TR-85-21

Optimal Stochastic Control of Linear Stochastic Systems with Poisson Process Coefficients

By

C.W. Li

G.L. Blankenship

TR-85-21

Optimal Stochastic Control of Linear Stochastic Systems with Poisson Process Coefficients

C.W. Li†
Department of Mathematical Studies
Hong Kong Polytechnic
Hung Hom, Kowloon, Hong Kong

G.L. Blankenship††
Electrical Engineering Department
University of Maryland
College Park, MD 20742

September 7, 1985

Abstract: We obtain results similar to those for LQG problems on the control system structure for optimal linear quadratic regulator problems with Poisson noise disturbances. If the coefficient matrices of the system dynamics and the performance index are constant, the optimal control of the finite time problem converges to the time-invariant control of the infinite time problem quasi-uniformly, almost surely. Both the long term average cost criterion and the discounted cost criterion are investigated for infinite time problems.

Key Words: Dynamic programming, stochastic Gronwall's inequality, Poisson noise.

---

†This research was conducted in the Applied Mathematics Program at the University of Maryland, College Park. It was supported in part by the Department of Energy under contract DE-AC01-79ET-29244.

††This research was supported in part by the Army Research Office under contract DAAG29-83-C-0028 with SEI, Greenbelt, MD. Address correspondence concerning the paper to this author.

Introduction.

Optimal stochastic control problems based on Gaussian white noise models have been studied by Wonham [1]-[3], Haussmannn [4] and Ahmed [5], among many others. Rishel [6][7] has studied the control problem for systems with jump Markov disturbances. In this paper we investigate the linear quadratic regulator control problem with state- and control-dependent Poisson noises in the state dynamics. As in Wonham's treatment of LQG problems [2] [3], the optimal control law is a combination of a linear feedback and a feed-forward control, which is similar to those for LQG problems. The optimal cost and optimal control are computed by solving a class of Riccati-like equations. When the coefficient matrices in the system dynamics and the performance index are constant, we consider the infinite time problem with average cost criterion. The optimal control exists and is also the combination of a constant linear feedback and a constant feed-forward control obtained by solving an algebraic Riccati-like equation. In addition, the optimal control of the finite time problem converges to the time-invariant control of the infinite time problem quasi-uniformly, almost surely. Similar results follow if we use a discounted cost criterion.

The optimal stochastic scheduling of systems with Poisson noise disturbances is treated in [8]. Almost sure stochastic stability of linear systems with Poisson noise disturbances is treated in [9]. See also [10].

1. Finite Time Control Problems.

In this section, we consider the stochastic control system

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + C(t)x(t)dN_1(t) + D(t)u(t)dN_2(t) + \xi(t)dN_3(t) \tag{1.1}$$
$$x(0) = x_0 \in \mathbb{R}^n \setminus \{0\}$$

where $A(t)$ and $C(t)$ are $n \times n$ matrices; $B(t)$ and $D(t)$ are $n \times m$ matrices; and $\xi(t)$ is an n-vector. For simplicity, we assume $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are piecewise continuous on $[0,T]$. $N_i(t)$, i=1, 2, 3, are independent Poisson processes with intensities λ_i, $i=1,2,3$, respectively, together with the performance index

$$J(u) = E\ \{\int_0^T L(t,x(t),u(t))dt\ +\ l(T,x(T))\} \tag{1.2}$$

Let the admissible control set be

$$U_{ad} \triangleq \{u(t) = \phi(t,x(t))\ |\ \phi: [0,T]\times \mathbb{R}^n \rightarrow \mathbb{R}^n \textit{ is piecewise continuous in } t \textit{ for each } x \in \mathbb{R}^n; \textit{ locally Lipschitz continuous and at most linear growth in } x \text{ for } \textit{each } t \in [0,T]\}$$

We want to minimize $J(u)$ with respect to all controls $u \in U_{ad}$.

Lemma 1.1 (Sufficient condition for optimality) *Suppose there exists a control* $u^0 \in U_{ad}$ *and a continuous functional* $V{:}[0,T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ *with continuous partial derivatives with respect to t and x such that*

$$V_t(t,x) + \min_{u \in \mathbb{R}^m} \{\mathbf{L}_u V(t,x) + L(t,x,u)\} = 0 \tag{1.3}$$

is achieved at $u^0(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$ *with*

$$V(T,x) = l(T,x),\ \forall\ x \in \mathbb{R}^n. \tag{1.4}$$

Here $\mathbf{L}_u$ *is the differential operator corresponding to the state process defined by*

$$\mathbf{L}_u V(t,x) = V_x{}'(A(t)x+B(t)u) + \lambda_1[V(t,x+C(t)x)-V(t,x)] + \lambda_2[V(t,x+D(t)u)-V(t,x)] + \lambda_3[V(t,x+\xi(t))-V(t,x)] \tag{1.5}$$

where $'$ *denotes matrix transpose. Then*

$$V(0,x_0) = J(u^0) = \inf_{u \in U_{ad}} J(u).$$

Proof. Let $x^0(s)$ and $x(s)$ be the trajectories corresponding to control laws ϕ^0 and ϕ, respectively, with initial state $x^0(t) = x = x(t)$. From the integration formula for point processes, we have

$$V(t,x) = -E_{t,x}\{\int_t^T [V_s(s,x^0(s)) + \mathbf{L}_{\phi^0} V(s,x^0(s))]ds + V(T,x^0(T))\}$$

$$= E_{t,x}\{\int_t^T L(s,x^0(s),\phi^0(s,x^0(s)))ds + l(T,x^0(T))\}.$$

But with arbitrary $u(t) = \phi(t,x(t))$, we have from (1.3)

$$V(t,x) = -E_{t,x}\{\int_t^T [V_s(s,x(s)) + \mathbf{L}_{\phi} V(s,x(s))]ds + V(T,x(T))\}$$

$$\leq E_{t,x}\{\int_t^T L(s,x(s),\phi(s,x(s)))ds + l(T,x(T))\}.$$

Thus, $V(0,x_0) = J(u^0) \leq J(u)$ for all $u \in U_{ad}$ which shows that u^0 is an optimal control and $V(0,x_0)$ is an optimal value.

QED

Remark. Equation (1.3) together with (1.4) make up the Hamilton-Jacobi-Bellman (HJB) dynamic programming equation for this problem.

Now, we consider our system as a regulator, i.e., we let

$$L(t,x,u) = x'Q(t)x + u'R(t)u \tag{1.6}$$

$$l(T,x) = x'P_T x$$

with symmetric piecewise continuous matrices

$$Q(t) \geq 0, \quad R(t) > 0, \quad t \in [0,T]$$

and

$$P_T \geq 0 \text{ constant.}$$

To solve the (HJB) equation, we use

$$V(t,x) = x'P(t)x + 2p(t)'x + q(t) \tag{1.7}$$

for some $n \times n$ symmetric, non-negative definite matrix $P(t)$, n-vector $p(t)$ and scalar function $q(t)$ satisfying the final conditions

$$P(T)=P_T, \quad p(T)=0, \quad q(T)=0. \tag{1.8}$$

Then

$$V_t(t,x) = x'\frac{d}{dt}P(t)x + 2\frac{d}{dt}p(t)'x + \frac{d}{dt}q(t)$$
$$V_x(t,x) = 2P(t)x + 2p(t). \tag{1.9}$$

Substituting (1.7) and (1.9) into (1.5), we get

$$\begin{aligned}
\mathbf{L}_u V(t,x) &= 2[P(t)x+p(t)]'[A(t)x+B(t)u] \\
&+ \lambda_1\{[x+C(t)x]'P(t)[x+C(t)x] \\
&+ 2p(t)'[x+C(t)x]+q(t) - [x'P(t)x+2p(t)'x+q(t)]\} \\
&+ \lambda_2\{[x+D(t)u]'P(t)[x+D(t)u]+2p(t)'[x+D(t)u]+q(t) \\
&- [x'P(t)x+2p(t)'x+q(t)]\} + \lambda_3\{[x+\xi(t)]'P(t)[x+\xi(t)] \\
&+ 2p(t)'[x+\xi(t)]+q(t) - [x'P(t)x+2p(t)'x+q(t)]\} \\
&= 2[P(t)x+p(t)]'[A(t)x+B(t)u] \\
&+ \lambda_1[x'C(t)'P(t)x+x'P(t)C(t)x \\
&+x'C(t)'P(t)C(t)x+2p(t)'C(t)x] \\
&+ \lambda_2[u'D(t)'P(t)x+x'P(t)D(t)u \\
&+u'D(t)'P(t)D(t)u+2p(t)'D(t)u] \\
&+ \lambda_3[\xi(t)'P(t)x+x'P(t)\xi(t)+\xi(t)'P(t)\xi(t)+2p(t)'\xi(t)].
\end{aligned} \tag{1.10}$$

Let the Hamiltonian functional be

$$H(t,x,u) \triangleq \mathbf{L}_u V(t,x) + L(t,x,u). \tag{1.11}$$

Then set

$$0 = \frac{\partial H}{\partial u} = 2\lambda_2[D(t)'P(t)x+D(t)'P(t)D(t)u+D(t)'p(t)]$$
$$+ 2B(t)'[P(t)x+p(t)]+2R(t)u$$

which implies

$$u(t) = -[R(t)+\lambda_2 D(t)'P(t)D(t)]^{-1}[B(t)+\lambda_2 D(t)]'[P(t)x+p(t)]. \tag{1.12}$$

In addition,

$$\frac{\partial^2 H}{\partial u^2} = 2[R(t) + \lambda_2 D(t)' P(t) D(t)] > 0, \ \forall \ t \in [0,T].$$

Thus, $u(t)$, defined in (1.12), minimizes (1.11). Now, set

$$\begin{aligned}
\hat{A}(t) &= A(t) + \lambda_1 C(t) \\
\hat{B}(t) &= B(t) + \lambda_2 D(t) \\
\hat{R}(t) &= R(t) + \lambda_2 D(t)' P(t) D(t) \\
\hat{K}(t) &= \hat{R}(t)^{-1} \hat{B}(t)' P(t) \\
\hat{k}(t) &= \hat{R}(t)^{-1} \hat{B}(t)' p(t).
\end{aligned} \tag{1.13}$$

Substituting (1.10) and (1.12) into (1.3), we obtain

$$\begin{aligned}
& x' \frac{d}{dt}P(t)x + 2\frac{d}{dt}p(t)'x + \frac{d}{dt}q(t) + 2[P(t)x + p(t)]'[A(t)x - B(t)\hat{K}(t)x - B(t)\hat{k}(t)] \\
& + \lambda_1\{x'C(t)'P(t)x + x'P(t)C(t)x + x'C(t)'P(t)C(t)x + 2p(t)'C(t)x\} \\
& + \lambda_2\{-[\hat{K}(t)x + \hat{k}(t)]'D(t)'P(t)x - x'P(t)D(t)[\hat{K}(t)x + \hat{k}(t)] \\
& + [\hat{K}(t)x + \hat{k}(t)]'D(t)'P(t)D(t)[\hat{K}(t)x + \hat{k}(t)] - 2p(t)'D(t)[\hat{K}(t)x + \hat{k}(t)]\} \\
& + \lambda_3\{\xi(t)'P(t)x + x'P(t)\xi(t) + \xi(t)'P(t)\xi(t) + 2p(t)'\xi(t)\} \\
& + x'Q(t)x + [\hat{K}(t)x + \hat{k}(t)]'R(t)[\hat{K}(t)x + \hat{k}(t)] = 0.
\end{aligned}$$

After simplifying, we obtain

$$\begin{aligned}
& x'\{\frac{d}{dt}P(t) + [\hat{A}(t) - \hat{B}(t)\hat{K}(t)]'P(t) + P(t)[\hat{A}(t) - \hat{B}(t)\hat{K}(t)] \\
& + \lambda_1 C(t)'P(t)C(t) + \hat{K}(t)'\hat{R}(t)\hat{K}(t) + Q(t)\}x \\
& + 2x'\{\frac{d}{dt}p(t) + [\hat{A}(t) - \hat{B}(t)\hat{K}(t)]'p(t) + \lambda_3 P(t)\xi(t)\} \\
& + \{\frac{d}{dt}q(t) - \hat{k}(t)'\hat{R}(t)\hat{k}(t) + \lambda_3\xi(t)'P(t)\xi(t) + 2\lambda_3 p(t)'\xi(t)\} = 0.
\end{aligned}$$

As we vary $x \in \mathbb{R}^n$, we obtain the following equations

$$\begin{cases} \frac{d}{dt}P(t)+[\hat{A}(t)-\hat{B}(t)\hat{K}(t)]'P(t)+P(t)[\hat{A}(t)-\hat{B}(t)\hat{K}(t)]+\lambda_1 C(t)'P(t)C(t) \\ \quad +Q(t)+\hat{K}(t)'\hat{R}(t)\hat{K}(t)=0 \\ P(T)=P_T \end{cases} \tag{1.14}$$

$$\begin{cases} \frac{d}{dt}p(t)+[\hat{A}(t)-\hat{B}(t)\hat{K}(t)]'p(t)+\lambda_3 P(t)\xi(t)=0 \\ p(T)=0 \end{cases} \tag{1.15}$$

$$\begin{cases} \frac{d}{dt}q(t)-\hat{k}(t)'\hat{R}(t)\hat{k}(t)+2\lambda_3\xi(t)'p(t)+\lambda_3\xi(t)'P(t)\xi(t)=0 \\ q(T)=0. \end{cases} \tag{1.16}$$

Since $\hat{R}$(t) and $\hat{K}$(t) only involve $P(t)$, (1.15) and (1.16) are easily solved once we solve (1.14). Equation (1.14) is well-known to have positive solutions if $D(t)\equiv 0$. To treat our case, we use the methods of quasi-linearization and successive approximation as in Wonham [2] to show existence and uniqueness of the solution of (1.14). Note that we always have a minimum property

$$\begin{aligned} &[\hat{A}(t)-\hat{B}(t)\hat{K}(t)]'P(t)+P(t)[\hat{A}(t)-\hat{B}(t)\hat{K}(t)]+\hat{K}(t)'\hat{R}(t)\hat{K}(t) \\ &\quad = [\hat{A}(t)-\hat{B}(t)K(t)]'P(t) + P(t)[\hat{A}(t)-\hat{B}(t)K(t)] \\ &\quad + K(t)'\hat{R}(t)K(t) - [K(t)-\hat{K}(t)]'\hat{R}(t)[K(t)-\hat{K}(t)] \end{aligned} \tag{1.17}$$

for any $m\times n$ matrix $K(t)$. Let

$$\begin{aligned} \Psi(P(t),R(t),K(t)) \triangleq\ &[\hat{A}(t)-\hat{B}(t)K(t)]'P(t) + P(t)[\hat{A}(t)-\hat{B}(t)K(t)] \\ &+ \lambda_1 C(t)'P(t)C(t) + K(t)'R(t)K(t). \end{aligned} \tag{1.18}$$

Then (1.14) becomes

$$\begin{cases} \frac{d}{dt}P(t) = -\Psi(P(t),\hat{R}(t),\hat{K}(t)) - Q(t) \\ P(T)=P_T. \end{cases} \tag{1.19}$$

For each $K(t)$, let $\Phi(t,s;K)$ be the transition matrix of $\hat{A}(t)-\hat{B}(t)K(t)$. To show that (1.19) has a solution, we choose $K_1(t)$ arbitrary and let

$$P^1(t) \triangleq \Phi(T,t;K_1)' P_T \Phi(T,t;K_1), \qquad t \in [0,T],$$

$$\begin{aligned} P^{k+1}(t) \triangleq \; & \Phi(T,t;K_1)' P_T \Phi(T,t;K_1) \\ & + \int_t^T \Phi(s,t;K_1)' [\lambda_1 C(s)' P^k(s) C(s) \\ & + K_1(s)' \hat{R}^k(s) K_1(s) + Q(s)] \, \Phi(s,t;K_1) \, ds \end{aligned} \tag{1.20}$$

where

$$\hat{R}^k(t) \triangleq R(t) + \lambda_2 D(t)' P^k(t) D(t).$$

Since all the functions involved are bounded on $[0,T]$, the sequence in (1.20) is easily seen to have a limit

$$P_1(t) \triangleq \lim_{k \to \infty} P^k(t).$$

Since $P_T \geq 0$ and is symmetric, from (1.20) we know $P^k(t) \geq 0$ and is symmetric for all $k = 1$. Thus, $P_1(t) \geq 0$ and is symmetric. Now, let

$$K_2(t) \triangleq \hat{R}_1(t)^{-1} \hat{B}(t)' P_1(t)$$

and

$$\hat{R}_1(t) \triangleq \lim_{k \to \infty} \hat{R}^k(t) = R(t) + \lambda_2 D(t)' P_1(t) D(t) > 0.$$

Using K_2(t), we can use an iteration similar to (1.20) to obtain P_2 and $\hat{R}_2$. Continuing this process, we have P_k, $\hat{R}_k$ and K_k such that

$$\begin{aligned} \hat{R}_k(t) &= R(t) + \lambda_2 D(t)' P_k(t) D(t) \\ K_{k+1}(t) &= \hat{R}_k(t)^{-1} \hat{B}(t)' P_k(t) \end{aligned} \tag{1.21}$$

and

$$\begin{aligned} & P_k(t) \geq 0, \quad \forall \; t \in [0,T] \\ & \frac{d}{dt} P_k(t) + \Psi(P_k(t), \hat{R}_k(t), K_k(t)) + Q(t) = 0 \\ & P_k(T) = P_T. \end{aligned} \tag{1.22}$$

From the minimum property (1.17), we have

$$\frac{d}{dt}P_k(t)+\Psi(P_k(t),\hat{R}_k(t),K_{k+1}(t))+Q(t)$$

$$+[K_k(t)-K_{k+1}(t)]'\hat{R}_k(t)[K_k(t)-K_{k+1}(t)]$$

$$=\frac{d}{dt}P_k(t) + \Psi(P_k(t),\hat{R}_k(t),K_k(t)) + Q(t)$$

$$=0$$

$$=\frac{d}{dt}P_{k+1}(t) + \Psi(P_{k+1}(t),\hat{R}_{k+1}(t),K_{k+1}(t)) + Q(t).$$

Let $S(t) = P_k(\mathrm{t}) - P_{k+1}(\mathrm{t})$, then

$$\frac{d}{dt}S(t)+\Psi(S(t),\hat{R}_k(t)-\hat{R}_{k+1}(t),K_{k+1}(t))$$

$$+[K_k(t)-K_{k+1}(t)]'\hat{R}_k(t)[K_k(t)-K_{k+1}(t)]=0$$

$$S(T)=0. \tag{1.23}$$

Since the last term in (1.23) is symmetric and ≥ 0, we know that $S(t) \geq 0$ as in (1.20). Thus, $P_k(t) \geq P_{k+1}(t) \geq 0$. By the monotone convergence theorem,

$$\hat{P}(t) \triangleq \lim_{k\to\infty} P_k(t)$$

exists and is symmetric so that $P_1(t) \geq \hat{P}(t) \geq 0$. Again

$$\hat{R}(t) \triangleq \lim_{k\to\infty} [R(t) + \lambda_2 D(t)' P_k(t) D(t)]$$

$$= R(t) + \lambda_2 D(t)'\hat{P}(t)D(t) > 0, \quad \forall\ t \in [0,T].$$

Thus $\hat{R}(t)^{-1}$ exists and

$$\hat{K}(t) \triangleq \lim_{k\to\infty} K_{k+1}(t) = \hat{R}(t)^{-1}\hat{B}(t)'\hat{P}(t).$$

If we let $\hat{\Phi}(t,s)$ be the transition matrix of $\hat{A}(t)$, then (1.22) implies

$$P_k(t) = \hat{\Phi}(T,t)'P_T\hat{\Phi}(T,t) + \int_t^T \hat{\Phi}(s,t)' \ [\lambda_1 C(s)'P_k(s)C(s)+Q(s)$$

$$-K_k(s)'\hat{B}(s)'P_k(s)-P_k(s)\hat{B}(s)K_k(s)$$

$$+K_k(s)'\hat{R}_k(s)K_k(s)]\ \hat{\Phi}(s,t)ds.$$

Since

$$||P_k|| \leq ||P_1||, \quad ||\hat{R}_k|| \leq ||R|| + \lambda_2 ||D||^2 ||P_1||$$

and

$$||K_k|| \leq ||R^{-1}|| \, ||\hat{B}|| \, ||P_1||,$$

then we apply the Dominated Convergence Theorem to get

$$\hat{P}(t) = \hat{\Phi}(T,t)' P_T \hat{\Phi}(T,t) + \int_t^T \hat{\Phi}(s,t)' \ [\lambda_1 C(s)' \hat{P}(s) C(s) + Q(s)$$

$$-\hat{K}(s)' \hat{B}(s)' \hat{P}(s) - \hat{P}(s) \hat{B}(s) \hat{K}(s) + \hat{K}(s)' \hat{R}(s) \hat{K}(s)] \ \hat{\Phi}(s,t) ds$$

after k tends to ∞. Thus, $\hat{P}$(t) is a solution of (1.19). Since $\Psi(\hat{P}, \hat{R}, \hat{K})$ is locally Lipschitz in $\hat{P}$, $\hat{P}$(t) is a unique solution of (1.19). In addition, let $K(t)$ be arbitrary and $P(t)$ be the solution of (1.19) with $K(t)$ replacing $\hat{K}$(t) and $P(T) = P_T$. Then by the minimum property (1.17), we have

$$0 = \frac{d}{dt} P(t) + \Psi(P(t), \hat{R}_P(t), K(t)) + Q(t)$$

$$= \dot{\hat{P}}(t) + \Psi(\hat{P}(t), \hat{R}(t), \hat{K}(t)) + Q(t)$$

$$= \dot{\hat{P}}(t) + \Psi(\hat{P}(t), \hat{R}(t), K(t)) - [K(t) - \hat{K}(t)]' \hat{R}(t) [K(t) - \hat{K}(t)] + Q(t),$$

where

$$\hat{R}_P(t) = R(t) + \lambda_2 D(t)' P(t) D(t).$$

Let $S(t) = P(t) - \hat{P}(t)$. We obtain

$$\frac{d}{dt} S(t) + \Psi(S(t), \hat{R}_P(t) - \hat{R}(t), K(t)) + [K(t) - \hat{K}(t)]' \hat{R}(t) [K(t) - \hat{K}(t)] = 0$$

$$S(T) = 0. \tag{1.24}$$

Since the last term in (1.24) is non-negative definite, we have $S(t) \geq 0$, so that $P(t) \geq \hat{P}(t)$. The solutions of (1.15) and (1.16), respectively, are thus easily shown to be

$$\hat{p}(t) = \lambda_3 \int_t^T \Phi(s,t;\hat{K})' \hat{P}(s)\xi(s)ds \quad \textbf{(1.25)}$$

and

$$\hat{q}(t) = \int_t^T [2\lambda_3 \xi(s)'\hat{p}(s) + \lambda_3 \xi(s)'\hat{P}(s)\xi(s) - \hat{k}(s)'\hat{R}(s)\hat{k}(s)]ds \quad (1.26)$$

with

$$\hat{k}(t) \triangleq \hat{R}(t)^{-1}\hat{B}(t)'\hat{p}(t). \quad (1.27)$$

Consequently, we have proved the following theorem.

Theorem 1.1. *The stochastic linear quadratic regulator* (1.1) *with* (1.2) *and* (1.6) *has an optimal control*

$$\hat{u}(t) = -[\hat{K}(t)\hat{x}(t) + \hat{k}(t)] \quad (1.28)$$

where

$$\hat{K}(t) = [R(t)+\lambda_2 D(t)'\hat{P}(t)D(t)]^{-1}[B(t)+\lambda_2 D(t)]\hat{P}(t) \quad (1.29)$$
$$\hat{k}(t) = [R(t)+\lambda_2 D(t)'\hat{P}(t)D(t)]^{-1}[B(t)+\lambda_2 D(t)]\hat{p}(t).$$

The optimal value is $J(\hat{u}) = V(0,x_0) = x_0'\hat{P}(0)x_0 + 2\hat{p}(0)'x_0 + q(0)$, *with* $\hat{P}(t) \geq 0$, $\hat{p}(t)$ *and* $\hat{q}(t)$ *being the unique solutions of* (1.14), (1.15) *and* (1.16), *respectively. And* $\hat{x}(t)$ *is the optimal trajectory of* (1.1) *corresponding to* $\hat{u}$ *as input control.*

2. Infinite Time Stochastic Control.

To investigate the infinite time problem (1.1) (1.2) as $T \rightarrow \infty$, we consider the case when the coefficient matrices in (1.1) and (1.6) are constant, i.e., $A(t) \equiv A$, $B(t) \equiv B$, $C(t) \equiv C$, $D(t) \equiv D$, $\xi(t) \equiv \xi$, $Q(t) \equiv Q$, $R(t) \equiv R$ and let $P_T \equiv 0$ for convenience. We would like to have a stationary feedback control, i.e., $\hat{P}(t) \equiv \hat{P}$, $\hat{p}(t) \equiv \hat{p}$. But $\hat{q}(0) \rightarrow \infty$ as $T \rightarrow \infty$ in (1.26); and so the cost $V(0,x_0) \rightarrow \infty$ as $T \rightarrow \infty$. This is to be expected since the noise dN_3 acts continuously on the system, so we modify the performance index (1.2) to be the average cost

$$J_{av}(u) \triangleq \lim_{T \to \infty} \frac{1}{T} E \int_0^T [x(t)' Qx(t) + u(t)' Ru(t)] dt \tag{2.1}$$

or the discounted cost

$$J_d(u) \triangleq E \int_0^\infty e^{-2\alpha t} [x(t)' Qx(t) + u(t)' Ru(t)] dt, \quad \alpha > 0 \tag{2.2}$$

We will discuss both. See [11] for a general discussion of this class of problems.

Since we want $\hat{P}_T(t) \longrightarrow \hat{P}$, a constant matrix, as $T \to \infty$ for each t, then $\hat{P}$ should be a solution of algebraic equation

$$\Psi(\hat{P}, \hat{R}, \hat{K}) + Q = 0 \tag{2.3}$$

Before investigating solutions of (2.3), we need some preliminary lemmas which are adapted from Wonham [2].

Lemma 2.1. *Let $G'G + H'H = F'F$ and M be an arbitrary matrix of compatible dimensions.*

(i) *If (G,A) is observable, then $(F,A+MH)$ is observable.*

(ii) *If (G,A) is detectable, then $(F,A+MH)$ is detectable.*

Proof. (i) Denote

$$<A \mid B> \triangleq \{B, AB, \ldots, A^{n-1}B\}.$$

If (G,A) is observable, then the range $R(<A' | G'>) = \mathbb{R}^n$. Since $x'F'Fx = 0$ implies $x'G'Gx = 0$ and $x'H'Hx = 0$, then the null space $N(F) \subset N(G) \cap N(H)$. We have

$$\begin{aligned} R(F') = N(F)^\perp &\supset [N(G) \cap N(H)]^\perp \\ &\supset N(G)^\perp \oplus N(H)^\perp = R(G') \oplus R(H'). \end{aligned} \tag{2.4}$$

Thus, $R(H'M') \subset R(H') \subset R(F')$. Since

$$R(<A'+F_1'|F'> = R(<A'|F'>)$$

for any F_1' such that range $R(F_1') \subset R(F')$, then

$$R(<A'+H'M'|F'>) = R(<A'|F'>) \supset R(<A'|G'>) = \mathbb{R}^n,$$

so that $R(<A'+H'M'|F'>) = \mathbb{R}^n$ and $(F, A+MH)$ is observable.

(ii) If (G,A) is detectable, then the unstable part of A' is controllable. From (2.4), we know $R(G'V'-H'M') \subset R(F')$ for any V'. Thus, there exists a matrix U' such that $G'V'-H'M' = F'U'$, so that

$$A' + G'V' = A' + H'M' + F'U'.$$

This shows that if we can find V' to reposition the unstable eigenvalues of A' to any desired locations, we can find U' which does the job for $A'+H'M'$. Hence, $(F, A+MH)$ is detectable.

QED

Lemma 2.2. *If (G,A) is detectable, then either A is stable or*

$$W(t;A,G) \triangleq \int_0^t e^{(A's)}G'G\,e^{(As)}\,ds \tag{2.5}$$

is unbounded as $t \to \infty$.

Proof. If A is not stable, then there exists an eigenvalue λ of A such that $\mathrm{Re}\lambda \geq 0$ and an eigenvector $\eta \neq 0$. Thus,

$$\eta^* W(t;A,G)\eta = \int_0^t e^{2s\,\mathrm{Re}\lambda}\,|\,G\eta\,|^2 ds, \tag{2.6}$$

where $*$ denotes transpose and complex conjugate. If (2.5) is bounded as $t \to \infty$, from (2.6), we must have $G\eta = 0$, so that $GA^k\eta = \lambda^k G\eta = 0$ for any $k \geq 0$. Thus,

$$\mathrm{Re}\,\eta \text{ and } \mathrm{Im}\,\eta \in N(<A',G'>')$$

Let E_A^+ and E_A^- be the generalized eigenspace of A corresponding to non-negative and negative eigenvalues, respectively. Then $E_A^+ \oplus E_A^- = \mathbb{R}^n$. Since (G, A) is detectable, $E_A^+ \subset R(<A'|G'>)$. Thus,

$$N(<A'|G'>') = R(<A'|G'>)^{\perp} \subset (E_A^+)^{\perp} = E_A^-.$$

Hence, Re η and Im $\eta \in E_A^+ \bigcap E_A^- = \{0\}$ which implies $\eta \equiv 0$, a contradiction.

QED

Lemma 2.3. *Let (G,A) be detectable and suppose*

$$A'P + PA + l(P) + G'G = 0 \tag{2.7}$$

has a solution $P \geq 0$ with $l(P)$ being linear in P. Then A is stable. Let Λ be formally defined as

$$\Lambda(S) \triangleq \int_0^{\infty} exp(A't) l(S) exp(At) dt \tag{2.8}$$

with $\Lambda^k(S) \triangleq \Lambda(\Lambda^{k-1}(S))$, $\Lambda^0(S) \triangleq$ S. *If (G,A) is observable, then*

$$\sum_{k=0}^{\infty} \Lambda^k(S) = (I - \Lambda)^{-1}(S) \tag{2.9}$$

for any $n \times n$ *symmetric matrix S, and P is the unique solution of* (2.7) *of the form*

$$P = (I - \Lambda)^{-1} \left[\int_0^{\infty} e^{(A't)} G' G e^{(At)} dt \right]$$

Proof. Since P is constant, $\frac{dP}{dt} \equiv 0$, then P is the solution of

$$\frac{d}{dt} S(t) + A'S(t) + S(t)A + l(S) + G'G = 0$$

$$S(T) = P.$$

Thus,

$$P = e^{(A'T)} P e^{(AT)} + \int_0^T e^{(A't)} [l(P) + G'G] e^{(At)} \, dt. \qquad \textbf{(2.10)}$$

Let

$$Z_t \triangleq \int_0^t e^{(A'\tau)} G'G \, e^{(A\tau)} \, d\tau.$$

Then (2.10) shows $P - Z_T \geq 0$ since $P \geq 0$. Thus, Z_T is bounded in T, $\forall\ T \geq 0$. Since (G, A) is detectable, by Lemma 2.2, we know A must be stable. From (2.7), P has the form

$$P = \Lambda(P) + Z_\infty \qquad (2.11)$$

$$= \Lambda^{k+1}(P) + \sum_{i=0}^{k} \Lambda^i (Z_\infty), \quad \forall\ k \geq 0. \qquad (2.12)$$

Since $Z_\infty \geq 0$ and Λ is linear, the series exists and

$$\sum_{k=0}^{\infty} \Lambda^k (Z_\infty) \leq P.$$

Suppose (G, A) is observable, then $Z_\infty \geq Z_t > 0$, $\forall\ t > 0$. If $S \geq 0$ and symmetric, then $S \leq \alpha Z_\infty$ for some $\alpha > 0$. Thus,

$$0 \leq \sum_{k=0}^{\infty} \Lambda^k (S) \leq \alpha \sum_{k=0}^{\infty} \Lambda^k (Z_\infty) \leq \alpha P$$

which shows that the series in (2.9) converges. If S is symmetric, but not non-negative definite, then $\exists$ symmetric matrices $S_1 \geq 0$ and $S_2 \geq 0$ such that $S = S_1 - S_2$. Hence, $\sum_{k=0}^{\infty} \Lambda^k (S) = \sum_{k=0}^{\infty} \Lambda^k (S_1) - \sum_{k=0}^{\infty} \Lambda^k (S_2)$ converges since both series converge. Thus (2.9) is established. In particular, $\Lambda^k (S) \to 0$ as k $\to \infty$. From (2.11), $P = (I - \Lambda)^{-1} (Z_\infty)$.

QED

Lemma 2.4 (Minimum principle). *Let $\hat{P} \geq 0$ satisfy $\Psi(\hat{P}, \hat{R}, \hat{K}) + Q = 0$ with $\hat{R} = R + \lambda_2 D'PD$ and $\hat{K} = \hat{R}^{-1} \hat{B}' \hat{P}$. Suppose $S \geq 0$ satisfies*

$\Psi(S, R_S, K) + Q = 0$ *for some matrix* K *with* $R_S \triangleq R + \lambda_2 D' SD$. *Let* $Q = G'G$. *If* $(G, \hat{A})$ *is detectable, then* $(\hat{A} - \hat{B}K)$ *is stable. In addition, if* $(G, \hat{A})$ *is observable, then* $\hat{P} \leq S$.

Proof. Let $G'G + K'R_SK = F'F$ with $H = R_S^{1/2}K$ and $M = -\hat{B}'R_S^{-1/2}$ for $R_S > 0$. By Lemma 2.1, $(F, \hat{A} - \hat{B}K)$ is detectable. From Lemma 2.3 with $l(P) = \lambda_1 C'PC$, we know $(\hat{A} - \hat{B}K)$ is stable. By the minimum property (1.17), we have

$$
\begin{aligned}
0 &= \Psi(S, R_S, K) + Q \\
&= \Psi(\hat{P}, \hat{R}, \hat{K}) + Q \\
&= \Psi(\hat{P}, \hat{R}, K) + Q - (K - \hat{K})'\hat{R}(K - \hat{K}).
\end{aligned}
$$

Let $V = S - \hat{P}$, we have

$$\Psi(V, R_S - \hat{R}, K) + (K - \hat{K})'\hat{R}(K - \hat{K}) = 0 \tag{2.13}$$

and $R_S - \hat{R} = \lambda_2 D'VD$. Consider a linear function

$$l(V) = \lambda_1 C'VC + \lambda_2 K'D'VDK \tag{2.14}$$

and Λ as defined in (2.8) with $\hat{A} - \hat{B}K$ replacing A. If $(G, \hat{A})$ is observable, then $(F, \hat{A} - \hat{B}K)$ is observable and the corresponding series in (2.9) converges. In particular, $\Lambda^k(V) \to 0$ as $k \to \infty$. From (2.13),

$$V = \Lambda(V) + Z$$

where

$$Z \triangleq \int_0^\infty e^{(\hat{A} - \hat{B}K)'t}(K - \hat{K})'\hat{R}(K - \hat{K})e^{(\hat{A} - \hat{B}K)t}\, dt.$$

Thus,

$$
\begin{aligned}
V &= \Lambda^{k+1}(V) + \sum_{i=0}^{k} \Lambda^i(Z) \\
&= \sum_{i=0}^{\infty} \Lambda^i(Z) \geq 0 \quad as \quad k \to \infty.
\end{aligned}
$$

Hence, $\hat{P} \leq S$.

QED

Now, we show that under certain conditions, (2.3) has a solution $\hat{P} \geq 0$. First, suppose $(\hat{A}, \hat{B})$ is stabilizable. Then $\exists\ K$ such that $(\hat{A} - \hat{B}K)$ is stable; and

$$\Psi(P, R_P, K) + Q = 0$$

is equivalent to

$$P = \int_0^\infty e^{(\hat{A} - \hat{B}K)'t}[\lambda_2 C' PC + Q + K' R_P K] e x^{(\hat{A} - \hat{B}K)t}\ dt.$$

$$\triangleq f(P, R_P, K). \tag{2.15}$$

To make f a contraction for each K, we have to impose a condition on l defined in (2.14).

Condition (I):

$$inf \left\{ \left|\left| \int_0^\infty e^{(\hat{A} - \hat{B}K)'t}\, l(I)\, e^{(\hat{A} - \hat{B}K)t}\ dt \right|\right| \text{ such that } (\hat{A} - \hat{B}K) \text{ is stable} \right\} < 1.$$

Since

$$-||P||l(I) \leq l(P) \leq ||P||l(I),$$

condition (I) implies for some K_1 and $\theta < 1$ such that

$$\left|\left| \int_0^\infty e^{(\hat{A} - \hat{B}K_1)'t}\, l(P)\, e^{(\hat{A} - \hat{B}K_1)t}\ dt \right|\right| \leq \theta ||P||, \ \forall\ P.$$

Thus,

$$||f(P_1, R_{P_1}, K_1) - f(P_2, R_{P_2}, K_1)|| \leq \theta||P_1 - P_1||.$$

By successive iteration

$$P^1 = 0$$

$$P^{k+1} = f\,(P^k, R^k, K_1),\ \ k \geq 1$$

with $R^k = R + \lambda_2 D' P^k D$, we have a fixed solution

$$P_1 = f\,(P_1, R_1, K_1)$$

with

$$R_1 = R + \lambda_2 D' P_1 D.$$

Since $P^k \geq 0$, we have $P_1 \geq 0$. Let

$$K_2 = R_1^{-1} \hat{B}' P_1.$$

By the minimum property (1.17), we have

$$0 = \Psi(P_1, R_1, K_1) + Q$$

$$= \Psi(P_1, R_1, K_2) + (K_1 - K_2)' R_1 (K_1 - K_2) + Q. \tag{2.16}$$

Let $G'G = Q$, $H = R_1^{1/2}(K_1 - K_2)$ and

$$F'F = G'G + H'H.$$

Suppose $(G, \hat{A})$ is detectable. By Lemma 2.1, $(F, \hat{A})$ is detectable. From Lemma 2.4, $(\hat{A} - \hat{B} K_2)$ is stable. Let

$$P_2^1 = 0$$

$$P_2^{k+1} = f\,(P_2^k, R_2^k, K_2)$$

with

$$R_2^k \triangleq R + \lambda_2 D' P_2^k D.$$

Since $P_2^2 \geq 0 = P_2^1$, and assuming $P_2^k \geq P_2^{k-1}$, we have

$$P_2^{k+1} - P_2^k = \int_0^\infty e^{(\hat{A} - \hat{B} K_2)' t} [\lambda_1 C' (P_2^k - P_2^{k-1}) C$$

$$+ \lambda_2 K_2' D' (P_2^k - P_2^{k-1}) D K_2]\, e^{(\hat{A} - \hat{B} K_2) t}\, dt \geq 0$$

by the induction hypothesis. Thus, $P_2^{k+1} \geq P_2^k \geq 0$, $\forall\ k \geq 1$. We want to show that $\{P_2^k\}$ is bounded. From (2.16),

$$P_1 = \int_0^\infty e^{(\hat{A} - \hat{B}K_2)'t} [\lambda_1 C' P_1 C + Q + K_2' R_1 K_2$$

$$+ (K_1 - K_2)' R_1 (K_1 - K_2)] e^{(\hat{A} - \hat{B}K_2)t} \, dt ,$$

so that

$$P_1 - P_2^{k+1} = \int_0^\infty e^{(\hat{A} - \hat{B}K_2)'t} [\lambda_1 C' (P_1 - P_2^k) C + \lambda_2 K_2' D' (P_1 - P_2^k) D K_2$$

$$+ (K_1 - K_2)' R_1 (K_1 - K_2)] e^{(\hat{A} - \hat{B}K_2)t} \, dt \geq 0$$

by induction, since $P_1 \geq 0 = P_2^1$. Thus,

$$P_2 \triangleq \lim_{k \to \infty} P_2^k$$

exists and $0 \leq P_2 \leq P_1$. Repeating the above procedure, we obtain sequences $\{P_k\}$, $\{R_k\}$ and $\{K_k\}$ such that

$$R_k = R + \lambda_2 D' P_k D ,$$

$$K_{k+1} = R_k^{-1} \hat{B}' P_k$$

and

$$0 \leq P_{k+1} \leq P_k \leq P_1, \ \forall \ k \geq 1.$$

Hence, $\hat{P} \triangleq \lim_{k \to \infty} P_k$ exists and P ≥ 0. Furthermore,

$$\hat{R} \triangleq \lim_{k \to \infty} R_k = R + \lambda_2 D' PD$$

$$\hat{K} \triangleq \lim_{k \to \infty} K_k = \hat{R}^{-1} \hat{B}' \hat{P} .$$

Since $\Psi(P_k, R_k, K_k) + Q = 0$, passing to the limit, we have $\Psi(\hat{P}, \hat{R}, \hat{K}) + Q = 0$. By Lemma 2.4, $(\hat{A} - \hat{B}\hat{K})$ is stable. If $(G, \hat{A})$ is observable, then $\hat{P}$ is the minimum non-negative solution of $\Psi(S, R_S, K) + Q = 0$, so $\hat{P}$ must be the unique solution of the class $S \geq 0$. To show $\hat{P} > 0$, we proceed as follows: If $\hat{P} = 0$, then $\hat{R} = R$ and $\hat{K}$

$= 0$, so then

$$0 = \hat{P} = \int_0^\infty e^{(\hat{A} - \hat{B}\hat{K})' t} [\lambda_1 C' \hat{P} C + Q + \hat{K}' \hat{R} \hat{K}] e^{(\hat{A} - \hat{B}\hat{K})t} \, dt,$$

$$= \int_0^\infty e^{(\hat{A}' t)} G' G e^{(\hat{A} t)} \, dt$$

Thus, $G e^{\hat{A} t} \equiv 0$ for all $t \geq 0$ which contradicts $(G, \hat{A})$ being observable. We summarize this result in the following theorem:

Theorem 2.5. *If* $(\hat{A}, \hat{B})$ *is stabilizable,* $(Q, \hat{A})$ *is detectable and condition* (I) *is satisfied, then* $\Psi(S, R_S, K_S) + Q = 0$ *has at least one solution* $\hat{P} \geq 0$. *The matrix* $(\hat{A} - \hat{B}\hat{R}^{-1}\hat{B}'\hat{P})$ *is stable with* $\hat{R} = R + \lambda_2 D' \hat{P} D$. *In addition, if* $(Q, \hat{A})$ *is observable, then* $\hat{P}$ *is unique among the solution set* $S \geq 0$ *and indeed* $\hat{P} > 0$.

Now, let $P_T(t)$ be the unique solution of

$$\begin{cases} \dfrac{d}{dt} P_T(t) + \Psi(P_T(t), R_T(t), K_T(t)) + Q = 0 \\ P_T(T) = 0 \end{cases} \tag{2.17}$$

with

$$R_T(t) = R + \lambda_2 D' P_T(t) D$$

$$K_T(t) = R_T(t)^{-1} \hat{B}' P_T(t).$$

Suppose the hypotheses in Theorem 2.5 are satisfied, then $\exists\, \hat{P} \geq 0$, a solution of the algebraic equation (2.3). From the minimum property (1.17),

$$\Psi(P_T(t), R_T(t), K_T(t)) + Q$$

$$= \Psi(P_T(t), R_T(t), \hat{K}) + Q - [\hat{K} - K_T(t)]' R_T(t) [\hat{K} - K_T(t)].$$

Let $S_T(t) = \hat{P} - P_T(t)$. Then

$$\frac{d}{dt} S_T(t) + \Psi(S_T(t), \hat{R} - R_T(t), \hat{K}(t)) + [\hat{K} - K_T(t)]' R_T(t) [\hat{K} - K_T(t)] = 0$$

$$S_T(T) = \hat{P}. \tag{2.18}$$

From the previous argument using quasi-linearization and successive approximations on (2.18), we know $S_T(t) \geq 0$, i.e., $P_T(t) \leq \hat{P}$ for all $0 \leq t \leq T$ which shows $P_T(t)$ is uniformly bounded $\forall\ T \geq 0$.

Lemma 2.6. *If P_1 and P_2 are solutions of*

$$\frac{d}{dt}P(t) + \Psi(P(t), R_P(t), K_P(t)) + Q = 0 \tag{2.19}$$

with terminal conditions P_1^0 and P_2^0 at $T > 0$, respectively. Suppose $0 \leq P_1^0 \leq P_2^0$. Then $0 \leq P_1(t) \leq P_2(t)$, $t \in [0, T]$.

Proof. By the minimum property (1.17),

$$\Psi(P_1(t), R_{P_1}(t), K_{P_1}(t)) = \Psi(P_1(t), R_{P_1}(t), K_{P_2}(t))$$
$$- [K_{P_2}(t) - K_{P_1}(t)]' R_{P_1}(t) [K_{P_2}(t) - K_{P_1}(t)].$$

Let $S(t) = P_2(t) - P_1(t)$. Then

$$\frac{d}{dt}S(t) + \Psi(S(t), R_{P_2}(t) - R_{P_1}(t), K_{P_2}(t))$$
$$+ [K_{P_2}(t) - K_{P_1}(t)]' R_{P_1}(t) [K_{P_2}(t) - K_{P_1}(t)]$$
$$S(T) = P_2^0 - P_1^0 \geq 0. \tag{2.20}$$

Since the last term in (2.20) is non-negative definite, we can show the solution $S(t) \geq 0$ as before by successive approximation. Thus $P_1(t) \leq P_2(t)$, $\forall\ 0 \leq t \leq T$.

QED

Suppose P_{T_1} and P_{T_2} are solution of (2.17) with $T_1 \leq T_2$. Since $P_{T_2}(T_1) \geq 0 = P_{T_1}(T_1)$, we have $P_{T_2}(t) \geq P_{T_1}(t)$, $t \in [0, T_1]$, by Lemma 2.6. Since $P_T(t) \leq \hat{P}$, then

$$P_\infty(t) \triangleq \lim_{T \to \infty} P_T(t)$$

exists pointwise and $0 \leq P_\infty(t) \leq \hat{P}$.

Suppose we can choose K so that $\hat{A} - \hat{B}K$ is stable. Thus, using the minimum property (1.17), the solution of (2.17) can be expressed as

$$P_T(t) = \int_t^T e^{(\hat{A} - \hat{B}K)'(\tau - t)}\{\lambda_1 C' P_T(\tau) C + Q + K' R_T(\tau) K$$

$$- [K - K_T(\tau)]' R_T(\tau) [K - K_T(\tau)]\} e^{(\hat{A} - \hat{B}K)(\tau - t)} \, d\tau. \qquad (2.21)$$

Since $P_T(t) \leq P_\infty(t) \leq \hat{P}$ and

$$\lim_{T \to \infty} R_T(t) = R + \lambda_2 D' P_\infty(t) D \triangleq R_\infty(t) > 0,$$

$$\lim_{T \to \infty} K_T(t) = R_\infty(t)^{-1} \hat{B}' P_\infty(t) \triangleq K_\infty(t)$$

are uniformly bounded, we can apply the Dominated Convergence Theorem to (2.21). Then as $T \to \infty$,

$$P_\infty(t) = \int_t^\infty e^{(\hat{A} - \hat{B}K)'(\tau - t)} \{\lambda_1 C' P_\infty(\tau) C + Q + K' R_\infty(\tau) K$$

$$- [K - K_\infty(\tau)]' R_\infty(\tau) [K - K_\infty(\tau)]\} e^{(\hat{A} - \hat{B}K)(\tau - t)} \, d\tau.$$

which shows that P_∞ satisfies (2.19).

Suppose P_T is the solution of (2.17). Set $\overline{P}_T(t) = P_T(T - t)$. Then $\overline{P}_T$ satisfies

$$\begin{cases} \dfrac{d}{dt} P(t) = \Psi(P, R_P, K_P) + Q \\ P(0) = 0 \end{cases} \qquad (2.22)$$

Since (2.22) has a unique solution, then $\overline{P}_{T_1}(t) = \overline{P}_{T_2}(t)$ or $P_{T_1}(T_1 - t) = P_{T_2}(T_2 - t)$. By Lemma 2.6, we have $P_T(t_1) \geq P_T(t_2)$ if $0 \leq t_1 \leq t_2 \leq T$. In addition,

$$P_\infty(t_1) = \lim_{T_1 \to \infty} P_{T_1}(t_1) = \lim_{T_1 \to \infty} \overline{P}_{T_1}(T_1 - t_1)$$

$$= \lim_{T_1 \to \infty} \overline{P}_{T_2}(T_2 - t_2) \quad with \quad T_2 - t_2 = T_1 - t_1$$

$$= \lim_{T_2 \to \infty} P_{T_2}(t_2) = P_\infty(t_2)$$

which shows that P_∞ is a constant matrix and is again a solution of the algebraic equation (2.3). Consequently, we have the following theorem.

Theorem 2.7. *Let $P_T(t)$ be the unique solution of* (2.17). *Then $P_T(t)$ is non-decreasing in T and non-increasing in t. If $(\hat{A}, \hat{B})$ is stabilizable, $(G, \hat{A})$ is detectable and condition* (I) *holds, then*

$$\lim_{T \to \infty} P_T(t) = P_\infty$$

Indeed, for any $\epsilon > 0$, $\exists\ d_0$ such that $||P_\infty - P_T(t)|| < \epsilon$ for all $t \in [0, T-d_0]$ and all $T \geq d_0$. P_∞ satisfies the algebraic equation

$$\Psi(P_\infty, R_\infty, K_\infty) + Q = 0 \tag{2.23}$$

with

$$R_\infty = R + \lambda_2 D' P_\infty D$$
$$K_\infty = R_\infty^{-1} \hat{B}' P_\infty \tag{2.24}$$

and $(\hat{A} - \hat{B} K_\infty)$ is stable. Furthermore, if $(G, \hat{A})$ is observable, then P_∞ is the unique solution of (2.23) in the solution class $S \geq 0$ and $P_\infty > 0$.

Proof. The only thing remaining to prove is the convergence of $P_T(t)$. Since $P_T(t) \uparrow P_\infty$ pointwise as $T \to \infty$, then for any $\epsilon > 0$, $\exists\ d_0 > 0$ such that $||P_T(0) - P_\infty|| < \epsilon$ for all $T \geq d_0$. By the invariance property of $P_T(t)$, $P_{T_1}(t_1) = P_{T_2}(t_2)$ if $T_1 - t_1 = T_2 - t_2$. Since $P_T(t)$ is non-increasing in t and $P_T(t) \leq P_\infty$, then

$$||P_T(t) - P_\infty|| < \epsilon \tag{2.25}$$

for all $t \in [0, T-d_0]$ and all $T \geq d_0$.

QED

Remark 2.1. Since B, D and R are constant matrices, we can show that for any $\epsilon > 0$, $\exists\ d_0 > 0$ such that

$$||R_T(t) - R_\infty|| < \epsilon$$

$$||K_T(t) - K_\infty|| < \epsilon \tag{2.26}$$

for all $t \in [0, T - d_0]$ and all $T \geq d_0$.

To prove the convergence of $P_T(t)$, we rewrite

$$\frac{d}{dt} p_T(t) + [\hat{A} - \hat{B} K_T(t)]' p_T(t) + \lambda_3 P_T(t)\xi = 0$$

or

$$\frac{d}{dt} p_T(t) + (\hat{A} - \hat{B} K_\infty)' p_T(t) + [K_\infty - K_T(t)]' \hat{B}' p_T(t) + \lambda_3 P_T(t)\xi = 0$$

so that with $p_T(T) = 0$,

$$p_T(t) = \int_t^T e^{(\hat{A} - \hat{B} K_\infty)'(\tau - t)} \{[K_\infty - K_T(\tau)]' \hat{B}' p_T(\tau) + \lambda_3 P_T(\tau)\xi\} d\tau. \tag{2.27}$$

Since $(\hat{A} - \hat{B} K_\infty)$ is stable, $\exists\ M, \beta > 0$ such that

$$||e^{(\hat{A} - \hat{B} K_\infty)'(t-s)}|| \leq M e^{-\beta(t-s)}, \quad all\ t \geq s.$$

From the convergent properties of (2.25) and (2.26), we know that $K_T(\mathrm{t})$ is uniformly bounded in t and T and $||P_T(t)|| \leq ||P_\infty||$. Thus $\exists\ M_1 > 0$ and $M_2 > 0$

$$|p_T(t)| \leq \int_t^T ||e^{(\hat{A} - \hat{B} K_\infty)'(\tau - t)}|| [(||K_\infty|| + ||K_T(\tau)||)||\hat{B}|| \ |p_T(\tau)|$$

$$+ \lambda_3 ||P_T(\tau)|| \ |\xi|] d\tau$$

$$\leq \int_t^T M e^{-\beta(\tau - t)} [M_1 |p_T(\tau)| + M_2] d\tau$$

$$\leq \frac{MM_2}{\beta} + \int_t^T MM_1 e^{-\beta(\tau - t)} |p_T(\tau)| d\tau.$$

By Gronwall's inequality, we have

$$| p_T(t) | \leq \frac{MM_2}{\beta} \exp\left(\int_t^T MM_1 e^{-\beta(\tau - t)} d\tau \right)$$

$$\leq \frac{MM_2}{\beta} e^{MM_1/\beta} \tag{2.28}$$

for all $t \in [0, T]$ and all $T \geq 0$. Let p_∞ be the algebraic solution of

$$(\hat{A} - \hat{B} K_\infty)' p + \lambda_3 P_\infty \xi = 0. \tag{2.29}$$

Then

$$p_\infty = - \lambda_3 [(\hat{A} - \hat{B} K_\infty)']^{-1} P_\infty \xi$$

$$= \lambda_3 \int_t^\infty e^{(\hat{A} - \hat{B} K_\infty)' (\tau - t)} P_\infty \xi d\tau.$$

From (2.27),

$$p_\infty - p_T(t) = \int_t^T e^{(\hat{A} - \hat{B} K_\infty)' (\tau - t)} \{ [K_T(\tau) - K_\infty]' \hat{B}' p_T(\tau) + \lambda_3 [P_\infty - P_T(\tau)] \xi \} d\tau$$

$$+ \int_T^\infty e^{(\hat{A} - \hat{B} K_\infty)' (\tau - t)} P_\infty \xi d\tau.$$

Since $(\hat{A} - \hat{B} K_\infty)$ is stable, then

$$\left| \int_T^\infty e^{(\hat{A} - \hat{B} K_\infty)' (\tau - t)} P_\infty \xi d\tau \right| < \epsilon$$

for all sufficiently large T. By the convergent properties in (2.25) and (2.26), there exists d_0 such that

$$\left| \int_t^{T - d_0} e^{(\hat{A} - \hat{B} K_\infty)' (\tau - t)} \{ [K_T(\tau) - K_\infty]' \hat{B}' p_T(\tau) + \lambda_3 [P_\infty - P_T(\tau)] \xi \} d\tau \right|$$

$$\leq \int_t^{T - d_0} M e^{-\beta(\tau - t)} M_3 \epsilon$$

$$\leq \frac{MM_3}{\beta}\epsilon$$

for some constant $M_3 > 0$ and all $t \in [0, T-d_0]$, all $T \geq d_0$. Since the integrand of the first integral of (2.28) is uniformly bounded in $t \leq T$, there exists a $M_4 > 0$ such that

$$\left| \int_{T-d_0}^{T} e^{(\hat{A} - \hat{B}K_\infty)'(\tau-t)} [(K_T(\tau)-K_\infty)'\hat{B}' p_T(\tau) + \lambda_3(P_\infty - P_T(\tau)\xi] d\tau \right|$$

$$\leq \int_{T-d_0}^{T} M e^{-\beta(\tau-t)} M_4 \, d\tau$$

$$\leq \frac{MM_4}{\beta}\left[e^{-\beta(T-d_0-t)} - e^{-\beta(T-t)}\right] < \epsilon$$

if $T-d_0-t$ is sufficiently large. From this analysis, we can conclude that for each $\epsilon > 0$, there exists $d_0 \leq d_1$ such that

$$| p_\infty - p_T(t) | < \epsilon \tag{2.30}$$

for all $t \in [0, T-d_0]$ and $T \geq d_1$. Furthermore, let

$$k_\infty = R_\infty^{-1}\hat{B}' p_\infty$$

$$u_\infty(t) = -[K_\infty x_\infty(t) + k_\infty].$$

In the same way, we can prove that for any $\epsilon > 0$, $\exists\ d_1 \geq d_0 > 0$ such that

$$| k_\infty - k_T(t) | < \epsilon \tag{2.31}$$

for all $t \in [0, T-d_0]$ and all $T \geq d_1$.

Recognizing that $q_T(t)$ tends to infinity as $T \to \infty$, we want to show that on the average it tends to a constant in the sense

$$\lim_{T\to\infty} \frac{1}{T} q_T(t) = \lim_{T\to\infty} \frac{1}{T} \int_t^T [2\lambda_3 \xi' p_T(\tau) + \lambda_3 \xi' P_T(\tau)\xi - k_T(\tau)' R_T(\tau) k_T(\tau)] d\tau$$

$$= 2\lambda_3\xi' p_\infty + \lambda_3 \xi' P_\infty \xi - k_\infty' R_\infty k_\infty \tag{2.32}$$

$$\triangleq q_\infty$$

With properties $P_T \to P_\infty$ and $p_T \to p_\infty$ as $T \to \infty$, we have for any $\epsilon > 0$, there exist $d_0 \leq d_1$, such that

$$||P_T(\tau) - P_\infty|| < epsilon, \quad |p_T(\tau) - p_\infty| < \epsilon$$

$$||R_T(\tau) - R_\infty|| < epsilon, \quad |k_T(\tau) - k_\infty| < epsilon,$$

for all $\tau \in [0, T-d_0]$ for all $T \geq d_1$. Thus,

$$\frac{1}{T} q_T(t) - q_\infty = \frac{1}{T}\left(\int_t^{T-d_0} + \int_{T-d_0}^{T} \right) \Big\{ 2\lambda_3 \xi' [p_T(\tau) - p_\infty]$$

$$+ \lambda_3 \xi' [P_T(\tau) - P_\infty] \xi - [k_T(\tau) - k_\infty]' R_T(\tau) k_T(\tau))$$

$$- k_\infty' [R_T(\tau) - R_\infty] k_T(\tau) - k_\infty' R_\infty [k_T(\tau) - k_\infty] \Big\} d\tau - \frac{t}{T} q_\infty .$$

In the second integral over finite interval $[T - d_0, T]$, the integrand is uniformly bounded, so that the average tends to 0 as $T \to \infty$ while in the first integral, the integrand is less than some constant multiplying ϵ, so that the average tends to 0 as $\epsilon \to 0$. Note that $\frac{t}{T} q_\infty \to 0$ as $T \to \infty$ for each finite t. Hence the limit in (2.32) is uniform on each compact interval. Thus, for the average cost

$$\lim_{T \to \infty} \frac{1}{T} J(u_T) = \lim_{T \to \infty} \frac{1}{T} V_T(0, x_0) = \lim_{T \to \infty} \frac{1}{T} q_T(0) = q_\infty .$$

If there exists a control $\hat{u}(t)$ defined on $[0, \infty)$ such that $J_{av}(\hat{u}) < q_\infty$, then for some T_0 such that $T \geq T_0$,

$$\frac{1}{T} [J(\hat{u}) - J(u_T)] < 0$$

which implies $J(\hat{u}) < J(u_T)$ for all $T \geq T_0$ and contradicts the hypothesis that $J(u_T)$ is the optimal value on $[0, T]$. Thus, q_∞ is the optimal value. We show

$$J_{av}(u_\infty) = q_\infty \tag{2.33}$$

in the next section.

3. Convergence of the Optimal Control and State

We shall directly prove the convergence of the optimal trajectory $x_T(t)$ with the optimal control $u_T(t)$ on $[0,T]$. In this way we avoid the difficulty of determining an ergodic probability measure for the process $x(t)$ with control $u(t)$ using a Lyapunov criterion as in [1], [3] and [4] - the usual method of constructing the optimal stationary control. Before proving (2.33), we need some lemmas.

Lemma 3.1 (Stochastic Gronwall inequality). *Let $g(t) \geq 0$, $\phi(t)$, $f(t)$ and $h(t)$ be real random functions such that*

$$\phi(t) \leq f(t) + \int_0^t g(\tau)\phi(\tau)d\tau + \int_0^t h(\tau)\phi(\tau)dN(\tau) \quad a.s. \tag{3.1}$$

where $N(t)$ is a Poisson counting process (which may be inhomogeneous) such that it counts the incidence during $[0,t]$. Then

$$\phi(t) \leq f(t) + \int_0^t g(\tau)f(\tau)exp(\int_\tau^t g(s)ds)d\tau$$
$$+ \int_0^t [f(\gamma)+\int_0^\gamma g(\tau)f(\tau)exp(\int_\tau^\gamma g(s)ds)d\tau]exp(\int_\gamma^t g(s)ds)$$
$$\cdot h(\gamma)exp(\int_\gamma^t h(s)dN(s))dN(\gamma) \quad a.s. \tag{3.2}$$

In addition, if $f(t) \equiv f$, a constant, then

$$\phi(t) \leq f \; exp(\int_0^t g(s)ds)\; exp(\int_0^t h(s)dN(s)) \quad a.s. \tag{3.3}$$

Proof. Denote $\{\tau_i\}$ the interarrival times of $N(t)$ and let $t_i = \tau_1 + \cdots + \tau_i$. Define

$$\phi_1(t) \triangleq \int_0^t g(\tau)\phi(\tau)d\tau + \int_0^t h(\tau)\phi(\tau)dN(\tau). \tag{3.4}$$

Then for $0 \leq t < t_1$,

$$\phi(t) \leq f(t) + \int_0^t g(\tau)\phi(\tau)d\tau$$

so that by the ordinary Gronwall inequality, we have

$$\phi_1(t) \leq \int_0^t g(\tau)f(\tau)exp(\int_\tau^t g(s)ds)d\tau \quad 0 \leq t < t_1$$

Suppose for $t_{i-1} \leq t < t_i$,

$$\phi_1(t) \leq \int_0^t g(\tau)f(\tau)exp(\int_\tau^t g(s)ds)d\tau$$

$$+ \sum_{j=1}^{i-1} [f(t_j)+\int_0^{t_j} g(\tau)f(\tau)exp(\int_\tau^{t_j} g(s)ds)d\tau]exp(\int_{t_j}^t g(s)ds)$$

$$\cdot\, h(t_j)exp(\sum_{k=j+1}^{i-1} h(t_k)) \quad a.s. \tag{3.5}$$

Then from (3.1) and (3.5)

$$\phi_1(t_i) \leq \Bigg\{ \int_0^{t_i} g(\tau)f(\tau)exp(\int_\tau^{t_i} g(s)ds)d\tau$$

$$+ \sum_{j=1}^{i-1} [f(t_j)+\int_0^{t_j} g(\tau)f(\tau)exp(\int_\tau^{t_j} g(s)ds)d\tau]exp(\int_{t_j}^{t_i} g(s)ds)$$

$$\cdot\, h(t_j)exp(\sum_{k=j+1}^{i-1} h(t_k)) \Bigg\}(1+h(t_i))+f(t_i)h(t_i) \quad a.s.$$

$$\triangleq M$$

so that for $t_i \leq t < t_{i+1}$

$$\phi(t) \le f(t) + M + \int_{t_i}^{t} g(\tau)\phi(\tau)d\tau$$

Again, by the ordinary Gronwall inequality,

$$\int_{t_i}^{t} g(\tau)\phi(\tau)d\tau \le \int_{t_i}^{t} g(\tau)(f(\tau)+M)exp(\int_{\tau}^{t} g(s)ds)d\tau \quad a.s.$$

$$= \int_{t_i}^{t} g(\tau)f(\tau)exp(\int_{\tau}^{t} g(s)ds)d\tau - M + M\ exp(\int_{t_i}^{t} g(s)ds) \quad a.s.$$

Thus, for $t_i \le t < t_{i+1}$,

$$\phi_1(t) \le M + \int_{t_i}^{t} g(\tau)\phi(\tau)d\tau$$

$$\le \int_{t_i}^{t} g(\tau)f(\tau)exp(\int_{\tau}^{t} g(s)ds)d\tau + M\ exp(\int_{t_i}^{t} g(s)ds)$$

$$\le \int_{t_i}^{t} g(\tau)f(\tau)exp(\int_{\tau}^{t} g(s)ds)d\tau$$

$$+ \sum_{j=1}^{i} [f(t_j) + \int_{0}^{t_j} g(\tau)f(\tau)exp(\int_{\tau}^{t_j} g(s)ds)d\tau] exp(\int_{t_j}^{t} g(s)ds)$$

$$\cdot h(t_j)exp(\sum_{k=j+1}^{i} h(t_k))].$$

By induction, (3.5) holds for any i and the result (3.2) follows. If $f(t) \equiv f$, (3.5) becomes

$$\phi_1(t) \le -f + f\,exp(\int_{0}^{t} g(s)ds) + \sum_{j=1}^{i-1} f\,exp(\int_{0}^{t} g(s)ds)h(t_j)exp(\sum_{k=j+1}^{i-1} h(t_k))$$

$$= f\,exp(\int_{0}^{t} g(s)ds)\left[1 + \sum_{j=1}^{i-1} h(t_j)exp(\sum_{k=j+1}^{i-1} h(t_k))\right] - f$$

$$\le f\,exp(\int_{0}^{t} g(s)ds)exp(\sum_{j=1}^{i-1} h(t_j)) - f \quad a.s.$$

for $t_{i-1} \le t < t_i$. Thus (3.3) follows.

QED

Remark. If

$$\phi(t)\leq f(t)+\int_0^t g(\tau)\phi(\tau)d\tau+\int_0^t h_1(\tau)\phi(\tau)dN_1(\tau)+\int_0^t h_2(\tau)\phi(\tau)dN_2(\tau) \tag{3.6}$$

where N_1(t) and N_2(t) are independent Poisson counting processes with zero probability of simultaneous jumps. Then we can define a Poisson process $N(t) = N_1(t) + N_2(t)$ and a random process $\{\mu(t)\}$ such that $\mu(t) = i$, $t_j \leq t < t_{j+1}$ if $N_i(t_j)$ increases, $i=1, 2$. Thus (3.6) is equivalent to

$$\phi(t)\leq f(t)+\int_0^t g(\tau)\phi(\tau)d\tau+\int_0^t h_{\mu(\tau)}(\tau)\phi(\tau)dN(\tau) \quad a.s.$$

By Lemma 3.1,

$$\begin{aligned}
\phi(t) &\leq f(t)+\int_0^t g(\tau)f(\tau)exp(\int_\tau^t g(s)ds)d\tau \\
&+\int_0^t [f(\gamma)+\int_0^\gamma g(\tau)f(\tau)exp(\int_\tau^\gamma g(s)ds)d\tau]exp(\int_\gamma^t g(s)ds) \\
&\cdot h_{\mu(\gamma)}(\gamma)exp(\int_\gamma^t h_{\mu(s)}(s)dN(s))dN(\gamma) \quad a.s. \\
&= f(t)+\int_0^t g(\tau)f(\tau)exp(\int_\tau^t g(s)ds)d\tau \\
&+\int_0^t [f(\gamma)+\int_0^\gamma g(\tau)f(\tau)exp(\int_\tau^\gamma g(s)ds)d\tau]exp(\int_\gamma^t g(s)ds) \\
&\cdot h_1(\gamma)exp(\int_\gamma^t h_1(s)dN_1(s)+\int_\gamma^t h_2(s)dN_2(s))dN_1(\gamma) \\
&+\int_0^t [f(\gamma)+\int_0^\gamma g(\tau)f(\tau)exp(\int_\tau^\gamma g(s)ds)d\tau]exp(\int_\gamma^t g(s)ds) \\
&\cdot h_2(\gamma)exp(\int_\gamma^t h_1(s)dN_1(s)+\int_\gamma^t h_2(s)dN_2(s))dN_2(\gamma) \quad a.s.
\end{aligned} \tag{3.7}$$

If $f(t) \equiv f$, then

$$\phi(t) \leq f \; exp(\int_0^t g(s)ds)\; exp(\int_0^t h_1(s)dN_1(s))\; exp(\int_0^t h_2(s)dN_2(s)) \quad a.s. \tag{3.8}$$

Lemma 3.2. *Suppose f(t,s) ≥ 0, $0 \leq s \leq t$, is a continuous real random function. Then* $\int_0^t f(t,s)ds$ *is uniformly bounded in t a.s. if and only if* $\int_0^t f(t,s)dN(s)$ *is uniformly bounded in t a.s.; where N(t) is a Poisson counting process.*

Proof. Let $\{\tau_i\}$ be the interarrival times corresponding to N(t) and let $t_i = \tau_1 + \cdots + \tau_i$ be the occurrence time. If $\int_0^t f(t,s)ds$ is uniformly bounded in t a.s. and $f(t,s) \geq 0$, $0 \leq s \leq t$, then there exists $M(\omega) > 0$ which is independent of t

$$\frac{1}{k}\sum_{j=1}^{[t/k]} f(t,\alpha_j) < M(\omega) \text{ for } \alpha_j \in [\frac{1}{k}(j-1), \frac{1}{k}j] \tag{3.9}$$

for all $t \geq 0$ and sufficiently large k, $[\frac{t}{k}]$ denotes the largest integer $\leq \frac{t}{k}$. Thus, the series

$$f(t,t_1) + f(t,t_2) + \cdots + f(t,t_l), \quad t_l \leq t < t_{l+1} \tag{3.10}$$

diverges as $t \rightarrow \infty$ only if $\exists$ subsequence $\{\tau_k\}$ such that $\tau_k \rightarrow 0$. However, $P(\tau_k < \frac{1}{k}) = 1 - e^{-\lambda/k} < 1$. Then $\tau_k \rightarrow 0$ with probability 0. Hence (3.9) converges a.s. and so $\int_0^t f(t,s)dN(s)$ is uniformly bounded in t. Conversely, if (3.10) converges a.s., then (3.9) diverges as $t \rightarrow \infty$ for any k only if $\exists$ subsequence $\{\tau_k\}$ such that $\tau_k \rightarrow \infty$. But, $P(\tau > T) = e^{-\lambda T} < 1$. Thus, $\tau_k < \infty$ a.s. Hence $\int_0^t f(t,s)ds$ is also uniformly bounded in t a.s.

QED

Now, we turn to our problem: To show that $x_T(t) \to x_\infty(t)$ with control $u_T(t) \to u_\infty(t)$, as $T \to \infty$. Since

$$dx_T(t) = Ax_T(t)dt + Bu_T(t)dt + Cx_T(t)dN_1(t) + Du_T(t)dN_2(t) + \xi dN_3(t)$$

with $u_T(t) = -[K_T(t)x_T(t) + k_T(t)]$, then

$$\begin{aligned} dx_T(t) &= [(A-BK_T(t))x_T(t) - Bk_T(t)]dt + Cx_T(t)dN_1(t) \\ &\quad - D[K_T(t)x_T(t) + k_T(t)]dN_2(t) + \xi dN_3(t) \\ &= [\hat{A} - \hat{B}K_T(t)]x_T(t)dt - Bk_T(t)dt - \lambda_1 Cx_T(t)dt + \lambda_2 DK_T(t)x_T(t)dt \\ &\quad + C\,x_T(t)dN_1(t) - DK_T(t)x_T(t)dN_2(t) - Dk_T(t)dN_2(t) + \xi dN_3(t) \\ &= [\hat{A} - \hat{B}K_\infty(t)]x_T(t)dt - \hat{B}(K_\infty - K_T(t))x_T(t)dt + [\lambda_2 DK_T(t) - \lambda_1 C]x_T(t)dt \\ &\quad - Bk_T(t)dt + Cx_T(t)dN_1(t) - DK_T(t)x_T(t)dN_2(t) - Dk_T(t)dN_2(t) \\ &\quad + \xi dN_3(t). \end{aligned} \tag{3.11}$$

Thus,

$$\begin{aligned} x_T(t) &= exp(\hat{A} - \hat{B}K_\infty)t\ x_0 + \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)\{\hat{B}[K_\infty - K_T(s)] \\ &\quad + \lambda_2 DK_T(s) - \lambda_1 C\}x_T(s)ds - \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)Bk_T(s)ds \\ &\quad + \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)[Cx_T(s)dN_1(s) - DK_T(s)x_T(s)dN_2(s)] \\ &\quad + \int_0^t exp(\hat{A} - \hat{B}k_\infty)(t-s)[-Dk_T(s)dN_2(s) + \xi dN_3(s)]. \end{aligned}$$

Since $(\hat{A} - \hat{B}K_\infty)$ is stable, $\exists$ M and $\beta > 0$ such that

$$|\,exp(\hat{A} - \hat{B}K_\infty)(t-s)\,| \le M\ e^{-\beta(t-s)},$$

so that

$$|\,x_T(t)\,| \le M\ e^{-\beta t}\ |\,x_0\,| + \int_0^t M\ e^{-\beta(t-s)}\,||B||\ |\,k_T(s)\,|\,ds$$

$$+ \int_0^t M\ e^{-\beta(t-s)} [\,||D||\ |k_T(s)|\ dN_2(s) + |\xi|\ dN_3(s)]$$

$$+ \int_0^t \{ Me^{-\beta(t-s)} [\,||B||\ ||K_\infty - K_T(s)||$$

$$+ \lambda_1 ||C|| + \lambda_2 ||D||\ ||K_T(s)||]\ |x_T(s)|\ \}\ ds$$

$$+ \int_0^t \{ Me^{-\beta(t-s)} [\,||C||\ |x_T(s)|\ dN_1(s)$$

$$+ ||D||\ ||K_T(s)||\ |x_T(s)|\ dN_2(s)].\ \} \tag{3.12}$$

Since $k_T(\mathrm{t})$ and $K_T(\mathrm{t})$ are uniformly bounded for $0 \le t \le T < \infty$ and

$$\int_0^t e^{-\beta(t-s)} ds = \frac{1}{\beta} e^{-\beta(t-s)} \Big|_0^t = \frac{1}{\beta}(1 - e^{-\beta t}) \le \frac{1}{\beta}, \quad \forall\ t \ge 0,$$

so that the third term on the right hand side of (3.12) is finite a.s. for every $t \ge 0$ by Lemma 3.2. The second term is easily shown to be finite $\forall\ t$. Thus, $\exists\ M_1(\omega) > 0$ and constant M_2, M_3 and M_4 such that (3.12) becomes

$$|x_T(t,\omega)| \le M_1(\omega) + \int_0^t M_2 e^{-\beta(t-s)} |x_T(s,\omega)|\ ds$$

$$+ \int_0^t M_3 e^{-\beta(t-s)} |x_T(s,\omega)|\ dN_1(s) + \int_0^t M_4 e^{-\beta(t-s)} |x_T(s,\omega)|\ dN_2(s).$$

By the Stochastic Gronwall Lemma 3.1,

$$|x_T(t,\omega)| \le M_1(\omega)\ exp(\int_0^t M_2 e^{-\beta(t-s)} ds)\ exp(\int_0^t M_3 e^{-\beta(t-s)} dN_1(s))$$

$$\cdot\ exp(\int_0^t M_4 e^{-\beta(t-s)} dN_2(s)) \quad a.s. \tag{3.13}$$

Since each integrand in each exponent is uniformly integrable over $[0,t)$, $\forall\ t$, by Lemma 3.2, $|x_T(t,\omega)|$ is uniformly bounded for $0 \le t \le T$, $\forall\ T \ge 0$. Furthermore,

$$dx_\infty(t) = Ax_\infty(t)dt + Bu_\infty(t)dt + Cx_\infty(t)dN_1(t) + Du_\infty(t)dN_2(t) + \xi dN_3(t)$$

with $u_\infty(t) = -[K_\infty x_\infty(t) + k_\infty]$. We have

$$\begin{aligned} dx_\infty(t) &= (A - BK_\infty)x_\infty(t)dt - Bk_\infty dt + Cx_\infty(t)dN_1(t) \\ &\quad - DK_\infty x_\infty(t)dN_2(t) - Dk_\infty dN_2(t) + \xi dN_3(t) \\ &= (\hat{A} - \hat{B}K_\infty)x_\infty(t)dt - Bk_\infty dt - Dk_\infty dN_2(t) + \xi dN_3(t) \\ &\quad - \lambda_1 Cx_\infty(t)dt + \lambda_2 DK_\infty x_\infty(t)\, dt \\ &\quad + Cx_\infty(t)dN_1(t) - DK_\infty x_\infty(t)dN_2(t) \end{aligned}$$

so that

$$\begin{aligned} d[x_\infty(t) - x_T(t)] &= [\hat{A} - \hat{B}K_\infty][x_\infty(t) - x_T(t)]dt - B[k_\infty - k_T(t)]dt \\ &\quad - D[k_\infty - k_T(t)]dN_2(t) + \hat{B}[K_\infty - K_T(t)]x_T(t)dt \\ &\quad - \lambda_1 C[x_\infty(t) - x_T(t)]dt + \lambda_2 D[K_\infty x_\infty(t) - K_T(t)x_T(t)]dt \\ &\quad + C[x_\infty(t) - x_T(t)]dN_1(t) \\ &\quad - D[K_\infty x_\infty(t) - K_T(t)x_T(t)]dN_2(t). \end{aligned}$$

Then

$$\begin{aligned} x_\infty(t) - x_T(t) &= -\int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)B[k_\infty - k_T(s)]ds \\ &\quad - \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)D[k_\infty - k_T(s)]dN_2(s) \\ &\quad + \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)B[K_\infty - K_T(s)]x_T(s)ds \\ &\quad + \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)[-\lambda_1 C + \lambda_2 DK_\infty][x_\infty(s) - x_T(s)]ds \\ &\quad + \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)C[x_\infty(s) - x_T(s)]dN_1(s) \\ &\quad - \int_0^t exp(\hat{A} - \hat{B}K_\infty)(t-s)DK_\infty[x_\infty(s) - x_T(s)]dN_2(s) \end{aligned}$$

$$- \int_0^t exp(\hat{A} - \hat{B} K_\infty)(t-s) D [K_\infty - K_T(s)] x_T(s) dN_2(s).$$

Thus,

$$\begin{aligned}
|x_\infty(t) - x_T(t)| \leq & \int_0^t M e^{-\beta(t-s)} ||B|| \ |k_\infty - k_T(s)| ds \\
& + \int_0^t M e^{-\beta(t-s)} ||D|| \ |k_\infty - k_T(s)| dN_2(s) \\
& + \int_0^t M e^{-\beta(t-s)} ||B|| \ ||K_\infty - K_T(s)|| \ |x_T(s)| ds \\
& + \int_0^t M e^{-\beta(t-s)} ||D|| \ ||K_\infty - K_T(s)|| \ |x_T(s)| dN_2(s) \\
& + \int_0^t \{ M e^{-\beta(t-s)} [\lambda_1 ||C|| + \lambda_2 ||D|| \ ||K_\infty||] \\
& \bullet |x_\infty(s) - x_T(s)| \} ds \qquad (3.14) \\
& + \int_0^t M e^{-\beta(t-s)} ||C|| \ |x_\infty(s) - x_T(s)| dN_1(s) \\
& + \int_0^t M e^{-\beta(t-s)} ||D|| \ ||K_\infty|| \ |x_\infty(s) - x_T(s)| dN_2(s).
\end{aligned}$$

Since (2.26), (2.31) and $x_T(s)$ is uniformly bounded, we argue as before that for each $\epsilon > 0$, $\exists \ d_0 \leq d_1$, such that for all $T \geq d_1$, the sum of the first four integrals of (3.14) is less than $\epsilon M_1(\omega)$ for some $M_1(\omega) > 0$, $t \in [0, T-d_0]$ and $\exists$ constants M_2, M_3 and M_4, such that

$$\begin{aligned}
|x_\infty(t) - x_T(t)| \leq \epsilon M_1(\omega) & + \int_0^t M_2 e^{-\beta(t-s)} |x_\infty(s) - x_T(s)| ds \\
& + \int_0^t M_3 e^{-\beta(t-s)} |x_\infty(s) - x_T(s)| dN_1(s) \\
& + \int_0^t M_4 e^{-\beta(t-s)} |x_\infty(s) - x_T(s)| dN_2(s). \qquad (3.15)
\end{aligned}$$

Again we apply the Stochastic Gronwall Lemma 3.1 and Lemma 3.2 to (3.15). We get the result

$$| x_{\infty}(t)-x_T(t) | \leq \epsilon M_1(\omega)\, exp\,(\int_0^t M_2\, e^{-\beta(t-s)}ds)\; exp\,(\int_0^t M_3\, e^{-\beta(t-s)}dN_1(s))$$

$$\cdot\, exp\,(\int_0^t M_4\, e^{-\beta(t-s)}dN_2(s))$$

$$\leq \epsilon M_5(\omega)$$

for some $M_5(\omega)$, all $t \in [0, T-d_0]$ and all $T \geq d_1$. Thus,

$$| u_T(t)-u_{\infty}(t) | \leq ||K_T(t)$$

$$-K_{\infty}||\; | x_T(t) | + ||K_{\infty}||\; | x_T(t)-x_{\infty}(t) | + | k_T(t)-k_{\infty} |$$

$$< \epsilon M_6(\omega)$$

for some $M_6(\omega)$, all $t \in [0, T-d_0]$ and all $T \geq d_1$. Hence,

$$\frac{1}{T}\,[J_T(u_{\infty})-J_T(u_T)]$$

$$= \frac{1}{T} E \int_0^T \{[x_{\infty}(t)'\, Q\, x_{\infty}(t)-x_T(t)'\, Q\, x_T(t)]$$

$$+ [u_{\infty}(t)'\, R\, u_{\infty}(t)-u_T(t)'\, R\, u_T(t)]\}dt \tag{3.16}$$

$$= \frac{1}{T} E \int_0^T \{[x_{\infty}(t)-x_T(t)]'\, Q\, x_{\infty}(t)+x_T(t)'\, Q\, [x_{\infty}(t)-x_T(t)]$$

$$+ [u_{\infty}(t)-u_T(t)]'\, R\, u_{\infty}(t)+u_T(t)'\, R\, [u_{\infty}(t)-u_T(t)]\}dt.$$

Since x_T, x_{∞}, u_T and u_{∞} are uniformly bounded, the integral of (3.16) can be partitioned into two parts; the first integral over $[0, T-d_0]$ is less than $\epsilon M_7(\omega)$ for some $M_7(\omega) > 0$ while the second integral over $[T-d_0, T]$ tends to zero as $T \rightarrow \infty$. Hence, (3.16) tends to 0 as $T \rightarrow \infty$ and (2.33) follows. We summarize the entire analysis in the following theorem.

Theorem 3.3. *Suppose all the coefficient matrices of* (1.1) *are constant. If* $(A+\lambda_1 C, B+\lambda_2 D)$ *is stabilizable,* $(Q, A+\lambda_1 C)$ *is observable and condition* (I) *in section* 2 *holds, then the optimal control exists and is of the form*

$$u_\infty(t) = -K_\infty x_\infty(t) - k_\infty \tag{3.17}$$

with

$$\begin{aligned} K_\infty &= [R + \lambda_2 D' P_\infty D]^{-1}(B + \lambda_2 D)' P_\infty \\ k_\infty &= [R + \lambda_2 D' P_\infty D]^{-1}(B + \lambda_2 D)' p_\infty \end{aligned} \tag{3.18}$$

where P_∞ and p_∞ *are the unique solutions of*

$$\begin{aligned} &(A+\lambda_1 C)'P + P(A+\lambda_1 C) + \lambda_1 C'PC + Q \\ &\quad + P(B+\lambda_2 D)[R+\lambda_2 D'PD]^{-1}(B+\lambda_2 D)'P = 0 \end{aligned} \tag{3.19}$$

and

$$[A+\lambda_1 C - (B+\lambda_2 D)(R+\lambda_2 D'PD)^{-1}(B+\lambda_2 D)'P]\, p + \lambda_3 P\,\xi = 0 \tag{3.20}$$

respectively. The optimal average cost is

$$J_{av}(u_\infty) = 2\lambda_3 \xi' p_\infty + \lambda_3 \xi' P_\infty \xi - k_\infty' R_\infty k_\infty . \tag{3.21}$$

Remark. The optimal control in (3.17) of the infinite time problem is a time invariant linear feedback control plus a stationary feed-forward control. The additive noise only affects the feed-forward control. Moreover, both gains in (3.18) are quite sensitive to the coefficients C, D of the state- and control-dependent noises, respectively. In general, large state dependent noise can destabilize the system (1.1) while large control dependent noise may diminish the effects of the gain K_∞ and k_∞, and increase the average cost in (3.21). Note that the matrices C and D should be small in norm to guarantee condition (I).

4. The Case of Discounted Cost.

If we use the discounted cost criterion (2.2), we can define a new state $\tilde{x}(t)$ and a new control $\tilde{u}(t)$ by

$$\tilde{x}(t) \triangleq e^{-\alpha t} x(t)$$
$$\tilde{u}(t) \triangleq e^{-\alpha t} u(t). \tag{4.1}$$

Then (2.2) becomes the limit of

$$J_T(\tilde{u}) \triangleq E \int_0^T [\tilde{x}(t)' Q \tilde{x}(t) + \tilde{u}(t)' R \tilde{u}(t)] dt \tag{4.2}$$

when $T \to \infty$. Now the new state dynamics are

$$\begin{aligned} d\tilde{x}(t) &= -\alpha e^{-\alpha t} x(t)dt + e^{-\alpha t}[A x(t)dt + B u(t)dt] \\ &+ \{e^{-\alpha t}[x(t)+C x(t)] - e^{-\alpha t} x(t)\} dN_1(t) \\ &+ \{e^{-\alpha t}[x(t)+D u(t)] - e^{-\alpha t} x(t)\} dN_2(t) \\ &+ \{e^{-\alpha t}[x(t)+\xi(t)] - e^{-\alpha t} x(t)\} dN_3(t) \\ &= (A-\alpha I)\tilde{x}(t)dt + B \tilde{u}(t)dt + C \tilde{x}(t)dN_1(t) \\ &+ D \tilde{u}(t)dN_2(t) + e^{-\alpha t} \xi dN_3(t). \end{aligned} \tag{4.3}$$

From Theorem 1.1, for each $T > 0$, the optimal control is

$$\tilde{u}_T(t) = -\tilde{K}_T(t)\tilde{x}_T(t) - \tilde{k}_T(t)$$

with

$$\tilde{K}_T(t) = [R + \lambda_2 D' \tilde{P}_T(t) D]^{-1} (B + \lambda_2 D)' \tilde{P}_T(t)$$
$$\tilde{k}_T(t) = [R + \lambda_2 D' \tilde{P}_T(t) D]^{-1} (B + \lambda_2 D)' \tilde{p}_T(t).$$

The optimal value is

$$J_T(\tilde{u}_T) = x_0' \tilde{P}_T(0) x_0 + 2\tilde{p}_T(0)' x_0 + \tilde{q}_T(0)$$

where $\tilde{P}_T(t) \geq 0$, $\tilde{p}_T(t)$ and $\tilde{q}_T(t)$ are unique solution of

$$\begin{cases} \frac{d}{dt}P(t) + (A + \lambda_1 C - \alpha I)' P(t) + P(t)(A + \lambda_1 C - \alpha I) + \lambda_1 C' P(t) C \\ \quad + Q - P(t)(B + \lambda_2 D)[R + \lambda_2 D' P(t) D]^{-1} (B + \lambda_2 D)' P(t) = 0 \\ P(T) = 0 \end{cases} \tag{4.4}$$

$$\begin{cases} \dfrac{d}{dt}p(t) + \{A + \lambda_1 C - \alpha I - (B + \lambda_2 D)[R + \lambda_2 D' P(t) D]^{-1}(B + \lambda_2 D)' P(t)\}' p(t) \\ \qquad + \lambda_3 P(t) e^{-\alpha t}\xi = 0 \\ p(T) = 0 \end{cases} \tag{4.5}$$

and

$$\begin{cases} \dfrac{d}{dt}q(t) + 2\lambda_3 e^{-\alpha t}\xi' p(t) + \lambda_3 e^{-2\alpha t}\xi' P(t)\xi \\ \qquad - p(t)'(B + \lambda_2 D)[R + \lambda_2 D' P(t) D]^{-1}(B + \lambda_2 D)' p(t) = 0 \\ q(T) = 0 \end{cases} \tag{4.6}$$

respectively. In the same manner as before, if $(A + \lambda_1 C - \alpha I, B + \lambda_2 D)$ is stabilizable, $(Q, A + \lambda_1 C - \alpha I)$ is detectable and condition (I) in section 2 holds with $\hat{A}$ being replaced by $A + \lambda_1 C - \alpha I$, then $\tilde{P}_T(t) \uparrow \tilde{P}_\infty$ uniformly in $[0, T - d_0]$ as $T \to \infty$ for some d_0 and $\tilde{P}_\infty \geq 0$ satisfies the algebraic equation

$$\begin{aligned} &(A + \lambda_1 C - \alpha I)' P + P(A + \lambda_1 C - \alpha I) + \lambda_1 C' P(t) C + Q \\ &\quad - P(B + \lambda_2 D)[R + \lambda_2 D' P(t) D]^{-1}(B + \lambda_2 D)' P = 0. \end{aligned} \tag{4.7}$$

In addition,

$$[A + \lambda_1 C - \alpha I - (B + \lambda_2 D)(R + \lambda_2 D' \tilde{P}_\infty D)^{-1}(B + \lambda_2 D)' \tilde{P}_\infty]$$

is a stable matrix. If $(Q^{1/2}, A + \lambda_1 C - \alpha I)$ is observable, then $\tilde{P}_\infty$ is the unique solution of (4.7) in the solution class $P \geq 0$.

Let

$$\tilde{R}_\infty = R + \lambda_2 D' \tilde{P}_\infty D$$

$$\tilde{K}_\infty = [R + \lambda_2 D' \tilde{P}_\infty D]^{-1}(B + \lambda_2 D)' \tilde{P}_\infty.$$

Let $\tilde{\Phi}_T(t, s)$ be the transition matrix of

$$A + \lambda_1 C - \alpha I - (B + \lambda_2 D)[R + \lambda_2 D' P(t) D]^{-1}(B + \lambda_2 D)' P(t).$$

Then the solution $P_T(t)$ of (4.5) becomes

$$\tilde{p}_T(t) = \lambda_3 \int_t^T \tilde{\Phi}_T(\tau,t)' \tilde{P}_T(\tau) e^{-\alpha\tau} \xi \, d\tau$$

$$\underset{T \to \infty}{\longrightarrow} \quad \lambda_3 \int_t^\infty exp\,[(A + \lambda_1 C - \alpha I - (B + \lambda_2 D)\tilde{K}_\infty)' \ (\tau - t)] \, \tilde{P}_\infty e^{-\alpha\tau} \xi \, d\tau$$

$$= - \ e^{-\alpha t} \lambda_3 [A + \lambda_1 C - 2\alpha I - (B + \lambda_2 D)\tilde{K}_\infty]'^{-1} \tilde{P}_\infty \xi$$

$$\triangleq \ e^{-\alpha t} \tilde{p}_\infty \tag{4.8}$$

where the convergence is uniform in $[0, T - d_0]$ as $T \to \infty$ for some $d_0 > 0$.

On the other hand, we want to show

$$\tilde{q}_T(t) = \int_t^T \{2\lambda_3 e^{-\alpha\tau} \xi' \tilde{p}_T(\tau) \ + \ \lambda_3 e^{-2\alpha\tau} \xi' \tilde{P}_T(\tau) \xi \ - \ \tilde{p}_T(\tau)' (B + \lambda_2 D)$$

$$\cdot \, [R + \lambda_2 D' \tilde{P}_T(\tau) D]^{-1} (B + D)' \tilde{p}_T(\tau)\} \, d\tau$$

converges to

$$\int_t^\infty \{2\lambda_3 e^{-2\alpha\tau} \xi' \tilde{p}_\infty \ + \ \lambda_3 e^{-2\alpha\tau} \xi' \tilde{P}_\infty \xi \ - \ e^{-2\alpha\tau} \tilde{p}_\infty' (B + \lambda_2 D)$$

$$\cdot \, [R + \lambda_2 D' \tilde{P}_\infty D]^{-1} (B + D)' \tilde{p}_\infty] \, d\tau$$

$$= \frac{1}{2\alpha} \ e^{-2\alpha t} \ [2\lambda_3 \xi' \tilde{p}_\infty \ + \ \lambda_3 \xi' \tilde{P}_\infty \xi$$

$$- \ \tilde{p}_\infty' (B + \lambda_2 D) \, (R + \lambda_2 D' \tilde{P}_\infty D)^{-1} (B + \lambda_2 D)' \tilde{p}_\infty]$$

$$\triangleq \ e^{-2\alpha t} \ \tilde{q}_\infty . \tag{4.9}$$

Thus,

$$| \, \tilde{q}_T(t) \ - \ e^{-2\alpha t} \tilde{q}_\infty \, |$$

$$\leq \left| \int_t^T \{2\lambda_3 e^{-\alpha\tau} \xi' [\tilde{p}_T(\tau) - e^{-\alpha\tau} \tilde{p}_\infty] \ + \ \lambda_3 e^{-2\alpha\tau} \xi' [\tilde{P}_T(\tau) - \tilde{P}_\infty] \xi \right.$$

$$- \ [\tilde{p}_T(\tau) - e^{-\alpha\tau} \tilde{p}_\infty]' \hat{B} \tilde{R}_T(\tau)^{-1} \hat{B}' \tilde{p}_T(\tau)$$

$$- e^{-\alpha\tau}\tilde{p}_{\infty}\hat{B}\tilde{R}_T(\tau)^{-1}[\tilde{R}_{\infty} - \tilde{R}_T(\tau)]\tilde{R}_{\infty}^{-1}\hat{B}'\tilde{p}_T(\tau)$$

$$- e^{-\alpha\tau}\tilde{p}_{\infty}\hat{B}\tilde{R}_{\infty}^{-1}\hat{B}'[\tilde{p}_T(\tau) - e^{-\alpha\tau}\tilde{p}_{\infty}]\}d\tau \Big| + e^{-2\alpha T}|\tilde{q}_{\infty}| . \qquad (4.10)$$

The last term in (4.10) tends to 0 as $T \to \infty$ which is independent of t. The integration in (4.10) over $[t, T]$ can be divided into two parts; the first integration over $[T - d_0, \mathrm{T}]$ can be made less than ϵ for each d_0 such that T is sufficiently large while the second integration over $[t, T - d_0]$ is less than $M_1\epsilon$ for some constant M_1, which is independent of t, since $||\tilde{P}_T(\tau) - \tilde{P}_{\infty}|| < \epsilon$, $||\tilde{R}_T(\tau) - \tilde{R}_{\infty}|| < \epsilon$, and $|\tilde{p}_T(\tau) - e^{-\alpha\tau}\tilde{p}_{\infty}| < \epsilon$ for all $\tau \in [0, T - d_0(\epsilon)]$ with some $d_0(\epsilon)$. Thus, (4.10) tends to 0 uniformly on $[0, T]$ as $T \to \infty$. Hence,

$$\lim_{T\to\infty} J_T(\tilde{u}_T) = \lim_{T\to\infty} [x_0'\tilde{P}_T(0)x_0 + 2\tilde{p}_T(0)'x_0 + \tilde{q}_T(0)]$$

$$= x_0'\tilde{P}_{\infty}x_0 + 2\tilde{p}_{\infty}'x_0 + \tilde{q}_{\infty}$$

$$\triangleq J^*. \qquad (4.11)$$

If $\exists$ u such that $J_d(\mathrm{u}) < J^*$, then for large T, we have $J_T(u) < J_T(\tilde{u}_T)$ which contradicts $J_T(\tilde{u}_T)$ being the optimal value for the problem restricted on $[0, T]$, so J^* must be the optimal value.

As before, we can also prove $\tilde{x}_T(t) \to \tilde{x}_{\infty}(t)$ as $T \to \infty$ and

$$\tilde{u}_T(t) = -\tilde{K}_T(t)\tilde{x}_T(t) - \tilde{k}_T(t)$$

$$\underset{T\to\infty}{\longrightarrow} -\tilde{K}_{\infty}\tilde{x}_{\infty}(t) - e^{-\alpha t}\tilde{k}_{\infty} \triangleq \tilde{u}_{\infty}(t)$$

where

$$\tilde{K}_{\infty} = [R + \lambda_2 D'\tilde{P}_{\infty}D]^{-1}(B + \lambda_2 D)'\tilde{P}_{\infty}$$

$$\tilde{k}_{\infty} = [R + \lambda_2 D'\tilde{P}_{\infty}D]^{-1}(B + \lambda_2 D)'\tilde{p}_{\infty}. \qquad (4.12)$$

The convergence is uniform on $[0, T - d_0]$ as $T \to \infty$ for some $d_0 > 0$. Thus the convergence of $x_T(t) \to x_{\infty}(t)$ and that of

$$u_T(t) = -\tilde{K}_T(t)x_T(t) - e^{\alpha t}\tilde{k}_T(t)$$

$$\underset{T \to \infty}{\longrightarrow} \; - \tilde{K}_{\infty} x_{\infty}(t) - \tilde{k}_{\infty} \triangleq u_{\infty}(t) \tag{4.13}$$

are uniform on each bounded interval. Thus,

$$| J_T(u_T) - J_T(u_{\infty}) |$$

$$= \left| \int_0^T e^{-2\alpha t} \{ [x_T(t)' Q \, x_T(t) - x_{\infty}(t)' Q \, x_{\infty}(t)] \right.$$

$$\left. + [u_T(t)' R \, u_T(t) - u_{\infty}(t)' R \, u_{\infty}(t)] \} dt \right|$$

$$\leq E \left| \int_0^T e^{-2\alpha t} \{ [x_T(t) - x_{\infty}(t)]' Q \, x_T(t) + x_{\infty}(t)' Q \, [x_T(t) - x_{\infty}(t)] \right.$$

$$\left. + [u_T(t) - u_{\infty}(t)]' R \, u_T(t) + u_{\infty}(t)' R \, [u_T(t) - u_{\infty}(t)] \} dt \right| . \tag{4.14}$$

Since the integrand of (4.14) is uniformly bounded in $0 \leq t \leq T$, then for each $\epsilon > 0$, we can find a sufficiently large $d > 0$ such that the integrand of (4.14) integrated over $[d, T]$ can be made less than $\epsilon M_2(\omega)$ for all $T > 0$. For this bounded interval $[0, d]$, there exists T_0, such that

$$| x_T(t) - x_{\infty}(t) | < \epsilon \quad \text{and} \quad | u_T(t) - u_{\infty}(t) | < \epsilon \quad a.s.$$

for all $t \in [0, d]$ and $T \geq T_0$, so that the integrand of (4.14) integrated over $[0, d]$ is less that $\epsilon M_3(\omega)$ for some $M_3(\omega) > 0$. Hence (4.14) can be made small as $T \to \infty$, so that

$$J_d(u_{\infty}) = \lim_{T \to \infty} \tilde{J}_T(u_{\infty}) = \lim_{T \to \infty} \tilde{J}_T(u_T)$$

which shows that u_{∞} in (4.13) is the optimal control.

We summarize these results as follows.

Theorem 4.1. *In the discounted cost case, if $(A + \lambda_1 C - \alpha I, \; B + \lambda_2 D)$ is stabilizable, $(Q^{1/2}, \; A + \lambda_1 C - \alpha I)$ is observable and condition* (I) *in section* 2 *holds with $(A + \lambda_1 C - \alpha I)$ instead of $\hat{A}$, then the optimal control exists and is of the form*

$$u_\infty(t) = -\tilde{K}_\infty x_\infty(t) - \tilde{k}_\infty \qquad \textbf{(4.15)}$$

where $\tilde{K}_\infty$ *and* $\tilde{k}_\infty$ *are defined in* (4.12). *The optimal discounted cost is*

$$J_d(u_\infty) = x_0{}' \tilde{P}_\infty x_0 + 2\tilde{p}_\infty{}' x_0 + \tilde{q}_\infty \qquad (4.16)$$

where $\tilde{P}_\infty$, $\tilde{P}_\infty$ *and* $\tilde{q}_\infty$ *are defined in* (4.7), (4.8) *and* (4.9) *respectively.*

Remark. The average cost criterion measures the long run performance on the average. It neglects the behavior of the system over any finite interval while the discount cost criterion emphasizes the initial performance, in particular, the initial condition x_0 as in (4.16). However, the optimal control involves a time-invariant linear feedback control and a stationary feed-forward control in both situations.

References

[1] Wonham, W.M., *Optimal stationary control of a linear system with state-dependent noise,* SIAM J. Control, Vol. 5, No. 3, 1967, pp. 486-500.

[2] Wonham, W.M., *On a matrix Riccati equation of stochastic control,* SIAM J. Control, Vol. 6, No. 4, 1968, pp. 681-697.

[3] Wonham, W.M., *Random differential equations in control theory,* Probabilistic Methods in Applied Mathematics, Vol. 2, Edited by A.T. Bharucha-Reid, Academic Press, New York, New York, 1970, pp. 131-212.

[4] Haussmann, U. G., *Optimal stationary control with state and control dependent noise,* SIAM J. Control, Vol. 9, No. 2, 1971, pp. 184-198.

[5] Ahmed, N. U., *Optimal control of stochastic systems,* Probabilistic Analysis and Related Topics, Vol. 2, Edited by A.T. Bharucha-Reid, Academic Press, New York, New York, 1979, pp. 1-68.

[6] Rishel, R., *Dynamic programming and minimum principles for systems with jump Markov disturbances,* SIAM J. Control, Vol. 13, No. 2, 1975, pp. 338-371.

[7] Rishel, R, *Control of systems with jump Markov disturbances,* IEEE Trans. Automatic Control, Vol. AC-11, 1975, pp. 241-244.

[8] Li, C.W. and Blankenship, G.L., *Optimal stochastic scheduling of systems with Poisson noise disturbances,* submitted for publication.

[9] Li, C.W. and Blankenship, G.L., *Almost sure stability of linear stochastic systems with Poisson process coefficients,* submitted for publication.

[10] Li, C.W., *Almost Sure Stability, Optimal Control and Scheduling of Stochastic Systems with Point Process Coefficients,* University of Maryland, College Park, Applied Mathematics Program, Ph.D. Dissertation, June 1984.

[11] Robin, M., *Long-term average cost control problems for continuous time Markov processes: A survey,* Acta Applicandae Mathematicae, Vol. 1, 1983, pp. 281-299.