ABSTRACT

Title of dissertation: ASYMPTOTIC PROBLEMS RELATED TO SMOLUCHOWSKI-KRAMERS APPROXIMATION

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According to the Smoluchowski-Kramers approximation, the solution $q_t^{\mu,\varepsilon}$, also referred to as "Physical" Brownian motion, of the Langevin's equation $\mu\ddot{q}_t^{\mu,\varepsilon} = -\dot{q}_t^{\mu,\varepsilon} + b(q_t^{\mu,\varepsilon}) + \sqrt{\varepsilon}\sigma(q_t^{\mu,\varepsilon})\dot{W}_t,\ q_0^{\mu,\varepsilon} = q,\ \dot{q}_0^{\mu,\varepsilon} = p$, where $\dot{W}_t$ is Gaussian white noise, converges to solution of the diffusion equation $\dot{q}_t^{\varepsilon} = b(q_t^{\varepsilon}) + \sqrt{\varepsilon}\sigma(q_t^{\varepsilon})\dot{W}_t,\ q_0^{\varepsilon} = q$ as $\mu \downarrow 0$ uniformly on any finite time interval for each fixed $\varepsilon > 0$. This is the main justification for describing the small particle motion by a diffusion equation. However, this relation is not sufficient for asymptotic problems when some parameter, say ε, approaches 0.

We consider two asymptotic problems related to this approximation.

First, we study relations between large deviations for these processes $q_t^{\mu,\varepsilon}$ and q_t^{ε} as $\varepsilon \downarrow 0$. In particular, we consider exit problems where relations between asymptotic exit position, asymptotic mean exit time and some other characteristics of the first exit of the trajectories $q_t^{\mu,\varepsilon}$ and q_t^{ε} from a bounded domain are of interest. Under the framework of Freidlin-Wentzell, these asymptotics can be represented by

quasi-potential, defined as the infimum of action functional over some set. Action functional and quasi-potentials for $q_t^{\mu,\varepsilon}$ are calculated in this paper. We establish that the asymptotics of $q_t^{\mu,\varepsilon}$ and q_t^{ε} are close for small particles when $0 < \mu \ll 1$. We pay special attention to the case when $b(q)$ is linear. Then the quasi-potentials can be calculated explicitly and they coincide for $q_t^{\mu,\varepsilon}$ and q_t^{ε}.

Second, we study the wavefront propagation for reaction-diffusion equations with diffusion governed by the infinitesimal generator of process $q_t^{\mu,\varepsilon}$ and q_t^{ε} and reaction term governed by a nonlinear function of KPP-type. In this case, the reaction-diffusion equation related to the process $q_t^{\mu,\varepsilon}$ is degenerate in terms of variable (p, q). When the diffusion coefficient and nonlinear term are space dependent but only changing slowly in space, we know as $t \to \infty$, the solution of the reaction-diffusion equation related to the process q_t^{ε} behaves like a running wave. Characterization of the position of wavefront for equations related to q_t^{ε} is well studied. In this work, we identify two characterizations of the position of the wavefront for the degenerate reaction-diffusion equation related to the process $q_t^{\mu,\varepsilon}$. We analyze two cases, under which we can obtain the convergence of the wavefronts of the degenerate reaction-diffusion equation related to $q_t^{\mu,\varepsilon}$to those of the non-degenerate one related to q_t^{ε}, for small $\mu > 0$.

ASYMPTOTIC PROBLEMS RELATED TO SMOLUCHOWSKI-KRAMERS APPROXIMATION

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DEDICATION

TO MY PARENTS AND SISTER

FOR THEIR LOVE AND SUPPORT

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Chapter 1

Introduction

1.1 Smoluchowski-Kramers Approximation

The motion of a particle in the force field $b(q) + \sqrt{\varepsilon}\sigma(q)\dot{W}$ with a friction proportional to the velocity (let, for brevity, the friction coefficient be equal to 1) is governed by the Newton law:

$$\mu \ddot{q}_t^{\mu,\varepsilon} = b(q_t^{\mu,\varepsilon}) - \dot{q}_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\sigma(q_t^{\mu,\varepsilon})\dot{W}_t, \tag{1.1}$$

$$q_0^{\mu,\varepsilon} = q,\ \dot{q}_0^{\mu,\varepsilon} = p;\ p, q \in \mathbb{R}^n.$$

Here $\mu > 0$ is the particle mass, $\varepsilon > 0$ is a positive parameter, $\sigma(q)$ is a non-degenerate $n \times n$-matrix, $\dot{W}_t$ is Gaussian white noise in $\mathbb{R}^n$; the functions $b(q)$ and $\sigma(q)$ are supposed to have continuous bounded derivatives.

This motion is also referred to as "Physical" Brownian motion that is defined in Langevin's model of Brownian motion after the construction of "Mathematical" Brownian motion. Langevin's model emphasizes that a particle moving due to random collisions with, say, gas molecules does not actually experience independent steps since its inertia tends to keep it moving roughly the same direction as its previous steps. Thus, it is considered to be a more realistic model than "Mathematical" Brownian motion, which treats the process as a random walk with independent identically distributed steps. Equation (1.1) due to Langevin's work is also called

Langevin's equation.

The Smoluchowski-Kramers approximation (see [12]) consists of the statement: For each $T > 0,\ \delta > 0$ and $(p, q) \in \mathbb{R}^{2n}$,

$$\lim_{\mu \downarrow 0} P\{\max_{0 \le t \le T} |q_t^{\mu,\varepsilon} - q_t^{\varepsilon}| > \delta\} = 0, \tag{1.2}$$

where q_t^{ε} is the solution of the equation

$$\dot{q}_t^{\varepsilon} = b(q_t^{\varepsilon}) + \sqrt{\varepsilon}\sigma(q_t^{\varepsilon})\dot{W}_t,\ q_0^{\varepsilon} = q. \tag{1.3}$$

This statement is the main justification for describing small particle motion by the first order diffusion equation (1.3).

However, an essential part of modern research related to equation (1.3) concerns asymptotic problems. For example, for fixed $\varepsilon = 1$, one can study behavior of stochastic process defined by (1.3) as $t \to \infty$ and its stationary distribution. Another example is given by the homogenization problem for equation (1.3). Various large deviation problems were considered in recent years: when $\varepsilon \downarrow 0$, exit problems and stochastic resonance for process q_t^{ε} are of interest. Wavefront propagation for reaction-diffusion equation of KPP type related to the diffusion process defined by (1.3) is widely studied from both the stochastic and PDE point of view. How are these results for q_t^{ε} defined by (1.3) and $q_t^{\mu,\varepsilon}$ defined by (1.1) related? In what cases can we describe the asymptotic behavior of small particle motion by results obtained for the diffusion equation (1.3)? Statement (1.2) concerning a finite time interval is not sufficient for results of these asymptotic problems.

In this work, we will consider two kinds of asymptotic problems: exit problems and wavefront propagation of reaction-diffusion equation. We will investigate the relations between system (1.1) and (1.3) in exit problems for the general vector field $b(q)$. In the problem of wavefront propagation, we'd like to compare the move of the wavefront of reaction-diffusion equations related to process $q_t^{\mu,\varepsilon}$ and q_t^ε. Other asymptotic problems such as stationary distributions, homogenization problems and exit problems in the case when vector field $b(q)$ is potential are treated in M. Freidlin's work [6].

1.2 Large Deviations: Exit from a Domain

1.2.1 Exit problem for the diffusion equation

The problem of diffusion exit from a domain for a process q_t^ε defined by (1.3) is studied in [7]. Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂G, which is attracted to an asymptotically stable equilibrium K for the field $b(q)$. The unperturbed trajectories q_t^0 of the deterministic system $\dot{q}_t^0 = b(q_t^0)$ issuing from the point $q \in G$ go to the equilibrium K as $t \to \infty$ and can't leave G. Due to the white noise, the perturbed trajectories q_t^ε issuing from $q \in G$ leave G with probability one (and in this case for every $\varepsilon \neq 0$). The perturbed trajectory follows the unperturbed trajectory (with small deviations) to a neighborhood of the asymptotically stable equilibrium K in finite time, stays there for a dominating amount of time, making excursions now and then, and finally leaves the domain G. Put $\tau^\varepsilon = \inf\{t : q_t^\varepsilon \notin G\}$. The first exit time of the diffusion process q_t^ε from domain G, τ^ε; the asymptotic exit position $q_{\tau^\varepsilon}^\varepsilon$ and some other characteristics of the first exit of the trajectory from the domain G are of interest in exit problems.

Let C_{0T} be the collection of continuous functions on interval $[0,T]$. Under the framework of the Freidlin-Wentzell theory ([7]), the action functional, which gives an estimate of the principal term of the logarithmic asymptotics of probabilities of events concerning the process q_t^ε, can be introduced. The action functional for the process q_t^ε, $0 \le t \le T$, in C_{0T} as $\varepsilon \downarrow 0$ has the form $(1/\varepsilon)S_{0T}(\varphi)$, where

$$S_{0T}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\sigma^{-1}(\varphi_s)(\dot{\varphi}_s - b(\varphi_s))|^2 ds, & \text{if } \varphi \in C_{0T} \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

It has the following three properties ([7],[17]):

1. the set $\Phi(s) = \{\varphi \in C_{0T} : S_{0T}(\varphi) \le s\}$ is compact;

2. for any $\delta > 0$, any $\gamma > 0$ and any $\varphi \in C_{0T}$,

$$P\{\|q_t^\varepsilon - \varphi\|_{C_{0T}} \le \delta\} \ge \exp\{-\varepsilon^{-1}[S_{0T}(\varphi) + \gamma]\}$$

for $\varepsilon \le \varepsilon_0$;

3. for any $\delta > 0$, any $\gamma > 0$ and any $s > 0$, there exists an $\varepsilon_0 > 0$ such that

$$P\{\|q_t^\varepsilon - \Phi(s)\|_{C_{0T}} > \delta\} \le \exp\{-\varepsilon^{-1}(s - \gamma)\}$$

for all $\varepsilon \le \varepsilon_0$.

Here

$$\|\varphi(s)\|_{C_{0T}} = \sup\{\varphi(s) : \ s \in [0,T]\}.$$

Introduce the quasi-potential $V(q)$, $q \in \mathbb{R}^n$, for the processes q_t^ε with respect to K:

$$V(q) = \inf\{S_{0T}(\varphi) : \varphi \in C_{0T},\ \varphi_0 = K,\ \varphi_T = q,\ T \ge 0\}. \tag{1.5}$$

The Hamilton-Jacobi equation for $V(q)$ has the form

$$\frac{1}{2}|\nabla V(q)|^2 + (b(q), \nabla V) = 0,\ V(q) > 0\ for\ q \neq K,\ V(K) = 0.$$

In [7], it is shown that the asymptotics of the first exit of the trajectory q_t^ε from the domain G can be expressed through the quasi-potential $V(q)$. For example, $q^\varepsilon_{\tau^\varepsilon} \to q_0$ in probability as $\varepsilon \downarrow 0$, where $V(q_0) = \min_{q\in\partial G} V(q)$, if q_0 is the only minimum of $V(q)$ on ∂G. Moreover, τ^ε is logarithmically equivalent to $\exp\{(1/\varepsilon)V(q_0)\}$ as $\varepsilon \downarrow 0$, i.e. $\varepsilon \ln \tau^\varepsilon \to V(q_0)$. Some other characteristics of the first exit can be expressed through the quasi-potential $V(q)$. In the case when the vector field $b(q) = -\nabla B(q),\ V(q) = 2B(q)$ for $q \in \{q \in G : V(q) \leq V(q_0)\}$.

1.2.2 Exit problem for Langevin's equation

We study the exit problem for the process $q_t^{\mu,\varepsilon}$ using the same approach as for the study of exit problem of process q_t^ε. The second order system (1.1) can be written as the first order system

$$\mu \dot{p}_t^{\mu,\varepsilon} = b(q_t^{\mu,\varepsilon}) - p_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\sigma(q_t^{\mu,\varepsilon})\dot{W}_t, \tag{1.6}$$

$$\dot{q}_t^{\mu,\varepsilon} = p_t^{\mu,\varepsilon};\ p_0^{\mu,\varepsilon} = p,\ q_0^{\mu,\varepsilon} = q.$$

If $K \in \mathbb{R}^n$ is an equilibrium of the vector field $b(q)$, then $(0, K) \in \mathbb{R}^{2n}$ is an equilibrium for (1.6) with $\varepsilon = 0$, and vice versa. Moreover, one can check that if K is an asymptotically stable equilibrium for system (1.3) with $\varepsilon = 0$, then $(0, K)$ is asymptotically stable for (1.6) with $\varepsilon = 0$, at least, if $\mu > 0$ is small enough. If $b(q) = -\nabla B(q),\ q \in \mathbb{R}^n$, and if K is asymptotically stable for the field $b(q)$, then

$(0, K)$ is asymptotically stable for $(p_t^{\mu,0}, q_t^{\mu,0})$ with any $\mu > 0$. (See Section 2.1 of Chapter 2)

Put $\tau^{\mu,\varepsilon} = \inf\{t : q_t^{\mu,\varepsilon} \notin G\}$. The asymptotic position $q_{\tau^{\mu,\varepsilon}}^{\mu,\varepsilon}$ at the exit time $\tau^{\mu,\varepsilon}$, the asymptotics of $\tau^{\mu,\varepsilon}$ as $\varepsilon \downarrow 0$ and some other characteristics of the first exit of the trajectory from G are of interest.

The relation (1.2) concerns finite time intervals, so that it is not sufficient for closeness of the asymptotics in the exit problems for processes $q_t^{\mu,\varepsilon}$ and q_t^{ε} . But taking into account that exit of q_t^{ε} from G occurs as a result of many trials and that in each of these trials the trajectory spends a bounded time outside any neighborhood of the equilibrium, one can expect that the asymptotics in exit problem for q_t^{ε} and $q_t^{\mu,\varepsilon}$ as $\varepsilon \downarrow 0$ are close, at least for small μ.

To study large deviations of the process $q_t^{\mu,\varepsilon}$ defined by the Langevin's equation, one must first calculate the action functional for the process $q_t^{\mu,\varepsilon}$ as $\varepsilon \downarrow 0$.

Theorem 1.2.1 (Freidlin-Wentzell [7])**.** *Let $(1/\varepsilon)S^{\lambda}(x)$ be the action functional for a family of measures λ^{ε} on a space X (with metric ρ_X) as $\varepsilon \downarrow 0$. Let F be a continuous mapping of X into a space Y with metric ρ_Y and let a measure ν^{ε} on Y be given by the formula $\nu^{\varepsilon}(A) = \lambda^{\varepsilon}(F^{-1}(A))$. The asymptotics of the family of measures ν^{ε} as $\varepsilon \downarrow 0$ is given by the action function $(1/\varepsilon)S^{\nu}(y)$, where $S^{\nu}(y) = \min\{S^{\lambda}(x) : x \in F^{-1}(y)\}$ (the minimum over the empty set is set equal to ∞).*

By virtue of Theorem 1.2.1, we are able to calculate the action functional for the Markov process $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ and the process $q_t^{\mu,\varepsilon}$. We introduce the quasi-potential $V^{\mu}(q)$ for the processes $q_t^{\mu,\varepsilon}$ and show that $V^{\mu}(q)$ under certain wide conditions is close to $V(q)$. This means that the Smoluchowski-Kramers approximation is good if we are interested in the exit problems and also in the problems related to stochastic resonance. Moreover, if $b(q) = -\nabla B(q)$, the quasi-potentials $V^{\mu}(q)$ and $V(q)$, in a sense, coincide for all $\mu > 0$ (compare with [2]).

1.3 Wavefront Propagation in Reaction-Diffusion Equations

1.3.1 KPP-type Reaction-Diffusion Equation

In 1937, Fisher [2] and Kolmogorov, Petrovskii and Piskunov (KPP) [15] started to study the existence of travelling waves of semi-linear reaction-diffusion equations that arise in physics, chemical kinetics and biology, and to investigate convergence of the solution of a Cauchy problem to a travelling wave as $t \to \infty$. The original equation is:

$$\frac{\partial u(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)), \text{ in } \mathbb{R} \times (0,\infty) \tag{1.7}$$

$$u(0,x) = \chi_{x<0}, \; x \in \mathbb{R}.$$

The nonlinear term $f(u)$ characterizing the multiplication and killing of particles in the absence of diffusion is of KPP-type, if it is continuously differentiable in $u \in \mathbb{R}^1$ such that $f(0) = f(1) = 0$, $f(u) > 0$ for $0 < u < 1$, $f(u) < 0$ for $u \notin [0,1]$ and $\sup_{0<u<1} u^{-1}f(u) = f'(0)$. Reaction-diffusion equations that have a KPP-type nonlinear term $f(u)$ are referred to as KPP equations.

It is proved in [15] that the solution $u(t,x)$ of (1.7) tends to 1 as $t \to \infty$, and the region where $u(t,x)$ is close to 1 is growing with speed $2\sqrt{Df'(0)}$.

Since then, the KPP equation has been extensively studied. When the diffusion coefficient and the nonlinear term depend on space and are slowly changing in space, the first generalized result on the KPP equation using a probabilistic treatment was given by Freidlin [9]. Freidlin separated the study of profile and speed of the travelling wave by introducing a small parameter. He considered the following Cauchy problem:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x^i x^j} a^{ij}(x) u^\varepsilon(t,x) + \sum_{i=1}^{n} b^i(x) \frac{\partial u^\varepsilon(t,x)}{\partial x^i} + \frac{1}{\varepsilon} f(x, u^\varepsilon(t,x)) \quad (1.8)$$

$$= L^\varepsilon u^\varepsilon + \frac{1}{\varepsilon} f(x, u^\varepsilon(t,x))$$

$$u^\varepsilon(0,x) = g(x) \geq 0, \ x \in \mathbb{R}^n, t > 0.$$

Here, the function $f(x,\cdot)$ satisfies the KPP assumption for all $x \in \mathbb{R}^n$. Put $c(x,u) = u^{-1} f(x,u)$ for $u > 0$ and $c(x,0) = \lim_{u \downarrow 0} u^{-1} f(x,u)$. The function $c(x,u)$, $x \in \mathbb{R}^n$, $u \geq 0$ is supposed to be continuous and satisfies a Lipschitz condition in u. Let $\max_{0 \leq u \leq 1} c(x,u) = c(x,0) = c(x)$. The $a^{ij}(x)$ are bounded functions having bounded second-order derivatives such that the form $\sum_{i,j}^n a^{ij}(x)\lambda_i \lambda_j$ does not degenerate uniformly in $\mathbb{R}^n$.

1.3.2 Characterization of Position of Wavefronts

In Freidlin [9], the first probabilistic methods for studying the generalized KPP-type reaction-diffusion equation (1.8) is undertaken within the framework of large deviation theory for stochastic differential equations.

Consider the Markov diffusion process $(X_t^\varepsilon,\ \mathbb{P}_x)$ in $\mathbb{R}^n$ governed by the operator L^ε. It solves the following stochastic differential equation:

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \sigma(X_t^\varepsilon)\dot{W}_t,\ X_0^\varepsilon = x. \tag{1.9}$$

Here W_t is a Wiener process in $\mathbb{R}^n$, $\sigma(x)$ is a $n \times n$ matrix such that $\sigma(x)\sigma^*(x) = (a^{ij}(x))$. Using the Feynman-Kac formula, the solution of problem (1.8) can be represented as:

$$u^\varepsilon(t,x) = E_x g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_t^\varepsilon, u(t-s, X_s^\varepsilon))ds\right\}, \tag{1.10}$$

where X_t^ε is the solution of equation (1.9).

To examine the behavior of the solution of equation (1.8) as $\varepsilon \downarrow 0$, he first finds the asymptotic formula for expression of the form:

$$E_x g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_t^\varepsilon)ds\right\},\ \varepsilon \downarrow 0,$$

by introducing an action functional for the family of processes $(X_t^\varepsilon, \mathbb{P}_x)$ as $\varepsilon \downarrow 0$. The action functional for process X_t^ε, $0 \le s \le t$, in C_{0t} as $\varepsilon \downarrow 0$ has the form $(1/\varepsilon)S_{0t}(\varphi)$, where

$$S_{0t}(\varphi) = \begin{cases} \frac{1}{2}\int_0^t |\sigma^{-1}(\varphi_s)(\dot{\varphi}_s - b(\varphi_s))|^2 ds, & \text{if } \varphi \in C_{0t} \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.11}$$

From properties of the action functional, the asymptotic formula for

$$E_x g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_t^\varepsilon)ds\right\},\ \varepsilon \downarrow 0$$

is obtained as:

$$\lim_{\varepsilon\downarrow 0}\varepsilon\ln E_x g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_t^\varepsilon)ds\right\} \tag{1.12}$$

$$= \sup\{\int_0^t c(\varphi_s)ds - S_{0t}(\varphi) : \varphi_0 = x, \varphi_t \in \text{supp } g\}$$

The proof of formula (1.12) and the properties of action functional can be found in Freidlin [8] and Freidlin and Wentzell [7]. From KPP assumption, we know that the relation for $c(x,u)$:

$$c(x,u) = u^{-1}f(x,u) \leq c(x)$$

holds. From the asymptotic formula (1.12) and the Feynman-Kac representation of solution (1.10), the following estimate is obtained:

$$0 \leq u^\varepsilon(t,x) \leq E_x g(X_t^\varepsilon)\exp\{\frac{1}{\varepsilon}\int_0^t c(X_t^\varepsilon)ds\} \tag{1.13}$$

$$\asymp \exp\left\{\frac{1}{\varepsilon}[\sup\{\int_0^t c(\varphi_s)ds - S_{0t}(\varphi) : \varphi \in C_{0t}, \varphi_0 = x, \varphi_t \in [G_0]\}]\right\},\ \varepsilon \downarrow 0$$

where the "$\asymp$" sign denotes logarithmic equivalence. Let

$$V(t,x) = \sup\left\{\int_0^t c(\varphi_s)ds - S_{0t}(\varphi) : \varphi_0 = x, \varphi_t \in [\text{supp } g] = [G_0]\right\}$$

where $[G_0]$ denotes the closure of the support of the function $g(x)$ in $\mathbb{R}^n$. Freidlin proved that, under a certain condition (N), from (1.13) it follows that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 0$ on the set $\{(t,x) : t > 0, x \in \mathbb{R}^n, V(t,x) < 0\}$. This convergence is uniform on every compactum lying in the region $\{(t,x) : t > 0, x \in \mathbb{R}^n, V(t,x) < 0\}$, and $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 1$ for $V(t,x) > 0$. Then the manifold $\Sigma_t = \{x \in \mathbb{R}^n : V(t,x) = 0\}$ can be considered as the position of the wavefront (i.e., the boundary between the excited and non-excited regions) at time t. Condition (N) is said to be fulfilled if the following relation

$$V(t,x) = \sup\{ \int_0^t c(\varphi_s)ds - S_{0t}(\varphi) : \\ \varphi \in C_{0t}, \varphi_0 = x, \varphi_t \in [G_0], V(t-s,\varphi_s) < 0 \text{ for } 0 < s < t\}$$

holds for any $t > 0$ and $x \in \Sigma_t$.

Theorem 1.3.1 (Freidlin [9]**).** *Suppose that $f(x,u)$ satisfies the KPP assumption and let condition (N) be fulfilled. Then for the solution $u^\varepsilon(t,x)$ of problem (1.8) the following relation holds:*

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x) = \begin{cases} 1, & \text{if } V(t,x) > 0 \\ 0, & \text{if } V(t,x) < 0. \end{cases}$$

This convergence is uniform on every compactum lying in the region $\{(t,x) : t > 0, x \in \mathbb{R}^n, V(t,x) > 0\}$ and $\{(t,x) : t > 0, x \in \mathbb{R}^n, V(t,x) < 0\}$, respectively.

Therefore, the equation

$$V(t,x) = 0$$

defines the wavefront which divides the regions where $u^\varepsilon(t,x)$ is close to 0 and is close to 1 for small $\varepsilon > 0$.

Inspired by Freidlin's work on reaction-diffusion equations, Evans and Souganidis ([3], [4]) proved the wavefront propagation of solution of equation (1.8) using analytical methods. They generalized Freidlin's result to the case when condition (N) is not satisfied. In their setup, the functional characterizing the wavefront is a viscosity solution of some variational inequality. Later Freidlin [10] and Freidlin and Lee [11] obtained and generalized their results using probabilistic methods.

Without condition (N), the position of the wavefront can be characterized by introducing a stopping time. A functional $\tau : C([0,t],\mathbb{R}^n) \to [0,t]$ is called a stopping time if τ depends only on $\varphi_s,\ 0 \le s \le u$, when restricted to $\{\tau \le u\}$. Let Γ_t be the collection of all stopping times not greater than t. If F is a closed subset of $[0,t] \times \mathbb{R}^n$ and $\{0\} \times \mathbb{R}^n \subset F$, then

$$\tau_F \equiv \min\{s : s \ge 0 \text{ and } (t-s, \varphi_s) \in F\}$$

is clearly a stopping not greater than t. Let Θ_t be the collection of such τ_F. Let

$$V_0(t,x) = \inf_{\tau \in \Gamma_t} \{ \sup \int_0^\tau c(\varphi_s)ds - S_{0\tau}(\varphi) : \ \varphi \text{ is absolutely continuous,}$$
$$\varphi_0 = x, \varphi_t \in G_0\},$$

$$V_1(t,x) = \inf_{\tau \in \Theta_t} \{ \sup \int_0^\tau c(\varphi_s)ds - S_{0\tau}(\varphi) : \ \varphi \text{ is absolutely continuous,}$$
$$\varphi_0 = x, \varphi_t \in G_0\},\ t > 0,\ x \in \mathbb{R}^n.$$

$$V^*(t,x) = \sup\{ \min_{0 \le a \le t} \int_0^a c(\varphi_s)ds - S_{0a}(\varphi) : \ \varphi \text{ is absolutely continuous,}$$
$$\varphi_0 = x, \varphi_t \in G_0\},\ t > 0,\ x \in \mathbb{R}^n.$$

Fredilin and Lee ([11]) proved that

$$V_0 = V_1 = V^*,$$

and they characterized the position of the wavefronts.

Theorem 1.3.2 (Freidlin [10] and Freidlin and Lee [11]). *Let* $u^\varepsilon(t,x)$ *be the solution of (1.8). Then* $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 0$ *uniformly for* (t,x) *belonging to any compact set* $F_1 \subset \{(s,y) : V^*(s,y) < 0\}$. *For any compact subset* F_2 *of the interior of the set* $\{(s,y),\ s>0,\ V^*(s,y)=0\}$, $lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 1$ *uniformly in* $(t,x) \in F_2$.

In 1999, Pradeilles [16], using representation of solutions with backward stochastic differential equations driven by Brownian motion (also see Pardoux and Peng [13], Pardoux,Pradeilles, Rao [14]) generalized the wavefront propagation result to the case when the parabolic operator L^ε is possibly degenerate. He established that when the parabolic operator L^ε satisfies a Hömander-type hypothesis, the wavefront location is given by the same formula as that in Freidlin and Lee [11] or Barles, Evans and Souganidis [4].

1.3.3 A Class of Degenerate Reaction-Diffusion Equation Related to "Physical" Brownian motion

In this work, we consider a class of degenerate reaction-diffusion equation related to the "Physical" Brownian motion $q_t^{\mu,\varepsilon}$ with zero drift, i.e $b(x) = 0$. Let $x = (p,q) \in \mathbb{R}^{2n},\ p \in \mathbb{R}^n,\ q \in \mathbb{R}^n$. Here, q is the position of a particle, p is the velocity of the particle. Consider the equation:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\varepsilon}{2\mu^2} \sum_{i,j=1}^{2n} A^{ij}(x) \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^i \partial x^j} + \sum_{i=1}^{2n} b^i(x) \frac{\partial u^\varepsilon(t,x)}{\partial x^i} + \frac{1}{\varepsilon} f(x, u^\varepsilon(t,x)) \quad (1.14)$$

$$u(0,x) = g(x),\ x = (p,q) \in \mathbb{R}^{2n},\ g(x) \geq 0,$$

where diffusion matrix $A(x)$ and vector $b(x)$ are:

$$A(x) = A(p,q) = \begin{pmatrix} (a(q))_{n\times n} & 0_{n\times n} \\ & \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix},\ b(x) = \begin{pmatrix} -(1/\mu)p \\ \\ p \end{pmatrix},\ p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

Assume that $f(x,u) = c(q,u)u$ satisfies the KPP assumption. We assume the $n \times n$ matrix $(a(q))$ is uniformly non-degenerate.

As is known, in this case, the operator

$$L^{\mu,\varepsilon} = \frac{\varepsilon}{2\mu^2} \sum_{i,j=1}^{2n} A^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{2n} b^i(x) \frac{\partial}{\partial x^i}$$

satisfies Hömander's hypothesis and is hyperelliptic.

Let us rewrite equation (1.14) as:

$$\frac{\partial u^\varepsilon(t,p,q)}{\partial t} = \frac{\varepsilon}{2\mu^2} \sum_{i,j=1}^{n} a^{ij}(q) \frac{\partial^2 u^\varepsilon}{\partial p^i \partial p^j} - \frac{1}{\mu} p \nabla_p u^\varepsilon + p \nabla_q u^\varepsilon + \frac{1}{\varepsilon} c(q,u^\varepsilon) u^\varepsilon \qquad (1.15)$$

$$u^\varepsilon(0,p,q) = g(p,q).$$

The operator governing the diffusion

$$L^{\mu,\varepsilon} = \frac{\varepsilon}{2\mu^2} \sum_{i,j=1}^{n} a^{ij}(q) \frac{\partial^2}{\partial p^i \partial p^j} - \frac{1}{\mu} p \nabla_p + p \nabla_q \qquad (1.16)$$

is degenerate in $x = (p,q)$.

When, for example, we put the initial condition $g(p,q) = \delta(p)\chi^{-1}(q)$, where $\delta(p)$ is a delta function centered at 0 and $\chi^{-1}(q)$ is the indicator function with support equal to the negative q-axis, by the maximum principle $u^\varepsilon(t,p,q)$ is a function between 0 and 1. Equation (1.15) can be considered as the reaction-diffusion equation to model the transition probability density of particles whose diffusion is governed by Langevin's equation and whose multiplication and killing is governed by $f(q,u)$. We consider the propagating wave type solution of (1.15) as $\varepsilon \downarrow 0$. For equations satisfying the Hömander hypothesis, from results of Pradeilles [16], we know that for each $\mu > 0$, the wave front location in the phase space (p,q) is given by the same formula as in Theorem 1.2.

The corresponding reaction-diffusion equation related to the process q_t^ε defined by diffusion equation (1.3) (with $b(q) = 0$) is defined as:

$$\frac{\partial u^\varepsilon(t,q)}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^{n} a^{ij}(q) \frac{\partial^2 u^\varepsilon(t,q)}{\partial q^i \partial q^j} + \frac{1}{\varepsilon} u^\varepsilon c(q, u^\varepsilon) \tag{1.17}$$

$$u^\varepsilon(0,q) = g(0,q), \;\; q \in \mathbb{R}^n$$

Equation (1.17) has been well studied. Our task in this part is to study equation (1.15) and its relation to (1.17). We would like to show that under certain conditions, as $\varepsilon \downarrow 0$, for small μ, the wavefronts of equation (1.15) and (1.17) are close.

1.4 Outline of the thesis

The thesis is organized as follows. In Chapter 2, we study relations between equations (1.1) and (1.3) in the exit problems. In particular, we investigate to obtain closeness of the asymptotic quantities, such as asymptotic exit position, asymptotic exit time for equations (1.1) and (1.3). We first study the relation between the unperturbed systems when ε, which characterizes the intensity of perturbation, is 0. We prove that when either of the conditions in Proposition 2.1.1 is satisfied, the equilibrium for the Langevin system is asymptotically stable. Then we calculate the action functional (Proposition 2.2.1) and introduce the quasi-potential for equation (1.1). It can be proved that the asymptotic exit position and time can be represented in terms of the quasi-potential for (1.1). A major theorem is given in the second section of this chapter, showing the convergence of the quasi-potential of (1.1) to that of (1.3) under a certain wide condition. Special attention is paid to linear systems. In this case, the corresponding quasi-potentials can be calculated explicitly.

In Chapter 3, we concentrate on the problem of wavefront propagation of equation (1.15) and (1.17) and the relation between their wavefronts. We will give a characterization of the position of the wavefronts for equation (1.15) in the general case and under an assumption that we call condition (N^μ) (Theorem 3.1.4). Then we show the convergence of the wavefronts of equations (1.15) and (1.17) in two settings. When both condition (N^μ) and (N) are satisfied, the location of the wavefront for the degenerate reaction-diffusion equation converges to that of the non-degenerate one, for each bounded initial position q and velocity p. An example is considered when the function $c(q)$ is linearly growing. When only condition (N) is satisfied, the wavefront of the degenerate reaction-diffusion equation is within a δ neighborhood of the non-degenerate one, here δ depends on μ. An example is given when $c(q)$ is a constant function.

Chapter 2

Large Deviations: Exit from a Domain

2.1 Relations between the Unperturbed Systems

In this section, relations between systems (1.3) and (1.6) for $\varepsilon = 0$ are considered. We will investigate some sufficient conditions such that if $K \in \mathbb{R}^n$ is an asymptotically stable equilibrium of system (1.3) with $\varepsilon = 0$, then $(0, K) \in \mathbb{R}^{2n}$ is asymptotically stable for system (1.6) with $\varepsilon = 0$.

Without loss of generality, one can assume that K is the origin. It is understood that the nonlinear system of (1.3) with $\varepsilon = 0$ can be expressed with a linear and nonlinear part as

$$\dot{q}_t^0 = Aq_t^0 + N(q_t^0)$$

for which $\dot{q}_t^0 = Aq_t^0$ is the linear approximation to this equation in the vicinity of the equilibrium K. From the assumption on $b(q)$, we know $N(q)$ is continuous for small $|q|$ and $N(q) = o(|q|)$ as $|q| \to 0$. Let A be stable: that is, *all eigenvalues of A have negative real part.* Then K is an asymptotically stable equilibrium position.

Similarly, the linear approximation to (1.6) with $\varepsilon = 0$ is

$$\begin{cases} \mu \dot{p}_t^{\mu,0} = -p_t^{\mu,0} + Aq_t^{\mu,0}, & p_0^{\mu,0} = p; \\ \dot{q}_t^{\mu,0} = p_t^{\mu,0}, & q_0^{\mu,0} = q. \end{cases}$$

It has an asymptotically stable equilibrium position at $(0, K) \in \mathbb{R}^{2n}$ when the matrix

$$A_\mu = \begin{bmatrix} -(1/\mu)E & (1/\mu)A \\ E & 0 \end{bmatrix}$$

is stable, where E is the $n \times n$ identity matrix.

Proposition 2.1.1. *Assume that A is stable. Let at least one of the following conditions hold:*

i. *All eigenvalues of A are real.*

ii. *The inequality $0 < \mu < \mu_0 = \min\{-a_k/b_k^2,\ k = 0, 1, \cdots, m,\ m \le n\}$ holds, where $a_k + ib_k,\ a_k < 0,\ b_k \neq 0,\ k = 0, 1, \cdots, m,\ m \le n$ are all complex eigenvalues of A.*

Then A_μ is stable.

Proof. Let λ_μ be an eigenvalue of A_μ. Since

$$\det(A_\mu - \lambda_\mu E) = \det \begin{bmatrix} (\lambda_\mu + 1/\mu)E & -(1/\mu)A \\ -E & \lambda_\mu E \end{bmatrix} = 0$$

is equivalent to

$$\det(\lambda_\mu(\mu\lambda_\mu + 1)E - A) = \det(\lambda E - A) = 0,$$

so that $\lambda_\mu(\mu\lambda_\mu + 1) = \lambda$ is an eigenvalue of A. Then we have

$$\lambda_\mu = \frac{-1 \pm \sqrt{1 + 4\lambda\mu}}{2\mu}.$$

Since A is stable, $Re(\lambda) < 0$. Consider the following two cases:

i When λ is real, then $Re(\lambda_\mu) < 0$ for any $\mu > 0$, i.e. A_μ is stable;

ii When $\lambda = a + bi,\ a < 0,\ b \neq 0$,

$$\lambda_\mu = \frac{-1 \pm \sqrt{z}}{2}, \tag{2.1}$$

where

$$z = (1 + 4a\mu) + 4b\mu i.$$

Formula (2.1) implies that λ_μ has negative real part if and only if $|Re(\sqrt{z})| < 1$, which is equivalent to

$$1 + 4a\mu + \sqrt{(1 + 4a\mu)^2 + (4b\mu)^2} < 2.$$

This implies that

$$\mu < -\frac{a}{b^2}.$$

Let $a_k + b_k i,\ k = 0, 1, \cdots, m,\ m \leq n$, be all complex eigenvalues of A, where $a_k < 0,\ b_k \neq 0$ for each k. Then if

$$0 < \mu < \mu_0 = \min\{-\frac{a_k}{b_k^2},\ k = 0, 1, \cdots, m,\ m \leq n\},$$

all eigenvalues of A_μ have negative real part, which means A_μ is stable. □

Assumption: From now on, we will assume that either of the two conditions in Proposition 2.1.1 is satisfied and we study the relation of the corresponding perturbed systems.

2.2 Action Functional and Convergence of Quasi-potentials

Consider the process $q_t^{\mu,\varepsilon}$ defined by the system (1.6). We assume $b(q)$ and $\sigma(q)$ are smooth enough and bounded, and $det(a(q)) \geq a_0 > 0$, where $a(q) = \sigma(q)\sigma^*(q)$. In order to study the exit problems of $q_t^{\mu,\varepsilon}$, we will first find the action functional for $q_t^{\mu,\varepsilon}$ as $\varepsilon \downarrow 0$.

Proposition 2.2.1. *The action functional for $q_t^{\mu,\varepsilon}$ in C_{0T} for fixed μ as $\varepsilon \downarrow 0$ has the form $\varepsilon^{-1} S_{0T}^{\mu}(\varphi)$, where*

$$S_{0T}^{\mu}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s + \dot{\varphi}_s - b(\varphi_s))|^2 ds & \text{if } \dot{\varphi} \text{ is absolutely continuous} \\ & \varphi_0 = q,\ \dot{\varphi}_0 = p; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

Proof. First, note that since $\dot{q}_t^{\mu,\varepsilon} = p_t^{\mu,\varepsilon}$ is continuous, system (1.6) can be written as follows:

$$\mu(p_t^{\mu,\varepsilon} - p) = \int_0^t b(q_s^{\mu,\varepsilon})ds - \int_0^t p_s^{\mu,\varepsilon} ds + \sqrt{\varepsilon} W_t \sigma(q_t^{\mu,\varepsilon}) - \sqrt{\varepsilon}\int_0^t W_s \frac{d}{ds}[\sigma(q_s^{\mu,\varepsilon})]ds, \tag{2.3}$$

$$q_t^{\mu,\varepsilon} - q = \int_0^t p_s^{\mu,\varepsilon} ds.$$

Let ψ_t be a continuous function on $[0, T]$ with values in $\mathbb{R}^n$. Consider the operator $F : \psi \to X$, where $X = X_t = (p_t, q_t) \in \mathbb{R}^{2n}$ is the solution of the system

$$\mu(p_t - p) = \int_0^t b(q_s)ds - \int_0^t p_s ds + \psi_t\sigma(q_t) - \psi_0\sigma(q) - \int_0^t \psi_s \frac{d}{ds}[\sigma(q_s)]ds, \quad p_0 = p,$$

$$q_t - q = \int_0^t p_s ds, \qquad q_0 = q,$$

$t \in [0,T]$. Let $X_1 = X_1(t) = (p_1(t), q_1(t)) = F\psi_1$, $X_2 = X_2(t) = (p_2(t), q_2(t)) = F\psi_2$. Since $b(q)$ and $\sigma(q)$ are Lipschitz continuous, for any $t \in [0,T]$, the norm of the difference will satisfy the the following inequality

$$\|X_1(t) - X_2(t)\| \leq K_1 \int_0^t \|X_1(s) - X_2(s)\| ds + K_2 T \|\psi_1 - \psi_2\|_{C_{0T}},$$

where $\|\psi\|_{C_{0T}} = \max_{t\in[0,T]} |\psi(t)|$, K_1, K_2 are some constants. From Gronwall's inequality,

$$\|X_1(t) - X_2(t)\| \leq e^{K_1 T} K_2 T \|\psi_1 - \psi_2\|_{C_{0T}},$$

which implies the continuity of operator F. Hence, the transformation $\tilde{F}_\mu : \psi \to q_t$ is also continuous, where by definition q_t solves the equation

$$\begin{aligned} q_t &= \tilde{F}_\mu \psi_t = \quad q + (1 - e^{-t/\mu})[\mu p - \sigma(q)\psi_0] + \int_0^t (1 - e^{(s-t)/\mu}) b(q_s) ds \\ &\qquad - \int_0^t \psi_s \frac{d}{ds}[\sigma(q_s)] ds + e^{-(t/\mu)} \int_0^t \psi_s \frac{d}{ds}[\sigma(q_s) e^{s/\mu}] ds. \end{aligned}$$

Moreover, $\tilde{F}_\mu$ has the inverse

$$(\tilde{F}_\mu^{-1} q)_t = \psi_t = \psi_0 + \int_0^t \sigma^{-1}(q_s)(\mu \ddot{q}_s + \dot{q}_s - b(q_s)) ds.$$

It follows from (2.3) that

$$q_t^{\mu,\varepsilon} = \tilde{F}_\mu(\sqrt{\varepsilon} W_t).$$

By theorem 1.2.1, the action functional for the family of the process $q_t^{\mu,\varepsilon}$ has the form $\varepsilon^{-1} S_{0T}^\mu(\varphi)$ where

$$S_{0T}^\mu(\varphi) = \min\{S_{0T}^w(\psi) : \tilde{F}_\mu \psi = \varphi\} = \min\{\frac{1}{2}\int_0^T |\dot{\psi}_s|^2 ds : \tilde{F}_\mu \psi = \varphi\}$$

$$= \frac{1}{2}\int_0^T |\frac{d}{dt}\tilde{F}_\mu^{-1}\varphi|^2 dt = \frac{1}{2}\int_0^T |\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s + \dot{\varphi}_s - b(\varphi_s))|^2 dt$$

if ψ_t is absolutely continuous, and $S^{\mu}_{0T}(\varphi) = +\infty$ otherwise in C_{0T}. Since

$$\dot{\varphi}_t = \dot{\varphi}_0 e^{-(t/\mu)} + \frac{1}{\mu} e^{-(t/\mu)} \int_0^t e^{(s/\mu)} b(\varphi_s) ds + \frac{1}{\mu} e^{-(t/\mu)} [e^{(s/\mu)} \sigma(\varphi_s)\psi_s]|_0^t$$
$$- \frac{1}{\mu} e^{-(t/\mu)} \int_0^t \psi_s \frac{d}{ds}[e^{(s/\mu)}\sigma(\varphi_s)] ds,$$

absolute continuity of ψ_t implies that $\dot{\varphi}_t$ is absolutely continuous. □

Now let $K \in \mathbb{R}^n$ be an asymptotically stable equilibrium for the dynamical system q_t^0 in $\mathbb{R}^n$ defined by the Equation (1.3) with $\varepsilon = 0$. The quasi-potential for the process q_t^{ε} with respect to the equilibrium $K \in \mathbb{R}^n$ is defined by (1.5), where the action functional assumes the form (1.4). Then $(0, K) \in \mathbb{R}^{2n}$ is asymptotically stable for system (1.6). We can define the quasi-potential $V^{\mu}(q)$ in a similar way as

$$V^{\mu}(q) = \inf\{S^{\mu}_{0T}(\varphi) : \varphi_0 = K,\ \dot{\varphi}_0 = 0,\ \varphi_T = q,\ T \geq 0,\ \varphi \in C_{0T}\}. \tag{2.4}$$

Theorem 2.2.2. *Let $V^{\mu}(q)$ and $V(q)$ be defined as above. Let $G \subset \mathbb{R}^n$ be compact. Then $V^{\mu}(q) \to V(q)$ for each $q \in G$ as $\mu \to 0$.*

Proof. Introduce the following quantities:

$$\begin{aligned} V^{\mu}(q,T) &= \inf\{S^{\mu}_{0T}(\varphi) : \varphi \in C_{0T},\ \varphi_0 = K,\ \varphi_T = q,\ \dot{\varphi}_0 = 0,\ q \in G\} \\ V(q,T) &= \inf\{S_{0T}(\varphi) : \varphi \in C_{0T},\ \varphi_0 = K,\ \varphi_T = q,\ q \in G\} \end{aligned}$$

$$V^{\mu}(q) = \inf_{T \geq 0} V^{\mu}(q,T), \qquad V(q) = \inf_{T \geq 0} V(q,T)$$

First, we show that for each $q \in G$,

$$\lim_{\mu \downarrow 0} V^{\mu}(q,T) = V(q,T). \tag{2.5}$$

This is equivalent to the following inequalities:

$$V(q,T) \geq \limsup_{\mu \downarrow 0} V^{\mu}(q,T) \tag{2.6}$$

$$V(q,T) \leq \liminf_{\mu \downarrow 0} V^{\mu}(q,T). \tag{2.7}$$

To show (2.6), let φ^* be an extremal of $S_{0T}(\varphi)$ such that

$$V(q,T) = S_{0T}(\varphi^*).$$

The Euler-Lagrange equations for extremals of $S_{0T}(\varphi)$ imply that they are in $C^2([0,T])$. Therefore

$$V^{\mu}(q,T) \leq S^{\mu}_{0T}(\varphi^*) \leq S_{0T}(\varphi^*) + \frac{1}{2}\mu^2 \int_0^T |\sigma^{-1}(\varphi^*_s)\ddot{\varphi}^*_s|^2 ds = V(q,T) + o(\mu).$$

This implies the limsup inequality (2.6).

To show (2.7), let $\hat{\varphi}$ be an extremal of $S^{\mu}_{0T}(\varphi)$ for fixed $\mu > 0$ such that $V^{\mu}(q,T) = S^{\mu}_{0T}(\hat{\varphi})$. Similarly, extremals of $S^{\mu}_{0T}(\varphi)$ are in $C^4([0,T])$. Let $a(x) = \sigma(x)\sigma^*(x)$, $x \in \mathbb{R}^n$ be strictly positive definite. Assume $b(x)$, $\sigma(x)$ have bounded derivatives. Then

$$\begin{aligned}
S^{\mu}_{0T}(\hat{\varphi}) &= \frac{1}{2}\int_0^T |\sigma^{-1}(\hat{\varphi}_s)(\mu\ddot{\hat{\varphi}}_s + \dot{\hat{\varphi}}_s - b(\hat{\varphi}_s))|^2 ds \\
&= \frac{1}{2}\int_0^T |\sigma^{-1}(\hat{\varphi}_s)\mu\ddot{\hat{\varphi}}_s|^2 ds + S_{0T}(\hat{\varphi}) + \mu\int_0^T (\sigma^{-1}(\hat{\varphi}_s)\ddot{\hat{\varphi}}_s, \sigma^{-1}(\hat{\varphi}_s)[\dot{\hat{\varphi}}_s - b(\hat{\varphi}_s)])ds \\
&\geq S_{0T}(\hat{\varphi}) + \frac{\mu}{2}\int_0^T \sum_{i=1}^n\sum_{j=1}^n a^{-1}_{ij}(\hat{\varphi}_s) d(\dot{\hat{\varphi}}^j_s\dot{\hat{\varphi}}^i_s) - \mu\int_0^T (a^{-1}(\hat{\varphi}_s)b(\hat{\varphi}_s), d\dot{\hat{\varphi}}_s) \\
&= S_{0T}(\hat{\varphi}) + \frac{\mu}{2}(a^{-1}(\hat{\varphi}_T)\dot{\hat{\varphi}}_T, \dot{\hat{\varphi}}_T) - \mu(a^{-1}(\hat{\varphi}_T)b(\hat{\varphi}_T), \dot{\hat{\varphi}}_T) + \frac{\mu}{2}(a^{-1}(\hat{\varphi}_T)b(\hat{\varphi}_T), b(\hat{\varphi}_T)) \\
&\quad - \frac{\mu}{2}(a^{-1}(\hat{\varphi}_T)b(\hat{\varphi}_T), b(\hat{\varphi}_T)) - \frac{\mu}{2}\int_0^T \sum_{i=1}^n\sum_{j=1}^n \dot{\hat{\varphi}}^j_s\dot{\hat{\varphi}}^i_s d[a^{-1}_{ij}(\hat{\varphi}_s)]
\end{aligned}$$

$$+ 2(\dot{\hat{\varphi}}_s, d[a^{-1}(\hat{\varphi}_s)b(\hat{\varphi}_s)])$$

$$= S_{0T}(\hat{\varphi}) + \frac{\mu}{2}|\sigma^{-1}(\hat{\varphi}_T)(\dot{\hat{\varphi}}_T - b(\hat{\varphi}_T))|^2 - \mu I$$

where

$$I = \frac{1}{2}|\sigma^{-1}(\hat{\varphi}_T)b(\hat{\varphi}_T)|^2 + \frac{1}{2}\int_0^T \sum_{i=1}^n \sum_{j=1}^n \dot{\hat{\varphi}}_s^j \dot{\hat{\varphi}}_s^i d[a_{ij}^{-1}(\hat{\varphi}_s)] - \int_0^T (\dot{\hat{\varphi}}_s, d[a^{-1}(\hat{\varphi}_s)b(\hat{\varphi}_s)])$$

Since $a(x)$ and $b(x)$ have bounded derivatives and $\|\hat{\varphi}\|_{W^{1,2}([0,T])} = (\int_0^T |\varphi(s)|^2 + |\dot{\varphi}(s)|^2 ds)^{1/2}$ is uniformly bounded, I is uniformly bounded for all $\mu > 0$. Therefore,

$$\begin{aligned} V^\mu(q,T) &= S_{0T}^\mu(\hat{\varphi}) \geq S_{0T}(\hat{\varphi}) + o(\mu) \geq V(q,T) + o(\mu) \\ &\Longrightarrow \liminf_{\mu \downarrow 0} V^\mu(q,T) = V(q,T). \end{aligned}$$

Thus, (2.5) is proved. It can be easily checked that the limit (2.5) is uniform in $T \geq T_0 > 0$.

It's easy to see that $V^\mu(q,T)$, and $V(q,T)$ are all decreasing functions in T. Therefore

$$V^\mu(q) = \lim_{T\to\infty} V^\mu(q,T) \tag{2.8}$$

$$V(q) = \lim_{T\to\infty} V(q,T). \tag{2.9}$$

From (2.8), and (2.9), we know $V^\mu(q)$ and $V(q)$ can be arbitrarily close as long as μ is small enough, thus

$$V^\mu(q) \longrightarrow V(q) \text{ for each } q \in G, \text{as } \mu \to 0.$$

□

2.3 Quasi-potentials for Linear Systems

In this section, we pay special attention to perturbations of processes defined by (1.6), when $b(q) = Aq$ and A is a constant stable matrix. We will see that quasi-potentials for the second order linear system and its Smoluchowski-Kramers approximation actually coincide for any $\mu > 0$ if the eigenvalues of A are real and for any stable A when μ is small enough.

2.3.1 Quasi-potential for the diffusion equatioin

Let us find an explicit formula for the quasi-potential of the first order linear systems defined as:

$$\dot{q}_t^\varepsilon = Aq_t^\varepsilon + \sqrt{\varepsilon}\sigma\dot{W}_t,\ q_0^\varepsilon = q. \tag{2.10}$$

We assume A is an $n \times n$ matrix, having the real parts of all eigenvalues negative; σ is a non-degenerate $n \times n$ constant matrix, $q_t^\varepsilon \in \mathbb{R}^n$. We eliminate the diffusion matrix σ by making a change of variable $Y_t = \sigma^{-1}q_t$. Then

$$\begin{aligned}\frac{d}{dt}\sigma^{-1}q_t &= (\sigma^{-1}A\sigma)\sigma^{-1}q_t + \sqrt{\varepsilon}\dot{W}_t \\ \dot{Y}_t &= (\sigma^{-1}A\sigma)Y_t + \sqrt{\varepsilon}\dot{W}_t.\end{aligned}$$

Since $\sigma^{-1}A\sigma$ has the same eigenvalues as A, any system of the form (2.10) can be reduced to a system of the form

$$\dot{q}_t^\varepsilon = Aq_t^\varepsilon + \sqrt{\varepsilon}\dot{W}_t,\ q_0^\varepsilon = q. \tag{2.11}$$

Thus the unperturbed linear system $\dot{q}_t = Aq_t \in \mathbb{R}^n$ has an asymptotical stable equilibrium position O, the origin of the coordinate system.

As is known ([7]), the action functional for the family q_t^ε in $C[0,T]$ as $\varepsilon \downarrow 0$ has the form $\varepsilon^{-1}S(\varphi)$, where

$$S(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\dot{\varphi}_t - A\varphi_t|^2 dt, & \varphi(0) = O,\ \varphi(T) = q,\ \varphi \text{ is absolutely continuous} \\ +\infty, & \text{otherwise.} \end{cases}$$

The quasi-potential for the process q_t^ε with respect to O is

$$V(q) = \inf\{S(\varphi) : \varphi \in C[0,T],\ \varphi(0) = O,\ \varphi(T) = q \in \mathbb{R}^n,\ T \geq 0\}$$

and the Hamilton-Jacobi equation for $V(x)$ is:

$$\frac{1}{2}(\nabla V, \nabla V) + (Aq, \nabla V) = 0,\ V(0) = 0,\ V(q) > 0\ for\ q \neq 0. \tag{2.12}$$

Lemma 2.3.1. *If there exists a symmetric positive definite matrix B solving the equation*

$$(B^2 q, q) = -(Aq, Bq), \tag{2.13}$$

then $V(q) = (Bq, q),\ q \in \mathbb{R}^n$.

Proof. We can simply check that if (2.13) holds and $B = B^*$, then $V(q) = (Bq, q)$ satisfies the Hamilton-Jacobi equation (2.12). Since B is positive definite, $V(q) = (Bq, q) > 0,\ for\ q \neq 0$. □

Example 2.3.2. When the matrix A is normal, that is, $A^*A = AA^*$ (see [7]), let

$B = -\frac{1}{2}(A + A^*)$, which is symmetric and positive definite. Then we have

$$\begin{aligned}
B + A &= -\frac{1}{2}(A^* - A), \text{ and } B = -\frac{1}{2}(A^* + A); \\
((B + A)q, Bq) &= \left(-\frac{1}{2}(A^* - A)q, -\frac{1}{2}(A^* + A)q\right) \\
&= \frac{1}{4}[(A^*q, Aq) + (A^*q, A^*q) - (Aq, A^*q) - (Aq, Aq)] \\
&= \frac{1}{4}[(A^*q, A^*q) - (Aq, Aq)] \\
&= 0.
\end{aligned}$$

Therefore,

$$((B + A)q, Bq) = (Bq, Bq) + (Aq, Bq) = 0$$

$$(B^2q, q) = -(Aq, Bq).$$

Thus B is the solution of (2.13). Then quasi-potential is

$$V(q) = -\frac{1}{2}((A + A^*)q, q).$$

In order to solve (2.13) for general, not necessarily normal A, we need the following result from matrix theory (see [18] for the proof).

Lemma 2.3.3. *Let A be a given matrix whose eigenvalues have negative real parts. Then the equation $AX + XA^* = Y$ has a unique solution X for every Y, and the solution can be expressed as*

$$X = \int_0^\infty e^{At}(-Y)e^{A^*t}dt.$$

Theorem 2.3.4. *The quasi-potential $V(q)$ for the processes q_t^ε defined by equation (2.11) is given by the formula:*

$$V(q) = \frac{1}{2}((\int_0^\infty e^{At}e^{A^*t}dt)^{-1}q, q).$$

Proof. Because of Lemma 2.3.1, we can look for the quasi-potential $V(q)$ in the form $V(q) = (Bq, q)$, where B satisfies Eq. (2.13). Since

$$(Aq, Bq) = (Bq, Aq) = \frac{1}{2}[(Aq, Bq) + (Bq, Aq)] = (\frac{1}{2}(B^*A + A^*B)q, q),$$

Eq. (2.13) becomes

$$(B^2q, q) = -(\frac{1}{2}(B^*A + A^*B)q, q).$$

The matrix $\frac{1}{2}(B^*A + A^*B)$ is symmetric, therefore

$$B^2 = -\frac{1}{2}(B^*A + A^*B). \qquad (2.14)$$

From the symmetry and non-degeneracy of B, (2.14) can be simplified to the following matrix equation

$$A(\frac{1}{2}B^{-1}) + (\frac{1}{2}B^{-1})A^* = -E.$$

From (2.14), solution of the simplified equation is

$$X = \frac{B^{-1}}{2} = \int_0^\infty e^{At}e^{A^*t}dt.$$

So

$$B = \frac{1}{2}\left(\int_0^\infty e^{At}e^{A^*t}dt\right)^{-1}.$$

For any nonzero vector $y \in \mathbb{R}^n$,

$$\begin{aligned} y^*Xy &= y^*\int_0^\infty e^{At}e^{A^*t}dty \\ &= \int_0^\infty (e^{A^*t}y)^*E(e^{A^*t}y)dt \end{aligned}$$

is positive, so B is positive definite. We get $V(q) = (Bq, q) > 0$ for $q \neq 0$. □

Example 2.3.5. Consider a general 2×2 Jordan matrix $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda < 0$.

The quasi-potential can be calculated explicitly and is equal to

$$V(q) = -\frac{2\lambda}{4\lambda^2+1}[2\lambda^2 q_1^2 + 2\lambda q_1 q_2 + (2\lambda^2+1)q_2^2],$$

where $q = (q_1, q_2)$.

From results in [7], an extremal φ_t solves the system of first order differential equations

$$\dot{\varphi}_t = (A + 2B)\varphi_t,$$

where

$$B = -\frac{2\lambda}{4\lambda^2+1}\begin{pmatrix} 2\lambda^2 & \lambda \\ \lambda & 2\lambda^2+1 \end{pmatrix}$$

is the symmetric matrix of the quadratic form $V(q)$.

From the Figure 2.1, one can see that the trajectories of the system $\dot{q}_t = Aq_t$ are logarithmic spirals winding in to the origin in the clockwise direction, while the trajectories of the extremal are also logarithmic spirals winding in to the origin but in the anti-clockwise direction. The level sets of the quasi-potential are ellipses.

Proposition 2.3.6. *The quasi-potential $V(q)$ for process q_t^ε defined by (2.10) is given by the formula*

$$V(q) = \frac{1}{2}\left((\int_0^\infty e^{At}\sigma\sigma^* e^{A^*t} dt)^{-1} q, q \right).$$

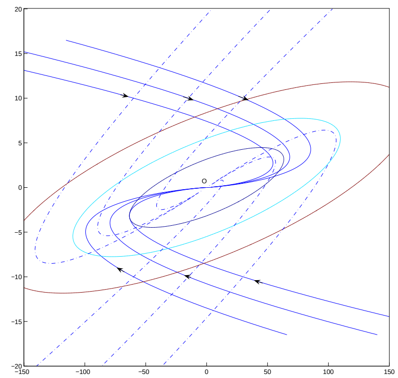

Figure 2.1: *Solid lines*: trajectories of the unperturbed system. *Dashed and doted lines*: trajectories of extremal of action functional. *Ellipses*: level sets of the quasi-potential.

Proof. Let $y_t^\varepsilon = \sigma^{-1} q_t^\varepsilon$, $y_0^\varepsilon = \sigma^{-1} q_0^\varepsilon = \sigma^{-1} q = y$. Then

$$\begin{aligned} V(y) &= \frac{1}{2}\left((\int_0^\infty \sigma^{-1} e^{At} \sigma\sigma^* e^{A^* t} (\sigma^{-1})^* dt)^{-1} y, y \right) \\ V(q) &= \frac{1}{2}\left(\sigma^* (\int_0^\infty e^{At} \sigma\sigma^* e^{A^* t} dt)^{-1} \sigma\sigma^{-1} q, \sigma^{-1} q \right) \\ V(q) &= \frac{1}{2}\left((\int_0^\infty e^{At} \sigma\sigma^* e^{A^* t} dt)^{-1} q, q \right). \end{aligned}$$

Since $\sigma\sigma^*$ is positive definite, $V(q) > 0$ for $q \neq 0$. □

2.3.2 Quasi-potential for Langevin's equation

To find an explicit representation of the quasi-potential for the second order linear system describing particle motion, let us first consider the case when the diffusion matrix σ is an identity matrix.

Consider the system

$$\begin{cases} \mu \dot{p}_t^{\mu,\varepsilon} = -p_t^{\mu,\varepsilon} + A q_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\dot{W}_t, & p_0^{\mu,\varepsilon} = p; \\ \dot{q}_t^{\mu,\varepsilon} = p_t^{\mu,\varepsilon}, & q_0^{\mu,\varepsilon} = q. \end{cases} \tag{2.15}$$

As we know, $(0, O)$ is an asymptotically stable equilibrium position for the system $q_t^{\mu,0}$ under the assumption in section 2.1.

From Proposition 2.2.1, the action functional for the family $q_t^{\mu,\varepsilon}$ in C_{0T} as $\varepsilon \downarrow 0$ has the form $\varepsilon^{-1}S^\mu(\varphi)$, where

$$S^\mu(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\mu\ddot{\varphi}_t + \dot{\varphi}_t - A\varphi_t|^2 dt, & \dot{\varphi} \text{ is absolutely continuous} \\ & \varphi_0 = q, \dot{\varphi}_0 = p; \\ +\infty, & \text{otherwise.} \end{cases}$$

Introduce the quasi-potential of $q_t^{\mu,0}$ with respect to the equilibrium O:

$$\begin{aligned} V^\mu(q) &= \inf\{S^\mu(\varphi) : \varphi \in C_{0T},\ \varphi_0 = O,\ \dot{\varphi}_0 = 0,\ \varphi_T = q,\ T > 0\} \\ &= \inf_{p\in\mathbb{R}^n} \mathcal{V}^\mu(p, q) \\ &= \inf_{p\in\mathbb{R}^n} \inf\{S^\mu(\varphi) : \varphi \in C_{0T},\ \varphi_0 = O,\ \dot{\varphi}_0 = 0,\ \varphi_T = q,\ \dot{\varphi}_T = p,\ T > 0\}. \end{aligned}$$

Let $z = (p, q)$. Then $\mathcal{V}^\mu(z) = \mathcal{V}^\mu(p, q)$. Let

$$\nabla\mathcal{V}^\mu = \begin{pmatrix} \nabla_p \mathcal{V}^\mu \\ \nabla_q \mathcal{V}^\mu \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{V}^\mu}{\partial p} \\ \frac{\partial \mathcal{V}^\mu}{\partial q} \end{pmatrix}.$$

The Hamilton-Jacobi equation for $\mathcal{V}^\mu(p, q)$ has the form:

$$(\nabla\mathcal{V}^\mu, Kz) + \frac{1}{2}(E_\mu \nabla\mathcal{V}^\mu, \nabla\mathcal{V}^\mu) = 0, \tag{2.16}$$

where

$$K = \begin{bmatrix} -(1/\mu)E & (1/\mu)A \\ E & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \; E_\mu = \begin{bmatrix} (1/\mu^2)E & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

E is the $n \times n$ identity matrix (see [19]).

Lemma 2.3.7. *Let the matrices K and E_μ and the vector z be defined as above. Let there exist a symmetric positive definite matrix D solving the equation*

$$(Dz, Kz) = -(DE_\mu Dz, z). \tag{2.17}$$

Then $\mathcal{V}^\mu(z) = (Dz, z)$, for all $z = (p, q) \in \mathbb{R}^{2n}$.

Proof. Similar to the proof of Lemma 2.3.1, we can simply check that if (2.17) holds and $D = D^*$, $\mathcal{V}^\mu(z) = (Dz, z)$ satisfies the Hamilton-Jacobi equation (2.16). Since D is positive definite, $\mathcal{V}^\mu(z) = (Dz, z) > 0, \; for \; z \neq 0.$ □

Theorem 2.3.8. *The quasi-potential for the process $q_t^{\mu,\varepsilon}$ defined by (2.15) is given by the formula:*

$$V^\mu(q) = \frac{1}{2}((\int_0^\infty e^{At}e^{A^*t}dt)^{-1}q, q).$$

Proof. Since

$$V^\mu(q) = \inf_{p\in\mathbb{R}^n} \mathcal{V}^\mu(p, q),$$

the proof is done if we can calculate $\mathcal{V}^\mu(z) = \mathcal{V}^\mu(p, q)$. Because of Lemma 3, we can look for the quasi-potential $\mathcal{V}^\mu(z)$ in the form $\mathcal{V}^\mu(z) = (Dz, z)$, where D satisfies Eq. (2.17). Since

$$(Dz, Kz) = (Kz, Dz) = \frac{1}{2}[(Dz, Kz) + (Kz, Dz)] = \left(\frac{1}{2}(D^*K + K^*D)z, z\right),$$

by symmetry of D, we have

$$(Dz, Kz) = \left(\frac{1}{2}(D^*K + K^*D)z, z\right).$$

By Eq. (2.17)

$$\left(\frac{1}{2}(D^*K + K^*D)z, z\right) = -(DE_\mu Dz, z).$$

Since the matrix $\frac{1}{2}(D^*K + K^*D)$ is symmetric,

$$\frac{1}{2}(D^*K + K^*D) = -DE_\mu D.$$

This is equivalent to

$$KD^{-1} + D^{-1}K^* = -2E_\mu. \tag{2.18}$$

By our assumption in Section 2, K is stable. From Lemma 2, we know that there exists a unique D^{-1} that solves (2.18). We show that the unique solution is given by the matrix:

$$D^{-1} = X = \begin{pmatrix} \frac{1}{\mu}E & 0 \\ 0 & G \end{pmatrix},$$

where

$$G = 2\int_0^\infty e^{At}e^{A^*t}dt.$$

This can be done simply by checking that X solves Eq. (2.18). We calculate the left-hand side of (2.18) and obtain

$$\begin{aligned} KX + XK^* &= \begin{pmatrix} -\frac{1}{\mu}E & \frac{1}{\mu}A \\ E & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\mu}E & 0 \\ 0 & G \end{pmatrix} + \begin{pmatrix} \frac{1}{\mu}E & 0 \\ 0 & G \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{\mu}E & E \\ \frac{1}{\mu}A^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{\mu^2}E & \frac{1}{\mu}(AG + E) \\ \frac{1}{\mu}(GA^* + E) & 0 \end{pmatrix} \end{aligned}$$

From Lemma 2, we know that $G = 2\int_0^\infty e^{At}e^{A^*t}dt$ is the unique solution of matrix equation

$$AG + GA^* = -2E. \tag{2.19}$$

Since $KX + XK^*$ is symmetric,

$$\frac{1}{\mu}(AG + E) = \frac{1}{\mu}(GA^* + E). \tag{2.20}$$

Equation (2.19) and (2.20) implies that

$$AG = GA^* = -E.$$

Therefore,

$$KX + XK^* = \begin{pmatrix} -\frac{2}{\mu^2}E & 0 \\ & \\ 0 & 0 \end{pmatrix} = -2E_\mu,$$

which means that $X = D^{-1}$ is the unique solution of matrix equation (2.18).

By inverting X, one can find D as:

$$D = X^{-1} = \begin{pmatrix} \mu E & 0 \\ & \\ 0 & \frac{1}{2}(\int_0^\infty e^{At}e^{A^*t}dt)^{-1} \end{pmatrix}.$$

Thus,

$$\mathcal{V}^\mu(p,q) = (Dz, z) = \mu|p|^2 + \frac{1}{2}((\int_0^\infty e^{At}e^{A^*t}dt)^{-1}q, q),$$
$$V^\mu(q) = \inf_{p\in\mathbb{R}^n} \mathcal{V}^\mu(p,q) = \frac{1}{2}((\int_0^\infty e^{At}e^{A^*t}dt)^{-1}q, q).$$

Obviously, the infimum is obtained when $\dot{\varphi}_T = p = 0$. □ □

Let us now consider the general particle motion defined by

$$\begin{cases} \mu\dot{p}_t^{\mu,\varepsilon} = Aq_t^{\mu,\varepsilon} - p_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\sigma\dot{W}_t, & p_0^{\mu,\varepsilon} = p; \\ & \\ \dot{q}_t^{\mu,\varepsilon} = p_t^{\mu,\varepsilon}, & q_0^{\mu,\varepsilon} = q, \end{cases} \tag{2.21}$$

where the diffusion matrix σ is not necessarily the identity. Let $V^\mu(q)$ be the quasi-potential for the process $q_t^{\mu,\varepsilon}$ defined by (2.21). Then

$$V^\mu(q) = \inf\{S^\mu(\varphi) : \varphi \in C_{0T},\ \varphi_0 = O,\ \dot{\varphi}_0 = 0,\ \varphi_T = q,\ T > 0\},$$

where $(1/\varepsilon)S^\mu(\varphi)$ is the action functional for process $q_t^{\mu,\varepsilon}$ as $\varepsilon \downarrow 0$. From proposition 1, $S^\mu(\varphi)$ has the form:

$$S^\mu(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\sigma^{-1}(\mu\ddot{\varphi}_t + \dot{\varphi}_t - A\varphi_t)|^2 dt, & \text{if } \dot{\varphi} \text{ absolutely continuous} \\ & \varphi_0 = q, \dot{\varphi}_0 = p; \\ +\infty, & \text{otherwise.} \end{cases}$$

By making a change of variable, $P_t = \sigma^{-1}p_t^{\mu,\varepsilon}$, $Q_t = \sigma^{-1}q_t^{\mu,\varepsilon}$, (2.21) becomes

$$\begin{cases} \mu\dot{P}_t = -P_t + \sigma^{-1}A\sigma Q_t + \sqrt{\varepsilon}\dot{W}_t, & P_0 = \sigma^{-1}p; \\ \dot{Q}_t = P_t, & Q_0 = \sigma^{-1}q, \end{cases} \tag{2.22}$$

which is a system with identity diffusion matrix. So quasi-potential $V^\mu(q)$ can also be defined in the following way:

$$V^\mu(q) = \inf\{\bar{S}^\mu(\varphi) : \varphi \in C_{0T},\ \varphi_0 = O,\ \dot{\varphi}_0 = 0,\ \varphi_T = \sigma^{-1}q,\ T > 0\},$$

where $(1/\varepsilon)\bar{S}^\mu(\varphi)$ is the action functional for process Q_t as $\varepsilon \downarrow 0$ and

$$\bar{S}^\mu(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\mu\ddot{\varphi}_t + \dot{\varphi}_t - \sigma^{-1}A\sigma\varphi_t|^2 dt, & \text{if } \dot{\varphi} \text{ absolutely continuous} \\ & \varphi_0 = \sigma^{-1}q,\ \dot{\varphi}_0 = \sigma^{-1}p; \\ +\infty, & \text{otherwise.} \end{cases}$$

Proposition 2.3.9. *The quasi-potential $V^\mu(q)$ for the process $q_t^{\mu,\varepsilon}$ defined by system (2.21) is given by the formula:*

$$V^\mu(q) = \frac{1}{2}((\int_0^\infty e^{At}\sigma\sigma^* e^{A^*t} dt)^{-1}q, q).$$

Proof. From Theorem 3,

$$\begin{aligned} V^{\mu}(q) &= \frac{1}{2}((\int_0^{\infty} e^{\sigma^{-1}A\sigma t}e^{(\sigma^{-1}A\sigma)^* t}dt)^{-1}\sigma^{-1}q, \sigma^{-1}q) \\ &= \frac{1}{2}((\sigma^{-1}\int_0^{\infty} e^{At}\sigma\sigma^* e^{A^* t}dt(\sigma^*)^{-1}\sigma^{-1}q, \sigma^{-1}q) \\ &= \frac{1}{2}((\sigma^{-1})^*\sigma^*(\int_0^{\infty} e^{At}\sigma\sigma^* e^{A^* t}dt)^{-1}\sigma\sigma^{-1}q, q) \\ &= \frac{1}{2}((\int_0^{\infty} e^{At}\sigma\sigma^* e^{A^* t}dt)^{-1}q, q). \end{aligned}$$

□ □

The coincidence of $V^{\mu}(q)$ for system (2.21) and $V(q)$ for system (2.10) is obvious from Propositions 3 and 4. With our assumptions made in Section 2, it occurs for all $\mu > 0$ if A has all eigenvalues real, and only for small μ, if A has some complex eigenvalues.

Chapter 3

Wavefront Propagation in the Reaction-Diffusion Equation

3.1 Wave Front Propagation for the Degenerate KPP-equation

3.1.1 General characterization of wavefronts

To study the wavefront propagation of equation (1.15), we first note that the operator $L^{\mu,\varepsilon}$ defined by equation (1.16) is the infinitesimal generator of the Markov process $(p_t^{\mu,\varepsilon},\ q_t^{\mu,\varepsilon})$ defined by the system

$$\begin{aligned} \mu\dot{p}_t^{\mu,\varepsilon} &= -\dot{q}_t^{\mu,\varepsilon} + \sigma(q_t^{\mu,\varepsilon})\dot{W}_t, \qquad & p_0^{\mu,\varepsilon} &= p \in \mathbb{R}^n \\ \dot{q}_t^{\mu,\varepsilon} &= p_t^{\mu,\varepsilon}, & q_0^{\mu,\varepsilon} &= q \in \mathbb{R}^n. \end{aligned} \tag{3.1}$$

This is equivalent to Langevin's equation defined by:

$$\mu\ddot{q}_t^{\mu,\varepsilon} = -\dot{q}_t^{\mu,\varepsilon} + \sigma(q_t^{\mu,\varepsilon})\dot{W}_t,\ q_0^{\mu,\varepsilon} = q,\ p_0^{\mu,\varepsilon} = p,\ p, q \in \mathbb{R}^n \tag{3.2}$$

where μ is the particle mass, $q_t^{\mu,\varepsilon}$ is the position of particle at time t, $p_t^{\mu,\varepsilon}$ is the velocity of the particle at time t and $-\dot{q}_t^{\mu,\varepsilon}$ is the friction exerted on the particle. We assume the diffusion coefficient $\sigma(q_t^{\mu,\varepsilon})$ is continuously differentiable and positive definite. Let $a(q) = \sigma(q)\sigma^*(q)$.

Lemma 3.1.1. *The action functional for the Markov process $(p_t^{\mu,\varepsilon},\ q_t^{\mu,\varepsilon})$ in C_{0t} for* *fixed* $\mu > 0$ *as* $\varepsilon \downarrow 0$ *has the form* $\varepsilon^{-1}S^{\mu}_{0t}(\phi),\ \phi = (\phi^1,\ \phi^2)$ *where*

$$S^{\mu}_{0t}(\phi^1,\ \phi^2) = \begin{cases} \frac{1}{2}\int_0^t |\sigma^{-1}(\phi^2_s)(\mu\dot{\phi}^1_s + \phi^1_s)|^2 ds, & \dot{\phi}^2_s = \phi^1_s,\ \phi^1_s \ \textit{absolutely continuous.} \\ & \phi^1_0 = p \in \mathbb{R}^n,\ \phi^2_0 = q \in \mathbb{R}^n \\ +\infty, & \textit{otherwise.} \end{cases}$$

Proof. Rewrite equation (3.1) as :

$$\begin{cases} \dot{p}^{\mu,\varepsilon}_t = -\frac{1}{\mu}p^{\mu,\varepsilon}_t + \frac{\sqrt{\varepsilon}}{\mu}\sigma(q^{\mu,\varepsilon}_t)\dot{W}_t, & p^{\mu,\varepsilon}_0 = p; \\ \dot{q}^{\mu,\varepsilon}_t = p^{\mu,\varepsilon}_t, & q^{\mu,\varepsilon}_0 = q. \end{cases}$$

Let $X^{\mu,\varepsilon}_t = \begin{pmatrix} p^{\mu,\varepsilon}_t \\ q^{\mu,\varepsilon}_t \end{pmatrix}$. Then the equation for $X^{\mu,\varepsilon}_t$ is:

$$\dot{X}^{\mu,\varepsilon}_t = \begin{pmatrix} \dot{p}^{\mu,\varepsilon}_t \\ \dot{q}^{\mu,\varepsilon}_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{\mu}p^{\mu,\varepsilon}_t \\ p^{\mu,\varepsilon}_t \end{pmatrix} + \begin{bmatrix} \frac{\sqrt{\varepsilon}}{\mu}\sigma(q^{\mu,\varepsilon}_t) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{W}_t \\ \dot{W}^1_t \end{pmatrix}.$$

Let $\psi_t = (\psi^1_t,\ \psi^2_t)^T$, the transpose of vector $(\psi^1_t,\ \psi^2_t)$, be a continuous function on $[0, T]$ with value on $\mathbb{R}^{2n}$. For each fixed $\mu > 0$, consider the transformation

$$J : \psi \longrightarrow X = \begin{pmatrix} p_t \\ q_t \end{pmatrix} \in \mathbb{R}^{2n}$$

defined by the system

$$p_t - p = -\frac{1}{\mu}\int_0^t p_s ds + \frac{1}{\mu}[\psi^1_t\sigma(q_t) - \psi^1_0\sigma(q)] - \frac{1}{\mu}\int_0^t \psi^1_s \frac{d}{ds}[\sigma(q_s)]ds,$$

$$q_t - q = \int_0^t p_s ds.$$

Let $X_1 = J\psi_1 = (p_1(t),\ q_1(t))^T,\ X_2 = J\psi_2 = (p_2(t),\ q_2(t))^T$. For any $t \in [0, T]$, the norm of the difference satisfies the inequality:

$$\|X_1(t) - X_2(t)\| \leq K_1 \int_0^t \|X_1(s) - X_2(s)\|ds + K_2 T\|\psi_1 - \psi_2\|_{C_{0T}}$$

where $\|\psi\|_{C_{0T}} = \max_{t\in[0,T]} |\psi(t)|$, K_1, K_2 are some constants. From Gronwall's inequality:

$$\|X_1(t) - X_2(t)\| \leq e^{K_1 T} K_2 T \|\psi_1 - \psi_2\|_{C_{0T}}.$$

which implies that the operator J is continuous.

From Theorem 1.2.1, the action functional for the process $(p_t^{\mu,\varepsilon},\ q_t^{\mu,\varepsilon})$ has the form $\varepsilon^{-1} S_{0t}^{\mu}(\phi)$ where

$$\begin{aligned}
S_{0t}^{\mu}(\phi) &= \min\{S_{0t}^{W}(\psi) : J\psi = \varphi\} \\
&= \min\{\frac{1}{2}\int_0^t |\dot{\psi}_s|^2 ds : \\
&\qquad \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} p + \frac{1}{\mu}[\psi_t^1 \sigma(\phi_t^2) - \psi_0^1 \sigma(q)] \\ q \end{pmatrix} + \int_0^t \begin{pmatrix} -\frac{1}{\mu}\phi_s^1 - \frac{1}{\mu}\psi_s^1 \frac{d}{ds}[\sigma(\phi_s^2)] \\ \phi_s^1 \end{pmatrix} ds\} \\
&= \min\{\frac{1}{2}\int_0^t |\dot{\psi}_s|^2 ds : \\
&\qquad \phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} + \int_0^t \begin{pmatrix} -\frac{1}{\mu}\phi_s^1 \\ \phi_s^1 \end{pmatrix} ds + \frac{1}{\mu}\int_0^t \begin{bmatrix} \sigma(\phi_s^2) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi}_s^1 \\ \dot{\psi}_s^2 \end{pmatrix} ds\}
\end{aligned}$$

if ψ_s^1 is absolutely continuous. When $\dot{\psi}_s^1 = \sigma^{-1}(\varphi_s^2)(\mu\dot{\phi}_s^1 + \phi_s^1)$, $\psi_s^2 = 0$, $\psi_s^2 = \psi_s^1$, $\frac{1}{2}\int_0^t |\dot{\psi}_s|^2 ds$ attains its the minimum. Thus the normalized action functional is given by:

$$S_{0t}^{\mu}(\phi^1,\ \phi^2) = \begin{cases} \frac{1}{2}\int_0^t |\sigma^{-1}(\phi_s^2)(\mu\dot{\phi}_s^1 + \phi_s^1)|^2 ds, & \dot{\phi}_s^2 = \phi_s^1,\ \phi_s^1 \text{ absolutely continuous} \\ & \phi_0^1 = p \in \mathbb{R}^n,\ \phi_0^2 = q \in \mathbb{R}^n \\ +\infty, & \text{otherwise.} \end{cases}$$

□

Let the functional $\tau : C([0,t], \mathbb{R}^{2n}) \to [0,t]$ be a stopping time which depends only on $\phi_s = (\phi_s^1,\ \phi_s^2)$, $0 \leq s \leq u$ when restricted to $\{\tau \leq u\}$. Let Γ_t be the collection of all stopping times not greater than t. If F is a closed subset of $[0,t]\times\mathbb{R}^{2n}$ and $\{0\}\times\mathbb{R}^{2n}\subset F$, then $\tau_F\equiv\min\{s: s\geq 0 \text{ and } (t-s,\phi_s)\in F\}$ is a stopping time not greater than t. Let Θ_t be the collection of such τ_F. Define:

$$\begin{aligned}
V_0^{\mu}(t,p,q) &= \inf_{\tau\in\Gamma_t}\sup\{\int_0^{\tau} c(\phi_s)ds - S_{0t}^{\mu}(\phi):\\
&\qquad \phi \text{ is abs. cont, } \phi_0=(p,q),\ \phi_t\in[G_0^{\mu}]\}\\
&= \inf_{\tau\in\Gamma_t}\sup\{\int_0^{\tau} c(\varphi_s)-\frac{1}{2}|\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s+\dot{\varphi}_s)|^2ds:\ \dot{\varphi}_s \text{ abs. cont.}\\
&\qquad \varphi_0=q,\dot{\varphi}_0=p,(\varphi_t,\dot{\varphi}_t)\in[G_0^{\mu}]\}.
\end{aligned}$$

Similarly, define:

$$\begin{aligned}
V_1^{\mu}(t,p,q) &= \inf_{\tau\in\Theta_t}\sup\{\int_0^{t} c(\phi_s) - S_{0t}^{\mu}(\phi):\\
&\qquad \phi \text{ abs. cont., } \phi_0=(p,\ q),\ \phi_t\in[G_0^{\mu}]\}\\
&= \inf_{\tau\in\Theta_t}\sup\{\int_0^{t} c(\varphi_s)-\frac{1}{2}|\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s+\dot{\varphi}_s)|^2ds:\ \dot{\varphi}_s \text{ abs. cont.}\\
&\qquad \varphi_0=q,\dot{\varphi}_0=p,(\varphi_t,\dot{\varphi}_t)\in[G_0^{\mu}]\}
\end{aligned}$$

and

$$\begin{aligned}
V^{*,\mu}(t,p,q) &= \sup\min_{0\leq a\leq t}\{\int_0^{a} c(\phi_s) - S_{0t}^{\mu}(\phi):\\
&\qquad \phi \text{ abs. cont., } \phi_0=(p,\ q),\ \phi_t\in G_0\}\\
&= \sup\min_{0\leq a\leq t}\{\int_0^{a} c(\varphi_s)-\frac{1}{2}|\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s+\dot{\varphi}_s)|^2ds:\\
&\qquad \dot{\varphi}_s \text{ abs. cont., } \varphi_0=q,\dot{\varphi}_0=p,(\varphi_t,\dot{\varphi}_t)\in[G_0^{\mu}]\}
\end{aligned}$$

It can be proved (Lemma 2.4 of [11]) that

$$V_0^{\mu}=V_1^{\mu}=V^{*,\mu}.$$

Theorem 3.1.2 ([16]). *The solution of equation (1.15) satisfies the following relation:*

$$\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(t,p,q) = 0,$$

uniformly on any compact subset of $\{(t,p,q) : V^{*,\mu}(t,p,q) < 0\}$. *There exists* $h > 0$ *such that*

$$\liminf_{\varepsilon \downarrow 0} u^{\varepsilon}(t,p,q) \geq h$$

uniformly on any compact subset of $\{(t,p,q) : V^{*,\mu}(t,p,q) = 0\}$.

3.1.2 Characterization of Wavefronts under Condition (N^{μ})

We will investigate in this section another characterization of wavefronts of equation (1.15) when it satisfies a certain condition (N^{μ}). To establish this characterization, we would first like to obtain an asymptotic formula similar to (1.12) as a lemma.

As is shown in Lemma 2.1, the action functional for the process $(p_t^{\mu,\varepsilon},\ q_t^{\mu,\varepsilon})$ as $\varepsilon \downarrow 0$ is $\varepsilon^{-1} S^{\mu}_{0t}(\phi)$. By the definition of action functional, the following estimates hold:

i. for any function $\phi = (\phi^{(1)},\ \phi^{(2)}) \in C_{0t}(\mathbb{R}^{2n}),\ \phi_0^{(1)} = p,\ \phi_0^{(2)} = q$ and arbitrary $\gamma,\ \delta > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$:

$$\mathbb{P}^{\mu,\varepsilon}_{p,q}\{\rho_{0t}(X^{\mu,\varepsilon}_s,\ \phi_s) < \delta\} \geq \exp\{-\frac{1}{\varepsilon}(S^{\mu}_{0t}(\phi) + \gamma)\};$$

ii. for any $r < \infty$, the set $\Phi_s = \{\phi \in C_{0t}(\mathbb{R}^{2n}) :\ \phi_0 = x = (p,q),\ S^{\mu}_{0t}(\phi) \leq r\}$ is compact in $C_{0t}(\mathbb{R}^{2n})$. Also for arbitrary $\gamma,\ \delta > 0$ one can find $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$

$$\mathbb{P}_{p,q}^{\mu,\varepsilon}\{\rho_{0t}(X_s^{\mu,\varepsilon},\ \Phi_s) \geq \delta\} \leq \exp\{-(1/\varepsilon)(r-\delta)\}.$$

Lemma 3.1.3. *Assume $g(x) = g(p,q),\ x = (p,q) \in \mathbb{R}^{2n}$, is a non-negative, bounded function and denote its support of $\{x \in \mathbb{R}^{2n} :\ g(x) > 0\}$ by G_0^μ. Let $c(q),\ q \in \mathbb{R}^n$, be bounded and uniformly continuous. Let $\phi = (\phi^{(1)},\ \phi^{(2)})$ and*

$$R_{0t}^\mu(\phi) = \int_0^t c(\phi_s^{(2)}) - \frac{1}{2}|\sigma^{-1}(\phi_s^{(2)})(\mu\dot{\phi}_s^{(1)} + \phi_s^{(1)})|^2 ds.$$

Then

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln E_{p,q}^{\mu,\varepsilon} g(p_t^{\mu,\varepsilon},\ q_t^{\mu,\varepsilon}) \exp\{\frac{1}{2}\int_0^t c(q_s^{\mu,\varepsilon})ds\}$$

$$= \sup\{R_{0t}^\mu(\phi) : \phi_0 = x,\ \phi_t \in [G_0^\mu]\}$$

Proof. Let $m = \sup\{R_{0t}^\mu(\phi) :\ \phi_0 = x,\ \phi_t \in [G_0^\mu]\}$. Since $c(q)$ is bounded, g is nonnegative and bounded and $m < +\infty$. The functional $R_{0t}^\mu(\phi)$ is upper semi-continuous. Thus, for any $\gamma > 0$, one can find $\hat{\phi} \in C_{0t}(\mathbb{R}^{2n})$ such that

$$\hat{\phi}_0 = x,\ \rho_{0t}(\hat{\phi}_s,\ \mathbb{R}^{2n} \setminus G_0) = \delta_1 > 0 \text{ and } R_{0t}^\mu(\hat{\phi}) > m - \gamma.$$

Let $\kappa > 0$ such that

$$\int_0^t |c(q_s^{\mu,\varepsilon}) - c(\hat{\phi}_s^{(2)})|ds < \frac{\gamma}{2}$$

provided

$$\rho_{0t}(X_s^{\mu,\varepsilon}, \hat{\phi}_s) < \kappa,\ \delta_2 = \kappa \wedge \frac{\delta_1}{2}.$$

Then we have estimates:

$$E_x^{\mu,\varepsilon} g(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon}) \exp\{\frac{1}{\varepsilon}\int_0^t c(q_s^{\mu,\varepsilon})ds\}$$

$$\geq E_x^{\mu,\varepsilon}\chi_{\{\rho_{0t}(X_s^{\mu,\varepsilon},\hat{\phi}_s)<\delta_2\}}g(X_t^{\mu,\varepsilon})\exp\{\frac{1}{\varepsilon}\int_0^t c(q_s^{\mu,\varepsilon})ds\}$$

$$\geq \min_{\{x:\rho_{0t}(x,\hat{\phi})<\delta_2\}}|g(x)|\exp\{\frac{1}{\varepsilon}\int_0^t c(\hat{\phi}_s^{(2)})ds-\frac{\gamma}{2\varepsilon}\}\times\mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X_s^{\mu,\varepsilon},\hat{\phi}_s)<\delta_2\}$$

From estimate (i) the inequalities continue as

$$\geq \min_{\{x:\rho_{0t}(x,\hat{\phi})<\delta_2\}}|g(x)|\exp\{\frac{1}{\varepsilon}\int_0^t c(\hat{\phi}_s^{(2)})ds-\frac{\gamma}{2\varepsilon}\}\exp\{-\frac{1}{\varepsilon}(S_{0t}^{\mu}(\hat{\phi})+\gamma)\}$$

$$\geq \min_{\{x:\rho_{0t}(x,\hat{\phi})<\delta_2\}}|g(x)|\exp\{\frac{1}{\varepsilon}(\int_0^t c(\hat{\phi}_s^{(2)})ds-S_{0t}^{\mu}(\hat{\phi}))-\frac{2\gamma}{\varepsilon}\}\cdot\exp\{\frac{\gamma}{2\varepsilon}\}$$

$$\geq \exp\{\frac{1}{\varepsilon}(\int_0^t c(\hat{\phi}_s^{(2)})ds-S_{0t}^{\mu}(\hat{\phi}))-\frac{2\gamma}{\varepsilon}\}$$

$$\geq \exp\{\varepsilon^{-1}(m-3\gamma)\}.$$

Considering all estimates above, we get

$$E_x^{\mu,\varepsilon}g(p_t^{\mu,\varepsilon},q_t^{\mu,\varepsilon})\exp\{\frac{1}{\varepsilon}\int_0^t c(q_s^{\mu,\varepsilon})ds\}\geq\exp\{\frac{1}{\varepsilon}(m-3\gamma)\}. \tag{3.3}$$

To derive an upper bound, put $s=|m|+t\sup_{q\in\mathbb{R}^n}|c(q)|+1$. Then

$$E_x^{\mu,\varepsilon}g(p_t^{\mu,\varepsilon},q_t^{\mu,\varepsilon})\exp\{\frac{1}{\varepsilon}c(q_s^{\mu,\varepsilon})ds\}=e_1+e_2 \tag{3.4}$$

where

$$e_1=E_x^{\mu,\varepsilon}g(p_t^{\mu,\varepsilon},q_t^{\mu,\varepsilon})\chi_{\{\rho_{0t}(X_s^{\mu,\varepsilon},\Phi_r)\geq\frac{\kappa}{2}\}}\cdot\exp\{\frac{1}{\varepsilon}\int_0^t c(q_s^{\mu,\varepsilon})ds\}$$

$$e_2=E_x^{\mu,\varepsilon}g(p_t^{\mu,\varepsilon},q_t^{\mu,\varepsilon})\chi_{\{\rho_{0t}(X_s^{\mu,\varepsilon},\Phi_r)<\frac{\kappa}{2}\}}\cdot\exp\{\frac{1}{\varepsilon}\int_0^t c(q_s^{\mu,\varepsilon})ds\}$$

$$e_1\leq\sup_{\{x:\rho_{0t}(x,\Phi_r)\geq\frac{\kappa}{2}\}}\exp\{\frac{1}{\varepsilon}t\sup_{q\in\mathbb{R}^n}|c(q)|\}\cdot\mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X_s^{\mu,\varepsilon},\Phi_r)\geq\frac{\kappa}{2}\}$$

$$\le \exp\{\frac{1}{\varepsilon}\, t \sup_{q\in\mathbb{R}^n} |c(q)| - \frac{1}{\varepsilon}(s-\gamma)\}$$

$$\le \exp\{-\frac{1}{\varepsilon}(|m|+1-\gamma)\}.$$

Choose a finite $\frac{\gamma}{2}$-net: $\phi_1,\ \phi_2,\ldots,\phi_n$. Then

$$e_2 \le \sup_{x\in\mathbb{R}^{2n}} |g(x)| \sum_{i=1}^{N} E_x^{\mu,\varepsilon} \chi_{\{\rho_{0t}(\phi_i,\ X_s^{\mu,\varepsilon})<\kappa\}} \exp\{\frac{1}{\varepsilon}\int_0^t c(q_s^{\mu,\varepsilon})ds\}$$

$$\le \sup_{x\in\mathbb{R}^{2n}} |g(x)| \sum_{i=1}^{N} \exp\{\frac{1}{\varepsilon}(\int_0^t c(\phi_{i_s})ds + \frac{\gamma}{2})\}\mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X_s^{\mu,\varepsilon},\phi_i)<\kappa\}.$$

Put $a_i = \inf\{S_{0t}^{\mu}(\phi) : \rho_{0t}(\phi,\phi_i) < \kappa\} - \gamma/4,\ i = 1,\ldots,N$. Since $S_{0t}^{\mu}(\phi)$ is semi-continuous, one can find $\alpha > 0$ such that $\rho_{0t}(\Phi_{a_i},\phi_i) > \kappa+\alpha$. Since

$$\rho_{0t}(X^{\mu,\varepsilon},\ \phi_i) \ge -\rho_{0t}(X^{\mu,\varepsilon},\Phi_{a_i}) + \rho_{0t}(\phi_i,\Phi_{a_i})$$

$$\mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X^{\mu,\varepsilon},\varphi_i)<\kappa\} \le \mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(\varphi_i,\Phi_{a_i}) - \rho_{0t}(X^{\mu,\varepsilon},\Phi_{a_i})\} \le \kappa\}$$

$$= \mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X^{\mu,\varepsilon},\Phi_{a_i}) \ge \rho_{0t}(\varphi_i,\Phi_{a_i}) - \kappa\}$$

$$= \mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X^{\mu,\varepsilon},\Phi_{a_i}) \ge \kappa+\alpha-\kappa\}$$

$$= \mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X^{\mu,\varepsilon},\Phi_{a_i}) \ge \alpha\},$$

we have

$$\mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X^{\mu,\varepsilon},\phi_i)<\kappa\} \le \mathbb{P}_x^{\mu,\varepsilon}\{\rho_{0t}(X^{\mu,\varepsilon},\Phi_{a_i}) \ge \alpha\}$$

$$\le \exp\{-\frac{1}{\varepsilon}(a_i - \frac{\gamma}{4})\}$$

This implies that

$$e_2 \le \sup_{x\in\mathbb{R}^{2n}} |g(x)| \sum_{i=1}^{N} \exp\{\frac{1}{\varepsilon}(\int_0^t c(\phi_{i_s})ds + \frac{\gamma}{2})\} \cdot \exp\{-\frac{1}{\varepsilon}(a_i + \frac{\gamma}{4})\}$$

$$= \sup_{x\in\mathbb{R}^{2n}} |g(x)| \sum_{i=1}^{N} \exp\{\frac{1}{\varepsilon}\int_0^t c(\phi_{i_s})ds - \frac{1}{\varepsilon}(a_i + \frac{\gamma}{4})\}$$

$$= \sup_{x\in\mathbb{R}^{2n}} |g(x)| \sum_{i=1}^{N} \exp\{\frac{1}{\varepsilon}[\sup\{R^{\mu}_{0t}(\phi) : \ \rho_{0t}(\phi,\phi_i) < \kappa\} + 2\gamma]\}$$

From equation (3.4), we obtain

$$E^{\mu,\varepsilon}_x g(p^{\mu,\varepsilon}_t, q^{\mu,\varepsilon}_t) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(q^{\mu,\varepsilon}_s)ds\} \le \exp\{\frac{1}{\varepsilon}(m+3\gamma)\right\} \tag{3.5}$$

Since γ is arbitrarily small, from estimate (3.3) and (3.5), we prove

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln E^{\mu,\varepsilon}_x g(p^{\mu,\varepsilon}_t, q^{\mu,\varepsilon}_t) \exp\{\frac{1}{\varepsilon}\int_0^t c(q^{\mu,\varepsilon}_s)ds\}$$

$$= \sup\{R^{\mu}_{0t}(\phi) : \ \phi_0 = x, \ \phi_t \in [\text{supp } g(p,q)] = [G^{\mu}_0]\}.$$

□

Let $\varphi = \phi^{(2)}$ be the second component of vector $\phi = (\phi^{(1)}, \phi^{(2)}) \in \mathbb{R}^{2n}$. Define $V^{\mu}(t,p,q)$ as:

$$V^{\mu}(t,p,q) = \sup\{R^{\mu}_{0t}(\phi) : \phi_0 = (p,q), \phi_t \in [G^{\mu}_0]\}$$

$$= \sup\{\int_0^t c(\phi_s)ds - S^{\mu}_{0t}(\phi) : \phi_0 = (p,q), \phi_t \in [G^{\mu}_0]\}$$

$$= \sup\{\int_0^t c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi)(\mu\ddot{\varphi}_s + \dot{\varphi}_s)|^2 ds :$$

$$\varphi_0 = q, \ \dot{\varphi}_0 = p, \ (\varphi_t, \dot{\varphi}_t) \in [G^{\mu}_0]\}$$

The Feynman-Kac formula implies that the function $u^\varepsilon(t,p,q)$ obeys the relation

$$u^\varepsilon(t,p,q) = E^{\mu,\varepsilon}_{(p,q)} g(p^{\mu,\varepsilon}_t, q^{\mu,\varepsilon}_t)\exp\{\frac{1}{\varepsilon}c(q^{\mu,\varepsilon}_s, u^\varepsilon(t-s, p^{\mu,\varepsilon}_s, q^{\mu,\varepsilon}_s)ds\} \tag{3.6}$$

where $c(q,u) = u^{-1}f(q,u)$, $c(q) = c(q,0) \geq c(q,u)$. Let

$$\Omega^\mu_- = \{(t,p,q):\ V^\mu(t,p,q) < 0\}.$$

We say that condition (N^μ) is fulfilled if

$$(N^\mu) : V^\mu(t,p,q) = \sup\{\int_0^t c(\varphi_s)ds - S^\mu_{0t}(\varphi) :$$

$$\varphi_0 = q,\ \dot\varphi_0 = p,\ (\varphi_t, \dot\varphi_t) \in [G^\mu_0],\ (t-s, \dot\varphi_s, \varphi_s) \in \Omega^\mu_- \text{ for } 0 < s < t\}$$

holds for any $t > 0$ and $(p,q) \in \Sigma_t = \{(p,q) \in \mathbb{R}^{2n} : V^\mu(t,p,q) = 0\}$.

Theorem 3.1.4. *Suppose $f(q,u)$ satisfies the KPP assumption for $q \in \mathbb{R}^n$ and let condition (N^μ) be fulfilled. Then for the solution $u^\varepsilon(t,p,q)$ of the Cauchy problem (1.15) the following relation holds:*

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,p,q) = \begin{cases} 1, & \textit{for } \Omega^\mu_+ = \{(t,p,q) : V^\mu(t,p,q) > 0\} \\ 0, & \textit{for } \Omega^\mu_- = \{(t,p,q) : V^\mu(t,p,q) < 0\} \end{cases}$$

This convergence is uniform on every compactum lying in the region

$$\{(t,p,q):\ t > 0,\ p,q \in \mathbb{R}^n,\ V^\mu(t,p,q) < 0\}$$

and

$$\{(t,p,q):\ t > 0,\ p,q \in \mathbb{R}^n,\ V^\mu(t,p,q) > 0\}$$

respectively.

Proof. From the KPP assumption, we know $c(q,u) \le c(q,0) = c(q)$. It follows that

$$0 \le u^\varepsilon(t,p,q) \le E^{\mu,\varepsilon}_{(p,q)} g(p^{\mu,\varepsilon}_t, q^{\mu,\varepsilon}_t) \exp\{\frac{1}{\varepsilon}\int_0^t c(q^{\mu,\varepsilon}_s)ds\}$$

$$\asymp \exp\{\frac{V^\mu(t,p,q)}{\varepsilon}\}.$$

Thus when $(t,p,q) \in \Omega^\mu_-$, that is, $V^\mu(t,p,q) < 0$,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \ln u^\varepsilon(t,p,q) \le V^\mu(t,p,q) < 0.$$

Therefore

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,p,q) = 0.$$

This convergence is uniform on the set $\Omega_\delta \cap \Omega^\mu_-$ where $\Omega_\delta = \{(t,p,q) : \ t \in [0,T], |(p,q)| < T, |V^\mu(t,p,q)| \ge \delta\}$.

To show $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,p,q) = 1$ whenever $V^\mu(t,p,q) > 0$, consider the strong Markov process

$$(Y^{\mu,\varepsilon}_s = (t_s, p^{\mu,\varepsilon}_s, q^{\mu,\varepsilon}_s) = (t-s, p^{\mu,\varepsilon}_s, q^{\mu,\varepsilon}_s),\ \mathbb{P}^{\mu,\varepsilon}_{(t,p,q)})$$

corresponding to the operator $L^{\mu,\varepsilon} - \partial/\partial t$. First we show that if (N^μ) holds, then for any $\delta > 0,\ T > 0$ there exists ε_0 such that when $0 < \varepsilon < \varepsilon_0$, for $(p,q) \in \Sigma_t,\ 0 < t < T$, we have

$$u^\varepsilon(t,p,q) > \exp\{-\delta/\varepsilon\}. \tag{3.7}$$

By virtue of condition (N^μ), let $\hat\varphi \in C_{0t}(\mathbb{R}^n),\ \hat\varphi_0 = q, \dot{\hat\varphi}_0 = p, (\hat\varphi, \dot{\hat\varphi}) \in [G^\mu_0]$. For some small number $\theta > 0$, suppose that when $s \in [\theta,\ t-\theta]$, the point $(t-s, \dot{\hat\varphi}_s, \hat\varphi_s)$ is at a positive distance κ from the complement of Ω^μ_- and $R^\mu_{0t}(\hat\varphi) = \int_0^t c(\hat\varphi_s)ds -$

$S^\mu_{0t}(\hat\varphi) > -\delta/4$. Since

$$0 \le u^\varepsilon(t,p,q) \le \exp\{\frac{V^\mu(t,p,q)}{\varepsilon}\}$$

$$0 \le u^\varepsilon(t-s,p,q) \le \exp\{\frac{V^\mu(t-s,p,q)}{\varepsilon}\}$$

$$0 \le u^\varepsilon(t-s,\dot{\hat\varphi}_s,\hat\varphi_s) \le \exp\{\frac{V^\mu(t-s,\dot{\hat\varphi}_s,\hat\varphi_s)}{\varepsilon}\} < 1,$$

for small $\varepsilon > 0$, $u^\varepsilon(t-s,\dot{\hat\varphi}_s,\hat\varphi_s)$ is close to 0, except for small parts near $s = 0$ and $s = t$. Therefore

$$\sup_{\theta < s < t-\theta}[c(\hat\varphi_s) - c(\hat\varphi_s, u^\varepsilon(t-s,\dot{\hat\varphi}_s,\hat\varphi_s))] < \frac{\delta}{4}$$

provided $\varepsilon > 0$ is small enough. Then one can find θ and κ_0 so small such that that

$$u^\varepsilon(t,p,q) = E^{\mu,\varepsilon}_{(p,q)} g(p^{\mu,\varepsilon}_t, q^{\mu,\varepsilon}_t)\exp\{\frac{1}{\varepsilon}\int_0^t c(q^{\mu,\varepsilon}_s, u^\varepsilon(t-s,p^{\mu,\varepsilon}_s,q^{\mu,\varepsilon}_s))ds\}$$

$$\ge E^{\mu,\varepsilon}_{(p,q)} g(p^{\mu,\varepsilon}_t, q^{\mu,\varepsilon}_t)\chi_{\rho_{0t}((\hat\varphi,\dot{\hat\varphi}),(q^{\mu,\varepsilon},p^{\mu,\varepsilon}))<\kappa_0}$$

$$\times \exp\{\frac{1}{\varepsilon}\int_0^t c(q^{\mu,\varepsilon}_s, u^\varepsilon(t-s,p^{\mu,\varepsilon}_s,q^{\mu,\varepsilon}_s))ds\}$$

$$\ge \mathbb{P}^{\mu,\varepsilon}_{(p,q)}\{\rho_{0t}((\hat\varphi,\dot{\hat\varphi}),(q^{\mu,\varepsilon},p^{\mu,\varepsilon})) < \kappa_0\}\exp\{\frac{1}{\varepsilon}(\int_0^t c(\hat\varphi_s)ds - \frac{\delta}{4})\}$$

$$\ge \exp\{-\frac{1}{\varepsilon}(S^\mu_{0t}(\varphi) + \frac{\delta}{2})\}\exp\{\frac{1}{\varepsilon}(\int_0^t c(\hat\varphi_s)ds - \frac{\delta}{4})\}$$

$$= \exp\{\frac{1}{\varepsilon}(\int_0^t c(\hat\varphi_s)ds - S^\mu_{0t}(\hat\varphi)) - \frac{\delta}{4\varepsilon} - \frac{2\delta}{4\varepsilon}\}$$

$$= \exp\{\frac{1}{\varepsilon}(R^\mu_{0t}(\hat\varphi) - \frac{3\delta}{4})\}$$

$$> \exp\{-\frac{\delta}{\varepsilon}\}.$$

Next we establish the inequality

$$\liminf_{\varepsilon\downarrow 0} u^\varepsilon(t,p,q) \geq 1.$$

Let λ be a small positive number. Introduce Markov times:

$$\tau_1^{\varepsilon,\lambda} = \tau_1 = \inf\{s:\ u^\varepsilon(t_s, p_s^{\mu,\varepsilon}, q_s^{\mu,\varepsilon}) \geq 1-\lambda\}$$

$$\tau_2^{\varepsilon,\lambda} = \tau_2 = \inf\{s:\ V^\mu(t_s, p_s^{\mu,\varepsilon}, q_s^{\mu,\varepsilon}) = 0\}$$

$$\tau^{\varepsilon,\lambda} = \tau = \tau_1^{\varepsilon,\lambda} \wedge \tau_2^{\varepsilon,\lambda} = \tau_1 \wedge \tau_2.$$

The strong Markov property and the Feynman-Kac formula imply:

$$u^\varepsilon(t,p,q) = E_{(t,p,q)}^{\mu,\varepsilon} u^\varepsilon(t_\tau, p_\tau^{\mu,\varepsilon}, q_\tau^{\mu,\varepsilon}) \exp\{\frac{1}{\varepsilon}\int_0^\tau c(q_s^{\mu,\varepsilon},\ u^\varepsilon(t-s, p_s^{\mu,\varepsilon}, q_s^{\mu,\varepsilon}))ds\}$$

$$= A_1 + A_2$$

where

$$A_1 = E_{(t,p,q)}^{\mu,\varepsilon} \chi_{\tau=\tau_1} u^\varepsilon(t_{\tau_1}, p_{\tau_1}^{\mu,\varepsilon}, q_{\tau_1}^{\mu,\varepsilon}) \exp\{\frac{1}{\varepsilon}\int_0^{\tau_1} c(q_s^{\mu,\varepsilon},\ u^\varepsilon(t_s, p_s^{\mu,\varepsilon}, q_s^{\mu,\varepsilon}))ds\}$$

$$A_2 = E_{(t,p,q)}^{\mu,\varepsilon} \chi_{\tau=\tau_2} u^\varepsilon(t_{\tau_2}, p_{\tau_2}^{\mu,\varepsilon}, q_{\tau_2}^{\mu,\varepsilon}) \exp\{\frac{1}{\varepsilon}\int_0^{\tau_2} c(q_s^{\mu,\varepsilon},\ u^\varepsilon(t_s, p_s^{\mu,\varepsilon}, q_s^{\mu,\varepsilon}))ds\}.$$

Since $c(q,u) \geq 0$ when $0 \leq u \leq 1-\lambda$,

$$A_1 \geq (1-\lambda) E_{(t,p,q)}^{\mu,\varepsilon} \chi_{\tau=\tau_1} = (1-\lambda)\mathbb{P}_{(t,p,q)}^{\mu,\varepsilon}\{\tau=\tau_1\}. \tag{3.8}$$

To bound A_2, let $V_0 = V^\mu(t,p,q) > 0$, choose $h > 0$ such that

$$\inf\{V^\mu(s,y,x): \ |s-t| < h, \ |y-p| < h, \ |x-q| < h\} > \frac{1}{2}V_0.$$

Select $\delta \in (0, \alpha/2)$, where

$$\alpha = h \cdot \min_{|x-q|\le h, \ 0\le u\le 1-\lambda} c(q,u).$$

By (3.7), for ε small enough

$$u^\varepsilon(t_{\tau_2}, p^{\mu,\varepsilon}_{\tau_2}, q^{\mu,\varepsilon}_{\tau_2}) > \exp\left\{-\frac{\delta}{\varepsilon}\right\}.$$

We denote $\tau_3 = \inf\{s : |q^{\mu,\varepsilon}_s - q| = h\}$. Let $D = \{x : |x-q| \le h\}$. Then

$$P\{\tau_3 \le t\} = P\{q^{\mu,\varepsilon}_s \text{ exits from } D \text{ for some } s \in [0,t]\}$$

$$\asymp \exp(-\frac{1}{\varepsilon}\inf\{S^\mu_{0t}(\varphi) : \varphi_0 = q, \dot\varphi_0 = p, (\varphi_s, \dot\varphi_s) \in \partial D,$$

$$\text{for some } s \in [0,t]\})$$

$$= \exp\{-\frac{C_1}{\varepsilon}\} \text{ for some } C_1 > 0.$$

Therefore, as $\varepsilon \downarrow 0$, $P\{\tau_3 \le t\} \to 0$ for any $t > 0$. Then A_2 can be bounded from below as follows

$$A_2 = E^{\mu,\varepsilon}_{(t,p,q)}\chi_{\tau=\tau_2} u^\varepsilon(t_{\tau_2}, p^{\mu,\varepsilon}_{\tau_2}, q^{\mu,\varepsilon}_{\tau_2}) \exp\{\frac{1}{\varepsilon}\int_0^{\tau_2} c(q^{\mu,\varepsilon}_s, \ u^\varepsilon(t_s, p^{\mu,\varepsilon}_s, q^{\mu,\varepsilon}_s))ds\}$$

$$\ge E^{\mu,\varepsilon}_{(t,p,q)}\chi_{\tau_2<\tau_3} \exp\{-\frac{\delta}{\varepsilon}\} \cdot \exp(\frac{\alpha}{h} \cdot \frac{\tau_2}{\varepsilon}) - E^{\mu,\varepsilon}_{(t,p,q)}\chi_{\tau_2\ge\tau_3}.$$

Since $\tau_2 > h$,

$$A_2 \ge \exp\{\frac{\alpha-\delta}{\varepsilon}\}\mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\tau = \tau_2 < \tau_3\} - \mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\tau_2 \ge \tau_3\}$$

$$\geq \exp\{\frac{\alpha}{2\varepsilon}\}\mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\tau_2 < \tau_3\} - o(\varepsilon). \tag{3.9}$$

Collecting estimates (3.8) and (3.9), we obtain

$$u^\varepsilon(t,p,q) \geq (1-\lambda)\mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\tau = \tau_1\} + \exp\{\frac{\alpha}{2\varepsilon}\}\mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\tau_2 < \tau_3\},$$

which implies that

$$u^\varepsilon(t,p,q) > 1-\lambda, \text{ for } \varepsilon \text{ small enough.}$$

This is true for any $\lambda > 0$, so

$$\liminf_{\varepsilon \downarrow 0} u^\varepsilon(t,p,q) \geq 1 \tag{3.10}$$

Finally we show that

$$\limsup_{\varepsilon \downarrow 0} u^\varepsilon(t,p,q) \leq 1.$$

Pick a small $\lambda > 0$. Denote $D^\varepsilon = \{(t,p,q) : t \geq 0, u^\varepsilon(t,p,q) \geq 1+\lambda\}$, and let $\tau_4 = \tau_4^{\mu,\varepsilon,\lambda} = \inf\{s : \ Y_s^{\mu,\varepsilon} \notin D^\varepsilon\}$, the first exit time of the process $Y_s^{\mu,\varepsilon}$ from D^ε. Then

$$u^\varepsilon(t,p,q) = E^{\mu,\varepsilon}_{(t,p,q)} u^\varepsilon(t-\tau_4, p^{\mu,\varepsilon}_{\tau_4}, q^{\mu,\varepsilon}_{\tau_4}) \exp\{\frac{1}{\varepsilon}\int_0^{\tau_4} c(q_s^{\mu,\varepsilon},\ u^\varepsilon(t_s, p_s^{\mu,\varepsilon}, q_s^{\mu,\varepsilon}))ds\}$$

$$= E^{\mu,\varepsilon}_{(t,p,q)} \chi_{\tau_4<t} u^\varepsilon(Y^{\mu,\varepsilon}_{\tau_4}) \exp\{\frac{1}{\varepsilon}\int_0^{\tau_4} c(q_s^{\mu,\varepsilon}, u^\varepsilon(Y_s^{\mu,\varepsilon}))ds\}$$

$$+ E^{\mu,\varepsilon}_{(t,p,q)} \chi_{\tau_4=t} g(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon}) \exp\{\frac{1}{\varepsilon}\int_0^{t} c(q_s^{\mu,\varepsilon}, u^\varepsilon(Y_s^{\mu,\varepsilon}))ds\}$$

When $u < 1+\lambda$, the KPP assumption implies that $c(q,u) = u^{-1}f(q,u) > 0$. Therefore

$$u^\varepsilon(t,p,q) \le (1+\lambda)\mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\tau_4 < t\} + \|g\| \exp\{-\frac{-t}{\varepsilon} \min_{1+\lambda \le u \le 2+\|g\|, |x-q| \le h} |c(x,u)|\}$$

$$\times \mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\tau_4 = t\} + \mathbb{P}^{\mu,\varepsilon}_{(t,p,q)}\{\inf\{s : |q^{\mu,\varepsilon}_s - q| = h < t\}\}$$

If we choose ε small enough, $u^\varepsilon(t,p,q) \le 1 + 2\lambda$, thus

$$\limsup_{\varepsilon \downarrow 0} u^\varepsilon(t,p,q) \le 1. \tag{3.11}$$

From (3.10) and (3.11), we get

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,p,q) = 1, \text{ when } (t,p,q) \in \Omega^\mu_+$$

□

3.2 Convergence of the Wavefronts

We first summarize the characterization of the wavefronts of equations (1.15) and (1.17). Let

$$G^\mu_0 = \{(p,q) : \ g(p,q) > 0\}$$

and let $[G^\mu_0]$ denote the closure of G^μ_0; let

$$G_0 = \{(0,q) : g(0,q) > 0\}$$

and let $[G_0]$ denote the closure of G_0.

For the degenerate reaction-diffusion equation (1.15):

i. When condition (N^μ) is satisfied, i.e.

(N^μ) :

$$V^{\mu}(t,p,q) = \sup\{R^{\mu}_{0t}(\varphi) = \int_0^t c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s + \dot{\varphi}_s)|^2 ds :$$

$$\varphi_0 = q,\ \dot{\varphi}_0 = p,\ (\varphi_t, \dot{\varphi}_t) \in [G^{\mu}_0],\ V^{\mu}(t-s, \dot{\varphi}_s, \varphi_s) < 0 \text{ for } 0 < s < t\}$$

the function

$$V^{\mu}(t,p,q) = \sup\{R^{\mu}_{0t}(\varphi) = \int_0^t c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s + \dot{\varphi}_s)|^2 ds :$$

$$\varphi_0 = q,\ \dot{\varphi}_0 = p,\ (\varphi_t, \dot{\varphi}_t) \in [G^{\mu}_0]\}$$

determines the position of the wavefront. In this case, the manifold

$$\Sigma^{\mu}_t = \{(p,q) \in \mathbb{R}^{2n} :\ V^{\mu}(t,p,q) = 0\}$$

separates the regions of Ω^{μ}_{+} and Ω^{μ}_{-}.

In

$$\Omega^{\mu}_{+} = \{(t,p,q) :\ V^{\mu}(t,p,q) > 0\}$$

the solution $u^{\varepsilon}(t,p,q)$ converges to 1 as $\varepsilon \downarrow 0$ uniformly in any compact subset of Ω^{μ}_{+}.

In

$$\Omega^{\mu}_{-} = \{(t,p,q) :\ V^{\mu}(t,p,q) < 0\}$$

the solution $u^{\varepsilon}(t,p,q)$ converges to 0 as $\varepsilon \downarrow 0$ and converges uniformly in any compact subset of Ω^{μ}_{-}.

ii. When condition (N^{μ}) is not satisfied, we know the function

$$V^{*,\mu}(t,p,q) = \sup \min_{0 \le a \le t} \{R^{\mu}_{0a}(\varphi) = \int_0^a c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s + \dot{\varphi}_s)|^2 ds :$$

$$\varphi_0 = q,\ \dot{\varphi}_0 = p,\ (\varphi_t, \dot{\varphi}_t) \in [G_0^\mu]\}$$

characterizes the position of the wavefronts. In this case, the solution $u^\varepsilon(t,p,q)$ converges to 1 as $\varepsilon \downarrow 0$ and converges uniformly in any compact subset of $\Omega_+^{*,\mu}$ defined as

$$\Omega_+^{*,\mu} = \{(t,p,q):\ V^{*,\mu}(t,p,q) = 0\}.$$

The solution $u^\varepsilon(t,p,q)$ converges to 0 as $\varepsilon \downarrow 0$ uniformly in any compact subset of $\Omega_-^{*,\mu}$ defined as

$$\Omega_-^{*,\mu} = \{(t,p,q):\ V^{*,\mu}(t,p,q) < 0\}.$$

iii. From the definition of $V^\mu(t,p,q)$ and $V^{*,\mu}(t,p,q)$, we know the following relation holds:

$$V^{*,\mu}(t,p,q) \leq V^\mu(t,p,q) \wedge 0.$$

This implies $\Omega_+^{*,\mu} \subseteq \Omega_+^\mu$ and $\Omega_-^{*,\mu} \supseteq \Omega_-^\mu$.

For the non-degenerate reaction-diffusion equation (1.17):

i. When condition (N) is satisfied, i.e.

(N) :

$$V(t,q) = \sup\{R_{0t}(\varphi) = \int_0^t c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)\dot{\varphi}_s|^2 ds :$$

$$\varphi_0 = q,\ \varphi_t \in [G_0],\ V(t-s,\varphi_s) < 0 \text{ for } 0 < s < t\},$$

the functional

$$V(t,q) = \sup\{R_{0t}(\varphi) = \int_0^t c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)\dot{\varphi}_s|^2 ds :$$

$$\varphi_0 = q,\ \varphi_t \in [G_0]\}$$

determines the position of the wavefront. In this case, the manifold

$$\Sigma_t = \{q \in \mathbb{R}^n :\ V(t,q) = 0\}$$

separates the region of Ω_+ and Ω_-.

In

$$\Omega_+ = \{(t,q) :\ V(t,q) > 0\}$$

the solution $u^\varepsilon(t,q)$ converges to 1 as $\varepsilon \downarrow 0$ and converges uniformly in any compact subset of Ω_+.

In

$$\Omega_- = \{(t,q) :\ V(t,q) < 0\}$$

the solution $u^\varepsilon(t,q)$ converges to 0 as $\varepsilon \downarrow 0$ and converges uniformly in any compact subset of Ω_-.

ii. When condition (N) is not satisfied, we know the function

$$V^*(t,q) = \sup \min_{0 \le a \le t} \{R_{0a}(\varphi) = \int_0^a c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)\dot{\varphi}_s|^2 ds :$$

$$\varphi_0 = q,\ \varphi_t \in [G_0]\}$$

characterizes the position of the wavefronts. In this case, the solution $u^\varepsilon(t,q)$ converges to 1 as $\varepsilon \downarrow 0$ uniformly in any compact subset of Ω_+^* defined as

$$\Omega_+^* = \{(t,q) :\ V^*(t,q) = 0\}.$$

The solution $u^\varepsilon(t,q)$ converges to 0 as $\varepsilon \downarrow 0$ uniformly in any compact subset of Ω_-^* defined as

$$\Omega_-^* = \{(t,q) : \; V^*(t,q) < 0\}.$$

iii. From the definition of $V(t,q)$ and $V^*(t,q)$, we know the following relation holds:

$$V^*(t,q) \leq V(t,q) \wedge 0.$$

Thus $\Omega_+^* \subseteq \Omega_+$ and $\Omega_-^* \supseteq \Omega_-$.

3.2.1 Convergence of Wavefronts Under Condition (N^μ) and N

In this section, we consider the convergence when both condition N^μ and condition N are satisfied for problem (1.15) and (1.17). In this case, the manifold

$$\Sigma_t^\mu = \{(p,q) \in \mathbb{R}^{2n} : \; V^\mu(t,p,q) = 0\}$$

can be considered as the position of the wave front for equation (1.15), and

$$\Sigma_t = \{q \in \mathbb{R}^n : \; V(t,q) = 0\}$$

can also be considered for equation (1.17).

Theorem 3.2.1. *Assume $f(q,u) = uc(q,u)$ satisfies the KPP assumption for $q \in \mathbb{R}^n$. Let conditions (N^μ) and (N) be fulfilled and let $D_p \subset \mathbb{R}^n$ and $D_q \subset \mathbb{R}^n$ be compact. Then for each $p \in D_p,\; q \in D_q$,*

$$\lim_{u \downarrow 0} V^\mu(t,p,q) = V(t,q)$$

for each $0 \leq t \leq T < \infty$.

Proof. It is equivalent to show that for each $p \in D_p \subset \mathbb{R}^n$, the following inequalities hold:

$$V(t,q) \geq \limsup_{\mu \downarrow 0} V^{\mu}(t,p,q) \tag{3.12}$$

$$V(t,q) \leq \liminf_{\mu \downarrow 0} V^{\mu}(t,p,q). \tag{3.13}$$

To show (3.13), take φ^* be an extremal of the functional

$$R_{0t}(\varphi) = \int_0^t c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)\dot{\varphi}_s|^2 ds$$

such that

$$V(t,q) = R_{0t}(\varphi^*) = \sup\{R_{0t}(\varphi) : \ \varphi_0 = q, \ \varphi_t \in [G_0]\}.$$

The Euler-Lagrange equation for extremals of $R_{0t}(\varphi)$ implies that they are in $C^2([0,t])$. Let $a(q) = \sigma(q)\sigma^*(q)$, $q \in D_q$ be strictly positive definite. Assume $\sigma(q)$, $q \in D_q$, have bounded derivatives. Then

$$\begin{aligned}
V^{\mu}(t,p,q) &\geq R^{\mu}_{0t}(\varphi^*) \\
&= \int_0^t c(\varphi^*_s) - \frac{1}{2}|\sigma^{-1}(\varphi^*_s)(\mu\ddot{\varphi}^*_s + \dot{\varphi}^*_s)|^2 ds \\
&= \int_0^t c(\varphi^*_s) - \frac{1}{2}|\sigma^{-1}(\varphi^*_s)\mu\ddot{\varphi}^*_s|^2 - \frac{1}{2}|\sigma^{-1}(\varphi^*_s)\dot{\varphi}^*_s|^2 \\
&\qquad - (\sigma^{-1}(\varphi^*_s)\mu\ddot{\varphi}^*_s, \ \sigma^{-1}(\varphi^*_s)\dot{\varphi}^*_s) ds \\
&= \int_0^t c(\varphi^*_s) - \frac{1}{2}|\sigma^{-1}(\varphi^*_s)\dot{\varphi}^*_s|^2 ds - \frac{\mu^2}{2}\int_0^t |\sigma^{-1}(\varphi^*_s)\ddot{\varphi}^*_s|^2 ds \\
&\qquad - \mu\int_0^t (a^{-1}(\varphi^*_s)\dot{\varphi}^*_s, \ddot{\varphi}^*_s) ds \\
&= R_{0t}(\varphi^*) - \frac{\mu^2}{2}\int_0^t |\sigma^{-1}(\varphi^*_s)\ddot{\varphi}^*_s|^2 ds - \mu\int_0^t \sum_{i=1}^n \sum_{j=1}^n a^{-1}_{ij}(\varphi^*_s)\dot{\varphi}^{*i}_s \ddot{\varphi}^{*j}_s ds
\end{aligned}$$

$$= R_{0t}(\varphi^*) - \frac{\mu^2}{2}\int_0^t |\sigma^{-1}(\varphi_s^*)\ddot{\varphi}_s^*|^2 ds - \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_j^n a_{ij}^{-1}(\varphi_s^*) d(\dot{\varphi}_s^{*j}\dot{\varphi}_s^{*i})$$

$$= R_{0t}(\varphi^*) - \frac{\mu}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}^{-1}(\varphi_s^*)\dot{\varphi}_s^{*j}\dot{\varphi}_s^{*i}|_0^t - \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\varphi}_s^{*j}\dot{\varphi}_s^{*i} d(a_{ij}^{-1}(\varphi_s^*))$$

$$- o(\mu)$$

$$= R_{0t}(\varphi^*) - \frac{\mu}{2}[(\sigma^{-1}(\varphi_t^*)\dot{\varphi}_t^*, \sigma^{-1}(\varphi_t^*)\dot{\varphi}_t^*) - (\sigma^{-1}(\varphi_0^*)\dot{\varphi}_0^*, \sigma^{-1}(\varphi_0^*)\dot{\varphi}_0^*)]$$

$$- \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\varphi}_s^{*j}\dot{\varphi}_s^{*i} d(a_{ij}^{-1}(\varphi_s^*)) - o(\mu)$$

$$= R_{0t}(\varphi^*) - \frac{\mu}{2}[|\sigma^{-1}(\varphi_t^*)\dot{\varphi}_t^*|^2 - |\sigma^{-1}(q)\dot{\varphi}_0^*|^2]$$

$$- \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\varphi}_s^{*j}\dot{\varphi}_s^{*i} d(a_{ij}^{-1}(\varphi_s^*)) - o(\mu).$$

Since $\varphi_s^* \in C^2([0,t])$, the derivatives in

$$A = -\frac{\mu}{2}[|\sigma^{-1}(\varphi_t^*)\dot{\varphi}_t^*|^2 - |\sigma^{-1}(q)\dot{\varphi}_0^*|^2] - \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\varphi}_s^{*j}\dot{\varphi}_s^{*i} d(a_{ij}^{-1}(\varphi_s^*))$$

are bounded. Thus $A \sim o(u)$ as $\mu \downarrow 0$, which implies that

$$V^\mu(t,p,q) \geq R_{0t}(\varphi^*) - o(\mu) = V(t,q) - o(\mu).$$

When $\mu \downarrow 0$, we have the following estimate:

$$\liminf_{\mu \downarrow 0} V^\mu(t,p,q) \geq V(t,q).$$

To prove (3.12), for each fixed $\mu > 0$ let $\hat{\varphi}$ be an extremal of the functional

$$R_{0t}^\mu(\varphi) = \int_0^t c(\varphi_s) - \frac{1}{2}|\sigma^{-1}(\varphi_s)(\mu\ddot{\varphi}_s + \dot{\varphi}_s)|^2 ds$$

such that $V^\mu(t,p,q) = R_{0t}^\mu(\hat{\varphi})$. Similarly, extremals of $R_{0t}^\mu(\varphi)$ are in $C^4([0,t])$. Then

$$R_{0t}^\mu(\varphi) = \int_0^t c(\hat{\varphi}_s) - \frac{1}{2}|\sigma^{-1}(\hat{\varphi}_s)(\mu\ddot{\hat{\varphi}}_s + \dot{\hat{\varphi}}_s)|^2 ds$$

$$= \int_0^t c(\hat{\varphi}_s) - \frac{1}{2}|\sigma^{-1}(\hat{\varphi}_s)\dot{\hat{\varphi}}_s|^2 ds - \frac{1}{2}\int_0^t \mu^2 |\sigma^{-1}(\hat{\varphi}_s)\ddot{\hat{\varphi}}_s|^2 ds$$

$$- \mu \int_0^t (\sigma^{-1}(\hat{\varphi}_s)\ddot{\hat{\varphi}}_s, \sigma^{-1}(\hat{\varphi}_s)\dot{\hat{\varphi}}_s) ds$$

$$= R_{0t}(\hat{\varphi}) - \frac{\mu^2}{2}\int_0^t |\sigma^{-1}(\hat{\varphi}_s)\ddot{\hat{\varphi}}_s|^2 ds - \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n a_{ij}^{-1}(\hat{\varphi}_s) d(\dot{\hat{\varphi}}_s^i \dot{\hat{\varphi}}_s^j)$$

$$= R_{0t}(\hat{\varphi}) - o(\mu) - \frac{\mu}{2}[(a^{-1}(\hat{\varphi}_t)\dot{\hat{\varphi}}_t, \dot{\hat{\varphi}}_t) - (a^{-1}(\hat{\varphi}_0)\dot{\hat{\varphi}}_0, \dot{\hat{\varphi}}_0)]$$

$$+ \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\hat{\varphi}}_s^i \dot{\hat{\varphi}}_s^j d[a_{ij}^{-1}(\hat{\varphi}_s)]$$

$$= R_{0t}(\hat{\varphi}) - o(\mu) - \frac{\mu}{2}[|\sigma^{-1}(\hat{\varphi}_t)\dot{\hat{\varphi}}_t|^2 - |\sigma^{-1}(q)p|^2]$$

$$+ \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\hat{\varphi}}_s^i \dot{\hat{\varphi}}_s^j d[a_{ij}^{-1}(\hat{\varphi}_s)]$$

$$\leq R_{0t}(\hat{\varphi}) + \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\hat{\varphi}}_s^i \dot{\hat{\varphi}}_s^j d[a_{ij}^{-1}(\hat{\varphi}_s)]$$

$$- \frac{\mu}{2}[|\sigma^{-1}(\hat{\varphi}_t)\dot{\hat{\varphi}}_t|^2 - \sigma^{-1}(q)p|^2].$$

Since $p \in D_p \subset \mathbb{R}^n$, $q \in D_q \subset \mathbb{R}^n$, D_p, D_q are compact and $\sigma(q)$ has bounded derivatives, the derivatives in the quantity

$$B = \frac{\mu}{2}\int_0^t \sum_{i=1}^n \sum_{j=1}^n \dot{\hat{\varphi}}_s^i \dot{\hat{\varphi}}_s^j d[a_{ij}^{-1}(\hat{\varphi}_s)] - \frac{\mu}{2}[|\sigma^{-1}(\hat{\varphi}_t)\dot{\hat{\varphi}}_t|^2 - \sigma^{-1}(q)p|^2]$$

are all bounded. Therefore, $B \sim o(\mu)$ as $\mu \downarrow 0$. Summarizing the above inequalities, we obtain

$$R_{0t}(\hat{\varphi}) \leq R_{0t}(\hat{\varphi}) + o(\mu) \leq V(t, q) + o(\mu).$$

Thus,

$$\limsup_{\mu \downarrow 0} V^\mu(t, p, q) \leq V(t, q).$$

From (3.12) and (3.13) we find that for fixed $p \in D_p \subset \mathbb{R}^n$, for each $t \in [0, T]$, $0 < T < \infty, q \in D_q \subset \mathbb{R}^n$, we have

$$\lim_{\mu \downarrow 0} V^\mu(t, p, q) = V(t, q). \tag{3.14}$$

□

Example 3.2.2. Consider the following example in $\mathbb{R}^1$. Recall that if the function $f(q, u)$ satisfies the KPP assumption, then it fulfills the relation

$$f(q, u) = uc(q, u), \ \max_{0 \le u \le 1} c(q, u) = c(q, 0) = c(q).$$

Assume that the function $c(q)$ is a linear function with slope of $k > 0$ for $q > 0$ and is 0 when $q < 0$, i.e.

$$c(q) = c(q, 0) = \begin{cases} kq, & q > 0 \\ 0, & q < 0 \end{cases}$$

We study the relation between the wavefront propagation of the following two equations. The first is

$$\frac{\partial u^\varepsilon(t, p, q)}{\partial t} = \frac{\varepsilon}{2\mu^2} \frac{\partial^2 u^\varepsilon}{\partial p^2} - \frac{1}{\mu} p \frac{\partial u^\varepsilon}{\partial p} + p \frac{\partial u^\varepsilon}{\partial q} + \frac{1}{\varepsilon} u^\varepsilon c(q, u^\varepsilon) \tag{3.15}$$

$$u^\varepsilon(0, p, q) = \delta(p) \chi^{-1}(q), \ q, \ p \in \mathbb{R}^1$$

where $\delta(p)$ is the delta function centered at 0, taking value 1 at 0 and 0 otherwise and $\chi^{-1}(q)$ is the indicator function such that

$$\chi^{-1}(q) = \begin{cases} 1, & q < 0 \\ 0, & q \ge 0 \end{cases}$$

The second equation is

$$\frac{\partial u^\varepsilon(t, q)}{\partial t} = \frac{\varepsilon}{2} \frac{\partial^2 u^\varepsilon}{\partial q^2} + \frac{1}{\varepsilon} u^\varepsilon c(q, u^\varepsilon) \tag{3.16}$$

$$u^\varepsilon(0,q) = \chi^{-1}(q),\ \ q \in \mathbb{R}^1.$$

We will prove in what follows that both condition (N^μ) and condition (N) are fulfilled when $c(q)$ is linearly growing as kq. Thus we can use the functionals $V^\mu(t,p,q)$ to characterize the wave front propagation for equation (3.15) and $V(t,q)$ for equation (3.16). Let

$$V^\mu(t,p,q) = \sup\{\int_0^t c(\varphi_s) - \frac{1}{2}|\mu\ddot{\varphi}_s + \dot{\varphi}_s|^2 ds,\ \varphi_0 = q,\ \dot{\varphi}_0 = p,\ \varphi_t = \dot{\varphi}_t = 0\}$$

$$V(t,q) = \sup\{\int_0^t c(\varphi_s) - \frac{1}{2}|\dot{\varphi}_s|^2 ds,\ \varphi_0 = q,\ \varphi_t = 0\}.$$

For the functional $V(t,q)$, the Euler-Lagrange equation has the form

$$\ddot{\varphi}_s = -k,\ \varphi_0 = q,\ \varphi_t = 0.$$

There exists a unique solution $\tilde{\varphi}_s,\ s \in [0,t]$, on which the supremum is attained. It has the form

$$\tilde{\varphi}_s = -\frac{1}{2}ks^2 - (\frac{q}{t} - \frac{kt}{2})s + q \tag{3.17}$$

and its derivative has the form

$$\dot{\tilde{\varphi}}_s = -ks - \frac{q}{t} + \frac{kt}{2}. \tag{3.18}$$

The functional $V(t,q)$ has the expression

$$V(t,q) = \int_0^t c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 ds$$

$$= -\frac{1}{2t}q^2 + \frac{1}{2}kqt + \frac{1}{24}k^2t^3. \tag{3.19}$$

By setting $V(t,q)=0$, we calculate the front position as

$$q(t)=\frac{1}{6}(3+2\sqrt{3})t^2k,\ \ q>0. \tag{3.20}$$

Since the front position $q(t)$ is a convex function and the extremal $\tilde{\varphi}_s$ is concave, condition (N) is satisfied. Thus characterization of the wave front position using function $V(t,q)$ is verified. Moreover, we also obtain the wave front position and extremals.

For equation (3.15), we will approach the functional $V^\mu(t,p,q)$ in the same way as we treat the functional $V(t,q)$. It is more complicated, but we can simplify the analysis somewhat by considering small μ.

Let

$$F(\varphi_s,\dot{\varphi}_s,\ddot{\varphi}_s)=c(\varphi_s)-\frac{1}{2}|\mu\ddot{\varphi}_s+\dot{\varphi}_s|^2.$$

The Euler-Lagrange equation for the functional $F(\varphi,\dot{\varphi},\ddot{\varphi})$ is calculated as

$$\begin{aligned}\frac{d}{d\varphi}F-\frac{d}{ds}\frac{d}{d\dot{\varphi}}F+\frac{d^2}{ds^2}\frac{d}{d\ddot{\varphi}}F&=c'(\varphi_s)-\frac{d}{ds}[-(\mu\ddot{\varphi}_s+\dot{\varphi}_s)]+\frac{d^2}{ds^2}[-\mu(\mu\ddot{\varphi}_s+\dot{\varphi}_s)]\\ &=c'(\varphi_s)+(\mu\varphi_s^{(3)}+\ddot{\varphi}_s)-(\mu^2\varphi_s^{(4)}+\mu\varphi_s^{(3)})\\ &=c'(\varphi_s)+\ddot{\varphi}_s-\mu^2\varphi_s^{(4)}=0\end{aligned}$$

The Euler-Lagrange equation is

$$\mu^2\varphi_s^{(4)}-\ddot{\varphi}_s=c'(\varphi_s)=k.$$

Let $\hat{\varphi}_s^\mu,\ s\in[0,t]$ be the solution of the Euler-Lagrange equation :

$$\mu^2\hat{\varphi}_s^{\mu,(4)}-\ddot{\varphi}_s^\mu=k,\ \hat{\varphi}_0^\mu=q>0,\ \dot{\hat{\varphi}}_0^\mu=p,\ \hat{\varphi}_t=\dot{\hat{\varphi}}_t^\mu=0. \tag{3.21}$$

It's easy to check that all solutions of (3.21) satisfy the equation

$$\ddot{\hat{\varphi}}^\mu(s) = c_1 \exp\left(-\frac{s}{\mu}\right) + c_2 \exp\left(\frac{s-t}{\mu}\right) - k.$$

Integrating with respect to s and using the boundary condition $\hat{\varphi}^\mu_t = \dot{\hat{\varphi}}^\mu_t = 0$ we find:

$$\hat{\varphi}^\mu(s) = -k(\frac{s^2}{2} - ts + \frac{t^2}{2}) + \mu c_1 \left[\mu(\exp(-\frac{s}{\mu}) - \exp(-\frac{t}{\mu})) + \exp(-\frac{t}{\mu})(s-t)\right] \tag{3.22}$$

$$+ \mu c_2[\mu(\exp(\frac{s-t}{\mu}) - 1) + t - s]$$

$$\dot{\hat{\varphi}}^\mu(s) = -k(s-t) + \mu c_1(\exp(-\frac{t}{\mu}) - \exp(-\frac{s}{\mu})) + \mu c_2(\exp(\frac{s-t}{\mu}) - 1) \tag{3.23}$$

From $\hat{\varphi}^\mu(0) = q$, we have

$$q = -\frac{kt^2}{2} + \mu c_1[\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu})] + \mu c_2[\mu(\exp(-\frac{t}{\mu}) - 1) + t],$$

which is equivalent to

$$\mu c_1[\mu\left(1 - \exp(-\frac{t}{\mu})\right) - t\exp(-\frac{t}{\mu})] + \mu c_2[\mu\left(\exp(-\frac{t}{\mu}) - 1\right) + t] = q + \frac{kt^2}{2}. \tag{3.24}$$

From $\dot{\hat{\varphi}}^\mu(0) = p$, we have

$$\dot{\hat{\varphi}}^\mu(0) = kt + \mu c_1\left(\exp(-\frac{t}{\mu}) - 1\right) + \mu c_2\left(\exp(-\frac{t}{\mu}) - 1\right) = p,$$

which implies that

$$(\mu c_1 + \mu c_2)\left(\exp(-\frac{t}{\mu}) - 1\right) = p - kt. \tag{3.25}$$

Solve (3.24) and (3.25) for μc_1 and μc_2, we obtain

$$\mu c_1 = \frac{(\exp(-\frac{t}{\mu}) - 1)(q + \frac{k}{2}t^2) - [\mu(\exp(-\frac{t}{\mu}) - 1) + t](p - kt)}{(\exp(-\frac{t}{\mu}) - 1)[2\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]} \tag{3.26}$$

$$\mu c_2 = \frac{[\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu})](p-kt) - (q+\frac{k}{2}t^2)(\exp(-\frac{t}{\mu})-1)}{(\exp(-\frac{t}{\mu})-1)[2\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]}. \quad (3.27)$$

Notice that when $\mu \downarrow 0$,

$$\mu c_1 \longrightarrow \frac{kt}{2} - p - \frac{q}{t} \quad (3.28)$$

$$\mu c_2 \longrightarrow \frac{kt}{2} + \frac{q}{t} \quad (3.29)$$

Thus, $\mu c_1 \sim O(1),\ \mu c_2 \sim O(1)$ as $\mu \downarrow 0$.

By substituting for $\hat{\varphi}_s^\mu,\ \dot{\hat{\varphi}}_s^\mu,\ \ddot{\hat{\varphi}}_s^\mu$, we calculate $V^\mu(t,p,q)$ as

$$\begin{aligned} V^\mu(t,p,q) &= \int_0^t c(\hat{\varphi}_s^\mu) - \frac{1}{2}|\mu\ddot{\hat{\varphi}}_s^\mu + \dot{\hat{\varphi}}_s^\mu|^2 ds \\ &= 2\mu^2(\exp(-\frac{t}{\mu})-1)c_2K_2 + k\mu^3(1-\exp(-\frac{t}{\mu}))c_1 + \frac{1}{2}t^2K_3 + tK_1 - \frac{1}{3}k^2t^3 \\ &+ [a\mu^3(\exp(-\frac{t}{\mu})-1) + 2k\mu^2 t]c_2 + \mu^3(\exp(-\frac{2t}{\mu})c_2^2 - \frac{1}{2}tK_2^2 + \frac{1}{2}kt^2K_2 \end{aligned}$$

where

$$K_1 = -\frac{k^2t^2}{2} + k\mu c_1\exp(-\frac{t}{\mu})\,(\mu+t) - k\mu c_2(\mu-t),$$

$$K_2 = \mu c_1\exp(-\frac{t}{\mu}) - \mu c_2 - k(\mu - t),$$

$$K_3 = k(\mu c_1\exp(-\frac{t}{\mu}) - \mu c_2 + kt).$$

We would like to solve $V^\mu(t,p,q) = 0$ for p,q to find the position of the wave front. In order to simplify the problem, it's helpful to write $V^\mu(t,p,q)$ as a function of p,q and find the dominating terms for small μ. First we write

$$\mu c_1 = l_1 q - l_2 p + l_{12}, \quad (3.30)$$

where $l_1,\ l_2,\ l_{12}$ are quantities not depending on p, q. From (3.26), we know

$$l_1 = \left[2\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t\right]^{-1}$$

$$\longrightarrow -\frac{1}{t} \text{ as } \mu \downarrow 0$$

$$l_2 = \frac{\mu(\exp(-\frac{t}{\mu})-1)+t}{(\exp(-\frac{t}{\mu})-1)[2\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]}$$

$$\longrightarrow 1 \text{ as } \mu \downarrow 0$$

$$l_{12} = \frac{l_1 k}{2}t^2 + l_2 kt$$

$$\longrightarrow \frac{kt}{2} \text{ as } \mu \downarrow 0$$

Similarly,

$$\mu c_2 = l_3 p - l_4 q + l_{34}, \tag{3.31}$$

where $l_3,\ l_4,\ l_{34}$ are quantities not depending on $p,\ q$. From (3.27) we know

$$l_3 = \frac{\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu})}{(\exp(-\frac{t}{\mu})-1)[2\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]}$$

$$\longrightarrow 0, \text{ as } \mu \downarrow 0$$

$$l_4 = \left[2\mu\left(1-\exp(-\frac{t}{\mu})\right) - t\exp(-\frac{t}{\mu}) - t\right]^{-1}$$

$$\longrightarrow -\frac{1}{t}, \text{ as } \mu \downarrow 0$$

$$l_{34} = -l_3 kt - \frac{l_4 k t^2}{2}$$

$$\longrightarrow \frac{kt}{2}, \text{ as } \mu \downarrow 0.$$

For the quantities $K_1,\ K_2,\ K_3$, we can write:

$$K_1 = \bar{l}_1 q + \bar{l}_2 p + \bar{l}_{12}, \tag{3.32}$$

where

$$\bar{l}_1 = k(\mu - t)l_4 - k\exp(-\frac{t}{\mu})(\mu + t)l_1$$

$$\longrightarrow k,\ \text{as}\ \mu \downarrow 0$$

$$\bar{l}_2 = k\exp(-\frac{t}{\mu})(\mu + t)l_2 - k(\mu - t)l_3$$

$$\longrightarrow 0,\ \text{as}\ \mu \downarrow 0$$

$$\bar{l}_{12} = -\frac{k^2t^2}{2} - l_{12}k\exp(-\frac{t}{\mu})(\mu + t) - k(\mu - t)l_{34}$$

$$\longrightarrow 0,\ \text{as}\ \mu \downarrow 0.$$

Also

$$K_2 = \bar{l}_3 q + \bar{l}_4 p + l_{34}, \tag{3.33}$$

where

$$\bar{l}_3 = l_1\exp(-\frac{t}{\mu}) + l_4$$

$$\longrightarrow -\frac{1}{t},\ \text{as}\ \mu \downarrow 0$$

$$\bar{l}_4 = -(l_3 + l_2\exp(-\frac{t}{\mu}))$$

$$\longrightarrow 0,\ \text{as}\ \mu \downarrow 0$$

$$\bar{l}_{34} = -k(\mu - t) + l_{12}\exp(-\frac{t}{\mu}) - l_{34}$$

$$\longrightarrow \frac{kt}{2}, \text{ as } \mu \downarrow 0.$$

Finally

$$K_3 = \bar{l}_5 q + \bar{l}_6 p + \bar{l}_{56}, \tag{3.34}$$

where

$$\bar{l}_5 = k(\exp(-\frac{t}{\mu})l_1 + l_4)$$

$$\longrightarrow -\frac{k}{t}, \text{ as } \mu \downarrow 0$$

$$\bar{l}_6 = -k(l_3 + l_4\exp(-\frac{t}{\mu}))$$

$$\longrightarrow 0, \text{ as } \mu \downarrow 0$$

$$\bar{l}_{56} = k\exp(-\frac{t}{\mu})l_{12} - kl_{34} + k^2 t$$

$$\longrightarrow \frac{k^2 t}{2}, \text{ as } \mu \downarrow 0.$$

Summarizing (3.32), (3.33), (3.34) we calculate the following asymptotics when $\mu \downarrow 0$:

$$K_1 \longrightarrow kq \tag{3.35}$$

$$K_2 \longrightarrow -\frac{1}{t}q + \frac{kt}{2} \tag{3.36}$$

$$K_3 \longrightarrow -\frac{k}{t}q + \frac{k^2 t}{2}. \tag{3.37}$$

Collecting (3.26), (3.27), (3.32), (3.33), (3.34), we calculate $V^\mu(t,p,q)$ as

$$V^\mu(t,p,q) = m_1 q^2 + m_2 q + m_3 p + m_4 \tag{3.38}$$

where

$$m_1 = \frac{\mu(\exp(-\frac{t}{\mu})-1)^2}{t^2} - \frac{1}{2t},$$

$$m_2 = \frac{1}{2}kt - \frac{2k\mu^2}{t} + k\mu + \frac{2k\mu^2}{t}\exp(-\frac{t}{\mu}) + k\mu\exp\left(-\frac{2t}{\mu}\right),$$

$$m_3 = k\mu^2\left(\exp(-\frac{t}{\mu})-1\right),$$

$$m_4 = \frac{1}{4}\mu k^2 t^2\left(\exp(-\frac{t}{\mu})+1\right)^2 + \frac{1}{24}k^2t^3.$$

Notice that when $\mu \downarrow 0$, the following limits hold for these quantities:

$$m_1 \longrightarrow -\frac{1}{2t}$$

$$m_2 \longrightarrow \frac{1}{2}kt$$

$$m_3 \longrightarrow 0$$

$$m_4 \longrightarrow \frac{1}{24}k^2t^3.$$

When $q>0$, there exists a $\mu_1>0$, such that whenever $0<\mu<\mu_1$, the wave front for q has the same concavity as:

$$\hat{q}(t) = \frac{2+\sqrt{3}}{2\sqrt{3}}kt^2; \tag{3.39}$$

and there exists a $\mu_2>0$ such that whenever $0<\mu<\mu_2$, the wave front for p has the same concavity as:

$$\hat{p}(t) = \frac{-\frac{1}{24}k^2t^3 - \frac{1}{2}qkt - \frac{1}{2t}q^2}{k\mu^2(\exp(-\frac{t}{\mu})-1)}$$

Therefore, there exists a $\mu_{12} = \mu_1 \wedge \mu_2$, when $0 < \mu < \mu_{12}$ the wavefronts $p^\mu(t)$ and $q^\mu(t)$ are close to the convex functions $\hat{p}(t)$ and $\hat{q}(t)$ respectively.

The extremals $\hat{\varphi}_s^\mu$ and $\dot{\hat{\varphi}}_s^\mu$ can be approximated in the same way. By plugging in the approximate quantities μc_1 and μc_2 for small μ into equations (3.22) and (3.23), we obtain:

$$
\begin{aligned}
\hat{\varphi}^\mu(s) \approx & -\frac{k}{2}(\frac{s^2}{2} - ts + \frac{t^2}{2}) \\
& + (\frac{kt}{2} - p - \frac{q}{t})[\mu(\exp(-\frac{s}{\mu}) - \exp(-\frac{t}{\mu})) + \exp(-\frac{t}{\mu})(1-t)] \\
& + (\frac{kt}{2} + \frac{q}{t})[\mu(\exp(\frac{s-t}{\mu}) - 1) + t - s].
\end{aligned}
$$

Its second derivative has the form:

$$
\ddot{\hat{\varphi}}^\mu(s) \approx -\frac{k}{2} + \frac{(\frac{kt}{2} - p - qt)\exp(-\frac{s}{\mu})}{\mu} + \frac{(\frac{kt}{2} + qt)\exp(\frac{s-t}{\mu})}{\mu}.
$$

For small μ, $\ddot{\hat{\varphi}}_s^\mu$ is close to $-k/2$ which is negative, and thus there exists a $\mu_3 > 0$ such that when $0 < \mu < \mu_3$, $\hat{\varphi}_s^\mu,\ s \in [O(\mu_3), t - O(\mu_3)]$, has the same concavity as its limit

$$
\hat{\varphi}_s = -\frac{1}{2}ks^2 - (\frac{q}{t} - \frac{kt}{2})s + q.
$$

For $\dot{\hat{\varphi}}^\mu(s)$, we have:

$$
\begin{aligned}
\dot{\hat{\varphi}}^\mu(s) = & -k(s-t) + (\frac{kt}{2} - p - \frac{q}{t})(\exp(-\frac{t}{\mu}) - \exp(-\frac{s}{\mu})) \\
& + (\frac{kt}{2} + \frac{q}{t})(\exp(\frac{s-t}{\mu}) - 1),
\end{aligned}
$$

and thus there exists a $\mu_4 > 0$ such that when $0 < \mu < \mu_4$, $\dot{\hat{\varphi}}^\mu(s),\ s \in [O(\mu_4), t - O(\mu_4)]$ has the same linearity as its limit

$$
\dot{\hat{\varphi}}(s) = -ks - \frac{q}{t} + \frac{kt}{2}.
$$

Therefore, there exists $\mu_{34} = \mu_3 \wedge \mu_4 > 0$ such that when $0 < \mu < \mu_{34}$, the extremal $\hat{\varphi}^\mu_s$ is concave, and $\dot{\hat{\varphi}}^\mu_s$ is linear on the interval $s \in [O(\mu_{34}), t - O(\mu_{34})]$.

Take $\mu_0 = \mu_{12} \wedge \mu_{34}$. Then for $0 < \mu < \mu_0$, the wave front $p^\mu(t)$, $q^\mu(t)$ is convex, the extremal $\hat{\varphi}^\mu(s)$ is concave, and $\dot{\hat{\varphi}}^\mu(s)$ is linear on the interval $s \in [O(\mu_0), t - O(\mu_0)]$. Therefore condition (N^μ) is satisfied for $0 < \mu < \mu_0$. Now we have justified the use of the functional $V^\mu(t, p, q)$ as a characterization of the wave front for equation (3.15).

From the above calculation of (3.19) and (3.38), we have proven $V^\mu(t, p, q) \longrightarrow V(t, q)$ for each bounded $p \in \mathbb{R}^1$. Moreover, from (3.20) and (3.39), we see that the asymptotic wave front positions are the same as $\mu \downarrow 0$. Therefore, we can use the wave front of equation (3.16) to approximate that of (3.15).

3.2.2 Convergence of Wavefronts in the General Case

Convergence of the wavefronts when neither (N^μ) nor (N) is satisfied so far can not be proved in general. However, we can still deal with some of the cases. In this section, we will consider a special case of equations (1.15) and (1.17) when the diffusion matrix $\sigma(q)$ is a unit matrix. For simplicity, let the initial condition be $g(0, p, q) = \delta(p)\chi^{-1}(q)$, $g(0, q) = \chi^{-1}(q)$, where $\delta(p)$ and $\chi^{-1}(q)$ are defined the same way as in the example of the previous section. We will study the relation between equations:

$$\frac{\partial u^\varepsilon(t, p, q)}{\partial t} = \frac{\varepsilon}{2\mu^2} \triangle_p u^\varepsilon - \frac{1}{\mu} p \nabla_p u^\varepsilon + p \nabla_q u^\varepsilon + \frac{1}{\varepsilon} u^\varepsilon c(q, u^\varepsilon) \qquad (3.40)$$

$$u^{\varepsilon}(0,p,q) = \delta(p)\chi^{-1}(q),\ \ q,\ \ p \in \mathbb{R}^n;$$

and

$$\frac{\partial u^{\varepsilon}(t,q)}{\partial t} = \frac{\varepsilon}{2} \triangle_q u^{\varepsilon} + \frac{1}{\varepsilon} u^{\varepsilon} c(q,u^{\varepsilon}) \tag{3.41}$$

$$u^{\varepsilon}(0,q) = \chi^{-1}(q),\ \ q \in \mathbb{R}^n.$$

Assume that for the given function $c(q,u)$, condition (N^{μ}) is not fulfilled for equation (3.40), while condition (N) is fulfilled for equation (3.41). Then functional

$$V^{*,\mu}(t,p,q) = \sup \min_{0 \le a \le t} \{R^{\mu}_{0a}(\varphi) : \varphi_0 = q, \dot{\varphi}_0 = p, \varphi_t = \dot{\varphi}_t = 0\},$$

where

$$R^{\mu}_{0a}(\varphi) = \int_0^a c(\varphi_s) - \frac{1}{2}|\mu\ddot{\varphi}_s + \dot{\varphi}_s|^2 ds,$$

is used to characterize position of the wavefronts for equation (3.40). Recall that the functional $V^{\mu}(t,p,q)$ is defined as:

$$V^{\mu}(t,p,q) = \sup\{R^{\mu}_{0t}(\varphi) : \varphi_0 = q, \dot{\varphi}_0 = p, \varphi_t = \dot{\varphi}_t = 0\}$$

where

$$R^{\mu}_{0t}(\varphi) = \int_0^t c(\varphi_s) - \frac{1}{2}|\mu\ddot{\varphi}_s + \dot{\varphi}_s|^2 ds.$$

The following result can be generalized for any $p \in D \subset \mathbb{R}^n$ where D is compact. For simplicity, fix $p = 0$. Define

$$\Omega^{*,\mu}_{+} = \{(t,q) : V^{*,\mu}(t,0,q) = 0\}$$

$$\Omega^{*,\mu}_{-} = \{(t,q) : V^{*,\mu}(t,0,q) < 0\}$$

We know that the solution of (3.40) converges to 1 as $\varepsilon \downarrow 0$ uniformly in any compact subset of $\Omega_+^{*,\mu}$, and converges to 0 as $\varepsilon \downarrow 0$ uniformly in any compact subset of $\Omega_-^{*,\mu}$. Similarly, define:

$$\Omega_+^{\mu} = \{(t,q) : V^{\mu}(t,0,q) > 0\}$$

$$\Omega_-^{\mu} = \{(t,q) : V^{\mu}(t,0,q) < 0\}$$

In general $V^{*,\mu}(t,p,q) \le V^{\mu}(t,p,q) \wedge 0$. When condition (N^{μ}) is not satisfied, the inequality is strict:

$$V^{*,\mu}(t,p,q) < V^{\mu}(t,p,q) \wedge 0. \tag{3.42}$$

It implies that

$$\Omega_+^{*,\mu} \subset \Omega_+^{\mu}$$

$$\Omega_-^{*,\mu} \supset \Omega_-^{\mu}$$

Since we assume condition (N) is satisfied for equation (3.41), the characterizations using functionals $V^*(t,q)$ and $V(t,q)$ are equivalent. Let

$$V^*(t,q) = \sup \min_{0 \le a \le t} \{R_{0a}(\varphi) : \ \varphi_0 = q, \ \varphi_t = 0\}$$

$$= \sup \min_{0 \le a \le t} \{\int_0^a c(\varphi_s) - \frac{1}{2}|\dot{\varphi}_s|^2 : \ \varphi_0 = q, \ \varphi_t = 0\}$$

$$V(t,q) = \sup\{R_{0t}(\varphi) : \ \varphi_0 = q, \varphi_t = 0\}$$

$$= \sup\{\int_0^t c(\varphi_s) - \frac{1}{2}|\dot{\varphi}_s|^2 ds : \ \varphi_0 = q, \ \varphi_t = 0\}$$

Define:

$$\Omega_+^* = \{(t,q) : V^*(t,q) = 0\}$$

$$\Omega^*_- = \{(t,q) : V^*(t,q) < 0\}$$

$$\Omega_+ = \{(t,q) : V(t,q) > 0\}$$

$$\Omega_- = \{(t,q) : V(t,q) < 0\}.$$

The functionals $V^*(t,q)$ and $V(t,q)$ are related by the equation

$$V^*(t,q) \le V(t,q) \wedge 0$$

in general. When condition (N) is satisfied, the inequality becomes the equality

$$V^*(t,q) = V(t,q) \wedge 0. \tag{3.43}$$

This implies

$$\Omega^*_+ = \Omega_+$$

$$\Omega^*_- = \Omega_-$$

As we know, the solution of (3.41) converges uniformly to 1 as $\varepsilon \downarrow 0$ in any compact subset of $\Omega^*_+ = \Omega_+$, and converges uniformly to 0 as $\varepsilon \downarrow 0$ in any compact subset of $\Omega^*_- = \Omega_-$.

Lemma 3.2.3. *Given $0 \le t \le T < \infty$, assume that the function $c(q)$ is sufficiently smooth. Let $\hat{\varphi}^\mu_s,\ s \in [0,t]$, be an extremal of $V^\mu(t,p,q)$ and let $\tilde{\varphi}_s,\ s \in [0,t]$, be an extremal of $V(t,q)$. Then*

$$\hat{\varphi}^\mu_s \longrightarrow \tilde{\varphi}_s$$

$$\dot{\hat{\varphi}}^\mu_s \longrightarrow \dot{\tilde{\varphi}}_s$$

as $\mu \downarrow 0$ for each $s \in [0,t]$.

Proof. Since $\hat{\varphi}^{\mu}_s$ and $\tilde{\varphi}_s,\ s \in [0,t]$ are extremals, they solve the Euler-Lagrange equations:

$$\mu^2 \hat{\varphi}^{\mu,(4)}_s - \ddot{\hat{\varphi}}^{\mu}_s = \nabla c(\hat{\varphi}^{\mu}_s),\ \hat{\varphi}^{\mu}_0 = q,\ \dot{\hat{\varphi}}^{\mu}_0 = p,\ \hat{\varphi}_t = \dot{\hat{\varphi}}_t = 0 \tag{3.44}$$

$$\ddot{\tilde{\varphi}}_s = -\nabla c(\tilde{\varphi}_s),\ \tilde{\varphi}_0 = q,\ \tilde{\varphi}_t = 0. \tag{3.45}$$

Consider equation (3.44) without boundary conditions. It can be written as the following system:

$$\dot{\hat{\varphi}}^{\mu} = v,\ \dot{v} = x,\ \mu\dot{x} = y,\ \mu\dot{y} = x + \nabla c(\hat{\varphi}^{\mu}).$$

Let $h(s) = \nabla c(\hat{\varphi}^{\mu}_s)$. Since $x(s) = \ddot{\hat{\varphi}}^{\mu}_s$, equation (3.44) can be written as

$$\mu^2 \ddot{x}(s) - x(s) = h(s),\ s \in [0,t]. \tag{3.46}$$

It can be calculated that for all $h \in C^1([0,t])$, any solution of (3.46) is given by

$$x(s) = \exp(-\frac{s}{\mu})c_1 + \exp(\frac{s-t}{\mu})c_2 + H(s) \tag{3.47}$$

where

$$H(s) = \frac{1}{2}\int_0^s \exp\left(\frac{r-s}{\mu}\right)\dot{h}(r)ds - \frac{1}{2}\int_s^t \exp\left(\frac{s-r}{\mu}\right)\dot{h}(r)dr - h(s).$$

Notice that the integral terms

$$\frac{1}{2}\int_0^s \exp\left(\frac{r-s}{\mu}\right)\dot{h}(r)ds - \frac{1}{2}\int_s^t \exp\left(\frac{s-r}{\mu}\right)\dot{h}(r)dr \longrightarrow 0,\ \text{as } \mu \downarrow 0.$$

Since $\ddot{\hat{\varphi}}^{\mu}(s) = x(s)$, replace $x(s)$ in (3.47) we find

$$\ddot{\hat{\varphi}}^{\mu}(s) = \exp(-\frac{s}{\mu})c_1 + \exp(\frac{s-t}{\mu})c_2 + H(s).$$

Integrate both sides of the equation using the boundary condition

$$\hat{\varphi}_t^{\mu} = \dot{\hat{\varphi}}_t^{\mu} = 0$$

to get

$$\hat{\varphi}^{\mu}(s) = \int_s^t \int_r^t H(z)dzdr + \mu c_1[\mu(\exp(-\frac{s}{\mu}) - \exp(-\frac{t}{\mu})) + \exp(-\frac{t}{\mu})(s-t)]; \tag{3.48}$$

$$+ \mu c_2[t - s + \mu(\exp(\frac{s-t}{\mu}) - 1)]$$

$$\dot{\hat{\varphi}}^{\mu}(s) = -\int_s^t H(r)dr + \mu c_1(\exp(-\frac{t}{\mu}) - \exp(-\frac{s}{\mu})) + \mu c_2(\exp(\frac{s-t}{\mu}) - 1). \tag{3.49}$$

From $\hat{\varphi}^{\mu}(0) = q,\ \dot{\hat{\varphi}}^{\mu}(0) = p$, we get

$$\mu c_1[\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu})] + \mu c_2[t + \mu(\exp(-\frac{t}{\mu}) - 1)] \tag{3.50}$$

$$= q - \int_0^t \int_r^t H(z)dzdr,$$

$$\mu c_1(\exp(-\frac{t}{\mu}) - 1) + \mu c_2(\exp(-\frac{t}{\mu}) - 1) = p + \int_0^t H(r)dr. \tag{3.51}$$

By solving (3.50) and (3.51) for $\mu c_1,\ \mu c_2$, we find

$$\mu c_1 = K_1\left[q - \int_0^t \int_r^t H(z)dzdr\right] + K_2\left[p + \int_0^t H(r)dr\right]$$

$$\mu c_2 = K_3\left[p + \int_0^t H(r)dr\right] - K_1\left[q - \int_0^t \int_r^t H(z)dzdr\right]$$

where

$$K_1(\mu, t) = \left[2\mu\left(1 - \exp(-\frac{t}{\mu})\right) - t\exp(-\frac{t}{\mu}) - t\right]^{-1},$$

$$K_2(\mu, t) = -\frac{\mu(\exp(-\frac{t}{\mu}) - 1) + t}{(\exp(-\frac{t}{\mu}) - 1)[2\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]},$$

$$K_3(\mu,t) = \frac{\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu})}{(\exp(-\frac{t}{\mu})-1)[2\mu(1-\exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]}.$$

When $\mu \downarrow 0$, we see that $\mu c_1 \sim O(1),\ \mu c_2 \sim O(1)$, moreover,

$$\mu c_1 \longrightarrow -\frac{1}{t}\left(q - \int_0^t \int_r^t H(z)dzdr\right) - \left(p + \int_0^t H(r)dr\right)$$

$$\mu c_2 \longrightarrow \frac{1}{t}\left(q - \int_0^t \int_r^t H(z)dzdr\right)$$

as $\mu \downarrow 0$.

We note that as $\mu \downarrow 0$

$$K_1(\mu,t) \sim [o(\mu) - t]^{-1} \longrightarrow -t^{-1}$$

$$K_2(\mu,t) \sim -\frac{o(\mu)+t}{o(\mu)-t} \longrightarrow -1$$

$$K_3(\mu,t) \sim o(\mu) \longrightarrow 0$$

$$H(s) \sim o(\mu) - h(s) = o(\mu) - \nabla c(\hat{\varphi}^\mu(s)).$$

Substituting $H(s),\ \mu c_1,\ \mu c_2$ into equations (3.48) and (3.49), we can rewrite the boundary value problem as a fixed point problem:

$$\varphi = T(\mu,\varphi),\ \varphi \in C^1([0,t]) \tag{3.52}$$

where the operator $T:\ [0,\infty)\times C^1([0,t]) \longrightarrow C^1([0,t])$ is defined by the right hand side of equation (3.48). When $\mu \downarrow 0$,

$$T(\mu,\varphi) \longrightarrow T(0,\varphi)$$

where

$$T(0,\varphi) = -\int_s^t \int_r^t \nabla c(\varphi(z))dzdr + \frac{t-s}{t}\left(q + \int_0^t \int_r^t \nabla c(\varphi(z))dzdr\right).$$

The problem

$$\varphi = T(0,\varphi)$$

is equivalent to

$$\ddot{\varphi}_s = -\nabla c(\varphi_s),\ \ s \in [0,t] \tag{3.53}$$

$$\varphi(0) = q,\ \ \varphi(t) = 0.$$

Let $\tilde{\varphi}_s,\ \ s \in [0,t]$ be a non-degenerate solution of Euler-Lagrange equation (3.45). By nondegeneracy we mean that the linearization of (3.53) is nonsingular. When $\mu > 0$ is small, from the implicit function theorem we know there exists a unique solution of problem (3.52). From (3.48) and (3.49), we can write the solution of problem (3.52) as

$$\hat{\varphi}^\mu(s) = -\int_s^t\int_r^t \nabla c(\hat{\varphi}^\mu(z))dzdr + \frac{t-s}{t-o(\mu)}\left(\int_0^t\int_r^t \nabla c(\hat{\varphi}^\mu(z)dzdr + q\right) + o(\mu) \tag{3.54}$$

$$\dot{\hat{\varphi}}^\mu(s) = \int_s^t \nabla c(\hat{\varphi}^\mu(r))dr - \frac{1}{t-o(\mu)}\left(\int_0^t\int_r^t \nabla c(\hat{\varphi}^\mu(z))dzdr + q\right) + o(\mu) \tag{3.55}$$

Thus, when $\mu \downarrow 0$, from (3.54) and (3.55) we obtain

$$\hat{\varphi}^\mu_s \longrightarrow \tilde{\varphi}_s$$

$$\dot{\hat{\varphi}}^\mu_s \longrightarrow \dot{\tilde{\varphi}}_s$$

for each $s \in [0,t]$. □

Theorem 3.2.4. *Assume that condition* (N^μ) *is not satisfied for equation (3.40), and condition* (N) *is satisfied for equation (3.41). Let* $V(t,q)$ *and* $V^*(t,q)$ *have the same extremals. Then there exists a* $\mu_0 > 0$ *such that when* $0 < \mu < \mu_0$,

$$\Omega_+^{*;\mu} \subseteq \{(t,q) : |(t,q) - \Omega_+| < \delta(\mu_0)\} \tag{3.56}$$

$$\Omega_-^{*;\mu} \subseteq \{(t,q) : |(t,q) - \Omega_-| < \delta(\mu_0)\} \tag{3.57}$$

where δ *is some constant depending on* μ_0,

$$|(t,q) - A| = \min\{\mathit{dist}\{(t,q),\ (s,y)\} :\ \textit{for all } (s,y) \in A \subset [0,T] \times D \subset \mathbb{R}^n\},$$

dist is the Euclidean distance in $\mathbb{R}^n$ *and* D *is compact.*

Proof. By the proof of Theorem 3.1, we obtain the following estimates:

i. if φ^* is an extremal of $R_{0t}(\varphi)$, that is, $V(t,q) = R_{0t}(\varphi^*)$, then

$$V^\mu(t,p,q) \geq R_{0t}^\mu(\varphi^*) \geq R_{0t}(\varphi^*) - o(\mu); \tag{3.58}$$

ii. if $\hat{\varphi}$ is an extremal of $R_{0t}^\mu(\varphi)$, that is, $V^\mu(t,p,q) = R_{0t}^\mu(\hat{\varphi})$, then

$$V^\mu(t,p,q) = R_{0t}^\mu(\hat{\varphi}) \leq R_{0t}(\hat{\varphi}) + o(\mu) \leq V(t,q) + o(\mu) \tag{3.59}$$

Let $\Omega_{-v} \subset \Omega_-$ be the complement of the δ neighborhood of Ω_+; that is,

$$\Omega_{-v} = \{(t,q) : |(t,q) - \Omega_+| < \delta\}^c \tag{3.60}$$

$$= \{(t,q) \in [0,T] \times D\} \setminus \{(t,q) : |(t,q) - \Omega_+| < \delta\}$$

Note that by the continuity of $V(t,q)$ in $(t,q) \in [0,T] \times D$, we can choose the number $\delta > 0$ such that $V(t,q) \leq -v_0 < 0$ for some small number $v_0 > 0$ and for all $(t,q) \in \Omega_{-v}$. Moreover, we have

$$-v_0 < V(t,q) < 0 \text{ for } (t,q) \in \Omega_- \setminus \Omega_{-v}.$$

Take (t_1, q_1) be any point in the set Ω_{-v} and $\hat{\varphi}$ be an extremal of $V^\mu(t_1, 0, q_1)$ such that $V^\mu(t_1, 0, q_1) = R^\mu_{0t}(\hat{\varphi})$. When condition (N^μ) is not satisfied, we have

$$V^{*,\mu}(t_1, 0, q_1) < V^\mu(t_1, 0, q_1) \wedge 0.$$

From inequality (3.59), we get

$$V^{*,\mu}(t_1, 0, q_1) < V^\mu(t_1, 0, q_1) = R^\mu_{0t}(\hat{\varphi}) \leq R_{0t}(\hat{\varphi}) + o(\mu) \leq V(t_1, q_1) + o(\mu).$$

Therefore

$$V^{*,\mu}(t_1, 0, q_1) < -v_0 + o(\mu).$$

Thus there exists a $\mu_1 > 0$ such that when $0 < \mu < \mu_1$, $V^\mu(t_1, 0, q_1) < 0$. Hence for $0 < \mu < \mu_1$,

$$\Omega_{-v} \subseteq \Omega^{*,\mu}_- \tag{3.61}$$

Similarly, let $\Omega_{+v} \subset \Omega_+$ be the complement of the δ neighborhood of Ω_+, that is,

$$\Omega_{+v} = \{(t,q) : |(t,q) - \Omega_-| < \delta\}^c \tag{3.62}$$

$$= \{(t,q) \in [0,T] \times D\} \setminus \{(t,q) : |(t,q) - \Omega_-| < \delta\}.$$

Again by the continuity of $V(t,q)$ in $(t,q) \in [0,T] \times D$, we can choose a number $\delta > 0$ such that $V(t,q) \geq v_1 > 0$ for some small number $v_1 > 0$ and for all $(t,q) \in \Omega_{+v}$. Moreover, we have

$$0 < V(t,q) < v_1 \text{ for } (t,q) \in \Omega_+ \setminus \Omega_{+v}.$$

Take (t_2, q_2) be any point in the set Ω_{+v} and let $\tilde{\varphi}$ be an extremal of $R_{0t}(\varphi)$ such that

$$V(t_2, q_2) = R_{0t}(\tilde{\varphi}) = \int_0^{t_2} c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 ds \geq v_1 > 0,\ \tilde{\varphi}_0 = q_2,\ \tilde{\varphi}_{t_2} = 0.$$

When condition (N) is satisfied, we know $V^*(t_2, q_2) = V(t_2, q_2) \wedge 0$. By assumption, $\tilde{\varphi}_s,\ s \in [0, t_2]$ is also an extremal of functional $V^*(t_2, q_2)$. Therefore

$$V^*(t_2, q_2) = 0 = \sup \min_{a \in [0, t_2]} \{\int_0^a c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 ds :\ \tilde{\varphi}_0 = q_2,\ \tilde{\varphi}_{t_2} = 0\},$$

which implies that

$$c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 \geq C_0 > 0, \text{ for all } s \in [0,\ t_2] \tag{3.63}$$

for some positive constant C_0. As is known, the extremal $\tilde{\varphi}_s$ solves the Euler-Lagrange equation

$$\ddot{\tilde{\varphi}}_s = \nabla c(\tilde{\varphi}_s),\ \tilde{\varphi}_0 = q_2,\ \tilde{\varphi}_{t_2} = 0,\ s \in [0, t_2].$$

Let $\hat{\varphi}^\mu$ be an extremal of $R^\mu_{0t}(\varphi)$, that is

$$V^\mu(t_2, 0, q_2) = \int_0^{t_2} c(\hat{\varphi}_s) - \frac{1}{2}|\mu\ddot{\hat{\varphi}}^\mu_s + \dot{\hat{\varphi}}^\mu_s|^2 ds,\ \hat{\varphi}^\mu_0 = q_2,\ \dot{\hat{\varphi}}^\mu_0 = 0,\ \hat{\varphi}^\mu_{t_2} = \dot{\hat{\varphi}}^\mu_{t_2} = 0.$$

Then $\hat{\varphi}^\mu$ solves the Euler-Lagrange equation:

$$\mu^2 \hat{\varphi}^{\mu,(4)}_s - \ddot{\hat{\varphi}}^\mu_s = \nabla c(\hat{\varphi}^\mu_s),\ \hat{\varphi}^\mu_0 = q_2,\ \dot{\hat{\varphi}}^\mu_0 = 0,\ \hat{\varphi}^\mu_{t_2} = \dot{\hat{\varphi}}^\mu_{t_2} = 0,\ s \in [0, t_2].$$

Since $\hat{\varphi}^\mu_s \in C^4([0, t_2])$, as $\mu \downarrow 0$, we have the following estimate:

$$\begin{aligned} & c(\hat{\varphi}^\mu_s) - \frac{1}{2}|\mu\ddot{\hat{\varphi}}^\mu_s + \dot{\hat{\varphi}}^\mu_s|^2 \\ = \ & c(\hat{\varphi}^\mu_s) - \frac{1}{2}|\dot{\hat{\varphi}}^\mu_s|^2 - \frac{1}{2}\mu^2|\ddot{\hat{\varphi}}^\mu_s|^2 - (\mu\ddot{\hat{\varphi}}^\mu_s, \dot{\hat{\varphi}}^\mu_s) \end{aligned}$$

$$= \quad c(\hat{\varphi}_s^\mu) - \frac{1}{2}|\dot{\hat{\varphi}}_s^\mu|^2 - o(\mu)$$

From the Lemma 3.2.3, we know that for each $s \in [0, t]$,

$$\hat{\varphi}_s^\mu \longrightarrow \tilde{\varphi}_s, \text{ as } \mu \downarrow 0$$

$$\dot{\hat{\varphi}}_s^\mu \longrightarrow \dot{\tilde{\varphi}}_s, \text{ as } \mu \downarrow 0.$$

Knowing $c(q)$ is smooth, as $\mu \downarrow 0$, we have

$$\begin{aligned}
& |[c(\hat{\varphi}_s^\mu) - \frac{1}{2}|\dot{\hat{\varphi}}_s^\mu|^2] - [c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2]| \\
= \quad & |c(\hat{\varphi}_s^\mu) - c(\tilde{\varphi}_s) - \frac{1}{2}(\dot{\hat{\varphi}}_s^\mu + \dot{\tilde{\varphi}}_s)(\dot{\hat{\varphi}}_s^\mu - \dot{\tilde{\varphi}}_s)| \\
\leq \quad & C(|\hat{\varphi}_s^\mu - \tilde{\varphi}_s| + |\dot{\hat{\varphi}}_s - \dot{\tilde{\varphi}}_s|) \\
\leq \quad & C\, o(\mu).
\end{aligned}$$

for some constant $C > 0$. This is equivalent to

$$c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 - o(\mu) \leq c(\hat{\varphi}_s^\mu) - \frac{1}{2}|\hat{\varphi}_s^\mu|^2 \leq c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 + o(\mu).$$

From (3.63), we conclude that there exists a $\mu_2 > 0$ such that when $0 < \mu < \mu_2$,

$$\begin{aligned}
c(\hat{\varphi}_s^\mu) - \frac{1}{2}|\mu\ddot{\hat{\varphi}}_s^\mu + \dot{\hat{\varphi}}_s|^2 &= c(\hat{\varphi}_s^\mu) - \frac{1}{2}|\dot{\hat{\varphi}}_s^\mu|^2 - o(\mu) \\
&\geq c(\tilde{\varphi}_s) - \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 - o(\mu) \\
&\geq C_0 - \mu_2 > 0.
\end{aligned}$$

Hence for any $(t_2, q_2) \in \Omega_{+v} \subset \Omega_+$, we have $V^{*,\mu}(t_2, 0, q_2) = 0$. This implies that when $0 < \mu < \mu_2$,

$$\Omega_{+v} \subseteq \Omega_+^{*,\mu} \tag{3.64}$$

Take $\mu_0 = \mu_1 \wedge \mu_2$. When $0 < \mu < \mu_0$, (3.61) and (3.64) hold. When $0 < \mu < \mu_0$, from (3.60) and (3.61), for some fixed δ depending on μ_0, we have the set inequalities

$$\{(t,q) : |(t,q) - \Omega_+| < \delta(\mu_0)\}^c \subseteq \Omega_-^{*,\mu}$$

$$(\{(t,q) : |(t,q) - \Omega_+| < \delta(\mu_0)\}^c)^c \supseteq (\Omega_-^{*,\mu})^c$$

$$\{(t,q) : |(t,q) - \Omega_+| < \delta(\mu_0)\} \supseteq \Omega_+^{*,\mu}.$$

Thus we have proved (3.56). By the same analysis, when $0 < \mu < \mu_0$, from (3.62) and (3.64), for some fixed δ depending on μ_0, we have the set inequalities

$$\{(t,q) : |(t,q) - \Omega_-| < \delta(\mu_0)\}^c \subseteq \Omega_+^{*,\mu}$$

$$(\{(t,q) : |(t,q) - \Omega_-| < \delta(\mu_0)\}^c)^c \supseteq (\Omega_+^{*,\mu})^c$$

$$\{(t,q) : |(t,q) - \Omega_-| < \delta(\mu_0)\} \supseteq \Omega_-^{*,\mu}.$$

Thus we have proved (3.57).

□

Example 3.2.5. Consider an example in $\mathbb{R}^1$ when the function $f(q,u) = f(u)$ depends only on u, that is

$$f(u) = uc(u), \ \max_{0 \le u \le 1} c(u) = c(0) = c.$$

where $c > 0$ is a constant not depending on q. We study the relation between the wavefront propagation of the following two equations. The first equation is

$$\frac{\partial u^\varepsilon(t,p,q)}{\partial t} = \frac{\varepsilon}{2\mu^2}\frac{\partial^2 u^\varepsilon}{\partial p^2} - \frac{1}{\mu}p\frac{\partial u^\varepsilon}{\partial p} + p\frac{\partial u^\varepsilon}{\partial q} + \frac{1}{\varepsilon}u^\varepsilon c(u^\varepsilon) \tag{3.65}$$

$$u^{\varepsilon}(0,p,q)=\delta(p)\chi^{-1}(q),\ \ q,\ \ p\in\mathbb{R}^1$$

where $\delta(p)$ is the delta function centered at 0, taking value 1 at 0 and 0 otherwise and $\chi^{-1}(q)$ is the indicator function such that

$$\chi^{-1}(q)=\begin{cases}1, & q<0\\ \\ 0, & q\geq 0\end{cases}$$

The second equation is defined as:

$$\frac{\partial u^{\varepsilon}(t,q)}{\partial t}=\frac{\varepsilon}{2}\frac{\partial^2 u^{\varepsilon}}{\partial q^2}+\frac{1}{\varepsilon}u^{\varepsilon}c(u^{\varepsilon}) \tag{3.66}$$

$$u^{\varepsilon}(0,q)=\chi^{-1}(q),\ \ q\in\mathbb{R}^1.$$

We will see later that when $c(q)=c>0$ is a constant not depending on q, condition (N^{μ}) for (3.65) is not fulfilled. As checked in Freidlin [8], we know condition (N) is fulfilled for equation (3.66). Thus we can use the functional $V^{*,\mu}(t,p,q)$ to characterize the wave front propagation for equation (3.65) and $V(t,q)$ for equation (3.66). From the sections above, it's easy to find that

$$V^{*,\mu}(t,p,q)=\sup\min_{0\leq a\leq t}\left\{\int_0^a c-\frac{1}{2}|\mu\ddot{\varphi}_s+\dot{\varphi}_s|^2ds,\varphi_0=q,\dot{\varphi}_0=p,\varphi_t=\dot{\varphi}_t=0\right\}$$

$$V(t,q)=\sup\left\{\int_0^t c-\frac{1}{2}|\dot{\varphi}_s|^2ds,\ \ \varphi_0=q,\ \ \varphi_t=0\right\}.$$

Recall the definition of the functional $V^{\mu}(t,p,q)$ as

$$V^{\mu}(t,p,q)=\sup\{\int_0^t c-\frac{1}{2}|\mu\ddot{\varphi}_s+\dot{\varphi}_s|^2ds,\varphi_0=q,\dot{\varphi}_0=p,\varphi_t=\dot{\varphi}_t=0\}.$$

It helps us to check that condition (N^{μ}) is not fulfilled.

Let $\hat{\varphi}_s^\mu$ be an extremal of the functional $V^\mu(t,p,q)$, such that

$$V^\mu(t,p,q) = ct - \int_0^t \frac{1}{2}|\mu\ddot{\hat{\varphi}}_s^\mu + \dot{\hat{\varphi}}_s^\mu|^2 ds$$

with

$$\hat{\varphi}_0^\mu = q,\ \dot{\hat{\varphi}}_0^\mu = p,\ \hat{\varphi}_t^\mu = \dot{\hat{\varphi}}_t^\mu = 0.$$

Let $\tilde{\varphi}_s$ be an extremal of $V(t,q)$ such that

$$V(t,q) = ct - \int_0^t \frac{1}{2}|\dot{\tilde{\varphi}}_s|^2 ds,\ \tilde{\varphi}_0 = q,\ \tilde{\varphi}_t = 0.$$

Since the Euler-Lagrange equation for $\tilde{\varphi}_s,\ s \in [0,t]$, has the form

$$\ddot{\tilde{\varphi}}_s = 0,\ \tilde{\varphi}_0 = q,\ \tilde{\varphi}_t = 0,$$

the extremal $\tilde{\varphi}_s$ can be easily calculated as

$$\tilde{\varphi}_s = -\frac{q}{t}s + q. \tag{3.67}$$

The functional $V(t,q)$ has the form

$$V(t,q) = ct - \frac{q^2}{2t}.$$

So the wavefront is

$$q = \sqrt{2c}\,t,\ t \geq 0. \tag{3.68}$$

First, to check that condition (N^μ) for equation (3.65) is not satisfied, we calculate the extremal $\hat{\varphi}_s^\mu,\ s \in [0,t]$. It satisfies the Euler-Lagrange equation

$$\mu^2 \hat{\varphi}_s^{\mu,(4)} - \ddot{\hat{\varphi}}_s^\mu = 0,$$

with

$$\hat{\varphi}_0^\mu = q,\ \dot{\hat{\varphi}}_0^\mu = p,\ \hat{\varphi}_t^\mu = \dot{\hat{\varphi}}_t^\mu = 0.$$

By similar calculation as in Example 3.2, we find

$$\ddot{\hat{\varphi}}_s^\mu = c_1 \exp(-\frac{s}{\mu}) + c_2 \exp(\frac{s-t}{\mu})$$

$$\hat{\varphi}_s^\mu = \mu c_1[\mu(\exp(-\frac{s}{\mu}) - \exp(-\frac{t}{\mu})) + \exp(-\frac{t}{\mu})(s-t)]$$

$$+ \mu c_2[\mu(\exp(\frac{s-t}{\mu}) - 1) + t - s]$$

$$\dot{\hat{\varphi}}_s^\mu = \mu c_1[\exp(-\frac{t}{\mu}) - \exp(-\frac{s}{\mu})] + \mu c_2(\exp(\frac{s-t}{\mu}) - 1)$$

for some constants $c_1,\ c_2$. From $\hat{\varphi}^\mu(0) = q,\ \dot{\hat{\varphi}}^\mu(0) = p$, we obtain the following equalities:

$$q = \mu c_1[\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu})] + \mu c_2[\mu(\exp(-\frac{t}{\mu}) - 1) + t]$$

$$p = \mu c_1(\exp(-\frac{t}{\mu}) - 1) + \mu c_2(\exp(-\frac{t}{\mu}) - 1).$$

We calculate the value of μc_1 and μc_2 from the above equations as:

$$\mu c_1 = K_1 q - K_2 p,\ \mu c_2 = K_3 p - K_1 q$$

where

$$K_1 = \frac{1}{2\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t} \xrightarrow{\mu\downarrow 0} -\frac{1}{t}$$

$$K_2 = \frac{\mu(\exp(-\frac{t}{\mu}) - 1) + t}{(\exp(-\frac{t}{\mu}) - 1)[2\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]} \xrightarrow{\mu\downarrow 0} 1$$

$$K_3 = \frac{\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu})}{(\exp(-\frac{t}{\mu}) - 1)[2\mu(1 - \exp(-\frac{t}{\mu})) - t\exp(-\frac{t}{\mu}) - t]} \xrightarrow{\mu\downarrow 0} 0$$

Note that as $\mu \downarrow 0$, we have $\mu c_1 \sim O(1)$ and $\mu c_2 \sim O(1)$. Moreover

$$\mu c_1 \longrightarrow -p - \frac{q}{t},\ \mu c_2 \longrightarrow \frac{q}{t}.$$

Plugging the values of $\ddot{\varphi}_s^\mu,\ \dot{\varphi}_s^\mu$ into functional $V^\mu(t,p,q)$, and solving for the value of p when $V^\mu(t,p,q)=0$, we obtain the position of the wavefront of p. It has the form

$$p^\mu(t)=\frac{-(X\pm\sqrt{Y})}{Z}$$

where

$$X=\mu q(\exp(-\frac{2t}{\mu})-2\exp(-\frac{t}{\mu})+1)$$

$$Y=-2\mu(1-\exp(-\frac{t}{\mu}))[(2\mu-t)-(2\mu+t)\exp(-\frac{t}{\mu})]$$

$$\times[\mu ct(1-4\exp(-\frac{t}{\mu})+3\exp(-\frac{2t}{\mu})+(2ct^2-q^2)\exp(-\frac{2t}{\mu})$$

$$Z=\mu(3\mu\exp(-\frac{2t}{\mu})-4\mu\exp(-\frac{t}{\mu})+2t\exp(-\frac{2t}{\mu})+\mu)$$

Since terms of $\exp(-\frac{2t}{\mu}),\ \exp(-\frac{t}{\mu})$ are relatively small compared with terms of $\mu,\ q,\ t$ when μ is small, the terms of $\mu,\ q,\ t$ dominate as $\mu\downarrow 0$. In this way, we can simplify the representation of X,Y,Z to find an approximation formula of $p^\mu(t)$ for small μ:

$$p^\mu(t)\approx-\frac{\mu q\pm\sqrt{-2\mu(2\mu-t)\mu ct}}{\mu^2}.$$

Differentiating twice we find

$$\ddot{p}^\mu(t)=-\frac{4c^2\mu^4}{(-2ct\mu^2(-t+2\mu))^{\frac{3}{2}}}<0.$$

This is a concave function for all $t\geq 0$. However, when μ is small,

$$\dot{\varphi}_s^\mu=\mu c_1[\exp(-\frac{t}{\mu})-\exp(-\frac{s}{\mu})]+\mu c_2(\exp(\frac{s-t}{\mu})-1)$$

is close to the convex function $\exp(\frac{s-t}{\mu})-1$, which implies that

$$V^\mu(t-s,\dot{\varphi}_s^\mu,\cdot)>0.$$

Condition (N^{μ}) is not satisfied.

It's easy to check that in this case, $V(t,q)$ and $V^{*}(t,q)$ have the same extremals. From the theorem, we conclude that the position of the wavefront of equation (3.65) is with in the δ neighborhood of wavefront (3.68), where δ is a function depends on μ.

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