SRC TR 85-10

A New Formula for the Structured Singular Value

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Abstract

The concept of *structured singular value* was recently introduced by Doyle as a tool for analysis and synthesis of feedback systems with structured uncertainties. It was found later to be a key to the design of control systems under joint robustness and performance specifications and to very nicely complement the H^∞ approach to control system design. This note proposes an equivalent definition of the structured singular value, leading to a fast algorithm for its computation. The speedup over previously proposed methods is at least of one order of magnitude, while the correct result is obtained on a larger class of problems.

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Summary

1. Introduction and preliminaries

The concept of *structured singular value* was recently introduced by Doyle [1] as a tool for analysis and synthesis of feedback systems with structured uncertainties. It was found later to be a key to the design of control systems under joint robustness and performance specifications and to very nicely complement the H^∞ approach to control system design [2]. This note proposes an equivalent definition of the structured singular value, leading to a fast algorithm for its computation. The speedup over previously proposed methods is at least of one order of magnitude, while the correct result is obtained on a larger class of problems.

We will make use of the following notation and nomenclature, largely inspired from that used in [1]. We will call *block-structure* of size m any m-tuple $\boldsymbol{k} = (k_1, \cdots, k_m)$ of positive integers[1]. Given a block-structure $\boldsymbol{k}$ of size m, we will invoke the projection matrices

$$P_i = \text{diag}\,(0 \cdot I_{k_1}, \cdots, 0 \cdot I_{k_{i-1}}, I_{k_i}, 0 \cdot I_{k_{i+1}}, \cdots, 0 \cdot I_{k_m}) , \tag{1}$$

the family of diagonal matrices

$$\boldsymbol{d} = \{\text{diag}(d_1 I_{k_1}, \cdots, d_m I_{k_m}) \mid d_i \in (0,\infty)\} , \tag{2}$$

the family of block unitary matrices

$$\boldsymbol{u} = \{\text{diag}(U_i) \mid U_i \text{ is a } k_i \times k_i \text{ unitary matrix}\} , \tag{3}$$

and, for any positive scalar δ, the family of block diagonal matrices

[1] This corresponds, in the terminology of [1], to structures with no repeated blocks.

$$X_\delta = \{\mathrm{diag}(\Delta_1, \cdots, \Delta_m) \mid \Delta_i \text{ is a } k_i \times k_i \text{ complex matrix satisfying } \bar{\sigma}(\Delta_i) \leq \delta\} , \quad (4)$$

where $\bar{\sigma}$ denotes the largest singular value. All of the above have dimension $n \times n$, where

$$n = \sum_{j=1}^{m} k_j . \quad (5)$$

The *structured singular value* of a complex $n \times n$ matrix M *with respect to block-structure* $\boldsymbol{k}$ is defined in [1] as the positive number μ having the property that

$$\det(I + M\Delta) \neq 0 \text{ for all } \Delta \in X_\delta \quad (6)$$

if, and only if,

$$\delta\mu < 1 . \quad (7)$$

Two approaches are considered in [1] for the computation of μ, respectively based on the identities

$$\mu = \max_{U \in \boldsymbol{u}} \rho(MU) , \quad (8)$$

where ρ denotes the spectral radius, and, for block-structures of size less than 4[2],

$$\mu = \inf_{D \in \boldsymbol{d}} \bar{\sigma}(DMD^{-1}) . \quad (9)$$

The first approach is discarded in [1] because the optimization problem it involves generally has non-global maxima. The function to be minimized in the second approach is convex and thus all stationary points are global minima. Algorithms are available for solving this kind of problems and this yields a reliable way of

[2] In many (but not all) cases, (9) is correct with block-structures of larger size.

computing μ for block-structures of size less than 4. However such algorithms are iterative in nature, thus requiring a large number of function evaluations, each of which involves a CPU demanding singular value decompositions.

In view of the above, there is need for a *fast*, reliable method for computing μ for *any block-structure*. The results presented in this note represent a significant step in that direction. They lead to an algorithm for computing μ that (i) is proven to yield the correct result for block-structures of size less than 4, (ii) in most (possibly all) cases is an order of magnitude faster than algorithms solving (9), and (iii) appears promising for block-structures of size 4 or more.

2. Main results

In all the theorems below, M is a complex $n \times n$ matrix and μ is its structured singular value with respect to structure k. The proofs of these theorems will be reported elsewhere.

Theorem 1[3]

$$\mu = \max_{x \in C^n} \{ \, ||Mx|| \; | \; ||P_i x|| ||Mx|| = ||P_i Mx||, \, i = 1, \cdots, m \, \} \tag{10}$$

Theorem 2

$$\mu = \max_{x \in C^n} \{ \, ||Mx|| \; | \; ||P_i x|| ||Mx|| \leq ||P_i Mx||, \, i = 1, \cdots, m \, \} \tag{11}$$

Corollary. If $D \in d$ and if $x \in C^n$ satisfies the constraints in (11) then

$$||Mx|| \leq \mu \leq \bar{\sigma}(DMD^{-1}) \tag{11'}$$

and the two extremes in (11') are equal if, and only if, they are both equal to μ.

[3] Throughout $||\cdot||$ denotes the Euclidean norm.

A major difference between expressions (8) and (9) on the one hand, and (10) and (11) on the other hand, is that the objective and constraints appearing in the latter are smooth and that their evaluation merely requires one matrix $\times$ vector multiplication. As seen below, this leads to substantial savings in computer time. Since (10) and (11) may have non-global maximizers, it is crucial to have the opportunity to check whether a given $x \in C^n$ solves (10) and (11) globally. This is provided by Theorems 3 and 4 below. In order to state them, we need to introduce some definitions.

Suppose that $x^* \in C^n$ solves (10), i.e., is a global maximizer for the optimization problem of Theorem 1. Then, from Theorem 2, it also solves the problem

$$\max_{x \in C^n} \{ \, ||Mx||^2 \;\; | \;\; ||P_i Mx||^2 - ||P_i x||^2 ||Mx||^2 \geq 0, \; i=1, \cdots, m \, \} \tag{12}$$

Consider now problem (12) as an optimization problem in $\mathbb{R}^{2n}$. Assuming that the Kuhn-Tucker constraint qualification holds[4], there must exist multipliers $\lambda_1, \cdots, \lambda_m$ which, together with x^*, satisfy the Kuhn-Tucker first order conditions of optimality for problem (12). After elementary manipulations, these conditions can be written as

$$M^H Mx + \sum_{i=1}^{m} \lambda_i (M^H P_i Mx - ||P_i x||^2 M^H Mx - ||Mx||^2 P_i x) = 0 \tag{13}$$

$$\lambda_j \geq 0, \; j=1, \cdots, m \tag{14}$$

$$||P_i Mx||^2 - ||P_i x||^2 ||Mx||^2 \geq 0, \; i=1, \cdots, m \tag{15}$$

where M^H denotes the conjugate transpose of M. Any pair $(x, \lambda) \in C^n \times \mathbb{R}^m$, with $\lambda = (\lambda_1, \cdots, \lambda_m)^T$, satisfying (13)-(15) will be called *Kuhn-Tucker pair* for (12). x is

[4] This is assumed throughout, but it can be relaxed.

called *Kuhn-Tucker point* and the λ_i's *Kuhn-Tucker multipliers associated with* x.

Theorem 3. Suppose that the size of the block-structure is less than 4 ($m < 4$). Then x^* solves the optimization problem (10) (or, equivalently, (11)) if, and only if, x^* is a Kuhn-Tucker point for (12) satisfying[5]

$$||Mx^*|| = \bar{\sigma}(DMD^{-1}) \tag{16}$$

where D is the only matrix in $\mathcal{d}$ for which

$$D^2 = \text{diag}(\lambda_i I_{k_i}) \tag{17}$$

where the λ_i's are the Kuhn-Tucker multipliers associated with x^*.

Suppose now that one attempts to solve (10), (11), or (12) using an optimization algorithm having the property that all the accumulation points of the sequence it constructs are Kuhn-Tucker points for (12). Theorem 3 can then be used to determine whether or not a Kuhn-Tucker point thus obtained is a global maximizer for (10). If this is the case, Theorem 1 (or 2) yields μ. Otherwise, one can use the matrix D provided by Theorem 3 as an initial guess for solving problem (9), thus obtaining μ. Luckily, if some care is taken in performing the former optimization, the latter is seldom necessary (see below).

Our last theorem applies to any block-structure. It may (when (9) yields μ) or may not provide a test for whether a Kuhn-Tucker point for (12) is a global maximizer.

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[5] If any λ_i is zero, (16) should be replaced by $||Mx^*|| = \lim_{j \to \infty} \bar{\sigma}(D_j M D_j^{-1})$ where $\{D_j\}$ is any sequence of nonsingular matrices converging to D.

Theorem 4. Suppose x^* is a Kuhn-Tucker point for (12) satisfying[6]

$$||Mx^*|| = \bar{\sigma}(DMD^{-1}) \tag{18}$$

where D is the only matrix in $\mathcal{d}$ for which

$$D^2 = \text{diag}(\lambda_i I_{k_i}) , \tag{19}$$

where the λ_i's are the Kuhn-Tucker multipliers associated with x^*. Then x^* solves (10). Moreover D is a minimizer for (9).

To conclude this section, let us mention that solutions U^* to (8), D^* to (9) when it yields μ, and x^* to (10) and (11), can easily be obtained from each other. In case (9) does not yield μ, the D^* obtained from U^* or x^* is such that μ is *some* singular value of D^*MD^{*-1}.

3. Numerical experiments

The method outlined above was tested on one thousand randomly generated complex matrices of size 8×8, with a block-structure of size 2. As the initial guess for the iterative solution of (12), we chose the normed unconstrained maximizer for (12), i.e., the first right singular vector of M.

In *all* cases, μ was obtained directly, without need to resort to problem (9). Typically the CPU time was 10 to 20 times smaller than that required when computing μ through problem (9). Promising preliminary tests were performed with block-structures of size 4. In particular, the correct value for μ was obtained on an example due to Doyle, where computation of μ through (9) fails.

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[6] Footnote 5 applies here as well.

Acknowledgments

Many thanks are due to Dr. J. Doyle who introduced the authors to the subject and gave them many enlightening insights. This research was supported by a grant from the Minta Martin Foundation, at the College of Engineering. University of Maryland, College Park.

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