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Stability of Multiparameter Singular Perturbation Problems with Parameter Bounds, I. Time-Varying Systems with Cone-Restricted Perturbations

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ABSTRACT

Explicit upper bounds are obtained for the singular perturbation parameters of a general uniformly asymptotically stable multiparameter singularly perturbed system. The study focuses on the linear time-varying case studied by H.K. Khalil and P.V. Kokotovic (SIAM J. Control Opt., 17, 56-65, 1979) in which the small parameters are constrained to have bounded mutual ratios. An upper bound is obtained on a weighted norm of the vector of singular perturbation parameters such that uniform asymptotic stability is ensured if this bound is met. The derivation makes liberal use of Liapunov function arguments. In a companion paper [2], it is shown that for linear time-invariant systems the 'bounded mutual ratios' assumption can be lifted, and typically less conservative parameter estimates are obtained.

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I. INTRODUCTION

Singular perturbation problems involving several small parameters have in the past attracted considerable attention, both from the engineering and the applied mathematics communities. This is due in part to the difficulty of these problems and in part to their wide applicability in power system dynamics, control of large scale systems, multi-modelling, differential games, and similar settings. The purpose of this paper is to derive explicit upper bounds on the singular perturbation parameters ensuring the uniform asymptotic stability of a general time-varying multiparameter singularly perturbed system. Parameter bounds for stable singularly perturbed systems have been obtained previously by Zien [36], Javid [18], Balas [5] and Abed [1] for problems with a single small parameter, and by Khalil [24] for problems containing several small parameters. The upper bound of [24] is derived for time-invariant multiparameter problems. In a companion paper [2] the present author also gives parameter bounds for stable time-invariant multiparameter singular perturbation problems which do not suffer from a certain conservativeness problem which is present in [24], and which occurs also in the results of the present paper.

The results of this paper are obtained for linear time-varying systems of the form

$$\dot{x} = A(t)x + B(t)y \tag{1a}$$

$$\epsilon_i \dot{y}_i = C_i(t)x + D_i(t)y, \qquad i=1,\ldots,M, \tag{1b}$$

where $x \in R^n$, $y = (y_1,\ldots,y_M) \in R^m$, $y_i \in R^{m_i}$, $\epsilon = (\epsilon_1, \ldots, \epsilon_M)$ with each ϵ_i a small real parameter, A, B, C_i, D_i are real matrices of appropriate dimension, and the dot signifies differentiation with respect to time t. Since the concern here is with the uniform asymptotic stability of the null solution of (1), the results will clearly apply as well to the nonlinear time-varying generalization of (1), if the nonlinear terms are decrescent [35].

The asymptotic analysis of (1) in the limit $\epsilon_i \to 0$, $i = 1,\ldots,M$ is an example of a ***singular perturbation*** problem. This means that the order of the system (1) differs for $\epsilon_i = 0$ and $\epsilon_i \neq 0$. For a discussion of results in singular perturbation theory, see the excellent review articles [27, 32, 26].

Most previous studies of singularly perturbed systems containing several small parameters begin by hypothesizing some type of relationship among the small parameters ϵ_i. The simplest of these reduces the problem to a standard single parameter singular perturbation problem. This is achieved by assuming that the parameters ϵ_i are known multiples of a single, auxiliary small parameter δ, so that $\epsilon_i = \alpha_i \delta$, $i=1,\ldots,M$. Khalil and Kokotovic have pointed out in [21] that this assumption is not justifiable in many cases of practical interest, if only because of the lack of knowledge of the coefficients α_i. As a more realistic assumption, [21] allows the parameters ϵ_i to be arbitrary subject to the requirement that the mutual ratios ϵ_i/ϵ_j, $i,j=1,\ldots,M$ are bounded from above and below by known positive constants. This is equivalent to constraining ϵ to lie in a (linear) cone in the positive orthant R_+^M. The authors of [21] refer to this as the "multiparameter assumption." This assumption is also invoked in [20, 22, 23, 24, 29] and in the present paper. A perhaps more common hypothesis is that of multiple time scales [17, 30, 11, 34, 29]. This requires that the small parameters ϵ_i are of ***different orders of magnitude,*** say $\epsilon_{i+1}/\epsilon_i \to 0$ as $\epsilon_i \to 0$. The multiparameter setting of [21] clearly differs drastically from this case; indeed only two time scales are present in the setting of [21], just as for single parameter singular perturbation problems.

Although the present paper employs Khalil and Kokotovic's assumption that the mutual ratios ϵ_i/ϵ_j are bounded, in the companion paper [2] similar results are obtained without any such restriction. The results in the companion paper [2] are derived for the time-invariant case, and yield each of the foregoing multiparameter singular perturbation set-ups as special cases. Similar results have previously been obtained by the author in [3], which contains some stability results for the time-invariant version of (1) without limiting *a priori* the way in which $\epsilon \to 0$. These include iterative expansions in the ϵ_i which show the dependence of the eigenvalues of (1) on ϵ. The present author is of the opinion that use of the term ***multiparameter singular perturbation*** should be expanded to include all possible assumptions on the way in which the small parameters ϵ_i are constrained, including even the multiple time scale setting discussed above.

The development of the paper is as follows. Section II introduces the problem and provides relevant background material. Several hypotheses used throughout the paper are also given in Section II. Section III is devoted to a constructive proof of the existence of a decoupling transformation which separates the fast and slow dynamics of (1). Two upper bounds on $|E(\epsilon)|$ are obtained in that section. These ensure the existence of a bounded decoupling transformation as well as the uniform asymptotic stability of the fast subsystem. In Section IV a further upper bound is obtained ensuring the uniform asymptotic stability of the slow subsystem. These three upper bounds are collected to yield an upper bound E_0 on $|E(\epsilon)|$ ensuring uniform asymptotic stability of the original multiparameter singularly perturbed system (1) (cf. Eq. (69)). This, the main result of the paper, appears as Theorem 1 of Section IV. Conclusions and some open questions are given in Section V.

Notation. Throughout the paper the Euclidean norm is used for vectors and the Frobenius norm for matrices. This is only for simplicity of the development, and does not represent a limitation of the method. The Euclidean and Frobenius norms are compatible. Recall that the Frobenius norm of a real matrix A is the square root of the sum of the squares of all the elements of A. It will be denoted by $|A|$. If A depends continuously on time t and is bounded, its norm is $\|A\| := \sup_{t \geq 0} |A(t)|$. With the hope that there will be no confusion, this is denoted simply by $|A|$. The transpose of A is indicated by A'. If A is a square matrix, $\sigma(A)$ denotes the spectrum or set of eigenvalues of A.

II. PRELIMINARY CONSIDERATIONS

It is useful to express Eq. (1) in the more compact form

$$\dot{x} = A(t)x + B(t)y \tag{2a}$$

$$E(\epsilon)\dot{y} = C(t)x + D(t)y. \tag{2b}$$

Here $C(t) :=$ *block col* $(C_1(t),...,C_M(t))$, $D(t) :=$ *block col* $(D_1(t),...,D_M(t))$, and $E(\epsilon) :=$ *block diag* $(\epsilon_1 I_{m_1}, \ldots, \epsilon_M I_{m_M})$, where I_k denotes the $k \times k$ identity matrix. With (2) one associates the *reduced system*

$$\dot{x} = A(t)x + B(t)y \tag{3a}$$

$$0 = C(t)x + D(t)y \tag{3b}$$

obtained by formally substituting $\epsilon = 0$ in (2).

The following assumptions are now made about the matrices A, B, C, D.

(H1) The matrices $A(t), B(t), C(t), D(t)$ are bounded and depend continuously on t for $t \geq 0$.

(H2) There is a $d > 0$ such that the eigenvalues of $D(t)$ all have magnitude $\geq d$ for all $t \geq 0$.

Hypothesis (H2) implies (3) is equivalent to the system

$$\dot{x} = [A(t) - B(t)D^{-1}(t)C(t)]\, x =: A_0(t)x. \tag{4}$$

It also implies that $D^{-1}(t)$ (and hence also $A_0(t)$) is bounded on $0 \leq t < \infty$. To see this, note that (H2) implies $|\det D(t)| \geq d^m$, so that Lemma 1 of Coppel ([13], p. 47) implies

$$|D^{-1}(t)| \leq (2^m - 1)d^{-m}\ |D(t)|^{m-1} \tag{5}$$

and the conclusion follows from (H1).

The following definitions are not used directly in the paper, but their relevance to the situation dictates that they be included for completeness. Recall ([33], p. 276) that a matrix F is said to be D*-stable* if the eigenvalues of DF have strictly negative real parts for any diagonal matrix D with strictly positive diagonal elements. The following generalization is due essentially to Khalil and Kokotovic [21].

Definition 1. The matrix $D(t)$ is said to be *block* D*-stable* relative to the multi-index $(m_1, \cdots, m_M)$ if for all $\theta_i > 0$, $i=1,\ldots,M$,

$$\mathrm{Re}\,\sigma(\,\Theta(\theta)D(t)\,) < 0 \tag{6}$$

for all $t \geq 0$, where $\Theta(\theta) :=$ *block diag* $(\theta_1 I_{m_1}, \ldots, \theta_M I_{m_M})$.

If $D(t)$ is not block D-stable, it may still be possible to find a set $H \subset R_+^M$ satisfying the next definition. Satisfaction of this definition was a main hypothesis in [21].

Definition 2. The matrix $D(t)$ *possesses Property* D relative to the set $H \subset R_+^M$ if there is a $\sigma_1 > 0$ such that

$$\mathrm{Re}\,\sigma(\,|\,\epsilon\,|\,E^{-1}(\epsilon)D(t)\,) < -\sigma_1 \tag{7}$$

for all $t \geq 0$, $\epsilon \in H$.

Remark 1. Note that if $D(t)$ possesses Property D relative to R_+^M then $D(t)$ is block D-stable according to Definition 1, and *vice versa*.

The results of this paper will apply as $\epsilon \to 0$ in ***any*** subset H of R_+^M for which all the mutual ratios ϵ_i/ϵ_j are bounded. These sets are cones of the form specified by Eq. (12) of hypothesis (H4) below. This generality is achieved based on hypothesis (H3) below, which was discussed but not enforced in [21].

(H3) There exists a continuously differentiable block diagonal positive definite matrix $P(t)$ with $\dot{P}(t)$ bounded,

$$P(t) = \textit{block diag}\,[P_1(t),\ldots,P_M(t)] \tag{8}$$

satisfying

$$c_1|y|^2 \leq y'\,P(t)y \leq c_2|y|^2, \quad \text{for all } y \in R^m,\ t \geq 0, \tag{9}$$

such that $Q(t)$ given by

$$P(t)D(t) + D'(t)P(t) = -Q(t) \tag{10}$$

is positive definite, and moreover satisfies

$$y'\,Q(t)y \geq c_3|y|^2, \quad \text{for all } y \in R^m,\ t \geq 0. \tag{11}$$

In (8), $P_i(t) \in R^{m_i}$, $i = 1,\ldots,M$. Note that Eq. (9) implies $P(t)$ is bounded.

This hypothesis implies that $D(t)$ is a block D-stable matrix (cf. Johnson [19], Khalil and Kokotovic [21]). It has been noted to yield the most interesting class of D-stable matrices [19]. It has also been employed by Khalil [24] to derive upper bounds on the small parameters for asymptotic stability of a class of nonlinear *autonomous* multiparameter singularly perturbed systems. Hypothesis (H3) is useful since it implies that $v(t,y) = y'\,P(t)E(\epsilon)y$ is a Liapunov function for the boundary layer system (13) (see [21]). The next hypothesis has been introduced in [21].

(H4) The parameters ϵ_i have bounded mutual ratios. That is, there exist positive numbers k_{ij}, K_{ij}, $i,j = 1,\ldots,M$ such that $\epsilon \in H$ where the cone $H \subset R^M$ is given by

$$H := \{ \epsilon \in R_+^M : k_{ij} \le \frac{\epsilon_i}{\epsilon_j} \le K_{ij} \}. \tag{12}$$

It will become apparent in the sequel that a natural boundary layer system associated with (1) is

$$\frac{dy}{dt} = E^{-1}(\epsilon)D(t)y. \tag{13}$$

Note that, contrary to the situation in single parameter singular perturbations, the boundary layer system depends on ϵ. This is the essence of the difficulties encountered in multiparameter perturbations. One can also define the boundary layer system in a suitable sped-up time scale, such as $\tau := t/|\epsilon|$.

Motivated by the single parameter theory, one attempts to find conditions under which the uniform asymptotic stability of (1) is ensured for sufficiently small $|\epsilon|$, $\epsilon \in H$, by that of the reduced system (4) and the boundary layer system (13). This is the spirit of the results in Khalil and Kokotovic [21] and of this work. The proof of this paper results in a computable upper bound E_0 on $|E(\epsilon)| = (m_1\epsilon_1^2 + \cdots + m_M\epsilon_M^2)^{1/2}$ ensuring uniform asymptotic stability of (1). This is of course equivalent to obtaining an upper bound on a weighted norm of ϵ. An (in general more conservative) upper bound on $|\epsilon|$ is easily obtained from these results (for any norm). An examination of the proof of [21] shows that it does not yield such an upper bound. This is mainly because [21] employs certain results of Coppel [12] which are based on compactness arguments. Note, however, that the result of [21] applies in case hypothesis (H3) above is not in force, if $D(t)$ possesses Property D (cf. Definition 2 above) relative to a conic set H of the form specified by Eq. (12). Hypothesis (H3) implies Property D, but the reverse implication does not hold [21].

III. DECOUPLING OF FAST AND SLOW DYNAMICS

It is well known [25, 10, 27, 32, 26] that for (nondegenerate) single parameter singularly perturbed systems it is possible to exhibit a nonsingular similarity transformation which exactly separates fast and slow dynamics. This transformation was presented by Chang [10] in the context of a general linear singularly perturbed boundary value problem. Chang's transformation is best understood as the composition of two simpler transformations. The first, derived by Chang in [9], results in block-triangularization of the system dynamics. The second transformation applied to the block-triangular system produces the desired block-diagonal (i.e., separated) form. A direct generalization of Chang's transformation to the multiparameter setting was applied to the stability analysis of multiparameter singularly perturbed systems by Khalil and Kokotovic [21]. A related transformation was used by the present author to study multiparameter singularly perturbed Hopf bifurcation in [4] as well as to obtain general results on stability of time-invariant multiparameter singularly perturbed systems in [3]. In [4] and [3] it was necessary to employ only the first step of Chang's transformation, yielding a block-triangular system. Also, the results of [4] and [3] apply regardless of the relative magnitudes of the small parameters. In [21] the full transformation was employed to completely separate the fast and slow dynamics of a linear time-varying multiparameter singularly perturbed system of the form (1). This is necessary because of the time-varying nature of (1). Thus a complete separation of fast and slow dynamics will also be used in this work. For an example illustrating the possible adverse effect of a (small) off-diagonal term on the stability of an otherwise stable linear time-varying system, see ([16], pp. 151-153).

The results of Chang [10] (see also [9, 21]) imply that the transformation

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} I - ME(\epsilon)L & -ME(\epsilon) \\ L & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{14}$$

applied to (1) will result in the (decoupled) system

$$\dot{\eta} = [A(t) - B(t)L(t,\epsilon)]\,\eta \tag{15a}$$

$$E(\epsilon)\dot{\xi} = [D(t) + E(\epsilon)L(t,\epsilon)B(t)]\,\xi \tag{15b}$$

provided $L(t,\epsilon)$ and $M(t,\epsilon)$ are solutions of the respective matrix differential equations

$$E(\epsilon)\dot{L} = DL - C - E(\epsilon)LA + E(\epsilon)LBL\,, \tag{16}$$

$$\dot{M}E(\epsilon) = (A - BL)ME(\epsilon) - M(D + E(\epsilon)LB) + B \tag{17}$$

defined for $0 \le t < \infty$.

The transformation (14) is easily verified to be a nonsingular similarity transformation for any value of ϵ, for any matrices L and M. See Eq. (57) of Section IV for the inverse transformation. Conditions will now be given for the equations (16) and (17) to have uniformly bounded solutions for all $\epsilon \in H$ with $|\epsilon|$ sufficiently small.

The next two Lemmas show that under (H1)-(H4), uniformly bounded solutions of (16) and (17) exist on $0 \le t < \infty$ for all $\epsilon \in H$ with $|\epsilon|$ sufficiently small. This fact follows from Lemmas 1 and 2 of [21]. Invoking hypothesis (H3), however, facilitates the constructive proofs of the Lemmas presented below which in addition yield explicit upper bounds on $\|E(\epsilon)\|$ ensuring the existence of these bounded solutions.

Lemma 1. Under hypotheses (H1)-(H4), there is a scalar $E_1 > 0$ such that Eq. (16) has a solution $L(t,\epsilon)$ which is uniformly bounded for $\|E(\epsilon)\| < E_1$, $\epsilon \in H$, $t \ge 0$. Moreover, the solution with initial condition $L(0,\epsilon) = -D^{-1}(0)C(0)$ is uniformly bounded for $\|E(\epsilon)\| < E_1$, $\epsilon \in H$.

Lemma 2. Let hypotheses (H1)-(H4) hold, and suppose $\|E(\epsilon)\| < E_1$, where E_1 is as in Lemma 1. Then there exists a scalar $E_2 > 0$ such that Eq. (17) has a solution $M(t,\epsilon)$ which is uniformly bounded for $\|E(\epsilon)\| < E_2$, $\epsilon \in H$, $t \ge 0$. Moreover, E_2 may be chosen so that $\|E(\epsilon)\| < E_2$ also implies the uniform asymptotic stability of the null solution of the fast subsystem (15b).

The proof of Lemma 1 will make use of the following elementary stability result, which is Lemma 1 in LaSalle and Lefschetz ([28], pp. 116-117). First some notation. Given a closed set $M \subset R^n$ and a positive scalar r, let M_r denote the set of all points whose distance from M is less than r. Also, let M^c, respectively M_r^c denote the set of points outside M (i.e., the complement of M), respectively M_r.

Proposition 1. Consider a system $\dot{x} = f(t,x)$, $t \ge 0$, $x \in R^n$, $t \ge 0$. Let $v(t,x)$ be a scalar function continuously differentiable in t and x for $t \ge 0$, $x \in R^n$, and let M be a closed set in R^n. If $\dot{v}(t,x) \le 0$ for all $x \in M^c$ and if $v(t_1,x_1) < v(t_2,x_2)$ for all $t_2 \ge t_1 \ge 0$, all $x_1 \in M$ and all $x_2 \in M_r^c$, then each solution of $\dot{x} = f(t,x)$ which at some time t_0 is in M can never thereafter leave M_r.

Remark 2. From the proof of Proposition 1 in [28] it is clear that the conclusion of Proposition 1 holds if $\dot{v}(t,x) \le 0$ is assumed to hold on $M_r - M$ rather than on all of M^c.

Proof of Lemma 1: It is straightforward to verify that the matrix differential equation (16) is equivalent to the vector differential equation

$$\overline{E}(\epsilon)\dot{\lambda}(t) = \overline{D}(t)\lambda(t) - \overline{C}(t) - \overline{E}(\epsilon)\overline{L}(t)\overline{A}(t) + \overline{E}(\epsilon)\overline{L}(t)\overline{B}(t)\lambda(t), \tag{18}$$

where the vector $\lambda \in R^{mn}$ is obtained from the $m \times n$ matrix $L = (L^1, \ldots, L^n)$ by concatenating the columns L^i, $i = 1,\ldots,n$ of L : $\lambda' := ((L^1)', \ldots, (L^n)')$. Denote the columns of $A(t)$, respectively $C(t)$, by $A^1(t),\ldots,A^n(t)$, respectively $C^1(t),\ldots,C^n(t)$. The matrices $\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}, \overline{L}$ appearing in Eq. (18) are defined as follows (here, *block diag* $(X,\ldots,X)$ implies n occurrences of X in the parentheses, for any matrix X):

$$\bar{A}(t) := block\ col\ (A^1(t),...,A^n(t)) \in R^{n^2}, \tag{19a}$$

$$\bar{B}(t) := block\ diag\ (B(t),...,B(t)) \in R^{n^2 \times mn}, \tag{19b}$$

$$\bar{C}(t) := block\ col\ (C^1(t),...,C^n(t)) \in R^{mn}, \tag{19c}$$

$$\bar{D}(t) := block\ diag\ (D(t),...,D(t)) \in R^{mn \times mn}, \tag{19d}$$

$$\bar{E}(\epsilon) := block\ diag\ (E(\epsilon),...,E(\epsilon)) \in R^{mn \times mn}, \tag{19e}$$

$$\bar{L}(t) := block\ diag\ (L(t),...,L(t)) \in R^{mn \times n^2}. \tag{19f}$$

Note that $\bar{L}(t)$ in (18) depends linearly on $\lambda(t)$, by (19f) and the definition of λ.

Define the Liapunov function candidate $v(t,\lambda)$ by

$$v(t,\lambda) := \lambda' \bar{P}(t)\bar{E}(\epsilon)\lambda \tag{20}$$

where $\bar{P}(t) :=$ *block diag* $(P(t),...,P(t))$. By (H3) $P(t)$ is block diagonal with the same structure as $E(\epsilon)$. Therefore $P(t)$ and $E(\epsilon)$ commute, implying that $\bar{P}(t)$ and $\bar{E}(\epsilon)$ also commute. Using this fact, $\dot{v}(t,\lambda)$ may be computed along trajectories of (18) as

$$\begin{aligned}\dot{v}(t,\lambda) = {} & \lambda' \{ \bar{D}'(t)\bar{P}(t) + \bar{P}(t)\bar{D}(t) \}\lambda + \lambda' \{ \frac{d}{dt}\bar{P}(t)\bar{E}(\epsilon) \}\lambda \\ & + \lambda' \{ \bar{B}'(t)\bar{L}'(t)\bar{P}(t)\bar{E}(\epsilon) + \bar{P}(t)\bar{E}(\epsilon)\bar{L}(t)\bar{B}(t) \}\lambda \\ & - \{ \bar{C}'\bar{P}(t)\lambda + \lambda'\bar{P}(t)\bar{C}(t) \} \\ & - \{ \bar{A}'(t)\bar{L}'(t)\bar{P}(t)\bar{E}(\epsilon)\lambda + \lambda'\bar{P}(t)\bar{E}(\epsilon)\bar{L}(t)\bar{A}(t) \}. \end{aligned} \tag{21}$$

Now using Eqs. (10), (11) and the fact that $|\bar{L}| = n^{1/2}|\lambda| = |\bar{L}'|$ (since $|\bar{L}|$ is the Frobenius norm of $\bar{L}$), (21) implies

$$\begin{aligned}\dot{v}(t,\lambda) \le {} & -c_3|\lambda|^2 + |\bar{E}(\epsilon)|\,|\frac{d}{dt}\bar{P}|\,|\lambda|^2 + 2n^{1/2}|\bar{E}(\epsilon)|\,|\bar{B}|\,|\bar{P}|\,|\lambda|^3 \\ & + 2|\bar{P}|\,|\bar{C}|\,|\lambda| + 2n^{1/2}|\bar{E}(\epsilon)|\,|\bar{A}|\,|\bar{P}|\,|\lambda|^2. \end{aligned} \tag{22}$$

Define $\mu := |\bar{E}(\epsilon)|$ and the parametrized cubic polynomial $p_\mu(\alpha)$ by

$$\begin{aligned}p_\mu(\alpha) := {} & \{ 2\mu n^{1/2}|\bar{B}|\,|\bar{P}| \}\alpha^3 + \{ \mu|\frac{d}{dt}\bar{P}| + 2\mu n^{1/2}|\bar{A}|\,|\bar{P}| - c_3 \}\alpha^2 \\ & + \{ 2|\bar{P}|\,|\bar{C}| \}\alpha. \end{aligned} \tag{23}$$

Note that for $\mu = 0$, $p_\mu(\alpha)$ reduces to a quadratic which takes negative values for all $\alpha > (2|\bar{P}|\,|\bar{C}|/c_3)$. For small $\mu > 0$ the cubic term dominates for large α and $p_\mu(\alpha)$ is positive for all sufficiently large α.

Recalling that $\bar{P}(t)$ and $\bar{E}(\epsilon)$ commute, it follows from (20) that $v(t,\lambda) = \lambda'\bar{E}^{1/2}(\epsilon)\bar{P}(t)\bar{E}^{1/2}(\epsilon)\lambda = \{\bar{E}^{1/2}(\epsilon)\lambda\}'\bar{P}(t)\{\bar{E}^{1/2}(\epsilon)\lambda\}$. Eq. (9) of hypothesis (H3) now implies that

$$c_1|\bar{E}^{1/2}(\epsilon)\lambda|^2 \le v(t,\lambda) \le c_2|\bar{E}^{1/2}(\epsilon)\lambda|^2. \tag{24}$$

Schwarz's inequality implies that $|\bar{E}^{1/2}(\epsilon)\lambda| \le |\bar{E}^{1/2}(\epsilon)|\,|\lambda|$ and that

$$|\lambda| = |\bar{E}^{-1/2}(\epsilon)\bar{E}^{1/2}(\epsilon)\lambda| \leq |\bar{E}^{-1/2}(\epsilon)|\,|\bar{E}^{1/2}(\epsilon)\lambda| \,. \tag{25}$$

Therefore

$$|\bar{E}^{1/2}(\epsilon)\lambda| \geq \frac{|\lambda|}{|\bar{E}^{-1/2}(\epsilon)|} \,. \tag{26}$$

Hence $v(t,\lambda)$ satisfies

$$\frac{c_1}{|\bar{E}^{-1/2}(\epsilon)|^2}|\lambda|^2 \leq v(t,\lambda) \leq c_2|\bar{E}^{1/2}(\epsilon)|^2\,|\lambda|^2. \tag{27}$$

A further inequality which will be employed below is

$$|\bar{E}^{-1/2}(\epsilon)|^2\,|\bar{E}^{1/2}(\epsilon)|^2 \leq n^2\bar{K}, \tag{28}$$

where $\bar{K}$ is given by

$$\bar{K} := \sum_{i=1}^{M} m_i^2 + \sum_{i<j} m_i m_j \,(K_{ij} + K_{ji}) \tag{29}$$

and the K_{ij} have been defined in Eq. (12) of (H4). The inequality (28) may be easily obtained from (11) if one recalls that $|\bar{E}^{\pm 1/2}| = n^{1/2}\,|E^{\pm 1/2}|$ since the matrix norm is the Frobenius norm.

One now applies Proposition 1. Define the set $M(\beta)$ by

$$M(\beta) := \{\, \lambda \in R^{mn} \,:|\lambda| \leq [\,\beta + \max\,(\,|\bar{D}^{-1}(0)|\,|\bar{C}(0)|,\, 2|\bar{P}|\,|\bar{C}|/c_3\,)\,] =: \alpha_1(\beta)\,\} \tag{30}$$

where $\beta > 0$ is arbitrary. Let the set $M_r(\beta)$ be defined as

$$M_r(\beta) := \{\, \lambda \in R^{mn} \,:|\lambda| \leq n\bar{K}^{1/2}\,(\frac{c_2}{c_1})^{1/2}\,\alpha_1(\beta) =: \alpha_2(\beta)\,\}. \tag{31}$$

Using inequalities (27), (28) above it is not difficult to show that $\lambda_1 \in M(\beta)$ and $\lambda_2 \in M_r^c(\beta)$ implies that $v(t_1,\lambda_1) < v(t_2,\lambda_2)$ for all $t_2 \geq t_1 \geq 0$, for *any* $\epsilon \in H$.

Next an upper bound $\mu_1(\beta)$ on μ will be obtained such that $p_\mu(\alpha) < 0$ for all $\alpha \in [\alpha_1(\beta),\, \alpha_2(\beta)]$ whenever $0 < \mu < \mu_1(\beta)$. The existence of such an upper bound, along with the preceding conclusions, implies that Proposition 1 applies so that any solution of Eq. (18) with initial condition in $M(\beta)$ will remain in $M_r(\beta)$ for all $t \geq 0$ if $|\bar{E}(\epsilon)| < \mu_1(\beta)$. This in will in turn imply the existence of solutions $L(t,\epsilon)$ to Eq. (16) bounded by $\alpha_2(\beta)$ (since $|L| = |\lambda|$).

Proceeding, it is easy to see that (23) implies that for any $\alpha > 2|\bar{P}|\,|\bar{C}|/c_3$, $p_\mu(\alpha) < 0$ for all $\mu \in [0,\, \mu^*(\alpha))$ where

$$\mu^*(\alpha) := \frac{c_3\alpha - 2|\bar{P}||\bar{C}|}{2n^{1/2}\,|\bar{B}|\,|\bar{P}|\alpha^2 + \{\,|\frac{d}{dt}\bar{P}| + 2n^{1/2}\,|\bar{A}|\,|\bar{P}|\,\}\,\alpha}. \tag{32}$$

From (30) and (31) it is clear that for any $\beta > 0$ and any $\alpha \in [\alpha_1(\beta),\, \alpha_2(\beta)]$, one has $\mu^*(\alpha) > 0$. Define

$$\mu_1(\beta) := \min_{\alpha_1(\beta)\leq\alpha\leq\alpha_2(\beta)} \mu^*(\alpha) \tag{33}$$

which is clearly positive for any $\beta > 0$.

An application of Proposition 1 and Remark 2 now implies that for any $\beta > 0$, all solutions of Eq. (18) with initial condition in $M(\beta)$ will remain in $M_r(\beta)$ for all $t \geq 0$, if $|\bar{E}(\epsilon)| < \mu_1(\beta)$. Noting that $M_r(\beta)$ is bounded and contains (by construction) the point $\lambda = -\bar{D}^{-1}(0)\bar{C}(0)$ (corresponding in Eq. (16) to $L = -D^{-1}(0)C(0)$) completes the proof of Lemma 1.

Q.E.D.

The preceding proof is constructive in that it also provides an explicit upper bound on $|E(\epsilon)|$ (actually on the related quantity $|\bar{E}(\epsilon)|$) ensuring the existence of bounded solutions to (16). Even more, it provides a family of upper bounds, one for each $\beta > 0$. The next corollary summarizes these observations to give an 'optimal' upper bound on $|E(\epsilon)|$.

Corollary 1. The upper bound E_1 on $|E(\epsilon)|$ in Lemma 1 may be taken as

$$E_1 := n^{-1/2} \sup_{\beta>0} \{ \mu_1(\beta) \} \tag{34}$$

where $\mu_1(\beta)$ is given by Eq. (33).

Next a proof will be given for Lemma 2. This proof relies on first ensuring the uniform asymptotic stability of the fast subsystem (15b) and then using an explicit representation for M to prove uniform boundedness.

Proof of Lemma 2: Let $Y(t,s,\epsilon)$, respectively $Z(t,s,\epsilon)$, denote the state transition matrices of systems (15a) and (15b). Consider the variation of the Liapunov function candidate $w(t,\xi)$ defined by

$$w(t,\xi) := \xi' P(t)E(\epsilon)\xi \tag{35}$$

along trajectories of Eq. (15b), the fast subsystem. Since $|E(\epsilon)| < E_1$, the foregoing proof of Lemma 1 implies $|L| < \alpha_2(\beta^*)$, where β^* achieves the supremum indicated in Eq. (34) ($\beta^* = \infty$ is a possibility). One has

$$\begin{aligned} \dot{w}(t,\xi) &= \xi' \{ D'(t)P(t) + P(t)D(t) \}\xi \\ &\quad + \xi' \{ B'(t)L'(t,\epsilon)P(t)E(\epsilon) + P(t)E(\epsilon)L(t,\epsilon)B(t) + \dot{P}(t)E(\epsilon) \}\xi \\ &\leq \{ -c_3 + |E(\epsilon)| (2|B||L||P| + |\dot{P}|) \} |\xi|^2 \\ &\leq \{ -c_3 + \nu (2\alpha_2(\beta^*)|B||P| + |\dot{P}|) \} |\xi|^2 \\ &=: -\bar{c}_3(\nu) |\xi|^2 \end{aligned} \tag{36}$$

where the additional constraint

$$|E(\epsilon)| < \nu \leq \frac{c_3}{2\alpha_2(\beta^*)|B||P| + |\dot{P}|} =: \nu^* \tag{37}$$

has been imposed on $|E(\epsilon)|$, and ν is an auxiliary parameter. Eq. (37) ensures that $\bar{c}_3(\nu)$ of (36) above will be positive implying the null solution of Eq. (15b) is uniformly asymptotically stable.

Note that, by Eq. (9),

$$\begin{aligned} w(t,\xi) &= (E^{1/2}(\epsilon)\xi)' P(t)(E^{1/2}(\epsilon)\xi) \\ &\leq c_2 |E^{1/2}(\epsilon)\xi|^2 \\ &\leq c_2 |E^{1/2}(\epsilon)|^2 |\xi|^2. \end{aligned} \tag{38}$$

Therefore, along trajectories of (15b),

$$|\xi|^2 \geq \frac{w(t,\xi)}{c_2 |E^{1/2}(\epsilon)|^2}. \tag{39}$$

Eq. (36) now implies the differential inequality

$$\dot{w}(t,\xi) \leq -\frac{\bar{c}_3(\nu) w(t,\xi)}{c_2 |E^{1/2}(\epsilon)|^2}, \tag{40}$$

so that $w(t,\xi)$ satisfies

$$w(t,\xi(t)) \le e^{-\frac{\bar{c}_3(\nu)}{c_2|E^{1/2}(\epsilon)|^2}(t-s)} w(s,\xi(s)) \tag{41}$$

for any $t \geq s \geq 0$. Recalling the definition (35) of w, this implies (by the Schwarz inequality)

$$w(t,\xi) \le |E(\epsilon)|\,|P(s)|\,|\xi(s)|^2 e^{-\frac{\bar{c}_3(\nu)}{c_2|E^{1/2}(\epsilon)|^2}(t-s)} \tag{42}$$

for any $t \geq s \geq 0$.

Eq. (9) of (H3) is now applied once more (the dependence of E on ϵ is now suppressed):

$$\begin{aligned} w(t,\xi) &= (E^{1/2}\xi)'\,P(t)(E^{1/2}\xi) \\ &\ge c_1\,|E^{1/2}\xi|^2 \\ &\ge \frac{c_1}{|E^{-1/2}|^2}|\xi|^2. \end{aligned} \tag{43}$$

Inequalities (42) and (43) together imply

$$|\xi(t)|^2 \le c_1^{-1}|P(s)|\,|\xi(s)|^2\,|E|\,|E^{-1/2}|^2\, e^{-\frac{\bar{c}_3(\nu)}{c_2|E^{1/2}|^2}(t-s)} \tag{44}$$

for all $t \geq s \geq 0$.

Next consider the implication of (44) for the state transition matrix $Z(t,s,\epsilon)$ of Eq. (15b). Since (44) is satisfied for each of the columns Z^i of Z and since $|Z|^2 = \sum|Z^i|^2$ (Frobenius norm), and noting that $Z(s,s,\epsilon) = I$, one obtains the inequality

$$|Z(t,s,\epsilon)|^2 \le c_1^{-1}|P(s)|\,|E|\,|E^{-1/2}|^2\, e^{-\frac{\bar{c}_3(\nu)}{c_2|E^{1/2}|^2}(t-s)} \tag{45}$$

However, note that by the Schwarz inequality and (28)

$$\begin{aligned} |E|\,|E^{-1/2}|^2 &= |E^{1/2}\,E^{1/2}|\,|E^{-1/2}|^2 \\ &\le |E^{1/2}|^2\,|E^{-1/2}|^2 \\ &\le \bar{K}^2 \end{aligned} \tag{46}$$

where $\bar{K}$ was defined in (29). Eqs. (45) and (46) now imply

$$|Z(t,s,\epsilon)| \le \bar{K} c_1^{-1/2}\,|P(s)|^{1/2}\, e^{-\frac{\bar{c}_3(\nu)}{2c_2|E^{1/2}|^2}(t-s)} \tag{47}$$

for all $t \geq s \geq 0$.

Note that the state transition matrix $Y(t,s,\epsilon)$ of Eq. (15a) satisfies

$$|Y(t,s,\epsilon)| \le \bar{Y}(\epsilon)e^{\sigma_0|t-s|} \tag{48}$$

for some $\bar{Y}$, $\sigma_0 > 0$, since the coefficient matrix is bounded. Indeed, a specific σ_0 is given as

$$\sigma_0 := |A| + |B|\,\alpha_2(\beta^*) \tag{49}$$

where the function $\alpha_2(\beta)$ is defined in Eq. (31) and β^* has been defined above as that β which yields E_1 in Eq. (34).

It can be verified by differentiation that

$$M(t,\epsilon) = -\int_t^\infty Y(t,s,\epsilon)B(s)Z(s,t,\epsilon)ds\;\; E^{-1}(\epsilon) \tag{50}$$

is a solution of Eq. (17) (cf. Chang [10]). Using Eqs. (47) and (48), one now has

$$|M(t,\epsilon)| \leq c_1^{-1/2}\overline{K}\overline{Y}(\epsilon)|B|\,|P|^{1/2}\,|E^{-1}(\epsilon)| \int_t^{\infty} e^{\sigma_0(s-t)} e^{-\frac{\overline{c}_3(\nu)}{2c_2|E^{1/2}|^2}(s-t)}\, ds$$

$$\leq c_1^{-1/2}\overline{K}\overline{Y}(\epsilon)|B|\,|P|^{1/2}\,|E^{-1}(\epsilon)|\frac{e^{-\sigma(\nu,\epsilon)t}}{\sigma(\nu,\epsilon)} \lim_{T\to\infty}\{e^{\sigma(\nu,\epsilon)T} - e^{\sigma(\nu,\epsilon)t}\} \tag{51}$$

where $\sigma(\nu,\epsilon)$ is defined as

$$\sigma(\nu,\epsilon) := \sigma_0 - \frac{\overline{c}_3(\nu)}{2c_2\,|E^{1/2}(\epsilon)|^2}. \tag{52}$$

Therefore $M(t,\epsilon)$ is bounded for all ϵ for which $\sigma(\nu,\epsilon) < 0$, which is equivalent to the inequality

$$|E^{1/2}(\epsilon)|^2 < \frac{\overline{c}_3(\nu)}{2\sigma_0 c_2}. \tag{53}$$

Eq. (53) can be used to yield an upper bound on $|E(\epsilon)|$ as follows. Suppose ϵ were constrained so that $|E(\epsilon)|^2 = m_1\epsilon_1^2 + \ldots + m_M\epsilon_M^2 < k$ where $k > 0$ is arbitrary. This would clearly imply $\epsilon_i < (k/m_i)^{1/2}$, $i = 1,\ldots,M$, so that $|E^{1/2}(\epsilon)|^2 = (\sum_{i=1}^{M} m_i\epsilon_i)^2 < k(\sum_{i=1}^{M} m_i^{1/2})^2$. Equating the right side of this last inequality with the right side of (53) and solving for k shows that (53) is implied by the following upper bound on $|E(\epsilon)|^2$:

$$|E(\epsilon)|^2 < \frac{\overline{c}_3(\nu)}{2\sigma_0 c_2}\left(\sum_{i=1}^{M} m_i^{1/2}\right)^{-2}. \tag{54}$$

Note that the auxiliary parameter ν is still arbitrary subject to $0 < \nu < \nu^*$ (Eq. (37)). To optimize the upper bound, one maximizes the lesser of the two upper bounds imposed by Eqs. (37) and (54). Thus it is required that

$$|E(\epsilon)| < E_2 \tag{55}$$

where the upper bound E_2 is given by

$$E_2 := \max_{0\leq\nu\leq\nu^*} \min\left\{\nu, \left(\frac{\overline{c}_3(\nu)}{2\sigma_0 c_2}\right)^{1/2}\left(\sum_{i=1}^{M} m_i^{1/2}\right)^{-1}\right\}. \tag{56}$$

Finally, the uniform boundedness of $M(t,\epsilon)$ can be shown by applying the inequality (28) to obtain an upper bound on (51). Such an upper bound will not be derived here. This is because the *uniform* boundedness of $M(t,\epsilon)$ is not needed for the stability considerations of this paper, and since ensuring uniform boundedness of $M(t,\epsilon)$ would require a further constraint on $|E(\epsilon)|$.

Q.E.D.

IV. UNIFORM ASYMPTOTIC STABILITY

The decoupling transformation (14) is invertible for any ϵ. Indeed, it is easy to check that the inverse transformation is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I & ME \\ -L & I - LME \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \tag{57}$$

From (14) and (57) it is clear that for bounded $L(t,\epsilon)$, $M(t,\epsilon)$, the uniform asymptotic stability of (1) is equivalent to the uniform asymptotic stability of (15a) and (15b).

An upper bound on $|\epsilon|$ ensuring uniform asymptotic stability of the fast subsystem (15b) has been derived in the foregoing analysis. It remains to find an upper bound ensuring the uniform asymptotic stability of the slow subsystem (15a).

It is useful to state the following basic Proposition, whose proof (essentially) may be found in Brockett ([6], p. 205).

Proposition 2. Let the equation $\dot{x}(t) = A(t)x(t)$ be uniformly asymptotically stable, and let $v(x) := x' R(t)x$ be a Liapunov function with $S(t) := -(A'(t)R(t) + R(t)A(t))$ positive definite. Then the null solution of the equation $\dot{x} = [A(t) + B(t)]x$ is also uniformly asymptotically stable for any $B(t)$ with $|B| < \delta$ where δ is given by

$$\delta := \frac{|S|}{2|R|}, \tag{58}$$

a necessarily positive quantity.

To apply this Proposition to Eq. (15a), it is necessary to find an estimate for the difference between the coefficient matrix in (15a) and the reduced system matrix $A_0(t)$ of Eq. (4). This estimate should depend on ϵ, and moreover vanish in the limit $\epsilon \rightarrow 0$, $\epsilon \in H$. The following mild hypothesis will be used in the derivation.

(H5) The matrix $D^{-1}(t)C(t)$ is continuously differentiable on $[0,\infty)$.

Defining the vector $l \in R^{mn}$ as

$$l := \lambda - \bar{D}^{-1}(t)\bar{C}(t). \tag{59}$$

Eq. (18) can be rewritten as

$$\bar{E}(\epsilon)\dot{l}(t) = -\bar{E}(\epsilon)\frac{d}{dt}\{\bar{D}^{-1}(t)\bar{C}(t)\} + \bar{D}(t)\,l(t) - \bar{E}(\epsilon)\bar{L}(t)\bar{A}(t) + \bar{E}(\epsilon)\bar{L}(t)\bar{B}(t)\lambda(t). \tag{60}$$

As in the proof of Lemma 1, the Liapunov function candidate

$$v(t,l) = l'\bar{P}(t)\bar{E}(\epsilon)l \tag{61}$$

is introduced. Evaluating $\dot{v}(t,l)$ along trajectories of (60) and proceeding as in the proof of Lemma 1, one obtains

$$\dot{v}(t,l) \leq -c_3|l|^2 + |\frac{d}{dt}\bar{P}||\bar{E}||l|^2 + 2|\bar{A}||\bar{P}||\bar{L}||\bar{E}||l| + 2|\bar{P}||\frac{d}{dt}(\bar{D}^{-1}\bar{C})||\bar{E}||l| + 2|\bar{P}||\bar{B}||\bar{L}||\lambda||\bar{E}||l|. \tag{62}$$

Recalling from the proof of Lemma 1 that $|E(\epsilon)| < E_1$, and that this implies $|\lambda| = |L| < \alpha_2(\beta^*)$, so that $|\bar{L}| < n^{1/2}\alpha_2(\beta^*)$, inequality (62) may be strengthened to

$$\dot{v}(t,l) \leq -\gamma(\kappa)|l|^2 + \rho|l||\bar{E}| \tag{63}$$

for $|E(\epsilon)| < \kappa$, where $0 < \kappa < \kappa^*$, κ^* is such that $\gamma(\kappa^*) = 0$, and where $\gamma(\kappa)$ and ρ are defined as

$$\gamma(\kappa) := c_3 - \kappa n^{1/2}|\frac{d}{dt}\bar{P}|, \tag{64}$$

$$\rho := 2n^{1/2}|\bar{A}||\bar{P}|\alpha_2(\beta^*) + 2|\bar{P}||\frac{d}{dt}(\bar{D}^{-1}\bar{C})| + 2n^{1/2}|\bar{P}||\bar{B}|(\alpha_2(\beta^*))^2, \tag{65}$$

and κ is an auxiliary parameter.

Application of Proposition 1 in a fashion similar to that in the proof of Lemma 1 now yields the following result, which is stated for Eq. (16) for convenience.

Lemma 3. Let (H1)-(H5) above hold, and moreover suppose $|E(\epsilon)| < E_1$ (Eq. (34)). Then for any $\delta > 0$ there exists a bounded solution $L(t,\epsilon)$ to Eq. (16) with $|L(t,\epsilon) - D^{-1}(t)C(t)| < \delta$, $t \geq 0$ whenever $|E(\epsilon)| < g(\kappa)\delta$ where $g(\kappa)$ is given by

$$g(\kappa) := n^{-3/2}\overline{K}^{-1/2}\frac{\gamma(\kappa)}{\rho}\left(\frac{c_1}{c_2}\right)^{1/2} \tag{66}$$

for any $0 \leq \kappa \leq \kappa^*$.

Proof: The Lemma follows by applying Proposition 1 to Eq. (60) using the Liapunov function v of (61) and the estimate (63) on $\dot{v}$. Define M_r of Proposition 1 as $\{ l \in R^{mn} : |l| < \delta \}$. Inequalities (27), (28) imply that the set M defined by $M := \{ l \in R^{mn} : |l| < n^{-1}\overline{K}^{-1/2}(c_1/c_2)^{1/2}\delta \}$ and M_r fit the set-up of Proposition 1. Eq. (63) is now used to show that if $|E(\epsilon)|$ ($= n^{-1/2}|\overline{E}(\epsilon)|$) $< g(\kappa)\delta$, then $\dot{v}(t,l) < 0$ on $M - M_r$. Proposition 1 and Remark 2 now assert the existence of a solution $l(t,\epsilon)$ of (60) with $|l| < \delta$. Recalling the one-to-one correspondence of solutions l of (60) and solutions L of (16) (see Eq. (59) and the definition of λ following (18)) completes the proof.

The reduced system (4) is now assumed uniformly asymptotically stable.

(H6) The null solution of the reduced system (4) is uniformly asymptotically stable.

To apply Proposition 2, note that (H4) implies the reduced system (4) has a quadratic Liapunov function $v(x) = x' R_0(t)x$ with $\dot{v}(x) = -x' S_0(t)x < 0$ along trajectories of $\dot{x} = A_0(t)$ (Eq. (4)). For a proof of this standard result and an explicit formula for $R_0(t)$ given any positive definite $S_0(t)$, see for instance Brockett ([6], Theorem 6, p. 203). Choosing a bounded $S_0(t)$, Proposition 2 and Lemma 3 now imply that the slow subsystem (15a) will be uniformly asymptotically stable if

$$|E(\epsilon)| < \min\left(\kappa, \frac{g(\kappa)\,|S_0|}{2\,|R_0|}\right). \tag{67}$$

Define E_3 as

$$E_3 := \max_{0 \leq \kappa \leq \kappa^*} \min\left(\kappa, \frac{g(\kappa)\,|S_0|}{2\,|R_0|}\right). \tag{68}$$

The main result of the paper may now be stated.

Theorem 1. Let hypotheses (H1)-(H6) hold. Then the null solution of the multiparameter singularly perturbed system (1) is uniformly asymptotically stable for all $\epsilon \in H$ with $|E(\epsilon)| < E_0$ where E_0 is the positive scalar given by

$$E_0 := \min(E_1, E_2, E_3) \tag{69}$$

and E_i, $i = 1,2,3$ are given in Eqs. (34), (56) and (68), respectively.

V. CONCLUSIONS

The paper has presented a derivation of an explicit upper bound on a weighted norm of the vector of singular perturbation parameters such that the multiparameter singularly perturbed system (1) is uniformly asymptotically stable if this upper bound is met, under certain technical assumptions. It is interesting to note that the assumption (H4) of bounded mutual ratios for the small parameters was crucial in the derivation, as was hypothesis (H3) on the matrix $D(t)$. Indeed, it is easily verified that the upper bound obtained here vanishes in the limit that the constraint (12) on the mutual ratios disappears. This can be checked by taking the limit as $\overline{K} \to \infty$ in (31), (33) to get $E_1 \to 0$, implying $E_0 \to 0$ by Eq. (69). The upper bound obtained by Khalil [24] in the (nonlinear) time-invariant case suffers from this same type of conservativeness.

Although hypothesis (H4) is valid for a large class of physical systems [21], it remains an interesting and open question as to the extent to which it can be relaxed in the time-varying case. In the companion paper [2], results similar to those given here are obtained for the time-invariant case. This is achieved with no *a priori* restriction on the small parameters ϵ_i, so that the results of [2] apply even under the multiple time scales hypothesis. Thus the conservativeness issue discussed above does not arise in the results presented in [2].

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