\prod(k_i(p_j)-1))^{1/2}} e^{+i\lambda \log h(p_j)},$$

equation (4.10) yields for each ℓ an equation of the form.

$$\begin{aligned} \sum_{j=1}^{N/2} [\Lambda_j (y_\alpha^\ell(p_j) A_j + y_\alpha^{\ell-1}(p_j) \ell B_j + y_\alpha^{\ell-2}(p_j) \ell(\ell-1) C_j) \\ + \Lambda_j^* (y_\alpha^\ell(p_j^*) A_j^* + y_\alpha^{\ell-1}(p_j^*) \ell B_j^* + y_\alpha^{\ell-2}(p_j^*) \ell(\ell-1) C_j^*)] = 0. \end{aligned}$$

Since $\Lambda_j + \Lambda_j^* = 0$ by (4.10) this equation can be rewritten as

$$\sum_{j=1}^{N/2} \Lambda_j [y_\alpha^\ell(p_j) (A_j - A_j^*) + y_\alpha^{\ell-1}(p_j) (\ell B_j - \ell B_j^*) + y_\alpha^{\ell-2}(p_j) \ell(\ell-1) (C_j - C_j^*)] = 0.$$

We thus have an infinite sequence of identities indexed by $\ell = 1, 2, 3, \dots$

Lemma For distinct real numbers y_j , $j=1, \dots, N/2$, if there exist a vector $(a_1, \dots, a_{N/2}, b_1, \dots, b_{N/2}, c_1, \dots, c_{N/2})$ satisfying

$$\sum_{j=1}^{N/2} (y_j^\ell a_j + \ell y_j^{\ell-1} b_j + \ell(\ell-1) y_j^{\ell-2} c_j) = 0 \quad \text{for } \ell = 1, 2, \dots$$

then $a_i = b_i = c_i = 0$; unless $\frac{N}{2} = 1$.

Proof Assume $\frac{N}{2} > 2$ then we may without loss of generality assume that

$|y_1| < |y_2| < |y_3| \dots < |y_{\frac{N}{2}}|$. It is clear by taking $\ell \rightarrow \infty$ in the equations

that $c_{\frac{N}{2}} = 0$. In this inductive manner, we conclude all $c_j = 0$. Next we see again that $b_{\frac{N}{2}} = 0$ and inductively all $b_j = 0$.

Applying the lemma to the above sequence of identities, it follows that unless $\frac{N}{2} = 1$, we will have $\Lambda_j(C_j - C_j^*) = 0$, which upon examination gives

$$\frac{h^2(p_j)}{k_\alpha(p_j)^{-1}} = \frac{h^2(p_j^*)}{k_\alpha^*(p_j)^{-1}} .$$

However the argument shows this identity works for all α . Taking the product over indices $\alpha = 1, \dots, n$, $e^L = 1$, which is a contradiction. This finishes the proof of Theorem 2.

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