AN ALGORITHM TO COMPUTE THE STRUCTURED
SINGULAR VALUE

by

Michael K.H. Fan
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System Research Center
University of Maryland
College Park, MD 20742

Abstract

The concept of structured singular value was recently introduced by Doyle as a tool for analysis and synthesis of feedback systems with structured uncertainties. It is a key to the design of control systems under joint robustness and performance specifications and it nicely complements the $H^\infty$ approach to control system design. This report proposes an algorithm to compute the structured singular value.
1. Introduction and preliminaries

The concept of \textit{structured singular value} was recently introduced by Doyle [1] as a tool for analysis and synthesis of feedback systems with structured uncertainties. It is a key to the design of control systems under joint robustness and performance specifications and it nicely complements the $H^\infty$ approach to control system design. [2]

Throughout the note, given any square complex matrix $M$, we denote by $\rho(M)$ its spectrum radius, by $\sigma(M)$ its largest singular value and by $M^H$ its complex conjugate transpose. Given any complex vector $x$, $x^H$ indicates its complex conjugate transpose and $\|x\|$ its Euclidean norm. We also make use of the following notation and nomenclature, largely inspired from that used in. [1] We will call \textit{block-structure} of size $m$ any $m$-tuple $\mb{k} = (k_1, \ldots, k_m)$ of positive integers. Given a block-structure $\mb{k}$ of size $m$, we will make use of the family of diagonal matrices

$$d = \{\text{block diag}(d_1 I_{k_1}, \ldots, d_m I_{k_m}) \mid d_i \in \mathbb{R}\};$$

(1.1)

and, for any positive scalar $\delta$ (possibly $\infty$), of the family of block diagonal matrices

$$X_\delta = \{\text{block diag}(\Delta_1, \ldots, \Delta_m) \mid \Delta_i \text{ is a } k_i \times k_i \text{ complex matrix satisfying } \sigma(\Delta_i) \leq \delta\};$$

(1.2)

All of the above have dimension $n \times n$, where

$$n = \sum_{j=1}^{m} k_j.$$

(1.3)

The following definition corresponds to the case of "no repeated blocks" in. [1]
Definition 1.1

The structured singular value \( \mu(M) \) of a complex \( n \times n \) matrix \( M \) with respect to block-structure \( \mathcal{K} \) is the positive number \( \mu \) having the property that

\[
\det(I + M\Delta) \neq 0 \quad \text{for all } \Delta \in \mathcal{X}_\delta
\]  

(1.4)

if, and only if,

\[
\delta \mu < 1 .
\]  

(1.5)

In other words, \( \mu(M) = 0 \) if there is no \( \Delta \) in \( \mathcal{X}_\infty \) such that \( \det(I + M\Delta) = 0 \), and \( (\min_{\Delta \in \mathcal{X}_\infty} \{ \sigma(\Delta) \mid \det(I + M\Delta) = 0 \})^{-1} \) otherwise.

It should be noted that \( \mathcal{D}, \mathcal{X}_\delta \), and \( \mu(M) \) all depend on the underlying block-structure. In most instances, we will not explicitly specify this block-structure.

We will make repeated use of the following easily derived fact. [1]

Fact 1.0 For all \( D \in \mathcal{D} \),

\[
\mu(M) = \mu(e^D M e^{-D})
\]  

(1.6)

In order to evaluate the structured singular value, more manageable expressions than those provided in Definition 1.1 are desirable. Such expressions are provided by the following fact. [1]

Fact 1.1 For block-structures of size less than \( \delta \),

\[
\mu(M) = \inf_{D \in \mathcal{D}} \sigma(e^D M e^{-D}) .
\]  

(1.7)
In fact, in many (but not all) cases, (1.7) is correct with block-structures of larger size. A counterexample, due to Doyle, [3] which shows that (1.7) is violated, is given in Appendix A.

In [1] Doyle proposed an algorithm, which is essentially based upon first derivatives, to solve problem (1.7). Since when the largest singular value of $e^D Me^{-D}$ is simple, the square of $\mathcal{H}(e^D Me^{-D})$ is continuously differentiable in $D$. Hence it is possible to express the first and second derivatives analytically such that, locally, Newton's method could be applied to solve problem (1.7). This report proposes a modified algorithm to compute the structured singular value based upon the first and second derivatives. In section 2, we will discuss a first order algorithm which mostly follows the line in [1] In section 3, we will discuss the continuity properties of hermitian matrices. Finally, in section 4, a second order algorithm is presented.
2. A first order algorithm

In this section, we will discuss algorithms to solve the right hand side of (1.7) or, equivalently,

\[ \inf_{D \in \mathcal{D}} \| e^{D} M e^{-D} \|_2 \]  \hspace{1cm} (2.1) \]

by means of (generalized) gradient search method. Since \( \| e^{D} M e^{-D} \|_2 \) is convex [4] in \( D \), it results that all stationary points are global minima. Since for any \( \alpha \in \mathbb{R} \), \( e^{D} M e^{-D} = e^{\alpha I + D} M e^{-\alpha I - D} \), without loss of generality, we assume that \( d_m = 0 \).

Recall that \( D = \text{blockdiag} \{ d_1 I_{k_1}, \ldots, d_m I_{k_m} \} \). Define \( d = [d_1 \cdots d_{m-1}]^T \), \( g(d) = \| e^{D} M e^{-D} \|_2 \) and \( H(d) = (e^{D} M e^{-D}) \). Note that \( g(d) \) is continuous but not always differentiable. However the following property holds

**Proposition 2.1** For any \( h \), the following expression exists

\[ \lim_{t \to 0^+} \frac{g(th) - g(0)}{t} \]  

**Definition 2.1** \( h \) is said to be a descent direction for \( g(d) \) at \( d = 0 \) if there exists a \( \delta > 0 \) such that for every \( t \in (0, \delta) \)

\[ g(th) < g(0) \]  

**Definition 2.2** A unit norm vector \( h \) is said to be a steepest descent direction for \( g(d) \) at \( d = 0 \) if \( h \) is a descent direction and a solution of

\[ \min_{h} \left\{ \lim_{t \to 0^+} \frac{g(th) - g(0)}{t} \mid \| h \| = 1 \right\} \]
Suppose that $H(0)$ has a simple largest eigenvalue $\lambda_1$, we denote $v_1$ the unit norm eigenvector corresponding to $\lambda_1$. Thus gradient of $g(\mathbf{d})$ at $\mathbf{d} = 0$ can be computed component-wise as follows. For $j = 1, \cdots, m-1,$

$$\nabla g_j(0) = v_1^H H_j v_1$$

(2.2)

where $H_j = 2 \text{Re} (H(0)^H \frac{\partial H(0)}{\partial d_j})$. So $-\nabla g(0)/\|\nabla g(0)\|$ is a (steepest) descent direction for $g(\mathbf{d})$ at $\mathbf{d} = 0$.

**Proposition 2.2** If $\lambda_1$ is simple and $\nabla g(0) = 0$, then $\mu(M) = \sigma(M)$.

*Proof. See. [1, 4]*

**Corollary 2.1** If (2.1) is achievable and the corresponding largest singular value is simple, then (1.7) holds ($m$ needs not to be less than four).

In the case that the largest eigenvalue of $H(0)$ has multiplicity $q$, $q > 1$, $g(\mathbf{d})$ is then not continuously differentiable at $\mathbf{d} = 0$. Therefore the gradient is not well defined. In order to find a descent direction for $g(\mathbf{d})$, a generalized gradient is introduced. Let $P_1$ denote the set containing all the unit norm eigenvectors corresponding to $\lambda_1$. Define

$$\nabla_2 = \{ \mathbf{y} = (y_1, \cdots, y_{m-1}) \mid y_j = \mathbf{z}^H H_j \mathbf{z}, \mathbf{z} \in P_1 \}$$

(2.3)

Clearly, if $\lambda_1$ is simple, $\nabla_2$ reduces to $\{\nabla g(0)\}$.

**Proposition 2.3** When $m \leq 3$, $\nabla_2$ is convex.

*Proof. See. [1]*
Proposition 2.4 \( \mu(M) = \bar{\sigma}(M) \) if and only if \( 0 \in \nabla_2 \).

Proof. See. [1]

Proposition 2.5 Suppose \( 0 \in \text{co} \nabla_2 \) and vector \( h \) has the property that

\[
\langle h, y \rangle < 0 \quad \text{for all } y \in \nabla_2,
\]

then \( h \) is a descent direction of \( g(d) \) at \( d = 0 \), where \( \text{co} \nabla_2 \) denotes the convex hull of \( \nabla_2 \).

Proof. See. [1]

Corollary 2.2 Assume that (2.1) is achievable and \( m \leq 3 \), then (1.7) holds.

Proof. Assume that \( D^* \) solves (2.1). By Proposition 2.5, we have \( 0 \in \text{co} \nabla_2 \) where \( \nabla_2 \) is defined in terms of \( e^{D^*} Me^{-D^*} \). By Proposition 2.3, since \( \nabla_2 \) is convex, \( 0 \in \nabla_2 \). Finally, by Proposition 2.4, since \( 0 \in \nabla_2 \), we conclude that

\[
\mu(M) = \mu(e^{D^*} Me^{-D^*}) = \bar{\sigma}(e^{D^*} Me^{-D^*}) = \inf_{D \in d} \bar{\sigma}(e^{D} Me^{-D}).
\] (2.5)

Proposition 2.6 Let \( h = -Nr(\text{co} \nabla_2) \), then \( h / \|h\| \) is a steepest descent direction of \( g(d) \) at \( d = 0 \), where \( Nr(\text{co} \nabla_2) \) denotes the nearest point to the origin in \( \text{co} \nabla_2 \).

We now are ready to state a first order algorithm for computing (2.1).

Algorithm 2.1
Step 1.
Data $M_0 = M$, $D_0 = 0$ ($d_0 = 0$).

$k = 0$.

Step 2.
Set $M_{k+1} = e^{D_k} M_k e^{-D_k}$.

Define search direction $h$ to be $-Nr (c_0 \nabla_2)$ where $\nabla_2$ is defined in terms of $M_{k+1}$.

Step 3.
Perform line search to find the step size $\alpha$.

Step 4.

$d_{k+1} = d_k + \alpha h$ ($D_{k+1}$ is therefore updated).

Set $k = k + 1$, go to step 2.

\[\square\]

**Proposition 2.7** Let $D^* = \sum_{k=1}^{\infty} D_k$ where $\{D_k\}$ is the sequence generated by Algorithm 2.1. Then

$$\bar{\sigma}(e^{D^*} Me^{-D^*}) = \inf_{D \in \mathcal{D}} \bar{\sigma}(e^D Me^{-D}) .$$

\[\square\]

Let $\{v_1, \ldots, v_q\}$ be a basis for $P_1$. (recall that $P_1$ denotes the set containing all the unite norm eigenvector corresponding to $\lambda_0$) Define $V = [v_1, \ldots, v_q]$, $H_j = V^H H_j V$ and $P_2 = \{z \in \mathbb{C}^q \mid z^H z = 1\}$. Note that $H_j$ is of size $q \times q$. By using these notation, $\nabla_2$ could be expressed in a more manageable way as follows
\[ \nabla_2 = \{ f(x) \mid f^j(x) = e^H_j x, \ j=1, \ldots, m-1, \ x \in P_2 \} \]  
(2.6)

The following algorithm, [1] which is based upon (2.6), is to find \( N_r(\text{co}\nabla_2) \).

**Algorithm 2.2**

**Step 1.**

Pick any \( x_0 \in P_2 \) and let \( x_0 = f(x_0) \). Set \( k = 0 \).

**Step 2.**

Set \( x_{k+1} = N_r(\text{co}[x_k, f(x_k)]) \).

**Step 3.**

Let \( x_{k+1} \) be any unit vector for \( \lambda_{\text{min}}(\sum_{j=1}^{m-1} x_j^j H_j) \), where \( \lambda_{\text{min}} \) denotes the smallest eigenvalue.

**Step 4.**

Set \( k = k + 1 \), go to step 2.

**Proposition 2.8** Let \( \{x_k\} \) be the sequence generated by Algorithm 2.2, then \( \{x_k\} \) converges to \( N_r(\text{co}\nabla_2) \). Furthermore, let \( x^* \) denote the limit and suppose that \( x^* \neq 0 \), then \( \sum_{j=1}^{m-1} x^j H_j \) is a strictly positive definite matrix.

**Proof.** It is shown in [1] that any convergent subsequence of \( \{x_k\} \) converges to \( N_r(\text{co}\nabla_2) \). Since the sequence \( \{\|x_k\|\} \) is bounded and \( N_r(\text{co}\nabla_2) \) has a unique solution, it is true that \( \{x_k\} \) itself converges and the limit is \( N_r(\text{co}\nabla_2) \). Furthermore, let \( x^* \) be any accumulation point of the sequence \( \{x_k\} \). By the algorithm and the definition of limit, we have \( \langle x^*, f(x^*) \rangle = \lambda_{\text{min}}(\sum_{j=1}^{m-1} x^j H_j) \). If \( \lambda_{\text{min}}(\sum_{j=1}^{m-1} x^j H_j) \) is not greater than
zero, we have $\not\exists N_r (\text{co}(z^+, f(z^+)))$ which leads to a contradiction.

As mentioned above, when $\lambda_1$, the largest eigenvalue of $H(0)$, is simple, i.e. $q = 1$, $\nabla_2$ reduces to $\{\nabla g(0)\}$ and, therefore, $N_r (\text{co} \nabla_2) = \nabla g(0)$. In the case that $q = 2$, it can be shown that the boundary of $\text{co} \nabla_2$ is a second order curve (or surface), possibly degenerate, in $\mathbb{R}^{m-1}$. Hence, $N_r (\text{co} \nabla_2)$ could also be solved analytically. Based on this observation, for any $q$, we will propose another algorithm to compute $N_r (\text{co} \nabla_2)$. Now we proceed this by giving more details about the case $q = 2$. For $j = 1, \ldots, m-1$, let

$$
\overline{H}_j = \begin{bmatrix} a_j & b_j \\ b_j^T & c_j \end{bmatrix}
$$

where $a_j, c_j \in \mathbb{R}$ and $b_j \in \mathbb{C}$. Define $\nabla_2$ accordingly. Also define $\underline{1}$ to be the vector in $\mathbb{R}^{m-1}$ such that the $j$th component of $\underline{1}$ is $(a_j + c_j)/2$, and, define $A$ to be the matrix in $\mathbb{R}^{(m-1)\times 3}$ with the $j$th row being $[(a_j - c_j)/2 \ \text{Re}(b_j) \ \text{Im}(b_j)]$. Recall that, for $q = 2$, $P_2 = \{z \in \mathbb{C}^2 \mid z^H z = 1\}$. Let $S = \{x \in \mathbb{R}^3 \mid x^T x = 1\}$. We define $g(z)$ to be an affine function such that $g(z) = A z + \underline{1}$.

**Proposition 2.9**  \[ \nabla_2 = g(S). \]

**Proof.** See [1]

By the Proposition 2.9, it becomes possible to image how the set $\nabla_2$ looks like for the case $q = 2$ and, fortunately, in this case the boundary of $\text{co} \nabla_2$ is either a point, an interval, an ellipse in $\mathbb{R}^2$ or an ellipsoid in $\mathbb{R}^3$. Therefore, finding the nearest point to the origin in $\text{co} \nabla_2$ is straightforward. Perform the singular value decomposition of $A$ such that
then \( N_r(\text{co} \nabla_2) = U_A \ N_r(Q_2 + U^T_1 L) \).

We now state another algorithm to compute \( N_r(\text{co} \nabla_2) \).

**Algorithm 2.3**

**Step 1.**

Pick any \( x_0 \in P_2 \) and let \( x_0 = f(x_0) \). Set \( k = 0 \).

**Step 2.**

Let \( u_k \) be any unit vector for \( \lambda_{\text{min}}(\sum_{j=1}^{m-1} H_j) \).

**Step 3.**

Define

\[
\overline{H}_j = \begin{bmatrix} u_k^H \\ u_k^H \end{bmatrix} H_j \begin{bmatrix} u_k \\ u_k \end{bmatrix} \quad j = 1, \ldots, m - 1.
\]

Define \( \overline{f} \) in terms of \( \overline{H}_j \). Analytically find the solution \( \overline{w} \) such that \( \overline{f}(\overline{w}) \) is the nearest point to the origin in set \( P_2 \). Set \( x_{k+1} = [u_k \ u_k] \overline{w} \) and \( x_{k+1} = f(x_{k+1}) \).

**Step 4.**

Set \( k = k + 1 \), go to step 2.

**Proposition 2.11** Proposition 2.8 also holds for Algorithm 2.3.
\[ A = U_A \Sigma_A V_A^T \]  

where \( U_A \) and \( V_A \) are orthogonal matrices in \( \mathbb{R}^{(m-1) \times (m-1)} \) and \( \mathbb{R}^{3 \times 3} \) respectively, and

\[
\Sigma_A = \begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
\vdots & \vdots & \sigma_3 \\
0 & 0 & 0
\end{bmatrix}
\]  

provided that \( \Sigma_A \) has appropriate dimension. The following Proposition gives the solutions for all cases in terms of the rank of \( A \).

**Proposition 2.10**

*case 1.* \( \text{rank}(A) = 0 \)

\[ Nr(\text{cov}_2) = I. \]

*case 2.* \( \text{rank}(A) = 1 \)

\[ Nr(\text{cov}_2) = U_A \cdot Nr(\text{cov}[(\sigma_1,0, \cdots ,0)^T + U_A^T I, (-\sigma_1,0, \cdots ,0)^T + U_A^T I]). \]

*case 3.* \( \text{rank}(A) = 2 \)

Let \( Q_1 \) denote the set

\[ \{ \mathbf{x} = (x_1, x_2,0, \cdots ,0) \mid \mathbf{x} \in \mathbb{R}^{m-1}, \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} \leq 1 \} , \]

then \( Nr(\text{cov}_2) = U_A \cdot Nr(\mathcal{Q}_1 + U_A^T I). \)

*case 4.* \( \text{rank}(A) = 3 \)

Let \( Q_2 \) denote the set

\[ \{ \mathbf{x} = (x_1, x_2, x_3,0, \cdots ,0) \mid \mathbf{x} \in \mathbb{R}^{m-1}, \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} + \frac{x_3^2}{\sigma_3^2} \leq 1 \} , \]
\[ g_i(d) = \lambda_{\max}(u_1, \ldots, u_q)^H D(u_1, \ldots, u_q) - (v_1, \ldots, v_q)^H D(v_1, \ldots, v_q) \]  \hspace{0.5cm} (2.10)

where \( \lambda_{\max}(A) \) denotes the largest eigenvalue of \( A \), and \( q \) is the multiplicity of \( H(0) \).

**Proposition 2.12** If \( g_i(h) < 0 \), then \( h \) is a descent direction. If \( g_i(h) > 0 \), then \( h \) is not a descent direction.

\[
\boxed{
\]

**Proposition 2.13** The following three statements are equivalent.

1. \( h \) is a steepest descent direction of \( g(d) \) at \( d = 0 \).

2. \( h = \frac{h_1}{||h_1||} \), where \( h_1 = -N_r(c_o \nabla_o) \).

3. \( h \) is a unit norm descent direction and \( h \) solves

\[ \min_{d} \{ g_i(d) \mid ||d|| = 1 \} . \]  \hspace{0.5cm} (2.11)

\[
\boxed{
\]
3. Continuity properties of hermitian matrix

Suppose $H(t)$ is hermitian and real analytic in $t$, it is well-known that by appropriate ordering of eigenvalues $\{\lambda_i\}$ and selection of eigenvectors $\{v_i\}$, it is possible to pair eigenvalues and eigenvectors $\{\lambda_i(t), v_i(t)\}$, such that both $\lambda_i(t)$ and $v_i(t)$ are analytic in $t$ and $H(t)v_i(t) = \lambda_i(t)v_i(t)$ for all $t$ and $i$. At values of $t$, where $H(t)$ has simple eigenvalues this is trivial. At degenerate points, it require selection of the $\lambda_i$ and $v_i$ such that analyticity is retained through (isolated) point where eigenvalues coalesce. Note that with this ordering, the $\{\lambda_i\}$ are not necessarily linearly ordered. In this section, we will explore the continuity properties of hermitian matrix such that by using these properties, we could derive a second order algorithm in the next section for computing (2.1).

Let $H(t)$ be an $n \times n$ hermitian matrix and suppose that it is real analytic in $t$. We denote $\lambda_1(t)$ the eigenvalue corresponding to the spectral radius of $H(t)$ and $v_1(t)$ the corresponding eigenvector such that,

$$H(t)v_1(t) = \lambda_1(t)v_1(t)$$  \hspace{1cm} (3.1)

We assume that $\lambda_1 = \lambda_1(0)$ is simple. Since $H(t)$, $\lambda_1(t)$ and $v_1(t)$ are analytic, we could express them in the form of Taylor series of $t$ at $t=0$ as follows

$$H(t) = H_0 + t \dot{H} + \frac{1}{2} t^2 \ddot{H} + o(t^3)$$  \hspace{1cm} (3.2)

$$\lambda_1(t) = \lambda_1 + t \dot{\lambda}_1 + \frac{1}{2} t^2 \ddot{\lambda}_1 + o(t^3)$$  \hspace{1cm} (3.3)

and

$$v_1(t) = v_1 + t \dot{v}_1 + \frac{1}{2} t^2 \ddot{v}_1 + o(t^3)$$  \hspace{1cm} (3.4)

where $\bar{x} \in \mathbb{C}^{n-1}$, $y \in \mathbb{C}^n$ and $[v_1 \mid V_{\parallel}]$ is a unitary matrix such that
\[ H_0 V_\perp = V_\perp \Lambda_\perp . \] (3.5)

where \( \Lambda_\perp \) is a diagonal matrix with all eigenvalues of \( H_0 \) except \( \lambda_1 \) in the diagonal.

Therefore we have

\[
(H_0 + t \dot{H} + \frac{1}{2} t^2 \ddot{H} + o(t^2)) (\varepsilon_1 + t V_{\perp \varepsilon} + \frac{1}{2} t^2 \varepsilon + o(t^2)) \\
= (\lambda_1 + t \dot{\lambda}_1 + \frac{1}{2} t^2 \ddot{\lambda}_1 + o(t^2)) (\varepsilon_1 + t V_{\perp \varepsilon} + \frac{1}{2} t^2 \varepsilon + o(t^2)) .
\] (3.6)

Since (3.6) is true for all \( t \), we could expand it and have equalities for its constant, \( t \)
and \( t^2 \) terms individually. Thus for constant term, we have

\[ H_{0 \varepsilon_1} = \lambda_{1 \varepsilon_1} ; \] (3.7)

for \( t \) term, we have

\[ \dot{H}_{\varepsilon_1} + H_0 V_{\perp \varepsilon} = \dot{\lambda}_{\varepsilon_1} + \lambda_1 V_{\perp \varepsilon} \] (3.8)

and for \( t^2 \) term, we have

\[ \ddot{H} V_{\perp \varepsilon} + \frac{1}{2} \ddot{H} \varepsilon_1 + \frac{1}{2} H_{0 \varepsilon} = \ddot{\lambda}_{\varepsilon_1} + \frac{1}{2} \ddot{\lambda}_{\varepsilon_1} + \frac{1}{2} \lambda_1 \varepsilon . \] (3.9)

Now we could express \( \dot{\lambda} \) and \( \ddot{\lambda} \) in terms of known quantities by performing some simple manipulations. Multiply \( t \) term on left by \( \varepsilon_1^H \) and yield

\[ \varepsilon_1^H (\dot{H}_{\varepsilon_1} + H_0 V_{\perp \varepsilon}) = \varepsilon_1^H (\dot{\lambda}_{\varepsilon_1} + \lambda_1 V_{\perp \varepsilon}) \] (3.10)

and

\[ \varepsilon_1^H \dot{H}_{\varepsilon_1} + \varepsilon_1^H H_0 V_{\perp \varepsilon} = \varepsilon_1^H \dot{\lambda}_{\varepsilon_1} + \varepsilon_1^H \lambda_1 V_{\perp \varepsilon} . \] (3.11)

Since
\[ \varepsilon_i H_0 V_\perp \varepsilon = \varepsilon_i H \lambda_1 V_\perp \varepsilon = 0 \]  

We then have
\[ \dot{\lambda} = \varepsilon_i \dot{H} \varepsilon_1 \]  

Multiply \( t \) term on left by \( V_i^H \) and yield
\[ V_i^H \dot{H} \varepsilon_1 + \Lambda_1 \varepsilon = \lambda_1 \varepsilon \]  

Hence
\[ \varepsilon = (\lambda_1 I - \Lambda_1)^{-1} V_i^H \dot{H} \varepsilon_1 \]  

Multiply \( t^2 \) term on left by \( \varepsilon_i^H \) and yield
\[ \varepsilon_i^H \dot{H} V_\perp \varepsilon + \frac{1}{2} \varepsilon_i^H \dot{H} \varepsilon_1 + \frac{1}{2} \varepsilon_i^H H_0 \varepsilon = \dot{\lambda} \varepsilon_i^H V_\perp \varepsilon + \frac{1}{2} \dot{\lambda} \varepsilon_i^H \varepsilon_1 + \frac{1}{2} \lambda_1 \varepsilon_i^H \varepsilon \]  

Since for any \( y \)
\[ \varepsilon_i^H H_0 y = \lambda_1 \varepsilon_i^H y \]  

thus we have
\[ \ddot{\lambda} = \varepsilon_i^H \dot{H} \varepsilon_1 + 2 \varepsilon_i^H \dot{H} V_\perp \varepsilon = \varepsilon_i^H \dot{H} \varepsilon_1 + 2 \varepsilon_i^H \dot{H} V_\perp (\lambda_1 I - \Lambda_1)^{-1} V_\perp \dot{H} \varepsilon_1 \]  

Note that, as long as that \( [\varepsilon_i | V_i] \) is unitary and (3.5) holds, \( \dot{\lambda} \) is independent of the choices of \( V_\perp \) and \( \Lambda_1 \). By the assumption that \( \lambda_1 \) is simple, (3.13) and (3.18) give the explicit expressions for \( \dot{\lambda} \) and \( \ddot{\lambda} \) respectively. Since the matrix \( (\lambda_1 I - \Lambda_1) \) in (3.18) is not invertible when \( \lambda_1 \) is not simple. As mentioned in the beginning of this section, it amounts to the choices of eigenvectors for the case when eigenvalues coalesce, such that, \( V_i^H \dot{H} \varepsilon_1 \) is in the range space of \( (\lambda_1 I - \Lambda_1) \). Thus a solution of \( \varpi \) could be
\[ \varpi = (\lambda_1 I - \Lambda_1)^* V_i^H \dot{H} \varepsilon_1 \]
where the superscript '+' denotes pseudo-inverse. Therefore, (3.18) becomes valid after \((\lambda_i I - \Lambda_i)^{-1}\) is replaced by \((\lambda_i I - \Lambda_i)^+\). For simplicity of discussion, we assume that

\[
\lambda_1 = \lambda_2 = \cdots = \lambda_q
\]  

(3.20)

and \(\phi_i, i = 1, \cdots, q\), are \(q\) mutually perpendicular unit norm eigenvectors associated with eigenvalue \(\lambda_i\).

**Proposition 3.1** There exists a choices of mutually perpendicular unit norm eigenvectors \(\Phi_i, i = 1, \cdots, q\), in the space spanned by \(\{\phi_1, \phi_2, \cdots, \phi_q\}\) such that for all \(i = 1, \cdots, q\), \(V_{\lambda_i}^H \hat{H}_{\Phi_i}\) lies in the range space of \((\lambda_i I - \Lambda_{\lambda_i})\) where \(V_{\lambda_i}\) and \(\Lambda_{\lambda_i}\) are defined similarly to \(V_L\) and \(\Lambda_L\).

**Proof.** It is easy to show that it suffices to prove that there exists a unitary matrix \(W, W \in \mathbb{C}^{q \times q}\), such that

\[
W^H [\phi_1 \phi_2 \cdots \phi_q]^H \hat{H} [\phi_1 \phi_2 \cdots \phi_q] W
\]  

(3.21)

is diagonal. Since matrix \([\phi_1 \phi_2 \cdots \phi_q]^H \hat{H} [\phi_1 \phi_2 \cdots \phi_q]\) is hermitian, it is always possible to change (3.21) to diagonal form by performing a unitary transformation.

It should be noted that the choice of matrix \(W\) is dependent of matrix \(\hat{H}\).

In this section, we will make use of the properties discussed in the previous section to derive a second order algorithm to solve (2.1). Define

\[ H(t) = (e^{Dt} Me^{-Dt})^H (e^{Dt} Me^{-Dt}) \]  \hspace{1cm} (4.1)

and let the singular value decomposition of \( M \) be

\[ M = U \Sigma V^H \]  \hspace{1cm} (4.2)

where

\[ U = [u_1 \cdots u_n] \]  \hspace{1cm} (4.3)

\[ V = [v_1 \cdots v_n] \]  \hspace{1cm} (4.4)

\[ \Sigma = \text{diag} \{ \sigma_1 \cdots \sigma_n \}. \]  \hspace{1cm} (4.5)

Then it is easy to get the following equalities

\[ H_0 = M^H M \]  \hspace{1cm} (4.6)

\[ \dot{H} = -DM^H M + 2M^H DM - M^H MD \]  \hspace{1cm} (4.7)

\[ \ddot{H} = D^2 M^H M - 4DM^H DM + 4M^H D^2 M + 2M^H MD - 4M^H DMD + M^H MD^2 \]  \hspace{1cm} (4.8)

If \( \lambda_1 \) is simple, we have

\[ \dot{\lambda}_1 = 2\sigma_1^2 (u_1^H Du_1 - v_1^H Dv_1) \]

\[ = 2 [d_1 \cdots d_m] \begin{bmatrix} \|P_1 Mv_1\|_F^2 - \|Mv_1\|_F^2 \|P_1 v_1\|_F^2 \\ \vdots \\ \|P_m Mv_1\|_F^2 - \|Mv_1\|_F^2 \|P_m v_1\|_F^2 \end{bmatrix} \]
\[
\begin{align*}
\dot{\lambda}_i &= \begin{bmatrix} \|P_1 u_i\|_F^2 - \|P_1 v_i\|_F^2 \\ \vdots \\ \|P_m u_i\|_F^2 - \|P_m v_i\|_F^2 \end{bmatrix} \\
&= 2\sigma_i^2 \begin{bmatrix} d_1 & \cdots & d_m \end{bmatrix} \\
\text{and} \\
\dot{\lambda}_1 &= \begin{bmatrix} u_1^H D & v_1^H D \end{bmatrix} E \begin{bmatrix} D u_1 \\ D v_1 \end{bmatrix} \\
&= \begin{bmatrix} 4\sigma_1 I & -4\sigma_1 M \\ -4\sigma_1 M^H & 2\sigma_1^2 I + 2M^H M \end{bmatrix} + \\
2 \begin{bmatrix} 2\sigma_1 U_1 \Sigma_\perp \\ -V_1(\sigma_1^2 I + \Sigma_\perp^2) \end{bmatrix} (\sigma_1^2 I - \Sigma_\perp^2)^{-1} \begin{bmatrix} 2\sigma_1 U_1 \Sigma_\perp \\ -V_1(\sigma_1^2 I + \Sigma_\perp^2) \end{bmatrix}^T \\
&= \begin{bmatrix} u_2 & \cdots & u_n \end{bmatrix}, \\
&= \begin{bmatrix} v_2 & \cdots & v_n \end{bmatrix}, \\
\Sigma_\perp &= \text{diag} \{\sigma_2, \ldots, \sigma_n\}. 
\end{align*}
\]

**Proposition 4.1**

\[
\lambda_i(t) = \lambda_i + t < \nabla g(0), \quad d > + \frac{1}{2} t^2 d^T B d + O(t^2 \|d\|^2) \\
\]  

where \(d = [d_1 \cdots d_{m-1}]^T\), \(\nabla g(0)\) is defined in Section 2 and

\[
B = \text{real part of} \begin{bmatrix} P_1 u_1 & \cdots & P_{m-1} u_1 \\ P_1 v_1 & \cdots & P_{m-1} v_1 \end{bmatrix}^H E \begin{bmatrix} P_1 u_1 & \cdots & P_{m-1} u_1 \\ P_1 v_1 & \cdots & P_{m-1} v_1 \end{bmatrix}. 
\]

Furthermore, \(B\) is non-negative definite.

The first three terms in the right hand side of (4.15) gives a second order model of \(\lambda_i(t)\) when \(\lambda_i\) is simple and \(t \|d\|\) is small. Since \(B\) is non-negative definite, the solu-
tion with minimum norm for the model is
\[ t \| d \| = -B^* \nabla g (0) \] (4.17)
and also \(-B^* \nabla g (0)\) is a descent direction of \(g (d)\) at \(d = 0\), where \(B^*\) denotes the pseudo-inverse of \(B\). We now state the algorithm to solve (2.1).

Algorithm 4.1

Step 1.

Data \(M_0 = M, D_0 = 0 (d_0 = 0)\).
\(k = 0\).

Step 2.

Set \(M_{k+1} = e^{D_k} M_k e^{-D_k}\).

If the largest singular value of \(M_{k+1}\) is simple, define search direction \(h\) to be \(-B^* \nabla g (0)\), otherwise define search direction \(h\) to be \(-N_{\tau} (e_0 \nabla g)\) where \(\nabla g, B\) and \(\nabla g (0)\) are defined in terms of \(M_{k+1}\).

Step 3.

Perform line search to find the step size \(\alpha\).

Step 4.

\(d_{k+1} = d_k + \alpha h\) (\(D_{k+1}\) is therefore updated).

Set \(k = k + 1\), go to step 2.
Appendix A. Counterexample of $\mu(M) \neq \inf_{D \in \mathcal{D}} (e^D Me^{-D})$

Let $a = (1 - (\frac{1}{3})^{1/2})^{1/2}$, $b = \frac{1}{2^{1/2}}$ and

$$M_1 = \begin{bmatrix} a & 0 \\ ab & ab \\ ab & ab(1-2a^2)^{1/2} & -\frac{a^2(1+i)}{2(1-2a^2)^{1/2}} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & a \\ ab & -ab \\ a^2(1-i) & a^2(1-i)^{1/2} \\ 2(1-2a^2)^{1/2} & (1-2a^2)^{1/2} \end{bmatrix}$$

Define $M = M_1M_2^D$ and structure $k = (1,1,1,1)$, then $\bar{\sigma}(M) = 1$ and

$$\bar{H}_1 = a^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \bar{H}_2 = a^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{H}_3 = a^2 \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$ 

It is easy to check that $\nabla_2$ is a circle with radius $a^2$ centered at origin. Thus $\nabla_2 \neq \text{co}\nabla_2$, $0 \in \text{co}\nabla_2$ and $0 \notin \nabla_2$. Therefore $\mu(M) < 1$ but

$$\inf_{D \in \mathcal{D}} (e^D Me^{-D}) = 1$$

For this example, by using the formula in, [5] we can show that $\mu(M) > 0.87$. 
References


