ABSTRACT

Title of Dissertation: DISCRETE INVERSE CONDUCTIVITY PROBLEMS ON NETWORKS

Farshad Foroozan, Doctor of Philosophy, 2006

Dissertation directed by: Carlos A. Berenstein, Professor,
Department of Mathematics and
The Institute for Systems Research

The purpose of this dissertation is to present a mathematical model of network tomography through spectral graph theory analysis. In this regard, we explore the properties of harmonic functions and eigensystems of Laplacians for weighted graphs (networks) with and without boundary. We prove the solvability of the Dirichlet and Neumann boundary value problems. We also prove the global uniqueness of the inverse conductivity problem on a network under a suitable monotonicity condition. As a physical interpretation to the discrete inverse conductivity problem, we define a variant of the chip-firing game (a discrete balancing process) in which chips are added to the game from the boundary nodes and removed from the game if they are fired into the boundary of the graph. We find a bound on the length of the game, and examine the relations between set of spanning weighted forest rooted in the boundary of the graph and the set of critical configurations of the chips.

DISCRETE INVERSE CONDUCTIVITY PROBLEMS ON NETWORKS

by

Farshad Foroozan

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2006

Advisory Committee

Professor Carlos Berenstein, Chairman
Professor Kenneth Berg
Professor David Mount
Professor Clyde Kruskal
Professor James Purtilo

To my daughter, Sarvnaz, and my wife, Lili.

ACKNOWLEDGEMENTS

I am indebted to my advisor, Professor Carlos Berenstein, for his vision, patience, guidance, and support. His original idea of discretization of the inverse connectivity problems and its contribution to the field of network tomography, inspired me to do this dissertation. This work would definitely have been impossible without him. I would like to express my gratitude to Professor Berg and Professor Mount whom I benefitted a lot from their very interesting comments and fruitful discussions over the last year. My warm thanks to Professor Levermore for his support and advice on many occasions. I would also like to thank my advisory committee members; Professor Kruskal, and Professor Purtilo for their helpful and stimulating discussions.

I am grateful to all the professors of the mathematics and the electrical engineering courses I took at University of Maryland over the years and who thereby shaped my understanding of applied mathematics; Professor Baras, Professor Benedetto, Professor Berg, Professor Freidlin, Professor Krishnaprasad, Professor Levine, Professor Narayan, Professor Papamarcou, Professor Schwartz, and Professor Warner.

It would have been impossible for me to survive without the financial support of the Mathematics department. I would like to thank the department for providing me with teaching assistantships.

Finally, I wish to express my deepest thanks to my parents and my family for their

patience, understanding, and encouragement.

Contents

1 Introduction **1**

1.1 Calculus on Graphs 3

1.2 Properties of Harmonic Functions 10

1.3 Spectra of the Normalized and the Combinatorial Laplacians 13

1.4 Dirichlet and Neumann Eigenvalues 17

1.4.1 Dirichlet Eigenvalues 17

1.4.2 Neumann Eigenvalues 21

1.5 The Diameter of a Weighted Graph 26

2 The Discrete Inverse Conductivity Problem **29**

2.1 Discrete Green's Function 31

2.2 Dirichlet and Neumann boundary Value Problems 40

2.3 Inverse Conductivity Problem on the Network 48

3 The Physical Interpretation of The Discrete Inverse Conductivity Problem **62**

3.1 Introduction 65

3.2 Basic Theory of the Dirichlet Game 67

3.3 Critical Configuration of the Dirichlet Game 70

3.4 Dirichlet Game as a Discrete Dynamical System 75

3.5 Basic Theory of Electrical Networks 79

3.6 Random Walk interpretation of Electrical Networks 88

3.7 Upper Bound for Run-Time Estimates 95

4 Algebraic Aspects of the Dirichlet Game **101**

4.1 The Determinant of the Dirichlet Laplacian 102

4.2 Relation of Green's Function to Dirichlet Game 106

4.3 Critical Configuration and Rooted Spanning Weighted Forest 111

5 REFERENCES **116**

Chapter I

1 Introduction

A network consists of interconnecting any pair of users or nodes by means of some links. Because of the complexity and the size of the network, it is desirable to study only a subset of the nodes (known as boundary nodes) to discover and detect certain problems in the interior of the network. These problems might include checking connectivity, finding largest components, tracking data traffic, assessing and dealing with a variety of security and reliability issues. From the practical point of view, this is normally done by setting up “monitors” at a relatively small subset of the nodes. From the monitors, data can be collected and examined. The problem of discovering the detailed inner structure of the network from a collection of “end-to-end” measurements can be seen as a type of inverse problem, analogous to those arising in tomography, but with a discrete flavor.

Motivated by this application, Berenstein and Chung [6] initiated the work of discretization of the inverse conductivity problem by means of the spectral graph theory. Their work led to a key result in the domain of discrete inverse conductivity problem which proves the uniqueness of weights (conductivity or connectivity) under certain monotonicity conditions. Further investigations include exploring some variant of chip-firing game as a physical interpretation of the discrete inverse conductivity problem. This is motivated in part by communication network models in which chips represent packets or jobs and the boundary nodes represent processors.

This thesis is divided into four chapters. Chapter I is a substantive introduction to the Laplacians of graphs, including properties of the spectra of Laplacians, and the development of calculus on graph. Chapter II introduces the discrete Green's function, and improves the result of Berenstein and Chung [6] under a weaker condition. Chapter III describes the Dirichlet chip-firing game on weighted graph as a physical interpretation of the discrete inverse conductivity problem and provides an upper bound estimate on the time for the configuration to reach a stable configuration in terms of the diameter of the weighted graph. Chapter IV combines the results of Chapter III and the discrete Green's function to allow fast computation of the upper bound run-time estimate.

Before presenting a more detailed summary of the contents of this chapter, we mention some important contributions to the field of spectral graph theory. An early fundamental problem in the study of the spectra of graphs was the question of whether a graph can be determined by its set of eigenvalues. The answer was found to be negative. That led to further research in the study of isospectral graphs [16, 29]. Motivated by the study of the eigenvalues of Laplacians of compact Riemannian manifolds, bounds for the eigenvalues of the discrete Laplacian have been studied [3, 21, 31, 50]. Further developments in this area include, properties of the second smallest eigenvalue [49, 50], expansion properties [2], isoperimetric number and Cheeger's constant [52], and the heat kernel of Laplacian [18].

This chapter is divided into five sections. The first section studies calculus on graphs. In the second section, we study the properties of harmonic function as a solution to the discrete Laplace's equation and formulate a theorem that gives a necessary and sufficient condition for a function to be harmonic. In the third section, the symmetric versions of the discrete Laplacian which are the combinatorial and the normalized Laplacians are introduced. This symmetrization will allow us to study the eigenvalues of the discrete Laplacian. The normalization of the Laplacian is mainly due to keep the subject parallel to the eigenvalues of the compact Riemannian manifold. Theorems involving properties of the eigenvalues of the weighted Laplacian have been developed. In section four, we study the Neumann and the Dirichlet boundary conditions along with the spectrum of the Neumann and the Dirichlet Laplacians. One of the objectives of this section is to formulate matrices whose eigenvalues are the Neumann and the Dirichlet eigenvalues. We also discuss the relationship between the Neumann eigenvalue and the eigenvalue of the transition probability matrix of the Neumann random walk. In the last section, we introduce the concept of the diameter of the weighted graph and its relation to the first eigenvalue of Laplacian. Interesting bounds on the diameter of the graph are also introduced.

1.1 Calculus on Graphs

We will begin with some definitions of graph theoretic terminologies which are largely derived from Berenstein-Chung [6].

By a graph $G = (V, E, \Theta)$, we mean a non-empty finite set V of vertices, a non-empty finite set E of edges, and an injective map Θ from E into the two element subsets of V. The elements of $\Theta(e)$ are called the endpoints of the edge e. For simplicity, we drop the notation Θ and write the graph as $G = (V, E)$.

A path in G is an ordered set of vertices $x_0, x_1, x_2, ..., x_n$ such that for each i, $1 \leq i \leq n$ implies $x_{i-1} \sim x_i$. Here, the notation $x_{i-1} \sim x_i$ means that two vertices x_{i-1} and x_i are connected (adjacent) by an edge in E. The graph G is connected if any two vertices are connected by a path.

A graph (S, T) is said to be a subgraph of $G = (V, E)$, if S and T are non-empty subsets of V and E respectively. If (S, T) consists of all edges of G which have both endpoints in S, then (S, T) is called an induced subgraph of G and is denoted by $G_S = (S, E_S)$. We define the boundary ∂S of S to be set of all vertices not in S but adjacent to some vertex in S, i.e., $\partial S = \{y \in (V \setminus S) \mid \exists x \in S, \text{ such that } x \sim y\}$. And the inner boundary is defined by $\partial^0 S = \{z \in S \mid \exists y \in \partial S, \text{ such that } y \sim z\}$.

A weighted graph $G = (V, E)$ has associated with it a non-negative function

$$w : V \times V \to R,$$

such that $w(x, y) = w(y, x)$, and $w(x, y) = 0$ if either $x = y$ or x and y are not connected by an edge in E.

For $x \in V$ and a non-empty subset U of V, we define a relative degree $d_{G_U}(x)$ of x with respect to the induced subgraph G_U of G as

$$d_{G_U}(x) = \sum_{y \in U} w(x, y),$$

if $U = V$, we call $d_G(x) = \sum_{y \in V} w(x, y)$ the degree of the vertex x.

The weighted discrete Laplacian Δ_w of a function $f : V \to R$ on a graph $G = (V, E)$ and a point $x \in V$ such that $d_G(x) \neq 0$ is defined as

$$\Delta_w f(x) = \sum_{y \in V} (f(x) - f(y)) \frac{w(x, y)}{d_G(x)},$$

if x is an isolated vertex of G, i.e., $d_G(x) = 0$ then we set $\Delta_w f(x) = 0$.

The symmetrized versions of the discrete Laplacian are the weighted combinatorial Laplacian L_w and the weighted normalized Laplacian $£_w$. The combinatorial Laplacian is related to the algebraic aspect of the graph theory, whereas the weighted normalized Laplacian is related to the geometric aspect of the spectral graph theory [18]. The weighted combinatorial Laplacian L_w of a function $f : V \to R$ is defined as

$$L_w f(x) = \sum_{y \in V} (f(x) - f(y)) w(x, y).$$

The weighted normalized Laplacian $£_w$ of a function $f : V \to R$ on a graph $G = (V, E)$ and a point $x \in V$ such that $d_G(x) \neq 0$ is defined as:

$$£_w f(x) = \sum_{y \in V} \left(\frac{f(x)}{\sqrt{d_G(x)}} - \frac{f(y)}{\sqrt{d_G(y)}} \right) \frac{w(x, y)}{\sqrt{d_G(x)}},$$

if $d_G(x) = 0$ then we set $£_w f(x) = 0$. The matrix formulation of the above Laplacians become:

$$\Delta_w(x, y) = \begin{cases} 1 & \text{if } x = y \text{ and } d_G(x) \neq 0 \\ -\dfrac{w(x, y)}{d_G(x)} & \text{if } x \text{ is adjacent to } y \\ 0 & \text{otherwise.} \end{cases}$$

$$L_w(x,y) = \begin{cases} d_G(x) & \text{if} \quad x = y \\ -w(x,y) & \text{if } x \text{ is adjacent to } y \\ 0 & \text{otherwise.} \end{cases}$$

$$£_w(x,y) = \begin{cases} 1 & \text{if } x = y \text{ and } d_G(x) \neq 0 \\ -\dfrac{w(x,y)}{(d_G(x)\, d_G(y))^{1/2}} & \text{if } x \text{ is adjacent to } y \\ 0 & \text{otherwise.} \end{cases}$$

Let T denote the diagonal matrix with the (x, x)-the entry having the value $d_G(x)$. Then the following relations hold between Δ_w, L_w, $£_w$:

$$\begin{aligned} L_w &= T^{1/2} £_w T^{1/2} = T\Delta_w, \\ £_w &= T^{-1/2} L_w T^{-1/2} = T^{1/2}\Delta_w T^{-1/2}, \\ \Delta_w &= T^{-1} L_w = T^{-1/2} £_w T^{1/2}, \end{aligned}$$

provided that G does not have an isolated vertex x, i.e., $d_G(x) \neq 0$.

We now develop the concept of differential and integral calculus on graphs. Let S be a non-empty subset of V and G_U an induced subgraph of the graph G. For a function f defined on S, the integration of f over S with respect to the relative degree $d_{G_U}(x)$ for each

$x \in S$ is defined as

$$\int_S f(x)\, d_{G_U}(x) = \sum_{x \in S} f(x)\, d_{G_U}(x).$$

If $U = V$ then the integration of f over S simply becomes:

$$\int_S f(x)\, d_G(x) = \sum_{x \in S} f(x)\, d_G(x).$$

The directional derivative of the function f defined on V for the graph G with no isolated vertices is defined as

$$D_{w,y} f(x) = (f(x) - f(y)) \left(\frac{w(x, y)}{d_G(x)} \right)^{1/2} ,$$

for each x and $y \in V$. The gradient ∇_w of a function f is defined to be a vector in R^n, where n is the number of vertices,

$$\nabla_w f(x) = (\, D_{w,y} f(x))_{y \in V}.$$

We now introduce the notion of outward normal derivative. For an induced subgraph $G_S = (S, E_S)$ of a graph G with non-empty boundary ∂S, the outward normal derivative $\frac{\partial f}{\partial_w n}(z)$ at $z \in \partial S$ is defined as

$$\frac{\partial f}{\partial_w n}(z) = \sum_{y \in S} (f(z) - f(y)) \frac{w(z, y)}{d_{G_S}(z)}.$$

As a consequence of the above definitions, it is easy to see that the following lemma is true.

Lemma 1.1.1

Let f be defined on the set of vertices V of the graph G with no isolated vertices. Then

$$\int_V |\nabla_w f(x)|^2 \, d_G(x) = 2\sum_{x\sim y} | f(x) - f(y) |^2 \, w(x, y),$$

where $\sum_{x\sim y}$ denotes the sum over all unordered pairs $\{x, y\}$ for which x and y are adjacent.

Proof: The proof follows easily from the definition of the integral.

$$\begin{aligned}\int_V \quad | \quad \nabla_w f(x) |^2 \, d_G(x) &= \sum_{x\in V}\sum_{y\in V} | f(x) - f(y) |^2 \, w(x, y) \\ &= 2\sum_{x\sim y} | f(x) - f(y) | \, w(x, y)\end{aligned}$$

QED

Theorem 1.1.2

For any pair of functions f, h defined on the set of vertices V of the graph G with no isolated vertices, we have:

$$2\int_V h(x)\Delta_w f(x) d_G(x) = \int_V (\nabla_w f(x)).(\nabla_w h(x)) d_G(x).$$

Proof:

$$\begin{aligned}2\int_V h(x)\Delta_w f(x) d_G(x) &= 2\sum_{x\in V} h(x)\Delta_w f(x) d_G(x) \\ &= 2\sum_{x\in V} h(x) \sum_{y\in V}(f(x) - f(y))\frac{w(x, y)}{d_G(x)} d_G(x)\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{x\in V} h(x)\sum_{y\in V}(f(x)-f(y))w(x,y)\\
&= \sum_{x\in V}\sum_{y\in V}(h(x)-h(y))(f(x)-f(y))w(x,y)\\
&= \sum_{x\in V}(\nabla_w f(x))\,.\,(\nabla_w h(x))\,d_G(x)\\
&= \int_V (\nabla_w f(x))\,.\,(\nabla_w h(x))\,d_G(x).
\end{aligned}
$$

QED

Theorem 1.1.3

Under the same hypothesis as in Theorem 1.1.2, we have the following identities

$a)$ $2\int_V f(x)\Delta_w f(x) d_G(x) = \int_V \mid \nabla_w f(x) \mid^2 d_G(x)$,

$b)$ $\int_V h(x)\Delta_w f(x) d_G(x) = \int_V f(x)\Delta_w h(x) d_G(x)$,

Furthermore, for an induced subgraph G_S of G with non-empty boundary ∂S , we have

$c)$ $\int_S (f(x)\Delta_w h(x) - h(x)\Delta_w f(x)) d_{G_{\overline{S}}}(x) = \int_{\partial S}(h(z)\frac{\partial f}{\partial_w n}(z) - f(z)\frac{\partial h}{\partial_w n}(z)) d_{G_S}(z)$,

where $\overline{S} = S \cup \partial S$. (c) is also known as Green's theorem.

Proof: (a) follows from Theorem 1.1.2, by substituting h for f. (b) also follows from Theorem 1.1.2 by symmetry. We prove part (c) as follows. Let w' be a real valued function on $\overline{S}\times\overline{S}$ defined by

$$
w'(x,y) = \begin{cases} w(x,y) & \text{if either } x \text{ or } y \text{ are in } S \\ 0 & \text{otherwise.} \end{cases}
$$

Let $\overline{E_S}$ be the subset of edges $E_{\overline{S}}$ such that $w'(x,y) > 0$ if $\{x,y\}$ is an edge in $E_{\overline{S}}$ with endpoints x and y. Then $\overline{G_S} = (\overline{S}, \overline{E_S})$ is a subgraph of G. Applying the Theorem 1.1.3

(b) to the graph $\overline{G_S}$ with w' as its weight function, we see that

$$\begin{aligned} 0 &= \int_{\overline{S}} (h(x)\Delta_{w'} f(x) - f(x)\Delta_{w'} h(x)) d'_{\overline{G_S}}(x) \\ &= \left[\int_S (h(x)\Delta_{w'} f(x) - f(x)\Delta_{w'} h(x)) d'_{\overline{G_S}}(x)\right] + \\ &\quad \left[\int_{\partial S} (h(z)\Delta_{w'} f(z) - f(z)\Delta_{w'} h(z)) d'_{\overline{G_S}}(z)\right] \quad (1) \end{aligned}$$

where $d'_{\overline{G_S}}(x)$ is the degree of the vertex x of the graph $\overline{G_S}$ with respect to the weight function w'. From the definitions of the degree and the discrete Laplacian, it is easily seen that if $x \in S$ then $d'_{\overline{G_S}}(x) = d_{G_{\overline{S}}}(x)$, $\Delta_{w'} f(x) = \Delta_w f(x)$, and $\Delta_{w'} h(x) = \Delta_w h(x)$. Also, if $z \in \partial S$ then $d'_{\overline{G_S}}(z) = d_{G_S}(z)$, $\Delta_{w'} f(z) = \frac{\partial f}{\partial_w n}(z)$, and $\Delta_{w'} h(z) = \frac{\partial h}{\partial_w n}(z)$. Substituting these equalities to (1) gives the required result:

$$\int_S (f(x)\Delta_w h(x) - h(x)\Delta_w f(x)) d_{G_{\overline{S}}}(x) = \int_{\partial S} (h(z)\frac{\partial f}{\partial_w n}(z) - f(z)\frac{\partial h}{\partial_w n}(z)) d_{G_S}(z).$$

QED

1.2 Properties of Harmonic Functions

Let $G_S = (S, E_S)$ be a connected induced subgraph of $G = (V, E)$ with non-empty boundary set ∂S . A function $f : \overline{S} \to R$ such that

$$\Delta_w f(x) = \sum_{y \in \overline{S}} (f(x) - f(y)) \frac{w(x,y)}{d_G(x)} = 0, \text{ for all } x \in S,$$

is said to be harmonic on G_S. Solving the above equation for $f(x)$, we get

$$f(x) = \frac{1}{d_G(x)} \sum_{y \in \overline{S}} f(y)w(x,y), \text{ for all } x \in S.$$

Therefore, a harmonic function is the weighted average of the values of function at its neighboring vertices. The following theorem shows that the maximum and minimum of a harmonic function cannot occur in the interior of the graph.

Theorem 1.2.1

Let G_S be a connected induced subgraph of G with non-empty boundary ∂S. For a non-constant function $f : \overline{S} \to R$, we have:

a) If $\Delta_w f(x) = 0$ for all $x \in S$, then f has no maximum or minimum value in S.

b) If $\Delta_w f(x) \geq 0$ for all $x \in S$, then f has no minimum value in S.

c) If $\Delta_w f(x) \leq 0$ for all $x \in S$, then f has no maximum value in S.

Proof: Part (a) follows from parts (b) and (c) together. Part (c) carries the same argument as (b). So we only prove part (b). Assume S has a vertex x such that $f(x)$ is minimum and there is a vertex $y_0 \in \overline{S}$ adjacent to x such that $f(x) \neq f(y_0)$. Such a choice is possible by the connectedness of G_S and the fact that f is a non-constant function on $\overline{S}$. Because $\Delta_w f(x) \geq 0$ then

$$\begin{aligned} f(x) &\geq \frac{1}{d_G(x)} \sum_{y \in \overline{S}} f(y)w(x,y) \\ &= \frac{1}{d_G(x)} \sum_{\substack{y \neq y_0 \\ y \in \overline{S}}} f(y)w(x,y) + f(y_0)w(x,y_0)\, \frac{1}{d_G(x)} \end{aligned}$$

$$> \frac{1}{d_G(x)} \sum_{\substack{y \neq y_0 \\ y \in \overline{S}}} f(x)w(x,y) + f(x)w(x,y_0) \frac{1}{d_G(x)} = f(x).$$

This is clearly a contradiction, therefore, f has no minimum value in S.

QED

Under the same hypothesis as in Theorem 1.2.1, the following statements are true.

Corollary 1.2.2

If $\Delta_w f(x) = 0$ and $\Delta_w g(x) \geq 0$ for all $x \in S$ then $g\mid_{\partial S} \leq f\mid_{\partial S}$ implies $g \leq f$ on S.

Corollary 1.2.3

If a function $f : \overline{S} \to R$ satisfies $\Delta_w f(x) = 0$ for all $x \in S$ and $|f|$ has a maximum on S, then f is a constant function.

The next theorem gives a necessary and sufficient condition for a function to be harmonic on G_S.

Theorem 1.2.4

Let G_S be a connected induced subgraph of G with non-empty boundary ∂S. Then the function $f : \overline{S} \to R$ is harmonic on G_S if and only if for every subset S' of S, we have $\int_{\partial S'} \frac{\partial f}{\partial_w n}(z) d_{G_{S'}}(z) = 0$.

Proof: Suppose that $\int_{\partial S'} \frac{\partial f}{\partial_w n}(z) d_{G_{S'}}(z) = 0$ for every subset S' of S. For $x \in S$, let $S' = \{x\}$. Since

$$\int_{\partial S'} \frac{\partial f}{\partial_w n}(z) d_{G_{S'}}(z) = \sum_{z \in \partial S'} (f(z) - f(x)) w(z,x),$$

therefore,

$$\sum_{z \in \partial S'} (f(z) - f(x))w(z, x) = 0.$$

This implies that f is harmonic on G_S . Conversely, suppose that $\Delta_w f(x) = 0$ for all $x \in S$. Clearly f is harmonic on any subset S' of S. Applying the Green's theorem , with h identically 1 and noting that $\Delta_w f(x) = 0$ on S', gives $\int_{\partial S'} \frac{\partial f}{\partial_w n}(z) d_{G_{S'}}(z) = 0$.

1.3 Spectra of the Normalized and the Combinatorial Laplacians

In this section, we discuss the spectra of the normalized and the combinatorial Laplacians for weighted graphs. We begin by characterizing the eigenvalues of the normalized and the combinatorial Laplacians in terms of the Rayleigh quotient, which for a given matrix A and vector x is defined as

$$R(x) = \frac{\langle x,\ Ax \rangle}{\langle x,\ x \rangle},$$

where the notation $\langle\ ,\ \rangle$ is the standard inner product. In particular, the Rayleigh quotient for a matrix A and eigenvector x evaluates to the corresponding eigenvalue. Let f denote an arbitrary function defined on the vertices V of the graph G . We can view f as a column vector. Then the Rayleigh quotient for L_w is

$$\frac{\langle f,\ L_w f \rangle}{\langle f,\ f \rangle} = \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x, y)}{\sum_x f^2(x)}.$$

And the Rayleigh quotient for $\pounds_w$ and the function g defined on V is

$$\begin{aligned}\frac{\langle g,\ \pounds_w g\rangle}{\langle g,\ g\rangle} &= \frac{\left\langle g, T^{\frac{-1}{2}}\ L_w T^{\frac{-1}{2}} g\right\rangle}{\langle g,\ g\rangle} \\ &= \frac{\langle f,\ L_w f\rangle}{\left\langle T^{\frac{1}{2}} f,\ T^{\frac{1}{2}} f\right\rangle} \\ &= \frac{\sum_{x\sim y}(f(x)-f(y))^2 w(x,y)}{\sum_x f^2(x) d_G(x)},\end{aligned}$$

where $g = T^{\frac{1}{2}} f$. Let $\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{n-1}$ be the eigenvalues of the real symmetric matrix $\pounds_w$. Then, as a consequence of the Courant-Fisher theorem [39, Theorem 4.2.11], we have

$$\lambda_0 = \min_{f\neq 0} \frac{\sum_{x\sim y}(f(x)-f(y))^2 w(x,y)}{\sum_x f^2(x) d_G(x)},$$

and

$$\lambda_{n-1} = \max_{f\neq 0} \frac{\sum_{x\sim y}(f(x)-f(y))^2 w(x,y)}{\sum_x f^2(x) d_G(x)}.$$

We can easily see from the above equations that all eigenvalues of $\pounds_w$ are non-negative and 0 is the minimum eigenvalue of $\pounds_w$. Let $\mathbf{1}$ denote the constant function which has a value 1 on each vertex. Then $T^{\frac{1}{2}}\mathbf{1}$ is an eigenfunction of $\pounds_w$ that corresponds to the eigenvalue 0. Applying the Courant-Fisher theorem again, we obtain the minimum nonzero eigenvalue to be:

$$\lambda_1 = \min_{f\neq 0,\ T^{\frac{1}{2}} f \perp T^{\frac{1}{2}}\mathbf{1}} \frac{\sum_{x\sim y}(f(x)-f(y))^2 w(x,y)}{\sum_x f^2(x) d_G(x)}.$$

In general, for $k = 0, \ldots, n-1$,

$$\lambda_k = \min_{f\neq 0,\ T^{\frac{1}{2}} f \perp \varphi_0,\ldots,\varphi_{k-1}.} \frac{\sum_{x\sim y}(f(x)-f(y))^2 w(x,y)}{\sum_x f^2(x) d_G(x)}.$$

where $\varphi_0, ..., \varphi_{n-1}$ are the corresponding eigenfunctions. Applying similar argument to the eigenvalues $\sigma_0 \leq \sigma_1 \leq ... \leq \sigma_{n-1}$ of L_w, we obtain $\sigma_0 = 0$ and for $k = 0, ..., n-1$, we have:

$$\sigma_k = \min_{f \neq 0,\ f \perp \psi_0, ..., \psi_{k-1}} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x, y)}{\sum_x f^2(x)},$$

where $\psi_0, ..., \psi_{n-1}$ are the corresponding eigenfunctions. The following important and relatively simple lemma of which the non-weighted version appears in Fan Chung [18, page 7].

Lemma 1.3.1

Let $\sigma_0 \leq \sigma_1 \leq ... \leq \sigma_{n-1}$ be the eigenvalues of the combinatorial Laplacian L_w, and $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{n-1}$ the eigenvalues of $£_w$. Then

(a) $\sum_i \sigma_i = \sum_x d_G(x)$ and $\sum_i \lambda_i \leq n$, where n is the number of vertices in G.

(b) For $n \geq 2$,

$$\lambda_1 \leq \frac{n}{n-1} .$$

Also for a graph G without isolated vertices, we have

$$\lambda_{n-1} \geq \frac{n}{n-1} .$$

(c) The multiplicity of zero eigenvalue is the same as the number of connected components of G.

(d) For all $i \leq n-1$, we have $\sigma_i \leq 2d_{\max}$ and $\lambda_i \leq 2$.

Proof: (a) follows by considering the traces of L_w and $£_w$.

For (b) suppose $\lambda_1 > \dfrac{n}{n-1}$, then it can be easily seen that $\sum_i \lambda_i > n$ which contradicts (a). And if there is no isolated vertex, we easily see that $\lambda_{n-1} \geq \dfrac{n}{n-1}$.

For (c) suppose that G is connected. Let ψ be an eigenfunction of L_w for the zero eigenvalue . Then

$$L_w(\psi) = 0,$$

or

$$\langle \psi, L_w \psi \rangle \ = \sum_{x \sim y} (\psi(x) - \psi(y))^2 w(x, y) = 0.$$

This implies that ψ must be a constant function on G and the eigenspace of L_w corresponding to zero eigenvalue must be one dimensional. When G is not connected, the result follows from block diagonalizing L_w in terms of the combinatorial Laplacian of the connected components of G. For the normalized Laplacian $\pounds_w$, the same argument applies.

To show (d), we use the Rayleigh quotient characterization of σ_{n-1} and λ_{n-1}. As argued above,

$$\sigma_{n-1} = \max_{f \neq 0} \frac{\langle f \ , \ Lf \rangle}{\langle \ f \ , \ f \ \rangle} = \max_{f \neq 0} \frac{\sum_{x \sim y} (f(x) \ - \ f(y))^2 w(x, y)}{\sum_{x \in V} f^2(x)}.$$

Since

$$\sum_{x \sim y} (f(x) - f(y))^2 w(x, y) \leq \sum_{x \sim y} 2(f^2(x) + f^2(y)) w(x, y),$$

therefore,

$$\sigma_{n-1} \leq \max_{f \neq 0} \frac{2 \sum_{x \in V} f^2(x) d_G(x)}{\sum_{x \in V} f^2(x)} \leq 2d_{\max}$$

The argument for λ_{n-1} is similar. The Rayleigh quotient for λ_{n-1} is

$$\begin{aligned}
\lambda_{n-1} &= \max_{f\neq 0} \frac{\sum_{x\sim y}(f(x) - f(y))^2 w(x,y)}{\sum_{x\in V} f^2(x) d_G(x)} \\
&\leq \max_{f\neq 0} \frac{\sum_{x\sim y} 2(f^2(x) + f^2(y)) w(x,y)}{\sum_{x\in V} f^2(x) d_G(x)} \\
&\leq \max_{f\neq 0} \frac{2\sum_{x\in V} f(x)^2 d_G(x)}{\sum_{x\in V} f^2(x) d_G(x)} \leq 2. \\
& QED
\end{aligned}$$

1.4 Dirichlet and Neumann Eigenvalues

1.4.1 Dirichlet Eigenvalues

Let G_S be an induced subgraph of the graph $G = (V, E)$ with the non-empty boundary ∂S such that $\overline{S} = V$. A function $f : \overline{S} \to R$ is called a Dirichlet function if $f(x) = 0$ for all $x \in \partial S$. We wish to study those Dirichlet functions f and g that satisfy

$$\begin{aligned}
L_w f(x) &= \sigma f(x) \qquad \text{for all } x \in S, \\
£_w g(x) &= \lambda g(x) \qquad \text{for all } x \in S.
\end{aligned}$$

In this case, f is a Dirichlet eigenfunction of L_w corresponding to a Dirichlet eigenvalue σ, and similarly g is a Dirichlet eigenfunction of $£_w$ corresponding to a Dirichlet eigenvalue λ. Viewing L_w as a matrix, we now define the Dirichlet Laplacian $L_{w,S}$ to be L_w restricted to the rows and columns of S. We define the Dirichlet normalized Laplacian and the Dirichlet discrete Laplacian similarly. The following lemma generalizes the Lemma 8.2 of [18] to graphs with arbitrary weights.

Lemma 1.4.1

A Dirichlet function $f : \overline{S} \to R$ is a Dirichlet eigenfunction if and only if $f|_S$ is an eigenfunction of $L_{w,S}$. Similarly, a Dirichlet function $g : \overline{S} \to R$ is a Dirichlet eigenfunction if and only if $g|_S$ is an eigenfunction of $£_{w,S}$. The eigenvalue σ of $L_{w,S}$ corresponding to f is given by

$$\sigma = \frac{\sum_{x\sim y}(f(x) - f(y))^2 w(x,y)}{\sum_{x\in V}(f(x))^2},$$

and the eigenvalue λ of $£_{w,S}$ corresponding to g is given by

$$\lambda = \frac{\sum_{x\sim y}(f(x) - f(y))^2 w(x,y)}{\sum_{x\in V}(f(x))^2 d_G(x)},$$

where $g = T^{1/2} f$.

Proof: Let f be a Dirichlet function on $\overline{S}$. Then for all $x \in S$,

$$\begin{aligned} L_{w,S} f|_S(x) &= f|_S(x) d_G(x) - \sum_{y\in S, x\sim y} f|_S(y) w(x,y) \\ &= f(x) d_G(x) - \sum_{y\in V, x\sim y} f(y) w(x,y) = L_w f(x). \end{aligned}$$

Therefore, for all $x \in S$ and f a Dirichlet function on $\overline{S}$, $L_w f(x) = \sigma f(x)$ if and only if $L_{w,S} f|_S(x) = \sigma f|_S(x)$. The Rayleigh quotient of $L_{w,S}$ for an eigenfunction f of $L_{w,s}$

shows that

$$
\begin{aligned}
\sigma &= \frac{\langle\, f\,|_S\ ,\ L_{w,s} f\,|_S \rangle}{\langle\, f\,|_S,\ f\,|_S \rangle} \\
&= \frac{\sum_{x\in S} f\,|_S\,(x)\left(f\,|_S\,(x) d_G(x) - \sum_{y\in S, x\sim y} f\,|_S\,(y) w(x,y)\right)}{\sum_{x\in S} f\,|_S^2\,(x)} \\
&= \frac{\sum_{x\sim y,\ x\in S,\ y\in \overline{S}} (f(x)\ -\ f(y))^2 w(x,y)}{\sum_{x\in V} f^2(x)} \\
&= \frac{\sum_{x\sim y} (f(x)\ -\ f(y))^2 w(x,y)}{\sum_{x\in V} (f(x))^2}.
\end{aligned}
$$

The proof for the normalized Laplacian proceeds analogously. Let g be a Dirichlet function on $\overline{S}$. Then for all x in S,

$$
\begin{aligned}
\pounds_{w,S}\, g\,|_S\,(x) &= \sum_{y\in S,\ x\sim y} \left(\frac{g\,|_S\,(x)}{\sqrt{d_G(x)}} - \frac{g\,|_S\,(y)}{\sqrt{d_G(y)}} \right) \frac{w(x,y)}{\sqrt{d_G(x)}} \\
&= g(x) - \sum_{y\in \overline{S},\ x\sim y} \frac{g(y)}{\sqrt{d_G(y) d_G(x)}} w(x,y) = \pounds_w g(x).
\end{aligned}
$$

Thus for all $x \in S$, $\pounds_{w,S}\, g\,|_S\,(x) = \lambda g\,|_S\,(x)$ if and only if $\pounds_w g(x) = \lambda g(x)$. The Rayleigh quotient for $\pounds_{w,S}$ shows that λ satisfies

$$
\begin{aligned}
\lambda &= \frac{\langle\, g\,|_S\ ,\ \pounds_{w,S}\, g\,|_S \rangle}{\langle\, g\,|_S\ ,\ g\,|_S \rangle} \\
&= \frac{\sum_{x\in S} g\,|_S\,(x)\left(g\,|_S\,(x) - \sum_{y\in S, x\sim y} \frac{g\,|_S\,(y)}{\sqrt{d_G(y) d_G(x)}} w(x,y)\right)}{\sum_{x\in S} g\,|_S{}^2\,(x)} \\
&= \frac{\sum_{x\sim y} (f(x)\ -\ f(y))^2 w(x,y)}{\sum_{x\in V} (f(x))^2 d_G(x)},
\end{aligned}
$$

where $g = T^{1/2} f$.

QED

The following example shows the difference between the Dirichlet eigenfunctions (eigenvectors) and the eigenvectors of the Laplacians.

Example 1

Consider the complete graph K_3 on the vertices $\{1, 2, 3\}$. Let $S = \{1, 2\}$ and $\partial S = \{3\}$. Then the Laplacians and the Dirichlet Laplacians are

$$L_w(x, y) = \begin{cases} 2 & \text{if} \quad x = y \\ -1 & \text{if} \quad x \neq y \end{cases} \qquad \text{and} \ \ \pounds_w = \frac{L_w}{2}$$

$$L_{w,s}(x, y) = \begin{cases} 2 & \text{if} \quad x = y \\ -1 & \text{if} \quad x \neq y \end{cases} \qquad \text{and} \ \ \pounds_{w,s} = \frac{L_{w,s}}{2}$$

L_w has eigenvectors $(1, 1, 1)$, $(1, -1, 0)$, $(0, 1, -1)$ with eigenvalues 0, 3, 3 respectively. However, only $(1, -1, 0)$ is a Dirichlet eigenvector for L_w with respect to S. The other Dirichlet eigenvector is $(1, 1, 0)$ with Dirichlet eigenvalue 1, since $L_{w,s}(1, 1) = (1, 1)$. But $(1, 1, 0)$ is not an eigenvector of L_w. Moreover, since $\pounds_w = \frac{L_w}{2}$ and $\pounds_{w,s} = \frac{L_{w,s}}{2}$, $\pounds_w$ has the same eigenvectors and Dirichlet eigenvectors with the corresponding eigenvalues and Dirichlet eigenvalues multiplied by $\frac{1}{2}$.

Letting $n = |S|$, we label the Dirichlet eigenvalues of L_w (or the eigenvalues of the Dirichlet combinatorial Laplacian $L_{w,s}$) by $\sigma_1^S \leq \sigma_2^S \leq \ldots \leq \sigma_n^S$. Similarly, we label the Dirichlet eigenvalues of $\pounds_w$ (or the eigenvalues of the Dirichlet normalized Laplacian $\pounds_{w,s}$) by $\lambda_1^S \leq \lambda_2^S \leq \ldots\lambda_n^S$. The following lemma describes the structure of the Dirichlet eigensystem of combinatorial and normalized Laplacian.

Lemma 1.4.2

The following holds for the Dirichlet eigensystem of the combinatorial and the corresponding eigensystem for the normalized Laplacian.

(a) $\sum_i \sigma_i^S = \sum_x d_G(x)$, and $\sum_i \lambda_i^S \leq n$, where $n = |S|$.

(b) $L_{w,S}$ and $£_{w,S}$ are positive semi-definite, therefore σ_i^S and λ_i^S are real and nonnegative. Furthermore, σ_1^S and $\lambda_1^S > 0$ if and only if every connected component of G has a vertex adjacent to a vertex in ∂S.

(c) For all $1 \leq i \leq n$, we have $\sigma_i^S \leq 2d_{\max}$ and $\lambda_i^S \leq 2$.

Proof: (a) follows from considering the traces of $L_{w,s}$ and $£_{w,s}$. For (b), according to the Lemma 1.4.1, σ_i^S and λ_i^S are expressed in terms of Rayleigh quotient of $L_{w,S}$ and $£_{w,S}$ respectively, which are all nonnegative. Furthermore, we easily see that $\sigma_1^S > 0$, since there does not exist a nonzero Dirichlet function with $f(x) = f(y)$ for all $\{x, y\} \subset E$, if and only if every connected component of G has at least one of its vertices adjacent to a vertex in the boundary ∂S. The proof of $\lambda_1^S > 0$ is similar. The proof of part (c) is similar to the proof of Lemma 1.3.1.

QED

1.4.2 Neumann Eigenvalues

Now, we consider the Laplacian $£_w$ act on functions $f : \overline{S} \to R$ such that the value of the function at a boundary vertex is the weighted average of the values of the function at adjacent vertices in S. In other words, f satisfies the following Neumann boundary condition:

$$\sum_{y \in S} (f(x) - f(y)) w(x, y) = 0, \text{ for all } x \in \partial S.$$

We are interested in studying those functions that satisfy Neumann conditions and

$$\pounds_w(f(x)) = \lambda f(x).$$

To do this, we will use the same idea as used for the description of the Dirichlet eigenvalues. We define the following matrix D_w with rows indexed by vertices in $\overline{S}$ and columns indexed by vertices in S,

$$D_w(x, y) = \begin{cases} 1 & \text{if } x = y \\ \dfrac{w(x, y)}{d_{G_S}(x)} & \text{if } x \in \partial S,\ y \in S \text{ and } x \sim y \\ 0 & \text{otherwise.} \end{cases}$$

Using the matrix D_w, we define the combinatorial Neumann Laplacian as

$$N_{w,S} = D_w^T L_w D_w\ .$$

Similarly, we define the normalized Neumann Laplacian as

$$\check{N}_{w,S} = T^{\frac{-1}{2}} D_w^T L_w D_w T^{\frac{-1}{2}}.$$

The action of the normalized Neumann Laplacian $\check{N}_{w,S}$ on the space of functions f on S is the same as the action of $\pounds_w$ on the space of functions f on $\overline{S}$ that satisfies the Neumann condition, that is,

$$\check{N}_{w,S}(f(x)) = \pounds_w(f(x)) \text{ for } x \in S\,.$$

Therefore the eigenvalues of $\check{N}_{w,S}$ and the corresponding eigenfunctions, f satisfy the following relation

$$\pounds_w(f(x)) = \lambda_{S,i} f(x),$$

where we define the values of f on ∂S in such a way that f would satisfy the Neumann condition on $\overline{S}$. Now, applying the Courant-Fisher theorem to the real symmetric matrix $\check{N}_{w,S}$, we see that

$$\begin{aligned}
\lambda_{S,1} &= \min_{g\,\perp T^{\frac{1}{2}}1} \frac{\langle g,\ \check{N}_{w,S} g\rangle}{\langle\, g\,,\, g\rangle} \\
&= \min_{g\,\perp T^{\frac{1}{2}}1} \frac{\sum_{x\in S} g(x)\check{N}_{w,S}(g(x))}{\sum_{x\in S} g^2(x)} \\
&= \min_{g\,\perp T^{\frac{1}{2}}1} \frac{\sum_{x\in S} g(x)\pounds_w(g(x))}{\sum_{x\in S} g^2(x)} \\
&= \min_{f\,:\,\sum f(x)d_G(x)=0} \frac{\sum_{x\in S} f(x)L_w(f(x))}{\sum_{x\in S} f^2(x)d_G(x)},
\end{aligned}$$

where $g = T^{\frac{1}{2}} f$. From the following relation

$$\sum_{x\in S} f(x)L_w(f(x)) = \sum_{\{x,y\}\in S'} (f(x) - f(y))^2 w(x,y),$$

where S' denotes the union of edges in S and the boundary edges (the edge with one endpoint in S and the other endpoint not in S), we define the first Neumann eigenvalue of an induced subgraph as follows

$$\lambda_{S,1}^N = \min_{\substack{f \\ \sum f(x)d_G(x)=0}} \frac{\sum_{\{x,y\}\in S'}(f(x)-f(y))^2 w(x,y)}{\sum_{x\in S} f^2(x)d_G(x)}, \quad (1)$$

where minimum is taken over all f that satisfies the Neumann condition. In general, we define the i-th Neumann eigenvalue $\lambda_{S,i}^N$ to be

$$\lambda_{S,i}^N = \min_{f\neq 0,\ f\ \perp\ \psi_0,\ldots,\psi_{i-1}} \frac{\sum_{\{x,y\}\in S'}(f(x)-f(y))^2 w(x,y)}{\sum_{x\in S} f^2(x)d_G(x)},$$

where ψ_k is an eigenfunction corresponding to $\lambda_{S,k}$. Clearly $\lambda_{S,0}^N = 0$. From the following lemma whose proof is simple and appears in [18], we easily see that the Neumann eigenvalue is the eigenvalue of the normalized Neumann matrix.

Lemma 1.4.3

Let $f : \overline{S} \to R$ be a function that satisfies (1). Then f satisfies,

a) For $x \in S$,

$$L_w f(x) = \sum_{\substack{y \\ \{x,\,y\}\in S'}} (f(x)-f(y))w(x,y) = \lambda_{S,1}^N f(x)d_G(x).$$

(b) For $x \in \partial S$,

$$L_w f(x) = 0.$$

(c) For any function $h : \overline{S} \to R$, we have

$$\sum_{x\in S} h(x)L_w f(x) = \sum_{\{x,\,y\}\in S'} (h(x)-h(y))(f(x)-f(y))w(x,y).$$

We close this section by presenting a special type of random walk that is related to Neumann matrix and call it the Neumann random walk. For an induced subgraph G_S with non-empty boundary ∂S, let the probability of moving from a vertex x in S to a neighbor y of x be $\frac{w(x,y)}{d_G(x)}$ if y is in S. If y is in ∂S, we then move from x to each neighbor z of y in S with the (additional) probability $\frac{w(x,y)w(y,\ z)}{d_G(x)d_{G_S}(y)}$. From $\sum_{\substack{z\sim y\\ z\in S}} w(z,y) = d_{G_S}(y)$, it implies that $\sum_{\substack{z\sim y\\ z\in S}} \frac{w(y,\ z)}{d_{G_S}(y)} = 1$. Hence $\sum_{\substack{z\sim y\\ z\in S}} \frac{w(x,y)w(y,\ z)}{d_G(x)d_{G_S}(y)} = \frac{w(x,y)}{d_G(x)}$ for y in ∂S. Using the relation $\sum_y w(x,y) = d_G(x)$, it follows that the probabilities of moving from a vertex x in S to the neighboring vertices or neighboring to the boundary vertices adjacent to x will add up to 1. The Neumann random walk is a little different from the random walk often used in which the probability of staying in x which has neighboring vertices in ∂S is $\frac{\sum_{y\in\partial S} w(x,y)}{d_G(x)}$. In other words the walker in the Neumann random walk, takes advantage of reflecting from the boundary imposed by the Neumann boundary condition. The transition matrix P_w for this walk is as follows:

$$P_w = T^{\frac{-1}{2}}(I - \check{N}_{w,S})T^{\frac{1}{2}},$$

The eigenvalues ρ_i of the transitional matrix P_w associated with the Neumann walk are closely related to the Neumann eigenvalues λ_i^S as follows [18] :

$$\rho_i \leq 1 - \ \lambda_i^S \ .$$

1.5 The Diameter of a Weighted Graph

In a connected graph G, the distance between two adjacent vertices x and y, is defined to be $\frac{1}{w(x, y)}$, where $w(x, y)$ is the weight of an edge connecting x to y. And the length of the path, P that connects a vertex x_0 to a vertex x_n through a sequence of vertices $\{x_i\}_{i=0}^{n}$ is defined to be $\sum_{i=0}^{n-1} \frac{1}{w(x_i, x_{i+1})}$. The distance between two vertices x and y, denoted by $dist(x, y)$, is defined to be the minimum over the length all possible paths connecting x and y. The diameter of G, denoted by D, is the maximum over the distance of all pairs of vertices in G. When graphs are used as models for communication networks, the diameter corresponds to delays in passing messages through the network, hence it plays an important role in performance analysis and cost optimization. In this section, we present a relationship between the diameter of the graph and the smallest positive eigenvalues of its Laplacians L_w and £$_w$. The following lemma generalizes the result of Chung [18] to graphs with arbitrary weights.

Lemma 1.5.1

Let G be a connected weighted graph with the weight function w. Let S be a connected induced subgraph of $G = (V, E)$. Then the eigenvalues σ_1 , λ_1 are related to the diameter D as follows

(a) $\lambda_1 \geq \frac{1}{D \sum_{x \in V} d_G(x)}$

(b) $\sigma_1 \geq \frac{1}{D \mid V \mid}$

Proof: For (a) let f be an eigenfunction achieving λ_1. Let x_0 denote a vertex with $\mid f(x_0) \mid = \max_{x \in V} \mid f(x) \mid$. Since f is orthogonal to the function $T\mathbf{1}$ (the eigenfunction of zero eigenvalue), we have $\sum_{x \in V} f(x) d_G(x) = 0$. Therefore, there exists a vertex x_k

such that $f(x_0)f(x_k) < 0$. Let P denote a shortest path in G joining x_0 to x_k with vertices $x_0, x_1, ..., x_k$. Then, from the Rayleigh quotient expression for λ_1, we have:

$$
\begin{aligned}
\lambda_1 &= \frac{\sum_{x\sim y}(f(x) - f(y))^2 w(x,y)}{\sum_{x\in V} f(x)^2 d_G(x)} \\
&\geq \frac{\sum_{i=1}^{k}(f(x_{i-1}) - f(x_i))^2 w(x_{i-1}, x_i))}{f(x_0)^2 \sum_{x\in V} d_G(x)} \\
&= \frac{\sum_{i=1}^{k}(f(x_{i-1}) - f(x_i))^2 w(x_{i-1}, x_i)\frac{\sum_{i=1}^{k}\left(\frac{1}{\sqrt{w(x_{i-1}, x_i)}}\right)^2}{\sum_{i=1}^{k}\left(\frac{1}{\sqrt{w(x_{i-1}, x_i)}}\right)^2}}{f(x_0)^2 \sum_{x\in V} d_G(x)} \\
&\geq \frac{\frac{1}{dist(x_0, x_k)}(\sum_{i=1}^{k}(f(x_{i-1}) - f(x_i))^2}{f(x_0)^2 \sum_{x\in V} d_G(x)}, \text{ by Cauchy-Schwarz} \\
&\geq \frac{\frac{1}{D}(f(x_0) - f(x_k))^2}{f(x_0)^2 \sum_{x\in V} d_G(x)} \\
&\geq \frac{1}{D\sum_{x\in V} d_G(x)}
\end{aligned}
$$

(b) follows an almost similar argument as that above. By Rayleigh quotient expression for σ_1, we have:

$$
\begin{aligned}
\sigma_1 &= \frac{\sum_{x\sim y}(f(x) - f(y))^2 w(x,y)}{\sum_{x\in V} f(x)^2} \\
&\geq \frac{\sum_{i=1}^{k}(f(x_{i-1}) - f(x_i))^2 w(x_{i-1}, x_i))}{f(x_0)^2 \mid V \mid} \\
&\geq \frac{\frac{1}{dist(x_0, x_k)}(\sum_{i=1}^{k}(f(x_{i-1}) - f(x_i))^2}{f(x_0)^2 \mid V \mid} \quad \text{by Cauchy-Schwarz} \\
&\geq \frac{\frac{1}{D}(f(x_0) - f(x_k))^2}{f(x_0)^2 \mid V \mid} \\
&\geq \frac{1}{D \mid V \mid}
\end{aligned}
$$

QED.

There are other interesting bounds are in the literature. Let M denote an $n \times n$ matrix with rows and columns indexed by the vertices of G such that $M(x, y) = 0$ if x and y are not adjacent. Furthermore, suppose there exists a polynomial $p_t(x)$ of degree t such that $p_t(M)(x, y) \neq 0$ for all x, y in V then we conclude that the diameter D satisfies $D \leq t$ [18]. In particular, if we take M to be sum of identity matrix and adjacency matrix of G, (i.e, $A(x, y) = 1$ if x is adjacent to y, and 0 otherwise) and the polynomial $p_t(x) = (1+x)^t$, then the following inequality for the diameter of regular graphs which are not complete graphs can be derived,

$$D \leq \left\lceil \frac{\log(\mid G \mid -1)}{\log(\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1})} \right\rceil \quad .$$

The above bound can be further improved by choosing p_t to be the Chebyshev polynomial of degree t. In this case, the logarithmic function is replaced by $cosh^{-1}$. According to [20], we have

$$D \leq \left\lceil \frac{cosh^{-1}(\mid G \mid -1)}{cosh^{-1}(\frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1})} \right\rceil \; .$$

Chapter II

2 The Discrete Inverse Conductivity Problem

Introduction

A network (weighted graph) represents a way of interconnecting any pair of users (nodes) by means of some links (edges). Thus, it is quite natural that its structure can be represented in a simplified form. When we have some problem on a part of network or try detecting such a problem, it is desirable to recognize this problem from measurements made on the boundary of the network since the network may be very large and its inner structure too complicated.

Problems involving graph identification have been previously studied by a number of researchers in the domain of realization of graphs with given distances [24], and on the reconstruction of graphs from vertex deleted problems [22, 31].

The most recent method introduced by Berenstein and Chung [6] is based on an analogy to the continuous version of the inverse conductivity problem. The inverse conductivity problem's original aim was to identify the conductivity coefficient in continuous media from boundary measurements, such as Dirichlet data, Neumann data. The discrete version of the inverse problem is to identify the connectivity of the nodes and the conductivity of the edges between each adjacent pairs of nodes from boundary measurements. In this chapter, we will provide an improved version of the following global uniqueness result due to Berenstein and Chung [6] for the inverse conductivity problem in a network satisfying the monotonicity condition:

Theorem:

Let w_1 and w_2 be weights with $w_1 \le w_2$ on $S \cup \partial S \times S \cup \partial S$ and f_i be functions satisfying for each $i = 1, 2$.

$$\begin{cases} \Delta_{w_i} f_i(x) = 0, & \text{for } x \in S \\ \dfrac{\partial f_i}{\partial_{w_i} n}(z) = \psi(z), & \text{for } z \in \partial S \\ \int_S f_i(x) d_{G_{w_i}}(x) = K \end{cases}$$

for a given function $\psi : \partial S \to R$ such that $\int \psi = 0$ and for suitably chosen $K > 0$. Furthermore, if we assume that $w_1(z, y) = w_2(z, y)$ on $\partial S \times \partial^0 S$ and $f_1 |_{\partial S} = f_2 |_{\partial S}$. Then we have:

$$f_1 = f_2 \text{ on } S,$$

and

$$w_1 = w_2 \text{ on } S \times S .$$

We organize this chapter as follows: We first discuss the Discrete Green's function in Section 1. In Section 2, we provide the solution of the Dirichlet and the Neumann value problems by means of the discrete Green's function. And finally in the last section, we arrive at the global uniqueness result discussed above and will give an improved version of this result under a weaker condition.

2.1 Discrete Green's Function

Discrete Green's functions are the inverse or pseudo-inverse of combinatorial Laplacians. The Green's function in the continuous case has been extensively used in solving differential equations [1, 3]. A treatment of Green's functions for partial differential equation can be found in [58]. The first major work on the discrete Green's functions as an inverse of the combinatorial Laplacians is [23]. Just as the Green's function in the continuous case depends on the domain and the boundary conditions, the discrete Green's functions are associated with underlying graph and boundary conditions.

We consider a weighted connected graph $G = (V, E)$. The Green's function for the case where the graph G has no boundary can be determined by brute force pseudo-inversion of the corresponding Laplacian. The following theorem explains this in greater detail.

Theorem 2.1.1

Let $f : V \to R$ be a function. Then the equation

$$\Delta_w f(x) = g(x), \text{ for } x \in V,$$

has a solution if and only if $\sum_{x\in V} g(x)d_G(x) = 0$. In this case, the solution is given by

$$f(x) = a_0 + \sum_{y\in V} Z_w(x, y)g(y),$$

where a_0 is an arbitrary constant and the matrix Z_w is given by

$$Z_w(x, y) = \sum_{i=1}^{n-1} \frac{1}{\lambda_i}\phi_i(x)\phi_i(y)\sqrt{\frac{d_G(y)}{d_G(x)}}.$$

Here the λ_i are the eigenvalues of $\pounds_w$ with the corresponding eigenfunctions ϕ_i for $i = 1, 2, ..., n-1$.

Proof: Suppose $\sum_{x \in V} g(x) d_G(x) = 0$. Then by considering $\phi_0(x) = \sqrt{\frac{d_G(x)}{\sum_x d_G(x)}}$ as an eigenfunction of $\pounds_w$ corresponding to the zero eigenvalue, we have

$$\begin{aligned}\langle T^{\frac{1}{2}} g, \phi_0 \rangle &= \sum_x \sqrt{d_G(x)} g(x) \sqrt{\frac{d_G(x)}{\sum_x d_G(x)}} \\ &= \sqrt{\frac{1}{\sum_x d_G(x)}} \sum_{x \in V} g(x) d_G(x) \ = 0.\end{aligned}$$

Now, consider the orthogonal expansion of $T^{\frac{1}{2}} f$,

$$(T^{\frac{1}{2}} f)(x) = \sum_{i=0}^{n-1} a_i \phi_i(x) \ ,$$

where $a_i = \langle T^{\frac{1}{2}} f \ , \phi_i \rangle$. Then since $\Delta_w = T^{-1/2} \pounds_w T^{1/2}$, and

$$\begin{aligned}\lambda_i a_i &= \lambda_i \langle T^{\frac{1}{2}} f, \phi_i \rangle \\ &= \langle T^{\frac{1}{2}} f, \lambda_i \phi_i \rangle \\ &= \langle T^{\frac{1}{2}} f, \pounds_w \phi_i \rangle \\ &= \langle \pounds_w T^{\frac{1}{2}} f, \phi_i \rangle \\ &= \langle T^{\frac{1}{2}} g, \phi_i \rangle \ .\end{aligned}$$

We, therefore, have:

$$a_i = \frac{1}{\lambda_i} \sum_x \sqrt{d_G(x)} g(x) \phi_i(x), \quad \text{for } i = 1, 2, ...n-1.$$

For $i = 0$, since

$$\begin{aligned}
\lambda_0 a_0 &= \lambda_0 \langle T^{\frac{1}{2}} f, \phi_0 \rangle \\
&= \langle T^{\frac{1}{2}} f, \lambda_0 \phi_0 \rangle \\
&= \langle T^{\frac{1}{2}} f, \pounds_w \phi_0 \rangle \\
&= \langle \pounds_w T^{\frac{1}{2}} f, \phi_0 \rangle \\
&= \langle T^{\frac{1}{2}} g, \phi_0 \rangle \\
&= 0.
\end{aligned}$$

and $\lambda_0 = 0$, we have a_0 that could be an arbitrary constant. From the orthogonal expansion of $T^{\frac{1}{2}} f$, we get:

$$\sqrt{d_G(x)} f(x) = a_0 \sqrt{\frac{d_G(x)}{\sum_x d_G(x)}} + \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \left[\sum_{y \in V} \sqrt{d_G(y)} g(y) \phi_i(y) \right] \phi_i(x),$$

or

$$\begin{aligned}
f(x) &= a_0 + \sum_{y \in V} \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \phi_i(x) \phi_i(y) \sqrt{\frac{d_G(y)}{d_G(x)}} g(y) \\
&= a_0 + \sum_{y \in V} Z_w(x, y) g(y).
\end{aligned}$$

Conversely, suppose that $\Delta_w f(x) = g(x)$ has a solution. Then $\sum_{x \in V} g(x) d_G(x) = 0$ because of Green's theorem (Theorem 1.1.3).

QED

The following corollary is a simple consequence of the above theorem.

Corollary 2.1.2

Under the same conditions as in Theorem 2.1.1, let G_S be an induced subgraph of G. Then every solution to

$$\Delta_w f(x) = 0 \text{ for all } x \in V \backslash S,$$

has the following form:

$$f(x) = a_0 + \sum_{y \in S} Z_w(x, y)\eta(y), \text{ for } x \in V,$$

where a_0 is an arbitrary constant and $\eta(y) = \Delta_w f(y)$, for $y \in S$.

The proof of Theorem 2.1.1 suggests the following definition of Green's function for graphs with no boundary in a closed form. We define G_w, $\mathbf{G}_w$, Z_w as Green's functions for L_w,$£_w$, and Δ_w respectively, as follows:

$$\begin{aligned} G_w &= \sum_{\sigma_i > 0} \frac{1}{\sigma_i} \psi_i \psi_i^T , \\ \mathbf{G}_w &= \sum_{\lambda_i > 0} \frac{1}{\lambda_i} \phi_i \phi_i^T , \\ Z_w &= T^{\frac{-1}{2}} \mathbf{G}_w T^{\frac{1}{2}} . \end{aligned}$$

The above definitions of G_w and $\mathbf{G}_w$ are equivalent to the following relations:

$$\begin{aligned} G_w L_w &= L_w G_w = I - \psi_0 \psi_0^T \text{ and } G_w \psi_0 \psi_0^T = 0, \\ \mathbf{G}_w £_w &= £_w \mathbf{G}_w = I - \phi_0 \phi_0^T \text{ and } \mathbf{G}_w \phi_0 \phi_0^T = 0, \end{aligned}$$

where

$$\phi_0\phi_0^T(x,y) = \frac{\sqrt{d_G(x)d_G(y)}}{\sum_{x\in G} d_G(x)},$$

and

$$\psi_0\psi_0^T(x,y) = \frac{1}{\mid G \mid}.$$

Also

$$\begin{aligned} Z_w\Delta_w &= (T^{\frac{-1}{2}}\,\mathbf{G}_w T^{\frac{1}{2}})(T^{\frac{-1}{2}}\,\pounds_w T^{\frac{1}{2}}) \\ &= T^{\frac{-1}{2}}(I - \phi_0\phi_0^T)T^{\frac{1}{2}} \\ &= I - \frac{D}{\sum_{x\in G} d_G(x)} \\ \text{and } Z_w D &= 0, \end{aligned}$$

where $D(x,y) = d_G(y)$.

In the case where the graph has a boundary, it is easier to define its Green's function. Let G_S be a subgraph of a connected graph G with non-empty boundary ∂S. Then $L_{w,S}$, $\pounds_{w,S}$, and $\Delta_{w,S}$ are invertible and the combinatorial Green's function $G_{w,S}$, the normalized Green's function $\mathbf{G}_{w,S}$, and the discrete Green's function $Z_{w,S}$ are defined as follows:

$$\begin{aligned} L_{w,S}\,G_{w,S} &= G_{w,S}\,L_{w,S} = I_S, \\ \pounds_{w,S}\mathbf{G}_{w,S} &= \mathbf{G}_{w,S}\pounds_{w,S} = I_S, \\ \Delta_{w,S}Z_{w,S} &= Z_{w,S}\Delta_{w,S} = I_S\,. \end{aligned}$$

Using the diagonal matrix T, as before, we have the following relations:

$$
\begin{aligned}
G_{w,S} &= T^{\frac{-1}{2}}\,\mathbf{G}_{w,S}T^{\frac{-1}{2}} = Z_{w,S}T^{-1}, \\
\mathbf{G}_{w,S} &= T^{\frac{1}{2}}G_{w,S} \;=\; T^{\frac{1}{2}}Z_{w,S}T^{\frac{-1}{2}}, \\
Z_{w,S} &= G_{w,S}T = T^{\frac{-1}{2}}\mathbf{G}_{w,S}T^{\frac{1}{2}}.
\end{aligned}
$$

There is a random walk interpretation of the discrete Green's function $Z_{w,S}$ corresponding to the Dirichlet Laplace operator $\Delta_{w,S}$ which is as follows. Let $P = \left[p_{xy}\right]$ be the transition probability matrix for the weighted transient random walk on S with absorbing states ∂S, where the probability p_{xy} of moving to state y from x is $\dfrac{w(x,y)}{d_G(x)}$. Then $\Delta_{w,S} = I - P$ and from the fact that $(I-P)^{-1} = I + P + P^2 + ...$, it follows

$$Z_{w,S} = I + P + P^2 + ...$$

where P^n is the n-step transition probability matrix .

We have studied several explicit formulas for the Green's function. We now consider a direct method for evaluating the Green's function for a path and a cycle [23]. For simplicity, we take the graph to be a standard graph, i.e., $w(x,y) = 1$ if and only if $x \sim y$.

Example 1: Green's function for a path

Theorem 2.1.4

Let the vertex set P_n be denoted by $S = \{1, 2, ..., n\}$ with boundary $\partial S = \{0, n+1\}$.

Then its Green's function satisfies:

$$\mathbf{G}_{w,S}(x, y) = \frac{2}{n+1} x(n + 1 - y),$$

for $1 \leq x \leq y \leq n$.

Proof: Since $\Delta_{w,S} = £_{w,S} = L_{w,S}/2$, we have $£_{w,S}\ \mathbf{G}_{w,S} = I$, and $\mathbf{G}_{w,S}£_{w,S} = I$. Here we assume that $1 \leq x < y \leq n$. From $£_{w,S}\ \mathbf{G}_{w,S} = I$, it follows that

$$\frac{1}{2}(2\mathbf{G}_{w,S}(x, y) - \mathbf{G}_{w,S}(x - 1, y) - \mathbf{G}_{w,S}(x + 1, y)) = 0.$$

From $\mathbf{G}_{w,S}£_{w,S} = I$, we have

$$\frac{1}{2}(2\mathbf{G}_{w,S}(x, y) - \mathbf{G}_{w,S}(x, y - 1) - \mathbf{G}_{w,S}(x, y + 1)) = 0.$$

with the condition that $\mathbf{G}_{w,S}(x, y) = 0$ if either x or y are the boundary point. Therefore, we have

$$\begin{aligned}
\mathbf{G}_{w,S}(x, y) - \mathbf{G}_{w,S}(x - 1, y) &= \mathbf{G}_{w,S}(x - 1, y) - \mathbf{G}_{w,S}(x - 2, y) \\
&= \mathbf{G}_{w,S}(x - 2, y) - \mathbf{G}_{w,S}(x - 3, y) \\
&= \ldots.. \\
&= \mathbf{G}_{w,S}(1, y).
\end{aligned}$$

This implies that

$$\mathbf{G}_{w,S}(x, y) = x\mathbf{G}_{w,S}(1, y). \qquad (1)$$

Using similar argument, we have

$$\mathbf{G}_{w,S}(1,y) = c(n+1-y), \qquad (2)$$

for some constant c. Now, we use the fact that

$$\frac{1}{2}(2\mathbf{G}_{w,S}(x,x) - \mathbf{G}_{w,S}(x-1,x) - \mathbf{G}_{w,S}(x+1,x)) = 1$$

to get $c = \dfrac{2}{n+1}$ and $\mathbf{G}_{w,S}(x,x) = cx(n+1-x)$. From (1) and (2), the result follows.

QED

Example 2 : Green's function for a cycle

Let the vertex set of the cycle C_n be denoted by $\{1, 2, ..., n\}$. Then the Laplacians are related by $\Delta_w = £_w = L_w/2$. Knowing the fact that a cycle is a boundary-less graph, the Green's function $\mathbf{G}_w$ is determined by the following relationship which was studied before,

$$\begin{aligned} \mathbf{G}_w £_w &= £_w \mathbf{G}_w = I - \frac{J}{n}, \\ \text{and } \mathbf{G}_w J &= 0, \end{aligned}$$

where J is the $n \times n$ matrix with all entries 1. Before stating the theorem regarding the Green's function for a cycle, we note the following facts regarding cycles. Since a cycle is invariant under translations, then the values $£_w(x,y)$ and $\mathbf{G}_w(x,y)$ depend only on the distance $| \ x - y \ |$ between x and y. Secondly, the distance between x and y on the cycle can be measured by travelling in either direction. So, we define

$$\mathbf{G}_w(|x-y|) = \mathbf{G}_w(n-|x-y|).$$

The following theorem describes the Green's function for the cycle.

Theorem 2.1.5

Let $n \geq 3$. Then the cycle C_n's normalized Green's function has the following form

$$\mathbf{G}_w(x,y) = \frac{(y-x)^2}{n} - |x-y| + \frac{(n+1)(n-1)}{6n}.$$

Proof: From $\mathbf{G}_w \pounds_w = \pounds_w \mathbf{G}_w = I - \frac{J}{n}$ and $\mathbf{G}_w J = 0$, we have the recurrence:

$$2\mathbf{G}_w(x,y) - \mathbf{G}_w(x,y-1) - \mathbf{G}_w(x,y+1) = \begin{cases} 2 - \frac{2}{n} & \text{if } x = y \\ \frac{-2}{n} & \text{if } x \neq y \end{cases}$$

or

$$2\mathbf{G}_w(z) - \mathbf{G}_w(z-1) - \mathbf{G}_w(z+1) = \begin{cases} 2 - \frac{2}{n} & \text{if } z = 0 \\ \frac{-2}{n} & \text{if } z > 0 \end{cases}$$

where $z = |x-y|$. The condition $\mathbf{G}_w J = 0$ implies that the sum of $\mathbf{G}_w$ across any row must be zero, i.e.,

$$\sum_{z=0}^{n-1} \mathbf{G}_w(z) = 0.$$

By considering the difference equation:

$$(\mathbf{G}_w(z+1) - \mathbf{G}_w(z)) - (\mathbf{G}_w(z) - \mathbf{G}_w(z-1) = \frac{2}{n} \text{ for } z > 0,$$

we conclude that $\mathbf{G}_w(z)$ is quadratic in z which we write as,

$$\mathbf{G}_w(z) = \frac{z^2}{n} + Bz + C.$$

Knowing the fact that $\mathbf{G}_w(z) = \mathbf{G}_w(n-z)$, we obtain $B = -1$. Now applying the condition:

$$\sum_{z=0}^{n-1} \mathbf{G}_w(z) = 0,$$

or

$$\sum_{z=0}^{n-1} (z - \frac{z^2}{n}) = nC.$$

Therefore,

$$C = \frac{(n+1)(n-1)}{6n}.$$

Hence, we obtain the desired result.

QED

2.2 Dirichlet and Neumann boundary Value Problems

In this section, we are interested in solving the equation $\Delta_w f(x) = g(x)$ for $x \in S$, such that f satisfies Dirichlet or Neumann boundary conditions on the boundary ∂S. The Dirichlet boundary value problem was solved by Chung [23] for standard graphs. (Although, there is an error in her proof, her idea of the proof as a whole is correct). We will extend her result to the graphs with arbitrary weights by using the same idea .

Theorem 2.2.1

Let G_S be a connected and induced subgraph of G with a non-empty boundary ∂S and $\sigma : \partial S \to R$ be a given function. Then the unique solution f to the Dirichlet boundary value problem (DBVP):

$$\begin{cases} \Delta_w f(x) = 0, \text{ for } x \in S, \\ f\mid_{\partial S} = \sigma \end{cases}$$

can be represented as:

$$f(x) = \sum_{y \in S} \sum_{z \in \partial S} \sum_{i=1}^{|S|} \frac{1}{\lambda_i} \phi_i^S(x) \phi_i^S(y) \sigma(z) \frac{w(y,z)}{\sqrt{d_G(x) d_G(y)}}, \text{ for } x \in S.$$

Proof: By considering the function $T^{\frac{1}{2}} f(x)$, we see that this function is the solution of the following equation:

$$£_w(T^{\frac{1}{2}} f(x)) = 0,$$

for $x \in S$. We also define the function $f_0 : S \cup \partial S \to R$ by

$$f_0(x) = \begin{cases} 0 & \text{if } x \in S \\ \sigma(x) & \text{if } x \in \partial S \end{cases}$$

then $T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0$ is a Dirichlet function on $S \cup \partial S$, since $(T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0) \mid_{\partial S} = 0$. Therefore, we can write $T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0$ as a linear combination of the eigenfunctions ϕ_i^S of $£_{w,S}$:

$$T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0 = \sum_i a_i \phi_i^S,$$

which implies that

$$a_i = \langle \phi_i^S, \ (T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0) \rangle.$$

Therefore, we have the following chain of equalities:

$$
\begin{aligned}
\lambda_i a_i &= \langle \lambda_i \phi_i^S, \;\; (T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0) \rangle \\
&= \langle £_{w,S} \phi_i^S, \;\; (T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0) \rangle \\
&= \langle \phi_i^S, T^{\frac{-1}{2}} L_w (f - f_0) \mid_S \rangle \\
&= \langle T^{\frac{1}{2}} \phi_i^S, \Delta_w (f - f_0) \mid_S \rangle \\
&= \langle T^{\frac{1}{2}} \phi_i^S, (-\Delta_w f_0) \mid_S \rangle \\
&= -\sum_{y \in S} \sqrt{d_G(y)} \phi_i(y) \frac{1}{d_G(x)} \sum_{z \in S} (f_0(y) - f_0(z)) w(x,z) \\
&= \sum_{y \in S} \sum_{z \in \partial S} \frac{1}{\sqrt{d_G(y)}} \phi_i^S(y) \sigma(z) w(y,z).
\end{aligned}
$$

Therefore,

$$
a_i = \frac{1}{\lambda_i} \sum_{y \in S} \sum_{z \in \partial S} \frac{1}{\sqrt{d_G(y)}} \phi_i^S(y) \sigma(z) w(y,z).
$$

Since we have,

$$
(T^{\frac{1}{2}} f - T^{\frac{1}{2}} f_0)(x) = \sum_i \frac{1}{\lambda_i} \sum_{y \in S} \sum_{z \in \partial S} \frac{1}{\sqrt{d_G(y)}} \phi_i^S(y) \sigma(z) w(y,z) \phi_i^S(x),
$$

we arrive at the following conclusion. Namely,

$$
f(x) = \sum_{y \in S} \sum_{z \in \partial S} \sum_{i=1}^{|S|} \frac{1}{\lambda_i} \phi_i^S(x) \phi_i^S(y) \sigma(z) \frac{w(y,z)}{\sqrt{d_G(x) d_G(y)}}, \quad \text{for } x \in S.
$$

QED.

By defining the function $B_\sigma : S \to R$ as

$$B_\sigma(y) = \sum_{z \in \partial S} \sigma(z) \frac{w(y,z)}{d_G(y)},$$

we can rewrite f as

$$f(x) = \langle \mathbf{G}_{w,S}(x, .), B_\sigma \rangle_S \text{ , for } x \in S,$$

where the notation $\langle \ , \rangle_S$ is the standard inner product in R^n, i.e., $\langle f, g \rangle_S = \sum_{x \in S} f(x)g(x)$. Note that the value of $B_\sigma(y)$ depends only on the value of σ on ∂S. In other words, $B_\sigma(y) = 0$ for all $y \in S \backslash \partial^0 S$. Also, two different boundary conditions σ_1 and σ_2 may give rise to the same solution f as long as $B_{\sigma_1} = B_{\sigma_2}$. By rewriting $f(x) = \langle \mathbf{G}_{w,S}(x, .), B_\sigma \rangle_{y \in S}$ as a matrix multiplication, i.e., $f = \mathbf{G}_{w,S} B_\sigma$, the Dirichlet boundary value problem is equivalent to the following equation:

$$\Delta_{w,S} f = B_\sigma \text{ on } S.$$

This relationship will allow us to identify uniquely the boundary values from a harmonic function f which satisfies $\Delta_w f(x) = 0$, for $x \in S$. The next theorem characterizes the harmonic functions with a set of singularities in a subgraph with non-empty boundary.

Theorem 2.2.2

Let G_S be a connected induced subgraph of a graph $G = (V, E)$ such that $V = S \cup \partial S$. If $\varnothing \neq T \subset S$ then every $f : V \to R$ satisfying

$$\Delta_w f(x) = 0, \ x \in S \backslash T$$

can be written uniquely as

$$f(x) = h(x) + \sum_{y \in T} \mathbf{G}_{w,S}(x,y)\Delta_w f(y),\ x \in V\ ,$$

where h is a harmonic function on S with the same boundary values as f, i.e., $h\ |_{\partial S} = f\ |_{\partial S}$.

Proof: For $x \in S \cup \partial S$, define f_1 and h as

$$\begin{aligned} f_1(x) &= \sum_{y \in T} \mathbf{G}_{w,S}(x,y)\Delta_w f(y), \\ h(x) &= f(x) - f_1(x). \end{aligned}$$

Then $h\ |_{\partial S} = f\ |_{\partial S}$. And for each $x \in S$,

$$\begin{aligned} \Delta_w h(x) &= \Delta_w f(x) - \Delta_w\ f_1(x) \\ &= \Delta_w f(x) - \Delta_w \left(\sum_{y \in T} \mathbf{G}_{w,S}(x,y)\Delta_w f(y) \right) \\ &= \Delta_w f(x) - \Delta_w \left(\sum_{y \in T} \sum_{i=1}^{|S|} \frac{1}{\lambda_i} \phi_i(x)\phi_i(y)\Delta_w f(y) \sqrt{\frac{d_G(y)}{d_G(x)}} \right) \\ &= \Delta_w \left(f(x) - \sum_{y \in T} \sum_{i=1}^{|S|} \frac{1}{\lambda_i} \frac{\phi_i(x)}{\sqrt{d_G(x)}} \phi_i(y)\Delta_w f(y)\sqrt{d_G(y)} \right) \\ &= \Delta_w \left(f(x) - \sum_{y \in T} \delta(x,y)\Delta_w f(y) \right) = 0. \end{aligned}$$

Hence h is a harmonic function. The uniqueness is evident.

QED

The following theorem formulates the solution to a nonhomogeneous Dirichlet boundary value problem which is a consequence of the Theorem 2.2.1. We will state this without proof.

Theorem 2.2.3

The solution to the following Dirichlet boundary value problem:

$$\begin{cases} \Delta_w f(x) = g(x), \; x \in S \\ f \mid_{\partial S} = \sigma \end{cases}$$

is given by

$$f(x) = \langle \mathbf{G}_{w,S}(x, .), B_\sigma \rangle_S + \langle \mathbf{G}_{w,S}(x, .), g \rangle_S.$$

We will now study the necessary and sufficient condition for the existence of the solution to the Neumann boundary value problem (NBVP). The next theorem discusses these conditions.

Theorem 2.2.4

Let G_S be an induced subgraph of a graph $G = (V, E)$ such that $V = \overline{S}$. For the real valued functions $f : \overline{S} \to R$, $g : S \to R$, and $\psi : \partial S \to R$, the necessary and sufficient condition for the solution to the NBVP

$$\begin{cases} \Delta_w f(x) = g(x), \; x \in S \\ \sum_{y \in S} (f(z) - f(y)) \dfrac{w(z, y)}{d_{G_S}(z)} = \psi(z), \; z \in \partial S \end{cases} \tag{1}$$

to exist, is

$$\int_S g(x) d_G(x) = -\int_{\partial S} \psi(z) d_{G_S}(z).$$

In this case, the solution is given by

$$f(x) = a_0 + \langle Z_{w'}(x, .), g\rangle_S + \langle Z_{w'}(x, .), \psi\rangle_{\partial S} ,$$

where a_0 is an arbitrary constant and $Z_{w'}$ is the Green's function with respect to the following weight function, w' :

$$w'(x, y) = \begin{cases} w(x, y) & \text{if either } x \text{ or } y \text{ are in } S \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We associate G with the following weight function, w' :

$$w'(x, y) = \begin{cases} w(x, y) & \text{if either } x \text{ or } y \text{ are in } S \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easily seen that $d'_{G'}(z) = d_{G_S}(z)$ for $z \in \partial S$ and $d'_{G'}(x) = d_{G_{\overline{S}}}(x)$ for $x \in S$ where $d'_{G'}(x)$ is the degree of the vertex x with respect to w' . Assume that $\int_S g(x)d_G(x) = -\int_{\partial S} \psi(z)d_{G_S}(z)$. Hence, the equation (1) can be written as:

$$\begin{cases} \sum_{y\in S\cup\partial S}(f(x) - f(y))\dfrac{w(x, y)}{d'_{G'}(x)} = g(x), \quad x \in \quad S \\ \sum_{y\in S\cup\partial S}(f(z) - f(y))\dfrac{w(z, y)}{d'_{G'}(z)} = \psi(z), \quad z \in \partial S \end{cases}$$

We can combine the above two equations into one equation to obtain

$$\sum_{y\in S\cup\partial S}(f(x) - f(y))\frac{w(x, y)}{d'_{G'}(x)} = \Psi(x), \quad x \in \quad S \cup \partial S ,$$

where,

$$\Psi(x) = \begin{cases} g(x), & x \in S \\ \psi(x), & x \in \partial S \end{cases}$$

Therefore, NBVP is equivalent to

$$\Delta_w f(x) = \Psi(x), \text{ for } x \in \overline{S}.$$

By the Theorem 2.1.1, we have:

$$\begin{aligned} f(x) &= a_0 + \langle Z_{w'}(x,.), \Psi \rangle_{\overline{S}} \\ &= a_0 + \sum_{y \in \overline{S}} Z_{w'}(x,y)\Psi(y) \\ &= a_0 + \sum_{y \in S} Z_{w'}(x,y)g(y) + \sum_{y \in \partial S} Z_{w'}(x,y)\psi(y) \\ &= a_0 + \langle Z_{w'}(x,.), g \rangle_S + \langle Z_{w'}(x,.), \psi \rangle_{\partial S} . \end{aligned}$$

where a_0 is an arbitrary constant. The converse is trivial.

QED

The above theorem shows that the solution to NBVP is uniquely determined by the Neumann data on the boundary of the graph up to an additive constant. Therefore, we get a unique solution once the value of f at some special vertex in S is defined.

2.3 Inverse Conductivity Problem on the Network

In this section, we study the inverse conductivity problem on the network (graph) S with non-empty boundary. The idea is to recover the conductivity w of the graph by using an input-output map, for example the Dirichlet data induced by the Neumann data (Neumann to Dirichlet map), with one boundary measurement. This Neumann to Dirichlet map is suggested by the previous section's result. As we have seen for a function $\psi : \partial S \to G$ with $\int_{\partial S} \psi(z) d_{G_S}(z) = 0$, the Neumann boundary value problem:

$$\begin{cases} \Delta_w f(x) = 0, \ x \in S \\ \sum_{y \in S} (f(z) - f(y)) \dfrac{w(z,y)}{d_{G_S}(z)} = \psi(z), \ z \in \partial S \end{cases}$$

has a unique solution up to an additive constant. Therefore the Dirichlet data (the boundary value of f) is well-defined up to an additive constant. But even though we are given all these data on the boundary, we are not guaranteed, in general, to be able to identify the conductivity w uniquely. To illustrate this, we consider the following example:

Example 1

Consider a graph $G = (\overline{S}, E)$ where $S = \{1, 2, 3\}$ forms a cycle and $\partial S = \{0, 4\}$ has the following weight conditions:

$$w(0,1) = 1 \text{ and } w(0,k) = 0, \text{ for } k = 2, 3, 4$$

and

$$w(3,4) = 1 \text{ and } w(k,4) = 0, \text{ for } k = 0, 1, 2\ .$$

Let $f : S \cup \partial S \to R$ be a function satisfying $\Delta_w f(x) = 0, k = 1, 2, 3$. Assume that

$$f(0) = 0,\ f(1) = 1,\ f(2) = \text{unknown},\ f(3) = 3,\ f(4) = 4$$

therefore, the boundary data $f\ |_{\partial S}$, $\frac{\partial f}{\partial_w n}\ |_{\partial S}$, and $w\ |_{\partial S \times \partial^0 S}$ are known. In fact

$$\begin{aligned} \frac{\partial f}{\partial_w n}(0) &= f(0) - f(1) = -1 \\ \frac{\partial f}{\partial_w n}(4) &= f(4) - f(3) = 1 \end{aligned}$$

The problem is to determine

$$w(1, 2) = x, w(2, 3) = y, w(1, 3) = z,\ \text{and}\ f(2)\ .$$

From $\Delta_w f(x) = 0$, for $x = 1, 2, 3$, we have:

$$\begin{aligned} f(1) &= \frac{f(0) + xf(2) + 3z}{1 + x + z} = 1 \\ f(2) &= \frac{xf(1) + yf(3)}{x + y} \\ f(3) &= \frac{xf(1) + yf(2) + f(4)}{z + y + 1} = 3 \end{aligned}$$

The above system is equivalent to

$$\begin{aligned} x(y-1)+y(x-1)+2z(x+y) &= 0 \\ \frac{x+3y}{x+y} &= f(2). \qquad (1) \end{aligned}$$

It is obvious that the above system has infinitely many solutions. Assuming $z = 0$, i.e., disconnect the edge between 1 and 3, we see that the above system reduces to

$$\begin{aligned} \frac{1}{x}+\frac{1}{y} &= 2 \\ \frac{x+3y}{x+y} &= f(2), \end{aligned}$$

where there are infinitely many pairs (x, y) of nonnegative numbers satisfying the equation. However, if we impose the following constraints

$$x \geq 1,\ y \geq 1,\ \text{and}\ z \geq 0,$$

then the equation (1) yields a unique triple solution $x = 1$, $y = 1$, $z = 0$ and $f(2) = 2$. Motivated by this example, we see that there must be certain monotonicity conditions imposed on the weights to yield the global uniqueness results. The following theorem is due to Berenstein and Chung [6]. For the sake of completeness, we present its proof.

Theorem 2.3.1

Let $\omega_1$and ω_2 be weights with $\omega_1 \leq \omega_2$ on $S \times S$ and $f_i : S \cup \partial S \rightarrow R$ be functions for $i = 1, 2$ satisfying:

$$\begin{cases} \Delta_{w_i} f_i(x) = 0, \ x \in S \\ \dfrac{\partial f}{\partial_{w_i} n}(z) = \sum_{y \in S}(f(z) - f(y))\dfrac{w_i(z, y)}{d_{G_{S,w_i}}(z)} = \psi(z), \ z \in \partial S \end{cases}$$

for a given function $\psi : \partial S \to R$ with $\int_{\partial S} \psi(z) d_{G_{S,w_i}}(z) = 0$, where $d_{G_{S,w_i}}(z)$ is the relative degree with respect to the weight w_i and S . Furthermore, we assume that

1) $w_1(z, y) = w_2(z, y)$ on $\partial S \times \partial^0 S$

2) $f_1 \mid_{\partial S} = f_2 \mid_{\partial S}$

we have:

$$\begin{aligned} f_1 &= f_2 \text{ on } S \cup \partial S \\ w_1(x, y) &= w_2(x, y) \text{ whenever } f_1(x) \neq f_1(y) \ or \ f_2(x) \neq f_2(y), \text{ for } x, y \in S. \end{aligned}$$

To prove this theorem, we use the method of energy functional which is mostly used for nonlinear partial differential equations. For a function $\sigma : \partial S \to R$, we define a functional by

$$I_w[h] = \int_{S \cup \partial S} \left(\frac{1}{4} \mid \nabla_w h \mid^2 - hg \right)(x) \, d_{G_w}(x)$$

for every function h in the set

$$A = \{h : S \cup \partial S \to R \mid \ h \mid_{\partial S} = \sigma\},$$

which is called the admissible set. In the continuous case, there is a well known Dirichlet principle which states that the energy minimizer in the admissible set is a solution of the Dirichlet boundary value problem. In the discrete version, there is a similar result which we will formulate in the next lemma.

Lemma 2.3.2 (Dirichlet's Principle) Assume that $f : S \cup \partial S \to R$ is a solution to the equations

$$\begin{cases} \Delta_w f(x) = g \text{ for } x \in S \\ f\mid_{\partial S} = \sigma \end{cases}$$

then

$$I_w\left[f\right] = \min_{h \in A} I_w\left[h\right].$$

Conversely, if $f \in A$ such that $I_w\left[f\right] = \min_{h\in A} I_w\left[h\right]$ then f is the unique solution to the above DBVP.

Proof: Let h be function in A. By the Theorem 1.1.2, Chapter 1, we have

$$\begin{aligned}
0 &= \int_{S\cup\partial S} (\Delta_w f - g)(f-h)(x) d_{G_w}(x) \\
&= \int_{S\cup\partial S} (\Delta_w f(f-h) - g(f-h))(x) d_{G_w}(x) \\
&= \int_{S\cup\partial S} \left(\frac{1}{2}\left((\nabla_w f).\nabla_w(f-h)\right) - g(f-h)\right)(x) d_{G_w}(x) \\
&= \frac{1}{2}\int_{S\cup\partial S} \mid \nabla_w f\mid^2 (x) d_{G_w}(x) - \frac{1}{2}\int_{S\cup\partial S} \left((\nabla_w f).(\nabla_w(f-h))\right)(x) d_{G_w}(x) \\
&\quad - \int_{S\cup\partial S} g(f-h)(x) d_{G_w}(x).
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
& \int_{S\cup\partial S} \left(\frac{1}{2} \mid \nabla_w f \mid^2 - gf\right)(x) d_{G_w}(x) \\
= & \int_{S\cup\partial S} \left(\frac{1}{2} ((\nabla_w f) . (\nabla_w h)) - gh\right)(x) d_{G_w}(x) \\
= & \frac{1}{2} \sum_{x\in S\cup\partial S} \sum_{y\in S\cup\partial S} (f(x) - f(y)) (h(x) - h(y)) \, w(x, y) \\
& - \int_{S\cup\partial S} (gh)(x) d_{G_w}(x) \\
\leq & \frac{1}{2} \sum_{x\in S\cup\partial S} \sum_{x\in S\cup\partial S} \frac{(f(x) - f(y))^2 + (h(x) - h(y))^2}{2} w(x, y) \\
& - \int_{S\cup\partial S} (gh)(x) d_{G_w}(x) \\
= & \frac{1}{4} \int_{S\cup\partial S} \mid \nabla_w f \mid^2 (x) d_{G_w}(x) + \frac{1}{4} \int_{S\cup\partial S} \mid \nabla_w h \mid^2 (x) d_{G_w}(x) \\
& - \int_{S\cup\partial S} (gh)(x) d_{G_w}(x).
\end{aligned}
$$

Therefore, it follows that

$$
\int_{S\cup\partial S} \left(\frac{1}{2} \mid \nabla_w f \mid^2 - gf\right)(x) d_{G_w}(x) \leq \int_{S\cup\partial S} \left(\frac{1}{4} \mid \nabla_w h \mid^2 - gh\right)(x) d_{G_w}(x),
$$

or

$$
I_w \left[f\right] \leq I_w \left[h\right], \quad h \in A.
$$

Since $f \in A$, we have:

$$
I_w \left[f\right] = \min_{h\in A} I_w \left[h\right].
$$

To prove the converse, Let χ_T be the characteristic function on T, the subset of vertices in

S, Then $f + \tau\chi_T \in A$ for each real number τ, since $\chi_T = 0$ on ∂S. Then

$$\begin{aligned} I_w\left[f + \tau\chi_T \right] &= \int_{S\cup\partial S} \left(\frac{1}{4} \mid \nabla_w f + \tau \nabla_w \chi_T \mid^2 - (f + \tau\chi_T) g \right)(x) d_{G_w}(x) \\ &= \frac{1}{4} \int_{S\cup\partial S} \left(\mid \nabla_w f \mid^2 + 2\tau \nabla_w f \nabla_w \chi_T + \tau^2 \mid \nabla_w \chi_T \mid^2 \right)(x) d_{G_w}(x) \\ &\quad - \int_{S\cup\partial S} ((f + \tau\chi_T) g)(x) d_{G_w}(x). \end{aligned}$$

Since the quantity $I_w\left[f + \tau\chi_T \right]$ is minimum when $\tau = 0$, therefore,

$$\frac{d(I_w\left[f + \tau\chi_T \right])}{d\tau} \mid_{\tau=0} = 0,$$

Hence

$$\begin{aligned} 0 &= \frac{1}{2} \int_{S\cup\partial S} (\nabla_w f . \nabla_w \chi_T)(x) d_{G_w}(x) - \int_{S\cup\partial S} (\chi_T g)(x) d_{G_w}(x) \\ &= \int_{S\cup\partial S} (\chi_T (\Delta_w f - g))(x) d_{G_w}(x) \\ &= \sum_{x\in T} (\Delta_w f - g)(x) d_{G_w}(x). \end{aligned}$$

By taking T as a singleton set $\{x\}$, for $x \in S$, we obtain the desired result, namely

$$\Delta_w f(x) = g(x).$$

We will use now the above lemma to prove the Theorem 2.3.1. We may assume that the boundary nodes are not connected by an edge since by letting $w_i(x, y) = 0$ for $x, y \in \partial S$, the Theorem 2.3.1 will be unchanged. Now, by letting $\sigma : \partial S \to R$ be a function such that

$$\sigma = f_1 \mid_{\partial S} = f_2 \mid_{\partial S},$$

we define I_{w_1} by:

$$I_{w_1}[h] = \frac{1}{4}\int_{S\cup\partial S} \left(\mid \nabla_w h \mid^2\right)(x) dG_{w_1}(x),$$

for every h in the admissible set

$$A = \{h : S \cup \partial S \to R \mid \quad h \mid_{\partial S} = \sigma\}.$$

Therefore, by the Theorem 1.1.2 of Chapter 1, we have:

$$\begin{aligned} I_{w_1}[h] &= \frac{1}{2}\int_{S\cup\partial S} h(x)\Delta_{w_1} h(x) dG_{w_1}(x) \\ &= \frac{1}{2}\int_S h(x)\Delta_{w_1} h(x) dG_{w_1}(x) + \frac{1}{2}\int_{\partial S} h(x)\Delta_{w_1} h(x) dG_{w_1}(x). \end{aligned}$$

Furthermore, for $z \in \partial S$, we have

$$\Delta_{w_1} f_1(z) = \Delta_{w_2} f_2(z).$$

It follows from the monotonicity condition of the weights, i.e., $w_1 \le w_2$ that

$$I_{w_1}[f_1] = \frac{1}{2}\int_S (f_1\Delta_{w_1} f_1)(x) dG_{w_1}(x) + \frac{1}{2}\int_{\partial S} (f_1\Delta_{w_1} f_1)(x) dG_{S,w_1}(x)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\partial S}(f_1\Delta_{w_1}f_1)(x)d_{G_{S,w_1}}(x)\\
&= \frac{1}{2}\int_{S}(f_2\Delta_{w_2}f_2)(x)d_{G_{S,w_2}}(x) + \frac{1}{2}\int_{\partial S}(f_2\Delta_{w_2}f_2)(x)d_{G_{S,w_2}}(x)\\
&= \frac{1}{2}\int_{S\cup\partial S}(f_2\Delta_{w_2}f_2)(x)d_{G_{S,w_2}}(x)\\
&= \frac{1}{4}\int_{S\cup\partial S}\mid\nabla_{w_2}f_2\mid^2(x)d_{G_{S,w_2}}(x)\\
&\geq \frac{1}{4}\sum_{x\in S\cup\partial S}\sum_{y\in S\cup\partial S}\left[f_2(x)-f_2(y)\right]^2 w_1(x,y),\ \text{since } w_1\leq w_2\\
&= \frac{1}{4}\int_{S\cup\partial S}\mid\nabla_{w_1}f_2\mid^2 d_{G_{w_1}}(x) = I_{w_1}\left[f_2\right].
\end{aligned}
$$

Since f_1 is the solution to the DBVP, by the Dirichlet principle f_1 must minimize the energy functional. On the other hand, we just proved that $I_{w_1}\left[f_1\right] \geq I_{w_1}\left[f_2\right]$, therefore $f_1 = f_2$ on $S\cup\partial S$. Now, by taking $f = f_1 = f_2$ on $S\cup\partial S$ and the fact that $I_{w_1}\left[f_1\right] = I_{w_2}\left[f_2\right]$, we get

$$
\sum_{x\in S\cup\partial S}\sum_{y\in S\cup\partial S}\left[f(x)-f(y)\right]^2 w_1(x,y) = \sum_{x\in S\cup\partial S}\sum_{y\in S\cup\partial S}\left[f(x)-f(y)\right]^2 w_2(x,y).
$$

or equivalently

$$
\sum_{x\in S\cup\partial S}\sum_{y\in S\cup\partial S}\left[f(x)-f(y)\right]^2\left[w_1(x,y)-w_2(x,y)\right] = 0.
$$

Hence,

$$
w_1(x,y) = w_2(x,y),\ \text{if } f_1(x)\neq f_1(y) \text{ or } f_2(x)\neq f_2(y).
$$

QED

From the proof of the above theorem, we see that if f_1 and f_2 are injective functions then we are able to show the uniqueness of weights, i.e, $w_1(x, y) = w_2(x, y)$ on $S \times S$. For a special class of graphs, for instance, paths, it is easy to see that harmonic functions f_1 and f_2 are injective and hence all the weights are identified. But, in general, most graphs do not admit an injective solution to either the DBVP or the NBVP. The objective of the next theorem is to impose an extra condition to yield uniqueness of weights. To better understand the idea of the next theorem, we examine the following example.

Example 2

Consider $S = \{1, 2, 3, 4, 5, 6\}$ with $\partial S = \{0, 7\}$ and the following weight conditions:

$$w_1(0, y) = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1(1, y) = \begin{cases} 1 & \text{if } y = 0 \text{ and } 2 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1(2, y) = \begin{cases} 1 & \text{if } y = 1,\ 3, \text{ and } 4 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1(3, y) = \begin{cases} 1 & \text{if } y = 2,\ 4, \text{ and } 5 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1(4, y) = \begin{cases} 1 & \text{if } y = 2,\ 3, \text{ and } 5 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1(5,y) = \begin{cases} 1 & \text{if } y = 3,\ 4, \text{ and } 6 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1(6,y) = \begin{cases} 1 & \text{if } y = 5,\ 6, \text{ and } 7 \\ 0 & \text{otherwise} \end{cases}$$

$$w_1(7,y) = \begin{cases} 1 & \text{if } y = 6 \\ 0 & \text{otherwise} \end{cases}$$

And $w_2 = w_1$ everywhere except $w_2(3,4) = k$, $k \geq 1$. Thus $w_1 \leq w_2$ throughout $S \cup \partial S$. Define $f : S \cup \partial S \to R$ by

$$\begin{aligned} f(0) &= a,\ f(1) = a - b,\ \ f(2) = a - 2b,\ f(3) = f(4) = \frac{(a+c)-(b+d)}{2}, \\ f(5) &= c - 2d,\ f(6) = c - d, f(7) = c, \end{aligned}$$

where a, b, c, d are any real numbers. It can be seen that f is a harmonic function with respect to both weights, w_2, w_1, Namely,

$$\Delta_{w_1} f(x) = \Delta_{w_2} f(x) = 0,\ \text{for } x \in S\,.$$

From the Dirichlet and the Neumann boundary data, we have:

$$\begin{aligned} f(0) &= a,\ f(7) = c, \frac{\partial f}{\partial_{w_1}}(0) = \frac{\partial f}{\partial_{w_2}}(0) = f(0) - f(1) = b,\ \text{and} \\ \frac{\partial f}{\partial_{w_1}}(7) &= \frac{\partial f}{\partial_{w_2}}(7) = f(7) - f(6) = d\,. \end{aligned}$$

We observe that f is uniquely determined by these data regardless of the weights. So $w_2(3,4) = k$ cannot be identified. But if we assume that the boundary values, $f(0) = a > 0$, and $f(7) = c > 0$ then by the maximum principle, f being a harmonic function, we will have $f(x) > 0$ for all $x \in S \cup \partial S$. Furthermore, if we suppose that

$$\int_S f(x) d_{G_{w_1}}(x) = \int_S f(x) d_{G_{w_2}}(x),$$

then

$$\int_S f(x) d_{G_{w_1}}(x) = 2f(1) + 3f(2) + 3f(3) + 3f(4) + 3f(5) + 2f(6),$$

and

$$\int_S f(x) d_{G_{w_2}}(x) = 2f(1) + 3f(2) + (2+k)f(3) + (2+k)f(4) + 3f(5) + 2f(6).$$

Therefore,

$$f(3) + f(4) = k(f(3) + f(4)).$$

This forces $k = 1$, since $f(3) + f(4) \neq 0$. Thus arriving at the conclusion that we have the uniqueness of weights. Using these conditions, we will improve Theorem 2.3.1 under weaker conditions.

Theorem 2.3.3

Let w_1 and w_2 be weights with $w_1 \leq w_2$ on $S \cup \partial S \times S \cup \partial S$ and f_i be functions satisfying for $i = 1, 2$

$$\begin{cases} \Delta_{w_i} f_i(x) = 0, \text{ for } x \in S \\ \dfrac{\partial f_i}{\partial_{w_i} n}(z) = \psi(z), \text{ for } z \in \partial S \end{cases}$$

for a given function $\psi : \partial S \to R$ such that $\int_{\partial S} \psi(z) d_{G_{S,w_i}}(z) = 0$. Furthermore, if we assume that

1) $w_1(z, y) = w_2(z, y)$ on $\partial S \times \partial^0 S$,

2) $f_1 |_{\partial S} = f_2 |_{\partial S} > 0$,

3) $\int_S f_1(x) d_{G_{w_1}}(x) = \int_S f_2(x) d_{G_{w_2}}(x)$.

Then we have

$$f_1 = f_2 \text{ on } S,$$

and

$$w_1 = w_2 \text{ on } S \times S.$$

Proof: By Theorem 2.3.1, $f_1 = f_2$ on S . So we define $f = f_1 = f_2$ on $S \cup \partial S$. Since $f_1 |_{\partial S} = f_2 |_{\partial S} > 0$, by Theorem 1.2.1 we must have $f = f_1 = f_2 > 0$ on S. It follows from the condition

$$\int_S f_1(x) d_{G_{w_1}}(x) = \int_S f_2(x) d_{G_{w_2}}(x),$$

that we have

$$\sum_{x \in S} f(x) d_{G_{w_1}}(x) = \sum_{x \in S} f(x) d_{G_{w_2}}(x),$$

or

$$\sum_{x \in S} f(x)(d_{G_{w_2}}(x) - d_{G_{w_1}}(x)) = 0. \qquad (1)$$

From the monotonicity condition of weights, i.e., $w_1(x, y) \le w_2(x, y)$ on $S \times S$ and the fact that weights are non-negative functions on $S \times S$, we have

$$d_{G_{w_1}}(x) \le d_{G_{w_2}}(x).$$

Using the fact that $f > 0$ on S, equation (1) implies that

$$d_{G_{w_2}}(x) - d_{G_{w_1}}(x) = 0, \quad \text{for } x \in S,$$

or

$$\sum_{y \in \overline{S}} (w_2(x, y) - w_1(x, y)) = 0, \ \text{for } x \in S.$$

Since $w_1 \le w_2$, we obtain the desired result.

QED

Chapter III

3 The Physical Interpretation of The Discrete Inverse Conductivity Problem

In the previous chapter, we studied the inverse conductivity problem as a means for identifying the inner structure of network by measurements made on the boundary. This method was initially proposed by Curtis and Morrow [26, 27] later developed by Carlos Berenstein into an important result in the domain of the discrete inverse conductivity problem [6]. The physical interpretation of the weighted Laplacian which is used here for the discrete inverse conductivity problem will be in terms of the chip-firing game. This is motivated in part by communication network models in which chips represent packets or jobs and the boundary nodes represent processors. Alternatively, the discretization of the inverse conductivity problem demands a discretization of an electrical network as a physical interpretation.

The chip-firing game (CFG) is a discrete dynamical model used in Physics, Computer Science, and Economics. It was introduced by Bjorner, Lovasz, and Shor in [11, 12]. The CFG is defined over an undirected graph G, called the support graph of the game. A configuration of the game is a mapping $s : V \to \mathbb{N}$ that associates a weight to each vertex, which can be considered as the number of chips stored in the vertex. The CFG is a discrete dynamical model, with the following firing rule: If, when the game is in a configuration s , a vertex v contains at least as many chips as its degree, one can transfer a chip from v along each of its edges to the corresponding neighboring vertex. CFGs are strongly convergent games [37], which means that, given an initial configuration, either the game can be played forever, or it reaches a unique stable configuration (where no firing is possible) independent on the order in which the vertices were fired. If for a given finite sequence of vertices of G, such that starting from s, this sequence of vertices is ready to fire to obtain a new configuration s', then the configuration s' is given by

$$s'(v) = s(v) - f(v)d_G(v) + \sum f(u)w(u,v),$$

where $f(v)$ is the number of times v occurs in the sequence, $d_G(v)$ is the degree of v, and $w(u,v)$ is the weight or the number of edges connecting u to v. This is true because each time v is fired it loses $d_G(v)$ chips, and each time $u \neq v$ is fired it gains $w(u,v)$ chips. The relationship between $s'$and s can be written more concisely in terms of the weighted Laplacian as follows:

$$s' = s - L_w(f).$$

CFG has been studied previously in terms of the classification of legal game sequences [11, 12], critical configurations [7,8,9], and by use of the chromatic polynomial [8], the Tutte polynomial [7,47], and matroids [48]. A parallel version of CFG, in which all ready vertices fire simultaneously, is studied in [35]. The chip-firing game is closely related to self-organized criticality [4, 5], and the sandpile model [36].

The discrete conductivity problem studied in Chapter 2 requires that $\Delta_w(f) = 0$ on S. This raises an interesting question of what the analogous condition in the context of CFG would be. In other words, since this condition requires the absence of sinks and sources in the interior of the graph, we would like to study a set of configurations that are stable.

Moreover, the necessary condition of $\int_{\partial S} \psi(z) d_{G_S}(z) = 0$ on the boundary of the graph forces us to allow the number of chips to be negative.

We consider a new variant of the chip-firing game, in which chips are fired in the game from the boundary, removed from the game when they are fired across a boundary, and the number of chips can be negative. Chung and Ellis [17] called their own variation of the chip-firing game, which is a special case of our version, the Dirichlet game. Their version considers only positive number of chips in the interior of a simple graph and the boundary nodes do not fire chips into the interior vertices after the game reaches a stable configuration. Because this difference is small we call our variant the Dirichlet game as well. We would like to point out that we are liberal about the usage of names either calling it Dirichlet game or simply CFG.

In this chapter, our objective is to obtain a bound on the length of the Dirichlet game, that is, how long it will take for a configuration to reach a stable and recurrent configurations, in terms of the initial number of chips in the game and the diameter of the graph. We start with the preliminaries in the first section of this chapter. The second section covers the fundamentals of CFG. The set of critical groups are discussed in the third section. In the fourth section, we explain how CFG is viewed as a discrete dynamical system. In Section 5, we discuss electrical networks in a naive way to attempt to understand the dynamics of CFG intuitively. The sixth and seventh sections give a probabilistic approach to the problem of CFG and electrical networks to obtain a bound on the time for the network to reach a stable configuration.

3.1 Introduction

The Dirichlet chip-firing game takes place in the setting of a connected graph $G = (V, E)$ with multiple edges such that $V = S \cup \partial S$ where we call S the interior of G and ∂S the boundary of G. Furthermore, we assume that the boundary nodes are not connected by any edges, i.e., $w(u, v) = 0$ if $u, v \in \partial S$. An instance of the Dirichlet game on the graph G starts with a number of chips on each of the vertices on the interior of G, where we allow the number of chips to be negative. The following steps are the rules of the Dirichlet game:

1) Choose a vertex v in the interior of G which has more than $d_G(v)$ chips, remove $d_G(v)$ chips from v and add $w(u, v)$ chips to each vertex u in the neighborhood of v, $N_G(v)$. Such a step is called *firing the vertex* v,

2) If there is no vertex v in the interior of G which has more than $d_G(v)$, then add $w(u, q)$ chips to each vertex u in $N_G(q)$ for every q in the boundary of G. In other words, all the nodes in the boundary of G fire simultaneously only when no firing is possible in the interior of G,

3) Chips fired from a vertex in the interior of G to a vertex in the boundary of G are instantly processed and removed from the game.

A configuration s of the Dirichlet game is a function defined on the vertices such that $s(v)$ is the number of chips in the vertex v. If $s(v) \geq d(v)$ for some v in the interior of G, then we say that v is ready in s. If $s(v) < d_G(v)$ for all v in the interior of G, then the boundary nodes are ready. Given a configuration s, a finite sequence $v_1, v_2, v_{3,}..., v_k$ of vertices is legal for s if v_1 is ready in s, v_2 is ready in the configuration obtained from s after firing v_1, etc.

A configuration s is said to be stable if $s(v) < d_G(v)$ for all v in the interior of G. It is called recurrent if there is a non-empty legal sequence which leads to the same configuration. And a configuration is called a critical configuration if it is both stable and recurrent. The following theorems due to Biggs [9] state that every configuration will eventually reach a critical configuration. For completeness, we state the proof with minor variations.

Theorem 3.1.1

For every configuration s, there is an upper bound on the length of a legal sequence of firings that does not contain the vertices in the boundary nodes.

Proof: Recall that vertices in the boundary of G fire only when no firing is possible in the interior of G and processes any chips that fire across the boundary nodes. Suppose that we start with finite number of chips and there is a vertex $v_1 \in S$ that is fired infinitely often. Let $P = v_{1,...,}v_n$ be a path with multiple edges according to the weight of G from v_1 to some vertex $v_n \in \partial S$, where all the vertices of the path except v_n belong to S. If some vertex in P, say v_i, for some $i \leq n - 1$, fires infinitely often, then v_{i+1} receives infinitely many chips and therefore, fires infinitely often also. Repeating this argument, we therefore see that the vertex v_n must process infinitely many chips. This is a clear contradiction to the fact that we started with a finite number of chips.

QED

Theorem 3.1.2

Let s be a configuration of the Dirichlet game. Then there is a critical configuration c which can be reached by a legal sequence of firings starting from s.

Proof: By Theorem 3.1.1, if we start from s and fire the vertices other than the boundary nodes then we must eventually reach a configuration where no vertex in G is ready to fire except the boundary nodes, that is, a stable configuration. If we then fire the boundary nodes and repeat the process, we reach another stable configuration. This procedure can be repeated as often as we would like to. Since the number of stable configurations is finite, at least one of them must recur, and this is the critical configuration.

QED

3.2 Basic Theory of the Dirichlet Game

In the following sections, we present the fundamentals of CFG from [9, 17, 63]. These results are still valid in our version of CFG with slight changes in the proofs. Suppose $\sigma = v_1, v_2, v_3, ..., v_k$ is a finite sequence which is legal for the configuration s. Then we denote the number of times v occurs in σ by $f_\sigma(v)$. As argued above, the relationship between s and the configuration s' which is obtained by applying σ to s can be written in terms of weighted Laplacian, i.e.,

$$s(v) - s'(v) = L_w(f_\sigma(v)).$$

The CFG is based on the confluence property. If we start with a given configuration s then there may be many different legal sequences starting from s, but they all lead to the same outcome in some sense. The following lemma and theorem explain this more precisely.

Lemma 3.2.1

Let $\sigma = v_1, ..., v_k$ and $\sigma' = v_1', ..., v_l'$ be legal sequences for a configuration s. Then there is a legal sequence $\tau = u_1, ..., u_j$ for s with the property that

$$f_\tau(u) = \max\,(f_\sigma(u), f_{\sigma'}(u)),$$

for all u in G.

Proof: Suppose the result is true for any positive integer less than $k+l$ (the sum of the lengths of the two legal sequences for the configuration s). If $k = 0$ or $l = 0$ we are done by setting $\tau = \sigma'$ or $\tau = \sigma$, respectively.

If $v_1 = v'_1$, then we can apply the induction hypothesis to the configuration obtained by applying v_1 on s with the sequences $v_2, v_3, ..., v_k$ and $v'_2, ..., v'_l$.

If $v_1 \neq v'_1$ and v_1 does not occur in σ', then $v_1, v'_1, v'_2, ..., v'_l$ is also legal for s. This is because by starting with v_1, we will be adding more chips to v'_1 and the sequence σ' was already legal for s. Now we can apply the induction hypothesis to the configuration obtained by applying v_1 to s with the sequences $v_2, v_3, ..., v_k$ and $v'_1, ..., v'_l$.

If $v_1 \neq v'_1$ and v_1 does occur in σ', then by shifting the first v_1 to the start of the σ', we will have a new sequence $v_1, v'_1, ...v'_{i-1}, v'_{i+1}, ..., v'_l$ which is still legal for s. We can now apply the induction hypothesis to the configuration obtained by applying v_1 to s with the sequences $v_2, v_3, ..., v_k$ and $v'_1, ..., v'_{i-1}, v'_{i+1}, ..., v'_l$.

QED

Theorem 3.2.2. Confluence property

Let σ and σ' be legal sequences for the configuration s which lead to new configurations s_1 and s_2. Then there is a configuration s_3 which is obtained from s_1 and s_2 by applying legal sequences, respectively.

Proof: According to the above lemma, there is a legal sequence τ for the configuration s where $f_\tau(u) = \max\ (f_\sigma\ ,\ f_{\sigma'})$. We can choose the initial part of τ to be the same as σ, and τ would still be legal for s. Then the remaining subsequence of τ will be legal for s_1. Similarly, there is a legal sequence τ' for the configuration s such that $f_{\tau'}(u) = \max\ (f_\sigma(u)\ ,\ f_{\sigma'}(u))$ and the initial part of τ'would be the same as that in σ', and the remaining subsequence of τ'will be legal for s_2. By Laplace's equation, both τ and τ' will lead s to the same configuration s_3 .

QED

Corollary 3.2.3

Using the same assumption as above, we have:

(a) If every vertex in σ and $\sigma^{'}$ appears at most once then the configuration s_3 can be obtained by firing the vertices at most once.

(b) If the boundary nodes do not appear in both σ and $\sigma^{'}$ and all the vertices appear at most once, then s_3 can be obtained by firing the vertices at most once and the boundary nodes do not appear in this sequence of firing.

Proof: Part (a) follows from the simple observation that if $f_\sigma \leq 1$ and $f_{\sigma'} \leq 1$ then $f_{\tau'} \leq 1$. And part (b) follows from the fact if $f_\sigma(q) = f_{\sigma^{'}}(q) = 0$, then $f_\tau(q) = 0$, for every boundary node q.

QED

3.3 Critical Configuration of the Dirichlet Game

Based on Berenstein and Chung's mathematical formulation of the discrete inverse conductivity problem, we require that $\Delta_w(f) = 0$ in the interior of the graph. This is a reasonable assumption if one wants to detect the problem of inner structure of the network through measurements made on the boundary of the graph. In the language of chip-firing game, we require that the net flow or net firing in the interior of the graph to be zero. In other words, we desire that the interior of the network be at a stable configuration. This is why the study of set of stable and recurrent configurations of the Dirichlet game becomes important in connection to the Berenstein and Chung's mathematical model of network tomography .

Lemma 3.3.1

Suppose $\sigma = v_1, ..., v_k$ is a legal sequence for a stable configuration s in which the boundary nodes appear only once. Then every vertex in G appears at most once in σ.

Proof: Since s is stable, the boundary nodes must appear at the beginning of the firing sequence σ. Let's suppose that some vertex appears more than once. Let v be the first vertex that appears more than once and v_i be the second appearance of v. Then v_i is ready after $v_1, ..., v_{i-1}$ has been applied to s. According to Laplace's equation, the number of chips on v_i after $v_1, ..., v_{i-1}$ is applied to s is

$$s(v_i) - f_\sigma(v_i)d_G(v_i) + \sum_u f_\sigma(u)w(u, v_i) = s(v_i) - d_G(v_i) + \sum_u f_\sigma(u)w(u, v_i).$$

Since v_i appears exactly once in $v_1, ..., v_{i-1}$. The upper bound for the number of chips on the vertex v_i after the sequence $v_1, ..., v_{i-1}$ has been applied to s is

$$s(v_i) - d_G(v_i) + \sum_{u \in N_G(v_i)} f_\sigma(u) w(u, v_i) \le s(v_i) - d_G(v_i) + d_G(v_i) = s(v_i).$$

But since s is stable, the number of chips at the vertex v_i, after the second appearance, will be less than the degree of v_i. This contradicts the fact that v_i is ready for the second firing time.

QED

Corollary 3.3.2

Let s be a stable configuration and $\sigma = v_1, ..., v_k$ be a legal sequence for s. Then every vertex in G appears in σ at most as often as the boundary nodes.

Proof: Partition $\sigma = v_1, ..., v_k$ into parts where the boundary nodes appear at the beginning of each part. Since s is stable, the boundary nodes must appear first. After applying the first part of σ to s we will obtain a new stable configuration. This is because the boundary nodes appear the second time in the second part. By applying Lemma 3.3.1 to each part, we obtain the result.

QED

The following result gives a necessary condition for a stable configuration to reappear.

Theorem 3.3.3

Let $\sigma = v_1, ..., v_k$ be a legal sequence for a stable configuration s such that after applying σ to s we return to the configuration s. Then every vertex in G appears the same number of times in σ.

Proof: It is easy to see that every vertex in G must appear in σ. Since s reappears after applying σ, if a vertex does not fire then it should not receive chips from neighboring vertices. Therefore, if a vertex v in G does not appear in σ, then all neighboring vertices do not appear in σ either. By the connectedness of G, none of the vertices of G appear in σ, which is a contradiction. So all the vertices of G must appear in σ. Let v be a vertex that appears a minimal number of times in σ and that is adjacent to a vertex v' that appears more often in σ than v itself (otherwise, since G is connected all the vertices must appear equally often). By Corollary 3.3.2, v cannot be a boundary node. After applying σ to s, v loses $pd_G(v)$ chips and gains at least $pw(u, v)$ chips from each u in the neighborhood of v and, in fact, at least $(p+1)w(v, v')$ chips from v', where p is the number of times v appears in σ. The lower bound on number of chips in v after applying σ is

$$s(v) - pd_G(v) + \sum_u pw(u, v) + w(v, v') = s(v) + w(v, v'),$$

contradicting the fact that s should reappear after applying σ.

QED

Theorem 3.3.4

Let $\sigma = v_1, ..., v_k$ be a legal sequence for a critical configuration s such that after applying σ to s, we get a stable configuration. If the boundary nodes appear exactly once in the firing sequence, then every vertex in G appears exactly once and the resulting configuration is s.

Proof: Since s is a critical configuration, there is a legal sequence of firing $\sigma' = v'_1, ..., v'_l$ for s such that after applying σ' the configuration s is returned. By Theorem 3.3.3, every vertex in G must appear in σ' the same number of times. And by Lemma 3.3.1

every vertex in G appears at most once in σ . This means that for all v in G

$$\max(f_{\sigma}(v), f_{\sigma'}(v)) = f_{\sigma'}(v).$$

According to the Theorem 3.2.2, we can form a legal sequence τ whose vertices appear the same number of times as in σ'. Therefore every vertex appears the same number times in τ and applying τ to s will yield s again. Also, we can choose τ in such a way that the initial part of τ will be the same as σ. Let τ' be the part of τ after the initial part σ. Then τ' is a legal sequence for a stable configuration resulted from applying σ to s. Therefore, we have a sequence τ in which each vertex appears equally often, and the initial part of τ is equal to σ in which each vertex appears at most once. In the remaining part, τ', each vertex appears at most as often as the boundary nodes. This is only possible if each vertex appears exactly once in σ, and the resulting configuration after σ is being applied to s is s again.

QED

Theorem 3.3.5

Let s be a configuration of the Dirichlet game. Then there is a unique critical configuration which can be reached by a legal sequence of firings.

Proof: By Theorem 3.1.2, one such critical configuration exists. Suppose c is the first critical configuration that is reached by the sequence of firings. Then by Theorem 3.3.4, once we activate the boundary nodes the first stable configuration will be a critical configuration which coincides with c. This proves the uniqueness.

QED

Corollary 3.3.6

Let c be a critical configuration. Then $0 \leq c(v) \leq d_G(v) - 1$ for all v in the interior of G.

Proof: The upper bound for $c(v)$ arises directly from the definition of critical configuration. For the lower bound, by above theorem, there is a legal sequence such that every vertex appears exactly once. v receives only $d_G(v)$ chips when applying the legal sequence. On the other hand, since v is fired as well, there must be a moment when v holds at least $d_G(v)$. This is only possible if $c(v) \geq 0$.

QED

The above theorem shows that a critical configuration is in some sense a saturated state, a state in which the network is neither in an active nor passive position. In other words, the total net flow in the interior of network is zero. Theorem 3.3.4 also provides a way to recognize the critical configuration. Once a stable configuration s is obtained, start with the boundary nodes to form a legal sequence of firing, until another stable configuration s' is obtained. By Lemma 3.3.1, this happens after at most n firings, where n is the number of vertices in G. If $s = s'$, then s' is a critical configuration, otherwise repeat the procedure starting with s'. The following theorem shows one can also recognize the critical configuration by knowing whether every vertex has been fired or not.

Theorem 3.3.7

Suppose s is an arbitrary configuration and $\sigma = v_1, ..., v_k$ is a legal sequence such that when it is applied to s results in a stable configuration s'. If all the vertices appear at least once in σ, then s' is a critical configuration.

Proof: If all the vertices appear exactly once in σ, then the configuration s' will be the same as s by Laplace's equation. Since s' is a stable configuration, so is s. This means that

s is a critical configuration. For the remaining cases, we use induction on k. If v_1 appears more than once then we apply the induction hypothesis to the configuration obtained by applying v_1 to s with the following legal sequence $v_2, ..., v_k$ for this new configuration. Therefore, assume that some vertex other than v_1 appears more than once. Let v_i be the vertex in σ that appears more than once and if v_j is the second appearance of v_i in σ then $\tau = v_1, ..., v_{i-1}, v_i, v_{i+1}, ..., v_{j-1}$ appears at most once in σ. We consider two cases. Assume first that v_i is not a boundary node. Since v_i is ready after applying τ to s, and v_i loses $d_G(v_i)$ chips and gains at most $\sum_u w(u, v_i) = d_G(v_i)$ chips, v_i must have been ready before the application of τ on s. Now apply the induction hypothesis to the configuration obtained by applying v_i to s with the legal sequence of $v_1, ..., v_{i-1}, v_{i+1}, ..., v_k$ in which each vertex appears at least once. For the second case, we assume v_i is one of the boundary nodes. Let s'' be the configuration obtained from s by adding $w(v, q)$ chips to all v in the neighborhood of q for every q in the boundary of G. Then the sequence $\sigma' = v_1, ..., v_{i-1}, v_l, ..., v_k$, in which the vertices from v_{i-1} to v_l in σ are all the boundary nodes and were removed from σ, is a legal sequence for s''. We can now apply the induction hypothesis to s'' and σ'. Note that applying σ' to s'' will result in the same configuration as applying σ to s.

QED

3.4 Dirichlet Game as a Discrete Dynamical System

In order to define time in a discrete sense, we have to find a way to get from one configuration to another irrespective of choices of the legal sequence within one unit of time.

Suppose we start with an unstable configuration s. If $\sigma = v_1, ..., v_k$ and $\sigma' = v'_1, ..., v'_l$ are two legal sequences such that all the vertices in the interior of G appear at most once in each of the sequences and, moreover, k and l are as large as possible with this property. Then, by the confluence property, there is a legal sequence whose vertices appear at most once and the its initial part is $\sigma = v_1, ..., v_k$. As k is the largest integer with the property that no vertices appear more than once, this new sequence must be σ. By a similar argument, there is a legal sequence whose vertices appear at most once and its initial part is σ'. Since l is the largest possible value with the property that no vertices appear more than once, this new sequence must be σ'. By the confluence property, $\sigma = v_1, ..., v_k$ and $\sigma' = v'_1, ..., v'_l$ will lead s to the same configuration. The above argument leads us to introduce the following definition:

Let s be a configuration, then a *cycle for s* is a legal sequence $\sigma = v_1, ..., v_k$ such that

(a) If the boundary nodes do not appear in σ, all the vertices appear at most once, and k is as large as possible.

(b) If s is a stable configuration, then the boundary nodes appear in the first part of $\sigma = v_1, ..., v_k$ and all the vertices appear at most once, and k is as large as possible.

Obviously, the cycles are not uniquely determined by s, but they all lead s to the same configuration. Let s be the initial configuration. We call s the configuration at time 0, denoting it by s_0. If s_t is the configuration at time t, then the configuration at time $t + 1$ is defined as configuration obtained by applying a cycle to s_t. The following theorem describes the dynamics of the CFG.

Theorem 3.4.1

Let s_0 be the starting configuration of a Dirichlet game on the connected graph G. Then for every v in the interior of G and positive integer t we have:

(a) if $s_t(v) < 0$, then $s_t(v) \leq s_{t'}(v) \leq d_G(v) - 1$ for all $t' \geq t$;

(b) if $0 \leq s_t(v) \leq d_G(v) - 1$, then $0 \leq s_{t'}(v) \leq d_G(v) - 1$ for all $t' \geq t$;

(c) if $s_t(v) \geq d_G(v)$, then $0 \leq s_{t'}(v) \leq s_t(v)$ for all $t' \geq t$.

Proof: Let v be in the interior of G, t a positive integer, and let σ be a cycle for s_t. Since every vertex in the interior of G appears at most once in σ, v receives at most $\sum_u w(u, v) = d_G(v)$ chips when applying the cycle σ to s_t. On the other hand, if v is fired, it loses $d_G(v)$ chips.

If $s_t(v) < 0$, then v is not fired in the cycle σ, therefore, $s_t(v) \leq s_{t+1}(v) \leq s_t(v) + d_G(v) \leq d_G(v)$. This proves part (a).

If $0 \leq s_t(v) \leq d_G(v) - 1$, then v gains at most $d_G(v)$ chips. If v is fired in the cycle σ then it loses $d_G(v)$ chips. Hence, we have $0 \leq s_{t+1}(v) \leq d_G(v) - 1$. And if v is not fired then we immediately have $0 \leq s_{t+1}(v) \leq d_G(v) - 1$. This proves part (b).

If $s_t(v) \geq d_G(v)$, then v is certainly fired in the cycle σ, hence it loses $d_G(v)$ chips but gains at most $d_G(v)$ chips. So we have $0 \leq s_{t+1}(v) \leq s_t(v)$ which proves part (c). The result follows by applying induction on k where $t' = t + k$.

QED

One of the difficulties in analyzing the discrete dynamics of the Dirichlet game arises when a vertex v is at a configuration s, such that $0 \leq s_t(v) \leq d_G(v) - 1$. In this case, it is not completely determined that the vertex v will fire after applying a cycle to s. Depending on the structure of the network and the number of chips at every vertex, the firing might or might not occur. The main tool in analyzing the dynamics of the Dirichlet game will be its continuous version, which is the electrical network. Recall the following equation for the

Dirichlet game:

$$s_{t+1}(v) - s_t(v) = -f_t(v)d_G(v) + \sum_u f_t(u)w(u, v),$$

where $f_t(v)$ is either 1 or 0 depending on whether v fires or not at time t. In the continuous version, $s_{t+1}(v) - s_t(v)$ is substituted by the differential of s, $s(v)$ is regarded as the amount of charge at the vertex v, which could be positive or negative, and $f_t(v) = 1$ or 0 depending on the charges being positive or negative at the vertex v. And when there is a zero charge at vertex v, we would like to have Kirchhoff's law applied to the vertex v. In other words, since the continuous dynamics obey the following differential equation:

$$\frac{ds}{dt} = -f_t(v)d_G(v) + \sum_u f_t(u)w(u, v),$$

we require that if a vertex v has zero charges, i.e, $s_t(v) = 0$ then $s_{t'}(v) = 0$ for all $t' \geq t$. This results in $\frac{ds}{dt} = 0$ or

$$-f_t(v)d_G(v) + \sum_u f_t(u)w(u, v) = 0,$$

which is exactly the Kirchhoff's law .

3.5 Basic Theory of Electrical Networks

As in the Dirichlet game we assume we are given a finite, undirected, connected graph with boundary nodes. At time t we assume at each vertex v in the interior contains a certain amount of charge $r_t(v)$, which can be negative. In the continuous version, firing is denoted by $\varphi_t(v)$ and the value of $\varphi_t(v)$ is determined by the following rule:

1) If $r_t(v) < 0$ for v in the interior of G then $\varphi_t(v) = 0$,

2) If $r_t(v) > 0$ for v in the interior of G then $\varphi_t(v) = 1$,

3) If $r_t(v) = 0$ for v in the interior of G then $\varphi_t(v)$ is obtained according to Kirchhoff's law, i.e.,

$$\varphi_t(v) = \sum_u \varphi_t(u) \frac{w(u, v)}{d_G(v)}.$$

4) If $r_t(u) > 0$ for some u in the interior of G then $\varphi_t(q) = 0$ for all q in the boundary of G,

5) If $r_t(u) \leq 0$ for all u in the interior of G then $\varphi_t(q) = 1$ for all q in the boundary of G.

Based on the theory of electrical networks [13, 26, 27, 34], φ_t is uniquely determined by the above rules.

We define a vertex v as being *passive* if $r_t(v) < 0$, *saturated* if $r_t(v) = 0$, and *active* if $r(v) > 0$. From the above rules, it is obvious that the boundary of G is active if no other vertices are active, and passive otherwise. For a certain configuration r, we call V_r^a the set of active vertices, and V_r^p the set of passive vertices.

Theorem 3.5.1

For a configuration r as above, we have $0 \leq \varphi_t(v) \leq 1$, for all vertices v.

Proof : Suppose there exists a vertex v such that $\varphi_t(v) > 1$. Choose v such that $\varphi_t(v)$ is maximum. Because of the rule (2), v cannot be in the boundary of G. From the equation $\varphi_t(v) = \sum_u \varphi_t(u)\frac{w(u,v)}{d_G(v)}$, since $\varphi_t(u) \leq \varphi_t(v)$ for all u in the neighborhood of G, we therefore have

$$\begin{aligned} d_G(v)\varphi_t(v) &= \sum_u \varphi_t(u)w(u,v) \\ &\leq \sum_u w(u,v)\varphi_t(v) = d_G(v)\varphi_t(v). \end{aligned}$$

The above inequality is forced to be an equality everywhere. Hence

$$\varphi_t(v) = \sum_u \varphi_t(u)\frac{w(u,v)}{d_G(v)},$$

for all u in the neighborhood of v. Since $\varphi_t(v)$ is maximum, $\varphi_t(v) = \varphi_t(u)$ for all u in the neighborhood of v. By repeating the same argument for the vertices neighboring u, and the fact that G is connected, we have $\varphi_t(u) = \varphi_t(v)$ for all u in G. But this contradicts the flow $\varphi_t(q) = 1$ for q in the boundary of G. By similar argument, we have $\varphi_t(v) \geq 0$ for all v in G.

QED

The following theorem shows that the dynamics of the continuous version of the Dirichlet game is almost the same as the discrete one.

Theorem 3.5.2

For any v in the interior of G and $t \geq 0$ we have

(a) If $r_t(v) = 0$, then $r_{t'}(v) = 0$ for all $t' \geq t$;

(b) If $r_t(v) < 0$, then $r_t(v) \leq r_{t'}(v) \leq 0$ for all $t' \geq t$;

(c) If $r_t(v) > 0$, then $0 \leq r_{t'}(v) \leq r_t(v)$ for all $t' \geq t$.

Proof:

If $r_t(v) = 0$, then $\varphi_t(v) = \sum_u \varphi_t(u)\frac{w(u,v)}{d_G(v)}$. Therefore, by the differential equation $\frac{dr_t}{dt} = -\varphi_t(v)d_G(v) + \sum_u \varphi_t(u)w(u,v)$, $\frac{dr_t}{dt} = 0$. Hence we must have $r_{t'}(v) = 0$ for all $t' \geq t$. This proves part (a).

If $r_t(v) < 0$, then $\varphi_t(v) = 0$. By Theorem 3.5.1, $\varphi_t(u) \geq 0$ for all u in G. Using the equation $\frac{dr}{dt} = -\varphi_t(v)d_G(v) + \sum_u \varphi_t(u)w(u,v)$, we have

$$\frac{dr}{dt} = -0(d_G(v)) + \sum_u \varphi_t(u)w(u,v) \geq 0.$$

Hence $r_t(v) \leq r_{t'}(v)$ for all $t' \geq t$ as long as $r_{t'}(v) < 0$. Once we reach $r_{t'}(v) = 0$, we obtain $r_{t''}(v) = r_{t'}(v) = 0$ for all $t'' \geq t'$ by part (a). This proves part (b).

If $r_t(v) > 0$, then $\varphi_t(v) = 1$. By Theorem 3.5.1, $\varphi_t(u) \leq 1$ for all u in G. Using the equation,

$$\begin{aligned} \frac{dr}{dt} &= -1(d_G(v)) + \sum_u \varphi_t(u)w(u,v) \\ &\leq -d_G(v) + \sum_u w(u,v) \ \leq 0, \end{aligned}$$

we have $r_{t'}(v) \leq r_t(v)$ for all $t' \geq t$ as long as $r_{t'}(v) > 0$. Once we reach $r_{t'}(v) = 0$, we get $r_{t''}(v) = r_{t'}(v) = 0$ for all $t'' \geq t'$ by part (a). This proves part (c).

QED

We say a configuration r_t is an active configuration, if there is a vertex v in the interior of G such that $r_t(v) > 0$. Otherwise, it is called inactive or passive. Note that the boundary nodes are active in an inactive configuration. A configuration is called recurrent if $\dfrac{dr_t(v)}{dt} = 0$, for all v in the interior of G. We consider a recurrent configuration, as analogous to a critical configuration for the Dirichlet game.

Although we have different critical configurations in the Dirichlet game, in electrical networks there is only one recurrent configuration which has zero charge at every vertex in the interior of the graph.

Theorem 3.5.3

For a connected graph G, if r_t is a recurrent configuration, then $r_t(v) = 0$ and $\varphi_t(v) = 1$ for all v in the interior of G.

Proof:

Suppose there is a vertex v in the interior of G such that $r_t(v) > 0$. Then $\varphi_t(v) = 1$ and $\varphi_t(q) = 0$ for all q, the boundary nodes. By the connectedness of G, we can choose v so that there would be a vertex u in the neighborhood of v with $\varphi_t(u) < 1$. From the equation

$$\frac{dr_t(v)}{dt} = -\varphi_t(v)d_G(v) + \sum_u \varphi_t(u)w(u, v),$$

we obtain $\dfrac{dr_t(v)}{dt} < 0$. Hence r_t cannot be a recurrent configuration. Furthermore, $r_t(v) \leq 0$ for all v in the interior of G. Now, if there is a vertex v in the interior of G such that $r_t(v) < 0$ then $\varphi_t(q) = 1$ for all boundary nodes, q. By the connectedness of G, we can choose v so that there would be a vertex u in the neighborhood of v with $\varphi_t(u) < 1$. From the equation, $\dfrac{dr_t(v)}{dt} = -\varphi_t(v)d_G(v) + \sum_u \varphi_t(u)w(u, v)$, we obtain $\dfrac{dr_t(v)}{dt} > 0$. Hence

r_t cannot be a recurrent configuration. We conclude that the only recurrent configuration is the configuration with $r_t(v) = 0$ for all v in the interior of G. It is obvious that $\varphi_t(v) = 1$ for all v in the interior of G.

QED

We now study the connection between the Dirichlet game and electrical networks. For a starting configuration r_0 of the electric network and a positive real number t, define

$$\Phi_t(v) = \int_0^t \varphi_x(v)dx,$$

because $0 \leq \varphi_x(v) \leq 1$, we have $0 \leq \Phi_t(v) \leq t$, for all v. Similarly, for a starting configuration s_0 and a positive integer t, define

$$F_t(v) = \sum_{x=1}^{t} f_x(v),$$

since $f_x(v) = 0$ or 1, therefore, we again have $0 \leq F_t(v) \leq t$. From the dynamics of the Dirichlet game and the electric network, we have

$$\begin{aligned} s_t(v) - s_0(v) &= -d_G(v)F_t(v) + \sum_u w(u,v)F_t(u) = -L_w(F_t(v)) && (1) \\ r_t(v) - r_0(v) &= -d_G(v)\Phi_t(v) + \sum_u w(u,v)\Phi_t(v) = -L_w(\Phi_t(v)) && (2) \end{aligned}$$

The following two theorems describe the connections between the Dirichlet game and the electrical networks.

Theorem 3.5.4

Let s_0 be a stable configuration of the Dirichlet game. Define r_0 to be the starting configuration of the electric network by $r_0(v) = s_0(v) - (d_G(v) - 1)$ for all v in the interior of G. If r_T is a recurrent configuration then for an integer $t > T$, s_t is a critical configuration of the Dirichlet game.

Proof:

By Theorem 3.3.7, it suffices to show that all the interior vertices will fire within the time interval between 0 and t, i.e., $F_t(v) > 0$ for all vertices v in G. This is done by showing that $\Phi_t(v) \leq F_t(v)$ for all vertices v in G. Since r_t is a recurrent configuration for $t \geq T$, we have $\phi_t(v) > 0$ for $t > T$ and for all vertices v in G. Now, suppose that there exists a vertex v such that $\Phi_t(v) - F_t(v) > 0$. Furthermore, we assume that the choice of v makes the quantity $\Phi_t(v) - F_t(v)$ maximum. From $F_t(q) = t$ and $\Phi_t(q) = t$, we conclude that v cannot be one of the boundary nodes. From Equations (1) and (2) above (see p. 83) and the fact that $\Phi_t(u) - F_t(u) \leq \Phi_t(v) - F_t(v)$ for all u in G, we have:

$$
\begin{aligned}
r_t(v) - s_t(v) &= r_0(v) - s_0(v) - (L_w(\Phi_t(v)) - L_w(F_t(v))) \\
&\leq r_0(v) - s_0(v) - \left(d_G(v) - \sum_u w(u,v) \right) (\Phi_t(v) - F_t(v)) \\
&= r_0(v) - s_0(v).
\end{aligned}
$$

Therefore, $r_t(v) - s_t(v) \leq -(d_G(v) - 1)$. Now, since $r_t(v) = 0$ we obtain $s_t(v) \geq d_G(v) - 1$. We also have $s_t(v) \leq d_G(v) - 1$,i.e., $s_t(v)$ is a stable configuration. Hence, we must have equality everywhere in the above inequality. In other words, $\Phi_t(u) - F_t(u) = \Phi_t(v) - F_t(v)$ for all u in the neighborhood of v. Repeat this argument for the neighborhood of u. By the connectedness of G, we finally get $\Phi_t(q) - F_t(q) > 0$ for all q, the boundary nodes. But this is clearly a contradiction.

QED

Theorem 3.5.5

Suppose s_0, the initial configuration of the Dirichlet game, is not a stable configuration and define r_0 to be the initial configuration of the electrical network as $r_0(v) = s_0(v)$ for all v in the interior of G. If r_t is a passive configuration for an integer t then s_t is a stable configuration of the Dirichlet game.

Proof: Suppose s_t is not a stable configuration. In this case, we claim that $\Phi_t(u) \geq F_t(u)$ for all vertices u in G. Now, choose a vertex x such that $s_t(x) \geq d_G(x)$, then $F_t(x) = t$ and $\Phi_t(x) \leq t$. By the above claim, we have $\Phi_t(x) - F_t(x) = 0$. Therefore,

$$\begin{aligned} r_t(x) - s_t(x) &= r_0(x) - s_0(x) - L_w(\Phi_t(x) - F_t(x)) \\ &\geq r_0(x) - s_0(x) = 0. \end{aligned}$$

Hence, $r_t(x) \geq s_t(x)$. But this clearly contradicts the fact that $r_t(x) \leq 0$ and $s_t(x) \geq d_G(x)$. Now, we will prove the claim. Suppose there exists a vertex v such that $\Phi_t(v) < F_t(v)$. Furthermore, we assume that the choice of v makes the quantity $F_t(v) - \Phi_t(v)$ maximum. Since s_t is not a stable configuration we have $F_t(q) = 0$ and $\Phi_t(q) \geq 0$ for all q, the boundary nodes. Hence v cannot be one of the boundary nodes. From Equations (1) and (2) (see p. 83), we have:

$$s_t(v) - r_t(v) = s_0(v) - r_0(v) - (L_w(F_t(v)) - L_w(\Phi_t(v)))$$

$$\begin{aligned} &\le s_0(v) - r_0(v) - \left(d_G(v) - \sum_u w(u,v)\right)(F_t(v) - \Phi_t(v)) \\ &= s_0(v) - r_0(v) = 0 \end{aligned}$$

Hence $s_t(v) - r_t(v) \le 0$. But r_t is a passive configuration so $s_t(v) \le 0$. On the other hand, from $F_t(v) > \Phi_t(v) \ge 0$, we conclude that $s_t(v) \ge 0$. But this forces equalities everywhere in the above inequality. Hence $\Phi_t(u) - F_t(u) = \Phi_t(v) - F_t(v)$ for all u in the neighborhood of v. Repeating the same argument for u and using the connectedness of G, we finally get $F_t(q) - \Phi_t(q) > 0$ for all q, the boundary nodes. But this is clearly a contradiction.

QED

Our objective is to find an upper bound for the time it takes for a configuration to reach a recurrent configuration. Intuitively, this time is bounded by a minimal path from the unique active vertex to the unique passive vertex. Since the active and passive vertices will change over time, the correct bound will be the diameter of the graph (the maximum of minimal paths between two vertices). To simplify the analysis, we will use the short circuits or contraction technique used in the theory of electrical networks [13].

Given a nonrecurrent configuration r on a connected graph G, the graph G' is obtained from G by contracting all active vertices, V_r^a, into one vertex a_0 and similarly all passive vertices, V_r^p, into one vertex p_0, and remove all the loops. If r is inactive then the boundary nodes become active, and we would contract all boundary nodes into one vertex, q_0. In an inactive position, we set $q_0 = a_0$, and $r'(p_0) = \sum_{v \in V_r^p} r(v)$; similarly in an active position, we set $q_0 = p_0$ and $r'(a_0) = \sum_{v \in V_r^a} r(v)$. For neither passive nor active vertices we set

$r'(v) = r(v)$. Now, if there exists an edge between two passive vertices then there is no flow between these two vertices. Similarly, if there is an edge between two active vertices then there is no flow between these vertices because the vertices have the same voltages. Hence, we can remove these edges. If we identify two active vertices into one vertex, we then see that this does not change the flow pattern of the network A similar argument applies on the set of passive vertices. Continuing this process, we see that if we contract all the active vertices into one vertex and all the passive vertices into one vertex then the flow pattern in the contracted graph will be the same as the original graph. The following theorem gives a mathematical translation of the above argument.

Theorem 3.5.6

Given a graph G with a nonrecurrent flow r_t , let φ'_t be the flow in the graph G' with configuration r'_t, then we have the following properties:

(a) If r_t is active then

$$\begin{aligned}
\frac{dr'_t(a_0)}{dt} &= -d_{G'}(a_0) + \sum_u w(u, a_0)\varphi'_t(u) \\
&= \sum_{v \in V^a_{r_t}} \left[-d_G(v) + \sum_u w(u, v)\varphi(u) \right] \\
&= \sum_{v \in V^a_{r_t}} \frac{dr_t(v)}{dt}.
\end{aligned}$$

(b) If r_t is inactive then

$$\begin{aligned}\frac{dr'_t(p_0)}{dt} &= \sum_u w(u,p_0)\varphi'_t(u)\\ &= \sum_{v\in V^p_{r_t}}\left[\sum_u w(u,v)\varphi(u)\right]\\ &= \sum_{v\in V^p_{r_t}}\frac{dr_t(v)}{dt}\,.\end{aligned}$$

The quantity $d_{G'}(a_0) - \sum_u w(u,a_0)\varphi'_t(u)$ is known as the effective conductance C_{EFF} of the graph G' in electrical networks theory. We will use the random walk interpretation of electrical networks [30, 31, 43] to better understand C_{EFF} of the electrical network.

3.6 Random Walk interpretation of Electrical Networks

As usual, we assume that G is connected with multiple edges. We assign to each edge a resistance 1. We can replace an edge with multiplicity w by one resistor with resistance $\frac{1}{w}$. This way, we can convert our graph to a simple graph such that a resistance R_{xy} is assigned to each edge connecting x to y. The conductance of an edge xy is $w(x,y) = \frac{1}{R_{xy}}$. We define a random walk on G to be a Markov chain with transition matrix P given by $P_{xy} = \frac{w(x,y)}{d_G(x)}$.

We choose two points a (an active vertex) and b (a passive vertex) and put a one-volt battery across these points establishing a voltage $\varphi(a) = 1$ and $\varphi(b) = 0$. Our objective now is to give a probabilistic interpretation of voltage and currents. By Ohm's law, the currents through an edge connecting x to y of the graph are given by

$$i_{xy} = \frac{(\varphi(x) - \varphi(y))}{R_{xy}} = (\varphi(x) - \varphi(y))w(x,y).$$

By Kirchhoff's law, we have:

$$\sum_y i_{xy} = 0,$$

therefore,

$$\varphi(x) = \sum_y \frac{w(x,y)}{d_G(x)} \varphi(y) = \sum_y P_{xy}\varphi(y), \text{ for } x \neq a, b.$$

Let $h(x)$ be the probability that starting from x, the state a is reached before the state b. Then $h(x)$ is harmonic in the interior and has the same boundary values as φ , i.e., $\varphi(a) = h(a) = 1$, and $\varphi(b) = h(b) = 0$. Therefore both φ and h are solutions to the Dirichlet problem for the Markov chain with the same boundary values. Hence $\varphi = h$. Therefore, we have the following theorem.

Theorem 3.6.1 (Probabilistic interpretation of voltage)

When unit voltage is applied between a and b, i.e., $\varphi(a) = 1$ and $\varphi(b) = 0$, the voltage $\varphi(x)$ equals the probability that a walker starting from the point x will return to a before reaching b.

QED

For the probabilistic interpretation of current, we assume the walker begins at a and ends at b. Let u_x be the expected number of times that the walker visits x before reaching b. Then for $x \neq a$, b, we have $u_x = \sum_y u_y P_{yx}$. Since $d_G(x)P_{xy} = d_G(y)P_{yx}$ we have

$$u_x = \sum_y u_y P_{xy} \frac{d_G(x)}{d_G(y)},$$

or

$$\frac{u_x}{d_G(x)} = \sum_y u_y \frac{P_{xy}}{d_G(y)}.$$

This means that $\varphi(x) = \dfrac{u_x}{d_G(x)}$ is harmonic for $x \neq a, b$, and $\varphi(x)$ is actually the voltage at x when we put a battery from a to b to establish a voltage $\varphi(a) = \dfrac{u_a}{d_G(a)}$ at a and $\varphi(b) = 0$ at b. Thus the current from x to y is

$$
\begin{aligned}
i_{xy} &= (\varphi(x) - \varphi(y))w(x, y) \\
&= \left(\frac{u_x}{d_G(x)} - \frac{u_y}{d_G(y)}\right) w(x, y) \\
&= u_x P_{xy} - u_y P_{yx} \ .
\end{aligned}
$$

Now $u_x P_{xy}$ is the expected number of times our walker will go from x to y and $u_y P_{yx}$ is the expected number of times she will go from y to x. Thus the current i_{xy} is the expected value (not necessarily an integer) for the net number of times the walker crosses along the edge from x to y. Hence, we have the following .

Theorem 3.6.2 (Probabilistic interpretation of currents)

When a unit current flows out of a into b, the current i_{xy} flowing through the edge connecting x to y is equal to the expected net number of times that a walker, starting at a and walking until he reaches b, will move along the branch from x to y. These currents are proportional to the currents that arise when a unit voltage is applied between a and b, the constant of proportionality being the effective resistance of the network.

QED

When we impose a voltage φ between points a and b, the voltage $\varphi(a) = \varphi$ is established at a and $\varphi(b) = 0$. And a current $i_a = \sum_x i_{ax}$ will flow out of a. The amount of current that flows depends upon the overall resistance of the network which is called the effective resistance R_{EFF}, defined by $R_{EFF} = \frac{\varphi(a)}{i_a}$, where the reciprocal quantity $C_{EFF} = \frac{1}{R_{EFF}} = \frac{i_a}{\varphi(a)}$ the effective conductance. We can interpret the effective conductance as an escape probability. When $\varphi(a) = 1$, The effective conductance equals the total current i_a flowing out of a. This current is

$$\begin{aligned} i_a &= \sum_y (1 - \varphi(y)) w(a, y) \\ &= \sum_y (1 - \varphi(y)) \frac{w(a, y)}{d_G(a)} d_G(a) \\ &= d_G(a)(1 - \sum_y P_{ay}\varphi(y)) \\ &= d_G(a) P_{escape} \ , \end{aligned}$$

where P_{escape} is the probability that starting at a, the walker reaches b before returning to a. Thus

$$C_{EFF} = d_G(a) P_{escape},$$

and

$$P_{escape} = \frac{C_{EFF}}{d_G(a)}.$$

The key element in finding the bound on the time it takes for the network to reach a recurrent configuration is finding an upper bound on the effective resistance R_{EFF} . Here, we use Rayleigh's Monotonicity Law [31] in finding the upper bound on R_{EFF} . For completeness, we will prove the Rayleigh's Monotonicity Law using the probabilistic interpretation of effective conductance.

Rayleigh's Monotonicity Law: If the resistances of a circuit are increased, the effective resistance R_{EFF} between any two points will increase. If they are decreased, it will decrease.

As usual we have a network of conductances (streets) and a walker moves from point x to point y with probability $P_{xy} = \dfrac{w(x,y)}{d_G(x)}$, where $w(x,y)$ is the conductance and $d_G(x)$ $= \sum_y w(x,y)$. As we have done earlier, we choose two points a and b. The walker that starts at a and walks until he reaches b. We say $\varphi(x)$ is the probability that the walker that starts at x, reaches a before b. Then $\varphi(a) = 1$, and $\varphi(b) = 0$. And the function $\varphi(x)$ is harmonic at every point $x \neq a, b$. As before, we denote by P_{escape} the probability that the walker , starting at a, reaches b before returning to a. Then $P_{escape} = 1 - \sum_y P_{ay}\varphi(y)$. As we have seen already, the effective conductance between a and b is the product $d_a P_{escape}$. We wish to show that if one of the conductances $w(r,s)$ is increased then the effective conductance increases. The case where r and s is either a or b is easy. Therefore, we assume that $r, s \neq a$ and $r, s \neq b$. Instead of increasing $w(r,s)$, we can think of it as adding a new edge (bridge) of conductance ϵ between r and s. Intuitively, we can think of it by adding a bridge, this opens up new possibilities of escaping. One might say, it also opens up new possibilities of returning to the starting point. It turns out the walker will cross the bridge more often in a *good* direction than a *bad* direction. Let rs be an edge with endpoints neither a nor b. We assume that $\varphi(r) \geq \varphi(s)$. There is therefore a better chance of escaping from s than r. A *good* direction means to cross the edge from r to s. We will show that the walker will cross the edge from r to s more often on the average than from s to r.

Let u_x be the expected number of times that the walker is at x, and u_{xy} the expected number of times he crosses the edge xy from x to y before he reaches b or returns to a. As we have seen in the random walk interpretation of currents (see p. 90) that $\frac{u_x}{d_G(x)}$ is harmonic for $x \neq a, b$ with boundary conditions $\frac{u_a}{d_G(a)}$, 0 at a and b respectively . But the function $\frac{\varphi(x)u_a}{d_G(a)}$ is also harmonic with the same boundary conditions as $\frac{u_x}{d_G(x)}$. By the uniqueness principle: $\frac{u_x}{d_G(x)} = \frac{\varphi(x)u_a}{d_G(a)}$. Now,

$$\begin{aligned} u_{rs} &= u_r P_{rs} \\ &= u_r \frac{w(r,s)}{d_G(r)} \\ &= \varphi(r)\frac{w(r,s)u_a}{d_G(a)}, \end{aligned}$$

and

$$\begin{aligned} u_{sr} &= u_s P_{sr} \\ &= u_s \frac{w(s,r)}{d_G(s)} \\ &= \varphi(s)\frac{w(s,r)u_a}{d_G(a)}. \end{aligned}$$

Since $w(r,s) = w(s,r)$ and by assumption $\varphi(r) \geq \varphi(s)$, this means that $u_{rs} \geq u_{sr}$.

Recall that we are denoting the conductance of the bridge by ϵ. Let us put δ superscripts on the quantities that refer to the walk on the graph with the bridge. Let E^δ denotes the expected net number of times the walker crosses the bridge from r to s. Therefore, we have $E^\delta = \left(\frac{u_r^\delta}{d_G(r)+\epsilon} - \frac{u_s^\delta}{d_G(s)+\epsilon}\right)\epsilon$.

Claim:

$$P_{escape}^\delta = P_{escape} + (\varphi(r) - \varphi(s))E^\delta.$$

The above claim is best explained through a game in which your fortune is $\varphi(x)$. Recall that $\varphi(x)$ is the probability of returning to a before reaching b. Then your expected final earning is $1(1-P^{\delta}_{escape})+0(P^{\delta}_{escape})=1-P^{\delta}_{escape}$. So the amount you would expect to lose is P^{δ}_{escape}. The first step you take away from a, you would expect to lose $1-\sum_x P^{\delta}_{ax}\varphi(x) = P_{escape}$. Every time you step away from r, you would expect to lose an amount

$$\varphi(r)-\left(\sum_x \varphi(x)\frac{w(r,x)}{d_G(r)+\epsilon}+\varphi(s)\frac{\epsilon}{d_G(r)+\epsilon}\right)=(\varphi(r)-\varphi(s))\frac{\epsilon}{d_G(r)+\epsilon}.$$

Similarly, every time you step away from s you expect to lose an amount

$$(\varphi(s)-\varphi(r))\frac{\epsilon}{d_G(s)+\epsilon}.$$

Hence, the total amount you expect to lose equals (expected loss at first step)+(expected loss at r)(expect number of times at r)+(expected loss at s)(expected number of times at s). Therefore,

$$\begin{aligned} P^{\delta}_{escape} &= P_{escape}+((\varphi(r)-\varphi(s))\frac{\epsilon}{d_G(r)+\epsilon}u^{\delta}_r+((\varphi(s)-\varphi(r))\frac{\epsilon}{d_G(s)+\epsilon}u^{\delta}_s \\ &= P_{escape}+(\varphi(r)-\varphi(s))E^{\delta}. \end{aligned}$$

This is exactly what we needed for the proof of Rayleigh's Monotonicity Law**.** From the Rayleigh's Monotonicity Law, we conclude that R_{EFF} is increased if one removes an edge from the graph. This is due to the fact that removal of an edge corresponds to replacing a unit resistor by an infinite resistor. Hence, R_{EFF} is bounded by the length of the minimal path connecting a to b. Moreover, it is bounded by the diameter of the graph.

3.7 Upper Bound for Run-Time Estimates

We now give an upper bound estimate for the time it would take for a configuration to reach a stable or recurrent configuration in the Dirichlet game and in the electrical network.

Theorem 3.7.1

(a) If r_0, the starting configuration for an electrical network, is active then for all values $t \geq D \parallel r_0 \parallel^+$, r_t is a passive configuration, where $\parallel r_0 \parallel^+ = \sum_{v \in S,\, r_0(v) > 0} r(v)$.

(b) If r_0, the starting configuration for electrical network, is passive then for all values $t \geq D \parallel r_0 \parallel^-$, r_t a recurrent configuration, where $\parallel r_0 \parallel^- = -\sum_{v \in S,\, r(v) < 0} r(v)$.

(c) For any $t \geq D \parallel r_0 \parallel$, r_t is the recurrent configuration, where

$$\parallel r_0 \parallel = \parallel r_0 \parallel^+ + \parallel r_0 \parallel^- .$$

Proof: By Theorem 3.5.2, If $r_t(v) < 0$ then $r_t(v) \leq r_{t'}(v) \leq 0$ for all $t' \geq t$ and if $r_t(v) > 0$ then $0 \leq r_{t'}(v) \leq r_t(v)$ for all $t' \geq t$. This means that for $t' \geq t$, we have $\{v : r_{t'}(v) > 0\} \subseteq \{v : r_t(v) > 0\}$ and if $r_t(u) > 0$ and $u \notin \{v : r_{t'}(v) > 0\}$ then $r_{t'}(u) = 0$. According to the Theorem 3.5.6, for a given t, we can obtain a graph G'_t by contracting all active vertices into a vertex a_0 and all passive vertices into a vertex p_0 such that $\sum_{v \in V^a_{r_t}} \frac{dr_t(v)}{dt} = \frac{dr'_t(a_0)}{dt}$. Since the total electrical currents away from a_0 to the neighboring vertices is the effective conductance of the graph G'_t, $\sum_{v \in V^a_{r_t}} \frac{dr_t(v)}{dt} = -C_{EFF}$. By Rayleigh's Monotonicity Law, $R_{EFF} \leq D'_t$, hence $C_{EFF} \geq \frac{1}{D'_t}$, where

D'_t is the diameter of the graph G'_t. Since the diameter of the graph decreases after the contraction we have $C_{EFF} \geq \dfrac{1}{D}$, where D is the diameter of the original graph G. Hence $\sum_{v \in V^a_{r_t}} dr_t(v)/dt \leq -\dfrac{1}{D}$ for all $t \geq 0$. Integrating both sides with respect to t, we find that

$$\begin{aligned}\sum_{v \in S,\, r_t(v) > 0} r_t(v) &\leq \sum_{v \in S,\, r_0(v) > 0} r_0(v) - \frac{t}{D} \\ &= \| r_0 \|^{+} - \frac{t}{D}.\end{aligned}$$

Since for an active configuration, we must have $\sum_{v \in S,\, r_t(v) > 0} r_t(v) > 0$, it is impossible for the configuration to be active for $t \geq D \| r_0 \|^{+}$. This proves part (a).

Part (b) follows the same argument. From the dynamics of the electrical network, we have $\{v : r_{t'}(v) < 0\} \subseteq \{v : r_t(v) < 0\}$ for all $t' \geq t$. And if $r_t(u) < 0$ and $u \notin \{v : r_{t'}(v) > 0\}$ then $r_{t'}(u) = 0$. Since r_0 is passive, the only active vertices must be the boundary nodes of G for all $t \geq 0$. Applying Theorem 3.5.6 again to the graph G'_t for a given $t \geq 0$, we have $\sum_{v \in V^p_{r_t}} \dfrac{dr_t(v)}{dt} = \dfrac{dr'_t(p_0)}{dt}$. As a consequence of Kirchhoff's law, the total of this currents away from a_0 into the neighboring vertices of a_0 is equal to the total of this currents into p_0 from the neighboring vertices of p_0. Hence $\dfrac{dr'_t(p_0)}{dt} = C_{EFF}$ or $\sum_{v \in V^p_{r_t}} \dfrac{dr_t(v)}{dt} = C_{EFF}$, where C_{EFF} is the effective conductance of G'_t. Using the fact $C_{EFF} \geq \dfrac{1}{D}$, we find that $\sum_{v \in V^p_{r_t}} \dfrac{dr_t(v)}{dt} \geq \dfrac{1}{D}$. Integrating both sides with respect to t, we obtain

$$\sum_{v \in S,\, r_t(v) < 0} r_t(v) \geq \sum_{v \in S,\, r_0(v) < 0} r_0(v) + \frac{t}{D}$$

$$= - \parallel r_0 \parallel^- + \frac{t}{D}.$$

Since for a passive configuration we must have $\sum_{v \in S,\, r_t(v)<0} r_t(v) < 0$, it is impossible for the configuration r_t to be passive for $t \geq D \parallel r_0 \parallel^-$. Since r_0 is a passive configuration r_t must be the recurrent configuration for $t \geq D \parallel r_0 \parallel^-$.

For part (c), suppose r_0 is not the recurrent configuration. Let t_0 be the time at which the configuration turns passive, then by part (b) for any $t \geq t_0 + D \parallel r_{t_0} \parallel^-$, r_t is the recurrent configuration. Hence, if $t \geq D \parallel r_0 \parallel^+ + D \parallel r_{t_0} \parallel^-$ then r_t is the recurrent configuration. By the dynamics of the electrical network, we have $\parallel r_{t_0} \parallel^- \leq \parallel r_0 \parallel^-$. So if $t \geq D \parallel r_0 \parallel$ then r_t is the recurrent configuration. And if r_0 is the recurrent configuration then r_t is recurrent for all $t \geq 0$. Hence r_t is the recurrent configuration for $t \geq D \parallel r_0 \parallel$.

Theorem 3.7.2

Given s_0 an initial configuration of the Dirichlet game which is not a stable configuration we have:

(a) for any integer $t \geq D \parallel s_0 \parallel^+$, s_t is a stable configuration,

(b) for any integer $t > D(\parallel s_0 \parallel + \sum_{v \in S} d_G(v) - \mid S \mid) + 1$, s_t is a critical configuration, where $\mid S \mid$ is the number of vertices in S.

Proof:

Define r_0 as in Theorem 3.5.5, $\parallel s_0 \parallel^+ = \parallel r_0 \parallel^+$. According to the Theorem 3.7.1, r_t is a passive configuration for $t \geq D \parallel s_0 \parallel^+$. By Theorem 3.5.5, s_t is a stable configuration for all integers $t \geq D \parallel s_0 \parallel^+$. This proves part (a).

For part (b), let t_0 be the first integer such that s_{t_0} is a stable configuration. Set $r_0(v) = s_{t_0}(v) - (d_G(v) - 1)$. According to the Theorem 3.7.1, for all $t \geq D \parallel r_0 \parallel^-$, r_t is a recurrent configuration. Therefore by Theorem 3.5.4, s_t is a critical configuration for all integers $t \geq t_0 + D \parallel r_0 \parallel^-$.

We now find an estimate for $\parallel r_0 \parallel^-$. If $s_0(v) < 0$ then $s_0(v) \leq s_{t_0}(v) \leq (d_G(v) - 1)$, therefore,

$$0 \leq -r_0(v) \leq -s_0(v) + d_G(v) - 1. \qquad (1)$$

If $s_0(v) \geq 0$, then $s_{t_0}(v) \geq 0$. Therefore,

$$-r_0(v) = -s_{t_0}(v) + (d_G(v) - 1) \leq (d_G(v) - 1). \qquad (2)$$

Putting the two inequalities (1) and (2) together, we obtain:

$$\parallel r_0 \parallel^- \leq \parallel s_0 \parallel^- + \sum_{v \in S} d_G(v) - \mid S \mid,$$

where $\mid S \mid$ is the number of vertices in S. By part (a), $t_0 < D \parallel s_0 \parallel^+ + 1$. Therefore we have that for all integers t such that

$$\begin{aligned} t &> D \parallel s_0 \parallel^+ + 1 + D \parallel s_0 \parallel^- + D \sum_{v \in S} d_G(v) - D \mid S \mid \\ &= D(\parallel s_0 \parallel + \sum_{v \in S} d_G(v) - \mid S \mid) + 1, \end{aligned}$$

s_t is a critical configuration.

QED

We now apply our result to the special case where our graph $G = S \cup \partial S$ is simple and connected with Chung and Ellis's version of the Dirichlet game. Their version is a special case of our version where there are no negative number of chips in the interior of G and the boundary nodes act only as processors. In other words, chips fired from a vertex in S to a vertex in ∂S are instantly processed and removed from the game. Thus a configuration s of this version of the Dirichlet game is a vector $s : V(G) \to \mathbb{Z}^+ \cup \{0\}$ which satisfies Dirichlet boundary condition $s(q) = 0$ for all $q \in \partial S$. A configuration s is stable, if $s(v) < d_G(v)$ for all v in S. Starting from an initial configuration s_0, we say that the game terminates when it reaches a stable configuration. In our version, this is equivalent to saying that the game has reached the first stable configuration. Let $f(v)$ be the number of times the vertex v in S is fired during the Dirichlet game and $\sum_{v \in S} f(v)$ is the total number of firings, which is finite by Theorem 3.1.1. Here, we call f the score vector of the game and is uniquely defined by

$$L_{w,S} f = s_0 - s_e,$$

where s_e is the end configuration of the game or the first stable configuration in our version. We state without proof the following result of Chung and Ellis [17] which computes a bound on the total number of firings $\sum_{v \in S} f(v)$ during the Dirichlet game by using the Dirichlet eigenvalues.

Chung and Ellis's result

Let f be the score vector of a chip-firing game with Dirichlet boundary conditions. Then the total number of firings in the game is bounded as follows:

$$\sum_{v \in S} f(v) \leq D \parallel s_0 \parallel | S |^{\frac{3}{2}}$$

where $\| s_0 \|$ is the total number of chips initially, $| S |$ is the number of interior vertices, and D is the diameter of the graph.

We will improve the above bound on the total number firings using Theorem 3.7.2 (a). For a real number a, let $\lceil a \rceil$ be the smallest integer greater than or equal to a. Now, for each integer t less than the time of the first stable configuration, there are at most $| S |$ vertex firings since the boundary nodes do not fire during this time. And by Theorem 3.7.2 (a), for any integer $t \geq D \| s_0 \|^+$, s_t is a stable configuration. Let t_0 be the time of the first stable configuration. Then there are at most $| S | t_0$ vertex firings before the first stable configuration. Since $t_0 \leq t$, we find that there are at most $| S | \left\lceil D \| s_0 \|^+ \right\rceil$ vertex firings before the first stable configuration. Using the fact that there are no vertices with negative number of chips in the interior of G, we find that $\| s_0 \|^+ = \| s_0 \|$. In the case of a simple graph, we have $| S | \lceil D \| s_0 \| \rceil = D \| s_0 \| | S |$, since D is an integer. Hence the total number of vertex firings in Chung and Ellis's version of the Dirichlet game is bounded by $D \| s_0 \| | S |$.

Chapter IV

4 Algebraic Aspects of the Dirichlet Game

The purpose of this chapter is to relate the Laplacian and the Green's function studied in Chapters 1, 2 to our version of the chip firing game studied in Chapter 3. As one of the applications of this relation is to obtain a bound on the total number of vertex firings to achieve a critical configuration from an arbitrary configuration s independent of the norm $\parallel s \parallel$ of s. The idea is to consider a class of configurations that leads to a unique critical configuration and choose a stable configuration in this class. In other words, given a configuration s with an arbitrary large number of chips, we seek a stable configuration s_0 such that s and s_0 belong to the same coset, i.e., s and s_0 will lead to the same critical configuration. This is done by first applying the Green function on s, to produce a sequence of legal firings that finds the stable configuration s_0.

We will organize this chapter as follows. In section 1, we generalize the classical result of the matrix-tree theorem [13] to weighted graphs. Namely, we will prove that the product of the eigenvalues of the Dirichlet Laplacian is the same as the number of spanning weighted forest rooted in the boundary of the graph. In the second section, we will obtain a bound on the total number of vertex firings to achieve a critical configuration from an arbitrary configuration s independent of the norm of s, $\parallel s \parallel$, by using Green's function. The last section discusses the fact that the of set of critical configurations has the same cardinality as the set of spanning weighted forest rooted in the boundary of the graph by introducing an algorithm to achieve this bijection.

4.1 The Determinant of the Dirichlet Laplacian

Recall from Chapter 1.1, the weighted combinatorial Laplacian is defined by

$$L_w(x,y) = \begin{cases} d_G(x) & if\ x = y \\ -w(x,y) & \text{if } x \text{ is adjacent to } y \\ 0 & \text{otherwise.} \end{cases}$$

For any function $f : V \to R$, we have

$$L_w f(x) = \sum_y (f(x) - f(y))w(x,y).$$

We consider the subgraph $\overline{G} = (\overline{S}, S')$ of $G = (\overline{S}, E)$ where $\overline{S} = S \cup \partial S$ and S' denotes the set of all edges in E excluding those edges whose endpoints are in ∂S. And let $L_{w,S}$ be the submatrix of L_w restricted to columns and rows indexed by vertices in S. Next, we consider the incidence matrix B with rows indexed by vertices in S and columns indexed by edges S' as follows:

$$B_w(x,e) = \begin{cases} \sqrt{w(x,y)} & if\ e = \{x,y\}, x < y \\ -\sqrt{w(x,y)} & if\ e = \{x,y\}, x > y \\ 0 & \text{otherwise} \end{cases}$$

We note that $L_{w,S} = B_w B_w^T$, where B_w^T denotes the transpose of B_w. We next define a weighted rooted spanning forest of S to be any subgraph F satisfying:

(1) F is an acyclic subgraph of G

(2) F has a vertex set $\overline{S}$,

(3) Each connected component of F contains exactly one vertex in ∂S.

Let us now define the weight of a rooted spanning forest of S. Each connected component of this rooted forest is a tree with its only root is at a vertex v in the boundary ∂S, and let T_v denote this tree in S . Recall that a tree is a connected subgraph with no cycles. Now, we define the weight of T_v as follows: For each edge $e = \{x, y\}$ in the edge set $E(T_v)$, the weight of this edge is defined as $w(e) = w(x, y)$, and

$$w(T_v) = \Pi_{e \in E(T_v)} w(e).$$

We also define

$$w(F) = \Pi_{v \in \partial S} w(T_v),$$

and

$$\kappa(S) = \sum_{F} w(F),$$

where the summation takes place over all possible rooted spanning forest F. We will now give a brief sketch of the proof of the following theorem which is quite similar to the original proof of the matrix-tree theorem [13].

Theorem 4.1.1

For an induced subgraph S in G with $\partial S \neq \emptyset$, the determinant of the Dirichlet weighted Laplacian $L_{w,S}$ is

$$\det L_{w,S} = \Pi_{i=1}^{s} \sigma_i = \kappa(S),$$

where $s = \mid S \mid$, the number of vertices in S.

Proof: The product of the eigenvalues

$$\begin{aligned}\Pi_{i=1}^{s}\sigma_i &= \det L_{w,S} \\ &= \det(B_w B_w^T) \\ &= \sum_X \det B_{w,X} \det B_{w,X}^T .\end{aligned}$$

where X ranges over all possible choices of s edges and $B_{w,X}$ denotes the square submatrix of B whose s columns correspond to the edges in X. This expansion over X, known as the Cauchy-Binet expansion, is described in [45].

Claim 1: If the subgraph with vertex set $\overline{S}$ and edge set X contains a cycle, then $\det B_{w,X} = 0$.

The proof is similar to that in the matrix-tree theorem [13, 22] and we just briefly mention that the columns restricted to those indexed by the cycle are dependent.

Claim 2 : If the subgraph formed by the edge set X contains a connected component having two vertices in ∂S, then $\det B_{w,X} = 0$.

Proof: Let Y denote a connected component of the subgraph formed by X. If Y contains more than one vertex in ∂S, then Y has no more than $|E(Y)| - 1$ vertices in S. The submatrix formed by the columns corresponding to edges in Y has rank at most $|E(Y)| - 1$. Therefore, $\det B_{w,X} = 0$.

From Claim 1 and Claim 2, we know that the edges of X form a forest and each connected component contains exactly one vertex in ∂S. Therefore, There is a column indexed by an edge with only one nonzero entry, say (x_1, e_1) with $x_1 \in S$. Therefore,

$$| \det B_{w,X} | = \sqrt{w(e_1)} \, | \det B_{x_1}^{(1)} |$$

where $B_{w,x_1}^{(1)}$ denotes the submatrix with rows indexed by $S - \{x_1\}$ and columns indexed by $X - \{e_1\}$. By removing $w(e)$ edges and one vertex at a time, we eventually obtain :

$$| \det B_{w,X} | = \Pi_{e \in X} \sqrt{w(e)}.$$

Combining the claims (1) and (2) , with the above result, we have:

$$\begin{aligned}
\Pi_{i=1}^{s} \sigma_i &= \det L_{w,S} \\
&= \sum_X \det B_{w,X} \det B_{w,X}^T \\
&= \sum_X \Pi_{e \in X} w(e) \\
&= \kappa(S).
\end{aligned}$$

QED

By the above discussion, the problem of evaluating the determinant or $\Pi_{i=1}^{s} \sigma_i$ of the Dirichlet eigenvalues is the same as enumerating rooted spanning weighted forests of an induced subgraph which is known to be a difficult problem. But since the eigenvalues can be computed in polynomial time, we can therefore say that there is a polynomial algorithm to evaluate $\kappa(S)$.

4.2 Relation of Green's Function to Dirichlet Game

The objective of this section is to show how the Dirichlet Laplacian and its Green's function can be used to develop an algorithm which produces the critical configuration corresponding to an arbitrary configuration. Let $C^0(\overline{S}; Z)$ and $C^1(S'; Z)$ denote the Abelian groups of integer valued functions defined on the vertices $\overline{S}$ and the edges S' of the graph $G = (\overline{S}, S')$ respectively. Considering the elements of these spaces as column vectors, the incidence matrix B_w and its transpose B_w^T can be regarded as homomorphisms

$$B_w : C^1(S'; Z) \to C^0(\overline{S}; Z),$$

and

$$B_w^T : C^0(\overline{S}; Z) \to C^1(S'; Z) .$$

We can also consider $L_w = B_w B_w^T$ as a homomorphism $C^0(\overline{S}; Z) \to C^0(\overline{S}; Z)$. Let the function $\sigma : C^0(\overline{S}; Z) \to Z$ be defined by $\sigma(f) = \sum_x f(x)$. Then we have the following lemma and theorem due to Biggs [9]. For completeness, we state the proof with minor variations.

Lemma 4.2.1

The image of L_w is a normal subgroup of the kernel of σ

Proof: Since each column of B has only two non-zero entries, $\sqrt{w(x, y)}$ and $-\sqrt{w(x, y)}$ that add up to 0, it follows that $\sigma B = 0$. Furthermore, if $x \in \operatorname{Im} L_w$, say $x = L_w y = B_w B_w^T y$, then $\sigma(x) = \sigma(B_w B_w^T y) = \sigma B_w (B_w^T y) = 0$. So $x \in Ker\ \sigma$. Hence the image of L_w is a subgroup of $Ker\ \sigma$. Since the groups are abelian, it is a normal subgroup.

Denote the set of critical configurations on a graph G by $K(G)$. For each configuration s there is a unique critical configuration $\gamma(s) \in K(G)$ determined by Theorem 3.3.5. The following theorem gives an abelian group structure to $K(G)$.

QED

Theorem 4.2.2

The set $K(G)$ of critical configurations on a connected graph G is in a one-to-one correspondence with the Abelian group $Ker\sigma \,/\mathrm{Im}\, L_w$.

Proof: First, we show that there is a configuration representing an element of $Ker\sigma \,/\mathrm{Im}\, L_w$. Given a $x \in Ker\ \sigma$, let s be the configuration defined on vertices in the S by

$$s(v) = \begin{cases} d_G(v) - 1 & if\ x(v) \geq 0 \\ d_G(v) - 1 - x(v) & if\ x(v) < 0 \end{cases}$$

and the value of s on the boundary is defined in such a way that $\sum_{q \in \partial S} s(q) = -\sum_{v \in S} s(v)$. Let $\theta = v_1, ..., v_n$ be a sequence that leads s to a stable configuration s_0 then $s_0 = s - L_w\theta$. Let $z = x + s - s_0$. Then $z = x + L_w\theta$, so $[z] = [x]$ and

$$z(v) = x(v) + s(v) - s_0(v) \geq \ d_G(v) - 1 - s_0(v) \geq 0.$$

Hence there is a configuration z representing the given coset $[x]$. We now show that the mapping

$$h : Ker\sigma \,/\mathrm{Im}\, L_w \to K(G),$$

given by $h(x) = \gamma(s)$, where s is any configuration in the coset $[x]$, is well-defined.

Suppose that s_1 and s_2 are configurations such that $[s_1] = [s_2] = [x]$. In this case $s_1 - s_2 = L_w f$ for some $f \in C^0(\overline{S}; Z)$. We can write $f = f_1 - f_2$ where f_1 and f_2 are nonnegative functions. Let $s_0 = s_1 - L_w f_1 = s_2 - f_2$. Then by Theorem 3.3.5, there is a unique critical configuration c is reached by a legal sequence of firings applied to s_0. So $c = \gamma(s_0)$. From the equation $s_0 = s_1 - L_w f_1$, we conclude that there is a legal sequence of firings that leads from s_1 to s_0. Hence, there is a legal sequence of firings that leads from s_1 to c, which is a critical configuration. By Theorem 3.3.5, we must have $\gamma(s_1) = c$. Using the same argument, we have $\gamma(s_2) = c$. Hence h is well-defined. To show h is injective, suppose that $h[s_1] = h[s_2]$. Then $\gamma(s_1) = \gamma(s_2) = c$, where c can be reached from s_1 and from s_2 by legal sequences of firings. So there are vectors x_1 and x_2 such that $c = s_1 - L_w x_1 = s_2 - L_w x_2$. Hence $s_1 - s_2 = L_w(x_1 - x_2)$ or $[s_1] = [s_2]$. It is easy to see that h is surjective. Given a critical configuration c in $K(G)$ then $\gamma(c) = c$ and $h([c]) = c$.

QED

Since there is an Abelian group structure on $Ker\sigma \,/\mathrm{Im}\, L_w$, defined by $[s_1] + [s_2] = [s_1 + s_2]$, we can translate this structure to $K(\overline{S})$ by Theorem 4.2.2 under the operation $\oplus$, where $h[s_1] \oplus h[s_2] = h[s_1 + s_2]$, that is $\gamma(s_1) \oplus \gamma(s_2) = \gamma(s_1 + s_2)$. Equivalently, for any two critical configurations c_1 and c_2, we have

$$c_1 \oplus c_2 = \gamma(c_1 + c_2).$$

For a real number a, let $\lfloor a \rfloor$ be the floor of a, i.e., the largest integer smaller than or equal to a. And for a vector $\mathbf{a}$, let $\lfloor \mathbf{a} \rfloor$ be the largest vector obtained by taking the floor of every coefficient of **a**. For the configuration s on the induced subgraph G_S define

$$s' = s - L_{w,S} \left\lfloor G_{w,S}(s) \right\rfloor,$$

where $G_{w,S}$ is the Green's function of the Dirichlet combinatorial Laplacian, i.e., $G_{w,S} = L_{w,S}^{-1}$. By the above argument, since $\left\lfloor G_{w,S}(s) \right\rfloor \in Z^{|\overline{S}|}$ and $s\ |_{\partial S}= 0$ we have $\gamma(s') = \gamma(s)$. The following lemma, extended from the case of a standard graph [17] to a weighted one describes how to obtain a configuration with a small number of chips from a configuration with arbitrary large number of chips with the same corresponding critical configuration.

Lemma 4.2.3

Given a configuration s and the discrete Green's function $G_{w,S}$, the configuration s' defined by

$$s' = s - L_{w,S} \left\lfloor G_{w,S}(s) \right\rfloor$$

satisfies

$$|\ s'(x)\ | < d_G(x).$$

Proof: Set $t = G_{w,S}(s) - \left\lfloor G_{w,S}(s) \right\rfloor$ then $0 \leq t(y) < 1$ for all $y \in S$, and

$$\begin{aligned} L_{w,S}(t) &= L_{w,S}G_{w,S}(s) - L_{w,S} \left\lfloor G_{w,S}(s) \right\rfloor \\ &= s - L_{w,S} \left\lfloor G_{w,S}(s) \right\rfloor \\ &= s'. \end{aligned}$$

Since

$$L_{w,S}(t) = t(x)d_G(x) - \sum_y t(y)w(x,y),$$

and $0 \leq t(y) < 1$ for all $y \in S$, we have $\mid s'(x) \mid < d_G(x)$.

QED

Applying Lemma 4.2.3 to any configuration s with an arbitrary large number of chips, we can obtain a configuration s' such that $\mid s'(x) \mid < d_G(x)$ and $\gamma(s') = \gamma(s)$, *i.e.*, both s and s' will reach the same critical configuration. This is done at the cost of $O(n^{\omega})$ arithmetic operations for matrix inversion, where $2 \leq \omega \leq 2.376$.

For real number a, let $[a]$ be the smallest integer greater than a. Since for each integer t there are at most n firings, where n is the number of vertices in G. By theorem 3.7.2, there is at most $n\left[D(\parallel s' \parallel + \sum_{x\in S} d_G(x) - \mid S \mid) + 1\right]$ number of firings to reach a critical configuration. Since $\parallel s' \parallel < \sum_{x\in S} d_G(x)$, there will be at most $n\left[D(2\sum_{x\in S} d_G(x) - \mid S \mid) + 1\right]$ firings to reach a critical configuration. This argument yields the complexity of determining a critical configuration corresponding to an arbitrary configuration, which we summarize in the following theorem.

Theorem 4.2.4

Given a configuration s in the Dirichlet game, computing the corresponding critical configuration requires at most

$$n\left[D(2\sum_{x\in S} d_G(x) - \mid S \mid) + 1\right]$$

vertex firings and $O(n^{\omega})$ arithmetic operations, where D is the diameter of G.

4.3 Critical Configuration and Rooted Spanning Weighted Forest

The set of critical configurations can be characterized as a set having the same cardinality as the number of spanning weighted forests rooted in ∂S. A bijection between the two sets is obtained algorithmically by playing a chip-firing game using a critical configuration as an initial point. The idea is motivated by a theorem due to Biggs [9] which states that the critical group $K(G)$ has order k , the number of spanning weighted tree in G rooted in a distinguished vertex q. We will form a new graph G' by connecting q to all the vertices in the boundary ∂S such that the number of edges connecting q to each vertex $u \in \partial S$ is exactly 1, then the set of a spanning weighted forest of G rooted in the boundary ∂S has the same cardinality as the set of spanning weighted trees of G' rooted in q such that the number of edges connecting q to each vertex $u \in \partial S$ is one. This is due to our basic assumption that $w(x, y) = 0$ for all $x, y \in \partial S$. Before, we state the next theorem which draws a bijection between the set of critical configurations in G and the set of critical configurations c_q in G' such that $c_q \mid_{\partial S} (x) = d_{G'} - 1$, we will present the following corollary which is a consequence of the Theorem 3.3.7.

Corollary 4.3.1

The configuration c is critical if and only if $c + \alpha_c$ yields c under some firing sequence which is a permutation of S, where α_c is the configuration defined as follows:

$$\alpha_c(v) = \sum_{u \in \partial S} w(u, v), \quad \text{for every } v \in S .$$

Theorem 4.3.2

Given a critical configuration c in G, define c_q, a configuration in G' with a single boundary node $\{q\}$, by

$$c_q(x) \quad = c(x)\chi_S + (d_{G'}(x) - 1)\chi_{\partial S}.$$

Then c_q is a critical configuration of G' with a single boundary node $\{q\}$. Conversely, any critical configuration c_q in G' with a single boundary node $\{q\}$ such that

$$c_q \mid_{\partial S} (x) = \ d_{G'}(x) - 1,$$

can be written in the above form for some critical configuration c in G .

Proof: It is easy to see that c_q is a stable configuration in G'. Since c is a critical configuration in G and the number of chips in the boundary nodes are at most $d_{G'}(x) - 1$. Now, the configuration

$$(c_q + \alpha_{c_q})(x) = c(x)\chi_S + \alpha_{c_q}(x),$$

will yield the configuration $c(x)\chi_S + (d_{G'}(x))\chi_{\partial S}$ after q is being fired. Also once the vertices in ∂S are fired exactly once, we get the configuration $(c(x) + \alpha_c(x))\chi_S$. Since c is a critical configuration, By Corollary 4.3.1 there is a sequence of firings which is a permutation of S such that after applying it to the configuration $c+\alpha_c$, we obtain c. Putting it all together, we have a legal sequence of firings which is a permutation of $\overline{S}$ such that it will lead the configuration $c_q + \alpha_{c_q}$ to c_q. Hence, by Corollary 4.3.1, c_q is a critical configuration of G'.

Conversely, given a critical configuration c_q in G' such that $c_q \mid_{\partial S} (x) = d_{G'}(x) - 1$, then by Corollary 4.3.1, there is a sequence of firings which is a permutation of $\overline{S}$ such that it will lead $c_q + \alpha_{c_q}$ to c_q . Since the vertices in G are fired only once after the vertex q is activated by this sequence of firings, by applying Theorem 3.3.7, the configuration $c = c_q\mid_S$ which reappears would be a critical configuration. Furthermore, $c_q(x) = c_q\mid_S(x)\chi_S$.

QED

Theorem 4.3.2 combined with the argument at the beginning of the previous paragraph would imply that the set of critical configurations in G is in a one-to-one correspondence with the set of spanning weighted forest rooted in the boundary ∂S. Now, we are ready to present an algorithm to find the spanning forest once a critical configuration in G is given. The idea behind this algorithm lies in Corollary 4.3.1. Given a critical configuration, once we activate the boundary nodes, the vertices are only fired once. So we construct an edge only when firing one vertex would make the adjacent vertex fire. In this way, since the vertices are only fired once, we would avoid making a cycle in the construction. Hence, the resulting construction is a tree. Since all the vertices are fired, it is a spanning tree. Now, give a critical configuration c_q in G' . Assign a total ordering of edges in G'. Add a chip to every vertex in ∂S as if q were fired. Then add the edges $\{q, u\}$ to the tree which was initially empty for each u adjacent to q. Fire the vertex u that is ready. If there are more than one vertex that are ready, fire the one that has a shortest path to q with respect to the total ordering given to the edges initially. Add $\{u, v\}$ to the tree if firing u causes v to be ready to fire. According to the number of chips, $w(u, v)$, that v receives add $d_G(v) - c_q(v)$ edges between u and v. Note that because c_q is a critical configuration then we must have

$d_G(v) - c_q(v) \geq 0$. Now, since u causes v to fire we have

$$d_G(v) - c_q(v) \leq w(u, v) \; .$$

Hence the weight of the edge of the constructed tree is less than or equal to the original weight. Repeat this process until all vertices are fired. By Corollary 4.3.1, the construction leads to a spanning weighted tree. If there are two critical configurations , c_q and c'_q such that $c_q(v) \neq c'_q(v)$ for some vertex $v \in S$ and firing a vertex u adjacent to v causes v to fire in both critical configurations c_q and c'_q , then the number of edges connecting u to v in construction of the corresponding spanning trees to c_q and c'_q will be different. This shows that the two different critical configurations yield two different spanning weighted trees. Since the set of critical configurations has order k , the number of spanning tree in G', this algorithm is actually a bijection from the set of critical configuration of G' to the set of spanning tree in G'. Now, given a critical configuration c in G, we find the corresponding critical configuration c_q in G' such that $c_q \mid_{\partial S} (x) = d_{G'}(x) - 1$. Once the corresponding spanning weighted tree with respect to c_q is constructed by the above algorithm , cut all the edges that connect q to the vertices of ∂S. Then the resulting subgraph is a spanning weighted forest rooted in ∂S that corresponds to the critical configuration c in G. From the condition $c_q \mid_{\partial S} (x) = d_{G'}(x) - 1$, it is readily seen that this algorithm draws a bijection from the set of critical configurations in G to the set of spanning weighted forest rooted in ∂S.

The chip-firing game can be used to model several aspects of Internet computing in connection with routing and fault tolerance. For instance, one such model assumes that chips are labeled by messages which they carry, and studies the propagation of messages in terms of informed and uninformed nodes of the network. In this regard, the above algorithm shows a way to geometrically reconstruct the network by means of playing the chip-firing game on the graph. Since, for a given critical configuration, the above algorithm constructs the corresponding spanning weighted forest rooted in ∂S. In this way, we can find out which pairs of nodes are connected and what their conductivities are.

5 REFERENCES

[1] D. Aldous, *Hitting Times for Random Walks on Vertex-Transitive Graphs,* Math. Proc. Cambridge Philos. Soc. **106(1)** (1989), 179-191.

[2] N. Alon, *Eigenvalues and Expanders,* Combinatorica **6(2)** (1986), 83-96.

[3] W. Anderson and D. Morley, *Eigenvalues of the Laplacian of a Graph,* Linear and Multilinear Algebra **18(2)** (1985), 141-145.

[4] P. Bak, How Nature Works, *The science of Self-Organized Criticality*, Copernicus, New York, 1996.

[5] P. Bak, C. Tang, and W. Kurt, *Self Organized Criticality*, Phys. Rev. **A(3)** (1988), 364-374.

[6] C. A. Berenstein and S-Y Chung, *Harmonic Functions and Inverse Conductivity Problems on Networks*, to appear in SIAM J. Appl. Math.

[7] N. L. Biggs, *The Tutte Polynomial as a Growth Function,* J. Algebraic Combin. **10(2)** (1999), 115-133.

[8] N. L. Biggs and P. Winkler, *Chip-Firing and the Chromatic Polynomial*, CDAM Research Report Series LSE-CDAM-97-03.

[9] N. L. Biggs, *Chip-Firing and the Critical Group of the Graph,* J. Algebraic Combin. **9(1)** (1999), 25-45.

[10] N. L. Biggs, *Algebraic Potential theory on graphs*, Bull. London Math. Soc. **29(6)** (1997), 641-682.

[11] A. Bjorner and L. Lovasz, *Chip-Firing Games on Directed Graphs,* J. Algebraic Combin. **1(4)** (1992), 305-328.

[12] A. Bjorner, L. Lovasz, and P. Shor, *Chip-Firing Games on Graphs*, European J. Combin. **12(4)** (1991), 283-291.

[13] B. Bollabas, *Modern Graph Theory*, Springer-Verlag 1998.

[14] A. Bondy and R.L. Hemminger, *Graph Reconstruction-A survey*, J. Combin. Th. **1(4)** (1997), 227-268.

[15] A.P. Calderon, *On an Inverse Boundary Value Problem*, Numerical Analysis and its Applications to Continuum, Phys. Soc. Braz. Math., (1980), 65-73.

[16] E. T. Chen, B. A. Gong, and L. J. Guo, *The isospectral Problem in a Graph Theory,* Natur. Sci. J. Harbin Normal Univ. **5** (1989), 50-57.

[17] F. Chung and R. Ellis, *A Chip-Firing Game and Dirichlet Eigenvalues*, J. Discrete Math. **257** (2002), 341-356.

[18] F. Chung, *Spectral Graph Theory*, AMS, 1997.

[19] F. Chung, *Spanning Trees in Subgraphs of Lattice*, Amer. Math. Soc., Providence, RI, (1999), 201-219.

[20] F. Chung, V. Faber, and T. A. Manteeuffel, *On the Diameter of a Graph from Eigenvalues Associated with its Laplacian*, S. J. Discrete Math. **7** (1994), 443-754.

[21] F. Chung, A. Grigorian, and S. T. Yau, *Upper Bound for Eigenvalues of the Discrete and Continuous Laplace Operator*, Advances in Math. **117** (1996), 165-178.

[22] F. Chung and R.P. Langlands, *A Combinatorial Laplacian with Vertex Weights*, Journal of Combinatorics Theory **75** (1996), 316-327.

[23] F. Chung, *Discrete Green's Functions*, Journal of Combinatorics Theory (A) **91** (2000), 191-214.

[24] F. Chung, *Distance Realization Problem with Applications to Internet Tomogra-*

phy, J. Computer and System Sciences **63(3)** (2001), 432-448.

[25] D. Coopersmith, U. Feige, and A. Schrijver, *Random Walks on Regular and Irregular Graph*, Siam J. of Discrete Math. **9** (1996), 301-308.

[26] E. B. Curtis and J. A. Morrow, *Inverse Problems for Electrical Networks*, Series on Applied Math. **13** (2000)

[27] E. B. Curtis and J. A. Morrow, *The Dirichlet to Neumann Problem for a Resistor Networks*, AMS 1990.

[28] M. Doob, *On Imbedding a Graph in an Isospectral Family*, In Proceedings of the Second Manitoba Conference on Numerical Mathematics (Univ. Manitoba, Winnipeg, Man.) (1972), 137-142.

[29] M. Doob, D. M. Dragosm, and H. Sachs, *Spectra of Graphs*, Johann Ambrosius Barth, Heidelberg, 3rd edition 1995.

[30] J. L. Doob, *Stochastic Processes*, John Wiley & Sons, New York, 1953.

[31] P. Doyle and J. L. Snell, *Random Walks and Electric Networks*, AMS, 1984.

[32] J. O. Durand-Lose, *Parallel Transient Time of One-Dimensional Sand-Pile*, Theoret. Comput. Sci. **1** (1998), 183-193.

[33] K. Eriksson, *No Polynomial Bound for the Chip-Firing Game on Directed Graphs*, Proc. Amer. Math. Soc. **4** (1991), 1203-1205.

[34] R. P. Feynman, *The Feynmann Lecture on Physics*, Addison-Wesley, 1964.

[35] A. Gabrielov, *Abelian Avalanches and Tutte Polynomia*l, Physics **A** (1993), 253-254.

[36] E. Goles and M. A. Kiwi, *Games on Line Graph and Sand Piles*, Theoret. Comput. Sci. **2** (1993), 121-134.

[37] E. Goles and M. Margenster, *Universality of the Chip-Firing Game*, Theoret. Comput. Sci. **1** (1997), 121-134.

[38] E. Dhoker and D.H. Phong, *On Determinants of Laplacians on Riemann Surfaces*, Comm. Math. Phys. **104** (1986), 537-545.

[39] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge 1985.

[40] V. Isakov, *Inverse Problem for Partial Differential Equations*, Springer-Verlag, New York, (1998).

[41] S. Kakutani, *Markov Processes and the Dirichlet Problem,* Proc. Jap. Acad. **21** (1945), 227-233.

[42] A. Kelmans and Y. Xuerong, On the Distribution of Eigenvalues of Graphs, Discrete Math.**199(1)** (1999) 251-258.

[43] F. Kelly, *Reversibility and Stochastic Networks*, John Wiley & Sons, New York, 1979.

[44] M. Kiwi, A. R. Ndoundam, M. Tchuente, and E. Gelos, *No Polynomial Bound for the Period of the Parallel Chip-Firing Game on Graphs*, Theoret. Comput. Sci. **2** (1994), 527-532.

[45] P. Lancaster, *Theory of Matrices*, Academic Press, New York, 1969.

[46] L. Lovasz and P. Winkler, *Mixing of Random Walks and other Diffusions on a Graph*, surveys in Combinatorics, (1995), 119-154.

[47] C. Meerino, *Chip-Firing and the Tutte Polynomial*, Ann. Comb. **1** (1997), 253-259.

[48] C. Meerino, *The Chip-Firing Game and Matroid Complexes,* Discrete Mathemat-

ics and Theoretical Computer Science Proceedings **AA**, (2001), 245-256.

[49] R. Merris, *Degree Maximal Graphs are Laplacian Integral. Linear Algebra Appl*. **197/198** (1994), 143-176.

[50] R. Merris, *A survey of Graph Laplacians*, Linear and Multilinear Appl. **199** (1994), 381-389.

[51] B. Mohar, The Laplacian Spectrum of Graphs, Graph Theory, Combinatorics, and Applications **2** (1991), 871-898.

[52] B. Mohar, *Isoperimetric Numbers of Graphs*, J. Combin. Theory **Ser. B 47** (1989), 274-291.

[53] C. Moore and M. Nilson, *The Computational Complexity of Sandpiles*, J. Statist. Phys. **96(1)** (1999), 205-224.

[54] C. St. J. A. Nash-Williams, *Random Walks and Electric Currents in Networks*, Proceedings of the Cambridge Philosophical Soc. **55** (1959), 181-194.

[55] G. Polya and G. Szego, *Isoperimetric Inequalities of Mathematical Physics*, Ann. of Math. Studies **no. 27** Princeton University Press, (1951).

[56] A. M. Polyakov, *Quantum Geometry of Bosonic Strings,* Phys. Letters **B 103** (1981), 207-210.

[57] D. Ray and I. M. Singer, *Analytic Torsion for Complex Manifolds,* Ann. Math. **98** (1973), 154.

[58] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, Springer-Verlag 1993.

[59] H. L. Royden, Harmonic Functions on Open Riemann Surfaces, Trans. Amer. Math. Soc. **73** (1952), 40-94.

[60] P. Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. **110** (1987), 113-120.

[61] R. Seidel, *On the all-Pairs-shortest-path problem*, Proc. Sym. Theo. on Computing, **ACM** (1992), 745-749.

[62] J. Sylvester and G. Uhlmann, *A global Uniqueness Theorem for an Inverse Value Problem*, Ann. Math., **125** (1987), 153-169.

[63] G. Tardos, *Polynomial Bound for a Chip-Firing Game on Graphs*, Siam J. Discrete Math. **1** (1988), 397-398.