SRC TR 85-11-r1

Local Feedback Stabilization and Bifurcation Control, I. Hopf Bifurcation

by

E.H. Abed and J.-H. Fu

Systems & Control Letters 7 (1986) 11–17
North-Holland

February 1986

Local feedback stabilization and bifurcation control, I. Hopf bifurcation *

Eyad H. ABED and Jyun-Horng FU

Department of Electrical Engineering and Systems Research Center, University of Maryland, College Park, MD 20742, USA

Received 10 July 1985
Revised 30 October 1985

Abstract: Local bifurcation control problems are defined and employed in the study of the local feedback stabilization problem for nonlinear systems in critical cases. Sufficient conditions are obtained for the local stabilizability of general nonlinear systems whose linearizations have a pair of simple, nonzero imaginary eigenvalues. The conditions show, in particular, that generically these nonlinear critical systems can be stabilized locally, even if the critical modes are uncontrollable. The analysis also yields a direct method for computing stabilizing feedback controls. Use is made of bifurcation formulae which require only a series expansion of the vector field.

Keywords: Control systems, Stabilization, Bifurcation, Hopf bifurcation, Nonlinear systems.

1. Introduction

Recently several authors have addressed the question of stabilizability of nonlinear control systems and its relation to controllability. Sussmann [20] shows that under very general assumptions a reasonable notion of controllability for areal analytic control system implies the existence of a piecewise analytic feedback control steering all points in the state space to any given point. He showed by example that an analytic stabilizing feedback control may fail to exist. Brockett [4] obtains both necessary conditions and sufficient conditions for the existence of a smooth feedback which renders an equilibrium point of a nonlinear system locally asymptotically stable. Although the results of [4] are applicable to general nonlinear control systems $\dot{x} = f(x, u)$, verification of the sufficiency conditions in specific examples is complicated by the need to construct a Liapunov function. It is noted in [4] and [3] that the only interesting situation is that in which the linearized system $\dot{x} = Ax + Bu$ has uncontrollable modes with zero real part and no unstable uncontrollable modes. This is the only case for which the linear theory is inadequate. Stability problems for systems in which the linearization has some eigenvalues with zero real part and the remaining eigenvalues with negative real part are referred to as *critical cases* in stability. Thus only the critical cases are interesting in questions of local stabilizability of nonlinear systems. Aeyels [3] studied this same problem for a class of critical nonlinear systems using center manifold reduction and a standard stability computation for the Hopf bifurcation in two dimensions. Although the results on stabilizability of a certain class of systems reported in [3] are useful and easy to apply, obtaining generalized results using this approach may prove difficult since the analysis depends on the computation of a center manifold. The series expansion of the center manifold in turn depends on the feedback control. This leads to a measure of trial and error in determining sufficient conditions for stabilizability. These remarks remain valid if one attempts a reduction of dimension via the method of Liapunov–Schmidt [7] and then applies reasoning analogous to that of [3].

This paper has two main goals. The first is to indicate the connection between local stabilization in critical cases and a seemingly distinct problem, that of (local) *bifurcation control*. This connection becomes transparent given some basic facts about bifurcations of equilibria of differential equations, and provided that a clear definition of local bifur-

* Research supported in part by the National Science Foundation under Grants ECS-84-04275 and OIR-85-00108, by a grant from the Minta Martin Fund for Aeronautical Research, Univ. of Maryland, and by the Power Engineering Program, Univ. of Maryland, through grants from BG&E, PEPCO, VEPCO, Bechtel Corp. and the General Electric Foundation.

cation control problems is given. The point to be noted here is that often results on stabilization in critical cases can be directly applied also to problems in the control of bifurcations. Correspondingly, local bifurcation control problems provide added motivation for the study of stabilization in critical cases. The connection is established here for the case of Hopf bifurcation; the case of stationary bifurcation will be considered elsewhere [8]. The second goal of this work is to obtain generally applicable results for the case in which the linearized system's state dynamics matrix possesses a pair of simple, pure imaginary eigenvalues. The case of a simple zero eigenvalue will be treated in a forthcoming paper [8] by viewing it as a problem of controlling a double-point bifurcation. The case of two pure imaginary eigenvalues is treated here as a Hopf bifurcation control problem. Although the work of [3] in some ways resembles the present approach, the former involves a preliminary state transformation and a reduction to a center manifold, neither of which is needed here. Moreover, the present approach allows the derivation of generally valid analytical criteria for stabilizability as well as specific stabilizing feedback controls. This is possible through use of bifurcation formulae which involve only Taylor series expansion of the vector field and eigenvector computations. These formulae are derived in [12,13,14] by appealing to the Fredholm Alternative. They significantly simplify similar formulae obtained by Hopf [11].

It is appropriate to note that work on bifurcation control has been reported by Mehra [18] and Mehra, Kessel and Carroll [17]. See also the account in Casti [6]. These results tend to be concerned with the problem of globally *removing* bifurcations by state feedback. They apply only to stationary bifurcations, since they are obtained by appealing to a global implicit function theorem. This differs markedly from the *local* bifurcation control problems considered here and in [8], in which one seeks only to modify the stability properties of the bifurcated solutions.

2. The local feedback stabilization problem

Consider the nonlinear control system

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in R^n$ is the state, $u \in R$ is the control, and f is smooth in x and u. A scalar control has been assumed for simplicity though the case of a vector control is easily handled by the same methods. Suppose that the origin is an equilibrium point of (1) in the absence of a control effort, i.e. $f(0, 0) = 0$. For this system, the *local smooth feedback stabilization problem* is to find a smooth feedback $u = u(x)$ with $u(0) = 0$ such that the origin is a locally asymptotically stable equilibrium point of the controlled system $\dot{x} = f(x, u(x))$.

The linearization of Eq. (1) at $x = 0$, $u = 0$ is given by

$$\dot{x} = A_0 x + bu \tag{2}$$

where

$$A_0 := \frac{\partial f}{\partial x}(0, 0) \quad \text{and} \quad b := \frac{\partial f}{\partial u}(0,0).$$

If the pair (A_0, b) is controllable, then a standard linear systems result asserts the existence of a linear feedback $u = -kx$ such that the resulting system $\dot{x} = (A_0 - bk)x$ is asymptotically stable. Applying this feedback in the original nonlinear system (1) renders the origin locally asymptotically stable. Moreover, the same conclusion applies if the uncontrollable modes of (2) are asymptotically stable. In contrast, if (A_0, b) has an unstable uncontrollable mode, then the origin of (1) remains unstable regardless of the applied feedback. These same considerations were used in [4] and [3] to determine that the only interesting and essentially nonlinear situation encountered in local feedback stabilization occurs when some uncontrollable modes of (2) are pure imaginary, and any other uncontrollable modes are asymptotically stable.

In the light of the foregoing discussion, it is appropriate to assume that the matrix A_0 of Eq. (2) possesses at least one eigenvalue with zero real part. These eigenvalues should correspond to uncontrollable modes of (2). The type of results obtained will depend heavily on the number of eigenvalues of A_0 which are assumed pure imaginary, their multiplicity, and whether they are zero or have nonzero imaginary parts. The results of this paper will apply in case the following hypothesis is satisfied.

(H) The matrix $A_0 = (\partial f/\partial x)(0,0)$ has a pair of simple, complex conjugate eigenvalues $\lambda_1 = i\omega_c$, $\lambda_2 = -i\omega_c$ on the imaginary axis, where $\omega_c \neq 0$. Moreover, all other eigenvalues of A_0 have negative real part.

In studying the local feedback stabilization problem, it will be convenient to view the system (1) as resulting from a one-parameter family of systems

$$\dot{x} = f_\mu(x, u) \tag{3}$$

upon setting $\mu = 0$. Here μ is an auxiliary real parameter, and the dependence of f_μ on μ is smooth and such that for the eigenvalue $\lambda_1(\mu)$ which is the continuous extension of the critical eigenvalue λ_1 of (H) above, Re $\lambda_1'(0) \neq 0$. This requirement along with satisfaction of (H) implies that Eq. (3) undergoes a Hopf bifurcation from the origin at $\mu = 0$. This means that a nonconstant periodic solution of Eq. (3) emerges from the origin for μ near zero [16,10,7,14].

It should be stressed that the parameter dependence in Eq. (3) need not be artificial. In many applications, one is interested in the qualitative dependence of Eq. (3) on an actual physical parameter μ. For instance, one might seek a feedback control $u(x)$ which stabilizes the bifurcated periodic solution of (3) for all sufficiently small values of $|\mu|$. A main observation of this paper is that this *local Hopf bifurcation control problem* and the local feedback stabilization problem defined above can be resolved by the same analysis.

To set the framework and notation of the paper, it is useful to review briefly some basic results from bifurcation analysis. In the next section the Hopf Bifurcation Theorem and an associated stability computation [12,13] are recalled. In Section 4 the connection between local feedback stabilization and local bifurcation control under hypothesis (H) is made precise, and the results of Section 3 are used to obtain stabilizability conditions for Eq. (1).

3. Hopf bifurcation and bifurcation formulae

In this section a stability formula from Hopf bifurcation theory is recalled. The formula was derived in [12,13]. It will first be necessary to fix the general setting and recall standard notation.

Consider a general one-parameter system of ordinary differential equations

$$\dot{x} = F_\mu(x) \tag{4}$$

where $F_0(0) = 0$ and F is smooth in x, μ. Suppose that hypothesis (H) applies to this system, i.e. that $D_x F_0(0)$ has a simple pair of nonzero pure imaginary eigenvalues $\pm i\omega_c$, with all other eigenvalues in the open left half complex plane. From (H) and the Implicit Function Theorem, it follows that for $|\mu|$ sufficiently small, Eq. (4) has a locally unique equilibrium point $x_0(\mu)$ near 0. The Jacobian matrix of (4) evaluated along this equilibrium path is

$$A(\mu) := \frac{\partial F_\mu}{\partial x}(x_0(\mu)). \tag{5}$$

Note that $A(0) = A_0$ and that $A(\mu)$ depends smoothly on μ. Thus $A(\mu)$ possesses a complex conjugate pair of simple eigenvalues $\lambda(\mu)$, $\overline{\lambda(\mu)}$ which depend smoothly [19] on μ and such that $\lambda(0) = i\omega_c$ (cf. (H)). Denote the real and imaginary parts of $\lambda(\mu)$ by $\alpha(\mu)$ and $\omega(\mu)$, respectively.

Under hypothesis (H), and assuming $\alpha'(0) \neq 0$, the Hopf Bifurcation Theorem asserts the existence of a one-parameter family $\{p_\varepsilon, 0 < \varepsilon \leq \varepsilon_0\}$ of nonconstant periodic solutions of Eq. (4) emerging from $x = 0$ at $\mu = 0$. Here ε is a measure of the amplitude of the periodic solutions and ε_0 is sufficiently small. The periodic solutions $p_\varepsilon(t)$ have period near $2\pi\omega_c^{-1}$ and occur for parameter values μ given by a smooth function $\mu(\varepsilon)$. Exactly one of the characteristic exponents of p_ε is near 0, and is given by a real, smooth and even function

$$\beta(\varepsilon) = \beta_2\varepsilon^2 + \beta_4\varepsilon^4 + \cdots. \tag{6}$$

Moreover, $p_\varepsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\beta(\varepsilon) < 0$ but is unstable if $\beta(\varepsilon) > 0$. Denote by β_{2K} the first nonvanishing coefficient in the expansion (6). Checking the sign of β_{2K} is sufficient for determining stability. Generically, $K = 1$ so that locally the stability of the bifurcated periodic solutions p_ε is typically decided by the sign of the coefficient β_2.

An algorithm for the computation of β_2 can be useful in the solution of local feedback stabilization problems under hypothesis (H). In [3] the evaluation of (a scaled version of) β_2 is performed using a formula which applies to two-dimensional systems. The original n-dimensional system is reduced to a two-dimensional system by appealing to the Center Manifold Theorem. Use is then made of the fact [5,7] that the stability properties of an equilibrium on the center manifold coincide with its stability in R^n. In fact, the value of $\beta(\varepsilon)$ is known [16,10,1] to be the same for the original and the reduced systems. The approach taken in this paper differs from that of [3] mainly in the choice of algorithm for computing β_2. The implications for the type of results one obtains are nontrivial. Next an algorithm for computing β_2 is given; the discussion follows closely that of Howard [13].

Step 1. Suppose for simplicity that $x_0(\mu) \equiv 0$; this has no effect on the final formula for β_2. Express $F_\mu(x)$ in the series form

$$F_\mu(x) = L_0 x + Q(x, x) + C(x, x, x) + \cdots \tag{7}$$

where the terms not written explicitly are higher order in μ and x than those which are. In (7), Q is generated by a vector valued *symmetric* bilinear form $Q(x, y)$ giving the second order (in x) terms at $\mu = 0$, and C is generated by a vector valued *symmetric* trilinear form $C(x, y, z)$ giving the third order (in x) terms at $\mu = 0$. Let r be the right (column) and l the left (row) eigenvector of L_0 with eigenvalue $i\omega_c$. Normalize by setting the first component of r to 1 and then choose l so that $lr = 1$.

Step 2. Solve the equations

$$-L_0 a = \tfrac{1}{2} Q(r, \bar{r}), \tag{8}$$

$$(2i\omega_c I - L_0) b = \tfrac{1}{2} Q(r, r) \tag{9}$$

for a and b.

Step 3. The coefficient β_2 is

$$\beta_2 = 2\,\mathrm{Re}\{2lQ(r, a) + lQ(\bar{r}, b) + \tfrac{3}{4} lC(r, r, \bar{r})\}. \tag{10}$$

4. Control of Hopf bifurcations

Now suppose $\beta_2 \neq 0$. Besides locally determining the stability of the bifurcated periodic solutions $p_\varepsilon(t)$, it is known that the sign of the coefficient β_2 also *determines the stability of the equilibrium* $x_0(\mu)$ *at criticality* (i.e. at $\mu = 0$). This fact implies that if a feedback control $u = u(x)$ can be found such that $\beta_2 < 0$ for the Hopf bifurcation occurring in the controlled system

$$\dot{x} = f_\mu(x, u(x)) \tag{11}$$

then the local feedback stabilization problem described in Section 2 is solved. Simply use the feedback $u = u(x)$ in Eq. (1). Indeed, such a feedback solves *both* the local smooth feedback stabilization problem for Eq. (1) *and* the local Hopf bifurcation stabilization problem for *any* parametrized version of (1) of the form (11). This establishes the connection between local feedback stabilization and Hopf bifurcation control. Recall that the appearance of μ in (11) is artificial in the local feedback stabilization problem but is an integral part of the problem formulation in the case of local Hopf bifurcation control.

Summarizing, a sufficient condition for the existence of a solution to the local smooth feedback stabilization problem for Eq. (1) *and* the local Hopf bifurcation control problem for Eq. (11) is that there exist a feedback $u(x)$, $u(0) = 0$ such that Eq. (11) undergoes a Hopf bifurcation at $\mu = 0$ with $\beta_2 < 0$. The remainder of this section is devoted to uncovering an easily verifiable condition for this to be the case. The first step is to evaluate β_2 for the controlled system (β_2^*) and determine its relationship to β_2 for the uncontrolled system; this is pursued next. In what follows, starred quantities pertain to the controlled system.

Proceeding, rewrite Eq. (1) in the series form

$$\dot{x} = L_0 x + u\gamma + uL_1 x + Q_0(x, x) + u^2 L_2 x + uQ_1(x, x) + C_0(x, x, x) + \cdots \tag{12}$$

where the notation is similar to that in Section 3, but here the terms $u\gamma$, $uL_1 x$, $u^2 L_2 x$ and $uQ_1(x, x)$ have been included since they occur in the computation of β_2 for the controlled system. The vector γ in Eq. (12) corresponds to b in Eq. (2). Note that, in this context, one cannot in general suppose that $\gamma = 0$. This would result from requiring $f(0, u) \equiv 0$ rather than merely $f(0, 0) = 0$. Ensuring that this is the case is quite complicated since a u-dependent change of coordinates would be needed, and u is a *control* intended to depend on x. This is in contrast to the standard assumption in the *analysis* of Hopf bifurcations that, for Eq. (4), $F_\mu(0) \equiv 0$.

It is well known that only the quadratic and cubic terms occurring in a nonlinear system undergoing a Hopf bifurcation influence the value of β_2. This fact is clear from Eq. (10) for β_2. Thus only the linear, quadratic and cubic terms in an applied feedback $u(x)$ have potential for influencing β_2. Therefore the feedback control $u(x)$ may be assumed to be of the form

$$u(x) = c^{\mathrm{T}}x + x^{\mathrm{T}}Q_u x + C_u(x, x, x), \tag{13}$$

where c is a real column vector, Q_u is a real *symmetric* $n \times n$ matrix, and C_u is a cubic form generated by a scalar valued *symmetric* trilinear form. The closed loop dynamics, upon application of a feedback control u of the form (13), become

$$\dot{x} = L_0^* x + Q_0^*(x, x) + C_0^*(x, x, x) + \cdots \tag{14}$$

where the matrix L_0^*, the quadratic form $Q_0^*(x, x)$ and the cubic form $C_0^*(x, x, x)$ are

$$L_0^* = L_0 + \gamma c^{\mathrm{T}}, \tag{15a}$$

$$Q_0^*(x, x) = (x^{\mathrm{T}}Q_u x)\gamma + Q_0(x, x) + (c^{\mathrm{T}}x)L_1 x, \tag{15b}$$

and

$$C_0^*(x, x, x) = C_u(x, x, x)\gamma + C_0(x, x, x) + (c^{\mathrm{T}}x)^2 L_2 x + (c^{\mathrm{T}}x)Q_1(x, x) + (x^{\mathrm{T}}Q_u x)L_1 x. \tag{15c}$$

By Eq. (15a), L_0^* differs from L_0 only in case $\gamma c^{\mathrm{T}} \neq 0$. Thus it is convenient to set

$$c = 0 \tag{16}$$

to simplify comparing β_2 for the controlled and uncontrolled systems. This ensures that the critical eigenvalues and the left and right eigenvectors of L_0 and L_0^* needed to compute β_2 by Howard's algorithm are identical. (Of course, if the critical eigenvalues are assumed uncontrollable, then they will be the same for L_0 and L_0^*, regardless of c.)

Note that the trilinear form C_0^* of the controlled system is not necessarily symmetric in the form indicated by Eq. (15c), even with c set to 0. Fortunately, however, no such difficulty arises in the case of the bilinear form Q_0^*:

$$Q_0^*(x, y) = (x^{\mathrm{T}}Q_u y)\gamma + Q_0(x, y). \tag{17}$$

To render $C_0^*(x, y, z)$ symmetric, write

$$C_0^*(x, y, z) = C_u(x, y, z)\gamma + C_0(x, y, z) + \tfrac{1}{3}\{(y^{\mathrm{T}}Q_u z)L_1 x + (x^{\mathrm{T}}Q_u y)L_1 z + (z^{\mathrm{T}}Q_u x)L_1 y\}. \tag{18}$$

Apply Eq. (10) to the controlled system (14) to get

$$\beta_2^* = 2\,\mathrm{Re}\{2lQ_0^*(r, a^*) + lQ_0^*(\bar{r}, b^*) + \tfrac{3}{4}lC_0^*(r, r, \bar{r})\}, \tag{19}$$

where a^*, b^* are obtained using Eq. (17) and the algorithm of Section 3 as

$$a^* = -\tfrac{1}{2}(L_0^*)^{-1}Q_0^*(r, \bar{r}) = -\tfrac{1}{2}L_0^{-1}Q_0(r, \bar{r}) - \tfrac{1}{2}(r^{\mathrm{T}}Q_u\bar{r})L_0^{-1}\gamma \tag{20}$$

and

$$b^* = \tfrac{1}{2}(2i\omega_c^* I - L_0^*)^{-1}Q_0^*(r, r) = \tfrac{1}{2}(2i\omega_c I - L_0)^{-1}Q_0(r, r) + \tfrac{1}{2}(r^{\mathrm{T}}Q_u r)(2i\omega_c I - L_0)^{-1}\gamma. \tag{21}$$

Equations (17)–(21) now imply that β_2^* is given in terms of β_2 by

$$\beta_2^* = \beta_2 + 2\,\mathrm{Re}\,\Delta \tag{22}$$

where Δ is given by

$$\Delta = 2l\left[r^{\mathrm{T}}Q_u a^*\gamma - Q_0\left(r, \tfrac{1}{2}(r^{\mathrm{T}}Q_u\bar{r})L_0^{-1}\gamma\right)\right] + l\left[\bar{r}^{\mathrm{T}}Q_u b^*\gamma + Q_0\left(r, \tfrac{1}{2}(r^{\mathrm{T}}Q_u r)(2i\omega_c I - L_0)^{-1}\gamma\right)\right] + \tfrac{3}{4}lC_u(r, r, \bar{r})\gamma + \tfrac{1}{4}l\left[2(r^{\mathrm{T}}Q_u\bar{r})L_1 r + (r^{\mathrm{T}}Q_u r)L_1\bar{r}\right]. \tag{23}$$

It remains to use Eq. (23) to find conditions under which β_2^* can be made negative. This will be achieved by determining criteria under which both the sign and magnitude of Re Δ can be set to any desired value by feedback control.

The case $l\gamma \neq 0$ deserves special consideration, since by the well known Popov–Belevitch–Hautus (PBH) test [15] the critical modes are then controllable for the linearized system. Hence a *linear* stabilizing feedback exists in this case. Interestingly, Eq. (23) shows that if $l\gamma \neq 0$ a *cubic* stabilizing feedback also exists. To see this, simply set $Q_u = 0$ and consider the effect of the cubic terms in the feedback control. The outcome is that since

Δ reduces to

$$\Delta = \tfrac{3}{4} C_u(r, r, \bar{r})\, l\gamma \tag{24}$$

and $C_u(r, r, \bar{r})$ can be assigned any complex value by appropriate choice of the trilinear form $C_u(x, y, z)$, the origin is certainly locally stabilizable if $l\gamma \neq 0$.

Theorem 1. *Let hypothesis* (H) *hold and assume that* $l\gamma \neq 0$. *That is, the critical eigenvalues are controllable for the linearized system. Then there is a feedback* $u(x)$ *with* $u(0) = 0$ *which solves the local smooth feedback stabilization problem for Eq.* (1) ***and*** *the local Hopf bifurcation control problem for Eq.* (11). *Moreover, this can be accomplished with only third order terms in* $u(x)$, *leaving the critical eigenvalues unaffected.*

Next the case in which $l\gamma = 0$ will be considered. By the PBH test, this corresponds to the critical modes being uncontrollable for the linearized system. Note that the class of nonlinear systems considered in [3] is of this category. Proceeding, suppose $l\gamma = 0$ and observe that the expression (23) for Δ now simplifies to

$$\begin{aligned}\Delta = & -2lQ_0\left(r, \tfrac{1}{2}\left(r^{T}Q_u\bar{r}\right)L_0^{-1}\gamma\right) \\ & + lQ_0\left(r, \tfrac{1}{2}\left(r^{T}Q_u r\right)(2i\omega_c I - L_0)^{-1}\gamma\right) \\ & + \tfrac{1}{4}l\left[2\left(r^{T}Q_u\bar{r}\right)L_1 r + \left(r^{T}Q_u r\right)L_1\bar{r}\right].\end{aligned} \tag{25}$$

Thus only the quadratic terms in the feedback impact β_2 in case $l\gamma = 0$. Note that for the special class of systems studied in [3], only quadratic terms were employed to stabilize the system. By employing the present approach, it has been possible to show that in general, when $l\gamma = 0$ only the quadratic terms in the feedback control can influence the value of β_2.

The value of Eqs. (22) and (25) in studying the local feedback stabilization and Hopf bifurcation control problems when $l\gamma = 0$ will now be illustrated more convincingly by obtaining an explicit condition for stabilizability in this case. Require the (real) matrix Q_u to be such that

$$\operatorname{Im} Q_u r = 0 \quad \text{and} \quad \operatorname{Re} Q_u r \neq 0.$$

A simple computation shows that this is possible; otherwise Re r and Im r would necessarily be proportional, which is absurd since the critical eigenvalues are nonzero. Introduce the real parameter

$$\rho := (\operatorname{Re} r)^{T} Q_u (\operatorname{Re} r) \tag{26}$$

and, recalling that Q_0 is a bi*linear* form, note that Eq. (25) becomes

$$\begin{aligned}\Delta = \rho\Big\{ & -2lQ_0\left(r, \tfrac{1}{2}L_0^{-1}\gamma\right) \\ & + lQ_0\left(r, \tfrac{1}{2}(2i\omega_c I - L_0)^{-1}\gamma\right) \\ & + \tfrac{1}{4}l\left[2L_1 r + L_1\bar{r}\right]\Big\}.\end{aligned} \tag{27}$$

Thus the following sufficient condition is obtained in the case $l\gamma = 0$.

Theorem 2. *Suppose that hypothesis* (H) *is satisfied and that* $l\gamma = 0$. *Then there is a feedback* $u(x)$ *with* $u(0) = 0$ *which solves the local smooth feedback stabilization problem for Eq.* (1) ***and*** *the local Hopf bifurcation control problem for Eq.* (11), *provided that*

$$\begin{aligned}0 \neq \operatorname{Re}\Big\{ & -2lQ_0\left(r, \tfrac{1}{2}L_0^{-1}\gamma\right) \\ & + lQ_0\left(r, \tfrac{1}{2}(2i\omega_c I - L_0)^{-1}\gamma\right) \\ & + \tfrac{1}{4}l\left[2L_1 r + L_1\bar{r}\right]\Big\}.\end{aligned} \tag{28}$$

The derivation of Theorem 2 also indicates how a stabilizing feedback might be chosen. Simply choose ρ of the proper sign (depending on the sign of the expression in (28)) and large enough magnitude.

Acknowledgments

The authors are grateful to Prof. J.C. Alexander for introducing them to Howard's paper [13], to Prof. P.S. Krishnaprasad for his important suggestions regarding Theorem 1 and for providing a copy of [17], and to Prof. C.A. Berenstein and the anonymous reviewer for their careful reading of the manuscript and for several useful suggestions.

References

[1] E.H. Abed, A simple proof of stability on the center manifold for Hopf bifurcation, *Siam Review*, to appear.

[2] E.H. Abed, Singularly perturbed Hopf bifurcation, *IEEE Trans. Circuits and Systems* **32** (1985) 1270–1280.

[3] D. Aeyels, Stabilization of a class of nonlinear systems by a smooth feedback control, *Systems Control Lett.* **5** (1985) 289–294.

[4] R.W. Brockett, Asymptotic stability and feedback stabilization, in: R.W. Brockett, R.S. Millman and H.J. Sussmann, Eds., *Differential Geometric Control Theory* (Birkhauser, Boston, 1983) pp. 181–191.

[5] J. Carr, *Applications of Centre Manifold Theory* (Springer, New York, 1981).

[6] J. Casti, *Connectivity, Complexity, and Catastrophe in Large-Scale Systems* (Wiley-Interscience, Chichester, 1979).

[7] S.-N. Chow and J.K. Hale, *Methods of Bifurcation Theory* (Springer, New York, 1982).

[8] J.-H. Fu and E.H. Abed, Local feedback stabilization and bifurcation control, II. Stationary bifurcation, Manuscript in preparation (1985).

[9] J. Guckenheimer, Persistent properties of bifurcations, *Physica D* **7** (1983) 105–110.

[10] B.D. Hassard, N.D. Kazarinoff, and Y.-H. Wan, *Theory and Applications of Hopf Bifurcation* (Cambridge Univ. Press, Cambridge, MA, 1981).

[11] E. Hopf, Bifurcation of a periodic solution from a stationary solution of a system of differential equations, *Ber. Math. Phys. Kl. Sacks, Akad. Wiss.* **94** (1942) 3–22 (translation to English with commentary in [16]).

[12] L.N. Howard and N. Koppell, Editorial comments (on Hopf's paper), Section 5A in [16].

[13] L.N. Howard, Nonlinear oscillations, in: F.C. Hoppensteadt, Ed., *Nonlinear Oscillations in Biology* (Amer. Math. Soc., Providence, RI, 1979) pp. 1–68.

[14] G. Iooss and D.D. Joseph, *Elementary Stability and Bifurcation Theory* (Springer, New York, 1980).

[15] T. Kailath, *Linear Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1980).

[16] J.E. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications* (Springer, New York, 1976).

[17] R.K. Mehra, W.C. Kessel, and J.V. Carroll, Global stability and control analysis of aircraft at high angles-of-attack, ONRCR-215-248-1, U.S. Office of Naval Research, Arlington, VA (June 1977).

[18] R.K. Mehra, Catastrophe theory, nonlinear system identification and bifurcation control, in: *Proc. of the Joint Automatic Control Conference* (1977) pp. 823–831.

[19] A.B. Poore, On the theory and application of the Hopf–Friedrichs bifurcation theory, *Arch. Rational Mech. Anal.* **60** (1976) 371–393.

[20] H.J. Sussmann, Subanalytic sets and feedback control, *J Differential Equations* **31** (1979) 31–52.