Abstract

Title of dissertation: ORBIT DISCONTINUITIES AND TOPOLOGICAL MODELS FOR BOREL SEMIFLOWS

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Let T_t be a Borel semiflow on a standard Polish space X. We say two distinct points x and y are "instantaneously discontinuously identified" (IDI) by the semiflow if $T_t(x) = T_t(y)$ for all $t > 0$. We define the concept of "orbit discontinuity", a generalization of IDI, and examine the prevalence and structure of orbit discontinuities for arbitrary Borel semiflows. In particular we show that points have only countably many orbit discontinuities and that the set of orbit discontinuities has measure zero with respect to any measure preserved by the semiflow. Additionally, if the semiflow preserves a Borel probability measure on X, we show that the Ambrose-Kakutani theorem can be adapted to find both an extension and a factor of the semiflow which are conjugate to the original semiflow except on a set of measure zero. Both the factor and extension are characterized by a Polish space called the "base" with a vertical semiflow consisting of repeated quotient maps onto successively larger closed subsets of the base together with a return-time transformation describing how points return to the base. The points where the conjugacy fails are the orbit discontinuities of the original semiflow.

Furthermore, we develop the concept of “orbit discontinuity” from a measure-theoretic perspective. Assuming T_t preserves a Borel probability measure μ on X, we show that for all points x in an invariant set of full $\mu-$measure, there exist two “measure paths” $\mu^+_{x,t}$ and $\mu^-_{x,t}$ which give, for almost every time t, a natural distribution on the set of points y with $T_t(x) = T_t(y)$. These measures are constructed by taking weak$^*-$limits of conditional expectations. We show that these measure paths coincide and are weak$^*-$continuous except at countably many times t. If the measure paths differ at $t = 0$ for some point x, then x has an orbit discontinuity at time 0.

ORBIT DISCONTINUITIES AND TOPOLOGICAL MODELS FOR BOREL SEMIFLOWS

by

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Introduction

1.1 Dynamical systems and ergodic theory

This dissertation is motivated by the problem of finding universal topological models for Borel semiflows on Polish spaces. In particular, the manner in which points are identified by such a semiflow must be accounted for when building such a model. The work here focuses on the analysis of such identifications from both a topological and measure-theoretic perspective.

This introductory chapter motivates the work in this thesis and provides some background in dynamical systems. In Chapters 2 and 3 we develop the concepts of "instantaneous discontinuous identification" and "orbit discontinuity" from a topological perspective. In Chapter 4 these notions are applied to improve some models for semiflows originally developed by Krengel. Finally in Chapter 5 we reexamine the same phenomena in a measure-theoretic context. Chapter 6 briefly discusses some further questions arising from this work.

This work is in the broad field of dynamical systems. A dynamical system consists of a phase space (which in this paper will usually be denoted X) together with an action on X which describes how points in the space move as time passes.

By an *action* of a group G on a space X we mean a map $G \times X \to X$ denoted by $(g, x) \mapsto gx$ such that

i. $g_1(g_2(x)) = (g_1 g_2)(x)$ for all $g_1, g_2 \in G$ and all $x \in X$, and

ii. If e is the identity element of G, $ex = x$ for all $x \in X$.

In such a setup time can either be measured discretely (for example, if the acting group is the integers) or continuously (for example, in an action of the real numbers). The time measurement in a dynamical system need not be one-dimensional; for instance if one has two commuting invertible transformations on the space X one obtains an action of the group $\mathbb{Z}^2$ on X. Dynamical systems arise naturally in astronomy, meteorology, and population biology, as well as many other fields of study.

One restriction of group actions is that the map $x \mapsto gx$ associated to any particular group element g must be invertible (its inverse is the action of g^{-1}). However, many examples of dynamical systems occurring in nature are not invertible; some examples of this are the logistic map (which has applications in population dynamics), solutions to the heat equation (which describes the flow of heat in a homogenous body), and Brownian motion (which models the "random" movement of dust particles floating in a liquid).

This leads one to consider actions of not groups but rather semigroups. We establish some notation here: given a semigroup G, for $g \in G$, we define $T_{-g}(A) = \{x : T_g(x) \in A\}$. This notation allows us to make sense of "pulling back" a set backwards in time even if the action is not invertible. In this paper we consider actions of the semigroup $\mathbb{R}^+ = [0, \infty)$; such systems are called *semiflows*. In this context we use T_t rather than T_g to represent the semiflow to emphasize that the subscripted variable represents the passage of time in the system.

We say that a dynamical system $(X, \mathcal{F}, \mu, T_g)$ is a *measure-preserving system* if the phase space X is a standard probability space and the action T_g is a measurable action that preserves a probability measure μ on X. To say that $(X, \mathcal{F}, \mu)$ is a standard probability space means that X can be thought of as the unit interval $[0, 1]$, and $\mathcal{F}$ and μ can be thought of as the Lebesgue-measurable subsets of $[0, 1]$ and Lebesgue measure, respectively. We call an action *measurable* (given some $\sigma-$algebra $\mathcal{G}$ of subsets of G) if for any measurable set $A \in \mathcal{F}$, the set of points in $G \times X$ which map into A under the action $(g, x) \mapsto gx$ is $\mathcal{G} \times \mathcal{F}-$measurable. For a discrete action this is equivalent to the action T_g of each element $g \in G$ being measurable (i.e. $T_{-g}(A) \in \mathcal{F}$ for any $A \in \mathcal{F}$). To say that a measurable action preserves μ means that for any $A \in \mathcal{F}$ and any $g \in G$, $T_{-g}(A) \in \mathcal{F}$ and $\mu(T_{-g}(A)) = \mu(A)$. From a probabilist's perspective, the elements of $\mathcal{F}$ are thought of as events, and μ measures the probability of an event occuring. To say an action is measure-preserving means that the probability of observing an event does not change as time passes (in probability such a process is called stationary).

The study of measure-preserving dynamics is called ergodic theory. This name comes from the "ergodic hypothesis" which states that for a measure-preserving system and a measurable function $f : X \to \mathbb{R}$, the "time averages" of f, which for a $\mathbb{Z}-$action are

$$\frac{1}{n}\sum_{k=1}^{n} f(T_k(x)),$$

and for an $\mathbb{R}-$action (flow) are

$$\frac{1}{t}\int_0^t f(T_s(x))ds,$$

converge as $n \to \infty$ (or as $t \to \infty$ in the case of a continuous time action) to the "space average" of f, namely

$$\int_X f\, d\mu.$$

Notice that in the case of a flow, the assumption that the action is measurable is important because it ensures that the integrand in each time average is integrable.

Much is known regarding this hypothesis; perhaps most famously Birkhoff showed in 1931 that time averages converge pointwise (almost surely in μ) to an invariant function $\widehat{f}$ (a function $\widehat{f}$ is *invariant* if $\widehat{f} = \widehat{f} \circ T_g$ a.s.-μ for all $g \in G$). This invariant function $\widehat{f}$ is necessarily the space average of f if the space X cannot be subdivided into nontrivial invariant pieces (a set $A \subseteq X$ is *invariant* if $\mu(A \bigtriangleup T_g(A)) = 0$ for all g). A system satisfying this condition is therefore called *ergodic*; it is known that any measure-preserving system which is not ergodic can be decomposed into an integral of disjoint ergodic invariant components. Consequently it is appropriate (and convenient) to study only ergodic systems, and we do just that here.

One fundamental problem in ergodic theory is to determine when two systems are "equivalent". In particular two systems $(X, \mathcal{F}, \mu, T_g)$ and $(Y, \mathcal{G}, \nu, S_g)$ are said to be *measurably conjugate* if there exists an invertible measure-preserving map $\phi : X \to Y$ such that $S_g \circ \phi = \phi \circ T_g\ \forall g \in G$ for $\mu-$almost every $x \in X$. Systems which are measurably conjugate have all the same dynamical properties, so they can be thought of as "the same". A second problem is to determine what kinds of behavior is "generic", i.e. occurs in "most" dynamical systems.

In order to solve both the equivalence and genericity problems, it is useful to develop "universal models" for such systems. We say that a system $(\mathbf{X}, \mathbf{F}, \mathbf{T}_g)$ is a *universal model* for all systems in some class if every system $(X, \mathcal{F}, \mu, T_g)$ is measurably conjugate to $(\mathbf{X}, \mathbf{F}, \nu, \mathbf{T}_g)$ for some measure ν on $(\mathbf{X}, \mathbf{F})$ that is preserved under $\mathbf{T}_g$. Consequently any system in the given class is represented by a $\mathbf{T}_g-$invariant measure. This gives an approach to both the problems discussed above. Two systems will be measurably conjugate precisely when they can be represented by the same $\mathbf{T}_g-$invariant measure. Second, a property can be said to be *generic* for a class of systems if it holds for the universal model on a weak$^*-$residual set of $\mathbf{T}_g-$invariant measures.

For measure-preserving $\mathbb{Z}-$actions, Sinai's countable generator theorem [5] guarantees that the system $(\Omega^{\mathbb{Z}}, \sigma)$ is universal where $\Omega^{\mathbb{Z}}$ is the set of biinfinite sequences of points $\omega = ... \omega_{-1} \, \omega_0 \, \omega_{-1} ...$ taking values in a countable alphabet $\Omega = \{1, 2, 3, ...\}$ and $\sigma : \Omega^{\mathbb{Z}} \to \Omega^{\mathbb{Z}}$ is the shift map defined by the formula $(\sigma\omega)_n = \omega_{n+1}$. Ornstein identified in 1970 a class of measure-preserving systems (the finitely-determined class) which can be represented by the same shift-invariant measure if and only if they have the same entropy . Thus he showed that any two such systems are measurably conjugate if and only if their entropies are equal (see [6], [7], and [8]). Moreover a host of properties have been found via this universal model to be generic for ergodic $\mathbb{Z}-$actions (weak mixing, not mixing, zero entropy, etc.).

Similar models have been developed for the actions of a large number of discrete groups as well as some actions of semigroups like $\mathbb{Z}^+$. For flows (actions of $\mathbb{R}$), the Ambrose-Kakutani theory of cross-sections [1] shows that any measure-preserving flow is measurably conjugate to a "flow under a function" or "suspension flow". Such a system is described as follows: first, take a standard probability space $(\widehat{X}, \widehat{\mathcal{F}}, \widehat{\mu})$, an invertible $\widehat{\mu}-$preserving transformation $T : \widehat{X} \to \widehat{X}$ and a measurable function $f : X \to \mathbb{R}^+$. Now let $X = \{(x, t) \in \widehat{X} \times \mathbb{R}^+ : 0 \leq f(x)\}/\sim$ where $(x, f(x)) \sim (\widehat{T}(x), 0)$, and define a flow on X by $T_t(x, s) = (x, t + s)$. This

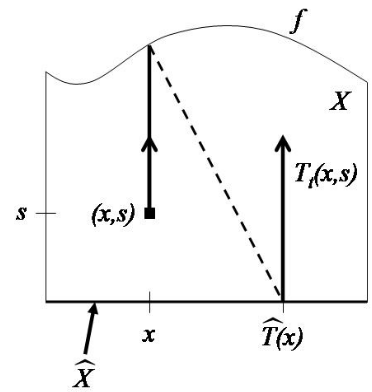

Figure 1.1: A suspension flow. The space X consists of the region bounded below by the base $\widehat{X}$ and above by the function f; the flow T_t takes points upward until they hit the graph of f.

flow takes points in X upward until they "hit" the graph of f, then they return to the base via the transformation $\widehat{T}$. Figure 1.1 describes this suspension flow construction.

However, no such models are known to exist for semiflows. The orbit structure of a semiflow is much more complicated than that of a discrete action or flow because points can be identified by a semiflow in a complicated fashion. Krengel showed that the Ambrose-Kakutani cross-sections can be constructed for a semiflow; unfortunately the resulting "suspension semiflow" (this is the same as a suspension flow save that the return map $\widehat{T}$ need not be invertible) is not necessarily conjugate to the original semiflow. Suspension semiflows are always $1-1$ away from the top and bottom of the space, and a given semiflow may not be invertible anywhere. It is natural to ask if the Ambrose-Kakutani-Krengel picture can be improved on; we in fact give an improvement in Chapter 4.

1.2 Polish spaces and Borel actions

Recently the ideas of ergodic theory have begun to be rephrased using tools of topology and descriptive set theory. Instead of supposing the phase space to be a probability space, one assumes that the phase space is Polish. A *Polish space* is a topological space which has some metric compatible with the given topology under which X becomes a complete, separable metric space. If a Polish space has a binary operation which is continuous and an identity element for the operation, we say the space is a *Polish semigroup*. Furthermore, if elements of a Polish semigroup have inverses under the operation and the function taking each element to its inverse is continuous, we say the semigroup is a *Polish group*. Examples of Polish spaces include $\mathbb{R}$ (a Polish group), $\mathbb{R}^+$ (a Polish semigroup), $[0, 1]$ and $(0, 1)$, each with the topology given by the usual distance. The natural class of measurable subsets of a Polish space are the Borel sets. A subset A of a Polish space is said to be *Borel* if any $\sigma-$algebra of subsets of X containing all the open sets must also contain A. Open and closed sets are Borel, and any set which can be written as the countable union of a countable intersection of a countable union of a ... etc. ... of open sets is Borel. An important subclass of Borel sets is the G_δ class; a set is a G_δ if and only if it is a countable intersection of open sets. The G_δ subsets of a Polish space are especially significant because of the following theorem [2]:

Theorem 1.2.1 *A subset Y of a Polish space X is Polish under the subspace topology if and only if Y is a G_δ in X.*

Suppose X and Y are two Polish spaces. We say that X and Y are *Borel isomorphic* if there is an injective map $\psi : X \to Y$ for which the preimage of every Borel set in Y is Borel in X and vice versa. In fact all Polish spaces of the same cardinality are Borel isomorphic [2]. We say a Polish space is *standard* if it has the cardinality of the continuum. In particular any standard Polish space is Borel isomorphic to the middle-thirds Cantor set.

Now we say the action $(g, x) \mapsto gx$ of a Polish group or semigroup G on a Polish space X is *Borel* if the inverse image of a Borel set in X is Borel in the product topology on $G \times X$. This thesis focuses on *Borel semiflows* as described below:

Definition 1.2.1 *A* Borel semiflow $(X, \mathcal{F}, \mu, T_t)$ *is a Polish space* X *with its* $\sigma-$*algebra* $\mathcal{F}$ *of Borel subsets and a probability measure* μ *on* $(X, \mathcal{F})$ *together with a Borel action* $T_t : X \times \mathbb{R}^+ \to X$ *which preserves* μ*, has no periodic points, and is surjective (i.e. for all* $x \in X$ *and* $t \in \mathbb{R}^+$*, there exists a* $y \in X$ *with* $T_t(y) = x$*).*

We assume throughout this paper the non-periodicity and surjectivity of all semiflows under consideration for technical reasons. The work here is motivated by the search for a universal model for Borel semiflows that preserve a Borel probability measure on X. (We mention here that in 1988 Wagh [10] gave a version of the Ambrose-Kakutani result for Borel *flows* on Polish spaces; all are conjugate to suspension flows over a standard Polish space with a Borel return-time function.)

The research presented in this dissertation is part of a larger project to describe all such semiflows as shift maps on path spaces. By a *path space* one means a set of functions from $[0, \infty)$ into the reals which pass through the origin (hypothetically there are other restrictions on the functions as well). The *shift map* σ_t for $t \geq 0$ on such a path space is defined by $\sigma_t(f)(\tau) = f(\tau + t) - f(t)$, i.e.

the shift "forgets" the piece of the graph of f over $[0, t)$ and then renormalizes so that the remaining path starts at the origin. Many examples of semiflows, most notably Brownian motion, can be described in this fashion.

To accomplish this goal, one can first measurably regard X as a subset of $[0, 1]$ (namely, the Cantor set since X is an uncountable Polish space) and then for each $x \in X$ define a function $\psi(x) : [0, \infty) \to \mathbb{R}$ by

$$\psi(x)(t) = \int_0^t T_s(x)ds.$$

This gives a mapping $x \to \psi(x)$ from X into a set Y of increasing, continuous functions from $[0, \infty)$ into $[0, \infty)$ satisfying $0 \leq \psi(x)(t) \leq t$.

Such a space Y is a Polish space, and the equivariance $\psi \circ T_t = \sigma_t \circ \psi$ holds so Y seems a good candidate for the desired universal model for semiflows. However, ψ is not necessarily injective. Consider two points $x \neq y$ in X such that $T_t(x) = T_t(y)$ for all $t > 0$; for these two points $\psi(x) = \psi(y)$. We say that the phenomenon exhibited in this example constitutes an "orbit discontinuity" of the semiflow because the forward orbits of x and y are "instantaneously discontinuously identified" (or "IDI") at time 0. Our work is centered on the examination of this phenomenon, as it is *the* obstacle to using the shift map on the space of continuous paths as a universal model for Borel semiflows. In fact our results can be used to modify the mapping $x \mapsto \psi(x)$ so that each x is associated to a different map (this is briefly discussed in Chapter 6).

To better explain what we mean by "instantaneous discontinuous identification", we consider an example. Take $0 < a < b$ and let $f : S^1 \to [a, b]$ be any continuous function. Then let (X, T_t) be the suspension semiflow over the transformation $\widehat{T} : S^1 \to S^1$ defined by $\widehat{T}(x) = 4x(\text{mod } 1)$ with return-time function f. More precisely, $X = \{(x, t) \in S^1 \times \mathbb{R}^+ : 0 \leq t \leq f(x)\}/\sim$ where

$(x, f(x)) \sim (\widehat{T}(x), 0)$. The action T_t defined by $T_t(x, s) = (x, t + s)$ is a Borel semiflow on a compact metric space. For this semiflow we never see any two points $x, y \in X$ for which $T_t(x) = T_t(y)$ for all $t > 0$ but $x \neq y$. In particular, any points instantaneously identified by the return map $x \mapsto 4x(\text{mod } 1)$ are already in the same $\sim$ −equivalence class so they are not distinct points. However, we can introduce IDI into this semiflow along a graph. Let $c \in (0, \min_{x \in S^1} f(x))$ and let $j : [0, 1/2) \to [0, \infty)$ be defined by $j(x) = f(x) - c$ (j is not defined on $[1/2, 1)$). Define an equivalence relation on X by

$$(x_1, t_1) \approx (x_2, t_2) \Leftrightarrow \begin{array}{c} 0 \leq x_1, x_2 < 1/2, |x_1 - x_2| = 1/4 \text{ and} \\ t_1 - j(x_1) = t_2 - j(x_2) > 0, \\ \text{OR} \\ x_1 = x_2 \text{ and } t_1 = t_2. \end{array}$$

The $\approx$ −equivalence classes are well-defined and closed since they each consist of at most 2 points. Let $X' = X/\approx$; X' is a Polish space. Also the semiflow T_t passes to a well-defined action $T_t : X' \to X'$. This new semiflow exhibits IDI; in particular points lying on the graph of j are discontinuously identified at time 0 by the semiflow because for any $t > 0$ they are mapped into the same $\approx$ −equivalence class. What has happened is that we have placed a graph across the Ambrose-Kakutani-Krengel picture and modified the semiflow so that it discontinuously identifies points on the graph (see Figure 1.2). This process could be done repeatedly, i.e. in a more complicated example one could introduce lots of graphs with varying domains into the Ambrose-Kakutani-Krengel picture and modify the semiflow to discontinuously identify points along the graphs as above. (Of course one would have to be careful to construct the graphs and identifications in a manner which respects the return-time function f and the

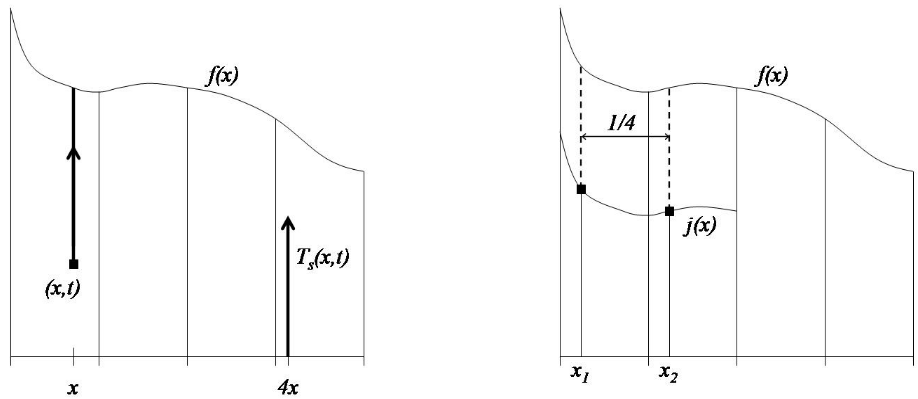

Figure 1.2: Introduction of IDI along a graph. The picture at left demonstrates the space X and the orbit of (x,t) under the suspension semiflow T_s. The picture at right shows the function $j(x)$. The two dashed lines are identified in X' so the points $(x_1, j(x_1))$ and $(x_2, j(x_2))$ (indicated by the squares in the right-hand figure) are "instantaneously discontinuously identified" by T_s at time 0.

Poincaré map $\widehat{T}$ to ensure that the resulting T_t is well-defined.)

In fact we show that for an arbitrary Borel semiflow this is the only way discontinuous identifications can occur. We show that all the orbit discontinuities of a Borel semiflow can be specified by giving a countable collection of Borel functions whose domains are Borel sets. The orbit discontinuities of the semiflow occur along the graphs as in the previous example.

In the next chapter we begin by modeling the original dynamical system by continuous maps. In particular we take a dense, countable, subsemigroup S of $\mathbb{R}^+$ and build an action of S on a Polish space by continuous maps which model the $S-$part of the original action. These continuous maps are defined as shifts on a space of sequences, so we call the action an "induced shift" of the semiflow.

Using these constructions, we in Chapter 3 define the notion of “orbit discontinuity”. Heuristically, a point x has an orbit discontinuity at time t if the set of points which are identified with x for times less than t is significantly smaller than the set of points identified with x for all times greater than t. We give two precise formulations of this phenomenon in Chapter 3; first a definition and then an equivalent condition laid out in Lemma 3.2.1. We construct the aforementioned “graphs” on which all orbit discontinuities must occur and as a consequence show that a point must have at most countably many orbit discontinuities. Consequently the set of points which have an orbit discontinuity at time 0 is a set of measure zero with respect to any measure preserved by the semiflow. We also prove that orbit discontinuities are “backward equivariant”, i.e. if a point x has an orbit discontinuity at time t_0 and y is such that $T_t(y) = x$, then y has an orbit discontinuity at time $t + t_0$. We also in this chapter show that an IDI is a special case of an orbit discontinuity, and that the IDI set of a Borel semiflow is invariant under time-changes.

Chapter 4 applies the idea of orbit discontinuity to improve the Krengel theorem referred to earlier. In particular, we replace the upward pointwise flow with an action thats take points in the base to repeatedly larger closed subsets of the base. We obtain two systems, one a factor of the original system and one an extension of the original system. These adaptations of the Krengel model are shown to be conjugate to the original semiflow except at the orbit discontinuities of the semiflow (a set of measure zero).

Next in Chapter 5 we examine the idea of “orbit discontinuity” from a measure-theoretic perspective. For each $x \in X$ and each $t \geq 0$, there are likely to be many points y identified with x by time t (y is *identified* with x at time t if

$T_t(y) = T_t(x)$). It is natural to ask whether a distribution exists on these points. In Chapter 5 we show that for an invariant set of full measure in X, there exist "measure paths" $\mu^+_{x,t}$ and $\mu^-_{x,t}$ which give these distributions for Lebesgue-almost every t. These measure paths are constructed by taking one-sided weak$^*-$limits of suitable conditional expectations. Most of the work is in verifying that the desired weak$^*-$limits actually exist. If either measure path has a weak$^*-$discontinuity at time t_0, we say x has a *measurable orbit discontinuity* then. We show that a point can only have countably many measurable orbit discontinuities and that if a point has a measurable orbit discontinuity at time 0, then it must have an orbit discontinuity (in the sense of Chapter 3) at time 0 as well.

Finally Chapter 6 discusses some further questions arising from this work, including a discussion of how the constructions made here could be used to obtain a universal model for Borel semiflows.

Chapter 2

Embedding in a shift map

A classical approach to the study of discrete measurable dynamical systems $(X, \mathcal{F}, \mu, T)$ is to take a partition for the space X which generates under the map T and label points in X by sequences which take values according to the partition. To study the map T one then needs only to study the corresponding shift map on the countable product of a finite space. Alternatively one could view the map T itself as a shift on $X^{\mathbb{Z}}$. In this section we adapt this idea to set up some machinery to help us deal with semiflows. However, we cannot encode the entire action of $\mathbb{R}^+$ into a shift (measurably) because in general the space $X^{\mathbb{R}^+}$ is not separable, hence is not Polish. We must therefore restrict ourselves to taking only countable products of X.

We say that a set $S \subset \mathbb{R}^+$ is *rationally generated* if there exists a finite list of nonnegative real numbers $r_1, ..., r_n$ with $r_1 = 1$ such that S consists of all sums

$$\sum_{i=1}^{n} q_i r_i$$

where the q_i are arbitrary nonnegative rational numbers. Note that any rationally generated set S is a countable sub-semigroup of $\mathbb{R}^+$ containing $\mathbb{Q}^+ = \mathbb{Q} \bigcap [0, \infty)$, so is dense in $\mathbb{R}^+$.

Start with a Borel semiflow $(X, \mathcal{F}, \mu, T_t)$ and a rationally generated set $S \subset$

$\mathbb{R}^+$. Let $\mathcal{T}$ be the given Polish topology on X. As X is a standard Polish space, it has a Borel structure isomorphic to that of the Cantor set $2^{\mathbb{N}}$ [2]. Consequently, there is a Polish topology $\mathcal{T}'$ on X for which $(X, \mathcal{T}')$ is homeomorphic to $2^{\mathbb{N}}$. Although $\mathcal{T}$ and $\mathcal{T}'$ may be different, their Borel sets are the same. Thus any action on X which is Borel with respect to $\mathcal{T}$ is Borel with respect to $\mathcal{T}'$, so we can use the $\mathcal{T}'$ topology on X without loss of generality. In particular we may assume $X = 2^{\mathbb{N}}$.

Now view the product space X^S in two ways: first, it is the set of all functions from S to X, and second, it is the set of sequences $\{x_s\}_{s \in S}$ of points in X indexed by elements of S. Endowing X^S with the product topology (i.e. product of the $\mathcal{T}'-$topologies), this makes X^S a product of Cantor sets, hence itself a Cantor space. We define a shift σ_s on X^S in the obvious way by setting $\sigma_s(f)(t) = f(t+s)$ for all $s \in S$.

Lemma 2.0.1 *For all $s \in S, \sigma_s : X^S \to X^S$ is uniformly continuous and closed.*

Proof: First, call a *cylinder* any set in X^S of the form $\{f : \exists$ a finite list $\{s_1, ..., s_N\}$ of elements of S and open sets $O_1, ..., O_N$ in X such that $f(s_j) \in O_j \, \forall j\}$. Observe that the product topology on X^S is generated by cylinders. Now if C is the cylinder associated to positions $\{s_1, ..., s_N\}$ and open sets $O_1, ..., O_N$, then $\sigma_{-s}(C)$ is a cylinder associated to the positions $\{s_1 + s, ..., s_N + s\}$ and the open sets $O_1, ..., O_N$. Hence σ_q is uniformly continuous; any continuous map between compact spaces is also closed. ■

Given a semiflow T_t and a rationally generated set S we have a natural embedding $i_T^S : X \to X^S$ defined by $i_T^S(x)(s) = T_s(x)$. Denote by π_S the projection map $X^S \to X$ taking a function $f \in X^S$ to $f(0)$; π_S is uniformly continuous.

These two maps yield a bijective equivalence:

$$X \underset{\pi_S}{\overset{i_T^S}{\rightleftarrows}} i_T^S(X)$$

Let X_1^S be the closure of $i_T^S(X)$ in X^S, and put the subspace topology on X_1^S so that it is a Polish space. Now for any $f \in X_1^S, \sigma_s(f)$ will also be in X_1^S for any $s \in S$ because of the equivariance $\sigma_s \circ i_T^S = i_T^S \circ T_s$. (In fact the image $i_T^S(X)$ is a forward invariant set in X_1^S under the shift.) Since we assume that the semiflow T_t is surjective, σ_s is also surjective for every t. Moreover the shift σ_s will preserve the measure $i_T^S(\mu)$ on X_1. Let $\mathcal{G}_S$ be the σ-algebra on X_1^S generated by cylinders; we call the measure-preserving S-action $(X_1^S, \mathcal{G}_S, i_T^S(\mu), \sigma_s)$ the *induced shift of* T_t *with respect to* S. It serves as a model for the S part of the original $\mathbb{R}^+$ action.

Of course, the entire $\mathbb{R}^+$ action still exists on $i_T^S(X)$; for $f \in i_T^S(X)$ and $t \geq 0$ one can define

$$T_t(f) = i_T^S \circ T_t \circ \pi_S(f).$$

The resulting semiflow $(i_T^S(X), i_T^S(\mathcal{F}), i_T^S(\mu), T_t)$ is measure-preserving and conjugate to the original action $(X, \mathcal{F}, \mu, T_t)$. If necessary, one can extend the $\mathbb{R}^+$ action to X_1^S by defining $T_t(f) = f$ for all $f \in X_1^S - i_T^S(X)$. This makes $X_1^S - i_T^S(X)$ a T_t-invariant set of $i_T^S(\mu)-$measure zero.

Chapter 3

Orbit discontinuities and instantaneous discontinuous identifications

3.1 Orbit discontinuities

We now describe what is meant by an orbit discontinuity of a semiflow and outline a purely topological approach to locating orbit discontinuities. Begin by choosing a rationally generated set S and constructing the induced shift (X_1^S, σ_s). For any $x \in X_1^S$ and any real number t, we define the equivalence class $[x]_t$ to be the set of points in X_1^S whose forward orbit agrees with the forward orbit of x at all times t in S greater than or equal to $\max(t, 0)$. More precisely,

$$[x]_t = \begin{cases} \bigcap_{s \geq t, s \in S} \sigma_{-s}\sigma_s(x) & \text{if } t \geq 0 \\ \{x\} & \text{if } t < 0 \end{cases}$$

(The extension of the definition of $[x]_t$ to negative times t is primarily a convenience which simplifies some later work, but is sensible because the only point in X_1^S with the same forward σ_s-orbit as x for all $s \geq 0$ is x itself.) For a fixed t, the sets $[x]_t$ partition X_1^S into disjoint closed sets. For a fixed x, the sets $[x]_t$ are an increasing sequence of closed sets. In particular, if $t \in S$, then $[x]_t = \sigma_{-t}\sigma_t(x)$.

First, recall that X_1^S is a closed subset of the Cantor space X^S; consequently X_1^S has a refining, generating sequence of finite partitions for which every atom of every partition is a clopen set. Let $\mathcal{P}_k$ be such a sequence of partitions and denote the individual atoms of $\mathcal{P}_k$ by $P_k^1, ..., P_k^N$. It is important to note here that "refining" and "generating" are meant in the topological sense, not the measure-theoretic, i.e. we require that *any* (not almost any) two distinct points x and y are separated by some $\mathcal{P}_k$ and that given $k_1 \leq k_2$, every atom of $\mathcal{P}_{k_1}$ is *equal to* the union of atoms of $\mathcal{P}_{k_2}$ (rather than equal to up to a set of measure zero). Let $\mathcal{C}(\mathcal{P}_k)$ be the collection of subsets of $\{P_k^1, ..., P_k^N\}$, i.e. an element of $\mathcal{C}(\mathcal{P}_k)$ is technically a collection of clopen sets of X_1^S, each set being an atom of the partition $\mathcal{P}_k$. However, we use abusive language and will also view an element $c \in \mathcal{C}(\mathcal{P}_k)$ as itself a clopen subset of X_1^S; i.e. given a set $A \subseteq X_1^S$ and a collection $c \in \mathcal{C}(\mathcal{P}_k)$ we may write "$A \bigcap c$" instead of the more precise "$A \bigcap (\bigcup_{P \in c} P)$".

Now given C, any subset of X_1^S, we define

$$Coll(\mathcal{P}_k, C) = \{P \in \mathcal{P}_k : P \bigcap C \neq \emptyset\}.$$

For each $t \geq 0$ we define a mapping $Coll_t^{\mathcal{P}_k} : X_1^S \to \mathcal{C}(\mathcal{P}_k)$ by

$$Coll_t^{\mathcal{P}_k}(x) = Coll(\mathcal{P}_k, [x]_t).$$

We think of $Coll_t^{\mathcal{P}_k}(x)$ as a collection of clopen sets each containing at least one point whose forward orbit is the same as that of x for times greater than t (or greater than 0 if t is negative). Given a collection of sets $c \in \mathcal{C}(\mathcal{P}_k)$, define the set of points that *eventually see* c as

$$E(c) = \{x \in X_1^S : Coll_t^{\mathcal{P}_k}(x) = c \text{ for at least two values } t \in [0, \infty)\}.$$

Notice that the definition of $E(c)$ implies that $Coll_t^{\mathcal{P}_k}(x) = c$ for some interval of times t, for certainly $Coll_{t_1}^{\mathcal{P}_k}(x) \subseteq Coll_{t_2}^{\mathcal{P}_k}(x)$ whenever $t_1 \leq t_2$.

Proposition 3.1.1 *For any $c \in \mathcal{C}(\mathcal{P}_k)$, $E(c)$ is a F_σ subset of X_1^S.*

Proof: For pairs $s, s' \in S$ with $s < s'$, define $A_{s,s'} = \{x \in X_1^S : Coll_s^{\mathcal{P}_k}(x) = Coll_{s'}^{\mathcal{P}_k}(x) = c\}$. Now to say $Coll_s^{\mathcal{P}_k}(x) = c$ means that $\sigma_s(x) \bigcap \sigma_s(P_k^j) \neq \emptyset$ for all $P_k^j \in c$ and $\sigma_s(x) \bigcap \sigma_s(P_k^l) = \emptyset$ for all $P_k^l \in \mathcal{P}_k - c$. Thus

$$Coll_s^{\mathcal{P}_k}(x) = c \Leftrightarrow x \in \sigma_{-s}\left(\bigcap_{P_k^j \in c} \sigma_s(P_k^j) \bigcap \left(X_1^S - \bigcup_{P_k^l \in \mathcal{P}_k - c} \sigma_s(P_k^l)\right)\right).$$

Now since $\mathcal{P}_k$ is finite, all the unions and intersections in the above expression are finite. As each P_k^j is clopen and since σ_s is closed and continuous, the condition $Coll_s^{\mathcal{P}_k}(x) = c$ is the intersection of the closed set

$$\bigcap_{P_k^j \in c} \sigma_{-s}\sigma_s(P_k^j)$$

and the open set

$$\sigma_{-s}\left(X_1^S - \bigcup_{P_k^l \in \mathcal{P}_k - c} \sigma_s(P_k^l)\right).$$

Similarly the condition $Coll_{s'}^{\mathcal{P}_k}(x) = c$ is the intersection of a closed set with an open set. Hence $A_{s,s'}$ is also the intersection of an open set $O_{s,s'}$ and a closed set $C_{s,s'}$. Of course open sets in a metric space are F_σ sets as well so $A_{s,s'}$ is a F_σ. Now since S is dense in $\mathbb{R}^+$,

$$E(c) = \bigcup_{s,s' \in S} A_{s,s'}$$

so $E(c)$ is a F_σ as desired. ■

Next, given $c_1, c_2 \in \mathcal{C}(\mathcal{P}_k)$ with c_1 contained in but not equal to c_2, define

$$\begin{aligned} J(c_1, c_2) = \{x \in X_1^S :\ & \exists \text{ time } j_{c_1,c_2}(x) \text{ and } \exists \epsilon > 0 \text{ such that} \\ & Coll_t^{\mathcal{P}_k}(x) = c_1 \text{ for } t \in (j_{c_1,c_2}(x) - \epsilon, j_{c_1,c_2}(x)) \text{ and} \\ & Coll_t^{\mathcal{P}_k}(x) = c_2 \text{ for } t \in (j_{c_1,c_2}(x), j_{c_1,c_2}(x) + \epsilon)\}. \end{aligned}$$

$J(c_1, c_2)$ is the set of points who see the collection c_1 before and up to some time j_{c_1,c_2} but see the collection c_2 immediately thereafter. We say that such points *jump from* c_1 *to* c_2 *at time* j_{c_1,c_2}. We note that this definition does not imply anything about the nature of $Coll^{\mathcal{P}_k}_{j_{c_1,c_2}}(x)$ other than the trivial fact that it contains c_1 and is contained in c_2.

Proposition 3.1.2 *For any* c_1, c_2 *as above,* $J(c_1, c_2)$ *is a* $G_{\delta\sigma}$ *in* X^S_1*.*

Proof: First notice that $J(c_1, c_2)$ is precisely the set of points that eventually see c_1 and eventually see c_2 but do not see any collections containing c_1 and contained in c_2 (except possibly at the one time $j_{c_1,c_2}(x)$). So

$$J(c_1, c_2) = \Big(E(c_1) \bigcap E(c_2)\Big) - \bigcup_{c' \in \mathcal{C}(\mathcal{P}_k),\ c_1 \subseteq c' \subseteq c_2, c' \neq c_1, c' \neq c_2} E(c').$$

By previous result, $E(c)$ is a F_σ for any c so $J(c_1, c_2)$ is the intersection of two F_σ sets and the complement of a F_σ set (i.e. a G_δ). Thus $J(c_1, c_2)$ is a $G_{\delta\sigma}$ (and a $F_{\sigma\delta}$ as well). ■

Lemma 3.1.3 *Fix* t *and suppose* $x_n \to x$ *in* X^S_1 *and* $Coll^{\mathcal{P}_k}_t(x_n) \supseteq c$ *for all* n*. Then* $Coll^{\mathcal{P}_k}_t(x) \supseteq c$*.*

Proof: Let $s \in S$ be greater than or equal to t. Then for each $n, Coll^{\mathcal{P}_k}_s(x_n) \supseteq c$. List the partition atoms that lie in the collection c as $P_1, ..., P_m$. For each of these, and for each n, there is a $x^j_n \in P_j$ such that $\sigma_s(x^j_n) = \sigma_s(x_n)$. Now each P_j is closed hence compact, so for each j, (after passing to a subsequence) we can assume that $x^j = \lim_{n\to\infty} x^j_n$ exists and is in P_j. By continuity of σ_s, we have $\sigma_s x^j_n \to \sigma_s x^j$ and $\sigma_s x_n \to \sigma_s x$ as $n \to \infty$. But these limits must be the same since $\sigma_s(x^j_n) = \sigma_s(x_n)$. Hence $\sigma_s x^j = \sigma_s x$ for all j. Thus $Coll^{\mathcal{P}_k}_s(x) \supseteq c$. Since this argument can be repeated for any $s \geq t$, $Coll^{\mathcal{P}_k}_t(x) \supseteq c$. ■

Proposition 3.1.4 *The function $j_{c_1,c_2}(x)$ is lower semi-continuous, hence Borel, as a function from $J(c_1, c_2)$ into $\mathbb{R}^+$.*

Proof: Let $\mathcal{P}_l$ be the partition such that $c_1, c_2 \in \mathcal{C}(\mathcal{P}_l)$. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in $J(c_1, c_2)$ converging to $x \in J(c_1, c_2)$. Take any $t > \liminf_{n\to\infty} j_{c_1,c_2}(x_n)$. Then there is a subsequence x_{n_k} with $j_{c_1,c_2}(x_{n_k}) < t\,\forall k$. Thus $Coll_t^{\mathcal{P}_l}(x_{n_k}) \supseteq c_2\,\forall k$ so by the previous lemma $Coll_t^{\mathcal{P}_l}(x) \supseteq c_2$. Therefore $j_{c_1,c_2}(x) < t$. ■

At this point all the sets and functions we have described depend on the particular choice of S as well as the choice of refining, generating sequence of partitions for X_1^S. However, we will now use the above constructions to describe what we mean by an orbit discontinuity; we will see that the existence and location of orbit discontinuities do not depend on the choice of rationally generated set or partitions.

Given a point $x \in X_1^S$ and a time $t \in [0, \infty)$, we say *x has an S-orbit discontinuity at time t* if there exists a clopen partition $\mathcal{P}_k$ (in a refining, generating sequence) of X_1^S and collections $c_1 \subseteq c_2, c_1 \neq c_2$ in $\mathcal{C}(\mathcal{P}_k)$ such that $x \in J(c_1, c_2)$ and $t = j_{c_1,c_2}(x)$. A point x in the original space X is said to have *an S-orbit discontinuity at time t* if $i_T^S(x) \in X_1^S$ has an S-orbit discontinuity at time t.

We establish some notation to be used in the sequel. Given two sets A and B in X_1^S, we define

$$d_M(A, B) = \inf_{a\in A, b\in B} d(a, b).$$

Note that d_M is not a metric; merely a function which measures how "separated" A and B are. In particular, $d_M(A, B) = 0$ if and only if $\overline{A} \bigcap \overline{B} \neq \emptyset$. Notice that if $x \in X_1^S$ has an S-orbit discontinuity at time t_0 with respect to some partition $\mathcal{P}_K$ of X_1^S, we must have a collection c_1 of atoms of $\mathcal{P}_K$, an atom A of $\mathcal{P}_K$, and a number $\epsilon > 0$ such that each of the following three conditions are satisfied:

i. $d_M(A, c_1) > 0$ (i.e. A is disjoint from c_1)

ii. $Coll_t^{\mathcal{P}_K}(x) = c_1$ for $t \in (t_0 - \epsilon, t_0)$

iii. $Coll_t^{\mathcal{P}_K}(x) \supseteq c_1 \bigcup \{A\}$ for $t \in (t_0, t_0 + \epsilon)$

Now for $k > K$, $\mathcal{P}_k$ is a refinement of $\mathcal{P}_K$ so in particular for any t, $Coll_t^{\mathcal{P}_k}(x) \subseteq Coll_t^{\mathcal{P}_K}(x)$. Therefore if x has an S-orbit discontinuity at time t_0 with respect to $\mathcal{P}_K$, we can conclude the following remark about the nature of $Coll_t^{\mathcal{P}_k}(x)$ for $k \geq K$: for any $t > t_0$, $Coll_t^{\mathcal{P}_k}(x)$ contains an atom A_k of $\mathcal{P}_k$ with

$$d_M(A_k, Coll_\tau^{\mathcal{P}_k}(x)) \geq d_M(A, c_1) > 0$$

for all $\tau < t_0$.

Proposition 3.1.5 *Suppose $x \in X_1^S$ has an S-orbit discontinuity at time t_0 with respect to some partition $\mathcal{P}_k$ in a refining, generating sequence. Then if $\mathcal{Q}_k$ is any other refining, generating sequence of finite partitions of X_1^S into clopen sets, x will have an S-orbit discontinuity at time t_0 with respect to some $\mathcal{Q}_{k'}$.*

Proof: By hypothesis there exists a collection c_1 of atoms of the partition $\mathcal{P}_k$, an atom A of $\mathcal{P}_k$, and $\epsilon > 0$ such that (i), (ii), and (iii) as above are satisfied. Given the sequence of partitions $\mathcal{Q}_k$, consider the function

$$m(k) = \max_{Q_k^j \in \mathcal{Q}_k} diam(Q_k^j).$$

As $\mathcal{Q}_k$ refines and generates (and since each atom of $\mathcal{Q}_k$ is compact), $m(k) \to 0$ as $k \to \infty$. Choose k' large enough so that and $m(k') < d_M(A, c_1)/4$. Consider a decreasing sequence of numbers $s_n \in S$ converging to t_0 from above. For each s_n, there is a point $y_n \in A$ such that $\sigma_{s_n} y_n = \sigma_{s_n} x$. Since the partition $\mathcal{Q}_{k'}$ is finite, there must be an atom A' of $\mathcal{Q}_{k'}$ with:

i. A' containins an infinite number of the y_n, and

ii. $A' \bigcap A \neq \emptyset$.

Thus for any number $s > t_0$ in S there is a $y_n \in A'$ such that $\sigma_s(y_n) = \sigma_s(x)$, so $Coll_s^{\mathcal{Q}_{k'}}(x) \supseteq A'$.

Let $d = \{Q_{k'}^j \in \mathcal{Q}_{k'} : Q_{k'}^j \bigcap c_1 \neq \emptyset\}$ and notice that $c_1 \subseteq d$. Now suppose $A' \in d$. Then there exists a $z \in A' \bigcap c_1$ and as $diam(A') < d_M(A, c_1)/4$, we have

$$d_M(A, c_1) < \frac{d_M(A, c_1)}{4}$$

which is impossible. Now let r_n be an increasing sequence of numbers in S converging to t_0 from below. Let $d_n = Coll_{r_n}^{\mathcal{Q}_{k'}}(x)$; $d_n \subseteq d$ for all n so $d' = \bigcup_n d_n \subseteq d$. We have shown that with respect to the partition $\mathcal{Q}_{k'}$, x jumps from the collection d' to a collection containing A' (which is not in d') at time t_0. This completes the proof. ■

Proposition 3.1.6 *Suppose $x \in X$ has an S-orbit discontinuity at time t_0. Then for any rationally generated set S' with $S' \supseteq S$, x has an S'-orbit discontinuity at time t_0.*

Proof: This proof is similar in vein to proof of the previous proposition. As before, there is a collection c_1 of atoms of a partition $\mathcal{P}_k$ of X_1^S, an atom A of $\mathcal{P}_k$, and $\epsilon > 0$ such that:

i. $d_M(A, c_1) > 0$,

ii. $Coll_t^{\mathcal{P}_K}(i_T^S(x)) = c_1$ for $t \in (t_0 - \epsilon, t_0)$, and

iii. $Coll_t^{\mathcal{P}_K}(i_T^S(x)) \supseteq c_1 \bigcup \{A\}$ for $t \in (t_0, t_0 + \epsilon)$.

As $S \subseteq S'$, any element $f \in X^{S'}$ can also be viewed as an element of X^S by restricting the domain. Let $\psi : X^{S'} \to X^S$ be this mapping which takes f to $f|_S$. In fact, ψ is a projection mapping from $X^{S'}$ onto a "subspace" X^S so ψ is uniformly continuous.

Let $\mathcal{Q}_k$ be any refining, generating sequence of clopen partitions for $X_1^{S'}$. Let $\mathcal{R}_k = \psi^{-1}(\mathcal{P}_k) \vee \mathcal{Q}_k$; $\mathcal{R}_k$ is a clopen sequence of partitions of $X_1^{S'}$ which refines and generates. Let $c_1' = \psi^{-1}(c_1)$; this set is a union of atoms of $\mathcal{R}_k$.

Let $A_1, ..., A_m$ be a list of atoms of $\mathcal{R}_k$ which are contained in $A' = \psi^{-1}(A)$. Let q_n be a decreasing sequence of rational numbers greater than t_0 converging to t_0 from above; for each of these there must be a point y_n in A' such that $\sigma_{q_n}(y_n) = \sigma_{q_n}(i_T^{S'}(x))$. So an infinite number of the y_n must lie in some single A_j. Hence for any $s \in S' \bigcap (t_0, \infty)$, $A_j \in Coll_s^{\mathcal{R}_{k'}}(i_T^{S'}(x))$.

Now let τ_n be an increasing sequence of rational numbers converging to t_0 from below. Let $d_n = Coll_{\tau_n}^{\mathcal{R}_{k'}}(i_T^{S'}(x))$; $d_n \subseteq c_1'$ for all n so $d = \bigcup_n d_n \subseteq c_1'$ as well. Since $c_1' \bigcap A' = \emptyset$, we have shown that at time t_0, $Coll_t^{\mathcal{R}_{k'}}(i_T^{S'}(x))$ jumps from the collection d to a collection containing A_j. Therefore x has an S'-orbit discontinuity at time t_0 as desired. ■

This result guarantees that if a semiflow has an orbit discontinuity with respect to $\mathbb{Q}^+$, it will have an orbit discontinuity with respect to all other rationally generated sets (as they must all contain $\mathbb{Q}^+$). We say then that a point $x \in X_1^S$ *has an orbit discontinuity at time* t (or that the semiflow *has an orbit discontinuity at* $T_t(x)$) if for the rationally generated set $\mathbb{Q}^+$, x has an $\mathbb{Q}^+$-orbit discontinuity (and similar for a point $x \in X$).

We now briefly consider the frequency of occurrences of orbit discontinuities along individual forward orbits.

Proposition 3.1.7 *Fix $x \in X$, and let $D(x)$ be the set of times at which x has an orbit discontinuity (call $D(x)$ the* discontinuity set *of x). Then $D(x)$ is a countable set.*

Proof: Consider a refining, generating sequence of partitions $\mathcal{P}_k$ for $X_1^{\mathbb{Q}^+}$. Now every $j \in D(x)$ corresponds to $i_T^{\mathbb{Q}^+}(x)$ belonging to some $J(c_1, c_2)$ with jump time j where $c_1, c_2 \in \mathcal{C}(\mathcal{P}_k)$ for some k. However, there are only countably many such pairs c_1, c_2 so there can be at most countably many $j \in D(x)$. ■

We summarize our results thus far in the following theorem:

Theorem 3.1.1 *Let X be a standard Polish space and T_t a Borel action of $\mathbb{R}^+$ on X. Then there exist a countable list of lower semi-continuous functions $j_{c_1,c_2}(x)$ whose domains are Borel subsets of $X_1^{\mathbb{Q}^+}$ taking values in $\mathbb{R}^+$ so that given any point $x \in X$, the orbit discontinuities along the forward orbit of x occur at the times $j_{c_1,c_2}(i_T^{\mathbb{Q}^+}(x))$. In particular for a point x, the set $D(x)$ of orbit discontinuities is countable.*

3.2 Instantaneous discontinuous identifications

In this section we show that the phenomenon of "instantaneous discontinuous identification" described in the introduction constitutes an orbit discontinuity. First we deal with the question of equivariance of orbit discontinuities. Obviously if x has an orbit discontinuity at time t, then we should see an orbit discontinuity at time $s + t$ for any point in $T_{-s}(x)$. We need the following preliminary lemma:

Lemma 3.2.1 *$x \in X$ has an orbit discontinuity at time t_0 if and only if for any rationally generated set S, there exists a $z \in X_1^S$ such that $z \in [i_T^S(x)]_t$ for all $t > t_0$ but z does not lie in the closure of $\bigcup_{t<t_0}[i_T^S(x)]_t$.*

Proof: ($\Rightarrow$) By definition $i_T^S(x)$ has an orbit discontinuity with respect to some clopen partition $\mathcal{P}_K$ of X_1^S in a refining, generating sequence. As before there must be a collection c_1^K of atoms of $\mathcal{P}_K$, an atom A of $\mathcal{P}_k$, and a number $\epsilon > 0$ such that each of the following is satisfied:

i. $d_M(A, c_1^K) > 0$

ii. $Coll_t^{\mathcal{P}_K}(i_T^S(x)) = c_1^K$ for $t \in (t_0 - \epsilon, t_0)$

iii. $c_2^K = Coll_t^{\mathcal{P}_K}(i_T^S(x)) \supseteq c_1^K \bigcup \{A\}$ for $t \in (t_0, t_0 + \epsilon)$

Choose a decreasing sequence of numbers $s_n \in \mathbb{Q}^+$ with s_n converging to t_0 from above. By the remark preceding Proposition 3.1.5, for each n and for all $k > K$ there exists an atom $A_{k,n}$ of $\mathcal{P}_k$ with $A_{k,n} \in Coll_{s_n}^{\mathcal{P}_k}(i_T^S(x))$ but

$$d_M(A_{k,n}, Coll_{s^-}^{\mathcal{P}_k}(i_T^S(x))) \geq d_M(A, c_1) > 0$$

for any $s^- < t_0$. Fix k; there must be an atom $\overline{A_k}$ of $\mathcal{P}_k$ with $\overline{A_k} = A_{k,n}$ for a infinite number of the n. Without loss of generality we can choose the $\overline{A_k}$ so that $\overline{A_{k+1}} \subseteq \overline{A_k}$ for all $k > K$. Since the $\mathcal{P}_k$ generate, $\bigcap_{k=1}^{\infty} \overline{A_k}$ must be a single point; call this point z ($\bigcap_{k=1}^{\infty} \overline{A_k}$ cannot be empty by the finite intersection property).

Let $t > t_0$, we claim that $z \in [i_T^S(x)]_t$. Notice that for any $k > K$ there is a point $z_k \in \overline{A_k} \bigcap [i_T^S(x)]_t$ since $\overline{A_k} \in Coll_t^{\mathcal{P}_k}(i_T^S(x))$ for $t > t_0$. The z_k must converge to z since they lie in the $\overline{A_k}$ (which refine to z). Since $[i_T^S(x)]_t$ is closed, $z \in [i_T^S(x)]_t$. However, for any $t < t_0$, $[i_T^S(x)]_t \subseteq Coll_t^{\mathcal{P}_k}(i_T^S(x))$ so if $y \in [i_T^S(x)]_t$, then

$$d(y, z) \geq d_M(Coll_t^{\mathcal{P}_k}(i_T^S(x)), z) \geq d_M(A, c_1) > 0.$$

Thus z is not in $\overline{\bigcup_{t<t_0} [i_T^S(x)]_t}$ as desired.

($\Leftarrow$) Let $S = \mathbb{Q}^+$. By hypothesis there exists a point z in the set

$$\bigcap_{t>t_0} [i_T^S(x)]_t - \overline{\bigcup_{t<t_0} [i_T^S(x)]_t}.$$

Let

$$\delta = d_M(z, \overline{\bigcup_{t<t_0} [i_S^T(x)]_t});$$

$\delta > 0$ so for all $t < t_0$, $\delta_t = d_M(z, [i_T^S(x)]_t) \geq \delta > 0$. Let $\mathcal{P}_k$ be a refining, generating sequence of partitions for X_1^S, choose k large enough so that the maximum diameter of a P_k-atom is less than $\delta/4$. $Coll_t^{\mathcal{P}_k}(i_T^S(x))$ must contain an atom $A \ni z$ for $t > t_0$. But for $t < t_0$, $Coll_t^{\mathcal{P}_k}(i_T^S(x))$ cannot contain an atom within d_M-distance $\delta/4$ of A. Thus $i_T^S(x)$ jumps from a collection c_1 to a collection containing A at time t_0 so there is an orbit discontinuity there as desired. ■

Notice that the point z is "uniformly" separated from $Coll_t^{\mathcal{P}_k}(i_T^S(x))$ for all $t < t_0$, in that the sense that there exists a $\delta_0 > 0$ so that for all $t < t_0$, $d(z, x') > \delta_0$ for all $x' \in Coll_t^{\mathcal{P}_k}(i_T^S(x))$.

Since we need only to find orbit discontinuities with respect to the single rationally generated set $\mathbb{Q}^+$, one might ask why we bothered to consider more general rationally generated sets at all. The reason is that this extension provides a nice proof of Proposition 3.2.2 below.

Proposition 3.2.2 *Suppose $x \in X$ has an orbit discontinuity at time t_0 and $\widehat{x} \in X$ is such that $T_s(\widehat{x}) = x$ for some $s > 0$. Then $\widehat{x}$ has an orbit discontinuity at time $t_0 + s$.*

Proof: Let S be the set generated by 1 and s. By Proposition 3.1.6, x must have an $S-$orbit discontinuity. We apply Lemma 3.2.1 to obtain a refining, generating sequence of partitions $\mathcal{P}_k$ of X_1^S, a $\delta_0 > 0$, and a point z with $z \in [i_S^T(x)]_t$ for $t > t_0$ but

$$d(z, x') \geq \delta_0 \text{ for any } x' \in \bigcup_{t<t_0} [i_S^T(x)]_t.$$

Since σ_s is uniformly continuous, there is a $\delta > 0$ so that if $d(x_1, x_2) < \delta$, then $d(\sigma_s(x_1), \sigma_s(x_2)) < \delta_0$. Choose K' large enough so that

$$\max_{P_{K'}^j \in \mathcal{P}_{K'}} diam(P_{K'}^j) < \delta/4;$$

we claim that for all $\tau < t_0$

$$d_M\left(Coll\left(\mathcal{P}_{K'}, \sigma_{-s}(z)\right), \sigma_{-s}\left(Coll_\tau^{\mathcal{P}_{K'}}(x)\right)\right) \geq \delta.$$

If not, there exist points z' and y' with $d(z', y') < \delta$, $\sigma_s(z') = z$ and $\sigma_s(y') = y \in Coll_\tau^{\mathcal{Q}_{K'}}(x)$. But since $d(z', y') < \delta$, $d(z, y) < \delta_0$ which is impossible.

Let A be any atom of $\mathcal{P}_{K'}$ intersecting $\sigma_{-s}(z)$ nontrivially; observe that $A \in Coll_t^{\mathcal{P}_{K'}}(i_T^S(\widehat{x}))$ for any $t > s + t_0$ since $z \in [x]_t$. However whenever $\tau < t_0$, $d_M(A, Coll_\tau^{\mathcal{P}_{K'}}(i_T^S(\widehat{x}))) \geq \delta$ so A cannot belong to $Coll_\tau^{\mathcal{P}_k}(i_T^S(\widehat{x}))$. Thus at time $s + t_0$, $Coll_t^{\mathcal{P}_{K'}}(i_T^S(\widehat{x}))$ jumps from some collection c^* to a collection containing A (which cannot be in c^*) so an orbit discontinuity exists then as desired. ■

It is natural to ask whether or not a converse to Proposition 3.2.2 holds, i.e. if x has orbit discontinuity at time t_0 and $s < t_0$, does $T_s(x)$ have orbit discontinuity at time $t_0 - s$? The answer to this question is unknown, and we suspect that it may be false. Fortunately, for a subclass of orbit discontinuities we are interested in, an analog of Proposition 3.2.2 and its converse does hold.

We say that a point $x \in X$ is *instantaneously and discontinuously identified (IDI)* if there is a $y \in X$ (with $y \neq x$) such that $T_t(x) = T_t(y)$ for all $t > 0$. The set of all $x \in X$ that are IDI with respect to the action T_t is denoted $IDI(T_t)$. Given any point $x \in X$, we define $IDI(x)$ to be the set of times $t \geq 0$ such that $T_t(x) \in IDI(T_t)$. Clearly, $IDI(x) \bigcap [t, \infty) = IDI(T_t(x)) + t$ for any $x \in X$ and any $t \geq 0$. (This is stronger than the corresponding statement for orbit discontinuities: we have only $D(x) \bigcap [t, \infty) \supseteq D(T_t(x)) + t$ by Proposition 3.2.2.)

Returning to the issue originally raised in the introduction, suppose that (X, T_t) is a Borel semiflow. Without loss of generality X can be taken to be a Cantor set in $[0, 1]$ since it is a Polish space; we then get a mapping ψ taking x to a continuous, increasing function passing through the origin by $\psi(x)(t) = \int_0^t T_s(x)ds$.

Proposition 3.2.3 *Let $(X, \mathcal{F}, T_t)$ be a Borel semiflow. Then ψ is not injective at x (i.e. $\#(\psi^{-1}\psi(x)) > 1$) if and only if $x \in IDI(T_t)$.*

Proof: ($\Leftarrow$) is immediate. For ($\Rightarrow$), we suppose that $\psi(x) = \psi(y)$. Then since $\psi(x)$ is an indefinite integral, it is differentiable almost everywhere so we obtain

$$(\psi(x))'(t) = T_t(x) = T_t(y) = (\psi(y))'(t)$$

for Lebesgue almost every t. But if $T_t(x) = T_t(y)$, then the forward orbits of x and y coincide for every time greater than t as well. Therefore $T_t(y) = T_t(x)$ for every $t > 0$. ■

Proposition 3.2.4 *If $x \in IDI(T_t)$, then x has an orbit discontinuity at time 0.*

Proof: Let $x \in IDI(T_t)$ and let $y \in X$ be such that $T_t(x) = T_t(y)$ for all $t > 0$. Choose a refining, generating sequence $\mathcal{P}_k$ of partitions of $X_1^{\mathbb{Q}^+}$ and choose k large enough so that the maximum diameter of a $\mathcal{P}_k$–atom is less than $d(i_T^{\mathbb{Q}^+}(x), i_T^{\mathbb{Q}^+}(y))/4$. With respect to $\mathcal{P}_k$, $i_T^{\mathbb{Q}^+}(x)$ jumps from the collection of the one atom of $\mathcal{P}_k$ containing $i_T^{\mathbb{Q}^+}(x)$ to a collection containing the atom containing $i_T^{\mathbb{Q}^+}(y)$ so x has an orbit discontinuity at time 0. ■

Corollary 3.2.5 *For any $x \in X$, $IDI(x) \subseteq D(x)$, so in particular $IDI(x)$ is countable.*

Proof: If $t \in IDI(x)$, then $T_t(x) \in IDI(T_t)$ so by the previous proposition $T_t(x)$ has an orbit discontinuity at time 0. Finally by Proposition 3.2.2 x has an orbit discontinuity at time t so $t \in D(x)$ as desired. ■

Notice that our results so far do not depend on the fact that T_t was a measure-preserving system; in fact they hold for any Borel action of $\mathbb{R}^+$ on a Polish space. If in fact T_t preserves some probability measure μ on X however, the tools developed thus far allow us to show that the set of orbit discontinuities has measure zero.

Proposition 3.2.6 *Suppose $(X, \mathcal{F}, \mu, T_t)$ be a measure-preserving semiflow, and let $D(T_t)$ be the set of points which have an orbit discontinuity at time 0 (call $D(T_t)$ the* discontinuity set *of T_t). Then if $\mu = \int \mu_z \, dz$ is the ergodic decomposition of μ, then with respect to every ergodic component μ_z, $D(T_t)$ has measure zero.*

Proof: Let (X_z, μ_z) be any ergodic component. Then for almost every $x \in X_z$, the ergodic theorem gives

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \chi_{D(T_t)}(T_s(x)) \, ds = \mu_z(D(T_t)).$$

But the integrals on the left-hand side of this expression are zero for all t since $D(x)$ is countable; thus $\mu_z(D(T_t)) = 0$. ■

Corollary 3.2.7 *Suppose $(X, \mathcal{F}, \mu, T_t)$ be a measure-preserving semiflow. Then if $\mu = \int \mu_z \, dz$ is the ergodic decomposition of μ, then with respect to every ergodic component μ_z, $IDI(T_t)$ has measure zero.*

We can more explicitly describe what it means for a semiflow to have an orbit discontinuity at a point in the case where the semiflow is always finite-to-1. In particular the following result guarantees that flows have no orbit discontinuities.

Proposition 3.2.8 *Suppose T_t is finite-to-1 for every t. Then:*

i. A point x has an orbit discontinuity at time t $\Leftrightarrow$ for any $\epsilon > 0$, we have $\#(T_{-(t-\epsilon)}T_{t-\epsilon}(x)) < \#(T_{-(t+\epsilon)}T_{t+\epsilon}(x))$.

ii. $t \in IDI(x) \Leftrightarrow$ for any $\epsilon > 0$, we have $\#(T_{-t}T_t(x)) < \#(T_{-(t+\epsilon)}T_{t+\epsilon}(x))$.

Proof: The second statement follows directly from the definition of IDI:

$$\begin{aligned} \#(T_{-t}T_t(x)) < \#(T_{-(t+\epsilon)}T_{t+\epsilon}(x)) \quad &\Leftrightarrow \quad \#(T_{-\epsilon}T_\epsilon(T_t(x))) > 1 \\ &\Leftrightarrow \quad T_t(x) \in D(T_t). \end{aligned}$$

It remains to verify the first statement.

($\Rightarrow$) Suppose not; then there exist non-negative rational numbers $q < t$ and $r > t$ such that $\#([i_T^S(x)]_q) = \#([i_T^S(x)]_r)$ for any rationally generated S. In fact this implies that $[i_T^S(x)]_q$ and $[i_T^S(x)]_r$ are equal as sets (as the first is contained in the second and they are both finite). Thus for any partition $\mathcal{P}_k$, $Coll_q^{\mathcal{P}_k}(i_T^S(x)) = Coll_r^{\mathcal{P}_k}(i_T^S(x))$ so x cannot have an orbit discontinuity at any time between q and r.

($\Leftarrow$) Let $S = \mathbb{Q}^+$ and let x' be an element contained in the nonempty set

$$\bigcap_{r>t, r\in S} [i_T^S(x)]_r \bigcap \left(X_1 - \bigcup_{s<t, s\in S} [i_T^S(x)]_s \right).$$

Since the set $\bigcup_{s<t,s\in S}[i_T^S(x)]_s$ is finite, it is equal to its closure so by setting $z = x'$, we can apply Lemma 3.2.1 to get the desired result. ■

3.3 An example: suspensions of the one-sided 2-shift

To better explain the concepts of this section, we consider two examples. Let

$$\begin{aligned}\Xi_R &= \{(g, c_g) : 0 \le c_g < 1 \text{ and } g \text{ is a function from } [0,\infty) \text{ into } \{0,1\} \text{ such} \\ &\qquad \text{that for every integer } i,\ g \text{ is constant on every interval of the form} \\ &\qquad [0,\infty) \bigcap [c_g + i, c_g + i + 1)\}\end{aligned}$$

and let

$$\begin{aligned}\Xi_L &= \{(g, c_g) : 0 \le c_g < 1 \text{ and } g \text{ is a function from } [0,\infty) \text{ into } \{0,1\} \text{ such} \\ &\qquad \text{that for every integer } i,\ g \text{ is constant on every interval of the form} \\ &\qquad [0,\infty) \bigcap (c_g + i, c_g + i + 1]\}.\end{aligned}$$

Both Ξ_R and Ξ_L can be thought of as spaces of $\{0,1\}-$valued functions which are constant on intervals of length 1; the c_g is a "marker" which indicates modulo 1 where the "jumps" in the graph of g can occur. Notice that for any nonconstant g in Ξ_R (or Ξ_L), the function g determines the constant c_g uniquely. We put the same metric on both Ξ_R and Ξ_L by

$$d(g, g') = \int_0^\infty \frac{|g(t) - g'(t)|}{e^t} dt \ + \ |c_g - c'_{g'}|.$$

Under this metric, both Ξ_R and Ξ_L are Polish spaces. The semiflow σ_t is defined on both Ξ_R and Ξ_L by shifting the function and translating the marker as follows:

$$\sigma_t(g, c_g) = (\sigma_t(g), (c_g - t) \text{ mod } 1)$$

where $\sigma_t(g)(s) = g(t+s)$; this is a Borel action. We have two semiflows (Ξ_R, σ_t) and (Ξ_L, σ_t) which are alike except in that Ξ_R is thought of as a set of right-continuous functions (by ignoring the marker) and Ξ_L is thought of as a set of left-continuous functions.

Semiflows of this type can be realized by taking suspensions of the map $\sigma : S^1 \to S^1$ defined by $\sigma : x \mapsto 2x (\text{mod } 1)$. Given the constant return-time function $f \equiv 1$, we construct two "suspension spaces":

$$Y'_R = S^1 \times [0, 1) \text{ and } Y'_L = S^1 \times (0, 1].$$

Usually the top and bottom of a suspension space are identified. Here we purposefully leave Y'_R as a suspension which is "open on top" and closed on the bottom and Y'_L as a suspension which is "closed on top" and "open on bottom". Denote by T_t the suspension semiflow on both Y'_L and Y'_R, and let π be the projection map $(\zeta, t) \mapsto \zeta$ from $S^1 \times \mathbb{R}^+$ to S^1.

Proposition 3.3.1 (Y'_R, T_t) *is topologically conjugate to* (Ξ_R, σ_t) *(and similarly* (Y'_L, T_t) *is topologically conjugate to* (Ξ_L, σ_t)*).*

Proof: Given any $(\zeta, t) \in Y'_R$, define a function $g(\zeta, t)$ by

$$g(\zeta, t)(s) = \begin{cases} 1 & \text{if } \pi(T_s(\zeta, t)) \in [1/2, 1) \\ 0 & \text{if } \pi(T_s(\zeta, t)) \in [0, 1/2) \end{cases}.$$

Notice that if we let $g = g(\zeta, t)$ and $c_g = c_{g(\zeta,t)} = 1 - t$, then $(g, c_g) \in \Xi_R$ as $g(\zeta, t)$ is necessarily constant on intervals of the form $[c_g + i, c_g + i + 1)$. Now

$$\begin{aligned} g(\zeta, t)(r + s) &= \begin{cases} 1 & \text{if } \pi(T_{r+s}(\zeta, t)) \in [1/2, 1) \\ 0 & \text{if } \pi(T_{r+s}(\zeta, t)) \in [0, 1/2) \end{cases} \\ &= g(T_s(\zeta, t))(r) \end{aligned}$$

so now for any $(\zeta, t) \in Y'_R$, we have

$$\begin{aligned} \sigma_s(g(\zeta, t), c_{g(\zeta,t)}) &= \big(g(T_s(\zeta, t)), (c_{g(\zeta,t)} - s) \text{ mod } 1\big) \\ &= \big(g(T_s(\zeta, t)), c_{g(T_s(\zeta,t))}\big) \end{aligned}$$

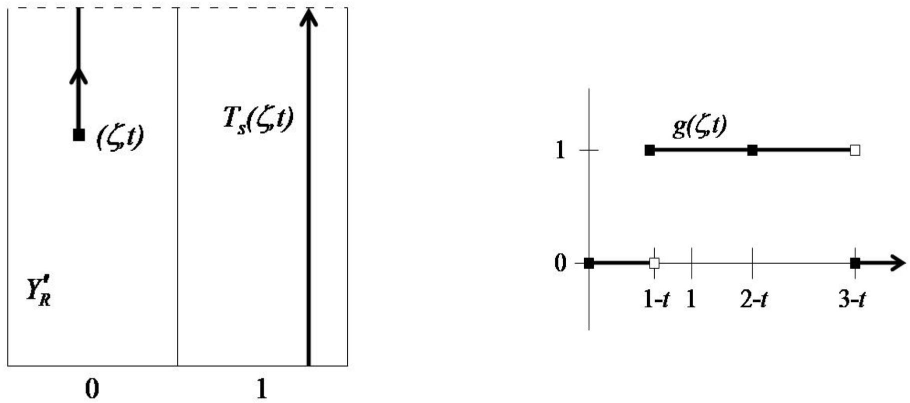

Figure 3.1: The orbit of a point (ζ, t) in Y'_R and its corresponding function $(g(\zeta,t), c_g) \in \Xi_R$.

so the two semiflows are equivariant. Now given a $(g, c_g) \in \Xi_R$, let

$$\zeta(g, c_g) = \sum_{i=0}^{\infty} \frac{g(i)}{2^i} \text{ and } t(g, c_g) = 1 - c_g.$$

This gives a mapping $(g, c_g) \mapsto (\zeta(g), t(g))$ which is an inverse to $(\zeta, t) \mapsto (g(\zeta, t), c_g)$. Thus the two semiflows are conjugate as desired (the conjugacy between (Y'_L, T_t) and (Ξ_L, σ_t) is given by the same formulas). ■

The conjugacy constructed in this proof is illustrated in Figures 3.1 and 3.2 below.

We now describe the orbit discontinuities and IDI in these examples. Consider first the right-continuous case; observe that $\sigma_t : \Xi_R \to \Xi_R$ is finite-to-1 for every t (in particular $\#(\sigma_{-t}(g, c_g)) \leq 2^t$ for every $t \geq 0$) so Proposition 3.2.8 applies. Given a $(g, c_g) \in \Xi_R$, the orbit discontinuities of (g, c_g) (with respect to σ_t) occur at times $c_g + \mathbb{Z}^+ = \{c_g, c_g + 1, c_g + 2, ...\}$. However, $IDI(\sigma_t) = \emptyset$ in this case. To see this, suppose (g, c_g) and (g', c'_g) are distinct elements of Ξ_R for which $\sigma_t(g, c_g) = \sigma_t(g', c'_{g'})$ for all $t > 0$. Then $g(0) = g'(0)$ since all functions in Ξ_R are

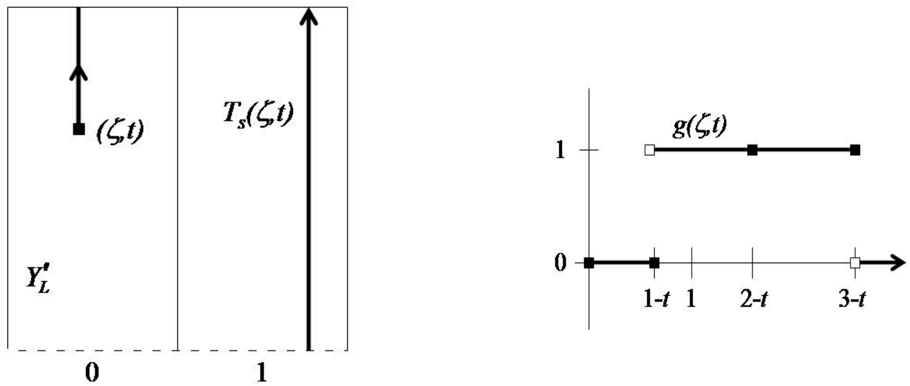

Figure 3.2: The orbit of a point (ζ, t) in Y'_L and its corresponding function $(g(\zeta,t), c_g) \in \Xi_L$.

right-continuous, and certainly $c_g = c'_{g'}$. Consequently, the suspension semiflow of the one-sided shift which is "open on top" and "closed on the bottom" has no IDI.

The orbit discontinuities in the left-continuous case (Ξ_L, σ_t) are the same as in the right-continuous case. However, this semiflow also exhibits IDI. Let $g_1 : [0,\infty) \to \{0,1\}$ be the constant function 1 and define $g_2 : [0,\infty) \to \{0,1\}$ by

$$g_2(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases} ;$$

clearly $(g_1, 0)$ and $(g_2, 0)$ are IDI by σ_t. In fact, every $(g, c_g) \in \Xi_L$ satisfies

$$IDI(g, c_g) = D(g, c_g) = c_g + \mathbb{Z}^+.$$

One can generalize this construction. Start with an endomorphism $\widehat{T}$ of $[0,1]$ which is nowhere invertible. Suppose there is a partition $\mathcal{P} = \{P_0, ..., P_n\}$ such that the partitions $\mathcal{P}, \widehat{T}^{-1}(\mathcal{P}), \widehat{T}^{-2}(\mathcal{P}), ...$ generate (this is true for any $\widehat{T}$ of finite entropy by the Krieger generator theorem). Now take a suspension semiflow T_t of

$\widehat{T}$ with return-time function f (actually we consider two suspensions: one which is "open on top" and "closed on the bottom" and one which is the opposite). Now any point in the suspension space maps to a function via the mapping $(x,t) \mapsto g(x,t)$ where

$$g(x,t)(s) = \frac{i}{n} \Leftrightarrow \pi(T_s(x,t)) \in P_i.$$

The resulting function $g(x,t)$ has range $\{0, 1/n, 2/n, ..., 1\}$, is either right- or left-continuous depending on the choice of suspension, and is constant on intervals. Call a function *allowable* if it is the image of some point in the suspension space under this mapping.

If one starts with a suspension which is "open on top" and "closed on the bottom", the allowable functions will be right-continuous and the shift σ_t restricted to the allowable functions will have no IDI. Conversely, if the suspension is taken to be "closed on top" but "open on the bottom", then any function which is the image of a point at the top of the suspension space will be IDI under the shift σ_t.

3.4 IDI and time changes

We end this section by verifying that the IDI set of a Borel semiflow is invariant under time changes. Given a Polish space X and an action T_t of $\mathbb{R}^+$ on X, we say that a function $\alpha : X \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *Borel cocycle for* (X, T_t) if:

i. for all $x \in X$, $\alpha_x(t) = \alpha(x,t)$ is a continuous bijection of $\mathbb{R}^+$,

ii. given any Borel set $B \subseteq \mathbb{R}^+$, the set $\{(x,t) : \alpha(x,t) \in B\}$ is a Borel subset of $X \times \mathbb{R}^+$, and

iii. for all $s, t \in \mathbb{R}^+$, and for all $x \in X$, α satisfies the cocycle relation

$$\alpha(T_t(x), s) = \alpha(x, t + s) - \alpha(x, t).$$

It is immediate that any Borel cocycle α also satisfies the following:

- For all $x \in X$, $\alpha(x, 0) = 0$.
- If $s < t$, then $\alpha(x, s) < \alpha(x, t)$.

In particular, suppose α is a Borel cocycle. Then α determines another Borel semiflow $\widetilde{T}_t$ on X by the action $\widetilde{T}_t(x) = T_{\alpha(x,t)}(x)$. In this case we say that $\widetilde{T}_t$ is a *time change* of T_t.

Theorem 3.4.1 *If $\widetilde{T}_t$ is a time-change of T_t, then $IDI(T_t) = IDI(\widetilde{T}_t)$.*

Proof: There is some Borel cocycle $\alpha : X \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\widetilde{T}_t(x) = T_{\alpha(x,t)}(x)$ for all x, t. Suppose $x \in IDI(T_t)$. Then there exists a $y \neq x$ with $T_t(y) = T_t(x)$ for all $t > 0$. Thus by the cocycle relation, we have

$$\begin{aligned}
\alpha_y(t+s) - \alpha_y(t) &= \alpha(y, t+s) - \alpha(y, t) \\
&= \alpha(T_t(y), s) \\
&= \alpha(T_t(x), s) \\
&= \alpha(x, t+s) - \alpha(x, t) \\
&= \alpha_x(t+s) - \alpha_x(t)
\end{aligned}$$

for all $s \geq 0, t > 0$. Since α_y and α_x are continuous, we can take the limit of both sides of this expression as $t \to 0^+$ and obtain $\alpha_y(s) = \alpha_x(s)$ for all $s \geq 0$. Therefore $\widetilde{T}_t(y) = T_{\alpha(y,t)}(y) = T_{\alpha(x,t)}(x) = \widetilde{T}_t(x)$ for all $t > 0$ so $x \in IDI(\widetilde{T}_t)$.

On the other hand, if $\widetilde{T}_t$ is a time-change of T_t, then T_t is a time-change of $\widetilde{T}_t$ by some other good cocycle β where $\beta(x,t)$ is defined as the solution to the implicit equation $\alpha(x, \beta(x,t)) = t$. So by the above argument $IDI(\widetilde{T}_t) \subseteq IDI(T_t)$ so the sets must be equal. ■

Chapter 4

Cross-sections and orbit discontinuities

In the 1940s Ambrose and Kakutani developed classical results regarding the existence of sections for measure-preserving flows. They applied these results to show that any measure-preserving flow (on a standard probability space) is measurably conjugate to a flow under a function. Later Krengel [3] showed that the Ambrose-Kakutani construction can be applied to semiflows. In particular, he showed that any measure-preserving semiflow has a measurable cross-section. This work is summarized in [4]. We repeat some of the basic definitions here for convenience.

Given a measure-preserving semiflow $(X, \mathcal{F}, \mu, T_t)$, we say that a set $F_0' \in \mathcal{F}$ is a *thick section* for the semiflow if there exist parameters $0 < \alpha < \beta$ and a measurable function $\gamma : X \to [0, \infty)$ with

i. $\gamma(T_{\gamma(x)}x) \geq \beta$,

ii. $\{T_t(x)\}_{\gamma(x) \leq t < \gamma(x)+\alpha} \subseteq F_0'$, and

iii. $\{T_t(x)\}_{\gamma(x)+\alpha \leq t < \gamma(x)+\gamma(T_{\gamma(x)}x)} \bigcap F_0' = \emptyset$.

Then a set F_0 is called a *section* or *cross-section* for the semiflow if F_0 consists of the left-endpoints of intervals of occurrence of some thick section F_0' on T_t-orbits.

F_0 is endowed with a $\sigma-$algebra $\mathcal{G}_0$ of measurable sets as follows: a set $A \subseteq F_0$ belongs to $\mathcal{G}_0$ if $\overline{A} = \{T_t(x) : x \in A, 0 \le t < \alpha\}$ is $\mathcal{F}-$measurable. Then $(F_0, \mathcal{G}_0)$ has measure μ_0 defined by setting $\mu_0(A) = \mu(\overline{A})/\mu(F_0')$ for each $A \in \mathcal{G}_0$. Given any section F_0, there is a return-time function $f(x) = \inf\{t \in \mathbb{R}^+ : T_t(x) \in F_0\}$; Lin and Rudolph [4] show that this map is measurable and that one can choose a section so that the return-time function is bounded above and below by any two values. If this is the case we say that the section is *bounded.* Define

$$\widehat{X} = \{(x,t) \in F_0 \times \mathbb{R}^+ : 0 \le t < f(x)\}$$

and notice that the map $\pi_X : \widehat{X} \to X$ defined by $\pi_X(x,t) = T_t(x)$ makes $\widehat{X}$ with its "semiflow under a function" a measurable extension of the original semiflow.

We will need the following condition on our sections. Call a section F_0 *good* if it is bounded and has the following two properties:

i. Whenever $T_{f(x)}(x) = T_{f(y)}(y)$ for some pair $x, y \in F_0$, then $f(x) = f(y)$.

ii. Given any $x \in F_0$ and $t < f(x)$, the set $T_{-t}T_t(x)$ is contained in F_0.

Proposition 4.0.1 *Let F be a good section. For each pair of points $x, y \in F$, define*

$$A_y(x) = \{t \in [0, f(x)) : T_t(x) = T_s(y) \textit{ for some } s \in [0, f(y))\}.$$

Then for all $x, y \in F, A_y(x) = A_x(y)$.

Proof: If $A_y(x)$ is empty, then the forward orbits of x and y are disjoint at least until they return to the section so $A_x(y)$ is also empty. Now if $t \in A_y(x)$, then there exists $s < f(y)$ so that $T_s(y) = T_t(x)$. By applying T to both sides, we get $T_{f(x)+s-t}(y) = T_{f(x)}(x)$ and $T_{f(y)}(y) = T_{f(y)+t-s}(x)$. But if the forward orbits of

x and y meet before they return to the section, then they must coincide when they return to the section, i.e. $T_{f(x)}(x) = T_{f(y)}(y)$. Since F is a good section, $f(x) = f(y)$ so we have $T_{f(y)+s-t}(y) = T_{f(y)}(y)$ and $T_{f(x)+t-s}(x) = T_{f(x)}(x)$. If $s \neq t$, one of x or y must hit the section before its return time. This is impossible so $s = t$ and $t \in A_x(y)$. Thus $A_y(x) \subseteq A_x(y)$; by symmetry these sets must therefore coincide. ■

This proposition ensures that for a good section F_0, any points which get identified before they return to the base must be identified at the same height, i.e. we cannot see points x, y in the base with $T_s(x) = T_t(y)$ (for $0 \leq s < f(x), 0 \leq t < f(y)$) but $s \neq t$. Also, if F_0 is a good section then for any point x in X there is a nonnegative number $\overline{f}(x)$ such that $T_{-\overline{f}(x)}(x) \subseteq F_0$ but $T_{-\alpha}(x) \bigcap F_0 = \emptyset$ for all $\alpha \in (0, \overline{f}(x))$ (recall that T_t is assumed to be surjective). So if one writes $x \in X$ as $T_t(y)$ where $y \in F_0$ and $0 \leq t < f(y)$, there is only one possible choice for t (of course there may be lots of choices for y).

Proposition 4.0.2 *Suppose that the semiflow* $(X, \mathcal{F}, \mu, T_t)$ *has a section* F *with bounded return time. Then the semiflow has a good section.*

Proof: Suppose the return time is bounded by some constant less than B. Then let $F_0 = T_{-B}(F)$. That F_0 is a section is obvious from the definition. Let f be the return time function for F_0; clearly $f < B$ so F_0 is a bounded section. Now suppose $x, y \in F_0$ are such that $T_{f(x)}(x) = T_{f(y)}(y)$. Assume $f(x) \geq f(y)$, and let $\delta = f(x) - f(y)$. Now $T_B(x)$ and $T_B(y) = T_{B+\delta}(x)$ are both in F. So $T_\delta(x) \in F_0$ and as $0 \leq \delta < f(x)$, $\delta = 0$.

Finally, to verify condition (ii) in the definition of good section, take $x \in F_0$ and $t < f(x)$ and consider some $y \in T_{-t}T_t(x)$. Then $z = T_t(x) = T_t(y)$ satisfies $T_{B-t}(x) \in F$ so $y \in F_0$ as desired. Thus F_0 is a good section. ■

4.1 Modifying the Ambrose-Kakutani constructions

Now we describe how to modify the Ambrose-Kakutani constructions to obtain better models for arbitrary semiflows. We start by establishing some notation. Let F_0 be a good section for $(X, \mathcal{F}, \mu, T_t)$ with return-time function $f : F_0 \to [b, B]$ (where $b > 0, B < \infty$); denote its $\sigma-$algebra of measurable sets by $\mathcal{G}_0$ and its measure by μ_0 (these were described earlier in this section). So the suspension semiflow arising from the section F_0 is denoted $(\widehat{X}, \mathcal{G}_0 \times \mathcal{L}, \mu_0 \times dt, \widehat{T}_t)$ ($\mathcal{L}$ and dt are the $\sigma-$algebra of Lebesgue measurable subsets of $[0, \infty)$ and Lebesgue measure, respectively). Finally denote the Poincaré return map from F_0 to itself by $\widehat{T}$ (to distinguish this discrete transformation from the suspension semiflow $\widehat{T}_t$, we use no subscript). In general $(\widehat{X}, \mathcal{G}_0 \times \mathcal{L}, \mu_0 \times dt, \widehat{T}_t)$ is an extension of the original system $(X, \mathcal{F}, \mu, T_t)$. The factor map π_X taking (x, t) to $T_t(x)$ may not be $1-1$ anywhere away from the base, however. We seek a model where the factor map is much "closer" to being $1-1$.

Let $F_1 = i_T^{\mathbb{Q}^+}(F_0)$ and $\nu_0 = i_T^{\mathbb{Q}^+}(\mu_0)$. We endow F_1 with the relative topology from $X^{\mathbb{Q}^+}$; ν_0 is then a Borel measure on F_1. Let $K(F_1)$ be the set of closed subsets of F_1. Since $i_T^{\mathbb{Q}^+}$ is injective (with one-sided inverse $\pi_{\mathbb{Q}^+}$) we can think of the return-time function f as a function on F_1 as well as on F_0. For each $x \in F_1$ and each $t \in [0, \infty)$, we define

$$[x]^t = \begin{cases} \overline{\bigcup_{q<t, q\in\mathbb{Q}^+} \sigma_{-q}\sigma_q(x)} & \text{if } t > 0 \\ \{x\} & \text{if } t \leq 0 \end{cases},$$

and

$$[x]_t = \begin{cases} \bigcap_{q>t, q\in\mathbb{Q}^+} \sigma_{-q}\sigma_q(x) & \text{if } t \geq 0 \\ \{x\} & \text{if } t < 0 \end{cases}.$$

(Note that this is not quite the same $[x]_t$ that was defined in Chapter 3.) For all x and t, the $[x]_t$ and $[x]^t$ are closed subsets of F_1 which increase in t for a fixed x and partition F_1 for a fixed t. Of course $[x]^t \subseteq [x]_t$ for all x, t; by Lemma 3.2.1 $[x]_t \neq [x]^t$ if and only if x has an orbit discontinuity at time t.

Lemma 4.1.1 *Let $x \in F_0$ and suppose $y \in F_0$ is such that $[i_T^{\mathbb{Q}^+}(x)]_t = [i_T^{\mathbb{Q}^+}(y)]_t$ where $t < f(x)$. Then $f(x) = f(y)$.*

Proof: Take any $s \in \mathbb{Q}^+ \bigcap (t, f(x))$; we have $T_s(x) = T_s(y)$ since $\sigma_s(i_T^{\mathbb{Q}^+}(x)) = \sigma_s(i_T^{\mathbb{Q}^+}(y))$. Let $\tau_0(y) = 0$ and for each $n > 0$ define

$$\tau_n(y) = \inf\{\tau : \#(\{T_t(y)\}_{0<t<\tau} \bigcap F_0) = n\}.$$

Since the section F_0 is bounded, we see that $\tau_n(y) \to \infty$ as $n \to \infty$ and also that the quantities $\tau_n(y) - \tau_{n-1}(y)$ are uniformly bounded away from zero. Now let N be such that $\tau_N(y) \leq s < \tau_{N+1}(y)$ and let $z = T_{\tau_N(y)}(y) \in F_0$. Then $T_{s-\tau_N(y)}(z) = T_s(x)$ but $s - \tau_N(y) < f(z)$. Since the section is good, this implies x and z are identified at the same height, i.e. $\tau_N(y) = 0$. Consequently $N = 0$ so in fact $y = z$ and since x and y are identified before they return to F_0, $f(x) = f(y)$ as desired. ■

At this point we can construct the spaces for our models. Lemma 4.1.1 tells us that whenever $x \in F_1$ is such that $f(x) > t$, any $y \in [x]_t$ must satisfy $f(y) = f(x) > t$. Thus we can define equivalence relations $\sim^t$ and $\sim_t$ on F_1 by

$$x \sim^t y \Leftrightarrow [x]^t = [y]^t \text{ and } f(x) = f(y) > t;$$

$$x \sim_t y \Leftrightarrow [x]_t = [y]_t \text{ and } f(x) = f(y) > t.$$

Now given $x \in F_1$ with $f(x) > t$, define the maps $\pi_t : x \mapsto [x]_t$ and $\pi^t : x \mapsto [x]^t$. We say that a set $A \subseteq F_1/\sim_t$ is $\mathcal{G}_t-$measurable if $\pi_t^{-1}(A) \in \mathcal{G}_0$. It is clear that the $\mathcal{G}_t-$measurable sets form a $\sigma-$algebra. Similarly we define the $\sigma-$algebra $\mathcal{G}^t$ to be the class of subsets of $F_1/\sim^t$ whose inverses under π^t are $\mathcal{G}_0-$measurable. Let $\nu_t = \pi_t(\mu_0)$ and $\nu^t = \pi^t(\mu_0)$; these are measures on the respective $\sigma-$algebras $\mathcal{G}_t$ and $\mathcal{G}^t$. It is important to note that the maximum ν_t (or ν^t) measure of a set is equal to $\nu_0(\{x \in F_1 : f(x) > t\})$; in particular we would not expect $\nu_t(A)$ to be equal to 1 for any set A if t is large enough. In fact the "largest" set (measure-theoretically) in the $\sigma-$algebras $\mathcal{G}_t$ and $\mathcal{G}^t$ is the set $\{x : f(x) > t\}$.

Now consider the two spaces

$$F^* = \{([x]^t, t) : x \in F_1, t \in [0, f(x))\} \text{ and}$$

$$F_* = \{([x]_t, t) : x \in F_1, t \in [0, f(x))\}.$$

One thinks of F^* and F_* as "copies" of $\widehat{X}$ where some identifications have taken place: in F^*, at each height t, points identified at any height less than t are collapsed to a single point; in F_* points identified by the semiflow at all heights greater than t are collapsed to a single point at height t.

Each of these can be thought of as a measurable factor of the original Ambrose-Kakutani extension $(\widehat{X}, \mathcal{G}_0 \times \mathcal{L}, \mu_0 \times dt, \widehat{T}_t)$. The idea is as follows: the measurable sets of the Ambrose-Kakutani extension are generated by rectangles in $\widehat{X}$. Consider a sub- $\sigma-$algebra of $\mathcal{G}_0 \times \mathcal{L}$ generated only by those rectangles which respect the $[\cdot]^t-$classes, i.e. sets of the form $A \times [a, b]$ where $A \in \mathcal{F}_0$ and given any $t \in [a, b]$, if $x \in A$ then $[x]^t \subseteq A$ (since A is a rectangle, it is sufficient that this condition hold when $t = b$). Now any points not separated by such rectangles lie at the same height and in the same $[\cdot]^t$−equivalence class, so the indecomposable elements of this sub-σ−algebra are naturally identified with the elements of F^*.

There is analogous structure associated to F_*. We consider a σ−algebra on $\widehat{X}$ generated by rectangles respecting the $[\cdot]_t$−classes; the atoms of this σ−algebra can be thought of as the points of F_*.

To make this precise, we define maps $\pi^* : \widehat{X} \to F^*$ and $\pi_* : \widehat{X} \to F_*$ by

$$\pi^*(x,t) = ([i_T^{\mathbb{Q}^+}(x)]^t, t) \text{ and } \pi_*(x,t) = ([i_T^{\mathbb{Q}^+}(x)]_t, t).$$

Let $\mathcal{G}^*$ be the σ−algebra of subsets of F^* whose inverse images under π^* are $\mathcal{G}_0 \times \mathcal{L}$−measurable sets; this makes π^* a measurable map from $(\widehat{X}, \mathcal{G}_0 \times \mathcal{L})$ to $(F^*, \mathcal{G}^*)$. Note that given any $A \in \mathcal{G}^*$, the set of elements of A at height t must be $\mathcal{G}^t$−measurable for Lebesgue almost-every t. Analogously define $\mathcal{G}_*$ to be the subsets of F_* whose inverse images under π_* are $\mathcal{G}_0 \times \mathcal{L}$−measurable. Let $\nu^* = \pi^*(\mu_0 \times dt)$ and $\nu_* = \pi_*(\mu_0 \times dt)$.

Proposition 4.1.2 *$\nu^* = \int_0^B \nu^t \, dt$.*

Proof: Given any $A \in \mathcal{G}^*$ let $A' = (\pi^*)^{-1}(A)$. Define $A'_t = \{x \in F_1 : (x,t) \in A'\}$. Then

$$\nu^*(A) = (\mu_0 \times dt)(A') = \int_0^B \mu_0 \left(A'_t\right) \, dt$$

and in particular, by Fubini's theorem we know that Lebesgue almost-every A'_t is μ_0−measurable and that the function $\mu_0\left(A'_t\right)$ is an integrable function of t. Now if we let $A^t \subset F^*$ be the set of points $([x]^t, t)$ at height t lying in A, we have

$(\pi^t)^{-1}(A^t) = A'_t$ so A^t is $\mathcal{G}^t-$measurable for almost every t and we have

$$\begin{aligned}
\int_0^B \nu^t(A^t)\,dt &= \int_0^B \mu_0\left((\pi^t)^{-1}(A_t)\right)\,dt \\
&= \int_0^B \mu_0\left(A'_t\right)\,dt \\
&= \nu^*(A) \text{ as desired. } \blacksquare
\end{aligned}$$

Proposition 4.1.3 $\nu_* = \int_0^B \nu_t\,dt$.

Proof: The proof is essentially the same as that of the previous result. Take $A \in \mathcal{G}_*$ and let $A' = (\pi_*)^{-1}(A)$; let A'_t be as in the previous proposition. We have

$$\nu_*(A) = (\mu_0 \times dt)(A') = \int_0^B \mu_0\left(A'_t\right)\,dt$$

and as before, A'_t is μ_0-measurable almost surely in t and $\mu_0\left(A'_t\right)$ is an integrable function of t. Let $A_t \subset F_*$ be the set of points $([x]_t, t)$ at height t lying in A. We see that $(\pi_t)^{-1}(A_t) = A'_t$ so

$$\begin{aligned}
\int_0^B \nu_t(A^t)\,dt &= \int_0^B \mu_0\left((\pi_t)^{-1}(A_t)\right)\,dt \\
&= \int_0^B \mu_0\left(A'_t\right)\,dt \\
&= \nu_*(A). \blacksquare
\end{aligned}$$

Next we define semiflows which closely model the original semiflow T_t. First, define $(T^*)_t : F^* \to F^*$ by

$$(T^*)_t([x]^s, s) = \begin{cases} ([x]^{s+t}, s+t) & \text{if } 0 \le t < f(x) - s \\ ([\widehat{T}(x)]^0, 0) & \text{if } t = f(x) - s \end{cases};$$

of course this definition suffices to determine the action of the semiflow for all t if we assume the semiflow obeys the formula $(T^*)_{t+\tau} = (T^*)_t \circ (T^*)_\tau$. Similarly define $(T_*)_t : F_* \to F_*$ by

$$(T_*)_t([x]_s, s) = \begin{cases} ([x]_{s+t}, s+t) & \text{if } 0 \le t < f(x) - s \\ ([\widehat{T}(x)]_0, 0) & \text{if } t = f(x) - s \end{cases}$$

and extend to larger times t as above. Notice $(T^*)_t$ and $(T_*)_t$ are factors of the semiflow $\widehat{T}_t$ under the measurable maps π^* and π_* respectively, so $(T^*)_t$ and $(T_*)_t$ are measurable, measure-preserving semiflows. Let us describe our construction heuristically. In the original Ambrose-Kakutani construction, points in the space are indexed by a base point x and a time t. The semiflow is prescribed by a vertical flow taking points (x, t) upward through points (x, τ) for $t < \tau < f(x)$, combined with a return map to the base. Here we still use the same return map to the base but the upward flow taking points to points is replaced with an action which maps x to the successively larger closed sets $[x]_t$ (or $[x]^t$ depending on which semiflow we use). The reason why these models are an improvement is that they are much closer to being conjugate to the original semiflow, as we see next.

Proposition 4.1.4 *Define* $\Psi : F^* \to X$ *by*

$$\Psi([x]^t, t) = T_t(\pi_{\mathbb{Q}^+}(x)).$$

Then:

i. Ψ *is well-defined.*

ii. Ψ *is measurable, i.e. if* $A \in \mathcal{F}$*, then* $\Psi^{-1}(A) \in \mathcal{G}^*$*.*

iii. $\Psi(\nu^*) = \mu$*.*

iv. For all $s \ge 0$*,* $T_s \circ \Psi = \Psi \circ (T^*)_s$*.*

v. Ψ *fails to be* $1-1$ *only at the orbit discontinuities of* T_t.

Proof: To verify (i), let $x, y \in F_1$ be such that $([x]^t, t) = ([y]^t, t)$. Then $\sigma_q(x) = \sigma_q(y)$ for some $q < t$ so $T_q(\pi_{\mathbb{Q}^+}(x)) = T_q(\pi_{\mathbb{Q}^+}(y))$. Consequently $\Psi([x]^t, t) = \Psi([y]^t, t)$ so Ψ is well-defined.

For the second statement, let $A \in \mathcal{F}$. Then $A' = \pi_X^{-1}(A)$ is a measurable set in the original Ambrose-Kakutani space $(\widehat{X}, \mathcal{G}_0 \times \mathcal{L})$. Define $A'_t = \{x \in F_0 : (x,t) \in A'\}$ and notice that if $x \in A'_t$, then $[x]^t \subseteq A'_t$ since any $y \in [x]_t$ satisfies $T_t(\pi_{\mathbb{Q}^+}(y)) = T_t(\pi_{\mathbb{Q}^+}(x))$. Therefore $(\pi^*)^{-1}(\pi^*(A') = A'$ so $A'' = \pi^*(A'))$ is $\mathcal{G}^*-$measurable. It is left to show that $A'' = \Psi^{-1}(A)$; we have the following:

$$([x]^t, t) \in A'') \Leftrightarrow (\pi_{\mathbb{Q}^+}(x), t) \in A' \Leftrightarrow \pi_X(\pi_{\mathbb{Q}^+}(x), t) \in A \Leftrightarrow T_t(\pi_{\mathbb{Q}^+}(x)) \in A$$

so $A'' = \Psi^{-1}(A)$ is $\mathcal{G}^*-$measurable as desired.

Statement (iii) follows from Proposition 4.1.2:

$$\mu(A) = (\mu_0 \times dx)(A') = \nu^t(A'').$$

For (iv), let $([x]^t, t) \in F^*$. If $s < f(x) - t$, then

$$\begin{aligned} T_s \circ \Psi([x]^t, t) &= T_s(T_t(\pi_{\mathbb{Q}^+}(x))) \\ &= T_{s+t}(\pi_{\mathbb{Q}^+}(x)) \\ &= \Psi([x]^{s+t}, s+t) \\ &= \Psi \circ (T^*)_s([x]^t, t). \end{aligned}$$

If $s = f(x) - t$, then

$$\begin{aligned} T_s \circ \Psi([x]^t, t) &= T_s(T_t(\pi_{\mathbb{Q}^+}(x))) \\ &= T_{f(x)}(\pi_{\mathbb{Q}^+}(x)) \\ &= \widehat{T}(\pi_{\mathbb{Q}^+}(x)) \\ &= \Psi([\widehat{T}(x)]^0, 0) \\ &= \Psi \circ (T^*)_s([x]^t, t). \end{aligned}$$

Last, we suppose $\Psi([x]^t, t) = \Psi([y]^s, s)$. Then $T_t(\pi_{\mathbb{Q}^+}(x)) = T_s(\pi_{\mathbb{Q}^+}(y))$ so $s = t$ since F_0 is a good section. Therefore $T_t(\pi_{\mathbb{Q}^+}(x)) = T_t(\pi_{\mathbb{Q}^+}(y))$ so $[x]^\tau = [y]^\tau$ for all $\tau > t$. If x does not have an orbit discontinuity at time t, then $[x]^t = [y]^t$ and Ψ is injective there. ■

Now we define a map $\Phi : X \to F_*$. First, any $x \in X$ can be written as $T_t(y)$ where $y \in F_0, 0 \leq t < f(y)$. In particular, recall that since F_0 is a good section there is only one choice for t in such a representation.

Proposition 4.1.5 *Define* $\Phi : X \to F_*$ *by*

$$\Phi(x) = \Phi(T_t(y)) = ([i_T^{\mathbb{Q}^+}(y)]_t, t).$$

Then:

i. Φ *is well-defined.*

ii. Φ *is measurable, i.e. if* $A \in \mathcal{G}_*$*, then* $\Phi^{-1}(A) \in \mathcal{F}$*.*

iii. $\Phi(\mu) = \nu_*$*.*

iv. For all $s \geq 0$*,* $\Phi \circ T_s = (T_*)_s \circ \Phi$*.*

v. Φ *fails to be* $1-1$ *only at the orbit discontinuities of* T_t*.*

Proof: The proof is like that of the previous result. To show Φ is well-defined, suppose we write $x \in X$ as $T_t(y)$ and $T_t(z)$ where $y, z \in F_0$ and $0 \leq t < f(y) = f(z)$. Then $T_t(y) = T_t(z) = x$ so $\sigma_q(i_T^{\mathbb{Q}^+}(y)) = \sigma_q(i_T^{\mathbb{Q}^+}(z))$ for all $q \in \mathbb{Q}^+$ greater than or equal to t. Thus $[i_T^{\mathbb{Q}^+}(y)]_t = [i_T^{\mathbb{Q}^+}(z)]_t$ so $\Phi(x)$ is well-defined.

To show (ii), suppose $A \in \mathcal{G}_*$. Then $A' = (\pi_*)^{-1}(A)$ is $\mathcal{G}_0 \times \mathcal{L}-$measurable so $\Phi^{-1}(A) = \pi_X(A') \in \mathcal{F}$.

Next, let $A \in \mathcal{G}_*$. Then by applying Proposition 4.1.3,

$$\nu_*(A) = (\mu_0 \times dt)(A') = \mu(\Phi^{-1}(A)).$$

For the fourth statement, let $x \in X$ and write $x = T_t(y)$ where $y \in F_0, 0 \leq t < f(y)$. Then if $0 \leq s < f(y) - t$,

$$\begin{aligned}
\Phi \circ T_s(x) &= \Phi \circ T_s(T_t(y)) \\
&= \Phi(T_{s+t}(y)) \\
&= ([i_T^{\mathbb{Q}^+}(y)]_{s+t}, s+t) \\
&= (T_s)_*([i_T^{\mathbb{Q}^+}(y)]_t, t) \\
&= (T_s)_* \circ \Phi(x).
\end{aligned}$$

If $s = f(y) - t$, then

$$\begin{aligned}
\Phi \circ T_s(x) &= \Phi \circ T_{f(y)}(y) \\
&= \Phi(\widehat{T}(y)) \\
&= ([i_T^{\mathbb{Q}^+}(\widehat{T}(y))]_0, 0) \\
&= (T_*)_s([i_T^{\mathbb{Q}^+}(y)]_t, t) \\
&= (T_*)_s \circ \Phi(x).
\end{aligned}$$

Last, suppose $\Phi(x) = \Phi(x')$ where $x \neq x'$. Then we write $x = T_t(y)$ and $x' = T_s(y')$ where $y, y' \in F_0$; since $\Phi(x) = \Phi(x')$ s must equal t and $[i_T^{\mathbb{Q}^+}(y)]_t =$

$[i_T^{\mathbb{Q}^+}(y')]_t$. Thus $T_\tau(y) = T_\tau(y')$ for all $\tau > t$, i.e. $T_\tau(x) = T_\tau(x')$ for all $\tau > 0$. So if $x \neq x'$, then x has an orbit discontinuity at time 0 (this means y has orbit discontinuity at time t). ■

What we have done is construct an extension $(T^*)_t$ and a factor $(T_*)_t$ of an arbitrary semiflow T_t which are conjugate to the original semiflow except on a set of measure zero. In particular both the factor and extension can be described as a semiflow consisting of repeated quotient maps onto closed equivalence classes of a cross-section of T_t together with a measurable return-time transformation describing how points return to the cross-section. We summarize the work of this section in the following result:

Theorem 4.1.1 *Given a measure-preserving Borel semiflow $(X, \mathcal{F}, \mu, T_t)$, we can construct measure-preserving semiflows $(F^*, \mathcal{G}^*, \nu^*, (T^*)_t)$ and $(F_*, \mathcal{G}_*, \nu_*, (T_*)_t)$ as in this section so that the following commutative diagram holds:*

$$\begin{array}{ccccc}
(F^*, \mathcal{G}^*, \nu^*) & \xrightarrow{\Psi} & (X, \mathcal{F}, \mu) & \xrightarrow{\Phi} & (F_*, \mathcal{G}_*, \nu_*) \\
\downarrow{\scriptstyle (T^*)_t} & & \downarrow{\scriptstyle T_t} & & \downarrow{\scriptstyle (T_*)_t} \\
(F^*, \mathcal{G}^*, \nu^*) & \xrightarrow{\Psi} & (X, \mathcal{F}, \mu) & \xrightarrow{\Phi} & (F_*, \mathcal{G}_*, \nu_*)
\end{array}$$

In particular, all maps are measurable and measure-preserving and Φ and Ψ are $1-1$ except at the orbit discontinuities of T_t (a set of measure zero). If T_t does not have an orbit discontinuity at x, then $\Psi^{-1}(x) = \Phi(x)$ as subsets of $K(F_1) \times [0, \infty)$.

Proof: The semiflows $(T^*)_t$ and $(T_*)_t$ are measure-preserving since they are equivariant with the measure-preserving semiflow T_t. The only other statement left to prove is the last one. Suppose T_t has no orbit discontinuity at x. Then if we write $x = T_t(y)$ where $y \in F_0$ and $0 \leq t < f(y)$, $\Psi^{-1}(x) = ([y]^t, t) = ([y]_t, t) = \Phi(x)$. ■

Chapter 5

A measure-theoretic approach to orbit discontinuities

The constructions in Chapter 3 were mostly topological in nature. We would like to know if similar constructions can be made using techniques from measure theory. To see why, we consider an "example" of a semiflow which is everywhere finite-to-1. Suppose $x \in X$ is such that $T_1(x)$ is IDI but no other $T_t(x)$ is IDI for $t \geq 0$. Suppose further that $\#(T_{-t}T_t(x)) = 1$ for $t \leq 1$. Thus there is some collection of points $z_1, ..., z_n \in X$ with $T_t(z_i) = T_t(x)$ for $t > 1$ but $T_t(z_i) \neq T_t(x)$ for $t \leq 1$. Heuristically, we ask the following question: given that $T_t(y) = T_t(x)$, what is the probability that $y = x$? For $t \leq 1$ in this example, the probability is 1 by assumption. It stands to reason, however, that this probability at times $t \in (1, 1+\epsilon)$ would be strictly less than 1 (and bounded above by a number strictly less than 1) since y could be one of the z_i when $t > 1$. Yet, if $T_t(y) = T_t(x)$ for t slightly greater than 1, it may still be the case that $y = x$ with probability 1. In this situation we do not want to regard the behavior of $T_t(x)$ at $t = 1$ as truly "discontinuous" with respect to x. On the other hand, if the probability is 1 for $t \leq 1$ but say $1/2$ for $t > 1$, we certainly want to regard the behavior at time 1 as "measurably discontinuous".

This chapter lays out a measure-theoretic approach to the study of orbit discontinuities along this line of thinking. In the next section, we use weak$^*-$limits of conditional expectations to define for each point x (in a T_t-invariant set of full $\mu-$measure) two "measure paths" $\mu^+_{x,t}$ and $\mu^-_{x,t}$ which heuristically give a distribution on the set of points which are identified with x by the semiflow at all times greater than t, and the set of points identified with x by T_t at some time less than t, respectively. We say x has a "measurable orbit discontinuity" at time t_0 if the two measures μ^+_{x,t_0} and μ^-_{x,t_0} differ; these are precisely the times at which $\mu^+_{x,t}$ or $\mu^-_{x,t}$ is discontinuous in t. We show that there exists an invariant set of full measure in X for which every point has only countably many measurable orbit discontinuities. We connect the ideas here with the ideas of Chapter 3 by showing that if a point has a measurable orbit discontinuity at time 0, then it must also have an orbit discontinuity there in the sense of Chapter 3 as well. We finally give some examples to show that the topological and measure-theoretic notions of orbit discontinuity may not coincide nicely in general.

5.1 Measurable orbit discontinuities

Here we begin with a semiflow $(X, \mathcal{F}, \mu, T_t)$ where $(X, \mathcal{F}, \mu)$ is a standard probability space (i.e. a Lebesgue space). The semiflow T_t is assumed to be a measurable, measure-preserving action on $(X, \mathcal{F}, \mu)$.

First denote for each $t > 0$ the $\sigma-$algebras

$$\mathcal{F}_t = T_{-t}(\mathcal{F}) = \{A \subseteq X : A = T_{-t}(B) \text{ for some } B \in \mathcal{F}\}.$$

A set A belongs to $\mathcal{F}_t$ if and only if $T_{-t}T_t(A) = A$. In particular observe that $\mathcal{F}_0$ is the original $\sigma-$algebra $\mathcal{F}$ and that as t increases, the $\mathcal{F}_t$ get smaller. Extend this notation to negative t by setting $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$.

Now consider the Rohklin decompositions of $(\mathcal{F}, \mu)$ over the subalgebras $\mathcal{F}_t$. By this we mean that for each t X can be conjugated measurably to the unit square $I^2 = [0,1] \times [0,1]$ where the $\mathcal{F}_t$−measurable sets correspond to subsets of I^2 of the form $A \times [0,1]$ where A is Lebesgue measurable in I [9]. The measure μ can then be written as

$$\mu = \int_0^1 \mu_{x,t}\, dx$$

where $\mu_{x,t}$ is the corresponding fiber measure for the point x with respect to $\mathcal{F}_t$ and dx is a Lebesgue measure on $[0,1]$ (not necessarily non-atomic). The measures $\mu_{x,t}$ (if they exist) are defined by

$$\int f\, d\mu_{x,t} = E(f|\mathcal{F}_t)(x).$$

Any μ−integrable function $f : X \to \mathbb{R}$ is therefore $\mu_{x,t}$−measurable for μ−almost every x by Fubini's theorem and satisfies

$$\int f\, d\mu \ = \int_0^1 \int f\, d\mu_{x,t}\, dx = \int_0^1 E(f|\mathcal{F}_t)(x)\, dx.$$

In particular $\mu_{x,0} = \delta_x$ (a point mass at x); for any $x \in X$ we extend the definition of $\mu_{x,t}$ to negative t by setting $\mu_{x,t} = \delta_x$ for $t < 0$.

Of course there is a problem concerning the *existence* of the fiber measures $\mu_{x,t}$. The conditional expectations $E(f|\mathcal{F}_t)$ are only guaranteed to exist for almost every x; consequently it is only immediate that given $t \geq 0$, for μ−almost every x, the measure $\mu_{x,t}$ exists. We would like to "reverse the quantifiers" and say that there exists a set of full measure in X such that for every x in that full-measure set, $\mu_{x,t}$ exists for every t. First for each $t \geq 0$ define

$$B_t = \{x \in X : \mu_{x,t} \text{ does not exist}\}$$

and notice that since $\mu(B_t) = 0$ for each t, $\mu(X - \bigcup_{q\in\mathbb{Q}^+} B_q) = 1$.

Lemma 5.1.1 *Let $x \in X$ and $t \geq 0$ be such that $\mu_{x,t}$ exists. Then for $0 \leq s \leq t$, $\mu_{T_s(x),t-s}$ exists and is equal to $T_s(\mu_{x,t})$.*

Proof: Let f be an integrable function. Then

$$\begin{aligned}\int f\, d(T_s(\mu_{x,t})) &= \int (f \circ T_s)\ d\mu_{x,t} \\ &= E(f \circ T_s | \mathcal{F}_t)(x) \\ &= E(f|\mathcal{F}_{t-s})(T_s(x)) \\ &= \int f\, d\mu_{T_s(x),t-s}. \ \blacksquare\end{aligned}$$

As a consequence we see that $T_s(\mu_{x,t})$ is a point mass for $s \geq t$.

Corollary 5.1.2 *$X_0 = \{x \in X : \mu_{x,t}$ exists for a dense set of $t \in \mathbb{R}\}$ is forward invariant under T_t.*

Proof: It suffices to show that $A = X - X_0$ is backward invariant. Let $x \in A$; there exists an interval $S \subseteq \mathbb{R}^+$ such that $\mu_{x,s}$ does not exist for all $s \in S$. Let $y \in T_{-t}(x)$. Given $s \in S, \mu_{y,t+s}$ cannot exist; otherwise $\mu_{T_t(y),s} = \mu_{x,s}$ exists by Lemma 5.1.1. Thus for any time in the set $S + t$ (which is of positive Lebesgue measure) y has no fiber measure, so $y \in A$. $\blacksquare$

Now X_0 contains $(X - \bigcup_{q \in \mathbb{Q}^+} B_q)$, so $\mu(X_0) = 1$. Therefore X_0 is an invariant set of full measure, so we can without loss of generality assume $X_0 = X$. So for any x, the fiber measure $\mu_{x,t}$ exists for a dense set $G(x)$ of t in $[0, \infty)$. Now we describe how to "fill in" the gaps where the measure is not guaranteed to exist from Rohklin. Notice that for each $t \in G(x)$, $E(f|\mathcal{F}_t)$ exists for any measurable f. Let $\{g_m(x)\}_{m=1}^{\infty}$ be a sequence of sequences $g_m(x) = \{g_{m,n}(x)\}_{n=1}^{\infty}$ with the following properties:

i. g_m is a strictly increasing sequence for all m (i.e. $g_{m,n}(x) < g_{m,n+1}(x)$);

ii. $g_{m,n}(x) \in G(x) \,\forall\, m, n$;

iii. $g_m(x)$ is a subsequence of $g_{m+1}(x)$ for all m;

iv. $\overline{\bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} g_{m,n}(x)} = [0, \infty)$

Such a sequence is guaranteed to exist; to construct one, first let $\widehat{g}_{1,n}(x) = n - 1$ then let

$$g_{1,n}(x) = \begin{cases} \widehat{g}_{1,n}(x) & \text{if } \widehat{g}_{1,n} \in G(x) \\ \widehat{g}_{1,n}(x) + \epsilon_{1,n}(x) & \text{otherwise} \end{cases}$$

where $\epsilon_{1,n}(x) \in (-1/8, 1/8)$ is chosen such that $\widehat{g}_{1,n}(x) + \epsilon_{1,n}(x) \in G(x)$. Then using induction define for $m > 1$

$$\widehat{g}_{m+1,n}(x) = \begin{cases} g_{m,\frac{n+1}{2}}(x) & \text{if } n \text{ is odd} \\ \\ \frac{1}{2}\left(g_{m,\frac{n}{2}}(x) + g_{m,\frac{n}{2}+1}(x)\right) & \text{if } n \text{ is even} \end{cases}$$

and then let

$$g_{m,n}(x) = \begin{cases} \widehat{g}_{m,n}(x) & \text{if } \widehat{g}_{1,n} \in G(x) \\ \widehat{g}_{m,n}(x) + \epsilon_{m,n}(x) & \text{otherwise} \end{cases}$$

where $\epsilon_{m,n}(x) \in (-1/2^{m+2}, 1/2^{m+2})$ is chosen such that $\widehat{g}_{m,n}(x) + \epsilon_{m,n}(x) \in G(x)$. This sequence of sequences obviously satisfies (i), (ii), and (iii) above, and since the maximum distance between two consecutive elements of the sequence $g_{m,n}(x)$ is a_m where a_m is defined by the recursive formula $a_0 = 1$, $a_m = a_{m-1}/2 + 1/2^{m+2}$. This distance a_m is less than $1/2^{m-1}$ so it approaches 0 as $m \to \infty$. Therefore the sequences $g_{m,n}$ are dense in $[0, \infty)$.

Given $x \in X$ and a sequence $g_{m,n}$ as above, let $\mathbb{G}_k(x) = \bigcup_{m=1}^{k} \bigcup_{n=1}^{\infty} g_{m,n}(x)$ and $\mathbb{G}(x) = \bigcup_{k=1}^{\infty} \mathbb{G}_k(x)$. For any $f : X \to [0, 1]$ that is $\mathcal{F}$-measurable, we can define a function $E_x(f) : \mathbb{G}(x) \to [0,1]$ by

$$E_x(f)(d) = E(f|\mathcal{F}_d)(x) = \int f \, d\mu_{x,t}.$$

We need an analog of Doob's classical lemma concerning downcrossings. A function f has n *downcrossings* of the interval $[a,b]$ if there exist lists of numbers $a_1, ..., a_n$ and $b_1, ..., b_n$ in the domain of f with $a_i < b_i < a_{i+1} \, \forall i$ and $f(a_i) \geq b, f(b_i) < a \, \forall i$. Similarly we say f has n *upcrossings* of the interval $[a,b]$ if there exist lists of numbers $a_1, ..., a_n$ and $b_1, ..., b_n$ in the domain of f with $a_i < b_i < a_{i+1} \, \forall i$ and $f(a_i) \leq b, f(b_i) > a \, \forall i$. If a function has n downcrossings of $[a,b]$, then it must have (at least) $n-1$ upcrossings of that interval, and vice versa.

Proposition 5.1.3 *Given any* $[a,b] \subseteq [0,1]$,

$$\mu\left(\{x : E_x(f) \text{ has } m \text{ downcrossings of } [a,b]\}\right) \leq \left(\frac{1-b}{1-a}\right)^m.$$

Proof: For each $d \in \mathbb{G}_k(x)$ define

$$A_{k,d} = \{x \in X : E_x(f)(d) \geq b \text{ and } E_x(f)(\delta) < b \text{ for all } \delta < d \text{ in } \mathbb{G}_k(x)\}.$$

$A_{k,d}$ is the set of points for which the function $E_x(f)|_{\mathbb{G}_k(x)}$ first crosses above the interval $[a,b]$ at time d. In particular, it is a stopping time which is $\mathcal{F}_d$–measurable. Now let

$$\overline{A_{k,d}} = \{x \in A_{k,d} : E_x(f)(\delta) < a \text{ for some } \delta > d \text{ in } \mathbb{G}_k(x)\}.$$

The set $\overline{A_{k,d}}$ indicates those points in $A_{k,d}$ which eventually cross beneath the interval $[a,b]$. For any $x \in \overline{A_{k,d}}$, there must be a least δ in $\mathbb{G}_k(x)$ greater than d for which $E_x(f)(\delta) \leq a$. Call this number $\Delta_{k,d}(x)$. Now

$$\int_{A_{k,d}} f \, d\mu = \int_{A_{k,d} - \overline{A_{k,d}}} f \, d\mu + \int_{\overline{A_{k,d}}} f \, d\mu; \tag{5.1}$$

manipulating the left-hand side of (5.1) we get

$$\int_{A_{k,d}} f \, d\mu = \int_{A_{k,d}} E_x(f)(d) \, d\mu \geq b \cdot \mu(A_{k,d}).$$

As for the right-hand side of (5.1), we note that since $f \leq 1$,

$$\int_{A_{k,d} - \overline{A_{k,d}}} f \, d\mu \leq \mu(A_{k,d}) - \mu(\overline{A_{k,d}}),$$

and by the definition of $\overline{A_{k,d}}$,

$$\int_{\overline{A_{k,d}}} f \, d\mu = \int_{\overline{A_{k,d}}} E_x(f)(d) \, d\mu \leq a \cdot \mu(\overline{A_{k,d}}).$$

Putting this all together, equation (5.1) becomes the inequality

$$b \cdot \mu(A_{k,d}) \leq \mu(A_{k,d}) - (1-a)\mu(\overline{A_{k,d}})$$

which can be rewritten to obtain

$$\frac{\mu(\overline{A_{k,d}})}{\mu(A_{k,d})} \leq \frac{1-b}{1-a}.$$

In particular, this means that only a fraction $(1-b)/(1-a)$ of the points x for which $E_x(f)$ crosses above b at time d can have $E_x(f)$ cross below a after time d. Using this fact, we proceed inductively. Given a finite list $d_1, ..., d_m$ of elements of $\mathbb{G}_k(x)$, we define the sets $A_{k,(d_1,...,d_m)}$ and $\overline{A_{k,(d_1,...,d_m)}}$ and the function $\Delta_{k,(d_1,...,d_m)} : \overline{A_{k,(d_1,...,d_m)}} \to \mathbb{G}_k(x)$ inductively as follows:

$$A_{k,(d_1,...,d_m)} = \begin{cases} A_{k,d_1} & \text{if } m = 1 \\ \{x \in \overline{A_{k,(d_1,...,d_{m-1})}} : E_x(f)(d_m) \geq b \text{ and} & \\ \quad E_x(f)(\delta) < b \text{ for all } \delta \in (\Delta_{k,(d_1,...,d_{m-1})}, d)\} & \text{if } m > 1 \end{cases}$$

$$\overline{A_{k,(d_1,...,d_m)}} = \{x \in A_{k,(d_1,...,d_m)} : E_x(f)(\delta) \leq a \text{ for some } \delta > d_m \text{ in } \mathbb{G}_k(x)\}$$

$$\Delta_{k,(d_1,...,d_m)}(x) = \min\{\delta > d_m : E_x(f)(\delta) \leq a\}$$

The set $A_{k,(d_1,\ldots,d_m)}$ is the set of points x for which $E_x(f)$ first becomes at least b at time d_1, then drops below a (at time $\Delta_{k,d_1}(x)$), then next becomes at least b at time d_2, then drops below a, then next becomes at least b at time d_3, etc. Inside each set $A_{k,(d_1,\ldots,d_m)}$ we pick out those points for which $E_x(f)$ drops below a again after time d_m and call them $\overline{A_{k,(d_1,\ldots,d_m)}}$. For any point in this set, there is a first time where $E_x(f) \leq a$; this time is called $\Delta_{k,(d_1,\ldots,d_m)}(x)$.

Now by the same argument as given above, we see that

$$\mu(\overline{A_{k,(d_1,\ldots,d_m)}}) \leq \left(\frac{1-b}{1-a}\right) \mu(A_{k,(d_1,\ldots,d_m)}).$$

Hence

$$\begin{aligned} \mu\left(\bigcup_{d_m > d_{m-1}} \overline{A_{k,(d_1,\ldots,d_m)}}\right) &\leq \left(\frac{1-b}{1-a}\right) \mu\left(\bigcup_{d_m > d_{m-1}} A_{k,(d_1,\ldots,d_m)}\right) \\ &\leq \left(\frac{1-b}{1-a}\right) \mu(\overline{A_{k,(d_1,\ldots,d_{m-1})}}). \end{aligned}$$

Applying the argument again we see

$$\begin{aligned} \mu\left(\bigcup_{d_{m-1} > d_{m-2}} \bigcup_{d_m > d_{m-1}} \overline{A_{k,(d_1,\ldots,d_m)}}\right) &\leq \left(\frac{1-b}{1-a}\right) \mu\left(\bigcup_{d_{m-1} > d_{m-2}} \overline{A_{k,(d_1,\ldots,d_{m-1})}}\right) \\ &\leq \left(\frac{1-b}{1-a}\right)^2 \mu(A_{k,(d_1,\ldots,d_{m-2})}) \\ &\leq \left(\frac{1-b}{1-a}\right)^2 \mu(\overline{A_{k,(d_1,\ldots,d_{m-2})}}) \end{aligned}$$

and inductively

$$\mu\left(\bigcup_{(d_1,\ldots,d_m) \in \mathbb{D}_k} \overline{A_{k,(d_1,\ldots,d_m)}}\right) \leq \left(\frac{1-b}{1-a}\right)^m.$$

Let

$$S_{m,k} = \{x : E_x(f)|_{\mathbb{G}_k(x)} \text{ has } m \text{ downcrossings of } [a,b]\};$$

in fact this set is equal to

$$\bigcup_{(d_1,\ldots,d_m)\in\mathbb{G}_k(x)} \overline{A_{k,(d_1,\ldots,d_m)}}$$

so $\mu(S_{m,k}) \leq \left(\frac{1-b}{1-a}\right)^m$ for all k, m. Finally if we let

$$S_m = \{x : E_x(f) \text{ has } m \text{ downcrossings of } [a,b]\},$$

we observe $S_m = \bigcup_k S_{m,k}$ and $S_{m,k} \subseteq S_{m,k+1}$ so

$$\mu(S_m) = \lim_{k\to\infty} \mu(S_{m,k}) \leq \left(\frac{1-b}{1-a}\right)^m. \blacksquare$$

Corollary 5.1.4 *Let f be $\mathcal{F}_0-$measurable and $E_x(f)$ defined as above. For $\mu-$almost every $x \in X$, the function $E_x(f)(d)$ has left- and right-hand limits at every $t \in \mathbb{R}^+$, i.e. there exist numbers L^- and L^+ so that*

- *Given any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $d \in (t-\delta, t)\bigcap\mathbb{G}(x)$, $|E_x(f)(d) - L^-| < \epsilon$.*
- *Given any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $d \in (t, t+\delta)\bigcap\mathbb{G}(x)$, $|E_x(f)(d) - L^+| < \epsilon$.*

Proof: Let $l^- = \liminf_{d\to t^-} E_x(f)(d)$ and $l^+ = \limsup_{d\to t^+} E_x(f)(d)$. Define

$$X_0(f) = \{x \in X : \forall\alpha,\beta \in \mathbb{Q}, E_x(f) \text{ has only finitely many downcrossings of } [\alpha,\beta]\}.$$

Let S_m be as in the previous proposition; then

$$X_0(f) = X - \bigcup_{\alpha\in\mathbb{Q}} \bigcup_{\beta\in\mathbb{Q},\beta>\alpha} \bigcap_{m=1}^{\infty} S_m.$$

By the previous proposition $\mu\left(\bigcap_{m=1}^{\infty} S_m\right) = 0$ so $X_0(f)$ has full measure in X. If $l^+ \neq l^-$, we can choose rational numbers α and β with

$$l^- + \frac{L^- - l^-}{4} < \alpha < \beta < L^- - \frac{L^- - l^-}{4}.$$

The function $E_x(f)$ must have infinitely many downcrossings of the interval $[\alpha, \beta]$ so x cannot lie in $X_0(f)$. ■

This result allows us to extend (for $\mu-$almost every x) the function $E_x(f) : \mathbb{G}(x) \to [0,1]$ to the reals in two ways by taking limits as in the previous corollary. First, we define $E_x^+(f) : \mathbb{R} \to [0,1]$ by setting

$$E_x^+(f)(t) = \lim_{d \to t^+} E_x(f)(d).$$

$E_x^+(f)$ is a right-continuous function in t. Similarly, define $E_x^-(f) : \mathbb{R} \to [0,1]$ by setting

$$E_x^-(f)(t) = \lim_{d \to t^-} E_x(f)(d).$$

The function $E_x^+(f)$ is left-continuous in t. The only discontinuities of either function are jump discontinuities; and at a time t of continuity of $E_x(f)$ we have $E_x^+(f)(t) = E_x(f)(t) = E_x^-(f)(t)$. Consequently $E_x^+(f)$ is continuous at t if and only if $E_x^-(f)$ is continuous at t.

Corollary 5.1.5 *For any f which is $\mathcal{F}_0-$ measurable, $E_x^+(f)$ and $E_x^-(f)$ have only countably many discontinuities for any $x \in X_0(f)$.*

Proof: Suppose t is a point of discontinuity for $E_x^+(f)$. Then there exist rational numbers α, β in between $\lim_{d \to t^-} E_x(f)(d)$ and $\lim_{d \to t^+} E_x(f)(d)$ such that $E_x(f)$ has either an upcrossing or downcrossing of $[\alpha, \beta]$. However, for $x \in X_0(f)$, every such rational interval can only be crossed by $E_x(f)$ a finite number of times.

Since there are only countably many choices for α and β, $E_x(f)$ can only have countably many discontinuities. ■

Take a countable family of continuous functions $\mathbf{F} = \{f_i\}_{i=1}^{\infty}$ mapping X into $[0, 1]$ whose linear span is dense in $L^1(X, \mu)$. By Corollary 5.1.5, for each $f_i \in \mathbf{F}$ there is a set X_i of full measure in X such that $E_x(f_i)$ has only countably many discontinuities. Let $X_0 = \bigcap_i X_i$ (this is a set of full measure in X); then for each $x \in X_0$ define

$$C(x) = \{t : E_x(f_i) \text{ is continuous at } t \text{ for every } f_i\};$$

the complement of $C(x)$ is countable.

We now have two mappings from $X \times \mathbf{F} \times \mathbb{R}$ into $[0, 1]$ defined by $(x, f, t) \mapsto E_x^+(f)(t)$ and $(x, f, t) \mapsto E_x^-(f)(t)$. Fix x and t; the resulting mappings $f \mapsto E_x^+(f)(t)$ and $f \mapsto E_x^-(f)(t)$ are bounded functionals since $|f_i| \leq 1$ and are linear by the linearity of the conditional expectation operator. Hence by the Riesz representation theorem they extend to measures $\mu_{x,t}^+$ and $\mu_{x,t}^-$ on X.

Proposition 5.1.6 *For every $x \in X_0$ and every t, $\mu_{x,t}^+$ and $\mu_{x,t}^-$ are the left- and right-hand weak*$-$ limits of the $\mu_{x,t}$. More precisely, fix x and t and let d^* be a metric for the weak*$-$ topology on X. Then:*

- *For every $\epsilon > 0$, there is a $\delta > 0$ so that for every $s \in (t, t + \delta)$ for which $\mu_{x,s}$ exists, $d^*(\mu_{x,s}, \mu_{x,t}^+) < \epsilon$.*
- *For every $\epsilon > 0$, there is a $\delta > 0$ so that for every $s \in (t, t - \delta)$ for which $\mu_{x,s}$ exists, $d^*(\mu_{x,s}, \mu_{x,t}^-) < \epsilon$.*

Proof: Let $f : X \to [0, 1]$ be continuous. Then there exists a sequence f_i with $f_i \to f$ in $L^1(X)$ and each f_i in the linear span of the $\mathbf{F}$. Fix $\epsilon > 0$ and let $s > t$.

Then:

$$\begin{aligned}
\left|\int f\,d\mu^+_{x,t} - \int f\,d\mu_{x,s}\right| &\le \left|\int (f-f_i)\,d\mu^+_{x,t}\right| + \\
&\quad \left|\int f_i\,d\mu^+_{x,t} - \int f_i\,d\mu_{x,s}\right| + \\
&\quad \left|\int (f-f_i)\,d\mu_{x,s}\right| \\
&\le \int |f-f_i|\,d\mu^+_{x,t} + |E^+_x(f_i)(t) - E_x(f_i)(s)| \\
&\quad + \int |f-f_i|\,d\mu_{x,s}.
\end{aligned}$$

The outer expressions in this final expression are less than $\epsilon/3$ if i is chosen large enough, and the interior summand is less than $\epsilon/3$ if s is chosen close enough to t by Corollary 5.1.4. Thus $\int f\,d\mu_{x,s} \to \int f\,d\mu^+_{x,t}$ as $s \to t^+$ as desired. The proof that $\mu^-_{x,t}$ is the left-hand limit is similar. ■

Now for each $x \in X_0$ we have two *measure paths* of x: the measures $\mu^+_{x,t}$ which are weak* right-continuous, and the measures $\mu^-_{x,t}$ which are weak* left-continuous. We say that x has a *measurable orbit discontinuity at time* t_0 if $\mu^+_{x,t_0} \neq \mu^-_{x,t_0}$.

Proposition 5.1.7 *The following are equivalent:*

i. x has no measurable orbit discontinuity at time t_0.

ii. The measure path $\mu^+_{x,t}$ is weak$-$continuous at t_0.*

iii. The measure path $\mu^-_{x,t}$ is weak$-$continuous at t_0.*

Proof: Notice $\mu^+_{x,t_0} = \mu^-_{x,t_0}$ if and only if the weak*-limits of $\mu_{x,t}$ as t approaches t_0 from both the right and left are the same. The assumption that either $\mu^+_{x,t}$ or $\mu^-_{x,t}$ is continuous at t_0 is equivalent to the equality of the left- and right-hand weak*-limits. ■

Notice that for $t \in C(x)$, $\mu^+_{x,t} = \mu_{x,t} = \mu^-_{x,t}$ so therefore x cannot have a measurable orbit discontinuity at time t. Consequently we immediately see the following:

Proposition 5.1.8 *Every $x \in X_0$ has only countably many measurable orbit discontinuities.*

Proposition 5.1.9 *Suppose $x \in X$ has a measurable orbit discontinuity at time t_0. Then for any $z \in T_{-s}(x)$, z has a measurable orbit discontinuity at time $s+t_0$.*

Proof: Recall that by Lemma 5.1.1 we know that $\mu_{x,t} = T_s(\mu_{z,t+s})$ so long as the first measure exists. Consequently by taking weak*−limits as $t \to t_0$ from both the left and right we obtain

$$\mu^+_{x,t_0} = T_s(\mu^+_{z,s+t_0})$$

and

$$\mu^-_{x,t_0} = T_s(\mu^-_{z,s+t_0}).$$

By assumption $\mu^+_{x,t_0} \neq \mu^-_{x,t_0}$. Therefore $\mu^-_{z,s+t_0} \neq \mu^+_{z,s+t_0}$ so z has a measurable orbit discontinuity at time t_0 as desired. ■

As a consequence, we see that the set of points x which have only countably many measurable orbit discontinuities is an invariant set. From Proposition 5.1.8 we know that this set is of full measure in X (it contains X_0) so we have the following theorem:

Theorem 5.1.1 *Given a measure-preserving semiflow $(X, \mathcal{F}, \mu, T_t)$ on a Lebesgue space, there exists an invariant set X' of full measure in X such that for every $x \in X'$, x has at most countably many measurable orbit discontinuities.*

5.2 Orbit discontinuities: measure theory versus topology

We now examine the relationship between orbit discontinuities in the sense of Chapter 3 and the measurable orbit discontinuities constructed here. Let X be a Polish space and let ν be any probability measure on X such that all the Borel subsets of X are $\nu-$measurable; we define the *support* of ν, denoted $supp(\nu)$, to be the complement of all open sets in X which have $\nu-$measure zero. Notice that for any open $A \subseteq X$ disjoint from $T_{-t}T_t(x)$, $\mu_{x,t}(A) = E(A|\mathcal{G}_t)(x) = 0$. Consequently the support of $\mu_{x,t}$ is contained in the closure of $T_{-t}T_t(x)$.

Lemma 5.2.1 *$supp(\mu^+_{x,t}) \subseteq \overline{\bigcap_{s>t} T_{-s}T_s(x)}$.*

Proof: Recall first that the support of each $\mu_{x,t}$ is contained in $\overline{T_{-t}T_t(x)}$. Let t_n be a decreasing sequence of numbers converging to t from above for which μ_{x,t_n} exists for every n; consequently $\mu^+_{x,t}$ is the weak$^*-$limit of the μ_{x,t_n}.

Let A be an open set in X with $A \bigcap \overline{\bigcap_{s>t} T_{-s}T_s(x)} = \emptyset$. Let A' be any closed set contained in A; by the Urysohn lemma there exists a continuous function f on X such that $f = 0$ on $\overline{\bigcap_{s>t} T_{-s}T_s(x)}$ and $f = 1$ on A'. Notice that $\int f\, d\mu_{x,t_n} = 0$ for every n; therefore $\int f\, d\mu^+_{x,t} = 0$ since $\mu^+_{x,t}$ is the weak$^*-$limit of the μ_{x,t_n}. But also

$$\int f\, d\mu^+_{x,t} \geq \mu^+_{x,t}(A')$$

so $\mu^+_{x,t}(A') = 0$. But since X is a metric space, A can be written as the increasing union of closed sets contained in A. Therefore $\mu^+_{x,t}(A) = 0$. Consequently any open set A of positive $\mu^+_{x,t}-$measure must intersect $\overline{\bigcap_{s>t} T_{-s}T_s(x)}$ nontrivially.

Thus $supp(\mu^+_{x,t}) \subseteq X - A$. Since A was chosen arbitrarily to be disjoint from $\overline{\bigcap_{s>t} T_{-s}T_s(x)}$, $supp(\mu^+_{x,t}) \subseteq \overline{\bigcap_{s>t} T_{-s}T_s(x)}$ as desired. ■

We now give a correspondence between measurable orbit discontinuities and orbit discontinuities as described in Chapter 3. Of course, measurable orbit discontinuities are defined for actions on Lebesgue spaces and orbit discontinuities are defined for Borel actions on Polish spaces, so we must assume here that the system under consideration has both a Lebesgue space and Polish space structure.

Proposition 5.2.2 *Let X be a Polish space; let $\mathcal{B}(X)$ be the $\sigma-$algebra of Borel subsets of X and suppose that $\mathcal{F}$ be a $\sigma-$algebra containing $\mathcal{B}(X)$ such that $(X, \mathcal{F}, \mu)$ is a Lebesgue space. Suppose T_t is an action of $\mathbb{R}^+$ on X such that $(X, \mathcal{B}(X), \mu, T_t)$ is a Borel semiflow and $(X, \mathcal{F}, \mu, T_t)$ is a measure-preserving semiflow. If $x \in X$ has a measurable orbit discontinuity at time 0, then x has an orbit discontinuity at time 0.*

Proof: By hypothesis $\mu^+_{x,0} \neq \delta_x$. Consequently $\mu^+_{x,0}$ must be supported on a set strictly larger than $\{x\}$. Let $z \in \text{supp}(\mu^+_{x,0}) - \{x\}$. Then by the preceding lemma $z \in \overline{\bigcap_{t>0} T_{-t}T_t(x)}$ so there exist a sequence of points $z_n \in X$ with $z_n \to z$ and $T_{1/n}(z_n) = T_{1/n}(x)$. Denote by i the inclusion $i_T^{\mathbb{Q}^+} : X \to X_1^{\mathbb{Q}^+}$ and consider the sequence $i(z_n)$ in $X_1^{\mathbb{Q}^+}$; let ζ be the limit of any subsequence $i(z_{n_k})$ which converges; $\zeta(0) = \lim_{k\to\infty} i(z_{n_k})(0)$ since the mapping from $X_1^{\mathbb{Q}^+}$ to X taking f to $f(0)$ is continuous. Therefore $\zeta(0) = z$ so in particular there is no subsequence of the $i(z_n)$ converging to $i(x)$. Consequently there exists a $\delta > 0$ and an $N > 0$ such that for all $n > N$, $d(i(z_n), i(x)) > \delta$.

Take a refining, generating sequence $\mathcal{P}_k$ of partitions for X. Choose k large enough such that the maximum diameter of a $\mathcal{P}_k-$atom is less that $\delta/4$. For every rational $q > 0$, $\sigma_{-q}\sigma_q(i(x))$ intersects an atom of $\mathcal{P}_k$ which is d_M-distance at least δ from the atom of $\mathcal{P}_k$ containing x, namely an atom containing an $i(z_n)$.

Such an atom cannot contain x, so we see x must have an orbit discontinuity at time 0. ■

It is unknown if anything more general can be said in this context. If a point x has a measurable orbit discontinuity at time $t_0 > 0$, we can conclude using reasoning along the lines of the proof of Proposition 5.2.2 that for every $s > t_0$ there is at least one point y_s with $T_s(y_s) = T_s(x)$ but $T_t(y_s) \neq T_t(x)$ for every $t < t_0$. However, it could be the case that the sequence $i_T^{\mathbb{Q}^+}(y_s)$ is the limit of points z_n in $X_1^{\mathbb{Q}^+}$ with $\sigma_{t_n}(z_n) = \sigma_{t_n}(i(x))$ for $t_n < t_0$, in which case x would not have an orbit discontinuity at time t_0.

Consider also this (admittedly trivial) example which illustrates that topological orbit discontinuities can occur where there is no measurable orbit discontinuity. Let Ω_L be the set of functions f from $[0, \infty)$ into $\{0, 1\}$ for which there exists a number $c_f \in [0, 1)$ such that $f(t)$ is constant on every interval of the form $[0, \infty) \bigcap (c_f + i, c_f + i + 1]$ for $i \in \mathbb{Z}$. (This is the same as the space Ξ_L constructed in Chapter 3 without the "marker".) We put a metric on Ω_L by

$$d(f, f') = \int_0^\infty \frac{|f(t) - f'(t)|}{e^t} dt;$$

this makes Ω_L a Polish space. The semiflow σ_t is defined on Ω_L by the shift $\sigma_t(f)(s) = f(t + s)$; this is a Borel action. Let δ_1 be the Dirac measure assigning mass 1 the σ_t−fixed point $g(x) \equiv 1$ and 0 to the rest of the space; our Borel measure-preserving semiflow is $(\Xi_R, \delta_1, \sigma_t)$.

The (topological) orbit discontinuities of this action do not depend on the measure; every $f \in \Xi_L$ has infinitely many orbit discontinuities at the times $c_f, c_f{+}1, c_f{+}2, \ldots.$ But the function $g \equiv 1$ has no measurable orbit discontinuities; for every t we have $\mu^+_{g,t} = \mu^-_{g,t} = \delta_1$. (The set of full measure on which the measure paths exist can be taken to be the fixed point g.)

Chapter 6

Further questions

Our original motivation for studying IDI was as a tool to obtain a representation of all Borel semiflows as shift maps on path spaces. However the representation we obtained was not a shift map on a path space but an improvement of the Ambrose-Kakutani model. In particular the models F^* and F_* have a well-defined measure-space structure under which they are measurably conjugate to the original semiflow except at its orbit discontinuities. Also, F^* and F_* can be endowed with a Polish topology as subsets of $K(F_1) \times [0, B]$. One would like to know if the Borel sets under this topology are $\mathcal{G}^*-$ (or $\mathcal{G}_*$)-measurable. If not, can one put a Polish topology on F^* and F_* such that the resulting action is Borel?

It also remains to show how one can adapt this work to build a path space representation. We have seen that the mapping ψ taking x to its path $\psi(x)(t) = \int_0^t T_s(x)ds$ fails to be injective only when $x \in IDI(T_t)$. Now take an $x \in X$ and let $I(x) = \{y \in X : T_t(y) = T_t(x) \, \forall \, t > 0\}$. Now there is an injection $\gamma : I(x) \to [0, 1]$ since X is Polish. We can then redefine ψ on $I(x)$ as follows:

$$\psi(y)(t) = \begin{cases} 0 & \text{if } t = 0 \\ \int_0^t T_s(y)\, ds + \gamma(y) & \text{if } t > 0 \end{cases}$$

Thus ψ maps $I(x)$ injectively into some set of functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ which are continuous except at $t = 0$ (but have right-hand limit as $t \to 0^+$). Of course this map is equivariant with the flow: $\sigma_t(\psi(y)) = \psi(T_t(y))$. This modification "fixes" the lack of injectivity of ψ at x.

More generally, it might be possible to assign for every $x \in X$ a function $\psi(x) : \mathbb{R}^+ \to \mathbb{R}^+$ which passes through the origin, is increasing, left-continuous, and has only countably many discontinuities (at the IDI set of x) by doing constructions across X motivated by the one described in the previous paragraph. The idea is to take the path $\psi(x)$ associated to x and add "gaps" in it at each time in the IDI set of x, adding a different-sized gap at time t for the different points identified with x at time t. The primary obstacle is that one has to be careful not to add an infinite amount of "gap" to the functions $\psi(x)$ in order to guarantee that the resulting paths are in fact well-defined functions, while at the same time making sure that the amount of gap added is consistent with the action of T_t (the action of σ_t must be equivariant with the original semiflow). In fact the amount of gap that is to be added at an IDI should be a value depending not only not only of the point which is IDI, but also on "which" IDI is occurring (i.e. the particular sets $J(c_1, c_2)$ to which the point belongs).

Many other unanswered questions about orbit discontinuities remain. For example, what kinds of behavior can be observed in a Borel semiflow at one of its orbit discontinuities? If x has an orbit discontinuity at time t, does $T_s(x)$ necessarily have an orbit discontinuity at time $t-s$? We saw in Chapter 3 that the IDI set of a Borel semiflow is invariant under Borel time-changes. Are the orbit discontinuities invariant under time-changes (i.e. are the orbit discontinuities of a time-change located at the time-change of the orbit discontinuities of the original semiflow)? Is the IDI set and orbit discontinuity set of a semiflow invariant under Kakutani shift equivalence (as defined in [4])?

Semiflows form a rich class of dynamical systems that have not been studied in great detail. The orbit of a point under an action of the continuous semigroup $\mathbb{R}^+$ can be very complicated in general. The methods here show that one can view these actions by studying the induced action of a countable semigroup $S \subseteq \mathbb{R}^+$ and taking limits to "fill in" the rest of the action. If the behavior of the semiflow is "continuous" then this approach is sufficient to control the behavior of the semiflow–our work shows that Borel semiflows are "continuous" in this sense most of the time. Hopefully this technique can be adapted to solve other problems related to non-invertible continuous-time dynamical systems.

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