A Variable-Step Double-Integration Multi-Step Integrator

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Overview

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- Motivation
- Derivation
- Preliminary Results
- Future Work

Background

- Naval Network and Space Operations Command is tracking over 12,000 objects in orbit.
- These objects may collide with the ISS or other US assets.
- Analytic methods no longer meet accuracy requirements, so numerical methods are used.
- Numerical methods require much more computation time.
- Planned sensor upgrades may increase the number of tracked objects to over 100,000.
- Faster numerical integrators are needed.

Integration Terminology

Integrators can be classified by several categories

- Single or Multi-Step - How many points are used to integrate forward, multi-step integrators need backpoints
- Fixed or Variable Step
- Single or Double Integration - whether they handle first or second order differential equations
- Summed or Non-Summed - Whether the integration is point to point, or from epoch (multi-step integrators only)

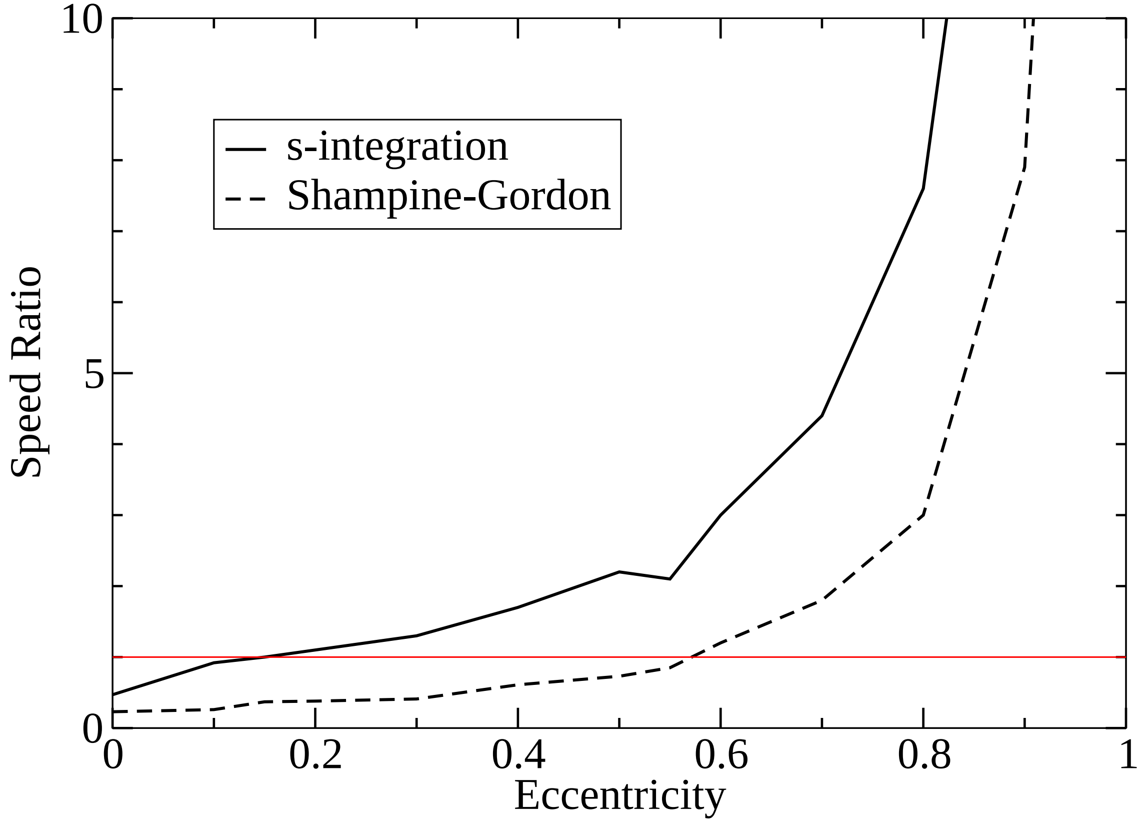
Integration Methods

Method	Single / Multi	Fixed / Variable	Non-Summed / Summed	Single / Double
Runge-Kutta	Single	Fixed	NA	Single
Runge-Kutta-Fehlberg	Single	Variable	NA	Single
Adams (non-summed)	Multi	Fixed	Non-Summed	Single
Summed Adams	Multi	Fixed	Summed	Single
Shampine-Gordon	Multi	Variable	Non-Summed	Single
Stormer-Cowell	Multi	Fixed	Non-Summed	Double
Gauss-Jackson	Multi	Fixed	Summed	Double
Proposed	Multi	Variable	Summed	Double

Variable-Step Integration

- Fixed-step integrators take more steps than needed at apogee.
- Variable-step integrators change the step size to control local error.
- An alternative to variable-step integration is to change the independent variable (s-integration)
  - Still a fixed-step method - no local error control.
  - Must integrate to find time - leads to in-track error.
- Test benefit of variable step by timing integrations of equivalent accuracy.

Speed Ratios at 400 km Perigee



Single / Double Integration

- Compare Adams and Störmer-Cowell
- Both use 30 sec step, 2 evaluations per step.
- Test by defining an error ratio:

$$\rho_r = \frac{1}{r_A N_{\text{orbits}}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\Delta r_i)^2}$$

where $\Delta r = |r_{\text{computed}} - r_{\text{ref}}|$.

- Comparisons are over 3 days.
- Reference is analytic solution (two-body).

Double vs. Single (Two Body)

Height (km)	Eccentricity	Störmer-Cowell	Adams
300	0.00	2.47×10^{-13}	2.66×10^{-12}
300	0.25	3.05×10^{-12}	7.90×10^{-12}
300	0.75	4.01×10^{-11}	2.66×10^{-10}
500	0.00	3.49×10^{-13}	7.90×10^{-13}
500	0.25	2.87×10^{-12}	9.21×10^{-12}
500	0.75	2.21×10^{-11}	1.69×10^{-10}
1000	0.00	9.63×10^{-14}	4.78×10^{-12}
1000	0.25	3.53×10^{-13}	9.58×10^{-12}
1000	0.75	9.70×10^{-12}	7.03×10^{-11}

Double vs. Single

- Similar results with perturbations.
- Without second evaluation, Adams is unstable.
- Störmer-Cowell is stable with one evaluation per step.
- Variable-step double-integration only needs one evaluation per step.
- Significant advantage over Shampine-Gordon.

Shampine-Gordon

- Solve the differential equation

$$y' = f(x, y)$$

  by approximating $f(x, y)$ with a polynomial $P(x)$ interpolating through the backpoints.

- $P(x)$ is written in **Divided Difference** form so the backpoints do not have to be equally spaced.

Divided Differences

n	x_n	$f[x_n]$	$f[x_n, x_{n-1}]$	$f[x_n, x_{n-1}, x_{n-2}]$
1	1	1		
2	3	5	2	
3	4	9	4	$2/3$

$$P(x) = 9 + (x - 4)(4) + (x - 4)(x - 3)(2/3)$$

Shampine-Gordon Predictor

- Integrating the polynomial:

$$p_{n+1} = y_n + \int_{x_n}^{x_{n+1}} P(x)\, dx$$

gives a predictor formula:

$$p_{n+1} = y_n + h_{n+1} \sum_{i=1}^{k} g_{i,1} \phi_i^*(n)$$

- The $g_{i,1}$ are integration coefficients.
- Coefficients must be calculated at each step.
- The $\phi_i^*(n)$ are modified divided differences.

Shampine-Gordon

- After the predictor an evaluation is performed.
- The corrector is derived using a polynomial that integrates through the backpoints plus the predicted value.
- A second evaluation follows the corrector.
- Step size is modified based on local error estimate:

$$r = \left(\frac{\epsilon}{\text{Error}} \right)^{\frac{1}{k+1}}$$

- r is bounded between 0.5 and 2, and not allowed to be between 0.9 and 2.

Double Integration - Predictor

- Solve the second order ODE

$$y'' = f(x, y, y')$$

- Replace f with $P(x)$ and integrate both sides twice:

$$p_{n+1} = y_n + h_{n+1}y'_n + \int_{x_n}^{x_{n+1}} \int_{x_n}^{\tilde{x}} P(x)\, dx\, d\tilde{x}$$

- To get rid of y' term, integrate backwards too:

$$p_{n+1} = \left(1 + \frac{h_{n+1}}{h_n}\right) y_n - \frac{h_{n+1}}{h_n} y_{n-1} + \int_{x_n}^{x_{n+1}} \int_{x_n}^{\tilde{x}} P(x)\, dx\, d\tilde{x} + \frac{h_{n+1}}{h_n} \int_{x_n}^{x_{n-1}} \int_{x_n}^{\tilde{x}} P(x)\, dx\, d\tilde{x}$$

Double Integration

- The coefficients $g_{i,2}$ from Shampine-Gordon can be used to find

$$\int_{x_n}^{x_{n+1}} \int_{x_n}^{\tilde{x}} P(x)\, dx\, d\tilde{x}$$

- New set of coefficents $g'_{i,2}$ needed for second integral.

- Predictor formula:

$$p_{n+1} = \left(1 + \frac{h_{n+1}}{h_n}\right) y_n - \frac{h_{n+1}}{h_n} y_{n-1} + h_{n+1}^2 \sum_{i=1}^{k} \left(g_{i,2} + \frac{h_{n+1}}{h_n} g'_{i,2}\right) \phi_i^*(n)$$

Double Integration - Implementation

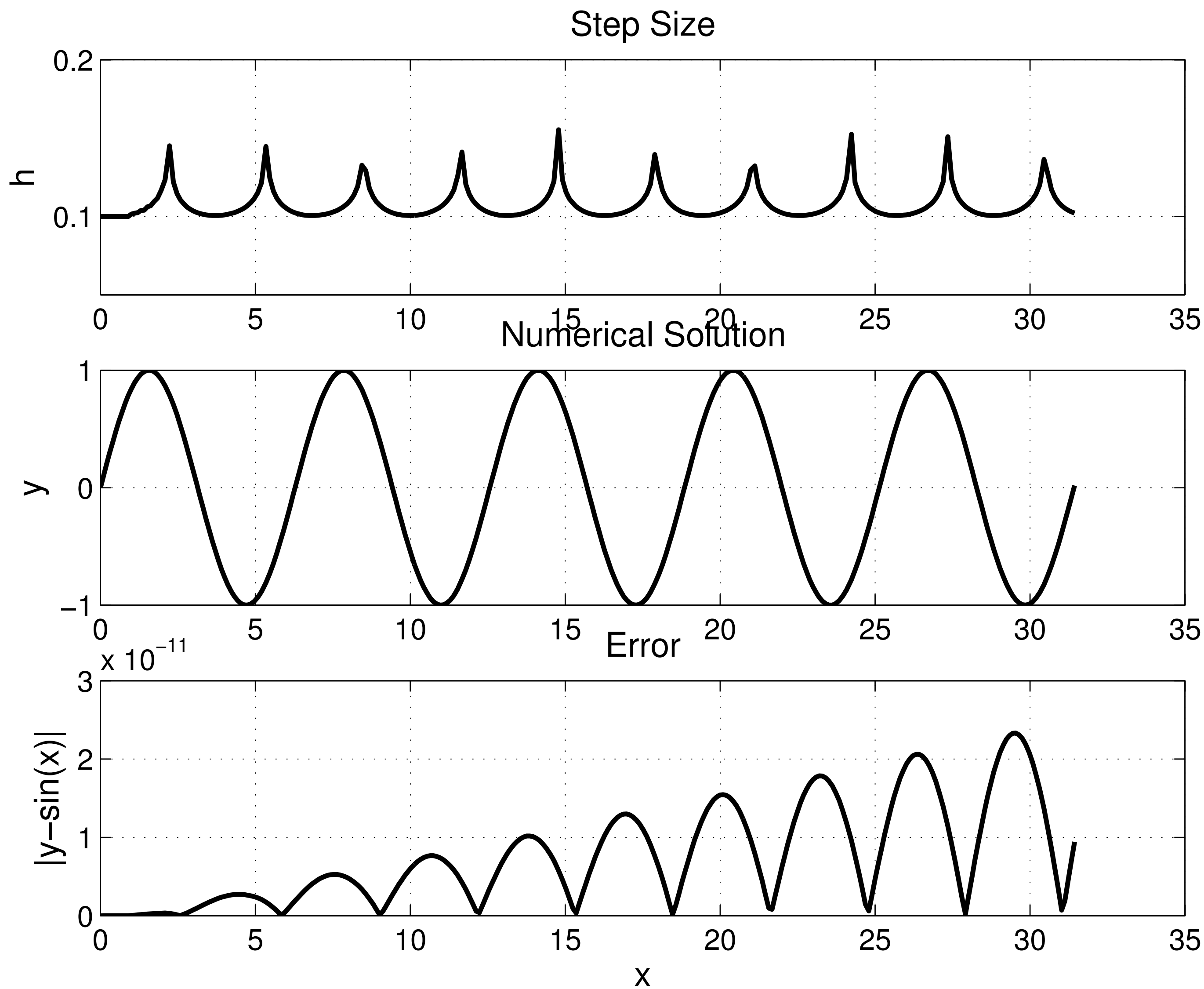
- Predictor is followed by an evaluation, and then the corrector.
- A second evaluation is **Not** performed.
- The factor r to change the step is calculated:

$$r = \left(\frac{0.5\epsilon}{\text{Error}} \right)^{\frac{1}{k+2}}$$

and bounded between 0.5 and 2.

Results

- Two implementations are tested, Matlab and Fortran.
- Implementations use 9 backpoints.
- Runge-Kutta used to start the integrator.
- Matlab test on $y'' = -y$
  Solution: $y = \sin(x)$
- Fortran test on two-body orbit propagation.
  - Implements single integration for velocity, double integration for position.


Step Size
h
0.2
0.1
Numerical Solution
y
1
0
−1
Error
x 10^−11
|y−sin(x)|
3
2
1
0
0 5 10 15 20 25 30 35
x

Fortran Results

Height (km)	Eccentricity	Error Ratio
300	0.00	6.41×10^{-10}
300	0.25	7.49×10^{-11}
300	0.75	1.98×10^{-11}
500	0.00	6.23×10^{-10}
500	0.25	5.99×10^{-11}
500	0.75	2.04×10^{-11}
1000	0.00	5.81×10^{-10}
1000	0.25	5.97×10^{-11}
1000	0.75	2.31×10^{-11}

Future Work

- Accuracy / Speed tests against other integrators.
- Start-up with variable-order implementation.
- Interpolation to get requested values.
- Choosing the best factor r from the two available: single and double-integration step-size control algorithms.