ABSTRACT

Title of Dissertation: BEYOND A RELATIONAL UNDERSTANDING OF FRACTIONS: ELEMENTS OF INSTRUCTION THAT CONTRIBUTE TO PRESERVICE TEACHERS' KNOWLEDGE AND MOTIVATION

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This study was undertaken in order to better understand preservice elementary teachers' knowledge of and motivations toward fractions before and after taking a course designed to promote relational understanding, as well as what teaching practices might be related to student outcomes. Students in five sections of the course were given a fraction assessment and a motivation questionnaire at the beginning and end of the semester, and observations were made of the nine days when fractions were taught. Students' knowledge of basic concepts improved, as did their computational skill and ability to solve word problems. However, their tendency to use inefficient algorithms did not change. Error patterns at the beginning of the semester revealed misconceptions about fractions, but errors at the end of the semester were largely reflective of low skill. Value and self-concept of ability increased while anxiety decreased, but these changes differed somewhat by instructor. In particular, having students explain their thinking instead of listen to lecture tended to have increased benefits for anxiety.

BEYOND A RELATIONAL UNDERSTANDING OF FRACTIONS: ELEMENTS OF INSTRUCTION THAT CONTRIBUTE TO PRESERVICE TEACHERS' KNOWLEDGE AND MOTIVATION

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland at College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2006

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DEDICATION

This work is dedicated to my father,

Harold Gene Jones,

For always telling me I could achieve

Whatever I set my mind to.

ACKNOWLEDGMENTS

There are many people I would like to thank for their encouragement and support during this journey. First, I would like to thank my mother, Dorothy Kaye McCoy, for all the thoughts and prayers. From the beginning, she was there to say “You can do it” whenever I felt discouraged or just needed someone to believe in me. I want to thank my partner and companion, Christopher Newton, for always putting things in perspective. He gave me strength when I had none, and he has made this journey worth taking. I would like to thank my best friend, Abbi Smith, for listening and listening and listening. I only hope I can be there for her the way she has been there for me.

I would also like to thank my advisor, Dr. Alexander, for pushing me beyond my limits and knowing better than me that it was not too far. I want to thank her for sharing her brilliance and wisdom, even when it was midnight before a deadline! I would like to thank my other mentor, Dr. Graeber, for all the opportunities she has given me. Without them, I would not be on the same path I now find myself. I would like to thank Dr. Fey, Dr. Croninger, and Dr. Wigfield for all their time and feedback and for asking the tough questions. I could not have asked for a more expert committee!

I want to thank the MAC-MTL group for welcoming me into the community-with special thanks to Sarah for all the stress-relieving laughter, to Grace, for asking “Who are you?” when I appeared on the scene, and to Ming, for asking “Where are you?” whenever I disappeared for a while. To Kathy and Mike, thanks for the advice on Thursdays. Thank you to Michelle, Liliana, and Emily, for welcoming me to the “family,” and to Daria, for the long talks and reminding me to rest. Finally, I want to thank my brother, Jason Jones, for making it easy to move across the country and for his interest in what I was doing.

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CHAPTER I

INTRODUCTION

Researchers have consistently commented on the large percentage of individuals lacking basic fraction skills (Ball, 1990a; Hecht, 1998; Mix, Levine, & Huttenlocher, 1999; Rittle-Johnson, Siegler, & Alibali, 2000; Saenz-Ludlow, 1994; Smith, 1995). This skill deficiency is evident by sixth grade, when instructional emphasis becomes increasingly procedural and mechanical in nature (Niemi, 1996). According to researchers, some algorithms pertaining to fractions are among the least understood in all of elementary school (Bulgar, 2003; Smith, 1995; Tirosh, 2000). In the middle grades, fractions constitute a significant part of mathematics instruction, and a weak background in fractions can lead to problems in future mathematics classes (Case, 1988). Ball (1990a) demonstrated that preservice teachers' knowledge of fractions is limited, and Ma (1999) extended these findings to inservice teachers. For these reasons, it is important to continue efforts to understand why individuals struggle with fractions and what can be done to change this phenomenon.

Research has shown that the quality of instruction is an important factor in student learning (Rice, 2003; Stigler & Hiebert, 1999). As such, it could be that characteristics of traditional fraction instruction are contributing to the persistent trend of poor performance in fractions for children, adolescents, and adults. Silver (1986) suggested that for common mathematical errors, we should "examine the possibility that our instructional procedures may be reinforcing the error rather than eradicating it" (p.190). In the United States, these instructional procedures have typically involved statements of definitions and processes followed by a great deal of practice (Stigler & Hiebert, 1999). However, researchers have established that when instruction focuses on linking concepts and procedures, students retain more and are better equipped to learn new ideas (NRC, 2001). More recently, Hill, Rowan, and Ball (2005) have shown that when teachers have this linked knowledge, it positively predicts student achievement gains as early as first grade.

Other researchers have suggested that when teachers emphasize learning and understanding fractions (rather than performance), students not only tend to perform better, but they also express more enjoyment of and interest in fractions (Schunk, 1996; Stipek et al., 1998). Stipek and her colleagues described these teachers as promoting a *learning orientation*, and they found that such an orientation was positively related to gains on non-routine fraction tasks but not routine ones. These results differ somewhat from Schunk's (1998) findings, in which teachers' manifestation of learning orientations also enhanced performance on routine tasks (e.g., adding and subtracting fractions). Research is needed to clarify these discrepancies, but it seems that instructional emphasis plays an important role in fraction knowledge.

In brief, fractions constitute a significant but difficult part of mathematics education, and there is clear evidence that students continue to perform poorly on fractions into the college years. Further, there is reason to believe that common instructional practices related to fractions (such as those emphasizing procedures and mechanics) contribute to poor fraction knowledge. There is also evidence that instruction emphasizing understanding (rather than performance) is related to both improved fraction knowledge and attitudes toward fractions. Still, far too little is known about the link between instructional practices and fraction performance, particularly for preservice teachers. Therefore, efforts to improve performance in fractions would be supported by a better understanding of how instructional practices within an elementary education course are related to the knowledge and attitudes of students taking such a course.

Statement of the Problem

Hill et al. (2005) suggested that teachers need more than mathematical skill; they need the kind of mathematical knowledge that enables them to provide students with explanations, to analyze student responses, and to use appropriate pictures to represent concepts. A question that remains is: How do teachers gain this knowledge? One place to begin addressing this question is with preservice teachers. As students who want to become teachers, this group represents a critical case in the education cycle. They are both learners of mathematics and learners of how to teach mathematics, which places them in a prime position to gain this knowledge. Some questions that follow are: What do preservice teachers know about fractions? What are their attitudes toward fractions? What instructional practices would help them improve their understandings and attitudes?

Given that children feel increasingly less competent in mathematics and value it less as they progress through elementary, middle, and high school (Jacobs, Lanza, Osgood, Eccles, & Wigfield, 2002), one might expect preservice teachers to also have poor attitudes. However, little is known preservice teachers' attitudes toward mathematics. One general finding is that elementary preservice teachers feel much less confident about and more anxious toward mathematics than do secondary preservice teachers (Ball, 1990a). In fact, females in general tend to feel more anxious about mathematics (Wigfield & Meece, 1988) and have lower mathematics self-concepts (Marsh, 1989). Since the majority of preservice elementary teachers are female, elevated levels of anxiety and decreased self-concepts might be expected for this group. However, not much is known about how elementary preservice teachers feel about fractions in particular. Marsh (1989) suggested that mathematics self-concepts begin to increase for university students in general, but given the findings of Ball (1990a), it is unclear whether this increase would exist for preservice elementary teachers.

Regarding elementary preservice teacher knowledge of fractions, the recent literature is primarily limited to division (Ball, 1990b; Tirosh, 2000). The mistakes these college students make with division of fractions are similar to the ones that younger students make. Two decades ago, Silver (1986) reported preservice teachers making addition mistakes similar to young students. Specifically, they tended to add numerators and denominators. Given the emphasis on reform over the last two decades, it is unclear whether such trends still hold.

While it well established that mathematical knowledge benefits from a conceptual understanding of procedures (e.g., Bruner, 1960/1977; Hiebert & Lefevre, 1986; NRC, 2001; Rittle-Johnson et al., 2001; Skemp, 1978), it is currently unclear how to best arrive at this understanding. Some researchers have demonstrated such an understanding of fractions is possible when students create their own solutions (e.g., Bulgar, 2003; Mack, 1990, 2000; Tzur, 2004), but these studies were primarily descriptive and lacked comparison groups. When comparison groups were used, researchers typically compared instruction with multiple beneficial elements (i.e., rich context, large amounts of discussion, encouragement of student-invented procedures) to instruction with none of these things (e.g., Morris, 1995). It is difficult to know the contributions of these elements when they are combined with an emphasis on conceptual understanding, which could conceivably occur through effective lecturing. If we want teachers to know how fraction concepts and procedures are related, we need to know more about how to accomplish this goal.

Purpose of the Study

The purpose of the current study was to extend the literature on elementary preservice teacher knowledge and motivation toward fractions. In particular, I examined these factors before and after the future teachers participated in a mathematics course designed to help them link elementary school concepts to their procedures.

Research Questions and Hypotheses

This study addressed the following research questions:

1. What is the level of elementary preservice teachers' knowledge of and motivations toward fractions at the beginning of a course designed to promote relational understanding in mathematics?

As reported in the literature, it was expected that the preservice teachers would perform at a level comparable to middle school students (Silver, 1986; Tirosh, 2000). Based on pilot test results, it was expected that the participants would have the least knowledge about division of fractions versus their knowledge of addition, subtraction, and multiplication. Further, Asku's (1997) research suggested that the participants would perform poorly on word problems included in the fraction knowledge measure. Finally, low motivation for fractions was expected for many of the teachers, based on the work of Ball (1990a) regarding motivation toward mathematics in general for elementary preservice teachers.

2. How will elementary preservice teachers' knowledge of and motivations toward fractions differ as a result of participating in a course designed to promote relational understanding in mathematics?

Based on the scant literature (e.g., Stipek et al., 1998), pilot tests results, and the goals of the course, it was expected that the course would contribute to significant improvement in participants' fraction knowledge. In addition, there should be evidence of enhanced concept/procedure linkages in the classification of error patterns at posttest. It was expected that the greatest improvements from pretest to posttest on this measure would be found for the word problems, which have been historically more demanding for students to solve (Asku, 1997). Consistent with the course activities, it was expected that the solutions for these word problems would involve the use of pictures rather than algorithms.

Along with improved knowledge, it was expected that there would be a significant rise in students' motivations toward fractions, in part due to their increased competence in this area (Alexander, 1997). Specifically, an emphasis on understanding should be related to lower anxiety and higher interest (Stipek et al., 1998; Schunk, 1996), as indicated in participants' self-reports from pretest to posttest.

3. What do preservice teachers' error patterns and ways of solving fraction problems reveal about their procedural and conceptual knowledge of fractions at the beginning and end of a course designed to promote relational understanding in mathematics?

Mack (1995) and others have reported that children make mistakes with fractions that are related to their knowledge of whole numbers. It was expected that some of these errors persisted into the college years (Silver, 1986). Considering the importance of linking concepts to procedures (NRC, 2001) and the prevalence of instruction that lacks such links (Stigler & Hiebert, 1999), it was expected that mistakes on the fraction knowledge measure would not be random. Rather, these errors would be classifiable and would reflect a particular lack of association between concepts and procedures.

4. Are there discernible profiles for instructors teaching different sections of a course designed to promote relational understanding in mathematics, and how do these profiles contribute to preservice teachers' fraction knowledge and motivation at the conclusion of the course?

Based on pilot study results, it was expected that the three course instructors would vary in their delivery of course content. For instance, some instructors were expected to ask questions to promote discussion, while others were expected to lecture. Based on Stipek (2002), it was expected that opportunities for students to discuss the material will be related to their improved motivation. Despite the goals of the targeted course, differences in the extent to which concepts are linked with procedures were expected. This difference may be related to the beliefs of the instructors, and it was expected that significant differences in the amount of linking would also be related to differences in knowledge (NRC, 2001) and motivation (Stipek, 2002) at posttest.

Definitions of Terms

In order to address these research questions, the following terms are used:

Anxiety has both a cognitive and an affective component (Wigfield & Meece, 1988), but since the affective component is the one shown to have debilitating effects on achievement, it is the component referred to in this study. As so defined, anxiety includes feeling scared and nervous, especially in testing situations.

Conceptual knowledge is “most clearly characterized as knowledge that is rich in relationships” (Hiebert & Lefevre, 1986, p. 3). For the purposes of this study, this term includes knowledge of definitions and principles, as well as ways of representing them.

Procedural knowledge in mathematics refers to a familiarity with symbols, conventions, and rules for solving problems (Hiebert & Lefevre, 1986).

Relational understanding indicates that a student’s knowledge includes both how to do something and why it is done that way (Skemp, 1978).

Procedural flexibility is an ability to use procedures in a way that is elegant and efficient. Students with procedural flexibility often depart from general algorithms when an alternate algorithm is more appropriate (Star, 2005).

For the purposes of this study, s*elf-concept of ability* represents both a person’s beliefs about his or her mathematical abilities, as well as his or her expectancies for success in the future. Although ability beliefs and expectancies are theoretically different, they are assessed together because they are linked empirically (Wigfield & Eccles, 2000).

Transfer items are novel problems that tap into targeted ideas but can not be solved with the acquired procedures (Hiebert & Wearne, 1988).

The v*alue* a person places on mathematics represents his or her beliefs about its importance and usefulness, as well as his or her interest in the domain (Wigfield & Eccles, 2000).

Significance of the Study

This study is expected to contribute to the literature in at least four ways:

1. Primarily, research on elementary preservice teachers' knowledge of fractions is limited to division (e.g., Ball, 1990a; Tirosh, 2000). The current study examined what these future teachers knew about adding, subtracting, multiplying, and dividing both fractions and mixed numbers.
2. Research on motivation for mathematics has largely focused on general mathematics (Middleton & Midgley, 2002; Pintrich, 2000; Wigfield & Meece, 1988). The current study extends the literature on mathematics motivation by examining elementary preservice teachers' motivations toward fractions.
3. The current study further validated an observation instrument previously used to investigate 4^{th} and 5^{th} grade classroom practices by employing that instrument in the observation of college classrooms.
4. The current study built on what is known about the benefits of linking conceptual and procedural knowledge by investigating what other elements of instruction may be important in realizing those benefits.

Limitations

While the present study is expected to contribute to the literature in a number of ways, there are a few apparent limitations. One limitation is that the small number of course instructors made it difficult to detect differences due to issues of statistical power. In addition, both the instructor and students samples were drawn from a single university setting. In particular, it is a setting where mathematics education is of great concern and efforts to improve it are relatively consistent with current with trends.

Another limitation is that the study occurred in natural settings. As such, the experiences of the participants, such as the types of instruction they received, was not controlled by the researcher. These limitations should be kept in mind while making any generalizations based on the study's findings.

CHAPTER II

REVIEW OF RELEVANT LITERATURE

One purpose of the present study was to explore older students' knowledge of and motivations toward fractions. The particular students selected for this exploration were elementary education majors enrolled in a course whose purpose was to promote a relational understanding of the mathematics they would someday teach. A primary goal of the study was to investigate the potential influence of this course on these students' fraction knowledge and motivations. In light of these interests, the current review is organized around three key bodies of research.

First, the literature on fraction knowledge will be considered. The goal is to understand what elementary preservice teachers know about fractions. As future teachers, their mathematical knowledge will likely have an impact on student achievement (Hill, Rowan, & Ball, 2005). But as students of education, I was interested in their knowledge and how it changed as a result of the instruction they received. Because the literature on preservice teacher knowledge of fractions is limited, much of the literature reported here involves research conducted with elementary and secondary students.

Because this study explored the contributions of instruction to students' knowledge and motivations, the literature on motivations toward mathematics in general and fractions in particular will be surveyed. This section of the review includes the research concerning elementary and secondary students' motivation toward fractions as well as what is known about elementary preservice teachers' motivation toward fractions. Given that the research on both age groups is limited, this facet of the literature review will be exhaustive with regard to fractions.

Finally, because a significant aspect of the present study concerned the influence of teachers' instructional actions on their students' knowledge and motivations, the literature on effective teaching of mathematics will also be reviewed. From the research on fraction knowledge, ideas about how to effectively teach fractions emerged. In order to place these ideas within a larger framework, some background on effective teaching literature will be provided. First, theoretical ideas related to effective teaching will be discussed. Next, the literature on teaching behaviors related to mathematics achievement will be mentioned. Finally, connections will be made between characteristics of quality instruction and motivation.

This overview of the relevant literature will conclude with a discussion of the unresolved issues pertaining to fractions knowledge and motivations and to the influence of instruction on such knowledge and motivations.

Fraction Knowledge

The vast amounts of research on rational number understanding may suggest that everything is known about how students learn and understand rational numbers. On the other hand, such extensive research literature may be evidence of the importance and difficult nature of learning about rational numbers. I have taken the latter stance and attempted to measure the progress made on understanding fraction knowledge. This knowledge includes procedural knowledge such as how to add, subtract, multiply, and divide common fractions, as well as how to compare and rename fractions. It also includes knowledge of basic fractions concepts and knowledge of when to apply the procedures.

To determine the progress researchers have made in understanding fraction knowledge, empirical articles from peer-reviewed journals during the period 1990-2004 were analyzed. The analysis focused on finding and describing themes that emerged from the literature. Five themes were identified, including the role of prior knowledge, the link between conceptual and procedural knowledge, the role of invented procedures, difficulties related to symbolism, and the limitations of the part-whole conceptualization of a fraction.

Fraction Knowledge of Students

Almost ten years ago, Pitkethly and Hunting (1996) reviewed the available research on *initial* fraction concepts, focusing on projects whose goals were to help children develop meaningful understandings about fractions. Although the present review will overlap somewhat with theirs, its primary purpose is to examine what is known about the acquisition of procedural knowledge, including knowledge of the four operations and prerequisites such as equivalent and improper fractions. As such, the participants in these studies are primarily upper elementary- and middle-school children. In some cases, however, researchers have introduced these skills to younger students, and their work has consequently been reviewed. Hence, this review extends the work of Pitkethly and Hunting not only by including research conducted since that time, but also by focusing on skills that develop after initial concepts are acquired.

Nearly twenty years ago, Robbie Case (1988) suggested a research agenda concerning middle-grade mathematics, of which he considered fractions to be a central part. Within this agenda, he suggested several lines of research that both continued and built upon the most current work at that time. Much empirical work has been added to the literature on fractions since then, and the five themes mentioned above have emerged from the current analysis of that work. The themes are consistent with those in Case's (1988) suggested research agenda for mathematics in the middle grades, and they will be presented and analyzed with respect this agenda. Careful attention will be given to the progress made on this agenda, and suggestions for an updated agenda will be made.

Prior Knowledge

According to Case (1988), "there is a pressing need for more detailed and clearly testable models of students' understandings" (p.266), and researchers have made some progress toward fulfilling this need by investigating how students make sense of fractions. According to Piaget (2001/1950), children actively seek to make meaning of the world around them. They find this meaning by relating new knowledge to prior knowledge, or knowledge gained from instructional experiences or experiences with one's environment. This basic idea from constructivism has been useful in understanding the fraction knowledge of students.

One major finding of the research on fraction knowledge is that children relate fractions to their knowledge of whole numbers. For example, Mack (1995) described students explaining 3/5 as three whole objects cut into five pieces. Byrnes and Wasik (1991) suggested students often over-generalize their knowledge of whole numbers when adding fractions. In other words, students tend to treat the fraction parts as if they were whole numbers, adding numerators together as well as denominators. This error was shown to be persistent even after students were warned not to do it.

Although some researchers have asserted that prior knowledge of whole numbers interferes with fraction learning, others have argued that whole number knowledge is actually the foundation upon which fraction knowledge is built (Olive, 1999; Saenz-Ludlow, 1994). For example, Olive described a student who found both 1/3 and 1/5 of a unit by counting by threes and by fives until he had a common multiple. He then split the unit into 15 parts and removed five pieces for 1/3 and three pieces for 1/5. According to Olive, he was able to arrive at this solution because he had and could coordinate his knowledge of whole numbers. Saenz-Ludlow (1994) also described a student using whole number knowledge to solve problems with fractions. While working with a money context, her student compared fractions by finding the content value and comparing those whole number amounts. For example, since 1/10 of 100 is 10 and 1/20 of 100 is 5, then 1/10 > 1/20.

Hunting, Davis, and Pearn (1996) found that not only did children use their knowledge of whole numbers to learn fractions, but their whole number knowledge was actually strengthened by learning about fractions. Vygotsky (1934/1986) described a similar interrelationship between arithmetic and algebra, claiming that learning algebra strengthens the skills of arithmetic. If so, then it also makes sense that learning fractions in a way that utilizes and builds on whole number knowledge has the result of improving that knowledge.

In her 1995 study, Mack also found interrelated knowledge between whole numbers and fractions. However, she demonstrated that these connections are not always smooth or accurate. Working with third and fourth graders to determine how children make sense of fraction procedures, she instructed them individually in 30-minute lessons, twice a week, for three weeks. The instruction varied with each student, depending on the students' skill level. In general, she focused on the relative size of fractions, the symbolic representation of fractions, adding, subtracting, and fraction equivalence. Not only did she find that students sometimes misapplied their ideas of whole numbers incorrectly to fractions, but she found that the reverse was also true. In other words, after some initial learning of fraction concepts, the students began to misapply their knowledge of fractions to whole numbers. For example, when learning to subtract a fraction from a whole number, they might refer to the whole number as if it were only a numerator. Specifically, one student suggested the “2” in 2 – 3/8 represented two eighths. In seeing this circular pattern of interference, Mack suggested that studies be conducted to determine if it would be helpful to develop students’ understanding of whole number and fraction symbols simultaneously.

Mack’s (1995) suggestion to develop fraction and whole number knowledge simultaneously is important in at least two ways. First, in Case’s (1988) research agenda, he claimed that the organization of topics should be based on what conceptual distinctions are natural for middle grade students, and more needed to be known to make such decisions. Evidence presented here suggests that the distinction between fractions and whole numbers is not a natural one. Second, Silver’s (1986) suggestion to re-evaluate practices where recurring errors are involved would aptly apply to this situation. If it is true that fractions and whole numbers are not naturally distinguishable to young learners and prior knowledge of one interferes with learning the other, then Mack’s suggestion should be a serious part of a future research agenda concerning instruction in fractions.

As a mathematician, Wu (2002) asserted that a major fault in usual presentations of fractions is the lack of “a clear definition of a fraction that includes whole numbers as a special kind of fraction” (p.16). Perhaps a simultaneous introduction of whole numbers and fractions would aid with such a definition. This does not imply, however, that all fraction skills should be learned at an early age.

Mix, Levine, and Huttenlocher (1999) provided some evidence for Mack's perspective. They found that early fraction ability emerges only slightly later than an ability to deal with whole numbers, but both are present at some level by age four. They also provided evidence that the two types of knowledge follow similar developmental paths. For example, the ability to calculate with fractions emerges before the skill of using symbols to represent fractions, just as the ability to calculate with whole numbers appears before the skill with whole number symbols.

In trying to reconcile whether prior knowledge interferes with or helps children learn new ideas about fractions, Tzur (2004) concluded that it does both. In fact, he claimed that whole number knowledge *and* prior fraction knowledge interfere, at least temporarily, with new fraction concepts. When students first try to make sense of a new concept, they may draw on prior knowledge in erroneous ways. Once they realize the discrepancy between the new situation and prior situations they have encountered, students begin to expand their original notions to include the new concept. Hence, the interference of prior knowledge may reflect both the unstable nature of new knowledge and the limited nature of prior knowledge.

In his study, Tzur (2004) asked a student to recreate a whole pizza given 5/8 of an unmarked pizza. Initially, the student relied on his prior knowledge of eighths to mistakenly divide the partial pizza into eight pieces. Before returning to the task a few days later, Tzur posed a similar task using a fraction that could be iterated to create the whole. For example, given 2/10, the student suggested iterating the piece five times to create the whole. Using this as a prompt, she was eventually able to solve the original task involving 5/8 of a pizza.

Besides whole number and prior fraction knowledge, informal knowledge may also affect fraction learning. According to Mack (2000), "informal knowledge can be characterized as applied, circumstantial knowledge constructed by the individual in response to his/her real-life experiences" (p. 308). In 2000, Mack investigated the long-term effects of students building on informal knowledge of fractions. In short, she found that informal knowledge may be able to assist with learning fractions procedures, even after algorithmic instruction.

During her investigation, Mack (2000) discovered several patterns regarding the use of informal knowledge. Prior to instruction in fraction multiplication, the students were encouraged to solve problems using their informal knowledge of sharing with peers. After learning a procedure in school, however, they relied only on the algorithm. When she asked about this reliance, the students claimed to value the algorithm's efficiency and the fact that they did not need to *understand* the problem. However, the students made some critical conceptual mistakes when relying solely on the algorithm. Furthermore, they no longer wanted to solve word problems, preferring a purely symbolic form of the problem. When pressed to justify solutions to word problems, however, the students reverted back to using their informal knowledge (Mack, 2000). Apparently, informal knowledge is a helpful for understanding problems with fractions, but instruction that focuses heavily on procedural skill discourages its use.

The idea of informal knowledge assisting formal knowledge might seem contrary to Vygotsky (1934/1986), who claimed that in general, scientific concepts reorganize spontaneous (or informal) ones, not the other way around. However, instruction focused heavily on procedural skill is probably not what Vygotsky had in mind when he made his claim. In fact, contemporary Vygotskians suggest that concepts learned in school can be subdivided into theoretical and empirical concepts (Kozulin, 1988). Empirical concepts are learned inductively by seeing many examples and arriving at some conclusion. On the other hand, theoretical concepts are learned in a deductive fashion, beginning with the general idea and applying the idea to various examples. It is doubtful that the algorithmic instruction that students received in Mack's (200) study was theoretically based, and it would be interesting to find out whether or not such instruction would result in different patterns.

In brief, research does seem to support the Piagetian (2001/1950) notion of linking new knowledge to old, but this link is complicated. It is not just prior knowledge of whole numbers that can interfere with new fraction knowledge (Byrnes & Wasik, 1991; Mack, 1995), but prior fraction knowledge can interfere as well (Tzur, 2004). And interference is not the only link between the old and new knowledge; researchers like Mack (2000), Olive (1999), and Saenz-Ludlow (1994) have shown that prior knowledge can sometimes assist in the acquisition of new fraction knowledge. Furthermore, the link is not unidirectional. As new knowledge is gained, prior knowledge can be reshaped (Tzur, 2004) or reinforced (Hunting, Davis, and Pearn, 1996). Given this strong but complex role of prior knowledge, its consideration in instructional practices needs to be more carefully examined.

Conceptual and Procedural Knowledge

Within discussions of prior knowledge, sometimes it becomes important to characterize the nature of that knowledge. Over the years, numerous such characterizations have surfaced, usually in the form of dichotomous distinctions such as empirical and theoretical concepts (Kozulin, 1988). However, a more cited distinction is that of conceptual and procedural knowledge (Hiebert & Lefevre, 1986). Simply put, conceptual knowledge might be thought of as “knowing that” whereas procedural knowledge might be thought of as “knowing how” (Byrnes & Wasik, 1999, p. 777). Other characterizations have attempted to describe links between two types of knowledge, such as Skemp’s (1978) relational and instrumental understanding. Instrumental understanding might be thought of as knowing “what to do” whereas relational understanding might be thought of as “knowing both what to do and why” (Skemp, 1978, p.9). Rather than being dichotomous, the former is actually a subset of the latter.

Rittle-Johnson, Siegler, and Alibali (2001) suggested that understanding *why* procedures work might strengthen the relationship between conceptual and procedural knowledge. Hiebert and Lefevre (1986) also made this suggestion and added that meaningful procedures are also more easily recalled. That understanding procedures would lead to better retention of them is consistent with Bruner’s (1960/1977) statements about the connection between memory and fundamental principles. He suggests that details can be more easily reconstructed when they are part of a greater structure and are more quickly forgotten when they are not.

Researchers have suggested that too many students lack such deep understandings of mathematical procedures, and as a result, Case’s (1988) agenda included the need for further clarification of the link between procedural and conceptual knowledge. Within the last two decades, several researchers have explored this link. Although there has not always been agreement about the nature of the relationship, researchers have tended to conclude that conceptual and procedural knowledge are both important in understanding fractions.

For example, Byrnes and Wasik (1991) examined whether or not conceptual knowledge, as measured using items such as comparing fractions and recognizing the appropriate drawing for a given fraction, is a sufficient condition for acquiring procedural knowledge, such as addition and multiplication of fractions. They found that it was not; some children with strong conceptual knowledge still use some procedures incorrectly. This result is consistent with Carpenter (1986), who claimed there are two situations in which procedures remain isolated from their related concepts. The first is when the concepts are not fully developed for the student, and the second is when the procedures are not linked to the concepts. Although Byrnes and Wasik did not test for Carpenter’s second situation in particular, they did provide evidence that a second situation must exist.

Hecht (1998) replicated findings of Byrnes and Wasik (1991) by showing that procedural and conceptual knowledge both contribute to fraction computation skills, but he extended their findings by showing that conceptual knowledge continues to predict fraction skills even after controlling for variables such as knowledge of math facts, procedural knowledge, and general verbal ability. In this study, conceptual knowledge was defined in a manner similar to Byrnes and Wasik. Though this type of knowledge is clearly important for success with fractions, more studies are needed that explore the relationship between a deep understanding of the procedures and the actual procedural skill. In other words, how does conceptual understanding *of* procedures aid in student learning of those procedures?

Aksu (1997) explored this relationship by comparing student performance on word problems with their performance on context-free problems involving all four operations with fractions. She used Skemp's characterization of relational versus instrumental understanding to analyze the results. After finding that students performed better on context-free problems than on word problems, she concluded the students had an instrumental understanding. In other words, the students had likely had memorized procedures without understanding them. These conclusions, however, are somewhat speculative. It might be helpful to repeat the experiment while controlling for important variables such as reading comprehension. Interviews of students and information about teaching practices would also be helpful in determining why the performance was lower for word problems.

Rather than simply using word problems that could be solved with an appropriately selected algorithm, Saxe and Gearhart (1999) chose problem-solving items that required a deep understanding of fractional concepts. These included finding fractions for unequal parts, estimating fractional parts of areas, and solving fair-share problems. They also included information about the teaching practices used with the students. They wanted to determine if there was a relationship between the degree to which instruction was aligned with reform practices and performance on these problem-solving items as well as computational items. By reform practices, the authors are referring to instruction that follows recommendations from groups such as the National Council of Teachers of Mathematics, such as encouraging sense-making and building on students' understandings.

Saxe and Gearhart (1999) reported findings separately for two different groups. For students who began with a very basic understanding of fractions, increases in alignment with reform practices were associated with increases in student performance on problem-solving items. For those who did not begin with a basic understanding of fractions, a relationship was apparent only at high levels of alignment. One explanation for this result is that, at low levels of alignment, students without a basic understanding of fractions do not have the support they need to become engaged in the task of understanding fractions. Indeed, these students may not even be aware that there is something more to fractions than memorization.

For computational items, no relationship was apparent for either group of students. Saxe and Gearhart (1999) explained this result by stating that in the short term, procedures can be simply memorized. They asserted that traditional practices support this type of memorization perhaps even better than reform practices. However, no long term measures were included in the study. Case's (1988) agenda called for a better understanding of how short term and long term change are related, and much still needs to be done to achieve such understanding. If we accept Bruner's (1960/1977) claim that knowledge learned without understanding is not likely to endure, then a long term study may produce different results for computational skills than was found by Saxe and Gearhart. Another important consideration is that the procedural knowledge required for these problems were limited to finding equivalent fractions and adding and subtracting fractions. The algorithms for multiplying and dividing fractions are quite different from those of adding and subtracting, and these students did not need to distinguish between them.

Niemi (1996) also tested limited amounts of procedural knowledge, but he studied conceptual knowledge of upper elementary school students from a variety of perspectives, including both representational fluency and conceptual understanding of procedures. Representational fluency was measured by asking students to recognize multiple representations of a given fraction. Then students were scored on their ability to explain and justify solutions to problem-solving tasks. The tasks included comparing fractions of objects, finding a fraction between two fractions, and evaluating the truth of statements involving equivalence and addition.

Niemi (1996) found that students who scored high on representational fluency also tended to explain fractions in terms of concepts and principles while avoiding misconceptions. However, they were no more likely to simply know the procedures than groups who scored low on representational fluency. But like Saxe and Gearhart (1999), Niemi had no follow-up measures, meaning that it is unclear whether representational fluency was related to retention of procedures. While many researchers agree that representational fluency is an indicator of conceptual understanding, some also contend that this understanding supports retention (NRC, 2001).

Although Rittle-Johnson et al. (2001) investigated knowledge of decimal fractions rather than common ones, they built upon Niemi's (1999) work by examining change over time. Rather than simply examining the relations between conceptual knowledge and procedural knowledge at a given time, Rittle-Johnson et al. wanted to know if conceptual knowledge was related to *improvements* in procedural knowledge. They found that it was, and being able to correctly represent problems was one link between them.

Altogether, this line of research demonstrates that conceptual and procedural knowledge are distinct and uniquely contribute to fraction skills (Byrnes & Wasik, 1999; Hecht, 1998). However, findings about the *link* between conceptual and procedural knowledge, which may best be characterized by the Skemp's (1978) relational understanding, are varied. While there seems to be agreement that relational understanding is helpful for solving novel problems (Niemi, 1996; Saxe and Gearhart, 1999), there is less agreement about its effects on computational items. In the short term, it appears that relational understanding makes no difference in computational skill (Saxe & Gearhart; Niemi, 1996). When examining change over time, however, relational understanding does seem to be important (Rittle-Johnson et al., 2001). This later finding needs to be further explored with common fractions rather than decimal fractions, but it is at least consistent with Bruner's (1960/1977) theory of learning.

Student-Invented Procedures

In an attempt to promote student understanding of procedures, many researchers have espoused the idea that students must actively create procedures in order to be successful at them. Case (1988) called for a "more careful examination of student's invented mathematics" (p. 267), and much has been accomplished with regard to this part of his agenda. In particular, multiplication of fractions has received extensive attention among researchers such as Mack (2000) and Olive (1999).

Like Mack (2000), Olive (1999) found that, although invented solutions to multiplication problems were better understood than taught algorithms, students claimed that the taught algorithm was easier. Another similar finding was that no connection was naturally made between the algorithm and the invented solutions; both researchers had to prompt students to make connections between the meaningful models and the rule of multiplying numerators and denominators. If this connection is not made naturally by students, then it seems that teachers would have to help them make it, and research is needed to determine how teachers can best help students make this connection.

In some cases, student–invented algorithms do not even resemble traditional algorithms. Sharp and Adams (2002) found this to be true in the case of the invert-and-multiply algorithm for division of fractions. Instead of inverting the second fraction and multiplying, students invented a common denominator method, which involved finding a common denominator and then dividing the numerators. Since dividing the denominators would give a quotient of one, the denominators can essentially be ignored. The researchers explained that the common denominator method builds more naturally on students' knowledge of division with whole numbers.

Bulgar (2003) also investigated what strategies students invent to solve fraction problems involving division, although her tasks only included a fraction for the divisor, not for the dividend. In a study with fourth graders who had not been introduced to an algorithm, three types of strategies emerged: reasoning with natural numbers, reasoning with measurement, and reasoning with fractions. Most of the tasks involved fractions with a numerator of one, but some of the tasks involved non-unit fractions. In the same article, Bulgar reported a study conducted years later, in which the teaching experiment was repeated with a group of fifth graders who were accustomed to inventing and justifying solutions. The results were similar, although most students used the reasoning with fractions strategy. For example, when asked to find how many bows of 1/3 meter could be made from six meters of ribbon, some students suggested that there were three one-thirds in one meter and multiplied three by six to find the total number of bows.

Although it was clear that these students could make sense of problems that involved division by both unit and non-unit fractions, it would have been interesting to investigate other distinctions. For example, do students perform similarly when both the dividend and the divisor are fractions? If so, can they do it once the context is removed? Can they outperform students who are carefully shown the meaning of division with fractions?

Carpenter (1986) suggested that when students invent correct procedures on their own, conceptual understanding has been demonstrated, but the same is not true for procedures that have been taught. Although it may be true that inventing correct procedures is evidence of conceptual understanding, it is debatable whether students *must* create the procedures in order to understand them. Little research has been conducted to understand what teachers can say to contribute to the understanding of procedures. Mack (1995, 2000), Warrington (1998), and Tzur (1999) each emphasized discovery on the part of the learner, with intervening prompts from the teacher. However, none of these researchers made any comparisons to other types of instruction, making it difficult to determine the effects of having students invent procedures. Morris (1995) compared different types of instruction, but she simply compared traditional teaching, in which no understanding was encouraged, to reform teaching, in which students were asked to invent procedures in order to gain understanding of them.

Although Case (1988) called for research on invented procedures, he also pointed to the lack of agreement on “the precise role of direct instruction in bringing about change” (p. 268). While it is clear that providing a rule without explanation does not generally lead to student sense-making, it is unclear whether or not sense-making can happen without invention. Research should be conducted that clarifies the role of the teacher in this regard. Can direct instruction that includes sense-making lead to deep student understanding, or do the students have to make sense of procedures by inventing them? Also, can students learn about fractions from listening to other students explain their strategies?

Tzur (2004) provided some evidence that it is insufficient to simply listen to another person’s strategy, even if it is a student-invented one. When one student recreated a whole pizza from 5/8 of an unmarked pizza, she was being watched by another student who seemed to follow the work and understand it. However, when pressed to do the same kind of task, the second student struggled. It was not until he completed the task himself that he seemed to internalize the new idea. As a result, Tzur suggests that “it is the teacher’s responsibility to make sure that each learner abstracts the intended mathematics beyond the level of emulating other students’ solutions” (p.110-111).

In an in-depth study of students who were competent with fractions, Smith (1995) found that multiple strategies were used to solve problems. Competent students seemed to understand the general algorithms learned from instruction, but they typically used them only when more efficient strategies could not be found. Instead, they students often invented strategies that were restricted to a certain set of fractions. Competent students seemed to prefer efficiency over general applicability, but they used general algorithms when in doubt of other strategies. This study points to both the importance of encouraging students to invent strategies, but also to the importance of acquiring reliable algorithms that can be taught. Unfortunately, no information was given as to how these algorithms were taught. Did the students make sense of the taught algorithms on their own, or did their teachers assist in the sense-making? Future research should also investigate what actions teachers can take to help students make sense of alternate strategies, since even the competent students in Smith's study sometimes tried to overuse them.

Although many researchers (e.g., Olive, 1999; Saenz-Ludlow 1994; Warrington, 1997, 1998) have demonstrated that students can accomplish some amazing things when asked to invent solutions to tasks with fractions, Morris (1995) is one of the few researchers who reported mistakes that students can make when inventing procedures. One such mistake was an overuse of the discovery that larger denominators indicate smaller pieces. Whereas many students (and adults) misuse their whole-number knowledge to suggest that $1/2 < 1/4$ because two is less than four (Biddlecomb, 2002), students encouraged to invent procedures often understand that the opposite is true. However, Morris reported students overextending this understanding. For example, they might suggest something like $3/4 < 2/3$ because four is greater than three. She also reported mistakes resulting from imperfectly drawn pictures, especially when the fractions were close in magnitude. Much more needs to be reported about incorrect strategies that students invent. If teachers are to incorporate student invention of algorithms into their classrooms, they also need to know what prompts would be most helpful when those inventions are not mathematically sound.

Making mistakes from imperfect pictures has been remedied by some researchers with the use of computer microworlds (Olive, 1999; Biddlecomb, 2002; Hunting et al. 1996; Tzur, 1995). A series of articles have resulted from the work of Steffe and Olive, who have encouraged students to invent procedures. The intent of these researchers was not so much to understand the effects of invention, but to explore the development of fractional knowledge (Olive, 1999). Through student exploration and invented ways of solving problems, support for the theoretical ideas of Steffe have been found with regard to the development of children's fractional knowledge (Olive, 1999). Using Steffe's model, mistakes that students make with fractions can be explained in terms of failure to reach increasingly abstract conceptualizations of units (i.e., iterable units, composite units, and iterable composite units.)

This work has been quite helpful in fulfilling the need for models of understandings described by Case (1988), as well as his request for research that would help clarify the role of computer technology in children's learning. Among other things, the technology has allowed these researchers to pose problems that cannot be easily drawn or easily solved with informal knowledge. For example, students used the microworld to reason through problems involving elevenths and fractional parts of elevenths (Olive, 1999). It remains to be seen, however, what long-term effects technology use has on fraction skills.

Using a computer program named Copycat, Hunting et al. (1996) also encouraged children to find their own solutions to problems. However, these children often chose not to use the program to solve problems, even though it was readily available to them. Instead, they relied on their knowledge of whole numbers. The precise role of computer technology is unclear based on these examples, but what does seem clear is that its role is not easy to define and may depend on the situation. Perhaps some uses of a computer would inherently result on the reliance of technology, while others would simply encourage its use as an aid, to be used or not used as needed. More research is needed to clarify not only what situations produce which outcome, but also which outcome is most desirable with respect to a student's understandings of and skill with fractions. Also, what difference does the purpose of the technology make in learning about fractions? For example, long term use of a calculator that provides answers to fraction problems will likely have completely different outcomes than the use of computers that provide context and ways of modeling fractions.

Like Olive (1999), Hunting et al. (1996), and Tzur (1999, 2004) with their computer worlds, many researchers provide contexts for tasks they ask of students. Sharp and Adams (2002) suggested that context plays an essential role in the invention of an algorithm. Saenz-Ludlow (1994), Mack, (1995, 2000), and Bulgar (2003) seem to concur. These researchers have each documented examples of students inventing algorithms by reasoning within a realistic context. The exact role of the context, however, is unclear, since no comparisons to context-free instruction were made. At the moment, the role of student-invention seems to be confounded with the role of context. In order to better understand the role of each, researchers should attempt teaching experiments that encourage invention without context.

Although Sharp and Adams (2002) claimed it was essential to begin with context-based problems, they also found that removing the context stimulated advances in thinking about fraction division and the development of a formal procedure. Research that encourages invention without context may find that context is simply one way to help students understand fractions, or they may determine that context is as essential as Sharp and Adams (2002) suggested. Or perhaps context provides a particular type of understanding (e.g., how to apply fractions), whereas an alternate approach may provide a different benefit (e.g., deductive reasoning). The more researchers understand about each component of effective teaching, the better they can inform teachers, curriculum writers, and other participants in the education process.

It is also unclear what role student explanations play in the understanding of fractions. In a study by Kazemi and Stipek (2001), four teachers were observed teaching lessons designed to expand students' understanding of part-whole relations and addition of fractions. Teachers were described as high press or low press, depending on the extent to which they pressed the students to justify their answers. They used the study to make a distinction between simply encouraging students to describe their steps for solving a problem and having students mathematically justify the processes they used. Evidence suggested that superficial norms such as working in groups and solving open-ended problems did not result in the same degree of conceptual engagement for high press and low press teachers. However, no data were collected to investigate the effects of high press on procedural knowledge.

Certainly, having students justify their answers can reveal student insights to the teacher, but what does it do for the student? Do students perform better if they regularly explain their thinking? Will they remember ideas for longer if they had verbalized them? Are there motivational benefits to having students explain their thinking? And if some benefits exist, are they similar regardless of whether a student invented a procedure or just explained one they learned in class? Case (1988) asked for research that examined the role of conscious reflection in change. It seems that the role of explanations would be to encourage this reflection, but asking for explanations is often coupled with inventing procedures (Kazemi & Stipek, 2001; Mack, 1995, 2000; Olive, 1999, Tzur, 1995; Warrington, 1997, 1998). Research should attempt to understand the effects of each component.

Despite confounding components, these articles provide evidence that students are *capable* of inventing ways of solving fraction problems, and as stated before, Carpenter (1986) believes that this is evidence of conceptual understanding. There is also evidence that inventing strategies may be a natural thing for students to do, at least for those who are competent with fractions (Smith, 1995). However, it is apparently not natural for students to connect their intuitive strategies to the generalized algorithms presented in school (Olive, 199; Mack, 2000). It is also not the case that students always produce algorithms that can be generalized or that they always generalize them correctly (Morris, 1995; Smith, 1995).

Meaning for Fraction Symbols

Another possible obstacle to understanding fractions might be their symbolic nature. Mack (1995) insisted that communication is one important reason to understand fraction symbols, claiming that students need to learn how the symbols are used in order to convey their meaning to others. In her study, the children did not seem to place importance on the correct use of symbols, as long as they could convey what they wanted. This arbitrary application of meaning complicates what is already a difficult task.

To illustrate that fraction symbols can be obstacles, some researchers have demonstrated what students can understand without symbols. For example, Mix, Levine, and Huttenlocher (1999) claimed that students can understand calculations with fractions much earlier if the symbols are removed. Saenz-Ludlow (1994) demonstrated that this knowledge can be quite extensive. She used the appropriate words for fractions in her case study, but she did not use the symbols. By using the words only, one third grader was able to understand the denominator as a *denomination*, or a relative size of a piece, rather than just a symbol. As a result, he was able to reason through problems such as one-third divided by one-twelfth and one-twelfth divided by one-third.

Kamii and Clark (1995) also emphasized reasoning in their study about student difficulties with equivalent fractions. They insist that relying too much on perceptual aids can inhibit reasoning about the symbols. Using one rectangle cut horizontally into fourths and another rectangle cut vertically into eighths, students in their study were asked how many of the eighths would be the same as three of the fourths. Although the sixth graders performed better than the fifth graders, neither group did well on the task. Despite having been taught how to find equivalent fractions, some of the students did not even accept "six" as the correct answer after it was suggested to them. Some might argue that providing a perceptual aid that cannot be easily used is misleading, but it at least demonstrated the reliance students seem to have on such aids. It would be helpful to compare these results to ones where students were given no perceptual aide at all and to ones where the perceptual aide was conducive to answering the question.

To help with symbolic reasoning, Sharp and Adams (2002) insisted it is important to include realistic or personal experiences before ever involving symbols. They asserted that this progression is essential to success with fractions, even with older students. As stated earlier, however, they also found that eventually removing the context promoted the formation of a formal procedure.

Rather than emphasizing context, Hiebert and Wearne (1988) emphasized the importance of connecting symbols to manipulatives before proceeding to any procedural instruction. They suggest a five stage process leading to symbol competence: connecting, developing, elaborating, routinizing, and building. During the connecting process, symbols are linked to a referent. During the developing process, students act on referents, rather than symbols, to carry out procedures. The elaboration process extends the actions in a way that can be generalized, thus requiring students to symbolically learn procedures. The routinizing process involves practicing the procedures until they become automatic. And finally, the building process involves learning to think and make arguments with the symbols. This last step seems especially important for later mathematics courses, where deductive reasoning and facility with symbols become important for success.

In their study of decimal fractions, Hiebert and Wearne (1988) attempted to provide meaning for procedures by making strong connections between decimal representations and base 10 blocks. By using novel problems that tapped into targeted ideas but could not be solved with the acquired procedures (i.e., transfer items), they attempted to measure what they called “cognitive change.” Although they failed to clearly define this term, the results were intriguing; they reported that students who attempted to use meanings on the transfer items were correct 80% of the time, whereas those who attempted to use rules were correct only 17% of the time. Both groups had performed well on the direct items, and the conclusion was that students who used meanings on transfer items had *learned* what was intended.

In 1995, Morris showed that these processes work with low-achieving students who were learning about common fractions. Like Hiebert and Wearne (1988), she used both direct and transfer items to assess student understanding. Low-achieving students in the experimental group were compared with both low- and middle-achieving students in a comparison group. Interviews were conducted in order to determine whether students were reasoning through problems or simply applying a rule. She found that not only did students in the experimental group outperform both comparison groups with regard to number of correct answers, but they also used quantitative reasoning much more often. Although the students did not progress through all five stages during the study, they were able to use the connections meaningfully when uncertainty arose. She also reported that these students were able to use this knowledge to recognize when students who were tutoring them made procedural mistakes. Because their knowledge of procedures was semantically grounded, they were able to notice when a syntactic approach just did not make sense.

In 1988, Case asked, “What, then, are the limits of conceptually-based transfer…?”(p.267). By extending the results of Hiebert and Wearne (1988) to low-achieving students, Morris (1995) partially addressed this question. However, no delayed assessments or interviews were included in the study to determine whether the effects endured. Future research should also investigate whether certain types of concepts are more likely to transfer and which instructional conditions make transfer more likely.

Taken together, the research concerned with fraction symbols suggests that successful fraction learning might include applying fractions to the real world (Sharp & Adams, 2002), connecting symbols to objects (Morris, 1995), and making sure these symbols are used consistently (Mack, 1995). Of the five processes describe by Hiebert and Wearne (1988), however, much more emphasis has been placed on the first two processes of connecting and developing. Significantly less is known about the third process (elaboration). And little is known about the fourth process (routinizing) when it follows the first three processes (as opposed to routinizing without first connecting, developing, and elaborating.) It is important to study this distinction since fraction skills that become routine without understanding limit the fifth process (building) described by Hiebert and Wearne.

In their review of multiplicative reasoning, Thompson and Saldanha (2003) insisted that it takes time to make deep connections between symbols and their meanings; however, few studies have examined how sense-making affects fraction skills over time. As Saxe and Gearhart (1999) demonstrated, sense-making may not be important for strong skills in the short term. More longitudinal studies are needed if researchers are to know how a deeper understanding of fraction symbols affects the fraction skills of students in the long term.

Limitations of a Part-Whole Conceptualization

Aside from difficulties related to fraction symbols, there may also be problems with how students conceptualize fractions. In their review of rational number concepts, Behr, Harel, Post, and Lesh (1992) insisted that conceptualizing fractions as part of a whole amount is not adequate for developing a complete understanding of fractions. Tzur (1999) supported this claim with his study of fourth grader's understandings (and limitations) of partitive fraction schemes, where he showed that conceptualizing fractions as part of a whole limits children's ability to understand improper fractions.

According to Tzur (1999), a partitive scheme is a kind of intermediate step between understanding a fraction as a part of a whole and understanding it as an iterable unit. After introducing unit fractions as the number of pieces that must be iterated to create a referent whole, students became quite adept at seeing fractions such as 3/5 as a unit comprised of three 1/5's. However, when Tzur asked the students to show fractions larger than the referent, they either said they could not do it, or they renamed the fractions in terms of the newly created "whole." For example, after iterating 1/8 nine times, the students wanted to refer to each piece as 1/9. It seemed that their understanding of the part-whole relationship prevented them from seeing fractions as greater than one whole.

In seeing this as an obstacle for the students, Tzur (1999) designed a lesson that involved iterating non-unit fractions. Because the children had no difficulty finding a fraction twice as long as 3/11, he built on this scheme to help the students conceptualize an improper fraction. Specifically, he asked the students what they would call a fraction twice as long as 6/11. Although they hesitated, they were able to overcome their part-whole understanding and see the unit 1/11 as having an invariant relation with the "whole" from which it was derived.

The approach used by Tzur (1999) was consistent with the theoretical suggestions described in Behr et al.'s (1992) review. Namely, he helped the students conceive of fractions like 5/7 as five iterable units of 1/7. However, it was inconsistent with the suggestions of Thompson and Saldanha (2003), who indicated that conceptualizing multiplication as repeated addition is necessarily problematic for students and should be avoided. When Tzur asked the students to iterate 1/5 three times, he was essentially asking them to conceptualize 1/5 times 3 as a repeated addition problem. But rather than being problematic, this conceptualization actually helped students overcome problems presented by the partitive scheme. Of course, repeated addition will not apply to situations where neither factor is a whole number, such as 1/5 times 1/3, but it can be a powerful tool in helping students understand some multiplicative situations.

The part-whole concept of a fraction might not have been the only limitation with the fourth graders in Tzur's (1999) study. Instead, his findings could be a reflection of Dienes's (1964) *Mathematical Variability Principle*. This principle essentially holds that, for students to understand concepts, irrelevant parts must vary so that relevant parts can be more easily seen. In the case of improper fractions, students who are only exposed to fractions equal to or less than one may perceive being less than one as an invariant aspect of fractions. Hence, they are forced to accommodate for parts greater than the whole by placing them within the whole and renaming the parts. A student might have a similar conception of a fraction as the ones in Tzur's study, namely that a fraction like 3/4 can be understood as three 1/4 units, but he might also have experience with multiple wholes, such as iterating 1/2 six times to make 3. In this case, conceiving of an improper fraction may not be so difficult.

Along these lines, Kamii and Clark (1995) suggested that proper and improper fractions should be introduced simultaneously "so that children will be able to think about the parts and whole at the same time” (p.375). Like Mack’s (1995) idea about introducing fractions along with whole numbers, these researchers seem to think that contrasting the ideas may serve to reinforce them. Research should be conducted to investigate the effectiveness of such approaches.

Another pedagogical issue with the part-whole conceptualization may be that important aspects of this conceptualization remain implicit in some classrooms. In particular, Yoshida and Sawano (2002) claim that teachers often assume that children understand the all the parts must be equivalent and consequently do not place much emphasis on that fact. Similarly, they believe that not enough emphasis is placed on the fact that two fractions cannot be compared unless their unit is the same.

Yoshida and Sawano (2002) found that when the ideas of equal parts and equal wholes are made explicit throughout fraction instruction, students often performed better than those without the explicit inclusion of these ideas. For drawing tasks and tasks comparing fractions with different denominators, the differences were significant. Ordering tasks using the same denominator resulted in similar performances from both groups, at least in the short term. For equal-partitioning tasks and computational tasks involving addition and subtraction of fractions, both groups performed equally on routine tasks. However, the experimental group outperformed the control group on the transfer tasks for both of these categories. No delayed assessments were given to determine if both groups equally retained their skills for the routine items.

If nothing else, the literature on the part-whole conceptualization continually supports the notion that fractions are complex. While some researchers have probed deep into student understanding (e.g., Tzur, 1999), others have focused more on how instruction effects fraction skill within the classroom (Yoshida & Sawano, 2002). In both cases, however, it seems true that a relational understanding is vital for student success with fractions, and the more teachers know about how to generate it, the better.

Limitations of the Fraction Knowledge Literature

In the past fifteen years, there has been much research attempting to understand fraction knowledge and how to further it, and the importance of generating a relational understanding might be a general conclusion from that research. In the many of studies reviewed, researchers suggested it is important for students to make sense of procedures. However, each study also emphasized other aspects as being important to student learning of fractions. For example, Kazemi and Stipek (2001) insisted that students need to justify their answers, and Tzur (1999) suggested that multiple meanings for fractions are necessary to be successful at certain procedures. Tzur (2004) also suggested that students must abstract concepts for themselves, and Sharp and Adams (2002) emphasized the need to introduce procedures through context. Most of these researchers also advocate students inventing their own algorithms as a way to better make sense of them (Mack, 1995, 2000; Olive, 1999; Saenz-Ludlow, 1994; Tzur, 1999, 2004). It seems these various elements could be confounded, and further research is needed to appreciate the contribution of each. We need to know, ultimately, how much each of these elements is contributing to student learning and sense-making. This is important because teacher training is time-consuming, difficult, and expensive, and knowing where to focus the most resources is crucial in the decision-making process.

An extensive gap in the literature exists concerning how fraction knowledge changes over time. Collectively, researchers have demonstrated that a deep conceptual understanding of fractions leads to better explanations and better performance on novel (or transfer) items, but it does not lead to better computational skills in the short term (Byrnes & Wasik, 1991; Hiebert & Wearne, 1988; Niemi, 1996; Saxe & Gearhart, 1999). However, few researchers have examined long-term effects of sense-making on fraction knowledge. Few studies of fraction knowledge even include students beyond upper elementary school. As students progress through school, they are introduced to more and more algorithms, making pure memorization an increasingly difficult task. However, teachers are under a great deal of pressure to cover large amounts of mathematics in short amounts of time, and it can be argued that it takes more time to teach why procedures work the way they do. If teachers are to take the necessary time to ensure understanding rather than memorization, they need to know it is worthwhile in the long term. At present, the evidence is lacking.

Another deficiency in the literature on fraction skills concerns the interaction of these skills. Although Tzur (2004) recently spoke of the role that certain fraction conceptions have on learning new concepts, little is known about how students keep (or fail to keep) the procedures separate in their minds. Does the knowledge of fraction addition interfere with the knowledge of fraction multiplication? Does fraction multiplication then interfere with fraction division? Does knowledge of reciprocals interfere with knowledge of equivalent fractions? Or does it help? How does the use of context, student invention, or the encouragement of explanations change these outcomes? Even if effective techniques are used to teach specific procedures, what happens when students are no longer focused on one in particular? For example, if they know how to add fractions appropriately when adding is the focus of instruction, will they still add correctly when they are asked to multiply just prior to an addition problem? Can they keep the various procedures separate when they are not separated? Does learning procedures meaningfully help them do so? It seems these are important questions to ask, given that these topics will not remain isolated as students progress through their mathematical careers.

To summarize, research on fractions suggests at least five things about the fraction knowledge of students. These are: prior knowledge can both hinder and help students learn fraction procedures; conceptual knowledge and procedural knowledge are related and both contribute to fraction skills; students are capable of finding meaning for fraction procedures by inventing them; understanding fraction symbols can aid in the understanding of procedures; and fractions should have multiple meanings to students. In general, this research seems to be supported by theories that address how structure and understanding affect learning. Much progress has been made since Case (1988) set his agenda over fifteen years ago, yet there is much still to learn before fraction difficulties are fully understood.

Fraction Knowledge of Preservice Teachers

The literature on what preservice elementary school teachers know about fractions is considerably less extensive than that of student knowledge. However, there is evidence that they tend to make some of the same mistakes as students, at least with respect to division of fractions (Tirosh, 2000).

Tirosh (2000) reported that 5 out of 30 prospective elementary teachers made mistakes when solving division of fraction problems. She also found that even the prospective teachers who knew the standard invert-and-multiply algorithm could not usually explain why it worked. These teachers were also surprised to find there were alternatives to the standard algorithm, even when the alternatives were intuitive. In particular, the teachers were hesitant to believe that dividing the numerators and denominators gave a correct answer, even though multiplying numerators and denominators gives a correct answer to a fraction multiplication problem.

Ball (1990b) also claimed that preservice teachers did not understand the division of fractions algorithm. Her study involved ten elementary and nine secondary preservice teachers who were asked to solve a problem involving division with fractions and then create a story that matched it. While most (but not all) could solve it, only five were able to give an appropriate representation of the division problem, none of which were elementary preservice teachers. Over half of the elementary teachers were unable to generate any representation, while a few gave inappropriate ones. Furthermore, the teachers seemed to make no connections to their knowledge of division, focusing instead on the fact that the problem involved fractions.

Tirosh and Graeber (1990) suggested that when connections are made to whole number division, they are often inappropriate. For example, they found that 15 out of 21 preservice elementary school teachers insisted that a quotient will always be less than a dividend. Those who were able to overcome this misconception also showed improvement in writing correct expressions for word problems. Although these word problems involved division with decimals rather than fractions, it would not be unreasonable to expect similar results with fractions.

Ma (1999) suggested that Chinese teachers do not hold the same misconceptions as those in the United States. Using the same division with fractions problem as Ball

(1990b), she found that practicing teachers in the United States performed worse than the preservice teachers described in Ball’s study; about one-third of them could not solve the problem correctly and another one-fifth seemed quite unsure of their approach. Also, the invert-and-multiply method was the only method described by these teachers. By contrast, all of the Chinese teachers correctly solved the problem, and collectively, they alluded to at least three alternate approaches to solving the problem. Ma suggested that teacher preparation may serve to break the cycle between ineffective mathematics instruction and low levels of teacher knowledge.

Motivation for Fractions

“There is no such thing as an unmotivated child. Children are motivated” (Middleton & Spanias, 1999, p. 67). This claim may be true for young children, but there is evidence of a decline in motivation as children go through the schooling process, particularly during the first year of middle school (Eccles et al., 1993). Mathematics is no exception; students who dislike mathematics tend to start disliking it around this same time (Middleton & Spanias, 1999). According to Wigfield and Eccles (2000), students’ beliefs about their ability to be successful in mathematics also decline significantly during this time, and they found these ability beliefs to be strong predictors of subsequent performance and anxiety. Marsh and Yeung (1997) added to these findings, suggesting a causal effect of mathematics self-concept on achievement.

Students’ Motivation for Fractions

Fraction instruction coincides with the transition to middle school, but little is known about motivation for fractions in particular. This section of literature will focus primarily on motivation for fractions, but this literature is limited and is mostly restricted to instructional influences.

Anand and Ross (1987) examined the role of context in learning how to divide fractions. Fifth and sixth graders who had been instructed in adding, subtracting, and multiplying fractions were randomly assigned to one of four groups regarding division: no instruction, instruction using personalized contexts, instruction using concrete (hypothetical) contexts, and instruction using no contexts. Students who received instruction were simply told how the algorithm worked. The posttest included personalized, concrete, and context-free problems as well as transfer items. The students also were given an attitude questionnaire that assessed their reactions to the instruction. In general, the researchers found that personalizing the context of word problems resulted in better performance and attitudes, particularly with middle- to low-achievers. A limitation of this study is that context was the only opportunity for sense-making given to the students. It is unclear whether these results would hold if other instructional attempts were also made.

Stipek et al.'s (1998) study is one of the few that attempted to merge ideas on what is considered good mathematics teaching and what is motivating to students. The purpose of the study was to examine relations between motivating teaching practices, student motivation to learn fractions, and student gains on non-routine and routine items. Non-routine items were meant to assess conceptual knowledge, while the routine items were meant to assess procedural knowledge.

A total of 624 fourth- through sixth- graders participated in the study, as well as 24 teachers. The teachers were selected based on how much they claimed to adhere to reform-based mathematics. This adherence was subsequently assessed through videotapes and questionnaires. A nine-dimension coding system was used with the videotapes to determine instructional practices. Three factors emerged from these dimensions: learning orientation, positive affect, and differential student treatment. The learning orientation scale reflected an emphasis on effort, learning, and autonomy. A teacher scoring high on this scale also de-emphasized performance. The positive affect scale reflected a teacher's positive affect, enthusiasm, and creation of a risk-free environment. The differential treatment scale reflected the level of social comparisons and emphasis on speed.

Questionnaires were administered to the students to assess motivation at the beginning of the year and after the fractions unit. The questionnaire assessed perceived ability, mastery orientation, performance orientation, help-seeking, positive emotions, negative emotions, and enjoyment. At the beginning of the year, the questionnaire measured student motivation for math in general, but after the unit on fractions, the questionnaire measured student motivation for fractions specifically. This inconsistency limited conclusions about change in motivation for fractions, since students may like math in general but dislike fractions. However, it allowed the researchers to predict motivation for fractions above and beyond general motivation for mathematics. A separate regression was run for each of the three teacher practice dimensions. It was found that a teacher's positive affect predicted students' positive emotions and help-seeking, and a teacher's learning orientation predicted students' positive emotions and enjoyment of fractions.

The researchers also calculated gains in fraction skill using pretest and posttest scores. The researchers found that a learning orientation was correlated with gains on non-routine items but not routine ones. These results are consistent the results of Saxe and Gearhart (1999) reported earlier, but like Saxe and Gearhart, Stipek et al. (1998) did not assess retention of skills. Future studies should examine the long-term effects of teaching practices that emphasize learning on students' ability to retain procedural information over time.

When examining the relation between student motivation and skill gains, Stipek et al. (1998) found an unexpected result. Gains related to non-routine items were not correlated with any of the student motivation components, while all but performance orientation were correlated with routine items at the end of the fraction unit. The researchers speculated that perhaps students who are more positive about mathematics are more attentive and practice more, resulting in gains on routine items.

To summarize the findings of Stipek et al. (1998), they found that a teacher's learning orientation predicted students' positive emotions and enjoyment of fractions and was related to gains in non-routine items. In contrast, they found that nearly all student motivation constructs related to gains on routine items. It should be noted, however, that the routine items were limited to finding equivalent fractions and adding and subtracting fractions. It is unclear whether the addition and subtraction problems included unlike denominators, a topic that is notoriously difficult for students (Byrnes & Wasik, 1991).

Schunk (1996) conducted two studies exploring relations between instruction emphasizing learning and student factors. Fourth graders were tested on their ability to add and subtract fractions with both like and unlike denominators. Student factors included self-efficacy, goal orientation, and self-satisfaction. Consistent with Stipek et al. (1998), he found that instruction emphasizing learning goals enhanced motivation. However, he found that it also enhanced procedural knowledge. He speculated that an emphasis on learning (rather than performance) influences self-efficacy, which in turn influences skill. The difficulty of skill assessed might explain why Schunk and Stipek et al. had different results with regard to skill. Not only did Schunk include both like and unlike denominators, he did so with fourth graders only, when it is more likely to be a new skill. By the time students reach the sixth grade, these skills have likely become routine. Future research should explore the level of difficulty when determining the effects of instructional emphasis on skills.

Schunk (1996) also examined the effects of self-evaluation on fraction skill and motivation. When students used self-evaluation on a daily basis, there was no longer an effect of instructional emphasis with regard to skill or self-efficacy. However, when students infrequently self-evaluated, students benefited from a learning goal emphasis. Schunk speculates that frequent self-evaluation makes progress clear to students, which results in increased self-efficacy and productivity. Shih and Alexander (2000) extend these results by showing that self-evaluation is more effective than socially-referenced evaluation for enhancing skills and self-efficacy for fractions. Their study also involved fourth graders and employed a skill assessment similar to the one used by Schunk (1996).

That these studies involved learning-oriented instruction is consistent with current mathematics education beliefs and efforts (NRC, 2001). However, they are simply a beginning. A next step is to explore how other elements of instruction are related to this orientation. As mentioned previously, elements such as context, discussion, and invented-procedures have been confounded in the fraction knowledge literature with an emphasis on understanding procedures.

Preservice Teachers' Motivation for Fractions

Even more so than the research on preservice teachers' fraction skills, the research on preservice teachers' motivation for fraction is extremely limited. While it has not been a primary focus of preservice teacher studies, it has been mentioned in a few. One conclusion to be made is that much variation exists between elementary and secondary preservice teachers (Ball, 1990a). Whereas all of the secondary candidates in Ball's (1990a) study reported they were good at mathematics and enjoyed it, only half of the elementary candidates did so. The elementary candidates were also more likely to feel anxious about mathematics, view mathematics a set of arbitrary facts rather than interconnected ideas, and blame their weak knowledge on this arbitrariness.

If teachers' attitudes toward fractions are negative, it could negatively impact the attitudes of students. Therefore, it is important to understand whether these attitudes can be influenced before teachers reach the classroom. It may be that improvements in preservice teachers' understandings of fractions are related to improvements in their attitudes toward fractions, and the proposed study will attempt to explore such relationships.

Effective Mathematics Teaching

Presumably, improvements in mathematical understanding (i.e., learning) can happen through effective mathematics teaching. However, it can be difficult to determine the link between teaching and learning. What factors are involved? Which factors contribute to learning and which ones interfere? Some theory and research addressing such questions will be briefly explored in an attempt to draw some conclusions about the relationship between teaching and learning. Then, some findings about effective mathematics teaching will be reported.

Theories of Teaching and Learning

When attempting understand how teaching and learning are related, some more basic (but difficult) ones immediately arise. In particular, what is meant by teaching? What is meant by learning? In the context of mathematics classrooms, one might equate teaching with instruction and learning with achievement as measured by grades or test scores. Although the first equation might be easy to agree upon, the second seems more controversial. For example, a student might duplicate information on a test that was not truly learned, while others may learn something well but not successfully demonstrate this learning on an assessment. Implied here is that test-measured achievement alone cannot measure learning.

Trying to distinguish between achievement and learning sounds strikingly familiar; for years, theorists have been trying to differentiate instruction (or teaching) and development. Whereas learning might take place in a short period of time, development requires a longer period of change (Case, 1988). James, Kaffka, Herbart, Thorndike, and Vygotsky have all stated various views on the subject, each in an attempt to understand ideas such as learning and transfer (Vygotsky, 1986). According to Vygotsky, some people would suggest that instruction and development are independent. Piaget would fall into this category, insisting that instruction must wait for development. As the child actively explores the world around him, development naturally occurs. Others such as Thorndike propose that instruction and development are identical because they are both based on “association and habit formation” (p.176). In trying to reconcile these opposing views, Kaffka suggested that learning is a part of development. The other part would be maturation, and the two parts influence each other. Vygotsky claims that this relationship allows for transfer to exist. He states, “We have given [the student] a pennyworth of instruction, and he has gained a small fortune in development” (p.177).

Did these theorists come to a consensus? Apparently not—decades later, Bruner (1960/1977) was still debating these ideas and Resnick (1987) continued the discussion more than a decade after that. Like Vygotsky, Bruner (1960/1977) believed that concepts can be transferred to new problems. In order for learning of this kind to happen, the structure of the subject must be taught rather than just isolated facts and techniques. If it is, then instruction can aid in the child’s intellectual development.

This view is entirely consistent with Vygotsky (1986), who believed that instruction in scientific concepts could help a child develop his higher mental functions. Three important things should be noted about this statement. First is the inclusion of the word *instruction*. For Vygotsky, the teacher plays a major role in learning and development. The second word of note is *scientific*. Vygotsky insists that it is the systematic structure of such concepts that creates the relationship between learning and development. Like Piaget, he believes the child can learn without instruction, but he would not expect the learning to be similar in nature, nor would he expect it to have the same effect on development. As such, Vygotsky distinguished *higher* mental functions from lower ones. Vygotsky would not suggest that memorized techniques and skills would be readily transferred by students from one context to another. He would, however, suggest that processes such as “awareness, abstraction, and control” (p. 79) could develop through scientific instruction. If transfer is to occur, a concept needs to be abstracted. It is on this point that he criticizes Thorndike's analysis of formal discipline. He suggests that Thorndike tried to discredit a theory concerning higher mental functions by examining lower functions such as habit formation and fact memorization.

Whether transfer exists is still debated. In 1987, Resnick suggested that while not all skills transfer to all other areas, the idea of transfer does exist. One problem might be about definitions—about whether transfer involves knowledge or "the skills for acquiring knowledge" (p.19). Carraher (2002) more recently argued that the transfer metaphor is fundamentally flawed as an explanation for learning and should therefore be abandoned. Instead, the relationship between prior knowledge and learning should focus on issues such as assimilation and accommodation, as described by Piaget. In any case, it seems clear that what is meant by learning is not so clear. For the purposes of discussing research on teaching and learning in this review, both aspects of learning – achievement and transfer –will be used. *Instruction* will be used interchangeably with *teaching*.

Descriptions of Effective Mathematics Instruction

Concerning effective mathematics teaching, Reynolds and Muijs (1999) described six elements consistently found in the scholarly literature. They involve opportunity, teacher emphasis, classroom management, teacher expectations, ratio of whole-class to individual work, and interactive dialogue. Each will be described in detail.

The authors describe opportunity to learn as the amount of exposure to mathematics instruction. This includes curriculum coverage, number of school days, hours spent teaching rather than managing activities or behavior, and use of homework.

In essence, this is a quantitative measure of opportunity, and more exposure is related to higher achievement (Reynolds & Muijs, 1999).

The second element states that quality of instruction is related to the academic orientation of the teacher. An effective teacher spends less time on management and personal matters and more time on academic engagement (Reynolds & Muijs, 1999). This seems consistent with Byrnes and Miller (2003), who found a business-like atmosphere to significantly predict achievement, but personal interest shown by the teacher did not. This is not to say that personal involvement would not add positively to the atmosphere of the class, but it did not seem to contribute to actual achievement.

Effective classroom management contributes to the academic time described in the first component, but it may also be a product of the second component. In other words, someone with a serious orientation seems not as likely to have management problems. Other important factors are being well-organized and being able to maintain attention. When behavior problems do arise, they are more often handled with positive language rather than harsh words (Reynolds & Muijs, 1999).

Having high expectations means believing that all students can be successful and treating them accordingly. Different than personal interest, this involves an emphasis on effort and control. The teacher encourages students and is not biased towards a particular group (Reynolds & Muijs, 1999).

The emphasis on whole-class instruction is meant to be contrasted with large amounts of time spent on individual work or free time. The material is also presented in a structured fashion, with overviews and summaries provided. Reynolds and Muijs (1999) do not emphasize more recent trends such as small-group work in their account of effective teaching, and they draw no firm conclusions about their effectiveness.

The last element stresses the importance of discussion. The teacher does not just lecture, and many of the questions require explanations rather than one-word responses (Reynolds & Muijs, 1999). Among other things, this allows the teacher to monitor the students and adjust to their needs.

Taken together, this list provides a comprehensive view of the research on effective teaching of mathematics in American schools. However, the list is primarily based on studies that correlate certain behaviors with achievement rather than detailed analyses of quality of instruction. Most of these elements are concerned with the atmosphere of the class. Mentioned briefly is the structure of the lesson and the level of questioning. Rather than just providing a structured lesson, which implies being organized and coherent, Bruner (1960/1977) would also suggest that the lesson emphasize the structure of the *subject*. In other words, the curriculum should be designed to reinforce major ideas, not just specific topics. These ideas should be interconnected where possible and revisited regularly. In this way, they are not as likely forgotten and are more likely to transfer to new situations.

Stipek (2002) addressed some of these issues in her article linking instruction and motivation. Rather than seeing motivation as a static student characteristic, she claimed that instruction can play a role in motivating students to do and enjoy mathematics. In her review, she noticed four recurring themes regarding instruction. They include emphases on conceptual thinking, learning and understanding, active participation, and authentic and meaningful tasks. Each of these will be described in detail.

Like Bruner, Stipek (2002) claimed that good instruction involves “big ideas” rather isolated topics. Research suggests that this type of instruction is more challenging and enjoyable for students. When students are involved in trying to understand something interesting rather than memorize meaningless rules, they are more motivated to learn.

The second theme (an emphasis on understanding) refers to where the focus lies in the classroom. If the focus is on the right answer rather than the solution strategies, students are not as motivated to learn. Instead, they are simply motivated to be correct (Stipek, 2002). This could mean more cheating, more looking up answers, and not showing work. More importantly, however, it can mean lower self-esteem or embarrassment when the answer is incorrect.

The third theme (an emphasis on active participation) is similar to the high involvement described earlier. Stipek (2002) claims that students do not enjoy listening as much as they do participating. In addition, autonomy can be created by allowing students to use their own strategies in solving problems rather than simply demonstrating one that was provided for them.

Providing authentic and meaningful tasks has also been related to motivation. If students are given choice and tasks are connected to their personal lives, they are more likely to attend and apply themselves (Stipek, 2002). Although it unclear how these four themes are related to achievement, they seem to at least provide a link between motivation and instruction. Unfortunately, not much research has examined the connection between these two areas of research, and more should be done to understand their relationship.

The relationship between achievement (rather than instruction) and motivation is clearer. Certain aspects of motivation have consistently been found to significantly predict achievement. For example, Byrnes and Miller (2003) found math self-concept to be a significant predictor. Interest in mathematics was a predictor in a study by Schiefele and Csikszentmihalyi (1995). It seems that interest in doing well and believing that one can do well are both predictive of achievement, but Stipek (2002) believes that these constructs can be influenced by instruction.

Unresolved Issues: Areas for Future Research

Based on current gaps in the literature on fraction knowledge, motivation toward fractions, and instructional influences on fraction knowledge and motivation, several unresolved issues can be identified and areas of future research outlined.

First, more studies should be conducted with older students (i.e., those beyond middle school). How does experience with higher levels of mathematics influence students' knowledge of fractions? How does experience with the technology often used at those levels influence fraction knowledge? We currently know little about how well the knowledge students gain in the middle grades is retained during (and after) those years.

Second, studies should examine the interactions among various types of fraction knowledge. For example, how does learning to cross-multiply with fractions influence students' knowledge of fraction division? Are students more likely than normal to make the mistake of adding numerators and denominators just after completing several fraction multiplication problems? Would teaching fraction topics in a different order result in different interactions? Studies involving only one or two operations can be misleading if indeed the various types of fraction knowledge interact strongly with each other.

Third, there has been the documented slump in students' general motivations toward mathematics during the middle grades (Eccles et al., 1993), but little is known about topic-specific motivation in mathematics during this time. Also, too little is understood about the motivational trajectory. While Jacobs et a. (2002) reported that motivation toward mathematics continues to decline throughout high school, little is known about motivations beyond high school. Although Marsh (1989) suggested that mathematics self-concepts begin to increase for university students, this may not be true for particular groups. Do those who plan on teaching mathematics to elementary students have and retain negative views of their abilities in this arena or espouse a limited value for fractions?

Finally, more studies should examine how instruction influences fraction knowledge. Is there one best way to achieve the relational understanding encouraged by NRC (2001) and others? Must the instruction be connected to realistic situations, as suggested by Sharp and Adams (2002)? Do teachers need to encourage discoveries as suggested by Tzur (2004), or can lecture be just as effective? Or is it just that certain fraction concepts need to be made more explicit, as suggested by Yoshida and Sawano (2002)? Or is a combination of these things necessary for fraction instruction to be effective?

While the proposed study cannot answer all of the aforementioned questions, it did attempt to address a number of them. Specifically, the knowledge and motivation of college students majoring in education was assessed before and after they took a course that promoted a relational understanding of fractions. Differences in the instructional style were recorded and examined as a factor in any knowledge or motivational gains. As a result, tentative conclusions are made about the role of instruction in learning about fractions.

CHAPTER 3

METHODOLOGY

In the following sections, four pilot studies and the method for the present study are described. The pilot studies focused on: a) the fraction knowledge pretest, b) the item distinguishing pretest and posttest, c) the motivation questionnaire, and d) observations of the proposed target course. The method for the present study will be described in four sections. Those sections include: a) participants, b) course overview, c) measures, and d) procedures.

Pilot Studies

Several pilot studies were conducted in preparation for the present study. First, the fraction knowledge assessment was piloted to determine: a) how much time it required, b) whether variability existed among preservice teachers with regard to their fraction knowledge and solution strategies, and c) whether the instructions on the measure were clear. The item distinguishing the pretest and posttest was piloted to determine: a) whether the instructions were clear, b) what methods might be used to solve it, and c) how well it discriminated conceptual understanding. The motivation questionnaire was piloted to determine: a) how much time it required, b) whether it was appropriate for people beyond high school, and c) whether the adapted version was clear. Finally, classroom observations were made to determine whether variability existed among instructors of the targeted course with regard to delivery style. Findings from each of these pilot studies are subsequently described.

Fraction Knowledge Assessments

Pretest

Four professors of mathematics education and four doctoral students in mathematics education met to discuss a draft of the fraction knowledge pretest. The primary purpose of the meeting was to determine the validity of the chosen items. One person commented that there were fewer addition and subtraction problems than multiplication and division. It was decided that since the addition and subtraction algorithms were similar, it was not necessary to have as many of each. Hence, no changes were made with respect to the number of problems for each operation. Another person suggested that using 5 1/2 and 1/2 in the division word problem would likely lead to solving the problem mentally, without using an algorithm. To motivate the use of an algorithm, the numbers were changed to 5 2/3 and1/6.

The secondary purpose of the meeting was to determine whether or not the problems could be used to reveal both misconceptions and knowledge of non-traditional solution methods. Only one change was suggested here. Specifically, the problem 9/10 ÷ 3/5 had been included to test whether students knew they could divide the numerators and denominators. The suggestion was made that the 5 be changed to a 10. In this way, the problem would be conducive to the "common denominator" method of dividing fractions. This alternate method suggests that when two fractions have the same denominator, the solution can be found by dividing the numerators. Another reason for the suggestion was that it could possibly reveal misconceptions. For example, a solution of 3/10 would suggest that a student believes you must keep the denominator the same. Based on these reasons, the suggested change was made.

The modified assessment was subsequently piloted with a group of eight undergraduates majoring in elementary education. Seven of these students finished the items in 18 to 20 minutes. The eighth person turned in her test within this time frame, but she had skipped four items. These problems were not consecutive, and three of them were division problems. Rather than not having enough time, it was likely that she did not know how to solve division problems with fractions, especially since she missed the one division problem that she did attempt.

Two changes were made as a result of this piloting. One question was changed because two students misunderstood what it was asking. This question had asked students to name the shaded part of a figure, but two students named the *shape* created by the shaded part rather than the fractional amount shown. This question was reworded to clarify that the fractional amount was needed from the students. Another change involved an attempt to reduce the number of calculation errors not related to fractions. For example, some students multiplied incorrectly when the problem involved numbers beyond basic facts (greater than 12). Smaller numbers were used to address this issue. The final version of the fraction knowledge pretest appears in Appendix A.

Posttest

The posttest was the same as the pretest except for the inclusion of an open-ended item. This item was adapted from Linn (1969) for a professional development workshop with middle-school mathematics teachers. One adaptation involved removing the word “piously” so as not to confuse or distract. Another adaptation involved changing the names Jenny and Jeff to Jenny and Kevin. In this way, the names could be represented by their first initial without confusion. The adapted version was piloted in both high school and undergraduate classrooms. Four experts in mathematics also completed the task, including a mathematics professor, a mathematics education professor, a high school mathematics teacher, and an engineer.

Results revealed at least six possible solution strategies. The high-school students were least successful at the task, while the experts were all successful. In general, the rate of success increased as the level of education increased. People who were unsuccessful at the task generally fell into one of two categories. One category included students who did not realize the need for fractions to be of the same whole amount when adding. The other group consisted of students who showed evidence of realizing they needed to rewrite the fractions in terms of the same whole amount, but they were unable complete this process. The only change made based on these pilots was to italicize the relevant information, since a few students had asked for clarification. The modified item appears in Appendix B.

Motivation Questionnaire

The draft motivation questionnaire was adapted from Eccles, Wigfield, and colleagues (Eccles et al., 1993; Wigfield & Eccles, 2000). According to these researchers, students' values and expectancies for success influence performance and persistence in a subject. For mathematics in particular, they are also related to anxiety (Wigfield & Meece, 1988). In the current study, the relations between anxiety, value, and expectancies are examined for fractions.

The domain-specific questionnaire was adapted to be topic-specific. For all but one item, the word *math* was replaced with *fractions*. For example, "How good at math are you?" was changed to "How good at fractions are you?" Only one item was changed in a more significant way. That item asked "How much do you like doing math?" and was changed to "How much do you like being given a set of fraction problems to solve?" A few of the anxiety scale items used by Wigfield and Meece (1988) were omitted because they did not seem applicable to college-level students or to fractions. Had these questions been adapted, they would have asked about intentions to take more fractions and being scared of advanced fractions.

To make sure the adapted questionnaire was feasible for use with those beyond high school, it was piloted with three young adults. It took each of them 4 to 5 minutes to complete. Only one item seemed to produce results that did not fit with its hypothesized scale. This item was also the only item that had been altered in a more significant way than changing the word *math* to *fractions*. The change had seemed necessary when first adapting the questionnaire, but a discussion with the three participants revealed that the original wording held a different connotation to them than the new one. Using the original wording (but making it specific to fractions) seemed to evoke the intended meaning and was therefore used in the present study. The final version appears in Appendix C.

Classroom Observations

Because the present study attempted to measure differences in instructional style for a particular course, I wanted to understand what sort of variations to expect. For example, I wanted to find out if some of the instructors lectured, despite being encouraged to promote discussions. For this reason, two different instructors of the course were observed during the semester prior to data collection. Observations revealed that not only were there were noticeable differences with regard to lecture, but also with rapport and enthusiasm. My discussions with a third instructor suggested that additional differences may exist. In particular, this instructor claimed to consistently spend the entire class time with the students. By contrast, the other two instructors finished the observed classes 20 minutes early. This instructor also claimed to encourage student discoveries regarding key mathematical ideas.

Method

The method for the present study will be reported in five sections. First, participants will be described. Second, the course will be overviewed. Third, details regarding the three measures will be provided. Fourth, the procedures will be explained. Finally, strategies for analyzing the data will be detailed.

Participants

Participants for the present study consisted of education majors and their course instructors. These students were enrolled in a mathematics course designed to help deepen their understanding of elementary school mathematics. There were 104 students signed up for the course at the beginning of the semester. Of these, 99 gave consent for their data to be used in the study. One student was late and therefore did not take the motivation pretest. A total of nine students dropped the course at various points during the semester and one person did not take the posttest. Attendance was recorded and considered as a covariate or as criteria for exclusion from the study. Three students were dropped from the analysis because they were absent for half or more of the targeted lessons on fractions. Only the 85 students with good attendance and complete data were included in the analysis. Additional demographic information about the students, such as gender, age, and ethnicity, was gathered on the same day as the pretest. The demographic sheet is included in Appendix D. The students were predominantly female and white, indicating they were a representative sample of elementary education majors. Of the 85 students included in the study, 20 were freshmen, 49 were sophomores, and 16 were upperclassmen.

There were five sections of the course and three instructors. Instructor A was a doctoral student in mathematics and had taught this course in the prior semester. He also taught a similar course one year earlier. He currently taught two sections of the course. Instructor B recently received her doctorate in mathematics education and had taught this course for 11 years. She taught one section of the course. Instructor C was also a doctoral student in mathematics and this was his first time teaching an education course. He taught two sections of the course. Consent forms were secured by all three course instructors who were willing to be observed during the teaching of fractions. Likewise, consent forms were secured for all students willing to have their knowledge and motivation data entered into analysis (see Appendix E).

Course Overview

An overview of the observed course will be presented in three sections. First, the purpose of the course will be provided. The student materials and instructor materials will be described in the second section. Finally, links between important features of this course and my research questions will be made.

The purpose of this course was to help students better understand the mathematics they will be teaching. The course description provided in the course syllabus (see Appendix F) was as follows:

> The course will review and extend topics of arithmetic and number theory that may be encountered in elementary school curricula. Students will actively investigate topics, working in groups on projects and writing explanations of their thinking as well as answers to problems.

The course was divided into four sections, and fractions were covered in each of these sections. The sections included: numbers and basic number theory, addition and subtraction, multiplication, and division. Students were expected to make connections between the meanings of the four operations and the corresponding fraction procedure. For example, the meaning of division in general was used to help explain why the algorithm for dividing fractions involves multiplying by the reciprocal of the divisor. Pictures were used extensively to illustrate these meanings. The philosophy statement, which appeared in the course syllabus, also referenced this emphasis on meaning:

> Many people think of math as a collection of meaningless procedures and rules that "magically" give the right answer when numbers from a problem are inserted correctly. Your experiences in this course will be very different from this! Throughout the course, in class, on projects, and on exams, you will be asked to "explain why or why not" or to "justify your answer." In other words, you will be expected to understand why the procedures you are using works or why the answer given is correct. You will be most successful this semester if you continually ask "why?" as you read, listen, and solve problems. Seeking connections and meaning can be a very rewarding way to learn–and someday teach–these math ideas.

The student materials included a text (Beckmann, 2005b), the class activities manual accompanying the text (Beckmann, 2005a), and a student packet created specifically for the course. A sample from the student packet is included in Appendix G. The instructors met regularly to receive teaching notes for the next segment of the course. These teaching notes included specific lesson plans and were provided by the course chair, who also instructed one of the sections. A sample of these teaching notes is included in Appendix H.

The primary feature of the course was that it attempted to help students make connections between the procedures they will be teaching and the concepts related to those procedures. This feature is directly related to the second research question of the study, which asked whether this kind of instruction has an impact on participants' knowledge of and motivation for fractions.

Another feature of the course was that the instructors used the same materials and followed the same lesson plans. This feature was instituted to ensure consistency in the *content* delivered across sections of the target course. Nonetheless, it was expected that the three course instructors would manifest marked differences in their *delivery* of course content. Question four asked about those expected differences in instructional delivery and whether they might impact the knowledge or motivation of the students.

Measures

Fraction Skills Assessments

Pretest. The fraction knowledge pretest contained 20 items. Of these, 15 were routine problems without context (e.g., 1/2 + 3/4), three were routine word problems, and two were non-routine problems. Each of these types will be described in detail.

The 15 routine problems without context involved three addition, three subtraction, four division, and five multiplication problems. The addition problems include adding fractions with the same denominator, adding fractions with different denominators, and adding mixed numbers with the same denominator. The subtraction problems were similar, except that a whole number minus a fraction was included rather than subtraction with the same denominator. The division problems included a mixed number divided by whole number, a whole number divided by fraction, a fraction divided by a fraction (same denominator), and a fraction divided by a fraction (different denominators). Finally, the multiplication problems included a fraction times a fraction (different denominators), a fraction times a fraction (same denominator), a fraction times whole number, a mixed number times a whole number, and a mixed number times a mixed number.

The three word problems could be solved using division, subtraction, and multiplication, respectively. Two non-routine items were adapted from Stipek et al. (1998) and were intended to assess knowledge of basic fraction concepts. In particular, one assessed knowledge of equal parts, and the other assessed knowledge of how fractions relate to units greater than one.

For all 20 problems, students were asked to show their work or explain their thinking. The purpose for students showing their work was to uncover any conceptual problems that were presumed to underlie incorrect answers or inefficient solutions. Thus, the work and explanations were carefully examined to identify such patterns. For example, the pilot test showed that for the problem 4 ÷ 1/4 three out of eight students multiplied by 1/4 rather than its reciprocal; a procedure that resulted in a quotient of 1 rather than 16. Two of them also failed to use the reciprocal for the problem 2 1/3 ÷ 9. However, none of these three students made the same error for the problem 4/9 ÷ 3/8 or the problem 9/10 ÷ 3/10. This pattern may suggest a conceptual problem with reciprocals rather than with division of fractions per se. If a student's understanding of reciprocals is simply that the numerator and denominator switch places, then he or she may not know what to do when the problem involves a whole number.

Viable solutions were also examined for patterns. For example, the pilot test showed that four out of eight students changed all mixed numbers to improper fractions no matter what problem was posed. While this procedure is efficient for multiplying mixed numbers, it is quite inefficient when adding mixed numbers with the same denominator. It can also be inefficient to change whole numbers to improper fractions. Of the seven students that successfully solved the problem 12 – 3/8 during the pilot test, only two did not use improper fractions. The pattern of changing mixed and whole numbers to improper fractions even when it is inefficient may suggest a lack of ability to reason with fractions. Smith (1995) found that students who were competent in reasoning with fractions used the general methods only when they did not know of a more efficient one.

Star (2005) has suggested that these competent students are demonstrating *procedural flexibility*.

Opportunities for students to demonstrate such flexibility were built into the pretest. For example, the students were asked to divide 9/10 and 3/10. Having common denominators in a division problem allows for a very efficient alternative to the traditional “flip and multiply” algorithm. Problems were also included that could reveal deficiencies in understanding. For example, students were asked to multiply with common denominators, namely 2/15 and 7/15. If students kept the denominator the same as they would with addition, it may suggest a reliance on perceptual cues rather than concepts, particularly if those students were to multiply correctly on other problems.

To validate the rubric for flexibility points, the fraction test was given to nine experts, including two engineers, two mathematicians, two mathematics educators, two secondary mathematics education majors, and one secondary mathematics teacher. These experts consistently used more elegant and efficient approaches on 11 different problems. As a result, there were 11 possible flexibility points awarded on the test.

The 15 routine problems without context were scored using a three level scoring system (0 to 2 points). Two points were awarded for methods that led to correct solutions. One point was awarded for responses containing minor mistakes. Such mistakes were not either careless or unrelated to fraction knowledge. Included here would be mistakes with whole numbers. For example, a student may know to use common denominators to add fractions but writes 22 + 34 = 58. No points were given for errors resulting from inappropriate solutions. An example of each score is presented in Table 1.

Table 1

Scoring Examples for the Routine Items without Context

	Point Values		
	0	1	2
Solution	$2\frac{3}{5}+3\frac{1}{5}=5\frac{4}{10}$	$2\frac{3}{5}+3\frac{1}{5}=7\frac{4}{5}$	$2\frac{3}{5}+3\frac{1}{5}=5\frac{4}{5}$
Computation	I added the numerators and denominators	$\frac{13}{5}+\frac{16}{5}=\frac{39}{5}=7\frac{4}{5}$	$\frac{13}{5}+\frac{16}{5}=\frac{29}{5}=5\frac{4}{5}$

For the non-routine problems, students were awarded no points for incorrect solutions and two points for correct ones. Word problems were scored using a four level system (0 to 3 points). No points were awarded for inappropriate solutions. One point was given if the student set up the problem appropriately but made more than minor errors when finding the solution. Two points were awarded for only minor errors, and three points were awarded for correct solutions.

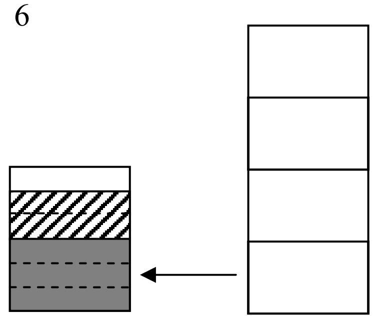
A second rater trained by the researcher scored ten percent of the tests to establish interrater agreement. For the knowledge pretest, interrater agreement was 98%. Thus, I scored all the remaining pretests independently.

Posttest. The posttest was the same as the pretest except for one additional item, adapted from Linn (1969). This item was an open-ended transfer item, or an item intended to test relevant concepts that cannot be directly solved with known algorithms. In this case, the problem involved mixtures of liquids, in which fraction algorithms could be used but not as directly as with the word problems. The primary purpose of this particular transfer item was to test the understanding that fractions cannot be added using the traditional algorithm unless they are fractions of the same whole amount.

If a student arrived at a correct solution, two points were awarded. If a student demonstrated that he or she knew the fractions must be rewritten in terms of the same whole amount but did not know what to do next, one point was awarded. No points were awarded for an inappropriate solution. An example of each of these scores appears in Table 2. The remainder of the posttest was scored in the same manner as the pretest. The same rater who scored 10 pretests also scored 10 posttests. Interrater agreement was 99%. Thus, I scored the remaining posttests independently.

Table 2

Sample Scoring for the Transfer Item

	Point Values		
	0	1	2
Solution	$\frac{7}{12}$	$\frac{5}{6}$	$\frac{5}{18}$
Computation	$\frac{1}{3}+\frac{1}{4}=$ $\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$	 If you pour Kevin's syrup into Jenny's glass, it will fill $\frac{1}{2}$ of it. Add that to Jenny's $\frac{1}{3}$ and you have $\frac{5}{6}$ of her glass filled with syrup.	Jenny's syrup is $\frac{1}{9}$ of the combined mixture. Kevin's syrup is $\frac{1}{6}$ of the combined mixture. $\frac{1}{9}+\frac{1}{6}=\frac{2}{18}+\frac{3}{18}$ $=\frac{5}{18}$

Motivation Questionnaire

The motivation questionnaire included 15 items adapted from the work of Eccles, Wigfield, and colleagues (Eccles et al., 1993; Wigfield & Eccles, 2000). Prior research has suggested three separate scales for these items. Two of the scales are based on the expectancy-value theory of achievement motivation: value and self-concept of ability. Interest, importance, and usefulness comprised the value scale, and ability and expectancy beliefs comprised the self-concept of ability scale. Although theoretically distinct, these subscales are empirically similar and have been placed together in prior studies (Anderman et al., 2001). The third scale assessed the affective component of anxiety. Wigfield and Meece (1988) have shown that anxiety can have both an affective and a cognitive component (fear and worry, respectively), but the affective component is the one that has debilitating effects on achievement. Therefore, only the affective component was used in the study.

Because the questionnaire has not been used with undergraduates or as a topic-specific measure, an exploratory factor analysis was conducted with the 15 items. Principal Axis Factoring (PAF) with oblique rotation was used with the pretest data to assess whether these items still suggested three scales. The results did suggest three scales, but one item was not loading strongly on any of them. This was item nine on the questionnaire and it asked, “How good would you be at learning something new about fractions?” Because undergraduates have most likely seen all combinations of fraction arithmetic, they may not have been certain how to answer this question. Thus, it made sense to drop the item from the questionnaire.

The test was run again with the remaining 14 items. Results suggested that three factors explained 62% of the variance among the items. These factors were similar to prior studies with one exception. The two items thought to test interest in fractions were loading strongly on the same factor as the items thought to test self-concept of ability. However, it made sense that undergraduates would have low interest in things they do not feel they are good at. Hence, these two items were included with self-concept of ability rather than value. The loadings for self-concept of ability ranged from .47 to .85. For value, they ranged from .68 to .80. For anxiety, they ranged from .62 to .90. Appendix I shows the factor loadings for the three factors.

For the motivation posttest, PAF with oblique rotation suggested only two factors. This time, the anxiety items were loading with the self-concept and interest items. The value scale continued to load on a separate factor. Appendix J shows the factor loadings for the two factors. Because the results at pretest were more similar to those from prior studies, in which extensive analyses were conducted, the three pretest factors were used. Implications for the change in factor structure over time are discussed in Chapter V.

Students were asked to answer the 14 items using a 7-point scale. Value now included four items, for a maximum of 28 points. Self-concept of ability included six items, for a maximum of 42 points. Anxiety included four items, for a maximum of 28 points. Alphas were computed to determine the reliability of the motivation factors. The alphas for value, anxiety, and self-concept of ability at pretest were .82, .86, and .90, respectively. The reliability posttest scores for value, anxiety, and self-concept were .84, .91, and .86 respectively.

Classroom Observation Instrument

The classroom observation instrument was designed specifically for a large, on-going longitudinal study on high quality teaching in upper elementary schools (Valli & Croninger, 2002). The instrument was used for more than three years of data collection, and data from this study demonstrated reliability. This investigator had been extensively trained to use the instrument and has served as an expert observer for reliability purposes. However, the instrument had never been used in college classrooms prior to the present study.

The dimensions of the instrument were created to be consistent with the literature on effective mathematics teaching, particularly to align with both traditional and reform ideas (Chambliss & Graeber, 2003). For example, codes for linking concepts and procedures, posing high-level tasks, making connections to real-world, and working in small groups were included based on reform ideas, whereas codes for reading from a text, lecturing, and focusing on procedures were included based on traditional ideas. A glossary was written to describe each code in detail. The instrument was chosen for the proposed study because it is designed to capture the features of instruction that are of interest in research question four, including organization of the class, kinds of questions, use of lecture, and links between concepts and procedures.

The first part of the observation instrument uses a time-sampled approach. Every three minutes, a screen opens and eight categories are coded, including teacher activity, student activity, organization of the class, attention of the teacher, content, context, classroom behavior, and use of technology. Nearly all categories are broken into sub-categories, but the coding is mutually exclusive within the eight broad categories. For example, a coder must decide if the organization of the class is whole group, small group, independent work, or a mixture of group and independent work. If one of the latter three sub-categories is appropriate, then the coder must also decide whether the students are focused on the same or different content. Hence, one choice is made from seven possible choices. An annotation can be made if additional comments or clarifications are needed.

The observation instrument also includes an attribution form, to be completed at the end of the lesson. The observed lesson is scored for evidence of five dimensions of effective pedagogical practice (Alexander & Murphy, 1998). Those five dimensions deal with: knowledge, strategic processing, development and individual differences, motivation, and context or situation. Each of these dimensions is examined using four specific instructional behaviors and rated on their centrality to the teachers' performance during the observed lesson. For example, the knowledge dimension assesses whether the teacher: promoted principled understanding, activated prior knowledge, manifested a deep understanding of the content, and illustrated the value or utility of the lesson. After all five dimensions are coded on a 4-point continuum ranging from not evident to pervasive, the lesson is rated for overall quality with 1 representing low quality and 4 indicating high quality. A complete list of codes for the time-sampling and attribution forms appears in Appendix J.

A second rater trained with this instrument observed two classes to establish interrater agreement. One class was led by Instructor A and one was led by Instructor C. Overall interrater agreement for the time-sampled data was 93%. The overall interrater agreement for the attribution scale, a more high-inference measure, was 79%.

Instructor Interviews

In order to gain insight into why instructional differences may exist, the instructors were interviewed about their beliefs and goals for the course at the beginning and end of the semester. The pretest and posttest contained similar questions, so that changes in attitudes and beliefs may be detected. Appendix K contains the interview questions.

Procedures

The previously described fraction knowledge and motivation measures were administered during the regular classroom times as part of the course. Including these assessments in the course enhanced the ecological validity of the present study and likely contributed to students' motivation to perform well. On the first day of class, pretests for fraction knowledge and motivation for fractions were administered to all the sections of the course. The posttests for fraction knowledge and motivation were administered near the end of the semester, after the last lesson involving fractions. Because of snow, the school opened late on the day planned for all posttests to be administered. As a result, three of the sections were administered their posttests on the next class day.

The pre-interviews for instructors were conducted at the beginning of the semester, and all instructors were interviewed on the same day. The post-interviews were conducted near the end of the semester, but because of the aforementioned weather conditions, two instructors were interviewed on the class day following the first instructor. All interviews were audiotaped and transcribed.

The classes were observed whenever the lesson involved fractions, which included nine instructional days. Rather than including all the fraction topics consecutively, the course was organized by the four operations. This organization resulted in the fraction instruction being spread throughout the semester. Fractions were covered during the unit on number theory, the unit on addition and subtraction, the unit on multiplication, and again during the unit on division. The instructors all used the same syllabus, exams, and other materials, so the same lesson was observed for each instructor during each day of fraction instruction.

Student absences were recorded on the days the classes were observed. If students were absent, they were assigned a value of 1 for the day. If they were late or left the class early, they were assigned a value of .25 for every 15 minutes of class they missed. In this way, a student who was consistently late to class would not have the same scores as someone who was always present for the entire class. Each class was 50 minutes long.

Analysis Strategy

Student Data

In the past few decades, it has been common practice to dichotomize knowledge of fractions into procedural and conceptual knowledge (Byrnes & Wasik, 1991; Carpenter, 1986; Case, 1988; Hiebert & Lefevre, 1986; Rittle-Johnson, Siegler, & Alibali, 2001). But in recent years, it has become apparent that these two classifications are not always straightforward (Asku, 1997; NRC, 2001; Star, 2005). Consequently, I decided to use five distinct measures of fraction knowledge and look for relations based on the data, rather than trying to dichotomize the data into measures of procedural and conceptual knowledge.

A computation score was created by totaling the scores for addition, subtraction, multiplication, and division. These scores were combined because they each required the students to carry out an algorithm in order to arrive at a correct answer. A basic concepts score was created using the non-routine items. These items were used because they require the students to understand the need for equal parts as well as how fractions relate to units greater than one. A word problem score was created by adding the points for correctly setting up the problem and the points for arriving at a correct solution. Although computation ability is necessary to correctly arrive at a solution, setting up the problem involves knowing which fraction procedure is appropriate. Although the National Research Council (2001) has suggested that computational skill and knowledge of when to use procedures are both parts of *procedural fluency*, research has suggested these skills are distinct from each other (e.g., Asku, 1997).

A flexibility score was used to assess whether students used efficient and elegant methods to find solutions to routine problems. Star (2005) argued that a "flexible solver…can navigate his or her way through this procedural domain, using techniques other than the ones that are overpracticed, to produce solutions that best match problem conditions or solving goals" (p.409). For fractions in particular, Smith (1995) showed that students who reasoned competently with fractions used alternate algorithms when they were more efficient. On the posttest, students were also tested on their ability to transfer learned concepts and procedures to novel situations.

Using correlations, relations between the five knowledge scores and the three motivation scores were examined. Using repeated measures analysis of variance (MANOVA), student scores were also examined for possible differences related to the instructor.

Teacher Data

In recent years, the work of those who research fraction knowledge has consistently supported the idea of relational understanding espoused by Skemp (1978). That is, researchers have generally agreed that students need to forge links between concepts and their related procedures, and that forging these links leads to better retention and better transfer of ideas to new situation (Carpenter & Lehrer, 1999; NRC, 2001). However, many teaching practices have been suggested as a means to this end. Some of the common practices deemed essential for learning fractions are the use of context (Sharp & Adams, 2002), the encouragement of student-invented procedures (Mack, 1995, 2000; Saenz-Ludlow, 1994; Tzur, 1999, 2004), and the practice of asking students to justify their solutions (Kazemi & Stipek, 2001). However, lecture has generally been associated with a lack of sense-making (Morris, 1995). Few studies if any have examined sense-making of fractions through lecture.

While the observation instrument used in this study was not particularly designed to capture whether students were inventing their own procedures, it was designed to capture whether or not the instructors were linking concepts to procedures, using context, asking students to justify their solutions, or lecturing. It also captured whether the activities were being led by the teacher and whether the students were engaged in high-level tasks as opposed to routine problems. In order to contribute to our understanding of the roles of these practices, the data were examined for evidence of them. In particular, the data were examined through three lenses: promotion of relational understanding, emphasis on high-level discourse, and the extent to which the class was teacher-directed. These lenses were chosen based on the goals of the course and on prior literature.

Because students in all classes were overwhelmingly on task and not using technology during the fraction lessons, the classroom behavior and technology categories were dropped from the analysis. Because the teacher's attention was overwhelmingly on the whole class for all classes, even when the students were working in groups or individually, this coding category was also dropped from the analysis. The remaining codes were from the following categories: teacher activity, student activity, class organization, content focus, and context. These categories were examined through the aforementioned lenses, and as a result, the data were recoded into nine variables. Of these, four were related to understanding, two related to discourse, and three showed evidence that the class was teacher-directed. These variables and their related codes will each be described.

Evidence of promoting a relational understanding was found in four of the coding categories: teacher activity, student activity, content, and context. Teacher activity related to understanding included posing a high level question or task, elaborating on a high level question or task, responding with a question back to the student, or responding with evaluation or feedback. Student activity related to understanding included working on a high level question or task. Linking concepts and procedures was considered a content focus relating to understanding. Context related to understanding included connections to prior mathematics, connections to another content area, and connections to the real world.

Evidence of high-level discourse was found in two of the coding categories: teacher activity and student activity. Teacher activity related to discourse included requests for an alternative method or requests for an elaboration of a student's response (i.e., having a student explain his or her thinking). Student activity related to discourse included responding with a conjecture, responding with an explanation or justification, or responding with an alternative method.

Evidence that the class was teacher-directed was found in three coding categories. Teacher activity related to a teacher direction included modeling with technology, modeling without technology, or lecturing mathematics content. Student activity related being teacher-led included students passively listening. Finally, organizing the class as a whole group (as opposed to small groups or individuals) was considered to be a more teacher-directed practice.

Analysis of variance (ANOVA) was used to determine whether teachers differed with regard to these teaching practices. Another ANOVA was used to determine differences on the attribution scales. Differences detected among teachers were used to make sense of differences found among the students when grouped by instructor.

CHAPTER IV

RESULTS AND DISCUSSION

This study was undertaken in order to better understand preservice elementary teachers' knowledge of and motivations toward fractions before and after taking a course designed to promote relational understanding, as well as what teaching practices might be related to student outcomes. Results will be reported in five sections. First, descriptive statistics will be presented. Then, each of the four research questions will be addressed.

To prepare the data for analyses, several steps were taken. First, student attendance was examined as criteria for exclusion from the study. Because I was interested in outcomes before and after a taking a course, it seemed appropriate that students with low attendance should not be considered. Three students were absent for half or more of the fraction lessons. Thus, these students were excluded from subsequent analyses.

Data for the remaining 85 students were examined for normality. At pretest, both the knowledge and motivation variables were statistically normal. Tests for homogeneity of variance were also satisfied, and an ANOVA was conducted to determine if the five sections of the course differed with regard to any of these scores at the beginning of the semester. None of the scores were significantly different. For subsequent analyses, classes were grouped by instructor.

Motivation variables were normal at posttest, but two of the knowledge variables (computation and basic concepts) were somewhat negatively skewed at posttest. Transformations did not help the distribution of these variables, as most students' scores were quite high at the end of the semester. Therefore, I decided to proceed with the analyses, interpreting outcomes concerning knowledge with some caution.

During 2 out of the 9 observations, Instructor A was absent. Instructor C taught his classes on both occasions. Because the intent of the observations was to capture teacher practices, those observations were dropped from the analysis. As a result, Instructor A had an average of 110 episodes for his two sections, Instructor B had 141 episodes for her one section, and Instructor C had an average of 142 episodes for his two sections.

Of the observational codes considered for analysis, only 59 were actually used during the semester. These codes belonged to five coding categories, which were examined through the three lenses described in Chapter III: relational understanding, high-level discourse, and teacher direction. As a result, nine variables were created. However, the discourse variable for teacher activity occurred less than 6% of the time for each teacher, causing it to be positively skewed. Because it is unlikely that a rarely occurring teaching practice would have a discernable effect on student outcomes, it was dropped from the analysis. The remaining variables were approximately normal. All six attribution variables were normally distributed.

Descriptive Statistics

Student Data

To answer the first research question, means and standard deviations were calculated for knowledge and motivation at the beginning of the semester. To answer the second question, means and standard deviations were determined for knowledge and motivation at the end of the semester. These descriptive statistics appear in Table 3.

Pearson correlations for knowledge and motivation were also computed, both for pretests and posttests. These can be seen in Tables 4.

Table 3

Descriptive Statistics for Knowledge and Motivation Variables

	Max Possible Value	M		SD	
Variable		Pre	Post	Pre	Post
Computation	30	23.25	27.60	7.26	3.92
Basic Concepts	4	2.87	3.72	1.17	.77
Word Problems	9	6.29	7.45	2.76	2.23
Flexibility	11	2.27	2.13	1.91	2.19
Transfer	2		.52		.63
Value	28	17.15	19.18	4.05	3.94
Anxiety	28	16.14	15.33	4.90	5.33
Self-Concept	42	22.02	24.73	6.29	5.59

Note. N=85. Transfer was only assessed at posttest.

Table 4

Correlations between Knowledge and Motivation Variables

Variable	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Pretest															
1. Computation	—														
2. Basic Concepts	.33**	—													
3. Word Problems	.58**	.40**	—												
4. Flexibility	.07	.15	. 33**	—											
5. Value	.16	-.03	.17	.02	—										
6. Anxiety	-.33**	-.24*	-.24*	-.15	-15	—									
7. Self-Concept	.36**	.00	.24*	.25*	.42**	-.61**	—								
Posttest															
8. Computation	.43**	.11	.34**	.05	-.02	-.31**	.25*	—							
9. Basic Concepts	.42**	.23*	.38**	.04	.16	-.22*	.17	.30**	—						
10. Word Problems	.40**	.17	.60**	.14	.10	-.21	.15	.27**	.35**	—					
11. Flexibility	.20	.12	.25*	.51**	.06	.02	.09	.00	.07	.24*	—				
12. Transfer	.26*	.17	.32**	-.04	.03	.34**	.16	.01	.21	.25*	.07	—			
13. Value	.09	.12	.16	.09	.32**	-.13	.16	.16	.12	.32**	-.04	.27*	—		
14. Anxiety	-.29*	-.15	-.18	-.16	-.01	.73**	-.39**	-.33**	-.28**	-.25*	-.14	-.36**	-.20	—	
15. Self-Concept	.39**	.18	.32**	.18	.14	-.54**	.59**	.39**	.25*	.29**	.08	.32**	-.43**	-.71**	—

Note. N=85. *p<.05, **p<.01.

Teacher Data

To answer the fourth research question, means and standard deviations for teacher practices were calculated for each instructor. The results are presented in Table 5. To create scores for each of the five learning principles, the four subscores under each principle were averaged. The overall quality score was coded as a separate variable. Means and standard deviations were calculated for each instructor and are displayed in Table 6.

Table 5

Proportion of Time Spent on Class Activities

		Understanding				Discourse	Teacher Direction		
Instructor		Teacher	Student	Content	Context	Student	Teacher	Student	Organization
A	*M*	.15	.20	.61	.25	.10	.15	.25	.64
(n=220)	*SD*	.36	.40	.49	.43	.29	.36	.43	.48
B	*M*	.21	.19	.57	.19	.23	.11	.16	.67
(n=141)	*SD*	.41	.39	.50	.39	.42	.32	.37	.47
C	*M*	.14	.21	.55	.20	.10	.22	.30	.76
(n=287)	*SD*	.35	.41	.50	.40	.33	.41	.46	.43

Table 6

Attribution Means by Instructor

Instructor		Knowledge	Strategic Processing	Development	Context	Motivation	Quality
A	*M*	2.96	2.84	2.09	2.59	2.21	3.2
(n=14)	*SD*	.80	.45	.48	.40	.37	.39
B	*M*	2.89	2.64	2.17	2.86	2.44	3.00
(n=9)	*SD*	.44	.36	.47	.36	.33	.50
C	*M*	2.54	2.42	1.94	2.31	2.11	2.61
(n=18)	*SD*	.56	.34	.34	.30	.37	.78

Knowledge and Motivation at the Beginning of the Semester

The first research question addressed in this study was: What is the level of elementary preservice teachers' knowledge of and motivations toward fractions at the beginning of a course designed to promote relational understanding in mathematics? Four knowledge variables and three motivation variables were used to answer this question.

Pretest Knowledge

As seen in Table 3, knowledge of fractions at the beginning of the semester was measured in four ways. The mean computation score was 23.25 (SD=7.26). For basic concepts, the mean score was 2.87 (SD=1.17). The mean word problem score was 6.29 (SD=2.76). These results mean that the average success rate was 78% for computation, 72% for basic concepts, and 70% for word problems. These scores can be regarded as the students' achievement for fractions at the beginning of the semester. In contrast, the flexibility score reflects a deeper knowledge of fractions. NRC (2001) used the term *procedural fluency* as a way of including both computation and flexibility. Out of 11 possible points, the mean flexibility score was 2.27 (SD=1.91). In brief, students at pretest correctly responded to about three-fourths of the problems, but they tended to use general algorithms to do so.

Many of the knowledge variables were related to each other at pretest. As shown in Table 4, computational skill was positively related to knowledge of basic concepts (.33, $p<.01$) and to the ability to solve word problems (.58, $p<.01$). Word problem success was also positively related to basic concepts (.40, $p<.01$) and flexibility (.33, $p<.01$). It is interesting to note that while word problems were related to both computation and concepts, flexibility was not related to either one. Star (2005) argued that computation and flexibility are both types of procedural knowledge, with the former being superficial and the latter being deep. This study supports the notion that the two types of knowledge are distinct.

Pretest Motivation

With regard to mathematics in general, Ball (1990a) reported that elementary preservice teachers tend to feel more anxious and less confident than secondary preservice teachers. As such, it was not surprising that the present study did not find high levels of motivation toward fractions in particular. As was seen in Table 3, the mean value score was 17.15 (*SD*=4.05) out of a possible 28 points. For anxiety, it was 16.14 (*SD*=4.90) out of a possible 28 points, and the mean score for self-concept of ability was 22.02 (*SD*=6.29) out of a possible 42 points. In other words, the students' motivation levels fell just to the right of the center of the scale. For value and self-concept of ability, this position is at least in the desired direction, but for anxiety, levels to the left of center would be desirable. In this sense, there were somewhat elevated levels of anxiety at pretest.

Correlations in Table 4 reveal that some of the motivation variables were related to one another. Students' self-concept of ability to do fractions was positively related to the value they placed on fractions (.42, $p<.01$) and negatively related to their anxiety toward fractions (-.61, $p<.01$). Anxiety was not related to value. Given these students' years of exposure to fractions, it seems feasible that they could understand the importance and usefulness of fractions whether or not they worry about how good they are at them.

Some of the motivation scores were also related to knowledge. In particular, anxiety was negatively related to computation (-.33, $p<.01$), basic concepts (-.24, $p<.05$), and word problems (-.24, $p<.05$). However, it was unrelated to flexibility. In other words, it seems that students worried about knowing answers, not about whether they could find the most efficient or elegant method for arriving at the answers. Self-concept of ability was positively related to computation (.36, $p<.01$), word problems (.24, $p<.05$), and flexibility (.25, $p<.05$), but not to basic concepts. It seems that students' perceived ability with fractions was related to both when and how to calculate answers, but not to their understanding of meaning.

Knowledge and Motivation at the End of the Semester

The second research question addressed in this study was: How will elementary preservice teachers' knowledge of and motivations toward fractions differ as a result of participating in a course designed to promote relational understanding in mathematics? This question was answered by examining posttest knowledge and motivation scores, as well as by examining how these outcomes compare to those at pretest.

Posttest Knowledge

In addition to the four pretest knowledge variables, an item assessing students' ability to transfer their knowledge to a novel situation was included in the posttest. As displayed in Table 3, the mean score for computation at the end of the semester was 27.60 (SD=3.01). For basic concepts, the mean score was 3.72 (SD=.77). The mean score for word problems was 7.45 (SD=2.23). These results indicate that the average success rate was 92% for computation, 93% for basic concepts, and 83% for word problems. In brief, students at posttest correctly responded to about nine-tenths of the problems.

In contrast, there was essentially no change in flexibility. The mean flexibility score was 2.13 (SD=2.19) out of 11 possible points. Implications of this finding will be discussed when the third research question is addressed. Transfer was also low. The mean for the transfer item was .52 (SD=.63), suggesting a 26% success rate on average. Details of this item will also be discussed with regard to the third research question.

It is not surprising that most of the knowledge posttest scores were moderately or strongly related to their corresponding pretest scores. One notable exception existed: basic concepts at posttest were only modestly correlated with basic concepts at pretest (.23, $p<.05$). One possible explanation is the same as was given for the performance increase in basic concepts: they were constantly reinforced as part of the course. It is likely that the course itself was more strongly related to basic concepts at posttest than was the students' knowledge of basic concepts at pretest.

Relations *between* knowledge variables at posttest were similar to pretest relations. One notable difference existed: at pretest, solving word problems was strongly related to computation (.58, $p<.01$), but at posttest, they were modestly related (.27, $p<.05$). Interestingly, word problems at posttest were more strongly related to computation at pretest (.40, $p<.01$) than to computation at posttest. One explanation is that, during the semester, students learned to solve word problems using pictures rather than algorithms. Perhaps those who struggled most with computation at the beginning of the semester were more likely to use pictures to solve word problems at the end of the semester. On the pretest, four students attempted to use pictures to help them solve word problems, but 30 students used them on the posttest. Apparently, students had more available tools for solving word problems at the end of the semester than at the beginning.

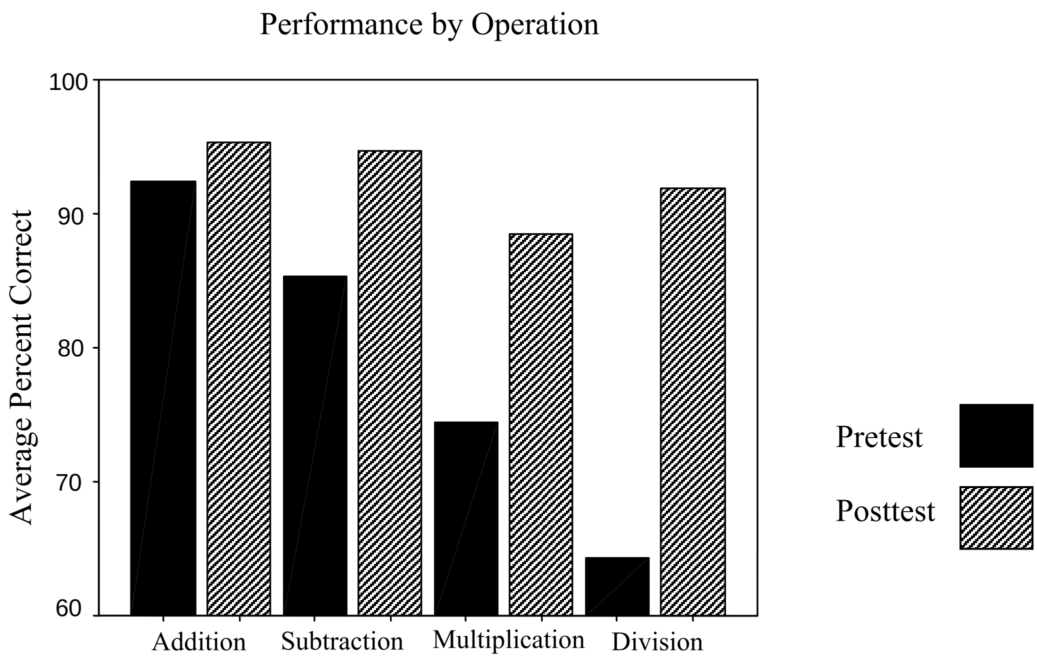
The ability to solve word problems was also related to knowledge of basic concepts (.35, $p<.01$), flexibility (.24, $p<.05$), and transfer (.25, $p<.05$). Computation and basic concepts were related to each other (.30, $p<.01$), but neither score was related to flexibility. Both pretest and posttest results suggest computation and flexibility were unrelated to each other. One explanation for the lack of significant correlation is the low tendency for these students to solve problems flexibly. Out of 11 possible points for flexibility, these students averaged less than three points. Another explanation is that general procedures can be memorized, but knowledge of when and how to deviate from those procedures takes, as Star (2005) suggested, a deeper understanding of the procedures.

Computation Changes from Pretest to Posttest

Performance on the routine computation items were examined separately for addition, subtraction, multiplication, and division. Because the number of problems was not equal for each operation, the scores were converted to percentages. Results revealed improvement on all four operations, with marked improvements in multiplication and division. Although students performed quite differently on the operations at pretest, the operations looked more similar at posttest. On average, students answered 92% of the addition problems correctly at pretest. This percent rose to 95% at posttest. For subtraction, the percent correct rose from 85% to 95%. For multiplication, the percent rose from 75% to 88%. For division, the percent correct rose from 64% to 92%. The results are graphically presented in Figure 1. The fact that students performed better on division than multiplication at posttest could, in part, be due to the fact that division was covered just prior to the posttest, whereas multiplication was covered one month prior. Other plausible reasons are discussed in the error patterns section.

Figure 1

Computation at Pretest and Posttest



I was also interested in how many students were making these errors. At pretest, 19 students made no errors on the routine computation problems. At posttest, 32 students made no errors on those same problems. In other words, despite high posttest performances on each of the operations (ranging from 88% to 95%), more than 3/5 of the students missed at least 1 of the 15 routine computation problems.

Some of the mistakes were considered minor, which meant they seemed to be careless or not related to knowledge of fractions (e.g., mistakes multiplying whole numbers). For scoring, these were the errors that were assigned a value of 1 instead of 0. There were 37 students who made no more than minor errors at pretest and 48 such people at posttest. Still, more than 2/5 of the students made at least one major fraction error on the 15 routine computation problems.

There were 48 students at pretest and 70 students at posttest who made no more than one major fraction error (e.g., 2/3 + 3/8 = 5/11). Clearly, more students performed better at posttest than pretest on the routine computation items. However, nearly 1/5 of the students still made more than one major fraction error when completing the 15 routine computation problems at posttest. As a result, I wanted to know where most of the students were having difficulties, so I examined the number of students making errors on each of the four operations.

With regard to specific operations at pretest, 22 students made addition errors, 26 students made subtraction errors, 49 students made multiplication errors, and 45 students made division errors. At posttest, 11 students made addition errors, 10 students made subtraction errors, 44 students made multiplication errors, and 17 students made division errors. Although the overall performance in multiplication improved, it is striking that more than half of the students made multiplication errors at posttest, particularly since only one-fifth of them made division errors. This finding is surprising because multiplying is the last step in the common algorithm for dividing fractions. Some possible explanations are provided in the section about error patterns.

Posttest Motivation

In general, motivation toward fractions changed in the desired direction from pretest to posttest. Statistical significance will be discussed in the next section. As was seen in Table 3, the mean value score was 19.11 (*SD*=5.47) out of a possible 28 points. The mean anxiety score was 15.33 (*SD*=5.33) out of a possible 28 points, and the mean score for self-concept of ability was 24.73 (*SD*=5.59) out of a possible 42 points. Similar to pretest, students' self-concept of ability was related to the value they placed on fractions (.43, $p<.01$) and to their anxiety toward fractions (-.71, $p<.01$). Value was again unrelated to anxiety. These results were seen in Table 4.

For anxiety, motivation was related to knowledge in similar ways to pretest. Anxiety was related to computation (-.33, $p<.01$), basic concepts (-.28, $p<.01$), word problems (-.25, $p<.05$), and transfer (-.36, $p<.01$). On the other hand, value was now related to word problems (.32, $p<.01$) and transfer (.27, $p<.05$). It seems that students at the end of the semester, students who knew when to apply procedures and concepts in context were also the ones who deemed fractions to be useful or important. Perhaps the extensive use of word problems during the course contributed to this shift.

Self-concept of ability was also related to knowledge somewhat differently at posttest than pretest. First, it was not related to flexibility, as it had been at pretest. In fact, none of the motivation variables were related to flexibility at posttest. Second, self-concept of ability was related to computation (.39, $p<.01$), basic concepts (.26, $p<.05$), word problems (.29, $p<.01$), and transfer (.32, $p<.01$). At pretest, self-concept of ability had not been related to basic concepts. Instead, students' perceptions of their ability to do fractions were related to their ability to add, subtract, multiply, and divide fractions, as well as to their knowledge of when to do these things. In contrast, ability perceptions at posttest were also related to their knowledge of both equal parts and the relationship between a fraction and its corresponding unit. Given the goals of the course, this change may not be surprising.

Pretest to Posttest Comparisons

Knowledge

Given that many of the pretest and posttest knowledge scores were significantly and moderately correlated, a repeated measures MANOVA was run to determine if student differences existed across instructors for the five knowledge variables. In other words, did students' knowledge change differently depending on who taught their class? To answer this question, time was used as a within-subjects factor and instructor was used as a between-subjects factor. The number of times a student was absent was unrelated to any of the knowledge variables at pretest or posttest and was therefore not used as a covariate in the analysis. The assumption of equality of covariance was met. Because transfer was only assessed at the end of the semester, it was not included in analyses for this section.

Results from the multivariate test showed a main effect for time, $\Lambda = .55$, $F(4, 79) = 16.13$, $p < .001$. As can be seen from the totals in Table 7, computation, basic concepts, and word problems tended to increase from pretest to posttest, whereas flexibility decreased. No main effect was found for instructor, $F(8, 158) = .882$, $p > .05$. There was also no instructor by time interaction, $F(8, 158) = .781$, $p > .05$. Although the slight skewness of computation and basic concepts at posttest may have concealed student differences across instructors, it seems more likely that there were no meaningful differences, given that the means improved greatly for all students. In sum, students gained significantly in knowledge from pretest to posttest, and these gains were similar across instructors.

Univariate tests revealed that the increases were significant for computation [$F(1, 82) = 34.80$, $p<.01$, $MSe = 21.46$], basic concepts [$F(1, 82) = 35.55$, $p<.01$, $MSe = .80$], and word problems [$F(1, 82) = 17.89$, $p<.01$, $MSe = 2.67$], but the decrease in flexibility was not significant, $F(1, 82) = .896$, $p>.05$. None of the five knowledge variables differed across instructors, *F*'s< 1.16, $p>.05$, and there were no time by instructor interactions for any of the five variables, *F*'s< 2.19, $p>.05$. In other words, the content of the course, rather than the delivery style of the instructor, seemed to contribute to knowledge gains.

Table 7

Student Fraction Knowledge across Instructors

		Computation		Basic Concepts		Word Problems		Flexibility	
Instructor		Pre	Post	Pre	Post	Pre	Post	Pre	Post
A	*M*	21.00	27.54	2.85	3.77	6.31	7.50	2.65	2.19
(n=26)	*SD*	7.59	4.57	1.16	.65	2.84	2.25	2.00	2.30
B	*M*	25.60	28.45	3.10	3.90	6.80	7.60	2.55	2.15
(n=20)	*SD*	6.20	1.91	1.02	.45	2.38	1.85	1.70	2.50
C	*M*	23.54	27.21	2.77	3.59	6.03	7.33	1.87	2.08
(n=39)	*SD*	7.28	4.23	1.27	.94	6.29	2.44	1.92	2.13
Totals	*M*	23.54	27.60	2.87	3.72	6.29	7.45	2.27	2.13
N=85	*SD*	7.26	3.92	1.17	.77	2.76	2.23	1.91	2.19

Motivation

Given that all pretest and posttest motivation variables were significantly correlated (moderate to strong), a repeated measures MANOVA was run to determine if student differences existed for motivation across the three instructors. In other words, did students' motivations change differently depending on who taught their class? To answer this question, time was used as a within-subjects factor and instructor was used as a between-subjects factor. The number of times a student was absent was unrelated to any of the motivation variables at pretest or posttest and was therefore not used as a covariate in the analysis. The assumption of equality of covariance was met.

Results from the multivariate test showed a main effect for time, Λ=.76, $F(3, 80) = 8.55$, $p<.001$. As seen by the totals in Table 8, value and self-concept of ability tended to increase from pretest to posttest, while anxiety tended to decrease. A main effect was also found for instructor, Λ=.84, $F(6, 160) = 2.41$, $p<.05$. Table 8 reveals that students in Instructor C's class changed the least during the semester. In fact, Instructor C's students showed an increase in anxiety rather than a decrease. Nonetheless, there was no overall interaction between instructor and time, $F(6, 160) = 1.47$, $p>.05$. In other words, the course content seemed to have a more powerful effect on student motivation than did the delivery style.

Table 8

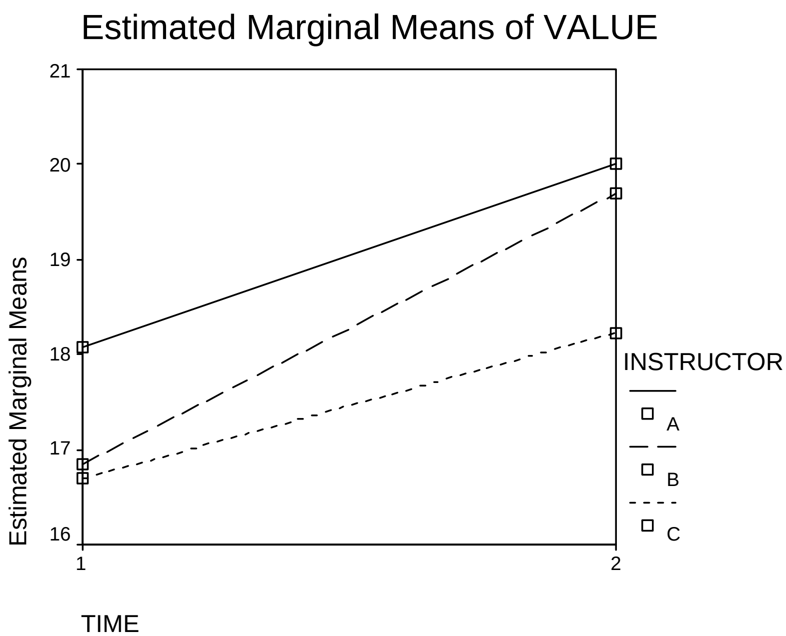
Student Fraction Motivation across Instructors

		Value		Anxiety		Self-Concept of Ability	
Instructor		Pre	Post	Pre	Post	Pre	Post
A	*M*	18.08	20.00	17.12	15.23	21.92	25.20
(n=26)	*SD*	3.49	3.57	5.07	5.39	6.01	5.11
B	*M*	16.85	19.70	14.95	13.10	24.50	27.95
(n=20)	*SD*	5.04	3.94	4.52	5.26	6.39	4.97
C	*M*	16.69	18.23	16.10	16.54	20.82	22.56
(n=39)	*SD*	3.83	4.09	4.95	5.09	6.21	5.36
Totals	*M*	17.15	19.12	16.14	15.33	22.02	24.73
N=85	*SD*	4.05	3.94	4.90	5.34	6.29	5.59

The univariate tests revealed that the increase for value over time was significant [$F(1, 82) = 15.85$, $p<.01$, $MSe = 11.01$]. There was no significant difference across instructors, $F(2, 82) = 1.91$, $p>.05$. There was no interaction between time and instructor, $F(2,82) = .52$, $p>.05$. As can be seen in Figure 2, all students' value tended to increase over time, and the increase was similar across instructors.

Figure 2

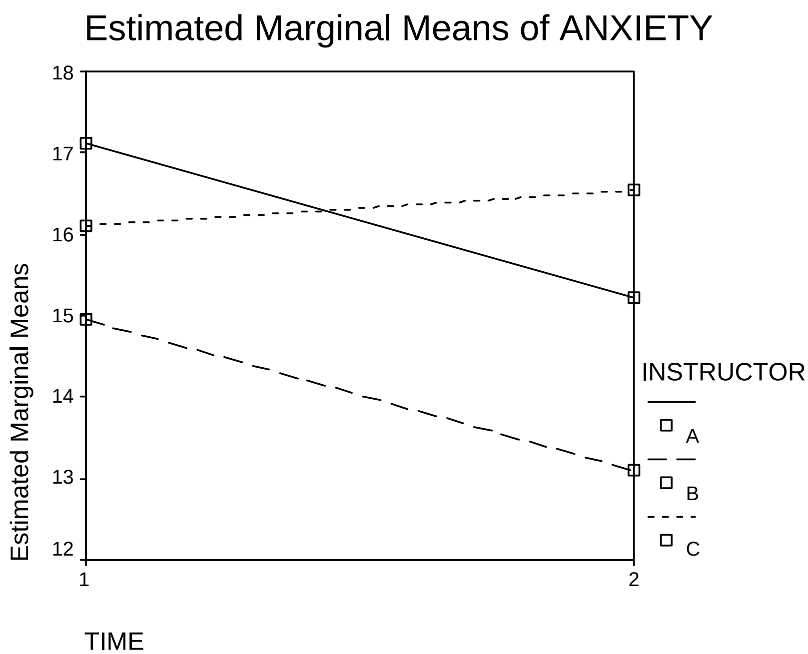
Students' Valuing of Fractions across Time



For anxiety, a significant interaction was found, [$F(2,82) = 4.23$, $p<.05$, $MSe = 6.64$]. There was no main effect for instructor, $F(2, 82) = 1.73$, $p>.05$, but there was a main effect for time, [$F(1,82) = 7.18$, $p<.01$, $MSe = 6.64$]. The main effect results for anxiety must be considered in light of the significant interaction. While student scores for anxiety were similar across instructors (i.e., no main effect for instructor), the slopes from pretest to posttest were different. Although anxiety decreased in general (i.e., main effect for time), this decrease was not apparent for Instructor C's students. These results are shown in Figure 3, and they will be further discussed in the last section of this chapter.

Figure 3

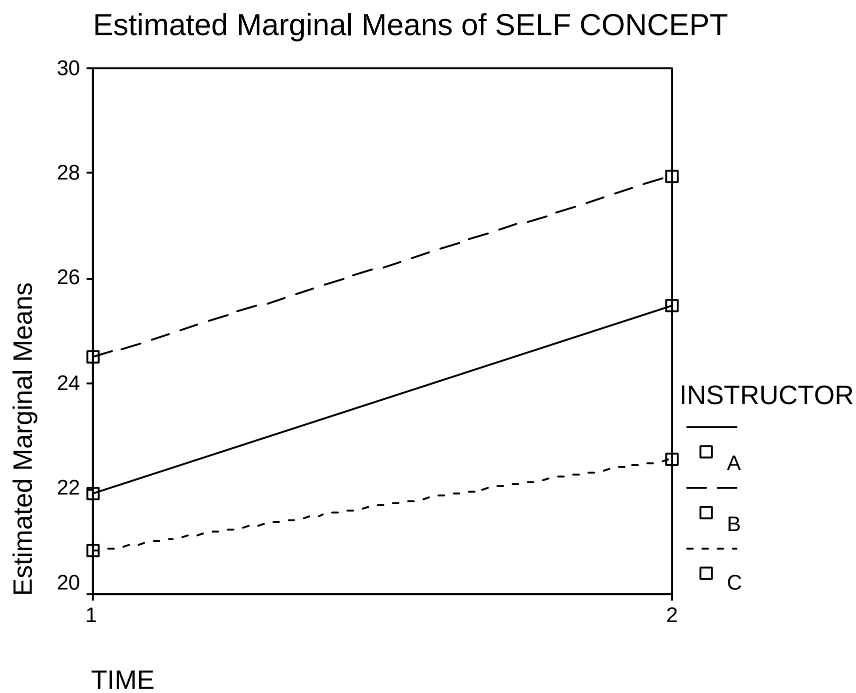
Students' Anxiety toward Fractions across Time



The increase from pretest to posttest for self-concept of ability was significant, [$F(1, 82) = 23.14$, $p<.01$, $MSe = 14.57$]. There was also a main effect for instructor, [$F(2, 82) = 5.43$, $p<.01$, $MSe = 50.77$]. Multiple comparisons using Bonferroni revealed that students in Instructor B's class had significantly higher self-concepts of ability than the students in Instructor C's class, $p<.05$. There was no time by instructor interaction, $F(2, 82) = 1.15$, $p>.05$. As seen in Figure 4, students' self-concepts of ability increased across instructors, but Instructor B's students began and ended with higher self-concepts of ability than did Instructor C's students.

Figure 4

Students' Self-Concept of Ability across Time



Errors and Flexibility

The third research question addressed in this study asked: What do preservice teachers' error patterns and ways of solving fraction problems reveal about their knowledge of fractions at the beginning and end of a course designed to promote relational understanding in mathematics? This question was answered primarily by examining errors on routine computation problems (i.e., errors patterns) and by examining problems that provide opportunity to show flexibility (i.e., ways of solving)

Error Patterns

Addition

Research has consistently suggested that young students make fraction mistakes based on their knowledge of whole numbers (Byrnes & Wasik, 1991; Mack, 1995), and

Silver (1986) has suggested similar patterns for preservice teachers. One common example is adding numerators and denominators (e.g., 2/3 + 5/6 = 7/9). In the present study, eight students added across numerators and denominators at pretest, while two made this mistake at posttest. Two things are noteworthy in this case. First, the error was not prevalent for these college students, as it tends to be with younger students. Less than ten percent of the students made the error at the beginning of the semester. Second, the error was likely related to prior fraction knowledge rather than knowledge of whole numbers. One reason for this claim is that none of the students added across for all addition problems. Many of the students only made the "adding across" mistake when the denominators were different. When denominators were alike, they tended to keep the denominator and only add the numerator.

Moreover, these students tended to follow this pattern with other operations, even when it was inappropriate (e.g., 2/15 × 7/15 = 14/15). In fact, 6 of the 8 students tended to keep like denominators the same no matter what operation was involved, just as they tended to operate on different denominators no matter what operation was involved. Rather than understanding that addition requires a common denominator while multiplication does not, these students seemed to believe that having like denominators requires keeping the denominator the same, while having different denominators requires using the given operation on the denominators.

Seven students made errors with whole numbers (e.g., mistakes with basic facts) at pretest, while three made these mistakes at posttest. Six students made errors changing the form of the fraction at pretest (e.g., changing a mixed number to an improper fraction), while three people made this mistake at posttest. For example, one student changed 3 1/5 to 7/5 instead of 16/5. Two students left at least one addition problem blank on the pretest, while no one did so on the posttest. Five students made miscellaneous errors (i.e., mistakes exclusive to one person) at pretest, while four students made such errors at posttest. These results can be seen in Table 9.

Table 9

Number of Students Making Addition Errors

	Pretest	Posttest
Added Across	8	2
Whole Number Errors	7	3
Changing Forms	4	3
Left Blank	2	0
Miscellaneous	5	4

Subtraction

For subtraction, nine students left at least one problem blank on the pretest, while only two left blanks on the posttest. For the problem 6 2/5 − 2 4/5, four students subtracted the first numerator from the second rather than regrouping on the pretest, giving 6 2/5 as their answer. One student made this mistake on the posttest. Students could make this error if they believed subtraction was commutative. They could also make the error if they believed that the smaller number is always subtracted from the larger number.

Other subtraction mistakes were similar to the ones made for addition. This finding is not surprising, considering that the two operations are closely related. Moreover, the procedures for addition and subtraction of fractions are quite similar.

Three students subtracted across numerators and denominators on the pretest, while only one student made this mistake on the posttest. Although it is less prevalent for subtraction, the reasons for this error are likely the same as the ones for addition. Four students made errors changing the form of the fraction at pretest, and six students made this mistake on the posttest. Two students made errors with whole numbers at pretest, and one student did so at posttest. There were nine miscellaneous errors at pretest, and there were only two such errors at posttest. See Table 10 for a summary of these errors.

Table 10

Number of Students Making Subtraction Errors

	Pretest	Posttest
Subtracted Across	3	1
Whole Number Errors	2	1
Changing Forms	4	6
Left Blank	9	2
Small from Large	4	1
Miscellaneous	9	2

Multiplication

Several of the errors for multiplication seemed related to prior fraction knowledge. For example, cross-multiplying is a technique that can be used to compare fractions. There were 14 students who mistakenly used the technique at pretest to multiply fractions (e.g., $2/3 \times 1/5 = 3/10$), whereas one student made this mistake at posttest. The improvement is not surprising, given that the reason why cross multiplication works when comparing fractions was explicitly discussed during the semester.

Also during the semester, students discussed why a common denominator is used to add or subtract fractions, as well as why the denominator remains in the answer. At pretest, 19 students mistakenly used this approach to multiply fractions. For example, several students solved 2/3 × 1/5 by finding a common denominator and then keeping that denominator in the product. Namely, they wrote 10/15 × 3/15 = 30/15. Most of them simplified this answer to "2", apparently not noticing that "2" was an unreasonable answer to the original problem.

At posttest, 11 students made the "common denominator" error. Although there was some improvement with regard to this error, it was only moderate. One reason the problem was persistent could be that one of the multiplication problems had a common denominator (i.e., 2/15 × 7/15), whereas the multiplication problems in class had different denominators. Out of the 11 people who kept the denominator at posttest, four of them did so only when the original fractions had common denominators. The students could have thought that a different algorithm was necessary when the denominators were the same, especially since they were shown such a method for dividing fractions.

When adding mixed numbers, it is valid to add the whole numbers separately from the fractional parts. However, eight students mistakenly multiplied "like parts" for problems involving mixed numbers on the pretest (e.g., 1 4/5 × 2 1/3 = 2 4/15). No students made this mistake on the posttest. During the semester, students were shown why changing the mixed numbers to improper fractions gave a correct solution, but they were also shown how the Distributive Property could be used to multiply mixed numbers.

This alternative approach may have helped students understand that multiplying mixed numbers without using improper fractions was more complicated than simply multiplying "like parts."

Five students left at least one multiplication problem blank on the pretest, while three left blanks on the posttest. There were 18 students who made mistakes with whole numbers on the pretest. Of these, 10 incorrectly multiplied 15 × 15. On the posttest, 16 students made errors with whole numbers. Of these, 12 multiplied 15 × 15 incorrectly. Oddly, multiplication with whole numbers did not seem to improve during the semester, despite the fact that it was included as part of the course. This problem had also been apparent during the pilot test, after which all 2-digit multiplication problems except 15 × 15 had been removed in order to reduce the number of errors not related to fraction knowledge.

Seven students either added numerators or denominators on the pretest (e.g., 2/3 × 1/5 = 3/15), while 14 students made such mistakes on the posttest. Six students made errors changing the form of the fraction, while 14 students made such mistakes on the posttest. There were five students who made miscellaneous mistakes on the pretest, while two made miscellaneous mistakes on the posttest. See Table 11 for a summary of the results.

Table 11

Number of Students Making Multiplication Errors

	Pretest	Posttest
Cross-Multiplied	14	1
Whole Number Errors	18	16
Changing Forms	6	14
Left Blank	5	3
Common Denominator	19	11
Added Numerators or Denominators	7	14
Like Parts Error	8	0
Miscellaneous	5	2

As stated earlier, more people made mistakes multiplying at posttest than dividing. One explanation for this difference could be that there were more opportunities for errors with whole numbers on the multiplication problems. For example, students had to multiply 15 × 15 as part of a multiplication problem, but they did not have to divide a 2-digit number by a 2-digit number. Out of 44 students who made errors on the multiplication problems, this 2-digit by 2-digit error was the only multiplication mistake for nine of them. Although limitations with whole number multiplication do limit fraction multiplication, they are not directly related to the fraction algorithm.

Another explanation for more students making errors with multiplication is related to the fraction algorithm. It could be that some students do not necessarily see the division algorithm as *containing* the multiplication algorithm. Instead, they could view the division algorithm as unconnected to the multiplication algorithm. For example, a student may know that one algorithm for dividing two fractions is to flip the divisor and then multiply numerators and denominators. On the other hand, that same student could believe the algorithm for multiplying two fractions is to find a common denominator and then multiply only the numerators, keeping the denominator the same. It is possible that the two processes are unrelated in some students' minds. There were five students who used common denominators at least once while multiplying, yet made no errors on the division problems.

Division

For division, 14 students left at least one problem blank on the pretest, and three students left blanks on the posttest. There were 15 students who made the "common denominator" mistake on the pretest, but about half of these students only did so when the denominators were already the same (i.e., for 9/10 ÷ 3/10). Two students kept the denominator the same for this problem on the posttest, but no one did so for problems with different denominators. As with addition and multiplication, students seem to believe there are different processes when the denominators are the same as compared to when they are different.

Seven students flipped the dividend instead of the divisor on the pretest (e.g., 2/9 ÷ 3/8 = 9/2 × 3/8). However, five of them only did so when the dividend was a whole number (i.e., for 4 ÷ 1/4). Only three students flipped the dividend on the posttest. On the pretest, 12 students wrote 4 ÷ 1/4 = 1, but it was not always clear whether they simply did not flip any number, or whether they had some other method for obtaining the "1." On the posttest, four students wrote that 4 ÷ 1/4 = 1.

The frequency of mistakes with 4 ÷ 1/4 is interesting. It could be that some students simply think of “1” whenever they see reciprocals. Or perhaps students are unsure what to do when whole numbers and fractions are being divided. Many of the students who made mistakes when the division problem involved a whole number had no problem dividing a fraction by a fraction. It could be that some students have a poor understanding of reciprocals. Rather than understanding two numbers are reciprocals when their product is 1, some students may just think the denominator and numerator simply switch places, leaving them uncertain of what to do when there is no denominator. Most of the students rewrote 4 as 4/1, but they may have felt that, in doing so, they were ready to multiply to the fractions.

There were 9 students who made whole number errors on the pretest, but no one made such mistakes on the posttest. Three students flipped correctly but then cross-multiplied on the pretest, but no one made this mistake on the posttest. There were seven students who either cancelled or cross-divided on the pretest (e.g., 2/9 ÷ 3/8 = 3/4) while only one student did so on the posttest. Three students either added or subtracted numerators or denominators on the pretest, and four people made such mistakes on the posttest. Three students made errors changing fraction forms on the pretest, while no one made this mistake on the posttest. There were thirteen mistakes categorized as miscellaneous on the pretest, but only five labeled this way on the posttest. A summary of these results is found in Table 12.

Table 12

Number of Students Making Division Errors

	Pretest	Posttest
Cross-Divided or Cancelled	7	1
Whole Number Errors	9	0
Changing Forms	3	0
Left Blank	14	3
Kept Denominator	15	2
Add/Sub. Numerators or Denominators	3	4
Flip Dividend	7	3
Cross-Multiplied	3	0
Reciprocals	12	4
Miscellaneous	13	5

Transfer

Yoshida and Sawano (2002) argued that the importance of equal parts and equals wholes needs to be made explicit in fraction instruction. They found that students who received such explicit instruction performed better on conceptual items and transfer items than did students where these ideas were only implicit. The transfer item for this study was created to assess students' knowledge of equal parts and equal wholes at the end of a course where these ideas were made explicit. To solve the problem, students had to understand that fractions cannot be combined (i.e., added together) unless they represented parts of a same-sized unit, and they had to know how to create same units.

Specifically, the problem asked what fractional part of a chocolate milk mixture would be syrup if a glass that was 1/3 syrup was combined with a glass that was 1/4 syrup, given that the second glass was twice as big.

There were 13 students who either left the problem blank or quit after a brief attempt. There were 40 students who seemed to understand that the fractions could not simply be added, but only seven who correctly solved the problem. There were 32 students who did not seem to understand this idea. Of these, 23 gave 7/12 as the solution, indicating they added 1/3 and 1/4. It could be argued that the students did not notice that the second glass was twice as big, but many of the students who gave 7/12 also drew two glasses, with one glass drawn twice as big as the other one. In other words, their drawings seem to indicate that they knew one glass was larger, but they did not know how to incorporate that information.

Of the 33 students who understood the fractions could not be added but did *not* answer the problem correctly, 20 gave either 10/12 or 5/6 as a solution. This answer indicated that the students "poured" all the syrup into the smallest glass, but they did not take the milk into account. Another seven students gave 5/12 as a solution, indicating they "poured" the syrup into the larger glass, but still did not account for the milk. In both cases, the students seemed to know that the fractions had to be part of the same whole, but they were uncertain of what whole was appropriate.

Conclusions about Error Patterns

The error analysis suggests several conclusions about preservice teachers' knowledge of fractions. It seems that the following statements can be made about preservice teachers' knowledge of fraction algorithms at the beginning of the semester:

- Students were most uncertain about dividing fractions, followed by subtracting, multiplying, and adding fractions. This claim is based on the number of people who did not attempt these problems.
- Students mostly made errors related to prior knowledge of fractions. They commonly misapplied algorithms. For example, they cross-multiplied when they should have multiplied across, added across when they should have found a common denominator, or kept the denominator the same when they should have multiplied by the reciprocal.
- Some students attended to superficial conditions, such as whether or not two fractions have the same denominator, to decide what algorithm was appropriate. Keeping the denominator the same when it was not appropriate was the most common error. Nearly 1/5 of the students made this error when dividing, and nearly 1/4 made the error when multiplying. For addition, students who added denominators generally did so only when the denominators were different.

At the end of the semester, mistakes that persisted were not directly related to the algorithms for adding, subtracting, multiplying, and dividing fractions. Many of these errors involved operations with whole numbers. The most persistent of these was the 2-digit by 2-digit multiplication problem. Another persistent error related to whole numbers was adding numerators or denominators that were the same, even though the problem involved multiplication. These students were likely doubling rather than squaring the number. More than half of the students who added numerators or denominators only did so when they were the same.

The last persistent error involved renaming fractions. Although algorithms for finding equivalent fractions and renaming mixed numbers as improper fractions were considered with some depth during the semester, these errors generally did not improve. It is not clear why the problem persisted.

Flexibility

Recall that procedural flexibility refers to the ability to notice and choose procedures that are elegant and efficient (Star, 2005). These procedures are often departures from general algorithms that tend to work in all cases. Generally speaking, flexibility was low and did not improve from pretest to posttest. For example, 70 students of the student who correctly solved 2/4 – 3/6 on the pretest used a general, but not necessarily efficient, approach. Namely, they found a common denominator and then subtracted the numerators. On the other hand, seven students solved the problem by noticing that both fractions were equivalent to 1/2. On the posttest, results were similar. There were 72 students who solved the problem using a common denominator, whereas 11 used the 1/2 – 1/2 method. All nine experts had used the 1/2 – 1/2 method.

On the pretest, 53 students solved 2 3/5 + 3 1/5 by changing the mixed numbers to improper fractions, while 28 added "like parts." Since the denominators were the same and the sum of the fractions was less than one, then adding the whole numbers and adding the numerators was an efficient alternative to the generalized approach. The posttest results were virtually the same. Whereas 51 used improper fractions, 30 used like parts. All nine experts had used the "like parts" method.

That there was little change in flexibility from pretest to posttest is likely a reflection of the course goals. The primary purpose of the course was to help students understand why the general algorithms worked, not how to find ways of solving that are more efficient that the general algorithms. A notable exception to the lack of improvement in flexibility involved the problem 9/10 ÷ 3/10. During the semester, students were explicitly taught the common denominator method of dividing fractions, in which a common denominator is found and the numerators are then divided. Using this method, the given problem can be solved using 9 ÷ 3 = 3. Eight students used this approach at posttest to solve the given problem, whereas one student used the approach at pretest. Because these fractions already had a common denominator, the common denominator method was certainly more efficient to use than the traditional algorithm. That eight students used this method is somewhat surprising, given that only one of the experts used it. It seems that for students to use their knowledge flexibly, it might be necessary to give explicit attention to flexibility. In other words, flexibility might improve if students are taught multiple ways to solve problems, as well as how to recognize what circumstances are appropriate for the various methods.

Differences between Instructors

The fourth research question addressed in this study was: Are there discernible profiles for instructors teaching the different sections of a course designed to promote relational understanding in mathematics, and how do these profiles contribute to preservice teachers' fraction knowledge and motivation at the conclusion of the course? This question was answered using observation data, attribution data, and teacher interviews, and by examining student outcomes grouped by instructor.

Time-Sampled Data

A one-way ANOVA was run to determine differences between instructors with regard to understanding, discourse, and teacher direction. The results showed no differences on any of the variables related to understanding. In other words, they spent similar proportions of time on activities such as linking concepts to procedures, using context, and having students work on high-level tasks. These results are not surprising given the goals of the course as outlined in the syllabus and the fact that all instructors followed the same curriculum. In fact, the primary goal of the course was to link concepts and procedures, and the proportion of time instructors spent doing so was quite high, ranging from .55 to .61. However, significant differences were found between instructors with regard to discourse and teacher direction. Details are reported separately for each.

For discourse, significant differences existed between the instructors with regard to student activity, [$F(2, 645) = 8.09$, $p<.001$, $\eta^2=.02$]. These activities included making conjectures, justifying answers, or suggesting alternate methods of solving problems. Post hoc tests using Bonferroni revealed that Instructor B encouraged such discourse more than both of the other instructors ($p<.01$). Specifically, the proportions in Table 5 show that Instructor B encouraged high-level discourse 23% of the time, or more than twice as often as each of the other instructors.

For teacher direction, significant differences were found for teacher activity, [$F(2, 645) = 3.90$, $p<.05$, $\eta^2=.01$], student activity, [$F(2, 645) = 5.01$, $p<.01$, $\eta^2=.02$], and class organization, [$F(2, 645) = 5.26$, $p<.01$, $\eta^2=.02$]. Post hoc tests using Bonferroni revealed that Instructor C was more teacher-directed than Instructor B with regard to both teacher activity ($p<.05$) and student activity ($p<.01$). In other words, he was more likely to model or lecture while the students passively watched or listened. Specifically, Table 5 showed that Instructor C performed these practices about twice as often as Instructor B. His class organization was also more teacher-directed than Instructor A ($p<.01$). Specifically, he organized the class in a whole group about 3/4 of the time, whereas Instructor A did so less than 2/3 of the time.

These results suggest the most apparent differences were between Instructor B and C. Although there were no differences between these instructors with regard to the amount of time they organized the class as a whole group, they did seem to use this time differently. Instructor C tended to lecture or model while the students passively listened, while Instructor B tended to use this time for students to discuss their thinking or discussing alternative methods of solving problems. Both used their respective instructional approaches to emphasize why procedures worked.

Attribution Data

A one-way ANOVA was run to determine differences between instructors with regard to the attribution data. Specifically, the teachers were tested for differences on each of the five learning dimensions as well as on their overall quality rating. The learning dimensions included knowledge, strategic processing, development and individual differences, context and situation, and motivation.

Based on the course goals, it was not surprising to find no significant differences on the knowledge principle. In other words, these teachers were similar in the degree to which their lessons promoted principled understanding, built on prior knowledge, were deep and accurate, and illustrated utility. In fact, Table 6 shows that all three teachers scored highest on the knowledge dimension. The table also shows that all three instructors scored lowest on the development and individual differences dimension and the motivation dimension. There were no significant differences between the instructors on these two dimensions. There were also no differences on the overall quality rating.

There were significant differences for strategic processing [$F(2, 38) = 4.51$, $p<.05$, $\eta^2=.19$], and context and situation [$F(2, 38) = 8.37$, $p<.01$, $\eta^2=.31$]. For Instructor A, the strategic processing dimension was the second highest score. Included in this dimension are: teaching or modeling domain-specific strategies, encouraging students to be reflective, and providing opportunities to engage in reasoning. Multiple comparisons using Bonferroni revealed that Instructor A scored significantly higher than Instructor C on this dimension, $p<.05$.

For Instructor B, the context and situation dimension was the second highest score. In other words, she tended to be caring and affirming, to welcome reactions, to use a variety of social interactions, and to effectively use (human or other) resources. Post hoc tests showed that Instructor B scored significantly than Instructor C on this dimension, $p<.01$. That Instructor B was welcoming and affirming of students' ideas is consistent with the finding that she encouraged discourse in her classroom.

Instructor Interviews

The interviews were examined in light of the time-sampling and attribution data because the primary purpose of the interviews was to further validate the results of the classroom observations. As such, many consistencies were found. For example, the finding that the instructors did not differ with regard to emphasizing why procedures work was consistent with the interviews. At the beginning of the semester, Instructor A stated that the purpose of the course was "to train our teachers so that they're not just passing on memorized information but really getting to the heart of why that we do the things we do in mathematics." At the end of the semester, he said his goals had been "to make sure that the people who are going to be teachers really understand the simple and basic things they are going to be teaching – that they really actually understand them and that they are not just following a cookbook or following the rules or whatever."

Similarly, Instructor B stated at the beginning of the semester that the goal of the course was for "the students to revisit the topics that they know how to do…but look at why we do the steps we do and hopefully, understand the topics at a much deeper level than they did when they first came into the class." At the end of the semester, she stated that her goal had been "to re-examine topics from arithmetic and number theory and think more deeply about why the procedures are what they are."

Finally, Instructor C responded to the same question at the beginning of the semester by saying, "the main purpose is to try to get them to think about the underlying 'why.'" At the end of the semester, he stated his particular goals for the course had been "to make sure they all learned something and that they had a better understanding of why this math works and how to explain it."

Differences in teacher practices were also supported by the interviews. For example, when asked at the beginning of the semester what instructional approaches he would use to meet his goals, Instructor C replied, "At this time, I honestly don't know." He added, "I am going to try and follow the teaching notes, work through the activities that are assigned and then from there, add in whatever extra flavor I can from my own experience." At the end of the semester, he claimed that used "lecture" and only "some group work." The group work was certainly part of the teaching notes, and given the common practice of lecture, it was likely a reflection of his experience. These responses are consistent with the findings that Instructor C lectured and modeled more that Instructor B, and he organized the class as a whole group more often than Instructor A.

Instructor B was no different than Instructor C with regard to whole class organization. This result was consistent with her post-interview, in which she referred to small group work by stating, “I probably did less of that this semester than I have in previous semesters. I was just presenting them with problems to reflect on and then having the whole class respond and summarize observations.”

On the other hand, Instructor B incorporated more high-level discourse than both of the other instructors. At the beginning of the semester, she said she often liked to pose a problem in which there were “a variety of ways to think about it, so that we can have a discussion about different ways people think and practice explaining our thinking to each other.” At the end of the semester, she stated that one course goals had been for the students “practice explaining.” This sort of welcoming of all student thoughts and reactions is also consistent with the context and situation principle, on which Instructor B scored significantly higher than Instructor C.

Connections to Student Differences

Although the limited number of teachers in this study makes the fourth question an exploratory one, the separate data streams seem to consistently suggest a relation between certain teaching profiles and student motivation. Instructor C had a more traditional style of instruction, which was more teacher-directed than the other two instructors. Specifically, he tended to model or lecture to the whole group while his students passively listened. On the other hand, Instructor A and Instructor B tended to have more reform-oriented practices. Specifically, Instructor A encouraged students to work together in small groups, and he encouraged the use of strategies, reasoning, and self-reflection. Instructor B tended to promote high-level discourse and was welcoming of students' thoughts and reactions.

Unlike the students in the other classes, the anxiety levels in Instructor C's class did not decrease over the course of the semester. This finding was true despite the fact that his traditional practices were to encourage relational understanding. Although no causal statements can be made with these data, it seems that reform-oriented practices may be related to lower levels of anxiety in students. Implications of this finding will be discussed in Chapter V.

CHAPTER V

SUMMARY, IMPLICATIONS, AND SUGGESTIONS FOR FUTURE RESEARCH

This study was undertaken to better understand preservice teachers' knowledge of and motivations toward fractions before and after taking a course to promote relational understanding in mathematics. I also attempted to find patterns in these students' errors and ways of solving fractions, and I examined what those patterns might reveal about their knowledge. Finally, I wanted to know how the delivery of instruction might be related to knowledge and motivation at the end of the semester. I begin this chapter by summarizing the findings of the study. Then, I discuss the educational implications of these findings. Finally, suggestions for future studies are made.

Summary of Findings

Four research questions were involved in this study. In this section, each of these questions will be answered and discussed. Specifically, I wanted to know:

1. What is the level of elementary preservice teachers' knowledge of and motivations toward fractions at the beginning of a course designed to promote relational understanding in mathematics?
2. How will elementary preservice teachers' knowledge of and motivations toward fractions differ as a result of participating in a course designed to promote relational understanding in mathematics?
3. What do preservice teachers' error patterns and ways of solving fraction problems reveal about their procedural and conceptual knowledge of fractions at the beginning and end of a course designed to promote relational understanding in mathematics?

4. Are there discernible profiles for instructors teaching different sections of a course designed to promote relational understanding in mathematics, and how do these profiles contribute to preservice teachers' fraction knowledge and motivation at the conclusion of the course?

Pretest Levels of Fraction Knowledge and Motivation

At the beginning of the semester, it was expected that elementary preservice teachers would have knowledge levels comparable to middle school students (Silver, 1986; Tirosh, 2000). For pretest computation, basic concepts, and word problems, the average performance rates ranged from 70% (word problems) to 78% (computation). The fact that students performed worse on word problems than on computation was not surprising, given that younger students produce similar results (Asku, 1997). Also as predicted, computation performance was not the same for all operations. The average performance on the four operations ranged from 64% for division to 92% for addition.

Compared to secondary preservice teachers, elementary preservice teachers tend to have lower confidence about and higher anxiety toward mathematics in general (Ball, 1990a). As such, similar results were expected for fractions in particular. At pretest, the preservice teachers scored near the center of the scale for value, self-concept of ability, and anxiety. These results are consistent with the trajectory suggested by prior studies of elementary, middle, and high school students, which have demonstrated motivational declines across these years of schooling (Jacobs et al., 2002). However, they are inconsistent with the findings of Marsh (1989), who suggested an increase in mathematics self-concept during the college years. It seems that those who want to teach elementary school have lower levels of motivation than the typical undergraduate student.

Posttest Levels of Fraction Knowledge and Motivation

As predicted, student knowledge at the end of the semester was higher than it had been at the beginning. In particular, the average success rate was 92% for computation, 93% for basic concepts, and 83% for word problems. Given the number of word problems involved in the course, it was expected that the greatest improvement would be with word problems. However, knowledge of basic concepts increased the most. At pretest, student knowledge of basic concepts was much lower than anticipated; hence, there was more room for growth than expected. However, basic concepts were re-visited with each new operation during the semester, making the increase unsurprising.

Although significant changes over time existed for value, anxiety, and self-concept of ability, these factors were only somewhat improved at posttest. The greatest increase was apparent for value of fractions, followed by self-concept of ability. The smallest change was apparent for anxiety. One reason for the smaller overall change in anxiety is that only students of Instructor A and Instructor B had decreased levels of anxiety at posttest.

Error Patterns and Ways of Solving Fractions

Researchers have suggested that preservice teachers make mistakes that are similar to those of younger students (Silver, 1986; Tirosh, 2000). Specifically, prior studies have indicated that they make mistakes such as adding across numerators and denominators (e.g., 2/3 + 5/7 = 7/10) and inverting the dividend rather than the divisor (e.g., 2/3 ÷ 5/7 = 3/2 × 5/7). Although I found some evidence of these errors, they were not prevalent. Some viable explanations follow.

Many researchers have considered the "adding across" error to be related to prior knowledge of whole numbers (e.g., Byrnes & Wasik, 1991), and it is not a surprising error for students who are early in their study of fractions. However, the error was not common for the undergraduate students in this study, even at the beginning of the semester. Less than 10% of the students made this error at pretest. In fact, it was more common for students to correctly add fractions (i.e., use a common denominator) but then to over-generalize the method, incorrectly applying it to multiplying (22%) and dividing fractions (18%). Further, many of the students who "added across" only did so for fractions with different denominators. Hence, it could be that they are misapplying their knowledge of the multiplication algorithm, for which multiplying numerators and denominators is appropriate. Given the number of years students have been exposed to whole numbers by the time they are in college, it makes sense that they are more likely to make mistakes based on their prior fraction knowledge than their prior knowledge of whole numbers.

Even fewer students inverted the dividend rather than the divisor at the beginning of the semester. Again, this mistake is not surprising for students just learning how to divide fractions (i.e., middle school students) because they are taught to "invert and multiply." Inverting the wrong fraction reflects an effort to imitate a procedure just learned but perhaps not fully understood. The participants in this study, however, were more likely to either have completely forgotten what method to use (as indicated by the 16% of students leaving the problem blank) or to have used an entirely inappropriate algorithm. The most common mistake, as with multiplication, was to keep the denominator the same.

It was also common for students to have trouble with division problems involving whole numbers or unit fractions. The fact that students were less likely to make division errors with two non-unit fractions (e.g., 2/9 ÷ 3/8) than with whole numbers or unit fractions suggests a poor understanding of reciprocals. If students are taught to "flip" the numerator and denominator to find a reciprocal, it does not follow that they understand the concept of a reciprocal. Furthermore, students who simply learn to "flip" numerators and denominators may not know what to do when there is no denominator (i.e., when dividing by a whole number). To complicate things further, students who are told to give the whole number a denominator of "1" may do so and think they are finished manipulating the fraction. For example, a few students rewrote 2 1/3 ÷ 9 as 7/3 × 9/1 rather than 7/3 × 1/9. Others seemed to know that the whole number can be given a denominator of "1" before it is inverted, but they seemed to believe this process was valid regardless of whether the whole number was the dividend or the divisor. For example, some students rewrote 4 ÷ 1/4 as 1/4 × 1/4 rather than 4 × 4, despite correctly inverting the divisor when the problem involved only non-unit fractions.

In general, many of the mistakes at pretest seemed to be related to students' prior knowledge of fractions. However, these errors were also the ones that diminished by the end of the semester. The most persistent errors seemed to be those involving whole numbers (e.g., errors with basic facts). Also, renaming fractions seemed to remain a source of many errors at posttest. It seems the errors at pretest revealed several key misconceptions about fractions (e.g., the appropriate procedure is dependent on whether the denominators are the same or different), whereas their errors at posttest were more reflective of inconsistent skill.

This shift in knowledge during the semester can be explained with the model of domain learning suggested by Alexander (1997). This model suggests that people go through three stages when learning a particular domain: acclimation, competence, and proficiency. According to Alexander, “a defining characteristic of acclimation is that this limited knowledge is also fragmented or unprincipled in its organization” (p. 224). Not only were these students limited in their knowledge of fractions at pretest, but they were also making errors that suggested poor understanding of basic fraction principles (e.g., the role of equal parts). I would suggest that while the undergraduates in this study were not mathematically uninformed at the beginning of the semester, they were still in the acclimation stage of learning about fractions. By the end of the semester, their increased knowledge and decreased misconceptions suggested a shift into competence, as signaled by their more principled pattern of responses.

Although fraction performance generally improved from pretest to posttest, the students’ ways of solving problems were procedurally consistent over time. In general, the students did not show evidence of procedural flexibility, as described by Star (2005). Instead, this study suggests that preservice teachers tend to rely on algorithms that work in all cases, using them even when they are inefficient. This finding was true both before and after taking a course designed to promote relational understanding of mathematics. Given these students’ low flexibility, as well as their continued errors and their low ability to transfer concepts to novel situations, I would suggest these students have not reached high levels of competence with fractions, but were, instead, in low to moderate levels of domain competence with regard to fractions.

Differences in Delivery and Relations to Student Outcomes

Three instructors taught the observed course, and each of them followed the same teaching notes and curriculum. However, discernible differences were detected in the delivery of the course content, particularly between Instructor B and Instructor C. Whereas Instructor B tended to use whole group time to encourage student discourse, Instructor C tended to use that time to lecture. Both instructors promoted relational understanding of the material, but Instructor B accomplished it by having her students explain their thinking and suggest alternate ways of solving problems. On the other hand, Instructor C tended to model or lecture to the students while they passively listened. Although he did have students provide solutions to problems, the ratio of student discourse to student listening was 1 to 3. In contrast, the ratio of student discourse to student listening in Instructor B's class was 3 to 2. The general finding that Instructor B was more welcoming and encouraging of student engagement was also supported by both the attribution data and the instructor interviews.

Some differences also existed between Instructor A and Instructor C. In particular, Instructor A tended to use more small groups, in which students worked together to solve problems. He also had a greater tendency to teach or model strategies and encourage reasoning and reflective thinking, as indicated by the higher ratings for strategic processing.

In general, Instructor C seemed to be more traditional in his delivery style while Instructor A and Instructor B seemed to align more, in various ways, with reform practices. Given these differences in teaching practices, it was not surprising to find differences between their students. Specifically, an interaction was found for anxiety. It is apparent in Figure 3 that the slopes for Instructor A and B were similar and negative, indicating a decrease in their students' levels of anxiety. This change was not apparent for Instructor C's students.

These findings must be considered in light of the study's limitations. The primary limitation was the low statistical power for detecting differences among the three teachers. As such, the fourth research question was exploratory. However, the various data streams in the present study suggest that reform practices may serve to lower anxiety for fractions.

Implications for Educational Practice

"It is a mistake to suppose that meaningful arithmetic is something new, something cut out of the whole cloth, as it were, during the past twenty or twenty-five years" (Brownell, 1947, p. 258). Brownell wrote these words nearly 60 years ago. When he spoke of "meaningful arithmetic," he was advocating instruction that made deliberate attempts to use mathematical relationships to help children make sense of mathematics. What was new, he asserted, was that the interest in sense-making had been extended to children beyond the primary grades. In the present study, I was interested in instruction that made deliberate attempts to use mathematical relationships to help preservice teachers make sense of mathematics. Is this progress? Let's hope so.

To situate the implications of this study, it is important to note the years of work that psychologists, mathematicians, and educators have done to bring to the forefront the importance of structures in teaching mathematics. These structures are the underlying principles that create coherence within the subject. "To learn structure, in short, is to learn how things are related" (Bruner, 1960/1977, p. 7). Resnick and Ford (1981) highlighted the contributions that both Bruner and Dienes made to the education community about how to best help students understand these structures, which are "seen as fundamental to meaningful learning" (p.125). Clearly, it has long been considered important for students to understand how concepts are related to each other and to procedures. As such, this study did not attempt to show whether or not a relational understanding was beneficial. Rather, it attempted to go beyond this premise and seek to better understand its benefits for preservice teachers as well as what instructional styles might contribute to those benefits.

Implications of Knowledge Findings

During the last two decades, the literature on fraction knowledge has been overwhelmingly concerned with helping students make sense of fractions. Consequently, there has been growing evidence that learning procedures with understanding contributes to greater knowledge of concepts and a greater ability to transfer those concepts to new situations (Hiebert & Wearne, 1988; Niemi, 1996; Saxe & Gearhart; 1999). On the other hand, there seems to be no difference in computation skill for students who learn procedures with understanding and those who do not (Niemi, 1996; Saxe & Gearhart, 1999). It is important to note, however, that these studies failed to distinguish between quantity and quality of errors. In the present study, both the number and nature of the errors decreased. At the end of the semester, students in this study seemed to have more principled knowledge of fractions. This finding suggests that when mathematics instruction explicitly links fraction concepts and procedures, that instruction results in decreased misconceptions and increased relational understanding.

Misconceptions revealed at the beginning of the semester also have implications for educators. Neither Brownell (1947) nor Bruner (1960/1977) suggested that it was impossible to learn procedures without meaning. Rather, a primary concern was that procedures learned without meaning would not be remembered. The current findings suggest a slightly different argument in the case of fractions: procedures learned without meaning will later be inappropriately applied. At the beginning of the semester, students in this study were remembering a number of procedures, such as cross-multiplying and finding a common denominator, but they were using them in inappropriate ways. This finding supports the notion that procedural knowledge unconnected to meaning deteriorates, but it does not suggest that students simply forget procedures. Rather, they forget the circumstances under which to use them. Wearne and Hiebert (1988) found similar results with middle-grade students' performance with decimals, and they cite others who did as well. They concluded, "Many students lack some essential conceptual knowledge and have memorized procedural rules they apply inappropriately" (p. 223).

This study extends those findings to fractions and preservice teachers. That similar results were found with fractions might have been anticipated, given that fractions and decimals are closely related. However, the finding that undergraduate students are also misapplying rules is less obvious. One might expect that years of using fraction knowledge in other mathematics classes would solidify the procedural knowledge. Instead, it seems that the procedures continue to be misunderstood and misapplied well beyond the time students first learn them. These findings have important implications for teacher educators, since these future teachers will likely continue the cycle of what Brownell (1947) called "meaningless arithmetic" if they do not gain a deeper understanding of the mathematics they will be teaching (Ma, 1999). The Conference Board of Mathematical Sciences (CBMS, 2001) recommended mathematics courses like the one in this study in order to help prospective teachers gain this understanding, and the class seems to be a promising attempt to break the cycle of teaching without understanding.

The fact that flexibility did not change from pretest to posttest also has implications for mathematics educators. NRC (2001) suggested that it is not enough to learn one procedure that works in all cases. “Students also need to be able to apply procedures flexibly” (p.121). The finding that these students were relatively inflexible in their solution methods at the beginning of the semester was evidence of their poor understanding. The fact that they were just as inflexible at the end of the semester suggests that this course was not enough to alter the long-standing patterns of these students. Wearne and Hiebert (1988) also noted that it was difficult to alter the patterns of students who had long used rules they did not understand. The current findings support their observation and extend it to preservice teachers.

One implication of this finding is that instilling flexibility should be an explicit part of mathematics instruction. In the current study, the students spent most of the time trying to understand the traditional algorithms, and to some extent, less traditional algorithms. Little time was spent discussing when the general algorithms were not efficient or necessary. At times, students shared alternate ways of illustrating a solution with pictures, but rarely did they discuss various ways to calculate an answer. Ma (1999) suggested that Chinese teachers valued being able to calculate an answer in multiple ways, and this practice contributes to their deep understanding of arithmetic.

A point could be made that it is not *necessary* to find the most efficient algorithm when solving fraction problems, and I understand this point. However, when students fail to notice situations in which applying basic concepts can make a problem easier, it may be evidence of poor understanding. Smith (1995) showed that students who are competent with fractions only use the most general algorithms when no easier or quicker approach is available. In the current study, experts used to validate the scoring rubric for flexibility often made comments to this effect. For example, in solving 2 1/2 × 4, one expert wrote "With a simple mixed number times a whole number, I tend to multiply the pieces and then add."

This approach can be contrasted with the more general approach of changing the mixed number to an improper fraction and then multiplying the numerators and denominators. One explanation for undergraduate students not following the same pattern of flexibility is that they are not fully competent with fractions. As has been suggested before, being able to correctly compute and answer is not sufficient evidence of competence (Alexander, 1997; Brownell, 1947). If these undergraduates are to some day teach fractions with meaning to elementary school students, it seems they need to be competent in them. Researchers are now finding empirical support for the common sense notion that when teachers posses a deep understanding of mathematics, their students do learn more (Hill et al., 2005).

Implications of Motivation Findings

Given that teaching procedures without meaning has continued to be the norm in this country (Stigler & Hiebert, 1999), it is likely students in the present study began the class without a relational understanding of fractions. The errors these students made at the beginning of the semester support this assumption. Instructor interviews suggested that these students were not only unaware of why procedures worked, but they were also unaware that this information was relevant. At the end of the semester, Instructor A commented that “the number one challenge is trying to get them to understand *why* they need to know why.” This disposition is evidence of a long-standing tradition in this country of viewing mathematics as a fragmented collection of procedures. As such, it was not surprising that self-concept of ability was related to computation skill at pretest but not to knowledge of basic concepts. The fact that self-concept of ability was related to basic concepts the end of the semester suggests that learning fraction procedures with meaning may contribute to a shift in the way these students perceive mathematics. In turn, this shift may positively influence the students they will later be teaching.

Given the current and past findings (e.g., Ball, 1990a), it is important that teacher educators understand and address both cognitive and affective effects of instruction. At the end of a course whose purpose was to encourage deep understanding of elementary school mathematics, students began to exhibit increased confidence as well as competence. Although much research exists on motivation and achievement, much less exists about motivation and instruction. Stipek (2002) suggested that instruction can motivate younger students when it emphasizes big ideas, learning rather than performance, active participation, and authentic and meaningful tasks. This study suggests that the same is true for preservice teachers. This finding is important because improvements in preservice teachers’ attitudes should benefit the attitudes of their students once they are in the classroom.

Results of this study also suggest reform-oriented practices, such as engaging students in discussions, may have added benefits for anxiety. However, these conclusions are tentative, given the small number of teachers involved in the study. If delivery of content does have differentiated effects on student outcomes, then educators designing methods courses should consider delivery as well as the content of their courses. Many researchers have suggested that students explain their thinking (e.g., Kazemi & Stipek, 2001), and the present study suggests the benefits may be more than cognitive.

Suggestions for Future Research

The findings and limitations of this study suggest several avenues for future research. One such avenue is the use of comparison groups. Although there is little reason to believe the observed changes in knowledge and motivation were the result of events outside of the course (given that the changes were aligned with course goals), it could be that other types of course would have similar results. As such, studies should be conducted that compare other types of mathematics courses for education majors to ones like the course in this study. Researchers and educators attempting to understand how to best prepare mathematics teachers will benefit from such comparisons.

More studies in the future should examine the interactions between knowledge for addition, subtraction, multiplication, and division of fractions. Currently, very few studies of fraction knowledge have included more than one operation. Given that several errors were not limited to a particular operation, more studies should include all four operations with fractions. Such error patterns are difficult to detect without comparing solution methods across operations, and it can be misleading to limit studies to a single operation. In the present study, there were some students who seemed to have “mastered” addition of fractions but used the same procedures for multiplication. Rather than understanding addition of fractions, it could be that these students have a misconception about the use of common denominators.

More studies are needed to understand the motivations of preservice teachers. In the present study, interest seemed to be more strongly related to self-concept of ability than to importance and usefulness of fractions. In the future, confirmatory factor analysis should be conducted with the questionnaire to determine if this finding holds. Also, prior studies have investigated how motivation changes over time, but more needs to be known about how motivation is related to changes in knowledge. At posttest, the motivation items loaded on two factors rather than three. One factor included low anxiety, high interest, and high self-concept of ability, perhaps reflecting a confidence that may come with increased knowledge. Alexander (1997) suggested that as people move into competence and toward proficiency in a domain, their interest shifts from being strongly situational or contextually-bound to increasingly more reliant on personal or deep-seated interest. Further, that personal interest becomes a driving force for the continued pursuit of expertise in a domain. Future studies should investigate this claim for the domain of fractions.

Future studies should explore ways to improve both flexibility with fractions and the ability to transfer to novel situations. Despite significant gains in conceptual knowledge, the preservice teachers in this study had low levels of flexibility and transfer. Alexander's (1997) model of domain learning would suggest that these preservice teachers lack proficiency with fractions, yet preservice teachers represent a critical case for the improvement of teacher knowledge of mathematics (CBMS, 2001; Ma, 1999).

More longitudinal studies are needed to understand what works with preservice teachers. The studies should include ongoing, extensive efforts to alter the current status of mathematics education. Following preservice teachers through content courses, methods courses, student teaching, and into their first year of teaching will help researchers better understand how their efforts are impacting what goes on in the classroom.

Research on relational understanding should certainly include topics other than fractions. Fractions were chosen for this study because they represent a critical turning point for many students (Case, 1988). However, the need for instruction that promotes relational understanding extends far beyond arithmetic. Algebra students are undoubtedly plagued with errors resulting from procedures learned without meaning and should therefore be investigated in a similar way. We also need to understand the cumulative effects of teaching with meaning. How would algebra students with a relational understanding of fractions compare to those without? Would these groups equally benefit from algebra instruction that emphasized meaning?

Finally, we need to continue to find ways to research instruction that attempts to link concepts and procedures. One reason this kind of research is difficult is that computation skill can be acquired without such links (Niemi, 1996; Saxe & Gearhart; 1999), making it difficult to detect any benefits of the instruction. Another reason is that a proposed benefit of this instruction is that procedures learned with meaning are better retained, yet it is difficult to follow students over time. Yet another reason is that teaching is complex. In order to control for the effects of other teacher practices and characteristics, it is helpful to have the same teachers teach both the “meaningful” and the “meaningless” instruction. Yet, this approach is unpractical and perhaps unethical. Why would a teacher with the knowledge, ability, and desire to teach with meaning not teach everyone in that manner? I am not certain it is even possible. If instead, one teacher teaches one way and a different teacher teaches another, then many other uncontrolled variables are introduced into the study. For these reasons and many others, we have perhaps more conviction about meaningful instruction than we do strong evidence. In the words of Brownell (1947), “Research on meaningful learning is extraordinarily difficult” (p.264). Yet, the pursuit must continue.

Appendix A

Fraction Knowledge Pretest

Name: ______________________

We are interested in what processes you use to solve these problems, so please show what you did or tell what you thought to get your answer.

A) $\frac{2}{3} \times \frac{1}{5} =$

B) $\frac{1}{8} \times 24 =$

C) $2\frac{1}{2} \times 4 =$

D) $2\frac{1}{3} \div 9 =$

E) $\frac{3}{8} + \frac{2}{8} =$

F) $\frac{2}{15} \times \frac{7}{15} =$

G) $4 \div \frac{1}{4} =$

H) $\frac{2}{3} + \frac{3}{8} =$

I) $5 - \frac{3}{8} =$

J) $\frac{9}{10} \div \frac{3}{10} =$

K) $\begin{array}{r} 6\frac{2}{5} \\ -\,2\frac{4}{5} \\ \hline \end{array}$

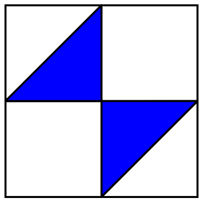
L) $\frac{2}{4} - \frac{3}{6} =$

M) $1\frac{4}{5} \times 2\frac{1}{3} =$

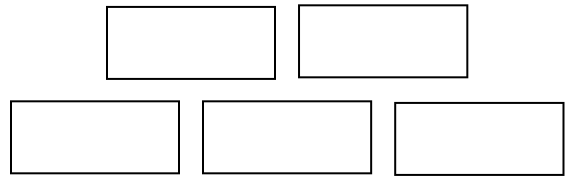
N) $2\frac{3}{5} + 3\frac{1}{5} =$

O) $\frac{2}{9} \div \frac{3}{8} =$

P) What fractional part of this figure is shaded?



Q) Color 2/3 of these rectangles.



R) Sandi has $5\frac{2}{3}$ yards of ribbon. How many $\frac{1}{6}$ yard pieces can she cut it into? *Show how you arrived at your answer.*

S) Derek can jump $5\frac{1}{3}$ feet. How many more feet does he need for a 9 foot jump? *Show how you arrived at your answer.*

T) A recipe requires $1\frac{2}{3}$ cups of sugar. How much sugar is needed for $1\frac{1}{2}$ batches? *Show how you arrived at your answer.*

Appendix B

Additional Item on Posttest

U) *Use pictures, numbers, or words to show your solution strategy.*

The Mathematical Preparation of Chocolate Milk

Jenny was mixing herself a glass of chocolate milk. “You sure have enough chocolate syrup in the glass,” remarked Kevin, who then found a glass of his own to drink.

“*Only a third of a glass of syrup,”* pointed out Jenny. “And you’re sure taking your share.”

“I only have one-fourth of a glass,” estimated Kevin.

“But Kevin, your glass holds twice as much!”

“Tell you what,” said Kevin after they both had mixed milk and syrup in their glasses. “Let’s combine our drinks in a larger pitcher, and then split the whole amount.”

While Jenny is trying to decide whether or not this arrangement is to her advantage, can you say **what part of the combined mixture would be syrup?**

Appendix C

Motivation Questionnaire

NAME: ____________________________

The following questions ask you to rate on a scale of 1 to 7 some of your feelings toward fractions. Answer each question by circling the number that best describes your particular feeling. There are no "right" or "wrong" answers, only answers that are most true for you.

1. In general, I find working on fractions:

Very boring	1	2	3	4	5	6	7	Very interesting

2. How good at fractions are you?

Not at all Good	1	2	3	4	5	6	7	Very Good

3. When someone asks you some questions to find out how much you know about fractions, how much do you worry that you will do poorly?

Not at all	1	2	3	4	5	6	7	Very much

4. Compared to other math topics, how good at fractions are you?

A lot worse than other math topics	1	2	3	4	5	6	7	A lot better than other math topics

5. How well would you expect to do if you were given a set of fraction problems?

Not at all Good	1	2	3	4	5	6	7	Very Good

6. When I am taking fraction test, I usually feel:

Not at all nervous and uneasy	1	2	3	4	5	6	7	Very nervous and uneasy

7. Compared to other math topics, how important is it to you to be good at fractions?

Not at all important	1	2	3	4	5	6	7	Very important

8. How much do you like fractions?

Not at all	1	2	3	4	5	6	7	Very much

9. How good would you be at learning something new about fractions?

Not at all Good	1	2	3	4	5	6	7	Very Good

10. Taking fraction tests scares me.

I never feel this way	1	2	3	4	5	6	7	I very often feel this way

11. I dread having to do fractions.

I never feel this way	1	2	3	4	5	6	7	I very often feel this way

12. In general, how useful is what you have learned about fractions?

Not at all Useful	1	2	3	4	5	6	7	Very useful

13. Compared to other students, how good at fractions are you?

Much worse than other students	1	2	3	4	5	6	7	Much better than other students

14. For me being good at fractions is:

Not at all important	1	2	3	4	5	6	7	Very important

15. Compared to other math topics, how useful is what you have learned about fractions?

Not at all useful	1	2	3	4	5	6	7	Very useful

Appendix D

Individual Demographic Sheet

Name: _________________________ **Individual Demographics**

Instructions: Please circle or fill in the appropriate response.

Gender: Male Female **Age:** ______

College Level:

____ Freshman

____ Sophomore

____ Junior

____ Senior

____ Other (please specify): _______________________________

Number of mathematics classes:

Taken in high school: ________

Taken/currently taking in college: ________

Ethnicity (check all that apply):

____ African American

____ American Indian

____ Asian/Pacific Islander American

____ European American

____ Hispanic American

____ Other (please specify): ________________________________

Appendix E

Consent Forms

STUDENT CONSENT FORM

Project Title	Beyond a Relational Understanding of Fractions: Elements of Instruction that Contribute to Preservice Teachers' Knowledge and Motivation
Purpose	This is a research project being conducted by Dr. Patricia Alexander at the University of Maryland, College Park. We are inviting you to participate in this research project because you are enrolled in MATH212. The purpose of this research project is to find out how instruction may affect individuals' understanding of and motivations toward fraction.
Procedures	As part of the course curriculum you will be administered a 20-minute fraction knowledge test and a 5-minute motivation questionnaire at the beginning and end of the semester. The fraction test will include addition, subtraction, multiplication, and division of fractions. The questionnaire will include questions like, "How good at fractions are you?" and "How much do you like fractions?" If you choose to participate in this study, the data you provide from the fraction knowledge and motivation measures will be analyzed in relation to instructional behaviors observed during the teaching of fractions. The researchers will have access to your class attendance records. You will also be given a demographics sheet to complete including information about gender, age, and race. You may refuse to answer any question.
Confidentiality	Participation is voluntary and all responses are confidential. The data you provide will be grouped with the data of others for reporting and presentation. Your name will be removed from the questionnaire and knowledge test and will not be used in the storage or reporting of information. Data will be stored in a locked cabinet in the office of the student investigator, on the University of Maryland campus. Access to this data will be limited to the project investigator and student assistant. After five years, the data will be shredded or boxed and moved to a secure storage facility. Your information may be shared with representatives of the University of Maryland, College Park or governmental authorities if we are required to do so by law.
Risks and Benefits	There are no known risks associated with participating in this research project. This research is not designed to help you personally, but the results may help the investigators learn more about how participation in this course is related to your fraction knowledge and motivation toward fractions.
Freedom to Withdraw	Your participation in this research is completely voluntary. Participation is not a course requirement and will not affect your grade. You may choose not to take part at all. If you decide to participate in this research, you may stop participating at any time. If you decide not to participate in this study or if you stop participating, you will not be penalized or lose any benefits to which you otherwise qualify.
Contact information	You have been informed that this research has been reviewed according to the University of Maryland, College Park IRB procedures for research involving human subjects. If you have any questions about the research study itself, you can contact Dr. Alexander (palexand@umd.edu; 301-405-2821) or Kristie Jones (kkjones@umd.edu; 301-405-1304) at: EDU 3304F, Department of Human Development, University of Maryland, College Park; If you have questions about your rights as a research subject or wish to report a research-related injury, you can contact: Institutional Review Board Office, University of Maryland, College Park, Maryland, 20742; irb@deans.umd.edu; (301-405-0678)
Statement of Age of Subject and Consent	The signature below indicates that: You are at least 18 years of age; the research has been explained to you; your questions have been fully answered; and you freely and voluntarily choose to participate in this research project.
Signature and Date	**NAME OF SUBJECT:** ______________________________________ **SIGNATURE OF SUBJECT:** ______________________________________ **DATE:** ____________________

INSTRUCTOR CONSENT FORM

Project Title

Beyond a Relational Understanding of Fractions: Elements of Instruction that Contribute to Preservice Teachers' Knowledge and Motivation

Purpose

This is a research project being conducted by Dr. Patricia Alexander at the University of Maryland, College Park. We are inviting you to participate in this research project because you are an instructor for MATH212. The purpose of this research project is to find out how instruction may affect individuals' understanding of and motivations toward fraction.

Procedures

You will be interviewed for 5 minutes at the beginning and end of the semester. The interview includes questions like, "What are your particular goals for the course?" and "What do you think the challenges will be for teaching this course?" You will be observed on days that fractions are taught and the researcher will be recording information about my instruction every three minutes. The coded categories include teacher activity, student activity, class organization, attention of the teacher, content and context of the lesson, use of technology, and classroom behavior. If you choose to participate in this study, data from those observations will be analyzed in relation to information collected on the fraction knowledge and motivation measures the students will take. You may refuse to answer any question.

Confidentiality

Participation is voluntary and all responses are confidential. The data you provide will be grouped with the data of others for reporting and presentation. Your name will not be used in the storage or reporting of information. Data will be stored in a locked cabinet in the office of the student investigator, on the University of Maryland campus. Access to this data will be limited to the project investigator and student assistant. After five years, the data will be shredded or boxed and moved to a secure storage facility. Your information may be shared with representatives of the University of Maryland, College Park or governmental authorities if we are required to do so by law.

Risks and Benefits

There are no known risks associated with participating in this research project. This research is not designed to help you personally, but the results may help the investigators learn more about how participation in this course is related to your fraction knowledge and motivation toward fractions.

Freedom to Withdraw

Your participation in this research is completely voluntary. If you decide to participate in this research, you may stop participating at any time. If you decide not to participate in this study or if you stop participating at any time, you will not be penalized or lose any benefits to which you otherwise qualify.

Contact information

You have been informed that this research has been reviewed according to the University of Maryland, College Park IRB procedures for research involving human subjects. If you have any questions about the research study itself, you can contact Dr. Alexander (palexand@umd.edu; 301-405-2821) or Kristie Jones (kkjones@umd.edu; 301-405-1304) at: EDU 3304F, Department of Human Development, University of Maryland, College Park;If you have questions about your rights as a research subject or wish to report a research-related injury, you can contact: Institutional Review Board Office, University of Maryland, College Park, Maryland, 20742; irb@deans.umd.edu; (301-405-0678)

Statement of Age of Subject and Consent

The signature below indicates that: You are at least 18 years of age; the research has been explained to you; your questions have been fully answered; and you freely and voluntarily choose to participate in this research project.

Signature and Date

NAME OF SUBJECT: ____

SIGNATURE OF SUBJECT: ____

DATE: ____________________

Appendix F

Course Syllabus

MATH 212: MATH FOR ELEMENTARY EDUCATION MAJORS

University of Maryland, College Park
Fall 2005

Textbooks: Mathematics for Elementary Teachers, by Sybilla Beckmann (bundled with Class Activities)
ISBN: 0-201-72587-8 – should include *both* text and Class Activities manual
Note: This text will also be used in Math 213 and 214
Student Packet (purchase at Armory 0127)

Course Description: This course will review and extend topics of arithmetic and number theory that may be encountered in elementary school curricula. Students will actively investigate topics, working in groups on projects and writing explanations of their thinking as well as answers to problems.

Schedule (subject to change):

Week of	**Topics**
Aug 31	Number Theory (Beckmann chap 12 + Packet as assigned)
Sep 7	(*Mon Sep 5 = Labor Day Holiday);* Number Theory, con't
Sep 14	Numbers (Beckmann chap 2 + Packet as assigned)
Sep 21	Numbers, con't
Sep 28	Numbers, con't; **Exam 1: Fri Sep 30**
Oct 3	Fractions (Beckmann chap 3 + Packet)
Oct 10	Fractions, con't; Addition & Subtraction (Beckmann chap 4 + Packet)
Oct 17	Addition & Subtraction, con't
Oct 24	**Exam 2: Wed Oct 26**; Multiplication (Beckman chaps 5, 6 + Packet)
Oct 31	Multiplication, con't
Nov 7	Multiplication, con't
Nov 14	Multiplication, con't
Nov 21	**Exam 3: Mon Nov 21**; Division; *(Friday Nov 25 = Thanksgiving Holiday)*
Nov 28	Division (Beckmann chap 7 + Packet), con't
Dec 5	Division, con't; Ratio and proportion
Dec 12	Review for Final Exam (*Last day of classes = Dec. 13)*

FINAL EXAM: Thursday, Dec 15, 1:30-3:30 pm; Location TBA

Grading:

Quizzes and/or homework**	100 pts
Projects* (2 @ 20 pts):	40 pts
Written Reflections (2 @ 10 pts):	20 pts
Exams (100 each):	300 pts
Final Exam:	150 pts
TOTAL:	**610 pts**

Letter grades will be given using the scale 90% - 100% = A; 80% - 89% = B, 70% - 79% = C, etc.

**Quizzes and/or homework are assigned by the individual instructor and may differ from section to section. Answers are usually in the text.

*Note: While collaboration is encouraged on projects, each student is to write up his or her own individual project without reference to any other person's writing. The *words and examples chosen should be different* from the others in your group. Direct copying of another's work is not acceptable under any circumstances.

Written reflections will be assigned two times during the semester. These are short (generally 1 – 2 pages) essays that address the assigned question or questions. They should be TYPED and organized in well-written English paragraphs. Examples should be included to illustrate the comments you make. The maximum grade for these essays is 10 points, which indicates an exceptionally well-thought-out response. An adequate response will receive an 8 or 9; responses which fall short in some way will be graded lower.

Projects: Projects do not need to be typed, but should be written up neatly—your write-up should not be your first draft! Answers should be thorough, fully supported by charts, examples, and explanations as needed. Write your report so that someone <u>not</u> in the class could follow your reasoning. The maximum grade for each project is 20 points.

Class Participation: Many class sessions will include time working with other students in groups, whole class discussions, and opportunities for students to explain their thinking. These experiences have been designed both to maximize opportunities to reflect on the content more deeply and to provide experience giving explanations—an important skill for future teachers to develop. Thus participating in class is very important to gain the most from the course.

Honor Code: The University has a nationally recognized Honor Code, administered by the Student Honor Council. The Student Honor Council proposed and the University Senate approved an Honor Pledge. The Pledge reads: "I pledge on my honor that I have not given or received any unauthorized assistance on this assignment/examination." In this course, the Pledge statement should be handwritten and signed on each exam. Students who fail to write and sign the Pledge will be asked to confer with the instructor.

Late Work and Make-up Exams:

Notification: If possible, the instructor should be notified *before* a due date or exam is missed. If this is impossible due to the nature of the emergency, the instructor must be contacted as soon as possible.

Excused Absences: The following are recognized as excused absences *if appropriate documentation is provided:*

*illness of a student or dependent
*death in the immediate family
*participation in a UM athletic team trip, documented by official letter from the Athletics Dept.
*religious observance which prevents class attendance, documented by a note from the leader of your congregation
*"compelling circumstance beyond the student's control," (considered on a case-by-case basis at the instructor's discretion).

Late work may be subject to a deduction of 20% per class period at the instructor's discretion.

Resources: Many students find the following resources helpful as they work to understand the material in this course:

Fellow students: Share phone numbers among group members and call or get together outside of class to discuss projects, homework, an upcoming quiz or exam.

Instructor's office hours: Be sure you know when and where to contact your instructor if you still have questions after individual study and discussion with classmates.

Department-sponsored tutoring: The math department schedules a Math 211 instructor to be available at specified times for walk-in tutoring. (Times to be announced.)

Learning Assistance Services: Located in Shoemaker, this center provides general resources for those experiencing difficulties in math due to math anxiety or lack of background in basic math. You may call the center at 301-312-7693 to find out what resources are available for your specific needs.

Philosophy of this Course:

Many people think of math as a collection of meaningless procedures and rules that "magically" give the right answer when numbers from a problem are inserted correctly. Your experiences in this course will be very different from this! Throughout this course, in class, on projects, and on exams, you will be asked to "explain why or why not" or to "justify your answer." In other words, you will be expected to understand <u>why</u> the procedure you are using works or why the answer you give is correct. You will be most successful this semester if you continually ask "why?" as you read, listen, and solve problems. Seeking connections and meaning can be a very rewarding way to learn—and someday teach—these math ideas.

Appendix G

Student Packet Sample

FRACTION WORD PROBLEMS

For each problem below, decide which approach or model of multiplication is best suited to interpreting it. Sketch an appropriate diagram or use fraction pieces (fraction strips or fraction circles) to illustrate the problem. Give the answer.

1. Jenny gives Megan 1/4 of a pizza. Megan gives 1/2 of her share to her roommate. What portion of the pizza does her roommate get?

2. Four families share a garden. The Abbotts plant tomatoes in 1/3 of their portion, and nobody else plants tomatoes. What part of the whole garden is planted in tomatoes?

3. Five roommates share a pan of brownies. Stuart eats 2/3 of his share immediately. What fraction of the whole pan does he eat?

4. Margot made a pan of brownies. She is willing to share with her three roommates, but she decides as the baker she is entitled to twice as much as they are.

   a. What fraction of the brownies does Margot get?

   b. If she plans to give 2/3 of her share to her sister, what fraction would her sister get?

   c. If she changes her mind and decides to give 3/4 of her share to her sister what fraction would her sister get?

5. A plot of ground is 2 1/2 yards by 3/4 yard. How many square yards is this? Be sure to connect your diagram and your algorithm.

6. A room is 8 3/4 feet by 7 1/2 feet. How many square feet is this? Calculate more than one way, using a diagram to support your answer.

Important Note for #5 and #6: What are the units on the factors? What are the units on the product? How do you see the different units in your diagram?

Appendix H

Sample of Teaching Notes

Teaching Notes – Chapter 3

In this chapter we will re-visit many facts students will recall about fractions. Remind them that, once again, we will be focusing on understanding and explaining *why* various procedures work, not just reviewing what those procedures are.

Day 1 (3.1)

Blank half-sheets of copy paper, about 8 per person

Intro the "Meaning of a Fraction" as the text does (top of p. 58). Do "Using Fraction Manipulatives" handout (attached). Whenever appropriate connect to procedures or operations students recall from elementary school.

Varying what equals one whole:
Note that for all of these we have been assuming one whole equals our piece of paper. What if we change that?? Look at Class Activity 3A #2 (p. 25). Have students think/pair/share for a minute or two until most everyone seems to have figured it out. Then ask students to explain how they thought about it. Some will think "This piece of paper should have four equal parts, so I'll fold it into fourths, each of which is 1/5 of the whole. Then I'll shade three of them." Others might fold the paper into fourths and then reason that each section is 1/4 of 4/5, which multiplies to 4/20 or 1/5. Note: Both multiplication and division have this feature of thinking in two ways about the "whole."

Continue in this problem/think/pair/share mode through 3A #3 (p. 25), #6, #7, (p. 27), 3B #1, 2 (p. 28). Note how we need to be careful to identify in each case what the "whole" is. In 3B #2, the whole is a collection of twelve circles! This is an important alternate model for fractions, called the "set model." It is more difficult for children to understand, but is important for them to begin to grapple with in the middle elementary grades. In a similar vein, return to 3A #6 and ask students why it is *not* appropriate to represent the diagram by writing 3/4 + 3/4 + . . . + 3/4 = 15/20. (When we add or subtract we need to keep the same "whole" throughout the number sentence; each 3/4 means 3/4 of one cup but 15/20 means 15/20 of 5 cups.)

What makes equal parts "equal"?:
Another tricky concept is that of "equal parts." It is essential that the "pieces" the whole is divided into be equal, but equal in what way? Is it necessary that they be identical in size and shape to be considered equal? Have students think/pair/share Class Activity 3C (p. 31).

What if the parts are to be equal in area, but do NOT need to be identical in shape. Look at the four rectangles at the top of p. 67. Which of these are legitimate divisions into fourths? Clearly the first and third are ok and the fourth is not. The second is tricky. It turns out to be ok, but takes some reasoning and appeal to the formula A = 1/2 bh for the area of a triangle to confirm. Challenge students to try to justify their answer.

More problems (do in class as time permits; assign remaining for homework): Packet pp. 70 – 75 #4, 6 (uses pattern blocks), 10, 12, 13.

HW: Read Beckmann 3.1

Do 3.1 (pp. 61-62) #3, 4, 6, 8

(If doing Packet #6 for homework, for the patterns blocks students can go online at http://www.arcytech.org/java/patterns/patterns/_j.shtml)

Using Fraction Manipulatives

Using the circles or squares or strips you are given as the "whole," make manipulatives to show halves, thirds, fourths, fifths, sixths, and eighths.
Discuss questions below. Be prepared to share your thinking with the class.

1) Susie thinks 1/3 is bigger than 1/2 because 3 is bigger than 2. How could you use these manipulatives to understand why 1/3 < 1/2?

2) Show how to find 1/2 of 1/3 using these tools; show how to find 2/3 of 3/4.

3) Show how to find how many eighths are in 3/4.

4) Alicia said that making 1/5 was hard. Brandon said it was easy because 1/5 is halfway between 1/4 and 1/6. Is he right? Explain. (from Bassarear, Explorations manual, p. 137)

Appendix I
Factor Loadings for Motivation Questionnaire

Factor loadings for PAF of the Motivation Questionnaire

Item	Pretest Factors			Posttest Factors	
	Self-Concept/ Interest	Anxiety	Value	Confidence	Value
1	.82			.43	
2	.85			.78	
4	.47			.55	
5	.85			.76	
8	.81			.59	
13	.48			.63	
3		.62		-.82	
6		.78		-.86	
10		.90		-.86	
11		.60		-.88	
7			.68		.78
12			.69		.67
14			.73		.79
15			.80		.71

Note. Items correspond to those in Appendix C. N = 85.

Appendix J

Classroom Observation Codes

Teacher Activity

- Requests
  - Student reflection on learning
  - Alternative method or strategy
  - Student self assessment
  - Elaboration of student response
  - Attention to a student’s response or idea
- Poses
  - Problem/task to solve/high order question
  - Routine exercise/low order question
- Elaborates on
  - Problem, task, or high order question posed previously
  - Routine exercise or low order task posed previously
- Responds with or states
  - A question back to student(s)
  - Evaluation with feedback
  - Evaluation with no feedback
  - A statement or answer
  - Extrinsic reward
  - Doesn’t; redirects conversation
- Models
  - With technology or tools
  - Without technology or tools
- Defines a Mathematical term
  - Elaborates; uses concept attainment
  - No or few examples
- Posts
  - Key idea
  - Student answer
- Lectures
  - Math content
  - Self-directed learning
  - Learning strategies
- Reads aloud from math text
- Listens to or watches student or reads student work
- Manages
  - An activity
  - Materials
  - Student behavior
- No obvious instruction or management in math

Student Activity

- Asks a question of
  - Another student or class
  - The teacher
- Responds with or states
  - Conjecture
  - Explanation or justification
  - Alternative method
  - Simple answer or statement
- Works on
  - Formal assessment
  - Problem or task
  - Extended writing
  - Routine exercises
- Write(s) on chalkboard
- Reads from math text
- Listen or watch
- Mixed
- Management
- No apparent academic behavior in math

Organization of Class

- Whole class instruction or discussion
- Small group
  - Same focus
  - Different focus
- Independent work
  - Same focus
  - Different focus
- Mixed group and independent
  - Same focus
  - Different focus

Attention of Teacher

- Whole class
- A small group
- An individual student
- Not attending to any student

Episode Content

Linking procedural and conceptual
Conceptual
Procedural
Learning strategies
  - Conceptual
  - Procedural

Mixed
Management
  - Resources or activity
  - Student behavior

Non-instructional activity
  - Teacher initiated
  - Other

Episode Context

Connects to
  - Other mathematics
  - Other content areas
  - Real world for scaffolding
  - Student's first language

Hooks or motivated students into topic
  - Cognitive or personal
  - Situational
  - Will be on test/future need

States agenda or objective
No specific context

Classroom behavior

On task:	almost all, many, half, some, almost none
In transition or waiting:	almost all, many, half, some, almost none
"Play" with work, socialize, unengaged:	almost all, many, half, some, almost none

Technology

Video, internet:	all, many, half, some, none
Other technology	
Models concept:	all, many, half, some, none
Tool use:	all, many, half, some, none

Attribution Scales

Knowledge

- Promotes principled understanding in students
- Activates and builds on students' knowledge and experiences
- Manifests a deep and accurate understanding of the lesson
- Illustrates the value or utility of the lesson content

Strategic Processing

- Explicitly teaches general or domain-specific strategies
- Models general or domain-specific strategies
- Encourages students to be reflective and self-regulatory
- Provides opportunities for engagement in reasoning or non-routine problem-solving

Development and Individual Differences

- Demonstrates understanding of developmental levels in lesson content and delivery
- Maintains high, but reasonable, expectations for all students
- Is cognizant of students' individual strengths and needs
- Plans for and adjusts to variations in students' thinking, behavior, or background

Motivation

- Incorporates student interest or choice in the lesson
- Manifests an interest or excitement in the content
- Involves all students actively in the lesson
- Promotes positive attributional and self-competency beliefs among all students

Context and Situation

- Creates a caring and affirming learning environment
- Makes effective use of available resources in planning and promotes their use
- Uses a variety of social interaction patterns during the lesson
- Welcomes and engages all students' thoughts and reactions

Appendix K

Instructor Interviews

Instructor Pre-Interview

1) Have you ever taught this class or one like it before?

2) Why are you teaching this class?

3) Are you looking forward to teaching it?

4) In your opinion, what is the purpose of MATH 212?

5) What are your particular goals for the course?

6) What instructional approaches will you use to meet those goals?

7) I understand that the students in MATH 212 are elementary education majors. How might that influence what you teach and how you teach?

8) What is your opinion of the curriculum materials?

9) What do you think the challenges will be for teaching this course?

Instructor Post-Interview

1) Will you teach this class or one like it again?

2) Looking back, how was the experience of teaching this class?

3) In your opinion, what was the purpose of MATH 212?

5) What were your particular goals for the course?

6) What instructional approaches did you use to meet those goals?

7) How did the fact that these students are elementary education majors influence what you taught and how you taught?

8) What is your opinion of the curriculum materials?

9) What were the challenges for teaching this course?

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