ABSTRACT

Title of Dissertation: Turaev Torsion of 3-Manifolds with Boundary

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We study the Turaev torsion of 3-manifolds with boundary; specifically how certain "leading order" terms of the torsion are related to cohomology operations. Chapter 1 consists mainly of definitions and known results, providing some proofs of known results when the author hopes to present a new perspective.

Chapter 2 deals with generalizations of some results of Turaev from [Tur02]. Turaev's results relate leading order terms of the Turaev torsion of closed, oriented, connected 3-manifolds to certain "determinants" derived from cohomology operations such as the alternate trilinear form on the first cohomology group given by cup product. These determinants unfortunately do not generalize directly to compact, connected, oriented 3-manifolds with nonempty boundary, because one must incorporate the cohomology of the manifold relative to its boundary. We define the new determinants that will be needed, and show that with these determinants enjoy a similar relationship to the one given in [Tur02] between torsion

and the known determinants. These definitions and results are given for integral cohomology, cohomology with coefficients in $\mathbb{Z}/r\mathbb{Z}$ for certain integers r, and for integral Massey products.

Chapter 3 shows how to use the results of Chapter 2 to derive Turaev's results for integral cohomology, by studying how the determinant defined in Chapter 2 changes when gluing solid tori along boundary components, and also how this determinant is related to Turaev's determinant when one glues enough solid tori along the boundary to obtain a closed 3-manifold. One can then use known gluing formulae for torsion to derive Turaev's results relating torsion and cohomology of closed 3-manifolds to the results in Chapter 2.

Turaev Torsion of 3-Manifolds with Boundary

by

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DEDICATION

First and foremost, I would like to dedicate this to my wife, Kathryn Rendall Truman. Without your love and support, this would not have been possible. Secondly, I dedicate this to the upcoming little Truman; perhaps I'll read excerpts of this document to you when you can't sleep.

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Chapter 1

Definitions and Notations

In this chapter we give definitions and set notations which will be used throughout. We will largely follow the notation in [Tur02].

1.1 The Algebraic Torsion of a Complex

In this section we define algebraic torsion of a chain complex, which will later be used to define the topological torsion of a CW-complex. We start with the easiest to define, the torsion of an acyclic complex over a field, and discuss a generalization to complexes which may not be acyclic. One may also generalize to complexes over rings (see [Tur01] or [Mil66]) though we will not need that here.

1.1.1 The Torsion of an Acyclic Complex Over a Field

First, let V be a finite-dimensional vector space over a field F. Let a and b be bases for V. We denote by (a/b) the matrix whose rows are obtained from expressing the vectors of the basis a in terms of the basis b, i.e. row i of (a/b) is the i^{th} vector of the basis a expressed in the basis b. Symbolically, if a_i is the i^{th} vector of a, and similarly for b, then $a_i = \sum_j (a/b)_{i,j} b_j$. We denote the determinant of (a/b) by $[a/b]$. Then

$$\begin{aligned} &[a/a] = 1 \\ &[a/b] = [b/a]^{-1} \\ &[a/c] = [a/b][b/c]. \end{aligned} \tag{1.1}$$

Furthermore, if a^*, b^* are the bases of $V^* = \operatorname{Hom}_F(V, F)$ dual to the bases a, b of V, then

$$(a^*/b^*) = ((a/b)^{-1})^{\mathsf{T}} \tag{1.2}$$

where the "T" denotes transpose, so in particular

$$[a^*/b^*] = [a/b]^{-1}. \tag{1.3}$$

To prove this, note if $a_1, \ldots, a_n$ are the vectors of a expressed in the b basis, and $a_1^*, \ldots, a_n^*$ are the vectors of a^* expressed in the b^* basis, we have the defining equation of a^*, $a_i^*(a_j) = \delta_{i,j}$, the Kronecker delta, so

$$\begin{aligned} \delta_{i,j} &= a_i^*(a_j) \\ &= \sum_{k=1}^n (a^*/b^*)_{i,k} b_k^*(a_j) \\ &= \sum_{k=1}^n (a^*/b^*)_{i,k} b_k^* \left(\sum_{\ell=1}^n (a/b)_{j,\ell} b_\ell \right) \\ &= \sum_{k=1}^n (a^*/b^*)_{i,k} \sum_{\ell=1}^n (a/b)_{j,\ell} b_k^*(b_\ell) \\ &= \sum_{k=1}^n (a^*/b^*)_{i,k} \sum_{\ell=1}^n (a/b)_{j,\ell} \delta_{k,\ell} \\ &= \sum_{k=1}^n (a^*/b^*)_{i,k} (a/b)_{j,k}. \end{aligned}$$

This means $(a^*/b^*)(a/b)^{\mathsf{T}}$ is the identity, proving (1.2).

The torsion of an acyclic complex generalizes these determinants in much the same way that Euler characteristic generalizes dimension.

Let C_* be a finite acyclic complex of finite dimensional F-vector spaces, with a collection of distinguished bases, c_i a basis of C_i for each i. For each i, choose a sequence of vectors b_i in C_i such that $\partial_i b_i$ is a basis of $\mathrm{im}(\partial_i : C_i \to C_{i-1})$. Denote the sequence of vectors obtained by appending the sequence b_i to the end of the sequence $\partial_{i+1} b_{i+1}$ by simply $\partial_{i+1} b_{i+1} b_i$. Then since the complex C_* is acyclic, $\mathrm{im}(\partial_{i+1})$ is precisely $\ker(\partial_i)$, so $\partial_{i+1} b_{i+1} b_i$ is a basis for C_i, hence we can make sense of the symbol $[(\partial_{i+1} b_{i+1} b_i)/c_i]$ for each i. Then we define the torsion of the complex C_* with distinguished bases c_* by

$$\tau(C_*, c_*) = \prod_i \; [(\partial_{i+1} b_{i+1} b_i)/c_i]^{(-1)^{i+1}} . \tag{1.4}$$

This definition does depend on the distinguished bases c_*, as suggested by the notation, but does not depend on the choices of b_*. To see why, note if β_* is another collection so that $\partial_i \beta_i$ bases $\mathrm{im}(\partial_i)$, then

$$[\partial_{i+1} b_{i+1} b_i / c_i] = [\partial_{i+1} \beta_{i+1} \beta_i / c_i] \cdot [\partial_{i+1} b_{i+1} b_i / \partial_{i+1} \beta_{i+1} \beta_i]$$

by (1.1). So we need to compute $[\partial_{i+1} b_{i+1} b_i / \partial_{i+1} \beta_{i+1} \beta_i]$. To compute this, we will need a bit more notation: let $k_i = \dim(\mathrm{im}(\partial_i))$, and denote the vectors in b_i by $b_i^1, \ldots, b_i^{k_i}$, and similarly for β_i. Then since $\partial_{i+1} \beta_{i+1} \beta_i$ is a basis for C_i, we can write each b_i^j as linear combinations from that basis, so we can define matrices A_i, B_i so that

$$b_i^j = \sum_{\ell=1}^{k_{i+1}} A_i^{j,\ell} \partial_{i+1} \beta_{i+1}^{\ell} + \sum_{p=1}^{k_i} B_i^{j,p} \beta_i^p . \tag{1.5}$$

Then we can write

$$(\partial_{i+1} b_{i+1} b_i / \partial_{i+1} \beta_{i+1} \beta_i) = \begin{pmatrix} (\partial_{i+1} b_{i+1} / \partial_{i+1} \beta_{i+1}) & 0 \\ A_i & B_i \end{pmatrix} . \tag{1.6}$$

But now, applying ∂_i to (1.5) tells us that $B_i = (\partial_i b_i / \partial_i \beta_i)$, so we can rewrite (1.6) as

$$(\partial_{i+1} b_{i+1} b_i / \partial_{i+1} \beta_{i+1} \beta_i) = \begin{pmatrix} (\partial_{i+1} b_{i+1} / \partial_{i+1} \beta_{i+1}) & 0 \\ A_i & (\partial_i b_i / \partial_i \beta_i) \end{pmatrix}.$$

Now we can compute the determinant

$$[\partial_{i+1} b_{i+1} b_i / \partial_{i+1} \beta_{i+1} \beta_i] = [\partial_{i+1} b_{i+1} / \partial_{i+1} \beta_{i+1}] \cdot [\partial_i b_i / \partial_i \beta_i].$$

When we compute the alternating products, we see

$$\begin{aligned} \prod_i [(\partial_{i+1} b_{i+1} b_i)/c_i]^{(-1)^{i+1}} &= \prod_i [(\partial_{i+1} \beta_{i+1} \beta_i)/c_i]^{(-1)^{i+1}} \cdot [\partial_{i+1} b_{i+1} b_i / \partial_{i+1} \beta_{i+1} \beta_i]^{(-1)^{i+1}} \\ &= \prod_i [(\partial_{i+1} \beta_{i+1} \beta_i)/c_i]^{(-1)^{i+1}} \cdot \Big([\partial_{i+1} b_{i+1} / \partial_{i+1} \beta_{i+1}] \cdot [\partial_i b_i / \partial_i \beta_i] \Big)^{(-1)^{i+1}} \\ &= \prod_i [(\partial_{i+1} \beta_{i+1} \beta_i)/c_i]^{(-1)^{i+1}}. \end{aligned}$$

The last equality holds since the product is alternating and each nonunity $[\partial_i b_i / \partial_i \beta_i]$ term occurs twice, but with alternately signed powers.

1.1.2 Generalization to Non-Acyclic Complexes

One may similarly define the torsion for a complex which is not acyclic, though we will not use this much. We must, as before, have a finite complex $C_* = (C_m \to C_{m-1} \to \cdots \to C_1 \to C_0)$ of finite dimensional vector spaces over a field F with c_i a distinguished basis of C_i for each i, but we must also have a distinguished sequence of vectors $h_i \in \ker(\partial_i) \subset C_i$ so that the sequence h_i projects to a basis of $H_i(C_*)$ under the projection $\ker(\partial_i) \to H_i(C_*)$. Then the torsion will depend on h_* as well, and we will reflect this in the notation. The rest of the definition is very similar; choose a sequence of vectors $b_i \in C_i$ so that $\partial_i b_i$ is a basis for $\mathrm{im}(\partial_i)$. Then if we concatenate the sequence $\partial_{i+1} b_{i+1}$ with the sequence h_i and then the sequence b_i, we get a basis for C_i ($\partial_{i+1} b_{i+1}$ together with h_i gives a basis for $\ker(\partial_i)$, and b_i is a lift of a basis for the image). Then we define the symbol

$(\partial_{i+1}b_{i+1}h_ib_i)$ to mean the basis of C_i obtained by the concatenation, and define the torsion to be

$$\tau(C_*, c_*, h_*) = (-1)^{|C|} \prod_{i=0}^{m} \; [(\partial_{i+1}b_{i+1}h_ib_i)/c_i]^{(-1)^{i+1}} \tag{1.7}$$

where

$$|C| = \sum_{i=0}^{m} \left[\left(\sum_{r=0}^{i} \dim(C_r) \right) \left(\sum_{r=0}^{i} \dim(H_r(C_*)) \right) \right] \pmod 2.$$

As above, this definition does not depend on the choices of b_i, and the proof is almost identical. The sign is included to guarantee invariance of sign-refined torsions under cellular subdivision (when this definition is used for CW-complexes). Also note that the sign vanishes for acyclic complexes.

1.2 Topological Torsion of a CW-Complex

Let X be a compact, connected CW-complex. Let $\hat{X}$ be the maximal abelian cover of X, i.e. the cover of X corresponding to the commutator subgroup of $\pi_1(X)$. Then $H = H_1(X)$ is the group of deck transformations of the cover. For each cell in X, choose a single lift in $\hat{X}$, and order and orient the cells arbitrarily. Then the chosen cells, with the order and orientations, are a basis for $C_*(\hat{X})$ as a $\mathbb{Z}[H]$-complex. For any ring homomorphism $\varphi : \mathbb{Z}[H] \to F$ where F is a field, we can consider F as a right $\mathbb{Z}[H]$-module, and then form the φ-twisted complex of X, $C_*^{\varphi}(X) = F \otimes_{\varphi} C_*(\hat{X})$. If this complex is not acyclic, we define $\tau^{\varphi}(X) = 0 \in F$. Otherwise, we have an acyclic F-complex with a distinguished basis coming from our distinguished basis of $C_*(\hat{X})$, so we can define $\tau^{\varphi}(X) \in F^{\times}$ as the torsion of that complex. This torsion is invariant under cellular subdivisions (see [Mil66]). Note that one can only have a nonzero torsion if the Euler characteristic $\chi(X) = 0$, so we will often impose that condition when we actually want to make a torsion calculation.

1.3 Refinements of the Topological Torsion

Unfortunately, because of the choices involved, φ-torsion is only defined up to sign (due to the arbitrary choices of order/orientation of the cells) and the action of H on F, due to the arbitrary choice of lifts of cells. We now discuss Turaev's refinements of the topological torsion. These refinements can be thought of as making specific choices which fix the choices of lifts of cells and the sign as part of the input.

1.3.1 Euler Structures

There are multiple ways one may think of Euler structures, but we will take as our starting point a definition of Euler structure that lends itself well to performing computations. Later, we will discuss another equivalent definition. The definition we choose comes from [Tur01], III.20.

Let X be a finite connected CW-complex. A family $\hat{e} = \{\hat{e}_i\}$ of open cells in the maximal abelian cover $\hat{X}$ of X will be called a *fundamental family* of cells if each open cell e_i of X has exactly one $\hat{e}_i$ in $\hat{e}$ lying over it, i.e. $\hat{e}$ is a choice of exactly one lift in $\hat{X}$ of each cell in X. Given two fundamental families $\hat{e}, \hat{e}'$, we know that for each i, there is a unique element $h_i \in H_1(X)$ with $\hat{e}'_i = h_i \hat{e}_i$. Then we will define

$$\hat{e}'/\hat{e} = \prod_i h_i^{(-1)^{\dim(e_i)}} \in H_1(X). \tag{1.8}$$

The index i runs over each open cell of X. It is clear that

$$\hat{e}/\hat{e} = 1, \quad \hat{e}/\hat{e}' = (\hat{e}'/\hat{e})^{-1}, \quad \hat{e}''/\hat{e} = (\hat{e}''/\hat{e}')(\hat{e}'/\hat{e}).$$

This implies that

$$\hat{e} \sim \hat{e}' \Leftrightarrow \hat{e}'/\hat{e} = 1 \tag{1.9}$$

is an equivalence relation on the set of fundamental families of cells. The set of equivalence classes is denoted $\mathrm{Eul}(X)$, and we will call its elements *combinatorial Euler structures* on X. Later we will define smooth Euler structures, which are equivalent to combinatorial Euler structures, and we will use the term *Euler structure* when there is no need to specify whether one uses smooth or combinatorial Euler structures. Let us now note that if we have chosen an Euler structure $e = [\hat{e}]$ with $\hat{e} = \{\hat{e}_i\}$, and if we construct a new Euler structure e' as the class of the fundamental family of cells where we shift *each* cell of $\hat{e}$ by the same element $h \in H_1(X)$, i.e. $\hat{e}_i' = h\hat{e}_i$ for each i, then it is clear that $\hat{e'}/\hat{e} = h^{\chi(X)}$, so $e' = e$ if and only if $\chi(X) = 0$.

There is a canonical free and transitive $H_1(X)$ action on $\mathrm{Eul}(X)$ defined as follows: $h[\hat{e}] = [\hat{e}]'$, where $[\hat{e}]$ denotes the equivalence class of the fundamental family $\hat{e}$ in $\mathrm{Eul}(X)$, if and only if $\hat{e}'/\hat{e} = h$. One may easily show such a thing exists by shifting a single 0-dimensional cell by h. One may also show (for a proof, see [Tur01] Lemma 20.1) that if X' is a cellular subdivision of X, then there is a natural $H_1(X)$-equivariant bijection $\mathrm{Eul}(X) \rightleftharpoons \mathrm{Eul}(X')$.

Now if $\varphi : \mathbb{Z}[H_1(X)] \to F$ is a homomorphism to a field, $C_*^{\varphi}(X)$ is acyclic, and $e \in \mathrm{Eul}(X)$ is an Euler structure, then we define $\tau^{\varphi}(X, e)$ to be the φ-torsion computed with respect to a fundamental family of cells whose equivalence class in $\mathrm{Eul}(X)$ is equal to e. One may easily see from (1.8), (1.9), and (1.4) (and the definition in 1.2 of the topological torsion) that the torsion only depends on the equivalence class in $\mathrm{Eul}(X)$ of a fundamental family, and that

$$\tau^{\varphi}(X, he) = \varphi(h)\tau^{\varphi}(X, e). \tag{1.10}$$

We can also still define $\tau^{\varphi}(X, e) = 0$ if $C_*^{\varphi}(X)$ is not acyclic, and (1.10) still holds.

1.3.2 Homology Orientations

A homology orientation of a connected finite CW-complex X is an orientation of the real vector space $\bigoplus_i H_i(X; \mathbb{R})$. Using this, we construct the sign-refined torsion as follows: choose an Euler structure $e \in \mathrm{Eul}(X)$, and a fundamental family of cells $\hat{e}$, with equivalence class e. Also choose an orientation ω of $\bigoplus_i H_i(X; \mathbb{R})$, i.e. a homology orientation of X, and a basis h_i of $H_i(X; \mathbb{R})$ for each i so that $(h_0, h_1, \dots)$ is a positively oriented basis with respect to ω. Then once we choose an order and orientation of the cells in $\hat{e}$, the basis h_* gives us the data we need to compute $\tau(C_*(X; \mathbb{R}), \hat{e}, h_*)$ via (1.7). Let τ_0 denote the sign of $\tau(C_*(X; \mathbb{R}), \hat{e}, h_*) \in \mathbb{R}$. Then for any field F and homomorphism $\varphi : \mathbb{Z}[H_1(X)] \to F$, we can define

$$\tau^{\varphi}(X, e, \omega) = \tau_0 \tau^{\varphi}(X, e)$$

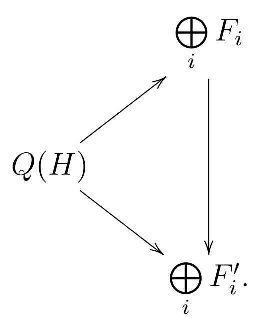
where we compute τ_0 and $\tau^{\varphi}(X, e)$ using the same order and orientation of the cells of $\hat{e}$. Then a change in the order/orientation will result in the same change in sign in τ_0 and $\tau^{\varphi}(X, e)$, i.e. the sign of $\tau^{\varphi}(X, e, \omega)$ is unaffected. One may easily see that

$$\tau^{\varphi}(X, e, -\omega) = -\tau^{\varphi}(X, e, \omega). \tag{1.11}$$

As with topological torsion and refinements by Euler structure, sign refined torsions are also invariant under cellular subdivisions.

1.4 The Turaev Torsion

In [Tur02] and [Tur01], Turaev proves that the quotient ring (i.e. the ring obtained by localizing at the multiplicative set of non-zerodivisors) of the integral group ring of a finitely generated abelian group splits as a direct sum of fields. This isomorphism provides ring homomorphisms from $\mathbb{Z}[H_1(X)]$ to various fields. Specifically, if we denote by $Q(H)$ the quotient ring of $\mathbb{Z}[H]$ where $H = H_1(X)$ and X is, as always, a finite CW complex, we have the inclusion $\mathbb{Z}[H] \hookrightarrow Q(H)$. There is an isomorphism $\Phi : Q(H) \xrightarrow{\approx} \bigoplus_i F_i$ where each F_i is a field and i ranges over a finite index set. This isomorphism is defined, for example, in [Tur02], and is unique up to unique isomorphism (which will decompose along the direct sum as a component-wise isomorphism $F_i \to F_i'$) making the following diagram commute:



Then denote by φ_i the map $\mathbb{Z}[H] \to F_i$ consisting of the inclusion to $Q(H)$ followed by the natural projection to F_i. Then for any homology orientation ω and Euler structure e, we define the Turaev torsion $\tau(X, e, \omega)$ by

$$\tau(X, e, \omega) = \Phi^{-1}\left(\bigoplus_i \tau^{\varphi_i}(X, e, \omega)\right) \in Q(H).$$

This definition does not depend on Φ (by the uniqueness of Φ). Henceforth, the symbol $\tau(X, e, \omega)$ will refer to the Turaev torsion of (X, e, ω) unless otherwise specified. The symbol $\tau^{\varphi}(X, e, \omega)$ will still refer to φ-torsion. It is clear that $\tau(X, he, \omega) = h\tau(X, e, \omega)$ and $\tau(X, e, -\omega) = -\tau(X, e, \omega)$ from (1.10) and (1.11). We will also use the notation $\tau(X, e)$ to refer to Turaev torsion without the sign refinement (hence that symbol does not have a well defined sign).

1.5 Refinements for Three-Manifolds with Boundary

In this section, all manifolds will be smooth, compact, connected, orientable 3-manifolds unless otherwise noted. When computing torsion, empty boundary versus nonempty boundary makes a difference in the cellular structure, so they will be treated separately. We will often use the notation $b_i(M)$ to denote the i^{th} Betti number of a manifold M. The torsion $\tau(M, e, \omega) \in Q(H_1(M))$ for $e \in \text{Eul}(M)$ and ω a homology orientation of M, is defined to be the torsion of a C^1 triangulation of M, and since torsion is invariant under cellular subdivision, and any two C^1 triangulations of M have a common subdivision, this torsion is actually an invariant of M.

1.5.1 Smooth Euler Structures

We briefly describe smooth Euler structures here; they are not essential for any constructions to follow, so we will note that all details may be found in [Tur02] and [Tur01]. The following definition will actually work for any odd compact connected oriented manifold M of dimension $m \geq 2$ and Euler characteristic equal to zero. A *regular vector field* on M is a nonsingular tangent vector field on M directed outside M on ∂M (transverse to ∂M). Regular vector fields u, v are *homologous* if for some point $x \in M$, the restrictions of u and v to $M - \{x\}$ are homotopic in the class of nonsingular vector fields on $M - \{x\}$ directed outside M on ∂M. The homology class of a regular vector field u is called a *smooth Euler structure* and denoted by $[u]$, and the set of homology classes is denoted $\mathrm{vect}(M)$.

There is a free and transitive action of $H_1(M)$ on $\mathrm{vect}(M)$, which we now describe. For regular vector fields u, v, the Poincaré dual of the first obstruction to constructing a homotopy from v to u lies in $H_1(M)$, and only depends on their homology classes $[u], [v] \in \mathrm{vect}(M)$. We will denote this element by $[u]/[v]$. One can show (see [Tur02] for an explicit construction) that for any $[u] \in \mathrm{vect}(M)$, $h \in H_1(M)$, there is a unique $[v] \in \mathrm{vect}(M)$ with $[v]/[u] = h$, so we will define $h[u] = [v]$ so that $h[u]/[u] = h$. This gives the free transitive $H_1(M)$ action on $\mathrm{vect}(M)$, and one can construct a canonical $H_1(M)$-equivariant bijection $\mathrm{Eul}(M) \rightleftharpoons \mathrm{vect}(M)$ which allows one to identify the two sets (again, see [Tur01] Theorem 20.2 or [Tur02] III.4.2 for details). Sometimes it is convenient to think of Euler structures in this way; for example, one can then construct a canonical $H_1(M)$-equivariant bijection $\mathrm{vect}(M) \rightleftharpoons \mathrm{spin}^c(M)$, see [Tur02] Chapter XI for details. This bijection is important when comparing the Turaev torsion to the Seiberg-Witten invariant.

1.5.2 Homology Orientations

In dimension three, if M is a closed, connected, oriented manifold, then the orientation on M induces a natural homology orientation (this is true more generally for any odd dimensional closed, connected, oriented manifold). To see why, note that if we choose any bases for $H_0(M;\mathbb{R})$ and $H_1(M;\mathbb{R})$, we may simply choose the dual bases of $H_3(M;\mathbb{R})$ and $H_2(M;\mathbb{R})$ respectively, with respect to the (non-degenerate) intersection pairing. So one easily sees that this is independent of the choices of bases of $H_0(M;\mathbb{R})$ and $H_1(M;\mathbb{R})$, since choosing different bases for either of these will result in different choices of bases for $H_2(M;\mathbb{R})$ and $H_3(M;\mathbb{R})$, and the result will be two canceling signs in the orientation. It is also easy to see how to generalize to any odd dimension greater than 1.

However, for compact oriented 3-manifolds with nonempty boundary, the orientation by itself does not naturally give a homology orientation. This means that it is not obvious that the sign refinement is useful here, since, to obtain the refinement, one must make a seemingly arbitrary choice at the beginning of calculation, rather than at the end. However, we will mention that a choice of homology orientation may be simply viewed as a choice of orientation of $H_1(M;\mathbb{R})$ and a choice of orientation of $H_1(M,\partial M;\mathbb{R})$. To see why, note that if we choose a basis $a_1,\ldots,a_{b_1(M)}$ of $H_1(M;\mathbb{R})$ which is positively oriented with respect to our chosen orientation, and a basis $\alpha_1,\ldots,\alpha_{b_2(M)}$ of $H_1(M,\partial M;\mathbb{R})$, then let $\alpha_1^*,\ldots,\alpha_{b_2(M)}^*$ be the basis of $H^1(M,\partial M;\mathbb{R})$ dual to $\alpha_1,\ldots,\alpha_{b_2(M)}$ under evaluation, and if we let [pt] be the homology class of a point in $H_0(M;\mathbb{R})$ and $[M]\in H_3(M,\partial M,\mathbb{R})$ be the fundamental class determined by the orientation, then we may define a homology orientation of M as the orientation determined by $([\mathrm{pt}],a_1,\ldots,a_{b_1(M)},\alpha_1^*\cap[M],\ldots,\alpha_{b_2(M)}^*\cap[M])$.

On the other hand, the exterior of an oriented link in a rational homology sphere does have a canonical homology orientation, which we repeat from [Tur02]. If E is the exterior of an oriented link $L=L_1\cup\cdots\cup L_m$ in an oriented rational homology sphere N, then the *natural homology orientation* ω_L of E is determined by the basis $([\mathrm{pt}], t_1, \ldots, t_m, g_1, \ldots, g_{m-1})$ where $[\mathrm{pt}] \in H_0(E; \mathbb{R})$ is the homology class of a point, $t_1, \ldots, t_m$ are homology classes of meridians, and $g_1, \ldots, g_{m-1}$ are the two-dimensional homology classes of the oriented boundaries of the tubular neighborhoods of $L_1, \ldots, L_{m-1}$ respectively.

1.5.3 General Computations of Three-Manifold Torsion

Three-Manifolds with Nonempty Boundary

Let M be a connected 3-manifold, with $\partial M \neq \varnothing$, and suppose $\chi(M) = 0$. Note that this is equivalent to $\chi(\partial M) = 0$. Then M admits a handlebody decomposition with 1 0-handle, m 1-handles, $m-1$ 2-handles, and 0 3-handles. This is dual to a handlebody decomposition of $(M, \partial M)$ which has 0 0-handles, $m-1$ 1-handles, m 2-handles, and 1 3-handle. Now $\chi(\partial M) = 0$ is satisfied if all of the boundary components of M are tori, but if one of the components of ∂M is not a torus, then at least one of the components must be homeomorphic to a 2-sphere. However, if that is the case, then the torsion is not interesting.

Proposition 1.1. *Let M be a compact connected oriented 3-manifold satisfying $\chi(M) = 0$ and ∂M containing at least one component homeomorphic to S^2. Then for any Euler structure e and homology orientation ω,*

$$\tau(M, e, \omega) = 0.$$

Proof. We will prove that for any $\varphi : \mathbb{Z}[H_1(M)] \to F$, $H_2^\varphi(M) \neq 0$, hence $C_*^\varphi(M)$ is not acyclic, and thus $\tau^\varphi(M, e, \omega) = 0$. We shall prove this by showing that there is a 2-handle of M the core 2-cell of which has nullhomotopic boundary map, which means that any lift of that cell to $\hat{M}$ has boundary equal to zero.

This implies that the boundary of the associated basis element in $C_2^{\varphi}(M)$ is also zero. Since there are no 3-handles, we can conclude $H_2^{\varphi}(M) \neq 0$.

First, note that the long exact sequence of the pair $(M, \partial M)$ tells us that $H_1(M, \partial M) \approx \mathrm{im}(H_1(M)) \oplus \mathbb{Z}^{\ell-1}$, where ℓ is the number of boundary components. The $\mathbb{Z}^{\ell-1}$ summand is generated by paths connecting distinct boundary components. We can explicitly see this by sliding handles in the relative decomposition of $(M, \partial M)$. Choose a base point in some boundary component; for convenience, choose a component other than the S^2 guaranteed by our assumption (there must be other boundary components since $\chi(M) = 0$). Let $*$ denote the base point chosen, and let $(\partial M)_*$ denote the boundary component containing $*$. Then since M is path-connected, and each path must be homotopic to a path factoring through the relative 1-skeleton, the relative 1-skeleton is path-connected, hence there must be 1-handles connecting the boundary components. In particular, there is at least a 1-handle connecting $(\partial M)_*$ to another boundary component. If we consider all of the components connected to $(\partial M)_*$ by a minimum of k 1-handles as components in the "k^{th} level" then we know that the first level is nonempty. We will now modify our handlebody structure so that every component is in the first level. If every boundary component is already in the first level, then we have nothing to do. Otherwise, the second level is nonempty. Then any component (say Σ') in the second level is connected to a component in the first level (say Σ) by one 1-handle, and Σ is connected to $(\partial M)_*$ by one 1-handle as well. Then slide the handle connecting Σ' to Σ along the handle connecting Σ to $(\partial M)_*$ to put Σ' in the first level. Then any third level components connected to Σ' are now second level. Proceed until all boundary components are first level. Then slide all other 1-handles along the paths connecting boundary components to $(\partial M)_*$ to get loops based at $*$. Then we have $\ell - 1$ 1-handles connecting the $\ell - 1$ boundary components other than $(\partial M)_*$ to $(\partial M)_*$, and the rest representing loops in M based at $*$, so we can explicitly see the decomposition of $H_1(M, \partial M)$ as the direct sum as given above. Now consider the decomposition of M dual to our new relative decomposition; in particular, notice that the boundary of the core 2-cell of the 2-handle dual to the relative 1-handle which connects $(\partial M)_*$ to our given S^2 is nullhomotopic, since it is freely homotopic to a loop on S^2. The result follows by the comments above. □

This means that the only interesting 3-manifolds with boundary (from the viewpoint of torsion) have each boundary component homeomorphic to a torus.

Now we give a computation for the Turaev torsion of a 3-manifold with nonempty boundary. We will choose a specific Euler structure by choosing a specific fundamental family of cells.

Let M be a 3-manifold with nonempty boundary, and we may as well assume each boundary component homeomorphic to a torus (the theorem below is true but trivial if not). Then, as above, M has a handlebody decomposition with 1 0-handle and 0 3-handles, and since $\chi(M) = 0$ it has one fewer 2-handle than it has 1-handles, so if M has m 1-handles, then it has $m - 1$ 2-handles. The core cells give us a presentation of $\pi = \pi_1(M)$ with m generators $g_1, \ldots, g_m$ and $m - 1$ relations $r_1, \ldots, r_{m-1}$ in the free group generated by $g_1, \ldots, g_m$. To choose a fundamental family of cells in $\widetilde{M}$, first choose any 0-handle $*$ lying over the 0-handle of M. Then choose lifts of the 1-handles of M so that one of the endpoints of the core 1-cells of each lift is the core 0-cell of $*$. Then the $\mathbb{Z}[\pi]$-boundary map of $C_*(\widetilde{M})$ from dimension 1 to dimension 0 is given by right multiplication by the column $\begin{pmatrix} g_1-1 \\ g_2-1 \\ \vdots \\ g_m-1 \end{pmatrix}$. Now we may homotope the boundary maps of the core 2-cells of the 2-handles of M so that they are based maps, and then choose our lifts of 2-handles in $\widetilde{M}$ so that they are all based at $*$ and the $\mathbb{Z}[\pi]$-boundary map of $C_*(\widetilde{M})$ from dimension 2 to dimension 1 is given by right multiplication by the matrix $\widetilde{\Delta}$ where $\widetilde{\Delta}_{i,j} = \partial r_i/\partial g_j$ is the matrix of Fox derivatives.

Now let $H = H_1(M)$, then the $\mathbb{Z}[H]$-complex of $\hat{M}$ is simply the projection of the above complex under the Hurewicz map $p : \pi \to H$ (or, more precisely, the induced map $\mathbb{Z}[\pi] \to \mathbb{Z}[H]$). So if we let $h_i = p(g_i)$ and $\Delta_{i,j} = p\left(\widetilde{\Delta}_{i,j}\right)$, then the complex for $\hat{M}$ is simply

$$(\mathbb{Z}[H])^{m-1} \xrightarrow{\Delta} (\mathbb{Z}[H])^m \xrightarrow{\begin{pmatrix} h_1-1 \\ h_2-1 \\ \vdots \\ h_m-1 \end{pmatrix}} \mathbb{Z}[H].$$

We are now almost ready to compute the torsion, after we introduce another notation. Henceforth, we will often need to strike a column from a matrix. We will use the notation $A(r)$ for the matrix obtained by striking the r^{th} column from the matrix A.

Theorem 1.1. *Let M be a 3-manifold with boundary $\partial M \neq \varnothing$ with handlebody decomposition as above and the Euler structure e equal to the class of the fundamental family of cells above, and $\chi(M) = 0$. Then for any homology orientation ω_M, and for any $1 \leq r \leq m$,*

$$\tau(M, e, \omega_M)(h_r - 1) = (-1)^{m+r}\tau_0 \det(\Delta(r)). \tag{1.12}$$

Here $\tau_0 = \pm 1$ is the sign of $\tau(C_*(M; \mathbb{R})) \in \mathbb{R} - \{0\}$; this torsion is also computed using the basis determined by our Euler structure and a positively oriented basis in homology with respect to our homology orientation.

Proof. Consider the splitting $Q(H) \approx \bigoplus_i F_i$ as a direct sum of fields, let φ_i be the projection to F_i. It is enough to show that for all i and all $1 \leq r \leq m$,

$$\tau^{\varphi_i}(M, e, \omega_M)\varphi_i(h_r - 1) = (-1)^{m+r}\tau_0\varphi_i(\det(\Delta(r))). \tag{1.13}$$

Note that φ_i is a ring homomorphism which extends in a natural way to matrices and that the extended homomorphism on matrices commutes with determinant and striking out columns, i.e. $\varphi_i(\det(\Delta(r))) = \det(\varphi_i(\Delta(r)))$ and $\varphi_i(\Delta(r)) = (\varphi_i(\Delta))(r)$. Then for each i, we will have the complex

$$F_i^{m-1} \xrightarrow{\varphi_i(\Delta)} F_i^m \xrightarrow{\begin{pmatrix} \varphi_i(h_1-1) \\ \varphi_i(h_2-1) \\ \vdots \\ \varphi_i(h_m-1) \end{pmatrix}} F_i.$$

We now proceed by cases:

1. $\varphi_i(h_r - 1) = 0$. The fact that $C_*^{\varphi_i}(M)$ is a complex gives a linear relation on the columns of $\varphi_i(\Delta)$. However, the identity $\varphi_i(h_r - 1) = 0$ tells us that the r^{th} column is not involved in this relation. Also, the relation is nontrivial, as we now explain: we know at least one of the h_k's is infinite order in H since $b_1(M) \geq 1$ (this is true for any compact 3-manifold with nonempty boundary and Euler characteristic equal to zero). Let us say h_j has infinite order. Then $h_j - 1$ is a unit in $Q(H)$ (h_j is infinite order implies $h_j - 1$ is not a zerodivisor in $\mathbb{Z}[H]$). Then $\varphi_i(h_j - 1) \neq 0$, so the coefficient of the j^{th} column is nonzero, hence we have a nontrivial linear relation on the columns of $\varphi_i(\Delta(r))$, hence $\det(\varphi_i(\Delta(r))) = 0$, so in this case (1.13) holds.

2. $\varphi_i(h_r - 1) \neq 0$, but the complex is not acyclic. This means

$$\tau^{\varphi_i}(M, e, \omega_M) = \tau_0\tau(C_*^{\varphi_i}(M)) = 0$$

by definition. Then let x be the row vector in F_i^m whose r^{th} entry is $(\varphi_i(h_r - 1))^{-1}$ and all other entries are zero. Then $\partial_1 x = 1$, so we have no 0^{th} homology in this complex. Also, $\dim(\ker(\partial_1)) = m - 1$. Our complex is not acyclic, which can only happen if the map $\varphi_i(\Delta)$ is not injective. Then note $\dim(\operatorname{im}(\varphi_i(\Delta(r)))) \leq \dim(\operatorname{im}(\varphi_i(\Delta))) < m - 1$ so $\varphi_i(\Delta(r))$ is not injective, hence $\det(\varphi_i(\Delta(r))) = 0$, so this case also satisfies (1.13).

3. $\varphi_i(h_r - 1) \neq 0$, and the complex is acyclic. This is actually the interesting case, when everything is nonzero. Then $\varphi_i(\Delta(r))$ is injective and we have a vector x as in the previous case; i.e. again let x be the row vector in F_i^m whose r^{th} entry is $(\varphi_i(h_r - 1))^{-1}$ and all other entries are zero. Note x spans a subspace of F_i^m which is a complementary subspace to the image of $\imath_r(\varphi_i(\Delta(r)))$, where $\imath_r : F_i^{m-1} \hookrightarrow F_i^m$ inserts a zero as the r^{th} coordinate. Now to compute the torsion we need to pick bases for the images of the boundary maps; for the image of ∂_2 we will just choose the images of the standard basis of F_i^{m-1}. Also we will choose $1 = \partial_1 x$ for the image of ∂_1. Then the change of basis matrices in the 2 and 0 position will just be the identity matrix, and we just have to figure out the change of basis matrix for the 1 position. This matrix will simply be $\left(\begin{smallmatrix}\varphi_i(\Delta)\\ x\end{smallmatrix}\right)$, i.e. the $(m \times m)$ matrix given by adjoining the row given by x onto the bottom of the matrix $\varphi_i(\Delta)$. Then the torsion is simply given by the determinant of this matrix, which is clearly $(-1)^{m+r}(\varphi_i(h_r - 1))^{-1}\det(\varphi_i(\Delta(r)))$.

In all of these cases,

$$\varphi_i(\tau(M, e, \omega_M)) = \tau_0(-1)^{m+r}(\varphi_i(h_1 - 1))^{-1}\det(\varphi_i(\Delta(r)))$$

$$\varphi_i(\tau(M, e, \omega_M))(\varphi_i(h_1 - 1)) = \tau_0(-1)^{m+r}\det(\varphi_i(\Delta(r))).$$

This proves (1.13) for all i which in turn proves (1.12). □

Chapter 2

Torsion vs. Cohomology for 3-Manifolds

In [Tur02] Chapters III and XII, Turaev describes how the Turaev torsion of closed, oriented, connected 3-manifolds is related to certain "determinants" in cohomology. Explicit formulae are derived for both integral cohomology and Mod-r cohomology for certain $r \geq 2$ (for example primes). In this chapter, we give analogues for (compact, connected, oriented) 3-manifolds with non-void boundary. The general strategy will be to define a purely algebraic determinant for certain forms on free R-modules, where R is some ring (our applications will have $R = \mathbb{Z}$ or $R = \mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$). We then relate the "leading term" of the Turaev torsion to this determinant arising from the form on $H^1(M; R)$ and $H^1(M, \partial M; R)$ defined by the particular cohomology product in which we are interested.

In the following chapter, all manifolds are compact, connected, oriented 3-manifolds with non-void boundary.

2.1 The Integral Cohomology Ring

2.1.1 Determinants

Let M be a 3-manifold with boundary $\partial M \neq \varnothing$, and suppose $\chi(M) = 0$. Also assume $b_1(M) \geq 2$ (b_1 denotes the first Betti number) so that $H^1(M, \partial M) \neq 0$. We now have a map $H^1(M, \partial M) \times H^1(M) \times H^1(M) \longrightarrow \mathbb{Z}$, defined by $(a, b, c) \mapsto \langle a \cup b \cup c, [M] \rangle$, where $[M]$ is the fundamental class in $H_3(M, \partial M)$ determined by the orientation. This is alternate in the sense that switching the last two variables costs a minus sign, i.e. $\langle a \cup b \cup c, [M] \rangle = -\langle a \cup c \cup b, [M] \rangle$. Since we assume $\chi(M) = 0$, we know $H^1(M)$ and $H^1(M, \partial M)$ will not have the same rank; they will differ by one. There is a notion for the determinant (see [Tur02], chapter III) of an alternate trilinear form (for example, the obvious analogue of the above form when M is closed), but because of the difference in rank, we must have a new concept of determinant for a mapping such as the one above. The determinant of an alternate trilinear form on a free R-module is independent of basis up to squares of units of R, so if $R = \mathbb{Z}$ it is independent of basis. This will not be true of our determinant; however we will present a sign-refined version based on a choice of homology orientation. For our usage, this is not more of a choice than we would normally make; if we want sign-refined torsion, then we have already chosen a homology orientation, and if we do not care about the sign of the torsion, we can ignore the sign here as well.

In more general terms, let R be a commutative ring with unit, and let K, L be finitely generated free R modules of rank n and $n-1$ respectively, where $n \geq 2$. For any module N, we can define the symmetric graded algebra $S(N) = \bigoplus_{\ell \geq 0} S^\ell(N)$ where $S^\ell(N)$ is the quotient of $\overbrace{N \otimes N \otimes \cdots \otimes N}^{\ell \text{ copies}}$ by the action of the symmetric group on ℓ objects. We note $S^0(N) = R$ and $S^1(N) = N$ to be precise. Multiplication in $S(N)$ is the image of tensor multiplication. For our purposes, we will let $S = S(K^*)$ where $K^* = \mathrm{Hom}_R(K, R)$. Note if $\{a_i^*\}_{i=1}^n$ is the basis of K^* dual to the basis $\{a_i\}_{i=1}^n$ of K then $S = R[a_1^*, \ldots, a_n^*]$, the polynomial ring on $a_1^*, \ldots, a_n^*$, and the grading of S corresponds to the usual grading of a polynomial ring. So now let $\{a_i\}_{i=1}^n, \{b_j\}_{j=1}^{n-1}$ be bases for K, L respectively, and let $\{a_i^*\}$ be the basis of K^* dual to the basis $\{a_i\}$ as above. Let $f : L \times K \times K \longrightarrow R$ be an R-module homomorphism which is skew-symmetric in the two copies of K; i.e. for all $y, z \in K, x \in L, f(x, y, z) = -f(x, z, y)$. Let g denote the associated homomorphism $L \times K \longrightarrow K^*$ given by $(g(x, y))(z) = f(x, y, z)$. Next we state a Lemma defining the *determinant* of f (d in the Lemma), but first we recall some notation from Chapter 1: $[a'/a] \in R^\times$ is used to denote the determinant of the change of basis matrix from a to a', and for a matrix A, we will let $A(i)$ denote the matrix obtained by striking out the i^{th} column

Lemma 2.1. *Let θ denote the $(n-1 \times n)$ matrix over S whose i, j^{th} entry, denoted $\theta_{i,j}$, is obtained by $\theta_{i,j} = g(b_i, a_j)$. Then there is a unique $d = d(f, a, b) \in S^{n-2}$ such that for any $1 \leq i \leq n$,*

$$\det\ \theta(i) = (-1)^i a_i^* d. \tag{2.1}$$

For any other bases a', b' of K, L respectively, we have

$$d(f, a', b') = [a'/a][b'/b]d(f, a, b). \tag{2.2}$$

Proof. Let β denote the $(n - 1 \times n)$ matrix with $\beta_{i,j} = g(b_i, a_j)a_j^*$. The sum of the columns of β is zero; indeed, for any i, the i^{th} entry (of the column vector obtained by summing the columns of β) is given by:

$$\sum_{j=1}^{n} \beta_{i,j} = \sum_{j=1}^{n} g(b_i, a_j)a_j^* = \sum_{j,k=1}^{n} f(b_i, a_j, a_k)a_j^* a_k^* = 0.$$

The last equality follows since the f term is anti-symmetric in j, k and the a term is symmetric. We now claim $(-1)^i \det \beta(i)$ is independent of i.

Claim. *Let Z be a $(n-1 \times n)$ matrix with columns c_i for $1 \le i \le n$ such that $\sum_{i=1}^{n} c_i = 0$. Then $(-1)^i \det(Z(i))$ for $1 \le i \le n$ is independent of i.*

The proof of this claim is as follows: think of det as a function on the columns; $\det(Z(i)) = \det(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n)$. Let $k \neq i$, then $c_k = -\sum_{p \neq k} c_p$, hence

$$\begin{aligned}
\det(Z(i)) &= \det(c_1, c_2, \ldots, c_{k-1}, -\sum_{p \neq k} c_p, c_{k+1}, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \\
&= \sum_{p \neq k} \det(c_1, c_2, \ldots, c_{k-1}, -c_p, c_{k+1}, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \\
&= \det(c_1, c_2, \ldots, c_{k-1}, -c_i, c_{k+1}, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n).
\end{aligned}$$

Here for notational convenience we have assumed $k < i$, but it clearly makes no difference. The last equality holds because in each term but the $p = i$ term, we will have two columns appearing twice. Now to move $-c_i$ to the i^{th} column, we will have to do $i - k - 1$ column swaps. Doing the column swaps and accounting for the negative sign of c_i, we get $\det(Z(i)) = (-1)^{i-k} \det(Z(k))$, which completes the proof of our claim.

This means $(-1)^i \det \beta(i)$ is independent of i. Now let $t_i = \det \theta(i) \in S^{n-1}$. It is clear that

$$\det(\beta(i)) = t_i \prod_{k \neq i} a_k^*.$$

Then for any $i, p \le n$, we have

$$\begin{aligned}
(-1)^i t_i a_p^* \prod_{k=1}^{n} a_k^* &= (-1)^i \det \beta(i) a_p^* a_i^* \\
&= (-1)^p \det \beta(p) a_i^* a_p^* \\
&= (-1)^p t_p a_i^* \prod_{k=1}^{n} a_k^*.
\end{aligned}$$

Now since the annihilators of a_k^* in S are zero, we must have

$$(-1)^i t_i a_p^* = (-1)^p t_p a_i^*.$$

This means that a_i^* divides $t_i a_p^*$ for all p, hence a_i^* divides t_i. Define s_i by $t_i = s_i a_i^*$. Note

$$(-1)^i s_i a_i^* a_p^* = (-1)^i t_i a_p^* = (-1)^p t_p a_i^* = (-1)^p s_p a_p^* a_i^*.$$

This means $(-1)^i s_i$ is independent of i. Let $d = (-1)^i s_i$. By definition,

$$(-1)^i \det \theta(i) = (-1)^i t_i = (-1)^i s_i a_i^* = a_i^* d.$$

This proves (2.1).

Now to prove the change of basis formula, note we do not have to change both bases simultaneously, but can instead first obtain the formula for $d(f, a', b)$ in terms of $d(f, a, b)$, and then do the same for b' and b. So let $\{a_i'\}$ be another basis for K. We show $d(f, a', b) = [a'/a]d(f, a, b)$. Let S_i be the $(n \times n-1)$ matrix obtained by inserting a row of zeroes into the $(n-1 \times n-1)$ identity matrix as the i^{th} row. Then one may easily see for any $(n-1 \times n)$ matrix A, the matrix $A(i)$ (obtained by striking out the i^{th} column) can also be obtained as $A(i) = AS_i$. Let S_i^+ denote the $(n \times n)$ matrix obtained by appending a column vector with a 1 in the i^{th} entry and zeroes otherwise on to the right of S_i, and let A_+^i denote the $(n \times n)$ matrix obtained by appending a row vector with a 1 in the i^{th} entry and zeroes otherwise on to the bottom of A. Note

$$\det(S_i^+) = (-1)^{n+i}$$

$$\det(A_+^i) = (-1)^{n+i} \det(A(i))$$

hence

$$\det(A_+^i S_i^+) = \det(AS_i) = \det(A(i)).$$

Now let (a'/a) denote the usual change of basis matrix so that $a'_i = \sum_{j=1}^{n} (a'/a)_{i,j} a_j$. Now $\theta_{i,j} = g(b_i, a_j)$, so let

$$\begin{aligned}
\theta'_{i,j} &= g(b_i, a'_j) \\
&= g(b_i, \sum_{k=1}^{n} (a'/a)_{j,k} a_k) \\
&= \sum_{k=1}^{n} g(b_i, a_k)(a'/a)_{j,k}.
\end{aligned}$$

Thus $\theta' = \theta \cdot (a'/a)^{\mathsf{T}}$. Now

$$\begin{aligned}
\det\left(\theta_+^i (a'/a)^{\mathsf{T}} S_i^+\right) &= \det\left(\theta_+^i\right) \det\left((a'/a)^{\mathsf{T}}\right) \det\left(S_i^+\right) \\
&= \det\left((a'/a)^{\mathsf{T}}\right) \det\left(\theta_+^i\right) \det\left(S_i^+\right) \\
&= \det\left(a'/a\right) \det\left(\theta_+^i \cdot S_i^+\right) \\
&= [a'/a] \det(\theta(i)) \\
&= [a'/a](-1)^i a_i^* d(f, a, b).
\end{aligned}$$

Now we will compute the same thing in a much longer way to complete our proof. Let e_i denote the row vector with a 1 in the i^{th} position and zeroes otherwise, i.e. the i^{th} basis vector of a as expressed in the a-basis, and let r_i denote the i^{th} row of $(a'/a)^{\mathsf{T}}$ and c_i denote the i^{th} column. Then

$$\begin{aligned}
\det(\theta_+^i (a'/a)^{\mathsf{T}} S_i^+) &= \det\left[\left(\begin{smallmatrix}\theta\\ e_i\end{smallmatrix}\right)(a'/a)^{\mathsf{T}}\left(\begin{smallmatrix}S_i & e_i^{\mathsf{T}}\end{smallmatrix}\right)\right]\\
&= \det\left[\left(\begin{smallmatrix}\theta\\ e_i\end{smallmatrix}\right)\left(\begin{smallmatrix}(a'/a)^{\mathsf{T}}(i) & c_i\end{smallmatrix}\right)\right]\\
&= \det\begin{pmatrix}\theta(a'/a)^{\mathsf{T}}(i) & \theta c_i\\ (a'/a)^{\mathsf{T}}(i)_i & (a'/a)^{\mathsf{T}}_{i,i}\end{pmatrix}\\
&= (-1)^{n-i}\det\left(\begin{smallmatrix}\theta(a'/a)^{\mathsf{T}}\\ r_i\end{smallmatrix}\right)\\
&= (-1)^{n-i}\det\left(\begin{smallmatrix}\theta'\\ r_i\end{smallmatrix}\right)\\
&= (-1)^{n-i}\sum_{k=1}^{n}(-1)^{n+k}(a'/a)^{\mathsf{T}}_{i,k}\det(\theta'(k))\\
&= (-1)^{n-i}\sum_{k=1}^{n}(-1)^{n+k}(a'/a)^{\mathsf{T}}_{i,k}(-1)^k(a'_k)^* d(f,a',b)\\
&= (-1)^{n-i}\sum_{k=1}^{n}(-1)^{n+k}(a^*/(a')^*)_{i,k}(-1)^k(a'_k)^* d(f,a',b)\\
&= (-1)^i d(f,a',b)a_i^*.
\end{aligned}$$

So $d(f,a',b) - [a'/a]d(f,a,b)$ annihilates a_i^* for each i, hence is zero.

The computation for a b change of basis is easier. Let b' be another basis for L and let (b'/b) denote the b to b' change of basis matrix. Let θ' denote the matrix $g(b'_i, a_j)$, then $\theta' = (b'/b)\theta$. So $\theta' S_i = (b'/b)\theta S_i$, hence $\det(\theta'(i)) = [b'/b]\det(\theta(i))$. This proves $d(f,a,b') = [b'/b]d(f,a,b)$, and completes the proof of (2.2). □

In the case, $R = \mathbb{Z}$, our determinant depends on the basis only by its sign. In this case, we can refine the determinant by a choice of orientation of the $\mathbb{R}$-vector space $(K \oplus L) \otimes \mathbb{R}$. Let ω be such a choice of orientation. Then define $\mathrm{Det}_\omega(f) = \det(f,a,b)$ where a, b are bases of K, L respectively such that the induced basis of $(K \oplus L) \otimes \mathbb{R}$ given by $\{a_1 \otimes 1, a_2 \otimes 1, \ldots, a_n \otimes 1, b_1 \otimes 1, b_2 \otimes 1, \ldots, b_{n-1} \otimes 1\}$ is positively oriented with respect to ω. Then $\mathrm{Det}_\omega(f)$ is well defined, and for any bases a', b', we have $\det(f,a',b') = \pm\,\mathrm{Det}_\omega(f)$ where the $\pm$ is chosen depending on whether a', b' induces a positively or negatively oriented basis of $(K \oplus L) \otimes \mathbb{R}$ with respect to ω. Note that for $K = H^1(M)$, $L = H^1(M, \partial M)$ where M is a compact connected oriented 3-manifold with non-void boundary, a choice of homology orientation will determine an orientation for $(K \oplus L) \otimes \mathbb{R}$. To see why, let ω_M be a homology orientation for M. Consider $\{a_1^*, \ldots, a_n^*\}$ a basis for $H^1(M; \mathbb{R})$ dual to a basis $\{a_1, \ldots, a_n\}$ of $H_1(M, \mathbb{R})$, and $\{b_1^*, \ldots, b_{n-1}^*\}$ a basis of $H^1(M, \partial M; \mathbb{R})$. We will say what it means for $\{a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_{n-1}^*\}$ to be a positively oriented basis for $H^1(M; \mathbb{R}) \oplus H^1(M, \partial M; \mathbb{R})$, and this will define our orientation. Let $[M]$ denote the fundamental class of M determined by the orientation of M (not the homology orientation). Then we will define an orientation of $H^1(M; \mathbb{R}) \oplus H^1(M, \partial M; \mathbb{R})$ by saying that $\{a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_{n-1}^*\}$ is a positively oriented basis if and only if $\{[\mathrm{pt}], a_1, \ldots, a_n, b_1^* \cap [M], \ldots, b_{n-1}^* \cap [M]\}$ is a positively oriented basis for $H_*(M; \mathbb{R})$ with respect to ω_M. We will denote the sign refined determinant with respect to this orientation by $\mathrm{Det}_{\omega_M}(f)$ (Note this is essentially the same thing as refining Det by the paired volume form associated to ω_M, as defined below in (2.11)).

2.1.2 Relationship to Torsion

We use the above to relate the torsion to the cohomology ring structure. Let $T = \mathrm{Tors}(H_1(M))$ denote the torsion subgroup of $H_1(M)$. Note that this is isomorphic to the torsion subgroup of $H_1(M, \partial M)$, so we will also denote the torsion subgroup of $H_1(M, \partial M)$ by T. Let $G = H_1(M)/T$, let $S(G)$ denote the graded symmetric algebra on G and let I denote the augmentation ideal in $\mathbb{Z}[H_1(M)]$. The filtration of $\mathbb{Z}[H_1(M)]$ by powers of I determines an associated graded algebra $A = \bigoplus_{\ell \geq 0} I^\ell / I^{\ell+1}$. Then there is an additive homomorphism $q_{H_1(M)}$:

$S(G) \longrightarrow A$ defined in [Tur02]. We repeat the definition here: The map $h \mapsto h - 1 \bmod I^2$ defines an additive homomorphism $H_1(M) \longrightarrow I/I^2$. This extends to a grading-preserving algebra homomorphism $\widetilde{q}_{H_1(M)} : S(H_1(M)) \longrightarrow A$. Any section $s : G \longrightarrow H_1(M)$ of the natural projection $H_1(M) \longrightarrow G$ induces an algebra homomorphism $\widetilde{s} : S(G) \longrightarrow S(H_1(M))$; set

$$q_{H_1(M)} = |T| \widetilde{q}_{H_1(M)} \widetilde{s} : S(G) \longrightarrow A.$$

Then $q_{H_1(M)}$ is grading preserving and is a $\mathbb{Z}$-module homomorphism, and obviously satisfies the multiplicative formula

$$q_{H_1(M)}(a) q_{H_1(M)}(b) = |T| q_{H_1(M)}(ab).$$

$q_{H_1(M)}$ does not depend on the choice of section s (see [Tur02]).

We are now ready to state the main result of this section:

Theorem 2.1. *Let $f_M : H^1(M, \partial M) \times H^1(M) \times H^1(M) \longrightarrow \mathbb{Z}$ be the $\mathbb{Z}$-module homomorphism defined by*

$$f_M(x, y, z) = \langle x \cup y \cup z, [M] \rangle.$$

Let $n = b_1(M) \geq 2$, let I be the augmentation ideal of $\mathbb{Z}[H_1(M)]$, and let e be any choice of Euler structure on M and ω_M be a homology orientation of M. Then $\tau(M, e, \omega_M) \in I^{n-2}$ and:

$$\tau(M, e, \omega_M) \bmod I^{n-1} = q_{H_1(M)}(\mathrm{Det}_{\omega_M}(f_M)) \in I^{n-2}/I^{n-1}. \qquad (2.3)$$

That $\tau(M, e, \omega_M) \in I^{n-2}$ is proved in [Tur02], Chapter II, the important thing here is its image modulo I^{n-1}; this is the "leading term" of the torsion in the associated graded algebra A. This proof is the method of [Tur02] Theorem 2.2 applied to this more general situation.

Proof. The first step is to arrange a handle decomposition coming from a C^1 triangulation to be in a convenient form. We will also arrange the relative handle decomposition for $(M, \partial M)$ that is Poincaré dual to the handle decomposition for M. First, we arrange our decomposition for M so that we have (0) 3-handles, $(m-1)$ 2-handles, (m) 1-handles, (1) 0-handle, and this is Poincaré dual to a relative handle decomposition for $(M, \partial M)$ with (0) 0-handles, $(m-1)$ 1-handles, (m) 2-handles, (1) 3-handle. With these decompositions, we have the following cellular chain complexes:

$$\begin{array}{rccccccc} C_*(M): & 0 & \longrightarrow & \mathbb{Z}^{m-1} & \longrightarrow & \mathbb{Z}^m & \xrightarrow{0} & \mathbb{Z} \\ & \downarrow\approx & & \downarrow\approx & & \downarrow\approx & & \downarrow\approx \\ C_*(M,\partial M): & 0 & \longleftarrow & \mathbb{Z}^{m-1} & \longleftarrow & \mathbb{Z}^m & \xleftarrow[0]{} & \mathbb{Z}. \end{array}$$

We will refer to the handles as "honest" handles and "relative" handles; honest handles being from the decomposition of M and relative ones from the relative decomposition of $(M, \partial M)$. Later, we will explicitly give the $(m-1 \times m)$ matrix for ∂_2 of the honest decomposition.

The core 0-cell of the honest 0-handle (of M) is a point, u, which we will say is positively oriented. At the same time we orient the relative 3-handle (of $(M, \partial M)$) with the positive orientation given by the orientation of M. Extend the core 1-disks of the honest 1-handles to obtain loops in M based at u, representing $x_1, \ldots, x_m \in \pi_1(M, u)$. We can arrange these to be convenient by sliding handles over each other and possibly reversing orientations of the core disks. Since sliding the i^{th} honest 1-handle over the j^{th} honest 1-handle replaces x_i with $x_i x_j$, and reversing orientation of the core 1-disk changes replaces x_i with x_i^{-1}, we may assume that the images of the homology classes of the first n of the x_i's form a basis of $G = H_1(M)/T$ and the rest of the classes end up in T. For $i = 1, \ldots, m$, set $h_i = [x_i] \in H_1(M)$ and $\widetilde{h_i} = h_i \bmod T$. Thus $\widetilde{h_1}, \ldots, \widetilde{h_n}$ is a basis of G and

$\widetilde{h}_i = 1$ for $i > n$. Denote the dual basis of $H^1(M)$ by $h_1^*, \ldots, h_n^*$, by definition, $\langle h_i^*, \widetilde{h}_j \rangle = \delta_{i,j}$, where $\langle \cdot, \cdot \rangle$ is evaluation pairing.

We now want to arrange the relative 1-handles in a similar way; so that the images of the first $n-1$ of them form a basis of $H_1(M, \partial M)/T$ and the other $m-n$ of them end up in the torsion group. Let c denote the number of components of ∂M, proceed as in Proposition 1.1 to get the first $c-1$ of the relative handles connecting boundary components and the rest represent loops with a common base point in one of the boundary components. Then we may proceed as before in the discussion of honest handles; we may arrange so that the first $n-c$ of these handles will give us the remaining free generators of $H_1(M, \partial M)/T$ and the rest of them simply end up in T (again by sliding handles, since the only handles that we need to slide represent loops all based at the same point). We will use similar notation, k_i will denote the homology class of the i^{th} handle and $\widetilde{k}_i = k_i \bmod T$. We will denote the dual basis of $H^1(M, \partial M)$ by $k_1^*, \ldots, k_{n-1}^*$. As before, the $\widetilde{k}_i$'s for $i \leq n-1$ are generators of $H_1(M, \partial M)/T$ and for $i > n-1, \widetilde{k}_i = 1$. Also, as before, $\langle k_i^*, \widetilde{k}_j \rangle = \delta_{i,j}$.

The attaching maps for the honest 2-handles determine (up to conjugation) certain elements $r_1, \ldots, r_{m-1}$ of the free group F generated by $x_1, \ldots, x_m$. We now have $\pi_1(M)$ presented by the generators $x_1, \ldots, x_m$ and the relations $r_1, \ldots, r_{m-1}$.

Now the cellular chain complex for M is in a particularly convenient form for our purposes. As usual, we use the notation ∂_p to denote the boundary map from dimension p to dimension $p-1$. Clearly ∂_1 is given by the zero map. Let us denote the matrix of ∂_2 by $(v_{i,j})$ where $1 \leq i \leq m-1$ and $1 \leq j \leq m$.

Now for $1 \leq i \leq n-1$, the core 2-disk of the i^{th} honest 2-handle represents a cycle in $C_2(M)$ (we have arrange for its homology class to be $k_i^* \cap [M] \in H_2(M)$).

As a homology class, it has boundary equal to zero, so $v_{i,j} = 0$ for $i \le n-1$ and all j. We apply the same argument to the relative handles as follows: the j^{th} relative 2-handle represents a homology class Poincaré dual to $h_i^* \cap [M]$, hence has boundary equal to zero, and $v_{i,j} = 0$ for all i and $j \le n$. The result is that $v_{i,j} = 0$ except for the bottom right hand $(m-n \times m-n)$ corner of the matrix; call this matrix v. This tells us that ∂_2 in the complex for M is given by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & v \end{smallmatrix}\right)$. This v is a square presentation matrix for the torsion group T, thus $\det(v) = \pm|T|$. Furthermore, $r_1, \ldots, r_{n-1} \in [F, F]$ since the first $n-1$ honest 2-cells are cycles.

Consider the chain complex $C_*(\widehat{M})$ associated to the induced handle decomposition of the maximal abelian cover $\widehat{M}$ of M. This is a free $\mathbb{Z}[H_1(M)]$-chain complex with distinguished basis determined by lifts of handles of M. For an appropriate choice of these lifts, we have (as before in Theorem 1.1) ∂_1 given by $x \mapsto x \cdot w$ where w is a column of height m whose i^{th} entry is $h_i - 1$. The map ∂_2 is (also as before in Theorem 1.1) given by the Alexander-Fox matrix for the presentation $\langle x_1, \ldots, x_m | r_1, \ldots, r_{m-1} \rangle$ for an appropriate choice of the r_i's in their conjugacy classes. This is an $(m-1 \times m)$ matrix whose $(i,j)^{\text{th}}$ entry is given by $\eta(\partial r_i / \partial x_j)$ where η is the projection $\mathbb{Z}[F] \longrightarrow \mathbb{Z}[\pi_1(M, u)] \longrightarrow \mathbb{Z}[H_1(M)]$. Let e_N be an Euler structure determined by the fundamental family of cells which gives this "nice" cellular structure to $\widehat{M}$. Clearly the $\mathbb{Z}[H_1(M)]$-complex for $\widehat{M}$ must augment to the $\mathbb{Z}$-complex for M, hence $\text{aug}(\eta(\partial r_i/\partial x_j)) = v_{i,j}$, hence $\eta(\partial r_i/\partial x_j) \in I$ for $i \le n-1, j \le n$. We claim for $i \le n-1, j \le n$,

$$|T|\eta(\partial r_i/\partial x_j) = -|T| \sum_{p=1}^{n} \langle k_i^* \cup h_j^* \cup h_p^*, [M] \rangle (h_p - 1) \bmod I^2. \tag{2.4}$$

Here I is the augmentation ideal. To see this, let $\widetilde{\eta}$ denote the composition of η with the projection $\mathbb{Z}[H_1(M)] \longrightarrow \mathbb{Z}[G]$. Let J denote the augmentation ideal in

$\mathbb{Z}[G]$. It is enough to show for $i \leq n-1, j \leq n$,

$$\widetilde{\eta}(\partial r_i/\partial x_j) = -\sum_{p=1}^{n} \langle k_i^* \cup h_j^* \cup h_p^*, [M] \rangle (\widetilde{h}_p - 1) \bmod J^2. \tag{2.5}$$

To prove (2.5), note that J/J^2 is isomorphic to the free abelian group G of rank n under the map $g \mapsto (g-1) \bmod J^2$, and is thus generated by $\widetilde{h}_1 - 1, \ldots, \widetilde{h}_n - 1$. For any $g \in G$, the expansion $g = \prod_{p=1}^{n} \widetilde{h}_p^{\langle h_p^*, g \rangle}$ gives

$$g - 1 = \sum_{p=1}^{n} \langle h_p^*, g \rangle (\widetilde{h}_p - 1) \bmod J^2. \tag{2.6}$$

Also, for any $\alpha \in F, j \leq n$,

$$\mathrm{aug}(\partial \alpha/\partial x_j) = \langle h_j^*, \eta(\alpha) \rangle. \tag{2.7}$$

Now $r_i \in [F, F]$ gives an expansion $r_i = \prod_{\mu} [\alpha_\mu, \beta_\mu]$ a finite product of commutators in F. Then

$$\eta(\partial r_i/\partial x_j) = \sum_{\mu} (\eta(\alpha_\mu) - 1)\eta(\partial \beta_\mu/\partial x_j) + (1 - \eta(\beta_\mu))\eta(\partial \alpha_\mu/\partial x_j).$$

Projecting to $\mathbb{Z}[G]$ we get

$$\widetilde{\eta}(\partial r_i/\partial x_j) \bmod J^2 = \\ \sum_{p=1}^{n} \left(\sum_{\mu} \langle h_p^*, \eta(\alpha_\mu) \rangle \langle h_j^*, \eta(\beta_\mu) \rangle - \langle h_p^*, \eta(\beta_\mu) \rangle \langle h_j^*, \eta(\alpha_\mu) \rangle \right) (\widetilde{h}_p - 1). \tag{2.8}$$

Now we consider the handlebody $U \subset M$ formed by the (honest) 0-handle and the (honest) 1-handles. The boundary circle of the i^{th} 2-handle lies in ∂U and represents r_i. The expansion $r_i = \prod_{\mu} [\alpha_\mu, \beta_\mu]$ tells us that the circle bounds a singular surface Σ_i', in U with meridians and longitudes homotopic to the α_μ's and β_μ's respectively. Let Σ_i be Σ_i' capped with the core disk of the i^{th} 2-handle. The orientation of the disk extends to an orientation of Σ_i and the fundamental class $[\Sigma_i]$ is represented in the chain complex for M by the core disk of the i^{th} 2-handle, hence $[\Sigma_i] = k_i^* \cap [M]$. Now for any 1-cohomology classes t_i, t_i' of Σ_i, we have

$$\langle t_i \cup t_i', [\Sigma_i]\rangle = \sum_{\mu} \langle t_i, \alpha_\mu\rangle\langle t_i', \beta_\mu\rangle - \langle t_i, \beta_\mu\rangle\langle t_i', \alpha_\mu\rangle. \tag{2.9}$$

Restricting h_j^* to Σ_i we get a 1-cohomology class whose evaluations on the meridians and longitudes are $\langle h_j^*, \eta(\alpha_\mu)\rangle$ and $\langle h_j^*, \eta(\beta_\mu)\rangle$ respectively. This proves (everything $\operatorname{mod} J^2$)

$$\begin{aligned}
\widetilde{\eta}(\partial r_i/\partial x_j) &= \sum_{p=1}^{n} \langle h_p^* \cup h_j^*, [\Sigma_i]\rangle(\widetilde{h}_p - 1) \qquad (2.10)\\
&= \sum_{p=1}^{n} \langle h_p^* \cup h_j^*, k_i^* \cap [M]\rangle(\widetilde{h}_p - 1)\\
&= \sum_{p=1}^{n} \langle h_p^* \cup h_j^* \cup k_i^*, [M]\rangle(\widetilde{h}_p - 1)\\
&= -\sum_{p=1}^{n} \langle k_i^* \cup h_j^* \cup h_p^*, [M]\rangle(\widetilde{h}_p - 1) \bmod J^2.
\end{aligned}$$

This proves (2.5) which in turn proves (2.4).

Recall by [Tur02] II.4.3, we have $\tau(M, e, \omega_M) \in \mathbb{Z}[H_1(M)]$. We have arranged our handles so that h_1 in particular has infinite order in $H_1(M)$, so by (1.12), we have

$$(h_1 - 1)\tau(M, e_N, \omega_M) = (-1)^{m+1}\tau_0 \det(\Delta(1)).$$

Recall e_N is chosen so that we may use (1.12). We now want to work out τ_0. For now we work in a very specific homology basis:

$$\{[\mathrm{pt}], h_1, \ldots, h_n, k_1^* \cap\ [M], \ldots, k_{n-1}^* \cap\ [M]\}.$$

Later, when we do the $\operatorname{Det}(f)$ calculation, we will use the bases for $H^1(M)$ and $H^1(M, \partial M)$ given by $\{h_1^*, \ldots, h_n^*\}$ and $\{k_1^*, \ldots, k_{n-1}^*\}$ respectively. We arbitrarily chose a homology orientation ω_M earlier; this basis will either be positively oriented or negatively oriented with respect to that choice of orientation. Using this homology basis, we compute $\tau(C_*(M;\mathbb{R})) = (-1)^{|C_*(M)|+n(m-n)} \det v$, where v is defined as above. This is a quick calculation; we may choose our bases of the images of the boundary maps so that the dimension 2 and dimension 0 change of basis matrices are the identity matrices. Then the dimension 1 change of basis matrix will be the block matrix $\left(\begin{smallmatrix} 0 & v \\ \mathrm{id} & 0 \end{smallmatrix}\right)$, where id represents the $(n \times n)$ identity matrix. This has determinant $(-1)^{n(m-n)} \det v$. Another quick calculation gives $|C_*(M)| = (mn + m + n) \bmod 2$. Hence $\tau_0 = \pm(-1)^m \operatorname{sign}(\det v)$ where the $\pm$ is chosen depending on whether our (most recently) chosen homology basis is positively or negatively oriented with respect to ω_M, respectively. This gives

$$(h_1 - 1)\tau(M, e_N, \omega_M) = \pm(-1)^{m+m+1} \operatorname{sign}(\det v) \det(\Delta(1)).$$

Let a denote the submatrix of Δ comprised of the first $n-1$ rows and n columns; thus a is the matrix whose i, j entry is given by $\eta(\partial r_i/\partial x_j)$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Let V denote the lower right hand $(m-n \times m-n)$ matrix $\eta(\partial r_i/\partial x_j)$ for $n \leq i \leq m-1$ and $n+1 \leq j \leq m$. Hence

$$(h_1 - 1)\tau(M, e_N, \omega_M) = \mp|\det v| \det a(1) = \mp|T| \ \det a(1) \bmod I^n.$$

Now the minus sign is chosen if our homology basis was positively oriented and the positive sign is chosen if our homology basis was negatively oriented with respect to our choice of homology orientations. Define

$$\theta_{i,j} = \sum_{p=1}^{n} \langle k_i^* \cup h_j^* \cup h_p^*, [M]\rangle \widetilde{h}_p.$$

Defining $q_{H_1(M)}$ as before, we see

$$(h_1 - 1)\tau(M, e_N, \omega_M) \bmod I^n = \mp q_{H_1(M)}(\det(\theta(1))).$$

Recall $\theta(1)$ denotes the $(n-1 \times n-1)$ matrix obtained by striking out the 1^{st} column of the $(n-1 \times n)$ matrix θ. Again, the minus sign is chosen if our homology basis chosen above was positively oriented with respect to ω_M, and the positive sign is chosen otherwise.

But now $\det(\theta(1)) = \mp \operatorname{Det}_{\omega_M}(f_M)\widetilde{h}_1$ where here the plus is chosen if our homology basis chosen above was negatively oriented with respect to ω_M and the minus is chosen otherwise. Then when we put this together, all of the signs will neatly cancel out, leaving

$$(h_1 - 1)\tau(M, e_N, \omega_M) \bmod I^n = (h_1 - 1)q_{H_1(M)}(\operatorname{Det}_{\omega_M}(f_M)).$$

Then, as in [Tur02], the map $\bigoplus_{\ell \geq 0} I^\ell / I^{\ell+1}$ defined by $x \in I^\ell/I^{\ell+1}$ maps to $(h_1 - 1)x \in I^{\ell+1}/I^{\ell+2}$ is injective, so

$$\tau(M, e_N, \omega_M) \bmod I^{n-1} = q_{H_1(M)}(\operatorname{Det}_{\omega_M}(f_M)).$$

But now recall $\tau(M, e, \omega_M)$ only differs from $\tau(M, e_N, \omega_M)$ by multiplication by an element of $H_1(M)$. They are both in I^{n-2}, so mod I^{n-1} they are equal. This completes the proof. □

2.2 The Cohomology Ring Mod-r

In this section, we will prove an analogous result to the one in Section 2.1 using cohomology modulo an integer r rather than integral cohomology. The integer r will have to be one such that the first cohomology group with Mod-r coefficients is a free $\mathbb{Z}_r$-module; for instance if r is prime. This will also imply that the first relative cohomology group is a free $\mathbb{Z}_r$-module, so we will still be able to compute a determinant as in Section 2.1, however will will need to refine that determinant slightly. To do so, we will first introduce the concept of a *paired volume form*, which will play a similar role to the square volume forms found in [Tur02], III.3

Before anything else, however, let us define the Mod-r torsion. This is defined when $b_1(M) \geq 2$ so that $\tau(M, e, \omega) \in \mathbb{Z}[H_1(M)]$ for any e, ω. Then $\tau(M, e, \omega; r)$ is the image of $\tau(M, e, \omega)$ under the projection $\mathbb{Z}[H_1(M)] \to \mathbb{Z}_r[H_1(M)]$ induced by the coefficient projection $\mathbb{Z} \to \mathbb{Z}_r$. Note that if $r = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$ where $p_1, \ldots, p_k$ are primes, then $\mathbb{Z}_r[H_1(M)]$ splits naturally as $\mathbb{Z}_{p_1^{e_1}}[H_1(M)] \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}[H_1(M)]$ and $\tau(M, e, \omega; r)$ splits as $\tau(M, e, \omega; p_1^{e_1}) + \cdots + \tau(M, e, \omega; p_k^{e_k})$, so understanding Mod-$r$ torsion when r is a power of a prime is sufficient to understand it for any r.

One may also define the Mod-r torsion when $b_1(M) = 1$ by using Turaev's "polynomial part" $[\tau]$ of the torsion; see [Tur02], II.3. Theorem 2.2 is true in this case as well, and one can use the argument in [Tur02] Theorem III.4.3 when the first Betti number is 1 (the last paragraph of the proof).

2.2.1 Determinants

Volume Forms

First we recall some definitions from [Tur02], III.3. If N is a finite rank free module over R, a commutative ring with 1, then a *volume form* ω on N is a map which assigns to each basis a of N a scalar $\omega(a) \in R$ such that $\omega(a) = [a/b]\omega(b)$ for any bases a, b. A *square volume form* is a map Ω which also assigns a scalar to each basis, but the change of basis formula is $\Omega(a) = [a/b]^2\Omega(b)$. Naturally, the square of a volume form is a square volume form. This notion is useful when working with closed manifolds as in [Tur02], III.3, but we must use a slightly different form in the case of a nonempty boundary, though in the same spirit. If K, L are two finite rank free R-modules, then a *paired volume form* on $K \times L$ is a map μ from (ordered) pairs of bases of K and L to R such that $\mu(a', b') = [a'/a][b'/b]\mu(a, b)$ where a, a' are bases of K and b, b' are bases of L. Note that the product of a volume form on K with a volume form on L is a paired volume form on $K \times L$, so this notion is very similar to the notion of a square volume form. We say a paired volume form is *non-degenerate* if its image lies in the units of R, or equivalently if there is a basis a of K and a basis b of L so that $\mu(a, b) = 1$. Note that we may easily construct a non-degenerate paired volume form given distinguished bases a, b of K, L respectively by *assigning* $\mu(a, b) = 1$, and extending to other bases by the change of basis formula.

Note the following properties of paired volume forms:

1. If $B : K \times L \to R$ is a bilinear form, where K and L are isomorphic R-modules, then $\mu(a, b) = \det(B_{a,b})$ is a paired volume form, where $B_{a,b}$ is the matrix of B with respect to the bases a and b of K and L respectively, and μ is non-degenerate if and only if B is a nondegenerate form, i.e. if B induces an isomorphism $K \to \mathrm{Hom}_R(L, R)$.

2. If K, L are free $\mathbb{Z}$-modules of finite rank r_K and r_L respectively, and ω is an orientation on $(K \times L) \otimes \mathbb{R}$, then there is a non-degenerate paired volume form μ_ω on $K \times L$ such that $\mu_\omega(a, b) = 1$ if the basis $a_1 \otimes 1, \ldots, a_{r_K} \otimes 1, b_1 \otimes 1, \ldots, b_{r_L} \otimes 1$ is positively oriented with respect to ω (and obviously μ_ω assigns -1 to bases which are negatively oriented with respect to ω).

3. If $0 \longrightarrow K_1 \longrightarrow K \longrightarrow K_2 \longrightarrow 0$ and $0 \longrightarrow L_1 \longrightarrow L \longrightarrow L_2 \longrightarrow 0$ are short exact sequences of finite rank free R-modules and μ_1, μ_2 are paired volume forms on $K_1 \times L_1, K_2 \times L_2$, then there is an induced paired volume

form on $K \times L$, which is non-degenerate if and only if μ_1 and μ_2 are both non-degenerate. To construct this, let a_i, b_i be bases of K_i, L_i respectively. Then we can construct the bases a_1a_2 and b_1b_2 of K and L respectively by concatenating the image of the basis a_1 with a lift of the basis a_2 in K, and similarly for b_1b_2 in L. Then for any bases a and b of K and L, define $\mu(a, b) = [a/a_1a_2][b/b_1b_2]\mu_1(a_1, b_1)\mu_2(a_2, b_2)$.

4. A non-degenerate paired volume form μ on $K \times L$ induces a non-degenerate paired volume form μ^* on $K^* \times L^* \approx (K \times L)^* = \mathrm{Hom}_R(K \times L, R)$ by $\mu^*(a^*, b^*) = (\mu(a, b))^{-1}$ where a^* is the basis of K^* dual to a basis a of K, and similarly for b, b^*.

5. If $\phi : R \to S$ is a surjection of rings, and μ is a nondegenerate paired volume form on the free R-modules $K \times L$, then there is an induced paired volume form μ_ϕ on $K \otimes_S S \times L \otimes_S S$ given by $\mu_\phi(a \otimes 1, b \otimes 1) = 1$ if a, b are bases of K, L such that $\mu(a, b) = 1$.

1 and 2 are clear, and 4 follows from (1.3). To prove 3, we first note that the constructed μ is clearly non-degenerate if μ_1 and μ_2 are, so let us show that it is well defined (and independent of the bases a_i, b_i). The first step is to notice that it suffices to show that the definition is independent of the choice of the bases K_i and the independence on the L_i bases will follow by symmetry. First we will show that this definition is independent of the lift of a_2 to K. To show this, suppose $\widetilde{a_1a_2}$ is the concatenation of the image of a_1 with another lift of a_2 to K. Then let $\widetilde{\mu}$ be defined using $\widetilde{a_1a_2}$ in the place of a_1a_2 in the definition of μ. To show that $\mu = \widetilde{\mu}$, we actually only need to show that $[\widetilde{a_1a_2}/a_1a_2] = 1$. But note $(\widetilde{a_1a_2}/a_1a_2)$ is a block matrix of the form $\left(\begin{smallmatrix} \mathrm{id} & 0 \\ A & \mathrm{id} \end{smallmatrix}\right)$ where A is some matrix and each "id" is

an identity matrix (we slightly abuse notation here, since they are possibly of different sizes), so $[\widetilde{a_1 a_2}/a_1 a_2] = \det\left(\begin{smallmatrix} \mathrm{id} & 0 \\ A & \mathrm{id} \end{smallmatrix}\right) = 1$. So now we actually know that this definition is independent of the splitting $K \approx K_1 \oplus K_2$. We use this aid in our proof of the independence of the definition of μ on the bases a_1 and a_2. Let α_1, α_2 be bases of K_1, K_2 and let $\mu_\alpha(a,b) = [a/\alpha_1\alpha_2][b/b_1 b_2]\mu_1(\alpha_1, b_1)\mu_2(\alpha_2, b_2)$. Then by the argument above, μ and μ_α are independent of the splitting $K \approx K_1 \oplus K_2$, so we may use the same splitting when we choose the lift of a_2 as when we choose the lift of α_2, i.e. we may arrange so that $(a_1 a_2/\alpha_1\alpha_2)$ is a block matrix of the form $\left(\begin{smallmatrix} (a_1/\alpha_1) & 0 \\ 0 & (a_2/\alpha_2) \end{smallmatrix}\right)$, and then $[a_1 a_2/\alpha_1\alpha_2] = [a_1/\alpha_1][a_2/\alpha_2]$ clearly. Now we compute

$$\begin{aligned}
\mu_\alpha(a,b) &= [a/\alpha_1\alpha_2][b/b_1 b_2]\mu_1(\alpha_1, b_1)\mu_2(\alpha_2, b_2) \\
&= [a/a_1 a_2][a_1 a_2/\alpha_1\alpha_2][b/b_1 b_2]\mu_1(\alpha_1, b_1)\mu_2(\alpha_2, b_2) \\
&= [a/a_1 a_2][b/b_1 b_2][a_1/\alpha_1]\mu_1(\alpha_1, b_1)[a_2/\alpha_2]\mu_2(\alpha_2, b_2) \\
&= [a/a_1 a_2][b/b_1 b_2]\mu_1(a_1, b_1)\mu_2(a_2, b_2) \\
&= \mu(a,b).
\end{aligned}$$

To prove 5, we merely need to note that such bases a, b exist since μ is nondegenerate, and then we may simply define μ_ϕ by the given formula and extend to other bases by the definition of a paired volume form.

The Refined Determinant

Now given free R-modules K, L of finite ranks n and $n-1$ respectively ($n \geq 2$), and given $f : L \times K \times K \to R$ an R-map as in Lemma 2.1, and given a paired volume form μ on $K \times L$, we can construct the μ-refined determinant, $\mathrm{Det}_\mu(f)$, to be

$$\mathrm{Det}_\mu(f) = \mu^*(a^*, b^*)d(f, a, b) \tag{2.11}$$

where d is defined as in Lemma 2.1. We can define this for any bases a, b of K, L respectively (and a^*, b^* the dual bases as usual), but by the properties of d and μ, this is independent of the chosen bases. Note that this will simply be the determinant taken with respect to any bases a, b with $\mu(a, b) = 1$ if such bases exist.

Constructing Paired Volume Forms

We now construct a paired volume form in a particular situation, which will be useful soon. Let H, H' be finite abelian groups which are isomorphic, though we will not fix a particular isomorphism. (These groups will appear later as the torsion groups $\mathrm{Tors}(H_1(M))$ and $\mathrm{Tors}(H_1(M, \partial M))$ which are isomorphic, though not necessarily in any natural way). Let $p \geq 2$ be a prime integer dividing $|H|$. Let $r = p^s$ for some $s \geq 1$ such that H/r is a direct sum of copies of $\mathbb{Z}_r$, so that we can think of H/r as a finite rank free $\mathbb{Z}_r$-module (and similarly for H'/r, since H, H' are isomorphic). We will now show how to construct a paired volume form on $H/r \times H'/r$ from a bilinear form $L : H \times H' \to \mathbb{Q}/\mathbb{Z}$. First, we repeat some definitions from [Tur02].

Let $H_{(p)}$ be the subgroup of H consisting of all elements annihilated by a power of p (similarly for $H'_{(p)}$). A sequence $h = (h_1, \ldots, h_n)$ of nonzero elements of $H_{(p)}$ is a *pseudo-basis* if $H_{(p)}$ is a direct sum of the cyclic subgroups generated by $h_1, \ldots, h_n$ and the order of h_i in H is less than or equal to the order of h_j for $i \leq j$. In other words, if the order of h_i is p^{s_i}, with $s_i \geq 1$, then $s_1 \leq s_2 \leq \cdots \leq s_n$. This sequence $(s_1, \ldots, s_n)$ is determined by $H_{(p)}$ and does not depend on h, and

$s \leq s_1$ since if we have a summand of order p^k for $k < s$, then projecting to H/r there is still a summand of order p^k, which contradicts our assumption that H/r is a sum of several $\mathbb{Z}_r$'s. Projecting a pseudo-basis to $H_{(p)}/r = H/r$ we get a basis $\overline{h}$ of the $\mathbb{Z}_r$-module H/r.

Let $L : H \times H' \to \mathbb{Q}/\mathbb{Z}$ be a bilinear form. We will say L is nondegenerate if the map induced by L from $H \to \mathrm{Hom}_{\mathbb{Z}}(H', \mathbb{Q}/\mathbb{Z})$ is an isomorphism (since everything is finite and of the same order, this is equivalent to the map being an injection or a surjection). Note if $z' \in H'_{(p)}$, then z' has order p^k for some $k \geq s$, and for any $z \in H, L(z, z') \in (p^{-k}\mathbb{Z})/\mathbb{Z}$, and therefore $p^k L(z, z') \in \mathbb{Z}/(p^k\mathbb{Z})$ (this is really simply saying that $L(z, z')$ is in the subgroup of $\mathbb{Q}/\mathbb{Z}$ isomorphic to $\mathbb{Z}/(p^k\mathbb{Z})$). Projecting this to $\mathbb{Z}_r$, we obtain an element which we will call $z \cdot z'$. Note we can do something similar if z has order p^k and z' does not necessarily, and that they clearly agree if both z, z' have order a power of p. Furthermore, $z \cdot z'$ is a $\mathbb{Z}_r$ pairing on $H \times H'$. Now we are ready to state the analogue of Lemma III.3.4.1 in [Tur02] (the proof is a direct generalization of the proof found there as well).

Lemma 2.2. *There is a unique paired volume form* μ^r_L *on* $H/r \times H'/r$ *such that for any pseudo-bases* $h = (h_1, \ldots, h_n), k = (k_1, \ldots, k_n)$ *of* $H_{(p)}, H'_{(p)}$ *respectively,*

$$\mu^r_L(\overline{h}, \overline{k}) = \det(h_i \cdot k_j) \in \mathbb{Z}_r. \tag{2.12}$$

Also, if L *is nondegenerate, then so is* μ^r_L*.*

Proof. It is clear that *given* pseudo-bases h, k then we can construct a paired volume form $\mu_{(h,k)}$ by $\mu_{(h,k)}(a, b) = [a/\overline{h}][b/\overline{k}] \det(h_i \cdot k_j)$ for any bases a, b of $H/r, H'/r$ respectively. Then $\mu_{(h,k)}(\overline{h}, \overline{k}) = \det(h_i \cdot k_j)$, so we would like to define $\mu^r_L = \mu_{(h,k)}$, so we now prove that the definition of $\mu_{(h,k)}$ does not actually depend on the chosen pseudo-bases. To prove this, it suffices to show that for any other pseudo-bases $x = (x_1, \ldots, x_n)$ of $H_{(p)}$ and $y = (y_1, \ldots, y_n)$ of $H'_{(p)}$,

$$\det(x_i \cdot y_j) = [\overline{x}/\overline{h}][\overline{y}/\overline{k}] \det(h_i \cdot k_j). \tag{2.13}$$

To prove (2.13), we can actually fix one pseudo-basis and check the formula by varying the other, by the symmetry of the construction, i.e. we only need to show

$$\det(x_i \cdot k_j) = [\overline{x}/\overline{h}] \det(h_i \cdot k_j). \tag{2.14}$$

Now, x is a pseudo-basis for $H_{(p)}$, so the order of x_i is equal to the order of h_i for each i. It is clear that if x is just a permutation of h (the permutation can only permute elements of the same order), then the basis $\overline{x}$ of H/r is the same permutation of the basis $\overline{h}$, and then (2.14) is clear. So now, we may assume that each x_i generates the same cyclic subgroup of $H_{(p)}$ as the corresponding h_i. Then for each i, there is some $c_i \in \mathbb{Z}$, with c_i coprime to p^{s_i}, hence coprime to $r = p^s$ (in fact, coprime to p), with $x_i = c_i h_i$. But then $x_i \cdot k_j = (c_i \pmod r) h_i \cdot k_j$, so $\det(x_i \cdot k_j) = \prod_i (c_i \pmod r) \det(h_i \cdot k_j)$. But clearly $[\overline{x}/\overline{h}] = \prod_i c_i \pmod r$, so the proof of (2.14) is completed, and (2.13) clearly follows from symmetry.

Now if L is nondegenerate, then to show that μ_L^r is nondegenerate, we just have to show that $\det(h_i \cdot k_j) \in \mathbb{Z}_r^\times$ for any pseudo-bases h, k of $H_{(p)}, H'_{(p)}$ respectively. Now L nondegenerate means that the map induced by L, $\widetilde{L} : H \to \mathrm{Hom}_{\mathbb{Z}}(H', \mathbb{Q}/\mathbb{Z})$, is a bijection. Then, in particular, the restriction of $\widetilde{L}$ to $H_{(p)}$ is also bijective on its image $\mathrm{Hom}_{\mathbb{Z}}(H'_{(p)}, \mathbb{Q}/\mathbb{Z})$. This means, for k any pseudo-basis of $H'_{(p)}$, for each k_j there is an $x_j \in H_{(p)}$ with $L(x_i, k_j) = \delta_{i,j} p^{-s_j}$, i.e. $x_i \cdot k_j = \delta_{i,j}$. Then (2.14) gives us our result, that $\det(h_i \cdot k_j) \in \mathbb{Z}_r^\times$ for any pseudo-bases h, k of $H_{(p)}, H'_{(p)}$ respectively. □

The $\mathbb{Q}/\mathbb{Z}$ linking form

There is a linking form on $\text{Tors}(H_1(M)) \times \text{Tors}(H_1(M, \partial M))$ defined as follows (we use a slightly different construction from the one in [Tur02]; our construction is more like the one in [Bre93] exercise VI.10.8).

From the Universal Coefficient Theorem, there is an exact sequence

$$0 \to H_2(M) \otimes \mathbb{Q}/\mathbb{Z} \to H_2(M; \mathbb{Q}/\mathbb{Z}) \to \text{Tor}(H_1(M), \mathbb{Q}/\mathbb{Z}) \to 0$$

but there is a canonical isomorphism $\text{Tors}(H_1(M)) \approx \text{Tor}(H_1(M), \mathbb{Q}/\mathbb{Z})$ given by

$$\begin{aligned} \text{Tors}(H_1(M)) &\approx \text{Tors}(H_1(M)) \otimes \mathbb{Z} \\ &\approx \text{Tor}(\text{Tors}(H_1(M)), \mathbb{Q}/\mathbb{Z}) \\ &\approx \text{Tor}(H_1(M), \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

With this in mind, our exact sequence becomes

$$0 \to H_2(M) \otimes \mathbb{Q}/\mathbb{Z} \to H_2(M; \mathbb{Q}/\mathbb{Z}) \to \text{Tors}(H_1(M)) \to 0.$$

Now choose elements $a \in \text{Tors}(H_1(M))$ and $b \in \text{Tors}(H_1(M, \partial M))$; we want to define their linking $L_M(a, b) \in \mathbb{Q}/\mathbb{Z}$. So choose $\overline{a} \in H_2(M; \mathbb{Q}/\mathbb{Z})$ mapping to a, then let $\alpha \in H^1(M, \partial M; \mathbb{Q}/\mathbb{Z})$ be Poincaré dual to $\overline{a}$, i.e. $\alpha \cap [M] = \overline{a}$. Then we define $L_M(a, b) = \langle \alpha, b \rangle \in \mathbb{Q}/\mathbb{Z}$. An important question one can ask at this point is whether there is a difference if we use the exact sequence for the Universal Coefficient Theorem for $H_2(M, \partial M; \mathbb{Q}/\mathbb{Z})$ instead of for $H_2(M; \mathbb{Q}/\mathbb{Z})$. We will defer the answer to this question until the proof of Theorem 2.2, during which we show why the definition is independent of whether one starts with $\text{Tors}(H_1(M))$ or $\text{Tors}(H_1(M, \partial M))$.

Constructing the Paired Volume Form for Cohomology

Let ω be a homology orientation. Then we have split exact sequences

$$0 \to \operatorname{Tors}(H_1(M)) \to H_1(M) \to H_1(M)/\operatorname{Tors}(H_1(M)) \to 0,$$

$$0 \to \operatorname{Tors}(H_1(M, \partial M)) \to H_1(M, \partial M) \to H_1(M, \partial M)/\operatorname{Tors}(H_1(M, \partial M)) \to 0.$$

Both of these sequences split, so they also split modulo r, and $H_1(M)/r \approx H_1(M; \mathbb{Z}_r)$, and similarly for $H_1(M, \partial M; \mathbb{Z}_r)$. The homology orientation induces a nondegenerate paired volume form on the free $\mathbb{Z}$-modules $H_1(M)/\operatorname{Tors}(H_1(M)) \times H_1(M, \partial M)/\operatorname{Tors}(H_1(M, \partial M))$ which induces a nondegenerate paired volume form on $(H_1(M)/\operatorname{Tors}(H_1(M)))/r \times (H_1(M, \partial M)/\operatorname{Tors}(H_1(M, \partial M)))/r$, and we have a nondegenerate paired volume form (induced by the $\mathbb{Q}/\mathbb{Z}$-linking form) on $\operatorname{Tors}(H_1(M))/r \times \operatorname{Tors}(H_1(M, \partial M))/r$, which we can then piece together as above to give a nondegenerate paired volume form on $H_1(M; \mathbb{Z}_r) \times H_1(M, \partial M; \mathbb{Z}_r)$, which in turn gives us a canonical nondegenerate paired volume form on the duals with which to refine our determinant. We will denote the canonical Mod-r paired volume form by on $H_1(M; \mathbb{Z}_r) \times H_1(M, \partial M; \mathbb{Z}_r)$ by μ_M^r and the refined determinant of the form f_M^r on $H^1(M, \partial M; \mathbb{Z}_r) \times H^1(M; \mathbb{Z}_r) \times H^1(M; \mathbb{Z}_r)$ by $\operatorname{Det}_r(f_M^r)$.

2.2.2 Relationship to Torsion

Now let I denote the augmentation ideal of $\mathbb{Z}_r[H_1(M)]$ instead of the augmentation ideal of $\mathbb{Z}[H_1(M)]$ as before (the augmentation ideal of $\mathbb{Z}_r[H_1(M)]$ is the image of the augmentation ideal of $\mathbb{Z}[H_1(M)]$ under the map induced by the coefficient projection $\mathbb{Z} \to \mathbb{Z}_r$). We now recall a definition from [Tur02] - we define

$q_r : S(H_1(M)/r) \to \bigoplus_{\ell \geq 0} I^\ell / I^{\ell+1}$ by

$$q_r(g_1, \ldots, g_\ell) = \prod_{i=1}^{\ell} (\widetilde{g_i} - 1) \pmod{I^{\ell+1}} \tag{2.15}$$

where $\widetilde{g_i}$ is a lift of g_i to $H_1(M)$ (the proof that this is independent of the lift is in [Tur02]).

Before we state the main theorem, we need to briefly discuss Mod-r surfaces. In particular, we need to give equivalent equations to (2.9). Some of the following statements (in particular Lemma 2.3, below) are used without proof in [Tur02] Theorem III.4.3, and an equivalent definition of Mod-r surfaces can be found in [Tur02] Section XII.3 (we use the definition below because it is a bit easier for our purposes).

Mod-r surfaces

Let $G(\mathcal{M}, \mathcal{N}; r)$ be a group generated by $\alpha_\mu, \beta_\mu, \gamma_\nu$ where μ, ν run over finite indexing sets $\mathcal{M}, \mathcal{N}$ respectively, with a single relator $\rho = \prod_\mu [\alpha_\mu, \beta_\mu] \prod_\nu \gamma_\nu^r$. Let $X(\mathcal{M}, \mathcal{N}; r)$ be a connected CW-complex with a single 0-cell, 1-cells a_μ, b_μ, c_ν (so that we can consider $\pi_1(X(\mathcal{M}, \mathcal{N}; r))$ to be generated by $\alpha_\mu, \beta_\mu, \gamma_\nu$), and a single 2-cell attached along ρ, so that $\pi_1(X(\mathcal{M}, \mathcal{N}; r)) \approx G(\mathcal{M}, \mathcal{N}; r)$ in an obvious way. Then $H_2(X(\mathcal{M}, \mathcal{N}; r); \mathbb{Z}_r) \approx \mathbb{Z}_r$, so let $[X(\mathcal{M}, \mathcal{N}; r)]$ be the generator of $H_2(X(\mathcal{M}, \mathcal{N}; r); \mathbb{Z}_r)$ given by the homology class of the two cell (whose boundary is zero modulo r). Now if $t, t' \in H^1(X(\mathcal{M}, \mathcal{N}; r); \mathbb{Z}_r)$, then let us compute

$$\langle t \cup t', [X(\mathcal{M}, \mathcal{N}; r)] \rangle = \varepsilon_* \left((t \cup t') \cap [X(\mathcal{M}, \mathcal{N}; r)] \right)$$

where $\varepsilon_* : H_0(X(\mathcal{M}, \mathcal{N}; r); \mathbb{Z}_r) \to \mathbb{Z}_r$ is simply augmentation, $[\text{pt}] \mapsto 1$. Let $a_\mu, b_\mu, c_\nu \in H_1(X(\mathcal{M}, \mathcal{N}; r); \mathbb{Z}_r)$ be the homology classes modulo r of $\alpha_\mu, \beta_\mu, \gamma_\nu$ respectively. We now claim

Lemma 2.3. *Let $t, t' \in H^1(X(\mathcal{M}, \mathcal{N}; r); \mathbb{Z}_r)$. If r is odd, then*

$$\langle t \cup t', [X(\mathcal{M}, \mathcal{N}; r)]\rangle = \sum_{\mu} \langle t, a_\mu\rangle\langle t', b_\mu\rangle - \langle t, b_\mu\rangle\langle t', a_\mu\rangle. \tag{2.16}$$

If r is even, then

$$\langle t \cup t', [X(\mathcal{M}, \mathcal{N}; r)]\rangle = \sum_{\mu} \langle t, a_\mu\rangle\langle t', b_\mu\rangle - \langle t, b_\mu\rangle\langle t', a_\mu\rangle + \frac{r}{2} \sum_{\nu} \langle t, c_\nu\rangle\langle t', c_\nu\rangle. \tag{2.17}$$

Proof. If we let $a_\mu^*, b_\mu^*, c_\nu^* \in H^1(X(\mathcal{M}, \mathcal{N}; r); \mathbb{Z}_r)$ be dual to a_μ, b_μ, c_ν under $\langle \cdot, \cdot \rangle$, then $1 = \langle a_\mu^* \cup b_\mu^*, [X(\mathcal{M}, \mathcal{N}; r)]\rangle = -\langle b_\mu^* \cup a_\mu^*, [X(\mathcal{M}, \mathcal{N}; r)]\rangle$. Clearly $c_\nu \cup c_\nu$ is 2-torsion for any r, and one can also show that all other cup products are zero (this follows from induction and a relatively simple Mayer-Vietoris argument). So the claim for r odd is completed. By the same Mayer-Vietoris argument, for even r, we only need to show the statement for $\mathcal{M}$ empty, and $\mathcal{N}$ only having one element, i.e. for even r, and a CW complex X with one 0-cell, one 1-cell c, and one 2-cell with boundary $r \cdot c$, we need to show $\langle c^2, [X]\rangle = \frac{r}{2}$. But this follows from simply noting that X is the 2-skeleton of a $K(\mathbb{Z}_r, 1)$. A more complete proof may be found in [Hat02] Chapter 3, Example 3.9. □

We are now ready to state the main theorem of this section.

Theorem 2.2. *Let r be a power of a prime such that $H_1(M)/r = H_1(M; \mathbb{Z}_r)$ is a free $\mathbb{Z}_r$-module of rank $b \geq 2$. Let T denote $|\operatorname{Tors}(H_1(M))|/r$. Then for any Euler structure e and homology orientation ω, $\tau(M, e, \omega; r) \in I^{b-2}$, and*

$$\tau(M, e, \omega; r) = T \cdot q_r(\operatorname{Det}_r(f_M^r)) \pmod{I^{b-1}}. \tag{2.18}$$

As in Theorem 2.1, that $\tau(M, e, \omega; r) \in I^{b-2}$ is proved in [Tur02] II.4.4, the important part of the theorem is its residue class modulo I^{b-1}.

Proof. The proof is similar to the proof of Theorem 2.1, and again is the method of the proof of [Tur02] Theorem 4.3 with modifications to apply it to manifolds with nonvoid boundary. Suppose $r = p^s$, where $p \geq 2$ is prime and $s \geq 1$. Let $n = b_1(M)$. Then $H_1(M)$ splits as $\mathbb{Z}^n \times (\mathrm{Tors}(H_1(M)))_{(p)} \times H'$ and $H_1(M, \partial M)$ splits as $\mathbb{Z}^{n-1} \times (\mathrm{Tors}(H_1(M, \partial M)))_{(p)} \times H''$ where the subscript of (p) denotes the maximal subgroup of a finite group whose order is a power of p and H', H'' are (isomorphic) subgroups of $\mathrm{Tors}(H_1(M))$ and $\mathrm{Tors}(H_1(M, \partial M))$ respectively with $|H'| = |H''| = T$. We again choose a handle decomposition of M and the dual relative handle decomposition of $(M, \partial M)$ with 1 honest 0-handle, m honest 1-handles, $m-1$ honest 2-handles, and no other handles, where $m \geq b \geq n$. Let $x_1, \ldots, x_m \in \pi_1(M)$ be the generators of $\pi_1(M)$ (based at the 0-cell) given by the core 1-cells of the honest 1-handles, and let $h_1, \ldots h_m$ denote their homology classes. Let $k_1, \ldots, k_{m-1}$ denote the classes in $H_1(M, \partial M)$ of the core cells of the relative 1-handles, and let $r_1, \ldots, r_{m-1}$ be the relators in $F = \langle x_1, \ldots, x_m \rangle$ given by the attaching maps of the honest 2-cells. Now, as in the proof of Theorem 2.1, we want to rearrange handles for a more convenient decomposition.

As in the proof of Theorem 2.1, we can arrange so that $h_1, \ldots, h_n$ are generators modulo $\mathrm{Tors}(H_1(M))$ and $h_{n+1} \ldots, h_m \in \mathrm{Tors}(H_1(M))$. We can also arrange so that $h_{n+1}, \ldots, h_b$ is a pseudo-basis of $(\mathrm{Tors}(H_1(M)))_{(p)}$. The argument, from [Tur02], is that given a surjection $\mathbb{Z}^m \to H_1(M)$ and a splitting of $H_1(M)$ as a direct sum of k cyclic groups, we may choose a basis $\alpha_1, \ldots, \alpha_m$ of $\mathbb{Z}^m$ such that α_i projects to a generator of the i^{th} cyclic group for $i \leq k$ and to $1 \in H_1(M)$ for $i > k$. We can realize this basis geometrically by handle moves, so that $h_{n+1}, \ldots, h_b$ is a pseudo-basis as desired, and $h_{b+1}, \ldots, h_m \in H'$. Let $p^{s_1}, \ldots, p^{s_{b-n}}$ be the orders of $h_{n+1}, \ldots, h_b$ respectively, and we may assume $s_1 \leq s_2 \leq \cdots \leq s_{b-n} \leq s$.

Now we will denote by $\widetilde{h}$ the projection of $h \in H_1(M)$ to $H_1(M)/r$, then $\widetilde{h}_1, \ldots, \widetilde{h}_b$ is a basis for $H_1(M)/r$ over $\mathbb{Z}_r$ and $\widetilde{h}_i = 1$ for $i > b$. Let $h_i^* \in H^1(M;\mathbb{Z}_r)$ for $i \leq b$ such that $\langle h_i^*, \widetilde{h}_j \rangle = \delta_{i,j}$.

Let k_i denote the class in $H_1(M, \partial M)$ of the i^{th} relative handle, using the methods in the proof of Theorem 2.1 and the methods above, we can arrange so that $k_1, \ldots, k_{n-1}$ are generators modulo $\mathrm{Tors}(H_1(M, \partial M))$, $k_n, \ldots, k_{b-1}$ form a pseudo-basis of $(\mathrm{Tors}(H_1(M, \partial M)))_{(p)}$ (they also have orders $p^{s_1}, \ldots, p^{s_{b-n}}$) and $k_b, \ldots, k_{m-1} \in T''$. This means, using $\widetilde{k}$ to denote projection of $k \in H_1(M, \partial M)$ to $H_1(M, \partial M)/r$, that $\widetilde{k}_1, \ldots, \widetilde{k}_{b-1}$ is a basis for $H_1(M, \partial M)/r$ over $\mathbb{Z}_r$ and $\widetilde{k}_i = 1$ for $i > b - 1$. As above, let $k_i^* \in H^1(M, \partial M; \mathbb{Z}_r)$ for $i \leq b - 1$ be such that $\langle k_i, \widetilde{k}_j \rangle = \delta_{i,j}$.

The matrix for the boundary map from dimension two to dimension one in $C_*(M)$ decomposes, as in the proof of Theorem 2.1, as $\left(\begin{smallmatrix} 0 & 0 \\ 0 & v \end{smallmatrix}\right)$, where v is a square presentation matrix for $\mathrm{Tors}(H_1(M))$. With the above setup, v can be split as the direct sum of a diagonal matrix (with $p^{s_1}, \ldots, p^{s_{b-n}}$ along the diagonal) and a square matrix v' which is a presentation matrix for H' (and its transpose a presentation matrix for H''), hence $\det(v') = \pm T$. Now let us think about how the diagonal submatrix of v (consisting of powers of p) arises. Let us take h_i for $n + 1 \leq i \leq b$; h_i has order $p^{s_{i-n}}$ according to the above argument, and in fact $p^{s_{i-n}} h_i$ is the boundary of the 2-cell transverse to k_i. This 2-cell has boundary zero in $\mathbb{Q}/\mathbb{Z}$, and its homology class in $H_2(M; \mathbb{Q}/\mathbb{Z})$ is Poincaré dual to the class of k_i^* in $H^1(M, \partial M; \mathbb{Q}/\mathbb{Z})$. This process is the precise process used in the construction of the linking pairing, first lifting an element of $\mathrm{Tors}(H_1(M))$ to $H_2(M; \mathbb{Q}/\mathbb{Z})$ and then using Poincaré duality to get an element dual (under evaluation) to an element of $\mathrm{Tors}(H_1(M, \partial M))$. The dual process, starting with an element of $\mathrm{Tors}(H_1(M, \partial M))$ and ending with an element dual under evaluation to an element in $\mathrm{Tors}(H_1(M))$, can also be read off of the submatrix of v with which we are currently concerned, so now one can easily see that constructing the linking form by either obvious method gives the same pairing, answering our question from the discussion of the $\mathbb{Q}/\mathbb{Z}$ linking form. Furthermore, we can see that for $n+1 \leq i \leq b$ and $n \leq j \leq b-1$,

$$(h_i \cdot k_j) = \delta_{i,j}. \tag{2.19}$$

Now as in the proof of Theorem 2.1, $r_1, \ldots, r_{n-1} \in [F, F]$, and the above argument shows $r_n, \ldots, r_{b-1}$ can each be expanded as $r_i = \prod_{\mu \in \mathcal{M}_i} [\alpha_\mu, \beta_\mu] \prod_{\nu \in \mathcal{N}_i} \gamma_\nu^r$, so we need to use Lemma 2.3. Henceforth, we will suppress the $\mathcal{M}_i, \mathcal{N}_i$ notation for simplicity.

Now let $\mathrm{pr} : \mathbb{Z}[H_1(M)] \to \mathbb{Z}_r[H_1(M)]$ be coefficient projection, $\eta : \mathbb{Z}[F] \to \mathbb{Z}[H_1(M)]$ be induced by the projection $F \to H_1(M)$ (through $\pi_1(M)$), and $p : \mathbb{Z}_r[H_1(M)] \to \mathbb{Z}_r[H_1(M)/r]$ be induced by $H_1(M) \to H_1(M)/r$. Finally, we will also denote by $\eta_r = p \circ \mathrm{pr} \circ \eta$. We now prove the analogue of (2.4) which is, for $i \leq b-1, j \leq b$

$$(\mathrm{pr} \circ \eta)(\partial r_i / \partial x_j) = -\sum_{p=1}^{b} f_M^r(k_i^*, h_j^*, h_p^*)(h_p - 1) \pmod{I^2}. \tag{2.20}$$

We will prove (2.20) by proving the analogue of (2.5), which is

$$\eta_r(\partial r_i / \partial x_j) = -\sum_{p=1}^{b} f_M^r(k_i^*, h_j^*, h_p^*)(\widetilde{h}_p - 1) \pmod{J^2}. \tag{2.21}$$

Note (2.20) follows from (2.21) since p induces an isomorphism $\mathbb{Z}_r[H_1(M)]/I^2 \to \mathbb{Z}_r[H_1(M)/r]/J^2$ (where J is the augmentation ideal in $\mathbb{Z}_r[H_1(M)/r]$). This follows from noting for any $h \in H_1(M)$, $h^r - 1 \in I^2$ since $(h^r - 1) = (h-1)(1 + h + \cdots + h^{r-1})$, and $(1 + h + \cdots + h^{r-1}) = (h-1) + (h^2 - 1) + \cdots + (h^{r-1} - 1)$

in $\mathbb{Z}_r[H_1(M)]$. To prove (2.21), we need to note that (2.6) and (2.7) can be used here *mutatis mutandis*; indeed, for $c \in H_1(M)/r$, we may use the same formula as (2.6) with slightly different meaning to the symbols

$$c - 1 = \sum_{p=1}^{b} \langle h_p^*, c\rangle (\widetilde{h}_p - 1) \pmod{J^2}. \tag{2.22}$$

Also, for any $\alpha \in F, j \leq b$, if we let aug_r denote $\mathrm{aug} \circ p \circ \mathrm{pr} \circ \eta$, $\mathrm{aug}_r : \mathbb{Z}[F] \to \mathbb{Z}_r$,

$$\mathrm{aug}_r(\partial\alpha/\partial x_j) = \langle h_j^*, \eta_r(\alpha)\rangle. \tag{2.23}$$

This follows, as before, from the fact that both sides are homomorphisms $F \to \mathbb{Z}_r$ sending x_i to $\delta_{i,j}$ for $i \leq b$.

For $1 \leq i \leq n-1$, we may compute $\partial r_i/\partial x_j$ by

$$\eta(\partial r_i/\partial x_j) = \sum_{\mu} (\eta(\alpha_\mu) - 1)\eta(\partial\beta_\mu/\partial x_j) + (1 - \eta(\beta_\mu))\eta(\partial\alpha_\mu/\partial x_j).$$

For $n \leq i \leq b-1$, we must add a term for the γ_ν's

$$\begin{aligned}\eta(\partial r_i/\partial x_j) = &\sum_{\mu} \big((\eta(\alpha_\mu) - 1)\eta(\partial\beta_\mu/\partial x_j) + (1 - \eta(\beta_\mu))\eta(\partial\alpha_\mu/\partial x_j)\big) \\ &+ \sum_{\nu} \big(\eta(\partial\gamma_\nu/\partial x_j)(1 + \gamma_\nu + \cdots + \gamma_\nu^{r-1})\big).\end{aligned} \tag{2.24}$$

For any r, any $c \in H_1(M)$, working modulo I^2,

$$\begin{aligned}\sum_{\ell=0}^{r-1} c^\ell &= \sum_{\ell=0}^{r-1} (1 + (c-1))^\ell \\ &= \sum_{\ell=0}^{r-1}\sum_{s=0}^{\ell} \binom{\ell}{s} 1^{\ell-s}(c-1)^s \\ &= \sum_{\ell=0}^{r-1} 1 + \ell(c-1) \pmod{I^2} \\ &= \sum_{\ell=0}^{r-1} \ell(c-1) \\ &= (c-1)r(r-1)/2.\end{aligned}$$

So if r is odd, then each extra γ_ν term is zero modulo I^2 in (2.24), and if r is even (in our case, a power of two), then for each ν (applying (2.22)),

$$\begin{aligned} 1 + \eta_r(\gamma_\nu) + \cdots + \eta_r(\gamma_\nu)^{r-1} &= -\frac{r}{2}(\eta_r(\gamma_\nu) - 1) \\ &= -\frac{r}{2}\sum_{p=1}^{b}\langle h_p^*, \eta_r(\gamma_\nu)\rangle(\widetilde{h}_p - 1) \pmod{J^2} \\ &= \frac{r}{2}\sum_{p=1}^{b}\langle h_p^*, \eta_r(\gamma_\nu)\rangle(\widetilde{h}_p - 1) \pmod{J^2}. \end{aligned} \tag{2.25}$$

The last line follows since in $\mathbb{Z}_r$ for an even r, $-\frac{r}{2} = \frac{r}{2}$.

Now, using maps from Mod-r surfaces (i.e. Lemma 2.3) instead of maps from surfaces, we can use the proof from Theorem 2.1 since (2.8) holds for odd r, so (2.10) holds for odd r, proving (2.21) for odd r. For even r, (2.8) holds with an additional term following from (2.23) and (2.25). Specifically, for an even r,

$$\begin{aligned} &\eta_r(\partial r_i/\partial x_j) \bmod J^2 = \\ &\quad \sum_{p=1}^{b}\Bigg(\sum_{\mu}\langle h_p^*, \eta_r(\alpha_\mu)\rangle\langle h_j^*, \eta_r(\beta_\mu)\rangle - \langle h_p^*, \eta_r(\beta_\mu)\rangle\langle h_j^*, \eta_r(\alpha_\mu)\rangle \\ &\quad + \frac{r}{2}\sum_{\nu}\langle h_j^*, \eta_r(\gamma_\nu)\rangle\langle h_p^*, \eta_r(\gamma_\nu)\rangle\Bigg)(\widetilde{h}_p - 1). \end{aligned}$$

This term also occurs in (2.10) for even r by Lemma 2.3, so (2.21) holds for even r as well, hence (2.20) holds for all r.

If we let a be the submatrix of $(\mathrm{pr} \circ \eta)(\partial r_i/\partial x_j)$ consisting of the $b \times b - 1$ upper left submatrix,

$$(h_1 - 1)\tau(M, e, \omega; r) = |\det(v')|\det(a(1)) = T\det(a(1)) \bmod I^b.$$

Now (2.20) tells us that computing $\det(a(1))$ is simply computing $q_r(\det(\Theta(1)))$ where $\Theta_{i,j} = \sum_{p=1}^{b} f_M^r(k_i^*, h_j^*, h_p^*)\widetilde{h}_p$.

$$\det(\Theta(1)) = -\widetilde{h}_1 d(f_M^r, h^*, k^*).$$

From here, we may follow the proof from Theorem 2.1, since $\mu_M^r(h^*, k^*)$ will simply be a sign just as in Theorem 2.1, since the linking form of the pseudo-bases is equal to the identity matrix, hence has determinant one. This follows from equation (2.19) which gives for $n+1 \leq i \leq b$ and $n \leq j \leq b-1$,

$$\det(h_i \cdot k_j) = \det(\delta_{i,j}) = 1.$$

□

2.3 Integral Massey Products

In this section, we give a generalization of Theorem 2.1 where we use Massey products rather than the cohomology ring. The results of this section are similar to results in Chapter XII Section 2 of [Tur02] for closed manifolds.

2.3.1 Determinants

First we obtain a new determinant. Let R be a commutative ring with 1, and let K, N be free R-modules of rank n,$n-1$ respectively, with $n \geq 2$ and let $S = S(K^*)$, the symmetric algebra on the dual of K, as in Lemma 2.1. Let $f : N \times K^{m+1} \to R$ be an R-map, with $m \geq 1$. Define $g : N \times K \to S$ by

$$g(x, y) = \sum_{i_1,\ldots,i_m=1}^{n} f(x, y, a_{i_1}, \ldots, a_{i_m}) a_{i_1}^* \cdots a_{i_m}^* \in S$$

where $\{a_i\}_{i=1}^n$ is a basis for K and $\{a_i^*\}$ is its dual basis. This definition for g looks dependent on the basis chosen, however note that the independence on the basis follows from linearity and (1.2).

Let $f_0 : N \to S$ be defined by

$$f_0(x) = \sum_{i_1,\ldots,i_{m+1}=1}^{n} f(x, a_{i_1}, \ldots, a_{i_{m+1}}) a_{i_1}^* \cdots a_{i_{m+1}}^* \in S.$$

Again, f_0 does not depend on the chosen basis, by precisely the same argument. Then we have the following lemma:

Lemma 2.4. *Suppose $f_0 = 0$. Let $a = \{a_i\}, b = \{b_j\}$ be bases of K, N respectively, and let θ be the $(n-1 \times n)$ matrix over S defined by $\theta_{i,j} = g(b_i, a_j)$. Then there exists a unique $d = d(f, a, b) \in S^{m(n-1)-1}$ such that*

$$\det(\theta(i)) = (-1)^i a_i^* d. \tag{2.26}$$

Furthermore, if a', b' are other bases for K, N respectively, then

$$d(f, a', b') = [a'/a][b'/b] d(f, a, b). \tag{2.27}$$

Proof. This is very similar to the proof of Lemma 2.1. Let β be the matrix over S given by $\beta_{i,j} = g(b_i, a_j)a_j^*$. Then the sum of the columns of β is zero; the i^{th} entry in that sum is $\sum_{j=1}^{n} \beta_{i,j} = f_0(b_i) = 0$ since our assumption is $f_0 = 0$. Now the same argument as given in Lemma 2.1 to prove (2.1) completes the proof of (2.26), and the argument given to prove (2.2) can be used to prove (2.27). □

Note that as before, over $\mathbb{Z}$ the determinant is well defined up to sign, and that one may also sign-refine this determinant to remove the sign dependence.

We may also define the condition that f is "alternate" in the K variables; let $\overline{f_0} : N \times K \to R$ be the R-map given by $\overline{f_0}(x, a) = f(x, \overbrace{a, a, \ldots, a}^{m+1\text{times}})$. Then $f_0(x) = 0$ for all x clearly implies $\overline{f_0}(x, a) = 0$ for all $x \in N, a \in K$. The converse is also true provided that every polynomial over R which only takes on zero values has all zero coefficients (this is true, for example, if R is infinite with no zero-divisors). To see why, consider $f_0(x)$ as a polynomial over R ($f_0(x) \in S$ which is isomorphic to the polynomial ring $R[a_1^*, \dots, a_n^*]$) and evaluate on the element $(r_1, \dots, r_n) \in R^n$; denote by α the resulting element of R. Then

$$\begin{aligned} \alpha &= \sum_{i_1,\dots,i_{m+1}=1}^{n} f(x, a_{i_1}, \dots, a_{i_{m+1}}) r_{i_1} \cdots r_{i_{m+1}} \\ &= \sum_{i_1,\dots,i_{m+1}=1}^{n} f(x, r_{i_1} a_{i_1}, \dots, r_{i_{m+1}} a_{i_{m+1}}) \\ &= f\left(x, \sum_{i_1=1}^{n} r_{i_1} a_{i_1}, \dots, \sum_{i_{m+1}=1}^{n} r_{i_{m+1}} a_{i_{m+1}} \right) \\ &= 0. \end{aligned}$$

The last equality holds since all of the entries after the first are identical.

The rest of the argument is very similar to the argument in [Tur02], section XII.2. Let M be a 3-manifold with nonempty boundary, and for $u_1, u_2, \dots, u_k \in H^1(M)$, let $\langle u_1, \dots, u_k \rangle$ denote the Massey product of $u_1, \dots, u_k$ as a subset of $H^2(M)$ (note in general this set may well be empty). See [Kra66] and [Fen83] for definitions and properties of Massey products. Now assume that $m \geq 1$ is an integer such that

$$(*)_m: \quad \text{for every } u_1, \dots, u_k \in H^1(M) \text{ with } k \leq m, \langle u_1, \dots, u_k \rangle = 0$$

Here $\langle u_1, \dots, u_k \rangle = 0$ means that $\langle u_1, \dots, u_k \rangle$ consists of the single element $0 \in H^2(M)$. This condition guarantees that for any $u_1, \dots u_{m+1} \in H^1(M)$, the set $\langle u_1, \dots, u_m \rangle$ consists of a single element; see [Fen83] Lemma 6.2.7. Define a $\mathbb{Z}$-map $f : H^1(M, \partial M) \times (H^1(M))^{m+1} \to \mathbb{Z}$ by

$$f(v, u_1, \dots, u_{m+1}) = (-1)^m \langle v \cup \langle u_1, \dots, u_{m+1} \rangle, [M] \rangle .$$

The outermost $\langle , \rangle$ is used to denote the evaluation pairing.

Lemma 2.5. *$f_0 = 0$, so f has a well-defined determinant (with the sign refinement as above).*

For $m = 1$, condition $(*)_m$ is void, and in fact the Massey product $\langle u_1, u_2 \rangle = -u_1 \cup u_2$, so this reduces to Lemma 2.1.

Proof. By the argument above, we only need to show that f is alternate. But this follows from [Kra66] Theorem 15, which gives that for any element $a \in H^1(M)$, the $m+1$ times Massey product of a with itself, $\langle \overbrace{a, \ldots, a}^{m+1 \text{times}} \rangle$, lies in $\mathrm{Tors}(H^2(M))$, hence cupping with an element of $H^1(M, \partial M)$ will give an element of $\mathrm{Tors}(H^3(M, \partial M))$, which is null. □

We will call this determinant $\mathrm{Det}(f)$, or if we care to introduce the sign-refined version with a homology orientation ω, $\mathrm{Det}_\omega(f)$. Since the change of basis formula (2.27) is identical to the change of basis formula (2.2), the sign refinement by homology orientation is the same.

2.3.2 Relationship to Torsion

Theorem 2.3. *Let M be a compact connected oriented 3-manifold with $\partial M \neq \varnothing, \chi(M) = 0$, $n = b_1(M) \geq 2$, and satisfying condition $(*)_m$ for some $m \geq 1$. Let e be an Euler structure on M, let ω be a homology orientation, and let $q_{H_1(M)}$ be defined as in Section 2.1. Define the form f as above. Then $\tau(M, e, \omega) \in I^{m(n-1)-1}$ and*

$$\tau(M, e, \omega) \bmod I^{m(n-1)} = q_{H_1(M)}(\mathrm{Det}_\omega(f)) \in I^{m(n-1)-1}/I^{m(n-1)}. \tag{2.28}$$

Proof. This proof is very much like the one in Section 2.1. In place of (2.3), we may use [Tur76] Theorem D, which gives the second line of the following string of equalities (all of which are $\bmod J^{m+1}$)

$$\begin{aligned}
\widetilde{\eta}(\partial r_i/\partial x_j) &= \sum_{i_1,\ldots,i_m=1}^{n} \operatorname{aug}(\widetilde{\eta}(\partial^{m+1} r_i/\partial x_{i_1}\ldots\partial x_{i_m}\partial x_j))(\widetilde{h_{i_1}}-1)\cdots(\widetilde{h}_{i_m}-1) \\
&= \sum_{i_1,\ldots,i_m=1}^{n} \left\langle \langle h^*_{i_1},\ldots,h^*_{i_m},h^*_j\rangle, (-[\Sigma_i])\right\rangle (\widetilde{h}_{i_1}-1)\cdots(\widetilde{h}_{i_m}-1) \\
&= \sum_{i_1,\ldots,i_m=1}^{n} \left\langle \langle h^*_{i_1},\ldots,h^*_{i_m},h^*_j\rangle, (-k^*_i\cap[M])\right\rangle (\widetilde{h}_{i_1}-1)\cdots(\widetilde{h}_{i_m}-1) \\
&= \sum_{i_1,\ldots,i_m=1}^{n} -\left\langle k^*_i\cup\langle h^*_{i_1},\ldots,h^*_{i_m},h^*_j\rangle, [M]\right\rangle (\widetilde{h}_{i_1}-1)\cdots(\widetilde{h}_{i_m}-1) \\
&= \sum_{i_1,\ldots,i_m=1}^{n} (-1)^{m+1}\left\langle k^*_i\cup\langle h^*_j,h^*_{i_m},\ldots,h^*_{i_1}\rangle, [M]\right\rangle (\widetilde{h}_{i_1}-1)\cdots(\widetilde{h}_{i_m}-1) \qquad (2.29) \\
&= \sum_{i_1,\ldots,i_m=1}^{n} (-1)^{m+1}\left\langle k^*_i\cup\langle h^*_j,h^*_{i_1},\ldots,h^*_{i_m}\rangle, [M]\right\rangle (\widetilde{h}_{i_1}-1)\cdots(\widetilde{h}_{i_m}-1) \\
&= \sum_{i_1,\ldots,i_m=1}^{n} -f(k^*_i,h^*_j,h^*_{i_1},\ldots,h^*_{i_m})(\widetilde{h}_{i_1}-1)\cdots(\widetilde{h}_{i_m}-1).
\end{aligned}$$

The line marked (2.29) follows from [Kra66] Theorem 8, and the next line is by symmetry. From here, the proof is identical to the proof of Theorem 2.1 after (2.10). □

Chapter 3

Gluing Formulae

We now examine the results of Chapter 2 under the gluing of solid tori, since by Proposition 1.1 the only manifolds of interest have each boundary component homeomorphic to a torus. Since there are known formulae for the Turaev torsion under the gluing of solid tori (stated below), we need to study how the determinants act under gluing. This will allow us to derive the results in Chapters III,XII of [Tur02] from the results of Chapter 2 above.

3.1 Known Gluing Results

First, we state known results, which can also be found in [Tur02]. The major difference in our approach will be that Turaev largely uses smooth Euler structures in his gluing constructions, whereas we prefer combinatorial, so there will be some differences in the constructions in 3.1.2.

3.1.1 Gluing Homology Orientations

This is based on [Tur02] Chapter V, with some changes in notation. Also, for simplicity, we will consider the solid tori being glued one-at-a-time, i.e. we will only give the definition for gluing one solid torus, and will consider the definition for gluing multiple solid tori to be given inductively; we can do this from [Tur02] Lemma V.2.3. First, we define a *directed solid torus* as a solid torus $Z = \mathbb{D}^2 \times S^1$ (where $\mathbb{D}^2$ is the standard 2-disk) with a distinguished generator of $H_1(Z) \approx \mathbb{Z}$, i.e. an orientation of the core S^1. Now if M is a compact connected 3-manifold with boundary consisting of tori and one boundary component T picked out, then we can consider $\overline{M} = M \cup_T Z$ (under some choice of homeomorphism $T \to \partial Z$). We can consider Z to be homology oriented by setting ω_Z to be the orientation of $([\mathrm{pt}], d)$ where d is the distinguished generator of $H_1(Z)$ (to be precise, we should note that we are extending scalars from $\mathbb{Z}$ to $\mathbb{R}$). This provides $H_*(Z, \partial Z; \mathbb{R})$ with an orientation via Poincaré duality by saying $a \in H_2(Z, \partial Z)$ and $b \in H_3(Z, \partial Z)$ give a positively oriented basis (a, b) of $H_*(Z, \partial Z)$ if and only if $(b^* \cap [Z], a^* \cap [Z])$ is a positively oriented basis of $H_*(Z)$ where $[Z]$ is either orientation class of Z. It is clear that the resulting homology orientation of $H_*(Z, \partial Z; \mathbb{R})$ does not depend on the (arbitrarily) chosen orientation of Z, but only depends on the distinguished direction of Z (i.e. the distinguished generator of $H_1(Z)$). This then provides $H_*(\overline{M}, M)$ with an orientation via excision; denote this orientation $\omega_{(\overline{M}, M)}$. Then we may define $\widetilde{\omega}$, an orientation of $H_*(\overline{M})$, from a given homology orientation ω of M and our earlier constructed $\omega_{(\overline{M}, M)}$. We define the orientation $\widetilde{\omega}$ of $H_*(\overline{M})$ by requiring that the torsion of the homology exact sequence with $\mathbb{R}$ coefficients of the pair $(\overline{M}, M)$ have a positive sign. Then we define the homology orientation of $\overline{M}$ induced from ω, $\omega^{\overline{M}}$, as

$$\omega^{\overline{M}} = (-1)^{b_3(\overline{M}) + (b_1(M)+1)(b_1(\overline{M})+1)} \widetilde{\omega}. \tag{3.1}$$

The sign in the equation is needed to guarantee that if we use this definition multiple times to glue on several directed solid tori, that the end result is independent of the order in which we perform our gluing, see [Tur02] Lemma V.2.3.

3.1.2 Gluing Euler Structures

This is based on [Tur02] Chapter VI. We describe the *distinguished Euler structure* on a directed solid torus in a slightly different manner from the construction in [Tur02] VI.2.1, where Turaev uses smooth Euler structures to make the definition. We present an alternate description here, using combinatorial Euler structures (the distinguished Euler structure described here is the image of the one described in [Tur02] under the canonical bijection $\mathrm{vect}(M) \rightleftharpoons \mathrm{Eul}(M)$).

Let $Z = \mathbb{D}^2 \times S^1$ be a directed solid torus with distinguished generator $h \in H_1(Z)$. Then $\hat{Z}$, the maximal abelian cover of Z, is actually the universal cover of Z, given by $\mathbb{D}^2 \times \mathbb{R}$. We can decompose Z as a single 0-handle and a single 1-handle, and then $\hat{Z}$ consists of all h-multiples of any lifts of the 0-handle and the 1-handle. The distinguished Euler structure is the equivalence class of any fundamental family of handles $\hat{e}_0, \hat{e}_1$ (where $\hat{e}_0$ lies over the single 0-handle of Z, and $\hat{e}_1$ lies over the single 1-handle) with the property that $\partial \hat{e}_1 = h\hat{e}_0 - \hat{e}_0$. Note that any two fundamental families $\hat{e}, \hat{e}'$ with that property have $\hat{e}'_0 = h^k \hat{e}_0$ and $\hat{e}'_1 = h^k \hat{e}_1$ for some $k \in \mathbb{Z}$ (the property guarantees that $\hat{e}'_0, \hat{e}_0$ and $\hat{e}'_1, \hat{e}_1$ differ by the same element of $H_1(M)$), and then $\hat{e}, \hat{e}'$ have the same equivalence class in $\mathrm{Eul}(M)$, and that equivalence class is the distinguished Euler structure of Z, denoted e_Z.

Now let us note that one may naturally take the Cartesian product of two fundamental families of cells to obtain a fundamental family of cells on a Cartesian product of complexes. It is clear that the equivalence relation (1.9) is preserved, so this tells us how to take a Cartesian product of combinatorial Euler structures.

Also note that the construction of the distinguished Euler structure on a solid torus also works for a directed S^1, and hence we may obtain a distinguished Euler structure e_T on the torus $T^2 = S^1 \times S^1$. A simple computation using [KS65] gives that the Turaev torsion $\tau(T^2, e_T) = \pm 1$ (this is, again, a slightly different construction than the equivalent construction in [Tur02] II.2.7).

Now we discuss how to glue Euler structures, in a more general way than we need. Let M be a compact, connected, oriented 3-manifold with ∂M consisting entirely of tori, and let $T \subset M - \partial M$ be a finite system of disjoint embedded tori splitting M into two 3-manifolds M_1, M_2 such that $T = M_1 \cap M_2 = \partial M_1 \cap \partial M_2$. Then we define a gluing map $\cup : \mathrm{Eul}(M_1) \times \mathrm{Eul}(M_2) \to \mathrm{Eul}(M)$. Choose a cellular decomposition of M so that M_1, M_2, T are all subcomplexes. This means, since $M = M_1 \cup_T M_2$, that each cell of M is a cell in at least one of M_1, M_2, T, and also that T is a subcomplex of each M_i. Now choose a zero cell $*$ of T (hence also a zero cell of each M_i and of M) to serve as a basepoint. Then choose a lift of $*$, say $\hat{*}$, in $\hat{M}$ (the maximal abelian cover of M) to serve as a basepoint of $\hat{M}$. Let $p : \hat{M} \to M$ be the projection, and let $M_1' \subset \hat{M}$ be the component of $p^{-1}(M_1)$ containing $\hat{*}$, and similarly for M_2', and let $T' = M_1' \cap M_2'$. Then M_1', M_2', T' are covers of M_1, M_2, T respectively with abelian deck groups, so $\hat{M}_i$ covers M_i' for each i. Then, given $e_i \in \mathrm{Eul}(M_i)$, we can choose fundamental families of cells for $\hat{M}_1, \hat{M}_2$ that represent the e_i such that the projections of the fundamental families to M_i' agree on cells of T' over T. Once we have done this, we have a fundamental family of cells of M, and hence its equivalence class is an Euler structure of M. Choosing fundamental families like this is reasonably easy due to the forgiving nature of the equivalence relation (1.9) on fundamental families giving rise to combinatorial Euler structures.

Then if we have e and Euler structure on M and $\overline{M}$ obtained by gluing a directed solid torus Z onto M along a boundary component of M, we can induce the Euler structure e to an Euler structure on $\overline{M}$ using the above constructions. We will denote the induced Euler structure by $e^{\overline{M}}$, and define it as

$$e^{\overline{M}} = e \cup e_Z. \tag{3.2}$$

3.1.3 The Turaev Torsion Under Gluing

Now we are ready to state how the Turaev torsion changes when we glue a solid torus along a boundary components. This is based on [Tur02] Chapter VII, where one can find the proof (Turaev gives the statement for multiple gluings at once, but for our purposes later we will state the theorem for a single torus).

Theorem 3.1 (Turaev, 2002)**.** *Let M be a compact connected 3-manifold whose boundary consists of tori, and let $e \in Eul(M)$ be an Euler class and ω be a homology orientation of M. Let $\overline{M}$ be a 3-manifold with $b_1(\overline{M}) \geq 1$ obtained by gluing a directed solid torus to M and let $h \in H_1(\overline{M})$ denote the image of the distinguished homology class of the directed torus in $\overline{M}$. Let* $\mathrm{in} : \mathbb{Z}[H_1(M)] \to \mathbb{Z}[H_1(\overline{M})]$ *be induced by the inclusion $M \hookrightarrow \overline{M}$. If $b_1(M) \geq 2$ then $\tau(M, e, \omega) \in \mathbb{Z}[H_1(M)]$, and*

$$\mathrm{in}(\tau(M, e, \omega)) = (h - 1)\tau(\overline{M}, e^{\overline{M}}, \omega^{\overline{M}}). \tag{3.3}$$

From this theorem, we can deduce quite a bit about how we would like to see the determinant changing under gluing. Note that the Mayer-Vietoris sequence for $\overline{M} = M \cup_T Z$ for a solid torus Z shows that $H_1(\overline{M})$ is obtained from $H_1(M)$ by "killing" the image of the element in $H_1(T)$ that is being identified with the meridian of Z. Clearly, multiplication by $|\operatorname{Tors}(H_1(\overline{M}))|$ is equivalent to multiplication by $\Sigma = \sum \sigma$ (where σ runs over $\operatorname{Tors}(H_1(\overline{M}))$) modulo $\overline{I}$ (the augmentation ideal of $\overline{M}$). Thus if h is finite order in $H_1(\overline{M})$ then multiplication by $|\operatorname{Tors}(H_1(\overline{M}))|$ kills $h-1$ modulo $\overline{I}$. By either applying Theorem 2.1 if $\overline{M}$ is not closed, or [Tur02] Theorem 2.2 if $\overline{M}$ is closed, one may then suspect that in this situation, we will see $(\imath_M)_*(\operatorname{Det}_\omega(f_M)) = 0$ for any homology orientation ω, where we let $(\imath_M)_*$ denote the map induced by $M \hookrightarrow \overline{M}$ from the symmetric algebra on $H_1(M)/\operatorname{Tors}(H_1(M))$ to the symmetric algebra on $H_1(\overline{M})/\operatorname{Tors}(H_1(\overline{M}))$. Let $g \in H_1(M)$ denote the element being killed in $H_1(\overline{M})$; if g is finite order and h is not, then $b_1(M) = b_1(\overline{M})$, but the multiplication by $(h-1)$ in (3.3) will mean that $(\imath_M)_*(\operatorname{Det}_\omega(f_M))$ will be in a higher power of the augmentation ideal than $\operatorname{Det}_{\omega^{\overline{M}}}(f_{\overline{M}})$ if $\overline{M}$ is not closed, but the same power if $\overline{M}$ is closed. One may then suspect that $(\imath_M)_*(\operatorname{Det}_\omega(f_M)) = 0$ in this case as well.

If, however, we either have $\overline{M}$ not closed, and both h, g infinite order elements in $H_1(\overline{M}), H_1(M)$ respectively, or $\overline{M}$ closed and h of infinite order, we would expect a interesting formulae relating the determinants. We state these formulae, and the results of the paragraph above, in a Theorem now, which we will use the remainder of the Chapter to prove.

Theorem 3.2. *Let M be a compact, connected, oriented 3-dimensional manifold with nonempty boundary consisting of tori and homology orientation ω. Let $\overline{M}$ be obtained by gluing a directed solid torus Z along one boundary component T of M, and let ℓ denote the image in $S(H_1(\overline{M})/\operatorname{Tors}(H_1(\overline{M})))$ of the distinguished generator of $H_1(Z)$. If $\overline{M}$ is closed, assume $b_1(\overline{M}) \geq 3$, and if not assume $b_1(\overline{M}) \geq 2$.*

1. *If* $\partial M \neq T$ *and the image of* $H_1(T)$ *in* $H_1(M)$ *is not rank 2, then*

$$\mathrm{Det}_\omega(f_M) = 0.$$

2. *If* $\partial M = T$ *and* $b_1(\overline{M}) \neq b_1(M)$ *then*

$$(\imath_M)_*(\mathrm{Det}_\omega(f_M)) = 0.$$

3. *If* $\partial M \neq T$ *and the image of* $H_1(T)$ *in* $H_1(M)$ *is of rank 2, then*

$$|\operatorname{Tors}(H_1(M))|(\imath_M)_*(\mathrm{Det}_\omega(f_M)) = |\operatorname{Tors}(H_1(\overline{M}))| \cdot \ell \cdot \mathrm{Det}_{\omega^{\overline{M}}}(f_{\overline{M}}).$$

4. *If* $\partial M = T$ *and* $b_1(\overline{M}) = b_1(M)$ *then let* s_0 *denote the sign of the orientation* $\omega^{\overline{M}}$ *with respect to the natural homology orientation of* $\overline{M}$ *induced by an orientation. Then*

$$|\operatorname{Tors}(H_1(M))|(\imath_M)_*(\mathrm{Det}_\omega(f_M)) = s_0 |\operatorname{Tors}(H_1(\overline{M}))| \cdot \ell \cdot \mathrm{Det}(f_{\overline{M}}).$$

Before the proof, however, we should note that Theorem 3.2 and Theorem 2.1 can be used to obtain [Tur02] Theorem 2.2 (which we used above to motivate Theorem 3.2, but which will not be used in the proof). We briefly outline the procedure: Start with a closed connected oriented 3-manifold $\overline{M}$ with $b_1(\overline{M}) \geq 3$. Then choose an infinite order $h \in H_1(\overline{M})$ and remove the interior of a tubular neighborhood of an embedded S^1 representing h. Call the resulting compact connected oriented 3-manifold with boundary M. Choose a homology orientation ω of M so that $\omega^{\overline{M}}$ is the canonical homology orientation. Then [Tur02] Theorem 2.2 follows from plugging the results of Theorem 3.2 into Theorem 2.1 and Theorem 3.1.

3.2 Integral Cohomology Determinants Under Gluing - The Proof of Theorem 3.2

3.2.1 General Remarks

Let M be a compact, connected, oriented 3-manifold with $\partial M = \coprod_i T_i$ where the index i runs over some nonempty finite set, and each T_i is a torus. We will also consider $T = T_1$ and $R = \coprod_{i>1} T_i$ so that $\partial M = T \coprod R$ (note if ∂M has one component T, then $R = \varnothing$). We will be gluing a solid torus along the boundary component T and will use $\overline{M}$ to denote the result, i.e. $\overline{M} = M \bigcup_T Z$ for a solid torus Z (the actual homeomorphism of T to $\partial(\mathbb{D}^2 \times S^1)$ will of course matter in the actual construction of $\overline{M}$, but we will not include it in our notation for simplicity). We will also assume that M, $\overline{M}$ are given consistent orientations. Since there is a difference in definition of the determinant for $\overline{M}$ closed, we will study the cases $R \neq \varnothing$ and $R = \varnothing$ separately. Here let us also set some notation for the rest of this chapter. We will often let λ, μ be a basis of $H_1(T)$ such that μ is the curve along which we will glue the meridian of our solid torus and λ is parallel to the distinguished generator of $H_1(Z)$. In other words, μ is killed in $H_1(\overline{M})$, and λ maps to $h \in H_1(\overline{M})$. The assumptions $b_1(\overline{M}) \geq 2$ if $\partial\overline{M} \neq \varnothing$ and $b_1(\overline{M}) \geq 3$ if $\partial\overline{M} = \varnothing$, will guarantee the appropriate ranges for $b_1(M)$ so that we will have well defined determinants for both M and $\overline{M}$.

Whether $\overline{M}$ is closed or not, we must analyze mappings in cohomology; there is an obvious and natural map $H^1(\overline{M}) \to H^1(M)$ induced by the inclusion $M \hookrightarrow \overline{M}$. However, ∂M does not map to $\partial\overline{M}$ under the inclusion, so it does not induce a map from $H^1(\overline{M}, \partial\overline{M})$ to $H^1(M, \partial M)$. This means we will require a way to work around this unfortunate detail.

Before we do so, however, we will give some results that we will be using throughout the chapter. First, note by excision:

$$H^i(\overline{M}, M) \approx H^i(Z, \partial Z) \approx \begin{cases} \mathbb{Z} & \text{if } i = 2, 3 \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Combining (3.4) with the cohomology exact sequence of the pair $(\overline{M}, M)$

$$H^1(\overline{M}, M) \to H^1(\overline{M}) \to H^1(M) \to H^2(\overline{M}, M) \tag{3.5}$$

we see that the cokernel of $H^1(\overline{M}) \to H^1(M)$ is rank 0 or rank 1, and the kernel is 0. This means $b_1(\overline{M})$ can either be $b_1(M)$ or $b_1(M) - 1$. Intuitively, the two cases correspond to either killing a finite order element or an infinite order element when we glue the solid torus along T.

We will also need to know something about how Poincaré duality compares before and after gluing. Intuitively, one would expect that "away from T" (whatever that means), duality should be largely unchanged. We now precisely state this intuitive idea. To set some convenient notation, we will use $\imath_M$ to denote the inclusion $M \hookrightarrow \overline{M}$, $\imath_R$ to denote the inclusion $R \hookrightarrow \partial M$, and finally $\imath_{\partial \overline{M}}$ to denote the inclusion $\partial \overline{M} \to \partial M$ (by itself, this is the same as $\imath_R$, but we will use the notation $\imath_{\partial \overline{M}}$ when we want to look at induced maps for the triple $(\overline{M}, \partial M, \partial \overline{M})$ and $\imath_R$ to look at induced maps for the triple $(M, \partial M, R)$). Note that the map induced on cohomology by $\imath_M$ maps $H^*(\overline{M}, \partial \overline{M})$ to $H^*(M, R)$.

Proposition 3.1. *For any $w \in H^1(\overline{M}, \partial \overline{M})$, if there is a $w' \in H^1(M, \partial M)$ such that $(\imath_M)^*(w) = (\imath_R)^*(w') \in H^1(M, R)$, then*

$$w \cap [\overline{M}] = (\imath_M)_*(w' \cap [M]).$$

Proposition 3.1 will allow us to work around the fact that the inclusion $\imath_M : M \hookrightarrow \overline{M}$ does not induce a map $\partial M \to \partial \overline{M}$, hence does not induce a map $H_3(M, \partial M) \to H_3(\overline{M}, \partial\overline{M})$. In particular, the inclusion does not induce anything so convenient as the map $[M] \mapsto [\overline{M}]$ of $H_3(M, \partial M) \to H_3(\overline{M}, \partial\overline{M})$. Furthermore, $\imath_M$ does not induce a nice map $H^1(\overline{M}, \partial\overline{M}) \to H^1(M, \partial M)$, so this Proposition helps us work around that as well.

Proof. Look at the commutative ladder induced by the inclusion $M \hookrightarrow \overline{M}$ with rows given by the cohomology sequences of the triples $(M, \partial M, R)$ and $(\overline{M}, \partial M, \partial\overline{M})$ (note $\partial\overline{M} = R$ and could be empty):

$$\begin{array}{ccccccc}
 & & H^2(\overline{M}, M) & = & H^2(\overline{M}, M) & & \\
 & & \uparrow & & \uparrow & & \\
H^0(\partial M, R) & \longrightarrow & H^1(M, \partial M) & \longrightarrow & H^1(M, R) & \longrightarrow & H^1(\partial M, R) \\
\| & & \uparrow & & \uparrow & & \| \\
H^0(\partial M, \partial\overline{M}) & \longrightarrow & H^1(\overline{M}, \partial M) & \longrightarrow & H^1(\overline{M}, \partial\overline{M}) & \longrightarrow & H^1(\partial M, \partial\overline{M}) \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{3.6}$$

A simple diagram chase, assuming a suitable w' exists, shows that there is a $\widetilde{w} \in H^1(\overline{M}, \partial M)$ mapping to $w \in H^1(\overline{M}, \partial\overline{M})$ and $w' \in H^1(M, \partial M)$. Now note by Alexander duality, $H_3(\overline{M}, \partial M)$ is free of rank 2, and we have the following diagram with any straight line exact:

$$\begin{array}{ccccccc}
H_2(\partial M, \partial\overline{M}) & & & & H_3(M, \partial M) \longrightarrow & H_2(\partial M) \longrightarrow & H_2(M) \\
 & \nwarrow & & \swarrow^{(\imath_M)_*} & & // & \swarrow \\
 & & H_3(\overline{M}, \partial M) & \longrightarrow & H_2(\partial M) & \longrightarrow & H_2(\overline{M}) \\
 & \swarrow & & \nwarrow^{(\imath_{\partial\overline{M}})_*} & & \nwarrow & \backslash\backslash \\
H_3(\overline{M}, M) & & & & H_3(\overline{M}, \partial\overline{M}) \longrightarrow & H_2(\partial\overline{M}) \longrightarrow & H_2(\overline{M})
\end{array} \tag{3.7}$$

We see $H_3(\overline{M}, \partial M)$ is generated by the images of the orientation classes $[M]$ and $[\overline{M}]$, and the difference of those images maps to (plus or minus) the generator of

$H_3(\overline{M}, M)$. So we now perform some simple computations:

$$\begin{aligned}
\widetilde{w} \cap (\imath_M)_*([M]) &= (\imath_M)_* \left((\imath_M)^*(\widetilde{w})\right) \cap [M]) \\
&= (\imath_M)_*(w' \cap [M]). \\
\widetilde{w} \cap (\imath_{\partial \overline{M}})^*([\overline{M}]) &= (\imath_{\partial \overline{M}})_* \left((\imath_{\partial \overline{M}})^*(\widetilde{w})\right) \cap [\overline{M}]) \\
&= (\imath_{\partial \overline{M}})_*(w \cap [\overline{M}]) \\
&= w \cap [\overline{M}].
\end{aligned}$$

The last equality follows since the map induced by $(\imath_{\partial \overline{M}})$ on $H_2(\overline{M})$ is equality in diagram (3.7).

So we want to compute the cap product of $\widetilde{w}$ with the difference of $(\imath_M)_*([M])$ and $(\imath_{\partial \overline{M}})_*([\overline{M}])$ and show that it is zero. The chain complex $C_*(\overline{M}, \partial M)$ consists of the chain complex $C_*(M, \partial M)$ with an additional two-handle and an additional three-handle, and the difference we are interested in is the class in $H_3(\overline{M}, \partial M)$ of the additional three-handle. To compute the cap product of $\widetilde{w}$ with this homology class, we evaluate $\widetilde{w}$ on a 1-front face, and this is the coefficient of the 2-back face. But each 1-front face of our 3-handle lies on T, and $\widetilde{w} \in H^1(\overline{M}, \partial M)$ means $\widetilde{w}$ is zero when restricted to ∂M, in particular when restricted to T. □

Recall that our determinants lie in the symmetric algebras $S = S\left((H^1(M))^*\right)$ and $\overline{S} = S\left((H^1(\overline{M}))^*\right)$ (for $M, \overline{M}$ respectively), so here we briefly comment on $S, \overline{S}$ and the map $S \to \overline{S}$ induced by the inclusion $M \hookrightarrow \overline{M}$. First, the map $H^1(\overline{M}) \to H^1(M)$ induced by inclusion induces a dual map $(H^1(M))^* \to (H^1(\overline{M}))^*$, and if we think of $(H^1(M))^*$ as simply $H_1(M)/\operatorname{Tors}(H_1(M))$ and $(H^1(\overline{M}))^*$ as simply $H_1(\overline{M})/\operatorname{Tors}(H_1(\overline{M}))$, then the map $S \to \overline{S}$ is the map induced by $H_1(M) \to H_1(\overline{M})$ (which maps $\operatorname{Tors}(H_1(M)) \to \operatorname{Tors}(H_1(\overline{M}))$). Now $H_1(M) \to H_1(\overline{M})$ is onto (its cokernel is contained in $H_1(\overline{M}, M) = 0$), and similarly $H^1(\overline{M}) \to H^1(M)$ is injective with free cokernel of rank 0 or 1. If the cokernel is rank 0, then $H^1(\overline{M}) \to H^1(M)$ and $H_1(M)/\operatorname{Tors}(H_1(M)) \to H_1(\overline{M})/\operatorname{Tors}(H_1(\overline{M}))$ are isomorphisms, as is $S \to \overline{S}$. If the cokernel is rank 1, then we may choose a basis $\alpha_1, \ldots, \alpha_{n-1}$ of $H^1(\overline{M})$ and then construct a basis $a_1, \ldots, a_n$ of $H^1(M)$ with $(\imath_M)^*(\alpha_i) = a_i$ for $1 \leq i \leq n-1$, and a_n having nonzero image in $H^2(\overline{M}, M)$. Then the induced map $(H^1(M))^* \to (H^1(\overline{M}))^*$ is the map $a_i^* \mapsto \alpha_i^*$ for $1 \leq i \leq n-1$, and $a_n^* \mapsto 0$ (and similarly $S \to \overline{S}$). We will slightly abuse notation and denote the map $S \to \overline{S}$ by $(\imath_M)_*$.

3.2.2 $R \neq \varnothing$

In this case, we know that $\overline{M}$ is also a 3-manifold with nonempty boundary, so we will use the determinant from 2.1.1. First, a preliminary result involving rank counting.

Lemma 3.1. *The following are all equal to zero:*

$$b_0(M,T) = b_0(M,R) = b_3(M,T) = b_3(M,R) = 0. \tag{3.8}$$

The following are all equal:

$$b_1(M,T) = b_2(M,T) = b_1(M,R) = b_2(M,R). \tag{3.9}$$

Proof. We first note $b_0(M,T) = 0$ and $b_0(M,R) = 0$ since $H_0(T) \to H_0(M)$ and $H_0(R) \to H_0(M)$ are both surjective, and then

$$b_3(M,R) = b^3(M,R) = b_0(M,T) = 0.$$

The first equation is by the universal coefficient theorem, the second is by Poincaré duality. We similarly conclude $b_3(M,T) = 0$.

Now $b_1(M,T) = b^2(M,R) = b_2(M,R)$ by the same reasoning as above, so it remains to show that $b_1(M,T) = b_2(M,T)$. This follows from counting ranks in the exact sequence of the pair (M,T) and noting that since $\chi(M) = \chi(T) = 0$ and $b_0(M,T) = b_3(M,T) = 0$, we must have $b_1(M,T) = b_2(M,T)$. $\square$

Now we will look at (the first few terms of) the exact sequence of the triple $(M, \partial M, R)$ and the (reduced) exact sequence of the pair (M,T) (both in cohomology):

$$0 \to H^0(\partial M, R) \to H^1(M, \partial M) \to H^1(M,R) \to H^1(\partial M, R), \tag{3.10}$$

$$0 \to H^1(M,T) \to H^1(M) \to H^1(T). \tag{3.11}$$

Note that $H^1(\partial M, R) \approx H^1(T)$ and in fact we can form a commutative square with the last two terms each of (3.10) and (3.11), where the right vertical arrow is an isomorphism:

$$\begin{array}{ccc} H^1(M,R) & \longrightarrow & H^1(\partial M,R) \\ \big\downarrow & & \big\downarrow \\ H^1(M) & \longrightarrow & H^1(T). \end{array} \tag{3.12}$$

Since $H^1(T) \approx \mathbb{Z}^2$, the maximum rank of the image of each horizontal arrow is two, and by commutativity and the fact that the right vertical arrow is an isomorphism, the rank of the image of $H^1(M,R)$ in $H^1(\partial M, R)$, which we will denote by $s = \text{rank}_{\mathbb{Z}}(\text{im}(H^1(M,R) \to H^1(\partial M,R)))$, is less than or equal to the rank of the image of $H^1(M)$ in $H^1(T)$, which we will denote by $r = \text{rank}_{\mathbb{Z}}(\text{im}(H^1(M) \to H^1(T)))$ (i.e. $r \geq s$). Now if $n = b_1(M)$ then $n - 1 = b_1(M, \partial M)$. Note $H^0(\partial M, R) \approx \mathbb{Z}$ so counting ranks in (3.10) gives

$$b_1(M,R) = n - 2 + s. \tag{3.13}$$

Case 1: $r = 2$

First, note that this can occur; for example the exterior of the Hopf link, where T is either boundary component, has $H^1(M) \to H^1(T)$ an isomorphism. So this case is not vacuous.

In this case, $b_1(M, T) = n - 2$, so by (3.13) and Lemma 3.1, $s = 0$. Each group in both (3.10) and (3.11) is free, and $H^1(M, \partial M) \to H^1(M, R)$ is onto hence splits, so given a basis $\beta_1, \ldots, \beta_{n-2}$ of $H^1(M, R)$, we may choose a basis $b_1, \ldots, b_{n-1}$ of $H^1(M, \partial M)$ such that $b_i \mapsto \beta_i$ for $1 \leq i \leq n-2$ and b_{n-1} is dual (under evaluation) to a path connecting T to one of the components of R. If we let $\imath_T : T \hookrightarrow M$ be the inclusion, then $(\imath_T)_*([T]) = b_{n-1} \cap [M]$.

We now similarly compare $H^1(M)$ to $H^1(M, T)$. Since $r = 2$, for any basis $\alpha_1, \ldots, \alpha_{n-1}$ of $H^1(M, T)$, we can choose a basis $a_1, \ldots, a_n$ of $H^1(M)$ with $\alpha_i \mapsto a_i$ for $1 \leq i \leq n-2$, and a_{n-1}, a_n mapping to linearly independent elements in $H^1(T)$. Thus if we choose any basis c_1, c_2 of $H^1(T)$, then $\imath_T^*(a_{n-1}) = a_{1,1}c_1 + a_{1,2}c_2$ and $\imath_T^*(a_n) = a_{2,1}c_1 + a_{2,2}c_2$ where $A = \left(\begin{smallmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{smallmatrix}\right)$ is an integral matrix with $\det(A) = D \neq 0$.

Now we look at the matrix over $S((H^1(M))^*)$ given by cup product as in Lemma 2.1. There will be a $n-2 \times n-2$ square matrix in the upper left hand corner composed of the cup products of the α's and β's, let us call this matrix $\mathfrak{M}$, and then the last two columns will be for cup products of the b_i with a_{n-1}, a_n and the last row for b_{n-1} cup the a_j. Recall the matrix θ from Lemma 2.1, $\theta_{i,j} = g(b_i, a_j)$; θ will have the form

$$\begin{pmatrix} \mathfrak{M} & v_1 & v_2 \\ w & \pm D a_n^* & \mp D a_{n-1}^* \end{pmatrix}. \tag{3.14}$$

Above, v_1, v_2 are dimension $n-2$ column vectors and w is a dimension $n-2$

row vector, and the signs are chosen depending on whether $c_1 \cup c_2$ is dual, under evaluation, to $\pm[T]$. Now since $(\imath_T)_*([T]) = b_{n-1} \cap [M]$, for any $u, v \in H^1(M)$, we can compute

$$\langle u \cup v \cup b_{n-1}, [M] \rangle = \langle u \cup v, b_{n-1} \cap [M] \rangle = \langle \imath_T^*(u) \cup \imath_T^*(v), [T] \rangle.$$

This explains the $\pm D$ terms in the matrix, and also allows us to note that if $a_i \cup b_{n-1} \neq 0$ for some i, then there is some $v \in H^1(M)$ such that

$$\langle a_i \cup v \cup b_{n-1}, [M] \rangle \neq 0,$$

so $\imath_T^*(a_i) \neq 0$. This means that the row vector w is equal to 0, since $\imath_T^*(\alpha_i) = 0$ for $1 \leq i \leq n-2$. And now it is easy to compute the determinant,

$$\det(\theta(n)) = (\pm D a_n^*) \det(\mathfrak{M}). \tag{3.15}$$

Now we know $b_1(M, R) = n - 2$, and we must have $b_1(\overline{M}, \partial \overline{M}) = n - 2$ as well since we must have $b_1(\overline{M}) = b_1(M) - 1$. Geometrically, this means if each generator of $H_1(T)$ is infinite order in $H_1(M)$ (which corresponds to $r = 2$), then by gluing a solid torus on T we must kill an infinite order element. Since $H^1(\overline{M}, \partial \overline{M})$ injects into $H^1(M, R)$ with a free cokernel and they have the same rank, $H^1(\overline{M}, \partial \overline{M}) \to H^1(M, R)$ an isomorphism,

We also look at the triple $(\overline{M}, M, T)$, using the following commutative diagram with exact rows and columns (we abuse notation and let T denote the image of

T in $\overline{M}$):

$$\begin{array}{ccccccc} & & H^2(\overline{M},M) & = & H^2(\overline{M},M) & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^1(M,T) & \longrightarrow & H^1(M) & \longrightarrow & H^1(T) \\ & & \uparrow & & \uparrow & & \| \\ 0 & \longrightarrow & H^1(\overline{M},T) & \longrightarrow & H^1(\overline{M}) & \longrightarrow & H^1(T) \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array} \tag{3.16}$$

This diagram is obtained by "pulling apart" (along the equalities) the braid diagram that gives rise to the exact sequence of the triple. By commutativity, we note that the image of $H^1(\overline{M})$ in $H^1(T)$ has rank 1, and thus $H^1(\overline{M},T)$ has rank $b_1(\overline{M})-1 = n-2 = b_1(M,T)$ and the map $H^1(\overline{M},T) \to H^1(M,T)$ is an injection with free cokernel of free $\mathbb{Z}$-modules of the same rank, hence is an isomorphism. So we may choose a basis of $H^1(\overline{M})$ by choosing a basis of $H^1(\overline{M},T)$ and a preimage of the generator of the image of $H^1(\overline{M})$ in $H^1(T)$, let us denote this element by $\alpha_{n-1} \in H^1(\overline{M})$.

We now have chosen bases of $H^1(\overline{M})$ and $H^1(\overline{M},\partial\overline{M})$ which are very similar to the bases of $H^1(M)$ and $H^1(M,\partial M)$, and the matrix we will want to study for the purposes of constructing the determinant, which we will call $\overline{\theta}$, will have the square $n-2 \times n-2$ block in the upper left corner $(\imath_M)_*(\mathfrak{M})$ (this follows from Proposition 3.1). This means

$$\det(\overline{\theta}(n-1)) = (\imath_M)_*(\det(\mathfrak{M})). \tag{3.17}$$

Now recall our notation of λ, μ as the basis of $H_1(T)$ introduced in 3.2.1. Then λ^*, μ^* is a basis of $H^1(T)$, and let $k \in \mathbb{Z}$ such that $\imath_T^*(\alpha_{n-1}) = k\lambda^*$ (we have no multiples of μ since $\langle \imath_T^*(\alpha_{n-1}), \mu\rangle = \langle \alpha_{n-1}, (\imath_T)_*(\mu)\rangle = 0$ since μ is killed in $\overline{M}$). We can take $(\imath_M)^*(\alpha_{n-1})$ to be one of our generators in $H^1(M)$ with nonzero image in $H^1(T)$ by commutativity of (3.16) and the fact that $(\imath_M)^*$ has free cokernel. Choose any suitable a_n for the final generator of $H_1(M)$, and let $m \in \mathbb{Z}$ such that $\langle a_n, (\imath_T)_*(\mu)\rangle = m$, so that the D given in (3.14) is simply $k \cdot m$ and note that since μ is killed in $H_1(\overline{M})$, we are introducing new m-torsion to $H_1(\overline{M})$, i.e. we have $|\operatorname{Tors}(H_1(\overline{M}))| = m \cdot |\operatorname{Tors}(H_1(M))|$. Also, for simplicity, if the D appearing in (3.15) is negative, we can change a_n to $-a_n$ to force the sign of D to be positive.

Now we are finally ready to compare the determinants of the forms f_M and $f_{\overline{M}}$. Let a be the basis of $H^1(M)$ consisting of $a_1 = (\imath_M)^*(\alpha_1), \ldots, a_{n-2} = (\imath_M)^*(\alpha_{n-2})$, followed by $a_{n-1} = (\imath_M)^*(\alpha_{n-1})$ and then a_n. Let b be the basis of $H^1(M, \partial M)$ consisting of $b_1, \ldots, b_{n-2}$ with $(\imath_R)^*(b_i) = (\imath_M)^*(\beta_i)$ (for $i \leq n-2$), followed by b_{n-1}. Then with θ expressed in this basis, $\det(\theta(n)) = (-1)^n a_n^* d(f_M, a, b)$ by Lemma 2.1. But by (3.15), we have (recalling we chose a_n so that D is positive)

$$D a_n^* \det(\mathfrak{M}) = (-1)^n a_n^* d(f_M, a, b).$$

This means

$$d(f_M, a, b) = (-1)^n D \det(\mathfrak{M}). \tag{3.18}$$

Furthermore, by (3.17) and Lemma 2.1, if we choose the basis α of $H^1(\overline{M})$ to be $\alpha_1, \ldots, \alpha_{n-2}$ followed by α_{n-1}, and the basis β of $H^1(\overline{M}, \partial\overline{M})$ to be $\beta_1, \ldots, \beta_{n-2}$, then

$$\begin{aligned}(\imath_M)_*(\det(\mathfrak{M})) &= \det(\overline{\theta}(n-1)) \\ &= (-1)^{n-1} \alpha_{n-1}^* d(f_{\overline{M}}, \alpha, \beta). \end{aligned} \tag{3.19}$$

Now plugging (3.19) into (3.18) we obtain

$$(\imath_M)_*(d(f_M, a, b)) = -D \alpha_{n-1}^* d(f_{\overline{M}}, \alpha, \beta).$$

We have constructed this so that $\langle \alpha_{n-1}, \lambda \rangle = k$ meaning $k\alpha_{n-1}^* \mapsto \ell$ in the canonical isomorphism $(H^1(\overline{M}))^* \to H_1(\overline{M})/\operatorname{Tors}(H_1(\overline{M}))$ where ℓ is the image of λ in $H_1(\overline{M})/\operatorname{Tors}(H_1(\overline{M}))$, so this means

$$(\imath_M)_*(d(f_M, a, b)) = (-1)m \cdot \ell \cdot d(f_{\overline{M}}, \alpha, \beta). \tag{3.20}$$

To complete the proof of Theorem 3.2 Item 3, we must see how the sign refined determinants work out using the induced homology orientation on $\overline{M}$. To do so, we first take the sign of the torsion of the exact sequence

$$H_3(\overline{M}, M; \mathbb{R}) \to H_2(M; \mathbb{R}) \to H_2(\overline{M}; \mathbb{R}) \to H_2(\overline{M}, M; \mathbb{R}) \to H_1(M; \mathbb{R}) \to H_1(\overline{M}; \mathbb{R}).$$

We do not need the end of the sequence since it contributes no sign. Note the sign in (3.1) is trivial, so $\omega^{\overline{M}}$ is simply an orientation of $H_*(\overline{M})$ making the torsion of the above sequence positive. A simple calculation shows us that if a, b, α, β are bases as above and we use them to compute the sign of the torsion of the above sequence, we obtain a negative answer. This means if a and b are bases such that $d(f_M, a, b) = \operatorname{Det}_\omega(f_M)$, then α and β are bases such that $d(f_{\overline{M}}, \alpha, \beta) = -\operatorname{Det}_{\omega^{\overline{M}}}(f_{\overline{M}})$. This proves Theorem 3.2 Item 3.

Case 2: $r = 1$

As in the earlier case, we first note that this case is not vacuous. An example would be the exterior of two disjoint unlinked S^1's embedded in S^3, with T as either boundary component. However, we will shortly see that the determinants in this case are as uninteresting as in our example.

We will first analyze the decompositions of $H^1(M, \partial M)$ and $H^1(M)$ with respect to $H^1(M, R), H^1(M, T)$, and $H^1(T)$ as before. Note if $r = 1$, then by (3.10) we know that $b_1(M, T) = n - 1$ and hence $s = 1$ as well. So now choose

$\alpha_1, \ldots, \alpha_{n-1}$ a basis of $H^1(\overline{M})$, and we can choose a basis of $H^1(M)$ with $\alpha_i \mapsto a_i$ for $1 \le i \le n-1$ and $\imath_T^*(a_n) \neq 0$.

Now $H_1(M, \partial M)$ still has a free summand of rank one generated by the dual of a path connecting T to any component of R, but now $H^1(M, R)$ splits as the image of $H^1(M, \partial M)$ plus another free generator, which must map to a_n under $H^1(M, R) \to H^1(M)$ since the cokernel of that map is free and everything else in $H^1(M, R)$ maps to zero in $H^1(T)$. So choose bases $b_1, \ldots, b_{n-1}$ of $H^1(M, \partial M)$ where b_{n-1} is as above, dual to a path connecting T to a component of R, and $b_i \mapsto \beta_i$ for $1 \le i \le n-2$ where $\beta_1, \ldots, \beta_{n-2}, \gamma$ is a basis of $H^1(M, R)$ and $\gamma \mapsto a_n$ under $H^1(M, R) \to H^1(M)$. Note we still have $(\imath_T)_*([T]) = b_{n-1} \cap [M]$.

In addition, we have, just as in our earlier case, for any $u, v \in H^1(M)$,

$$\langle u \cup v \cup b_{n-1}, [M] \rangle = \langle \imath_T^*(u) \cup \imath_T^*(v), [T] \rangle.$$

Since $1 = r = \text{rank}_{\mathbb{Z}}(\text{im}(\imath_T^*))$, we know $\imath_T^*(u), \imath_T^*(v)$ are both multiples of the same element in $H^1(T)$, so their cup product is zero. This means that the last row of the matrix θ consists entirely of zeros, so $\det(\theta(i)) = 0$ for any $1 \le i \le n$, proving Theorem 3.2 Item 1.

Case 3: $r = 0$

Unlike the first two cases, this case is vacuous; it cannot occur since $r = 0$ means $b_1(M, T) = b_1(M) = n$ hence $b_1(M, R) = n$, and then (3.13) gives $s = 2$, contradicting our earlier claim that $r \ge s$. Geometrically, this would correspond to the case that $H_1(T) \to H_1(M)$ has image entirely in $\text{Tors}(H_1(M))$. One can verify that this cannot happen by letting M' denote the result of gluing solid tori along each component of R in any way one likes, thus $T = \partial M'$. Note the following commutative diagram (the cokernel of $H_1(M) \to H_1(M')$ is contained in $H_1(M', M)$, which is zero by the homology analogue of (3.4), and similarly for $H_1(M,T) \to H_1(M',T)$):

$$\begin{array}{ccccc} H_1(T) & \longrightarrow & H_1(M) & \longrightarrow & H_1(M,T) \\ \| & & \downarrow & & \downarrow \\ H_1(T) & \longrightarrow & H_1(M') & \longrightarrow & H^1(M',T) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

From this diagram, we note that the image of $H_1(T)$ cannot be rank zero in $H_1(M)$, because it is rank one in $H_1(M')$.

3.2.3 $R = \varnothing$

In this case, we must compare the determinant from 2.1.1 (when we are looking at M, before the gluing) to the determinant from [Tur02] III.1. First, we know $H^1(M, \partial M) \to H^1(M)$ is an injection with free cokernel, which must be of rank 1 since $b_1(M, \partial M) = b_1(M) - 1$; we will still use n to denote $b_1(M)$. Now we still have $H^i(\overline{M}, M) \approx H^i(\mathbb{D}^2 \times S^1, S^1 \times S^1)$, so we still have $b_1(\overline{M})$ either equal to n or $n-1$ depending on whether the element in $H_1(M)$ killed is finite or infinite order, and each of these cases can occur. So let us examine both. Also note that the reasoning from above (in **Case 2:** $r = 1$) that led us to conclude that θ had a row consisting entirely of zeroes does not apply here, since that row corresponded to an element of $H^1(M, \partial M)$ connecting T to another boundary component, and such a thing does not exist if $\partial M = T$ (in fact, the image of $[T]$ is zero in $H_2(M)$, so we will not be pulling back cohomology elements along the inclusion of T into M at all).

Case 1: $b_1(\overline{M}) = b_1(M)$

If $b_1(\overline{M}) = n$, then we have $(\imath_M)^* : H^1(\overline{M}) \to H^1(M)$ is an isomorphism, and $H^1(M, \partial M) \to H^1(M)$ has kernel of rank 1. If we choose a basis $b_1, \ldots, b_{n-1}$ of $H^1(M, \partial M)$, we can choose a_n so that the images of the b_i, which we will call a_i, in $H^1(M)$ combined with a_n forms a basis of $H^1(M)$, and we will let $\alpha_i \in H^1(\overline{M})$ with $(\imath_M)^*(\alpha_i) = a_i$. Then the matrix $(\imath_M)_*(\theta)$ will be all but the last row of the matrix $\overline{\theta}$ by Proposition 3.1, and $(\imath_M)_*(\det(\theta(n))) = \det(\overline{\theta}(n,n))$, and hence

$$\begin{aligned}(\imath_M)_*((-1)^n a_n^* d(f_M, a, b)) &= (\imath_M)_*(\det(\theta(n))) \\ &= \det(\overline{\theta}(n,n)) \\ &= \alpha_n^* \alpha_n^* d(f_{\overline{M}}, \alpha, \alpha).\end{aligned}$$

Since $\alpha_n^* = (\imath_M)_*(a_n^*)$, the conclusion for determinants is that

$$(\imath_M)_*(d(f_M, a, b)) = (-1)^n (\imath_M)_*(a_n^*) d(f_{\overline{M}}, a, a).$$

Now if we have chosen λ, μ as above, a basis of $H_1(T)$ so that μ is the basis element along which the meridian of our solid torus is glued, then $(\imath_T)_*(\mu)$ is finite order in $H_1(M)$ since gluing does not change the first Betti number, so let us say that $(\imath_T)_*(\mu)$ has order $k \in H_1(M)$; this means

$$\operatorname{Tors}(H_1(\overline{M})) \cdot k = \operatorname{Tors}(H_1(M)).$$

Then since $\operatorname{Tors}(H_1(M)) \approx \operatorname{Tors}(H_1(M, \partial M))$, and $(\imath_T)_*(\mu)$ maps to zero in $H_1(M, \partial M)$, we must have a k^{th} root of $(\imath_T)_*(\lambda)$ in $H_1(M)$, which we can choose a_n to be dual to i.e. $\langle a_n, \lambda \rangle = k$. Finally, if we let ℓ denote the image of λ in $H_1(M)/\operatorname{Tors}(H_1(M))$, and since $(H^1(M))^*$ is (as discussed above in 3.2.1) naturally isomorphic to $H_1(M)/\operatorname{Tors}(H_1(M))$, then we can write $k\alpha_n^* = \ell$, so multiplying through by k we have

$$\begin{aligned} k \cdot (\imath_M)_*(d(f_M, a, b)) &= (-1)^n (k\alpha_n^*) d(f_{\overline{M}}, \alpha, \alpha) \\ &= (-1)^n \ell \cdot d(f_{\overline{M}}, \alpha, \alpha). \end{aligned}$$

To complete the proof of Theorem 3.2 Item 4, we must once again analyze signs. First, the sign of $\omega^{\overline{M}}$ is equal to $(-1)^n$ times the sign of an orientation ω' of $H_*(\overline{M})$ giving positive torsion of the exact sequence

$$H_2(M) \to H_2(\overline{M}) \to H_2(M, \overline{M}) \to H_1(M) \to H_1(\overline{M}).$$

This time, we have truncated the sequence both on the left and right since the truncated parts did not contribute to the sign. Another simple torsion calculation tells us that the sign s_0 in Theorem 3.2 Item 4 is simply $(-1)^n$ times the sign of $d(f_M, a, b)$ with respect to $\mathrm{Det}_\omega(f_M)$. This proves Theorem 3.2 Item 4.

Case 2: $b_1(\overline{M}) = b_1(M) - 1$

In this case, we may use the diagram (3.16), with $T = \partial M$, and we see that $H^1(\overline{M}, \partial M) \to H^1(M, \partial M)$ is an isomorphism, as is $H^1(\overline{M}, \partial M) \to H^1(\overline{M})$. So we may choose a basis $b_1, \ldots, b_{n-1}$ of $H^1(M, \partial M)$ and additional element $a_n \in H^1(M)$ with the images of the b_i, which we call a_i, combined with a_n is a basis of $H^1(M)$, and then $\overline{\theta} = (\imath_M)_*(\theta(n))$. Now we can choose $\alpha_1, \ldots, \alpha_{n-1}$ a basis of $H^1(\overline{M})$ with $(\imath_M)^*(\alpha_i) = a_i$ for $1 \leq i \leq n-1$. Using this basis, we compute the determinant $\det(\theta(1)) = (-1)a_1^* d(f_M, a, b)$ by running down the n^{th}

column:

$$
\begin{aligned}
(\imath_M)_*(\det(\theta(1))) &= \sum_{i<n}(-1)^{i+n}(\imath_M)_*(g(b_i,a_n))(\imath_M)_*(\det((\theta(1))(i,n))) \\
&= \sum_{i<n}(-1)^{i+n}(\imath_M)_*(g(b_i,a_n))\det(\overline{\theta}(i,1)) \\
&= \sum_{i<n}(-1)^{i+n}(\imath_M)_*(g(b_i,a_n))(-1)^{i+1}\alpha_i^*\alpha_1^* d(f_{\overline{M}},\alpha,\alpha) \\
&= (-1)^{n+1}\alpha_1^* d(f_{\overline{M}},\alpha,\alpha)\sum_{i<n}(\imath_M)_*(g(b_i,a_n))\alpha_i^* \\
&= (-1)^{n+1}(\imath_M)_*(a_1^*)d(f_{\overline{M}},\alpha,\alpha)\sum_{i<n}\sum_{k=1}^{n-1}(\imath_M)_*(f_M(b_i,a_n,a_k)a_k^*)\alpha_i^* \\
&= (-1)^{n+1}(\imath_M)_*(a_1^*)d(f_{\overline{M}},\alpha,\alpha)\sum_{i<n}\sum_{k=1}^{n-1}\langle b_i\cup a_n\cup a_k,[M]\rangle\alpha_k^*\alpha_i^* \\
&= (-1)^{n+1}(\imath_M)_*(a_1^*)d(f_{\overline{M}},\alpha,\alpha)\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}-\langle b_i\cup a_k,a_n\cap[M]\rangle\alpha_k^*\alpha_i^* \\
&= (-1)^{n+1}(\imath_M)_*(a_1^*)d(f_{\overline{M}},\alpha,\alpha)\sum_{i,k=1}^{n-1}-\langle b_i\cup b_k,a_n\cap[M]\rangle\alpha_k^*\alpha_i^* \\
&= 0.
\end{aligned}
$$

The last equality is true since we are summing over i,k and the $b_i\cup b_k$ is antisymmetric in i,k and $\alpha_k^*\alpha_i^*$ is symmetric. The line before that follows from noting that a_k is the image of b_k under $H^1(M,\partial M)\to H^1(M)$.This proves Theorem 3.2 Item 2, and in fact completes the proof of Theorem 3.2. □

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