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Stability of Multiparameter Singular Perturbation Problems with Parameter Bounds, II. Time-Invariant Systems with Arbitrary Perturbations

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ABSTRACT

The asymptotic stability of a general linear time-invariant multiparameter singular perturbation problem is studied with no *a priori* assumptions on the relative magnitudes of the small parameters. Thus the results apply, for example, to multiple time scale problems as well as to multiparameter singular perturbation problems possessing only two time scales. An explicit upper bound is obtained for a weighted norm of the vector of singular perturbation parameters such that asymptotic stability of the system is ensured if this bound is respected. In a companion paper [1], the time-varying case is studied using Liapunov stability theory, assuming that the small parameters have bounded mutual ratios (two time scales). The present paper makes use of fixed-point methods, in addition to Liapunov techniques, to arrive at the desired upper bound.

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I. INTRODUCTION

The purpose of this paper is to obtain stability results for singular perturbation problems containing several *independent* small parameters. That is, no restriction is imposed on the relative magnitudes of the singular perturbation parameters. An explicit upper bound is obtained on the singular perturbation parameters of stable, time-invariant multiparameter singular perturbation problems. In a companion paper [1], similar results are obtained for the linear time-varying case. As will be seen below, however, the results of the present paper are not a special case of those in [1], and indeed apply under much more general assumptions on the small parameters ϵ_i. For example, the results of the present paper apply simultaneously to multiparameter singular perturbation problems possessing multiple time scales as well as to multiparameter problems with only two time scales. The reader is referred to [1, 17, 3, 20] for a detailed discussion of multiparameter singular perturbations. In the companion paper [1] will also be found references to earlier work on parameter bounds for stable singularly perturbed systems. The results in the companion paper [1] are derived for the time-varying case assuming that the mutual ratios ϵ_i/ϵ_j are bounded away from 0 and ∞.

This paper is organized as follows. In Section II the problem is stated, background material of relevance is recalled and some assumptions are made. In Section III the Brouwer fixed point theorem is used to establish an upper bound on a weighted norm of the parameter vector ϵ ensuring the existence of a similarity transformation which exactly separates the fast and slow modes of the system under study. In Section IV two further upper bounds are derived ensuring stability of the separated fast and slow subsystems. In Section V the results are summarized yielding a single upper bound ensuring asymptotic stability of the multiparameter singularly perturbed system. Conclusions are collected in Section VI.

Notation: For a square matrix A, $\sigma(A)$ denotes the spectrum, or set of eigenvalues, of A. The vector norm which is employed in the paper is the Euclidean norm, indicated by $|\,|$. The matrix norm can be any matrix norm compatible with the Euclidean norm; examples include the spectral (or L_2) norm and the Frobenius norm. The formulae obtained can of course be modified to apply to arbitrary vector and matrix norms, but this is not pursued here. Matrix norms are also indicated by $|\,|$. The transpose of an arbitrary matrix A is indicated by A'.

II. PROBLEM STATEMENT AND BACKGROUND

In this section several requisite facts on time-invariant multiparameter singular perturbation problems are reviewed, and the overall strategy of the paper is outlined. The section begins with a precise statement of the problem to be considered.

A. Problem Statement

The system of interest is

$$\dot{x} = Ax + By \tag{1a}$$

$$\epsilon_i \dot{y}_i = C_i x + D_i y, \quad i = 1,\ldots,M. \tag{1b}$$

The notation in (1) is $x \in R^n$, $y = (y_1,\ldots,y_M) \in R^m$, $y_i \in R^{m_i}$, $\epsilon = (\epsilon_1, \ldots, \epsilon_M)$ with each ϵ_i a small real parameter, A, B, C_i, D_i are real matrices of appropriate dimension, and the dot denotes differentiation with respect to time t. The system dimension is evidently $n + \sum_{i=1}^{M} m_i$. The results obtained here for the linear system (1) apply as well to the the study of local asymptotic stability of an equilibrium point of the direct nonlinear generalization of (1).

It is convenient to express (1) more succinctly by defining matrices C and D by

$C := block\ col(C_1,...,C_M)$, $D := block\ col(D_1,...,D_M)$ and writing

$$\dot{x} = Ax + By \tag{2a}$$

$$E(\epsilon)\dot{y} = Cx + Dy. \tag{2b}$$

Here $E(\epsilon) := block\ diag(\epsilon_1 I_{m_1}, \ldots, \epsilon_M I_{m_M})$ where I_k denotes the $k \times k$ identity matrix.

The eigenvalues of (2) (or (1)) are of course the eigenvalues of the Jacobian matrix

$$J(\epsilon) = \begin{pmatrix} A & B \\ E^{-1}(\epsilon)C & E^{-1}(\epsilon)D \end{pmatrix} \tag{3}$$

of (2).

Two related problems are dealt with in this paper. The first concerns finding sufficient conditions ensuring asymptotic stability of the null solution of (1) (equivalently, (2)) for *all* sufficiently small $|\epsilon|$, $\epsilon \in R_+^M$. (Recall that R_+^M is the positive orthant of R^M.) The second problem is, given that the sufficient conditions for asymptotic stability are indeed satisfied, to exhibit a specific upper bound E_0 on $|E(\epsilon)|$ such that $|E(\epsilon)| < E_0$, $\epsilon \in R_+^M$ implies the asymptotic stability of (1). Note that the small parameters are not constrained to lie in a proper subset of R_+^M; all that is assumed is $\epsilon_i > 0$, $i = 1,...,M$. Thus the results will apply uniformly as $\epsilon \to 0$ along any path in R_+^M.

The basic strategy by which these problems will be addressed is that of ensuring that the eigenvalues of $J(\epsilon)$ have strictly negative real parts for all $\epsilon \in R^m$ with $|\epsilon|$ sufficiently small, using a transformation known to separate fast and slow dynamics. Explicit estimates will be obtained using perturbational stability results and fixed point techniques.

B. Hypotheses and Definitions

The following hypotheses will be employed in the paper.

(H1) The matrix D is nonsingular.

(H2) All eigenvalues of the matrix

$$A_0 := A - BD^{-1}C \tag{4}$$

have strictly negative real parts.

(H3) There exists a block diagonal positive definite matrix P,

$$P = block\ diag[P_1, \ldots, P_M] \tag{5}$$

satisfying

$$c_1|y|^2 \le y'Py \le c_2|y|^2, \text{ for all } y \in R^m, \tag{6}$$

such that Q given by

$$PD + D'P = -Q \tag{7}$$

is positive definite, and moreover satisfies

$$y'Qy \ge c_3|y|^2, \text{ for all } y \in R^m. \tag{8}$$

Hypothesis (H3) clearly implies (H1), which was stated explicitly only for clarity of the presentation. Indeed, (H3) also has implications for stability. It implies that D is a *block D-stable* matrix, a notion which has been employed previously in the stability analysis of multiparameter singular perturbation problems. See [1] for some references. The definitions of D-stability and block D-stability are as follows.

Definition 1. A matrix F is *D-stable* if Re $\sigma(\Theta F) < 0$ for any diagonal matrix Θ with strictly positive diagonal elements.

Definition 2. A matrix F is said to be *block D-stable* relative to the multi-index $(m_1, \ldots, m_M)$, if for all $\theta_i > 0$, $i=1,\ldots,M$, Re $\sigma(\Theta(\theta)F) < 0$, where $\Theta(\theta) =$ *block diag* $(\theta_1 I_{m_1}, \ldots, \theta_M I_{m_M})$.

C. A Riccati Equation

The next lemma was introduced in [3]. It gives an algebraic matrix Riccati equation whose solution is useful in exhibiting a transformation which decouples the fast and slow modes of (2).

Lemma 1 [3]. Suppose det $D \neq 0$ and denote $E = E(\epsilon)$. Then the Riccati equation

$$D\Gamma + E(D^{-1}C - \Gamma)A_0 - E\Gamma B\Gamma + ED^{-1}CB\Gamma = 0 \tag{9}$$

for the $m \times n$ matrix Γ has a locally unique solution $\Gamma(\epsilon)$ near $0 \in R^{m \times n}$ for $|\epsilon|$ sufficiently small.

The Riccati equation

$$DL - \epsilon LA + \epsilon LBL - C = 0 \tag{10}$$

for the $m \times n$ matrix L arises in the single parameter theory (cf. [25, 21] for a discussion and some references). The use of such equations to produce transformations which exactly decouple fast and slow modes was first proposed by Chang [9, 10] in the context of singular perturbation of a general boundary value problem for time-varying linear systems. Eq. (9) yields (10) upon setting $E(\epsilon) = \epsilon I_m$ and $L = D^{-1}C - \Gamma$. Here it is convenient to deal with Γ rather than L as the unknown since $\Gamma = O(|\epsilon|)$. Note that Eq. (10) above is the stationary, single parameter version of the differential equation ([1], Eq. (16)) used in the companion paper as part of a time-varying transformation to separate fast and slow dynamics.

D. Eigenvalues and Fast-Slow Decomposition

The next theorem gives an exact expression for the eigenvalues of $J(\epsilon)$ in terms of the eigenvalues of matrices associated with appropriate fast and slow subsystems of (1). This theorem was derived in [3] by using Lemma 1 above to exhibit a similarity transformation rendering $J(\epsilon)$ in block upper triangular form.

Theorem 1. Let $|\epsilon|$ be sufficiently small so that Lemma 1 applies. Then

$$\sigma(J(\epsilon)) = \sigma(A - BD^{-1}C + B\Gamma(\epsilon)) \cup \sigma(E^{-1}(\epsilon)D + D^{-1}CB - \Gamma(\epsilon)B) \tag{11}$$

if $\epsilon_i \neq 0$, $i = 1,\ldots,M$.

This result motivates the following definition of fast and slow subsystems associated with (1).

Definition 3. If a solution $\Gamma(\epsilon)$ of Eq. (9) exists, then the *slow subsystem* of (1) corresponding to Γ is

$$\dot{x} = (A - BD^{-1}C + B\Gamma(\epsilon))x, \tag{12}$$

and the *fast subsystem* is

$$\dot{y} = (E^{-1}(\epsilon)D + D^{-1}CB - \Gamma(\epsilon)B)y. \tag{13}$$

E. The Strategy

The strategy which will be followed in the remainder of the paper may now be briefly summarized. The derivation consists of two main steps. First, the Brouwer fixed point theorem is used to obtain an initial upper bound E_1 on $|E(\epsilon)|$ which ensures the existence of a solution to the algebraic matrix Riccati equation (9). This initial upper bound is parametrized by the magnitude of a norm constraint imposed on a solution to the Riccati equation. Satisfaction of this upper bound then ensures that the fast and slow subsystems introduced above are meaningful, and moreover that asymptotic stability of (1) is equivalent to that of the fast and slow subsystems. Next, Liapunov's direct method is applied to obtain a bound on the norm of the solution to the Riccati equation which ensures asymptotic stability of the fast and slow subsystems. An explicit upper bound on the weighted norm $|E(\epsilon)|$ of the vector of singular perturbation parameters ϵ readily follows. Note that a similar approach was used by the present author in [2] to obtain a parameter bound ensuring the stability of time-invariant singularly perturbed systems containing a single parameter.

III. FIXED-POINT ANALYSIS OF THE RICCATI EQUATION

Rewrite Eq. (9) in the fixed point form

$$F_\epsilon(\Gamma) = \Gamma \tag{14}$$

where the parametrized mapping $F_\epsilon : R^{m\times n} \to R^{m\times n}$ is defined for any $\epsilon \in R^M$ by

$$F_\epsilon(\Gamma) := D^{-1}E(\epsilon)\,\{\, (\Gamma - D^{-1}C)A_0 \;+\; \Gamma B \Gamma \;-\; D^{-1}CB\Gamma \,\}. \tag{15}$$

A parametrized upper bound $E_1(\alpha)$ on $|E(\epsilon)|$ will now be derived such that for any $\alpha>0$ and if $|E(\epsilon)| < E_1(\alpha)$, $|\Gamma| \leq \alpha$ implies $|F_\epsilon(\Gamma)| \leq \alpha$. Proceeding, let $\alpha>0$ be given and let $\Gamma \in R^{m\times n}$ satisfy $|\Gamma| \leq \alpha$. From (15) one has

$$|F_\epsilon(\Gamma)| \leq |E(\epsilon)|\,|D^{-1}|\,\{\, (\,|\Gamma| \;+\; |D^{-1}C|\,)\,|A_0| \;+\; |B|\,|\Gamma|^2 \;+\; |D^{-1}CB|\,|\Gamma|\,\}$$

$$\leq |E(\epsilon)|\,|D^{-1}|\,\{\, (\alpha + |D^{-1}C|\,)\,|A_0| \;+\; \alpha^2|B| \;+\; \alpha\,|D^{-1}CB|\,\}. \tag{16}$$

From (16) follows immediately that for $|F_\epsilon(\Gamma)| \leq \alpha$ to hold it suffices that $|E(\epsilon)| \leq E_1(\alpha)$ where $E_1(\alpha)$ is given by

$$E_1(\alpha) := \frac{\alpha}{|D^{-1}|\,\{\, |D^{-1}C|\,|A_0| \;+\; \alpha(\,|D^{-1}CB| + |A_0|\,) \;+\; \alpha^2\,|B|\,\}}. \tag{17}$$

Thus for any $\alpha>0$, F_ϵ is a continuous map of the closed ball $B_\alpha := \{\, \Gamma \in R^{m\times n} : |\Gamma| \leq \alpha\}$ into itself whenever $0 \leq \epsilon \leq E_1(\alpha)$. Note that B_α is homeomorphic to the closed unit ball in R^{mn}. The Brouwer fixed point theorem ([6], p. 54 and [14], p. 10) may now be invoked to establish the existence of a solution $\Gamma(\epsilon)$ to (14) in B_α for any ϵ with $|E(\epsilon)| < E_1(\alpha)$.

Theorem 2. For any $\alpha > 0$ and for all $\epsilon \in R^M$ with $|E(\epsilon)| \leq E_1(\alpha)$, the Riccati equation (9) has at least one solution $\Gamma(\epsilon)$ with $|\Gamma(\epsilon)| \leq \alpha$.

Remark 1. Note that for any system (1) with $B \neq 0$ (the generic and nontrivial case), the upper bound $E_1(\alpha)$ approaches 0 as $\alpha \to \infty$. It is clear, however, that a solution of (9) in the ball B_α for a given $\alpha = \alpha_0$ will also belong to any ball B_α with $\alpha > \alpha_0$. Therefore the upper bound of Theorem 2 can be made less conservative by using instead of $E_1(\alpha)$ the revised estimate

$$E_1^*(\alpha) := \max_{0\leq\beta\leq\alpha} E_1(\beta)\,. \tag{18}$$

Remark 2. It is natural to attempt to factorize the quadratic appearing in the denominator of the expression (17) for $E_1(\alpha)$. In this regard, note that replacing $|D^{-1}CB|$ in (17) by $|D^{-1}C|\,|B|$ results in the new (in general *more conservative*) upper bound E_1^{**} given by

$$E_1^{**}(\alpha) := \frac{\alpha}{|D^{-1}|\,(\,|D^{-1}C| \;+\; \alpha\,)\,(|A_0| \;+\; \alpha|B|\,)}. \tag{19}$$

Remark 3. Note that in the derivation of Theorem 2 only *existence* of a solution Γ to (9) needed to be ensured. A further upper bound on $|E(\epsilon)|$ to ensure uniqueness could easily be derived by a contraction mapping argument as in ([11], pp. 16-18). However, this could only result in a more conservative final estimate and would thus be counterproductive.

IV. STABILITY ANALYSIS

A. A Robustness of Stability Estimate

In the stability analysis of the slow subsystem (12) a basic perturbational result on stability of linear systems (Brockett [7], p. 205) will prove useful. The time-varying version of this result appeared as Proposition 2 of [1].

Proposition 1. Let A be a stable matrix. Assume given a positive definite matrix T and the (unique) positive definite solution R of the Liapunov matrix equation $A'R + RA = -T$. Then $A + B$ will be a stable matrix for any B with $|B| < \delta$, where δ is given by

$$\delta := \frac{|T|}{2\,|R|}\,. \tag{20}$$

It is worth noting that the matrix R has the explicit representation

$$R = \int_0^\infty e^{A't}\, T\, e^{At}\, dt\ , \tag{21}$$

even though this in general does not indicate an efficient means of computing R (see Laub [23]).

B. Stability of the Slow Subsystem

From (12), the eigenvalues of the separated slow subsystem are precisely the eigenvalues of the matrix (the superscript S indicates *slow* variables)

$$A_0 + B\Gamma =: A^S \tag{22}$$

if a solution Γ of (9) exists. Since by (H2) A_0 is a stable matrix, there exist positive definite matrices T^S, R^S such that

$$A'_0 R^S + R^S A_0 = -T^S. \tag{23}$$

Proposition 1 now implies that A^S is a stability matrix (i.e. the slow modes are stable) if there is a solution Γ of (9) with $|B\Gamma| < \dfrac{|T^S|}{2\,|R^S|}$. This will be true if

$$|\Gamma| < \frac{|T^S|}{2\,|B|\,|R^S|} \tag{24}$$

is satisfied, where Γ is some solution to Eq. (9).

C. Stability of the Fast Subsystem

Recall from Eq. (13) that the fast subsystem is given by

$$\dot{y} = (E^{-1}(\epsilon)D + D^{-1}CB - \Gamma(\epsilon)B)y =: A^F\, y\,, \tag{25}$$

assuming of course that a solution Γ of (9) exists. Rather than attempt to apply Proposition 1 to the fast subsystem (13), one uses hypothesis (H3) to study the stability. Proceeding, define the Liapunov function candidate $v(y)$ for (25) as

$$v(y) := y'\, PE(\epsilon) y \tag{26}$$

where P is from Eq. (5) of (H3). Evaluating $\dot{v}$ along trajectories of (25), one obtains

$$\dot{v}(y) = y' \{ D'\, E^{-1}(\epsilon)PE(\epsilon) + PD \}y$$

$$+\ y' \{ B'\, C'\, (D^{-1})'\, PE(\epsilon) + PE(\epsilon)D^{-1}CB \}y$$

$$- y' \{ B' \Gamma' PE(\epsilon) + PE(\epsilon)\Gamma B \}y. \quad \textbf{(27)}$$

Since by (H3) P is block diagonal with the same structure as $E(\epsilon)$, it follows that P and $E(\epsilon)$ commute, so that the expression $D' E^{-1}(\epsilon)PE(\epsilon) + PD$ in (27) can be rewritten as $D' P + PD = -Q$ by Eq. (7). Now Eq. (8) of (H3) and the Schwarz inequality are used to obtain an upper bound on the right side of (27). Thus

$$\dot{v}(y) \leq -c_3 |y|^2 + 2 |E(\epsilon)| |P| |D^{-1}CB| |y|^2$$

$$+ 2 |E(\epsilon)| |B| |\Gamma| |P| |y|^2. \quad (28)$$

Therefore to ensure that $\dot{v}(y) < 0$ it is sufficient to require that $|E(\epsilon)|$ be bounded from above as follows:

$$|E(\epsilon)| < \frac{c_3}{2 |P| (|D^{-1}CB| + |B| |\Gamma|)}, \quad (29)$$

where Γ is any solution of (9).

Define $E_2(\alpha)$ by

$$E_2(\alpha) := \frac{c_3}{2 |P| (|D^{-1}CB| + \alpha |B|)}. \quad (30)$$

V. THE UPPER BOUND

The foregoing results may now be combined to yield an upper bound E_0 on $|E(\epsilon)|$ such that for $|E(\epsilon)| < E_0$, $\epsilon \in R_+^M$, the asymptotic stability of the multiparameter singularly perturbed system (1) is certain. By the results of Section IV-B, the slow subsystem is well defined and its stability is ensured if there is a solution $|\Gamma|$ of Eq. (9) satisfying the inequality (24). By Theorem 2, this will be the case if $|E(\epsilon)| \leq E_1^*(\alpha)$ where α is *any* positive number satisfying

$$\alpha < \frac{|T^S|}{2 |B| |R^S|} =: \bar{\alpha}. \quad (31)$$

Similarly, the fast subsystem will be well defined and stable if the last remark holds and if $|E(\epsilon)| < E_2(\alpha)$ where $E_2(\alpha)$ is given by Eq. (30), and where, again, α is any positive scalar satisfying (31). By Eq. (4) stability of the fast and slow subsystems implies that of the singularly perturbed system (1).

For any given $\alpha > 0$ satisfying (31), clearly an upper bound on $|E(\epsilon)|$ ensuring stability of (1) is min ($E_1^*(\alpha)$, $E_2(\alpha)$). To optimize the upper bound, one takes the maximum of this quantity over all α satisfying Eq. (31). These remarks are summarized in the following theorem, in which the upper bound E_0 of this paper is stated explicitly.

Theorem 3. Let hypotheses (H1)-(H3) hold. Then the null solution of (1) is asymptotically stable for all $\epsilon \in R_+^M$ with $|E(\epsilon)| < E_0$ where E_0 is given by

$$E_0 := \max_{0 \leq \alpha \leq \bar{\alpha}} \min \{ E_1^*(\alpha), E_2(\alpha) \}, \quad (32)$$

and where $E_1^*(\alpha)$, $E_2(\alpha)$ and $\bar{\alpha}$ are given by Eqs. (18), (30) and (31) respectively.

VI. CONCLUSIONS

The paper has presented a derivation of a new upper bound on a weighted norm of the vector of small parameters ϵ ensuring stability of the multiparameter singularly perturbed system (1). The results apply *uniformly* as the small parameters approach 0 *independently*. Thus both the multi-time scale setting [15] and the bounded mutual ratios setting [17] are addressed. The upper bound obtained here is uniform and so does not display the conservativeness which occurred in Khalil [20] and in the companion paper [1]. The analysis consisted of applying the Brouwer fixed point theorem and Liapunov's direct method to obtain explicit estimates for well known results on time scale separation and on regular perturbation of stable linear systems. An expression for E_0

was then obtained by considering the implications of these estimates for the stability of fast and slow subsystems associated with system (1).

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