SRC TR 85-15

On The Stability of Multiple Time-Scale Systems

by

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INT. J. CONTROL, 1986, VOL. 00, NO. 0, 0000–0000 Paper No. 85042

On the stability of multiple time-scale systems

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The stability of time-invariant multiparameter singular perturbation problems is considered and the implications of two time-scale stability results for multiple time-scale systems are clarified. An example shows that the asymptotic stability of a multiparameter singular perturbation problem under the 'bounded mutual ratios' assumption for arbitrary bounds on the ratios of the small parameters does not imply asymptotic stability under the multiple time scales assumption for any ordering of the smallness of the parameters. However, this conclusion does apply when only two small parameters are present and the fast variables are scalar-valued. A multiparameter singularly perturbed system may be asymptotically stable for all sufficiently small (and positive) values of the perturbation parameters, even though the boundary layer system does not satisfy the D-stability criterion. These examples are discussed in the light of the 'strong D-stability' condition which must be imposed to obtain results that are robust to small perturbations in the model. Necessary and sufficient conditions for robustness of a stability property that holds for all sufficiently small values of the singular perturbation parameters are given.

1. Introduction

This paper resolves several questions regarding the asymptotic stability of time-invariant multiparameter singular perturbation problems of the form

$$\dot{x} = Ax + By \tag{1 a}$$

$$\varepsilon_i \dot{y}_i = C_i x + D_i y, \quad i = 1, \dots, M \tag{1 b}$$

Here $x \in \mathbb{R}^n$, $y = (y_1, \dots, y_M) \in \mathbb{R}^m$, $y_i \in \mathbb{R}^{m_i}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_M)$ with each $\varepsilon_i > 0$ a small parameter, A, B, C_i, D_i are real matrices of appropriate dimension, and the dot denotes differentiation with respect to time t.

Recall (Abed 1985 b, Khalil and Kokotovic 1979 b and Ladde and Siljak 1983) that (1) possesses two time scales if the ratios $\varepsilon_i/\varepsilon_j$ are bounded, and it possesses multiple time scales otherwise. The multiple time-scale case is typified by the assumption (Tokhonov 1952, Wasow 1965, Hoppensteadt 1971 and O'Malley 1969) $\varepsilon_{i+1}/\varepsilon_i \to 0$ as $\varepsilon_i \to 0$, $i = 1, \dots, M-1$. In the sequel, it will be convenient to refer to systems with two time scales by stating that they possess 'bounded mutual ratios.' In this way no confusion will arise when the bounds on the ratios of the small parameters are discussed.

The following three questions are resolved in this paper.

Question 1

Does the asymptotic stability of (1) under the bounded mutual ratios assumption for arbitrary bounds on the ratios $\varepsilon_i/\varepsilon_j$ imply asymptotic stability even under the

Received 4 November 1985.

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1. Introduction

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$$\dot{x} = Ax + By \tag{1a}$$

$$\epsilon_i \dot{y}_i = C_i x + D_i y, \quad i = 1,\ldots,M. \tag{1b}$$

Here $x \in R^n$, $y = (y_1, \ldots, y_M) \in R^m$, $y_i \in R^{m_i}$, $\epsilon = (\epsilon_1, \ldots, \epsilon_M)$ with each $\epsilon_i > 0$ a small parameter, A, B, C_i, D_i are real matrices of appropriate dimension, and the dot denotes differentiation with respect to time t.

Recall [2, 5, 7] that (1) possesses two time scales if the ratios ϵ_i / ϵ_j are bounded, and it possesses multiple time scales otherwise. The multiple time-scale case is typified by the assumption [11, 13, 3, 8] $\epsilon_{i+1}/\epsilon_i \to 0$ as $\epsilon_i \to 0$, $i = 1,\ldots,M-1$. In the sequel it will be convenient to refer to systems with two time scales by stating that they possess 'bounded mutual ratios.' In this way no confusion will arise when the bounds on the ratios of the small parameters are discussed.

The following three questions are resolved in this paper:

Question 1. Does the asymptotic stability of (1) under the bounded mutual ratios assumption for arbitrary bounds on the ratios ϵ_i / ϵ_j imply asymptotic stability even under the multiple time scales hypothesis, for arbitrary ordering of the smallness of parameters?

Question 2. Is the so-called block D-stability condition necessary for Eq. (1) to be asymptotically stable for all sufficiently small $|\epsilon|$, $\epsilon_i > 0$, $i = 1,\ldots,M$, regardless of the ratios ϵ_i / ϵ_j ?

Question 3. What are the necessary and sufficient conditions for the robustness of the stability property of Question 2 to small perturbations in the matrices A, B, C_i, D_i ?

The difficulty one encounters in Question 1 is that the upper bounds on the ϵ_i guaranteeing stability might vanish in the limit that the upper and lower bounds on the ratios of the ϵ_i approach ∞ and 0, respectively. Question 1 is resolved by presenting an example. It turns out that the answer to Question 1 is negative except in the case $M = m = 2$. This observation shows, in particular, that general results on systems (1) containing two small parameters ϵ_1, ϵ_2 need *not* apply to systems with three or more small parameters. Question 2 is the easiest of the questions above to resolve and is included mainly as motivation for Question 3, whose resolution depends on the recently proposed concept of strong D-stability [1].

Khalil and Kokotovic [4] have also considered the relationship between the stability properties of system (1) under the bounded mutual ratios assumption and the multiple time scales hypothesis. They obtained explicit conditions under which a certain block D-stability condition which implies stability assuming bounded mutual ratios also implies stability in the multiple time-scale set-up. The results on Question 1 given below indicate that some such additional conditions are indeed necessary to guarantee stability in the multiple time scales setting

As in [2], it is useful to define matrices C and D by

$$C = \text{block diag}\,(C_1,\ldots,C_M) \tag{2a}$$

$$D = \text{block col}\,(D_1,\ldots,D_M) \tag{2b}$$

and rewrite (1) as

$$\dot{x} = Ax + By \tag{3a}$$

$$E(\epsilon)\dot{y} = Cx + Dy \tag{3b}$$

where $E(\epsilon) :=$ block diag $(\epsilon_1 I_{m_1},\ldots,\epsilon_M I_{m_M})$.

In the sequel $\sigma(F)$ for a square matrix F denotes the spectrum or set of eigenvalues of F, and R_+^M denotes the positive orthant of R^M, i.e. the set $\{\epsilon \in R^M : \epsilon_i > 0,\ i = 1,\ldots,M\}$. It will at times be convenient to use the notation $A_0 := A - BD^{-1}C$, and to denote the Jacobian matrix of (3) by $J(\epsilon)$, i.e.

$$J(\epsilon) := \begin{bmatrix} A & B \\ E^{-1}(\epsilon)C & E^{-1}(\epsilon)D \end{bmatrix}. \tag{4}$$

2. Background

In this section several theorems on the stability of multiparameter singular perturbation problems are recalled. The first follows from a time-varying version proved by Khalil and Kokotovic [5]. It applies under the bounded mutual ratios assumption.

Theorem 1. *Let H denote the set*

$$H = \{\epsilon \in R_+^M : c_{ij} \le \frac{\epsilon_i}{\epsilon_j} \le C_{ij},\ i,j = 1,\ldots,M\} \tag{5}$$

where the c_{ij} and C_{ij} are fixed positive numbers. Then the null solution of (3) is asymptotically stable for all $\epsilon \in H$ with $|\epsilon|$ sufficiently small if: (i) the reduced system obtained by formally setting $\epsilon = 0$ is asymptotically stable, i.e.

$$\text{Re}\,\sigma(A - BD^{-1}C) < 0, \tag{6}$$

and (ii)

$$\text{Re}\,\sigma(E^{-1}(\epsilon)D) < 0 \tag{7}$$

for all $\epsilon \in H$.

Recall [10] that a square matrix F is said to be D-*stable* if for any diagonal matrix D with strictly positive diagonal elements, DF is stable. Assumption (*ii*) can be viewed as a special instance of the following definition [5].

Definition 1. The matrix $F \in R^{m \times m}$ is *block D-stable* (relative to the multi-index $(m_1,...,m_M)$) if for all $d_i > 0$, $i = 1,...,M$,

$$\operatorname{Re} \sigma(D(d)F) < 0 \tag{8}$$

where

$$D(d) := \text{block diag}(d_1 I_{m_1}, \ldots, d_M I_{m_M}). \tag{9}$$

Thus (ii) of Theorem 1 holds if D is block D-stable relative to the multi-index $(m_1,...,m_M)$. Before stating the next theorem it is useful to introduce the following terminology [1].

Definition 2. The matrix $F \in R^{m \times m}$ is *strongly D-stable* if (i) F is D-stable, and (ii) there is a $\mu > 0$ such that $F+G$ is D-stable for each $G \in R^{m \times m}$ with $|G| < \mu$.

Definition 2 is a special case of the following more general notion of 'strong block D-stability.' A general class of strongly block D-stable matrices has been identified in [1].

Definition 3. The matrix $F \in R^{m \times m}$ is *strongly block D-stable* (relative to the multi-index $\overline{m} := (m_1,...,m_M)$) if (i) F is block D-stable (relative to $\overline{m}$), and (ii) there is a $\mu > 0$ such that $F+G$ is block D-stable (relative to $\overline{m}$) for each $G \in R^{m \times m}$ with $|G| < \mu$.

The next theorem states that if the reduced system is stable and if D is strongly block D-stable, stability of the multiparameter singular perturbation problem (3) is guaranteed for all sufficiently small $|\epsilon|$, $\epsilon \in R_+^M$. Note the removal of the constraint (5) on the relative magnitudes of the singular perturbation parameters which was needed in Theorem 1. Thus, assuming *strong* block D-stability, a significant generalization is realized. Both the two time-scale setting and the multiple time-scale case are treated in the same framework.

Theorem 2. *Suppose that all eigenvalues of $A_0 = A - BD^{-1}C$ have strictly negative real parts, and let D be strongly block D-stable, relative to the multi-index $(m_1,...,m_M)$. Then there is a $\mu > 0$ such that the null solution of system* (1) *(or, equivalently, of* (3)*) is asymptotically stable for all $\epsilon = (\epsilon_1, \ldots, \epsilon_M)$ with $|\epsilon| < \mu$ and $\epsilon_i > 0$, $i = 1,...,M$.*

Theorem 2 is a special case of the following result.

Theorem 3. *Suppose A_0 is a stable matrix, and let the set $H \subset R_+^M$ be such that*

$$\operatorname{Re} \sigma(E^{-1}(\epsilon)D) < 0 \tag{10}$$

for all $\epsilon \in H$. Moreover, assume that (10) *also holds if D is replaced by any sufficiently small perturbation of D, for all $\epsilon \in H$. Then the null solution of* (3) *is asymptotically stable for all $\epsilon \in H$ with $|\epsilon|$ sufficiently small.*

The next lemma was introduced in [2]. It gives an algebraic matrix Riccati equation whose solution is useful in exhibiting a transformation which decouples the fast and slow modes of (3).

Lemma 1. *Suppose* $\det D \neq 0$ *and denote* $E = E(\epsilon)$, $A_0 := A - BD^{-1}C$. *Then the Riccati equation*

$$D\Gamma + E(D^{-1}C - \Gamma)A_0 - E\Gamma B\Gamma + ED^{-1}CB\Gamma = 0 \qquad (11)$$

for the $m \times n$ *matrix* Γ *has a locally unique solution* $\Gamma(\epsilon)$ *near* $0 \in R^{m \times n}$ *for* $|\epsilon|$ *sufficiently small.*

The next theorem gives an exact expression for the eigenvalues of $J(\epsilon)$ in terms of the eigenvalues of matrices associated with appropriate fast and slow subsystems of (3). This theorem was derived in [2] by using Lemma 1 to exhibit a similarity transformation rendering $J(\epsilon)$ in block upper triangular form. It will be the main tool in the stability analysis of the examples studied below.

Theorem 4. *Let* $|\epsilon|$ *be sufficiently small so that Lemma* 1 *applies. Then*

$$\sigma(J(\epsilon)) = \sigma(A - BD^{-1}C + B\Gamma(\epsilon)) \cup \sigma(E^{-1}(\epsilon)D + D^{-1}CB - \Gamma(\epsilon)B) \qquad (12)$$

if $\epsilon_i \neq 0$, $i = 1,\ldots,M$.

3. Stability of multiple time-scale systems

Condition (*ii*) of Theorem 1 holds if D is block D-stable. Hence if D is block D-stable and the eigenvalues of A_0 all have negative real parts, asymptotic stability of the multiparameter singularly perturbed system (3) is certain for all sufficiently small $|\epsilon|$, $\epsilon \in H$. Indeed, this statement applies for *any* set H of the form (5), i.e. for arbitrary bounds $0 < c_{ij} < C_{ij} < \infty$. In the limit that $c_{ij} \to 0$ and $C_{ij} \to \infty$ the set H approaches the positive orthant R_+^M.

These considerations might lead one to assert that if, for any set H of the form (5) (i.e., for arbitrary c_{ij}, $C_{ij} > 0$), the multiparameter problem (3) is asymptotically stable for all $\epsilon \in H$ with $|\epsilon|$ sufficiently small, then (3) is also asymptotically stable under the multiple time scales hypothesis, for all possible orderings of the smallness of the parameters $\epsilon_1, \ldots, \epsilon_M$. Unfortunately the claim is false. The claim fails because the upper bound on $|\epsilon|$ guaranteeing stability may vanish in the limit $c_{ij} \to 0$ and $C_{ij} \to \infty$, $i,j = 1,\ldots,M$.

Recall that the authors of [4] obtain explicit conditions under which block D-stability also implies stability in the multiple time-scale set-up. In the light of the first example given below, it is clear that some such additional condition on the system is necessary to guarantee stability in the multiple time-scale setting. Another approach is to invoke Theorem 2 using strong D-stability. Now on to the example.

Consider a multiparameter singular perturbation problem (3) with $n = 3$, $m = M = 3$, and with matrices A, B, C, D given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

$$C = I_3 , \qquad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{13}$$

The matrix D is easily checked to be D-stable. However D is not strongly D-stable, since for any $\mu > 0$ the perturbed matrix

$$D_\mu = \begin{pmatrix} \mu & 0 & 1 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{14}$$

is not D-stable. To see this, note that the characteristic polynomial of $\Theta(\theta) D_\mu$, where $\Theta(\theta)$ is a diagonal matrix diag $(\theta_1, \theta_2, \theta_3)$, $\theta_i > 0$, $i = 1,2,3$, is

$$\lambda^3 + (\theta_2 - \theta_1 \mu) \lambda^2 + (\theta_1 \theta_3 - \theta_1 \theta_2 \mu + \theta_2 \theta_3) \lambda$$
$$+ (1 - \mu) \theta_1 \theta_2 \theta_3 . \tag{15}$$

Clearly the coefficient of λ^2 can be made nonpositive by a suitable choice of θ_1, $\theta_2 > 0$, for any $\mu > 0$.

Use Theorem 4 to approximate the eigenvalues of the system for small $|\epsilon|$: one has $A - BD^{-1}C = -I_3$, so that the three slow eigenvalues are $O(|\epsilon|)$ close to -1, and

$$D^{-1}CB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

implying the fast eigenvalues are $O(|\epsilon|)$ close to the eigenvalues of

$$K(\epsilon) := E^{-1}(\epsilon) D + D^{-1}CB = \begin{pmatrix} 1 & 0 & \epsilon_1^{-1} \\ 0 & -\epsilon_2^{-1} & \epsilon_2^{-1} \\ -\epsilon_3^{-1} & -\epsilon_3^{-1} & 1 \end{pmatrix}. \tag{17}$$

The characteristic polynomial $p_K(\lambda)$ of $K(\epsilon)$ may be computed as

$$p_K(\lambda) := \lambda^3 + (\frac{1}{\epsilon_2} - 2) \lambda^2 + (1 - \frac{2}{\epsilon_2} + \frac{1}{\epsilon_1 \epsilon_3} + \frac{1}{\epsilon_2 \epsilon_3}) \lambda$$
$$+ (\frac{1}{\epsilon_2} - \frac{1}{\epsilon_2 \epsilon_3} + \frac{1}{\epsilon_1 \epsilon_2 \epsilon_3}). \tag{18}$$

Recall that all the roots of a cubic polynomial $\lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0$ have negative real parts if and only if $p_i > 0$, $i = 0,1,2$ and $p_1 p_2 > p_0$. The coefficients of $p_K(\lambda)$ will clearly be positive for all sufficiently small positive $\epsilon_1, \epsilon_2, \epsilon_3$. Now the condition $p_1 p_2 > p_0$ is easily checked to translate into the inequality

$$\frac{1}{\epsilon_2^2 \epsilon_3} + \frac{4}{\epsilon_2} > 2 + \frac{2}{\epsilon_1 \epsilon_3} + \frac{1}{\epsilon_2 \epsilon_3} + \frac{2}{\epsilon_2^2}. \tag{19}$$

Multiplication of (19) by $\epsilon_1 \epsilon_2$ gives the equivalent condition

$$4\epsilon_1 + \frac{1}{\epsilon_2} (\frac{\epsilon_1}{\epsilon_3}) > 2\epsilon_1 \epsilon_2 + 2\frac{\epsilon_1}{\epsilon_2} + \frac{\epsilon_1}{\epsilon_3} + 2\frac{\epsilon_2}{\epsilon_3}. \tag{20}$$

Clearly, inequality (20) can be satisfied given any (finite) bounds on the ratios ϵ_i / ϵ_j by choosing ϵ_2 sufficiently small. However, it is not satisfied for all $\epsilon_1,\epsilon_2,\epsilon_3 > 0$ with $\epsilon_i < \mu$ for any $\mu > 0$. This can be seen by noting that the left side of inequality (19) does not depend on ϵ_1. Thus, even though the left side of (19) is of higher order than the right side, it can not be made larger than the right side for all sufficiently small $|\epsilon|$. To see that even under the multiple time-scale set-up this system need not be stable, choose the ordering

$$\epsilon_1 = \epsilon_2^{\,3}, \quad \epsilon_3 = \epsilon_2^{\,2} \tag{21}$$

of the small parameters. Denoting ϵ_2 by δ for simplicity, (19) now becomes

$$\frac{1}{\delta^4} + \frac{4}{\delta} > 2 + \frac{2}{\delta^5} + \frac{1}{\delta^3} + \frac{2}{\delta^2}. \tag{22}$$

Since the right side of (22) is larger than the left side for all sufficiently small values of δ, clearly the multiple time-scale problem for this example with the given ordering of smallness of parameters is *unstable* for all sufficiently small ϵ_2!

Remark 1. In the analysis of this example, terms in the characteristic polynomial of order $O(|\epsilon|)$ have been justifiably disregarded, as they can not change the stability of the system for sufficiently small $|\epsilon|$ unless one is dealing with a case in which the analysis shows the system is marginally stable. The strictness of satisfaction of the inequalities above (or of their opposites in case of instability) shows that no marginal stability or marginal instability arises here.

Remark 2. One can easily check that it is *not* possible to construct an example with only two small parameters ϵ_1, ϵ_2 and scalar fast variables y_1, y_2 which yields the same conclusion as the foregoing example. Simply use Theorem 4 and the fact that a monic quadratic polynomial is stable precisely when the coefficients are positive. Hence the answer to Question 1 is yes only for $M = m = 2$.

4. D-Stability is not necessary for stability

From the results stated in Section 2 one might suspect that *strong* D-stability is a necessary condition for stability of system (3), if it is required that stability holds for all sufficiently small $|\epsilon|$, $\epsilon_i > 0$, $i = 1,\ldots,M$ regardless of the ratios ϵ_i/ϵ_j. The next example shows that such is not the case, and moreover that the stated property might hold even if the milder D-stability condition does not apply.

Consider Eq. (3) where

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; D^{-1}CB = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \tag{23}$$

and where A is chosen such that $A - BD^{-1}C$ is stable. Since D has a pair of pure imaginary eigenvalues $\pm i$, it is clearly not D-stable. The 'fast eigenvalues' of (3) will by Theorem 4 be $O(|\epsilon|)$ close to the roots of the characteristic polynomial of $E^{-1}(\epsilon)D + D^{-1}CB$, which is

$$\lambda^2 + \lambda + \frac{1}{\epsilon_1\epsilon_2}. \tag{24}$$

The coefficients in the characteristic polynomial are positive for all sufficiently small positive values of ϵ_1, ϵ_2.

Thus for the present example system (3) is stable for all sufficiently small $|\epsilon|$, $\epsilon_i > 0$, $i = 1,\ldots,M$, even though D is *not* strongly D-stable, indeed not even D-stable.

5. Robust stability

Two examples have been presented resolving questions on the relationship of two time-scale and multiple time-scale singular perturbation problems. Although both examples are self-explanatory, some remarks regarding the second example are appropriate. This example shows that D-stability is *not* a necessary condition for the asymptotic stability of the multiparameter singular perturbation problem (3) for all sufficiently small values of the small parameters, regardless of their relative magnitudes. This at first glance is surprising since the first example did not possess this desirable stability property, even though it did satisfy the D-stability hypothesis. The resolution of this issue lies in the realization that it is often important to study the *robustness* of stability properties under small perturbations of the system model. With this viewpoint, the stability property of the system of the second example does not persist under small perturbations of the matrix D, since the perturbed D matrix may possess an eigenvalue with positive real part. A similar statement holds for the first example since for some ϵ the matrix $E^{-1}(\epsilon)D$ may have positive real part if a small perturbation is allowed in D.

This line of reasoning leads to the following necessary and sufficient condition for the robust stability of the multiparameter singular perturbation problem (3) for all sufficiently small $|\epsilon|$, $\epsilon_i > 0$, $i = 1,\ldots,M$.

Theorem 5. *A necessary and sufficient condition such that*

(i) The null solution of the multiparameter singular perturbation problem (3) *is asymptotically stable for all sufficiently small* $|\epsilon| \in R_+^M$, *and*

(ii) Property (i) holds for all sufficiently small perturbations of the matrices A, B, C, D.

is that A_0 *be a stable matrix and* D *be strongly block* D*-stable.*

Acknowledgment

The authors are grateful to the reviewer for his important and insightful suggestions.

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