A Variable-Step Double-Integration Multi-Step Integrator

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Overview

- Background
- Motivation
- Derivation
- Preliminary Results
- Future Work
Naval Network and Space Operations Command is tracking over 12,000 objects in orbit.

These objects may collide with the ISS or other US assets.

Analytic methods no longer meet accuracy requirements, so numerical methods are used.

Numerical methods require much more computation time.

Planned sensor upgrades may increase the number of tracked objects to over 100,000.

Faster numerical integrators are needed.
Integration Terminology

Integrators can be classified by several categories

- Single or Multi-Step - How many points are used to integrate forward, multi-step integrators need backpoints
- Fixed or Variable Step
- Single or Double Integration - whether they handle first or second order differential equations
- Summed or Non-Summed - Whether the integration is point to point, or from epoch (multi-step integrators only)
## Integration Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Single / Multi</th>
<th>Fixed / Variable</th>
<th>Non-Summed / Summed</th>
<th>Single / Double</th>
</tr>
</thead>
<tbody>
<tr>
<td>Runge-Kutta</td>
<td>Single</td>
<td>Fixed</td>
<td>NA</td>
<td>Single</td>
</tr>
<tr>
<td>Runge-Kutta-Fehlberg</td>
<td>Single</td>
<td>Variable</td>
<td>NA</td>
<td>Single</td>
</tr>
<tr>
<td>Adams (non-summed)</td>
<td>Multi</td>
<td>Fixed</td>
<td>Non-Summed</td>
<td>Single</td>
</tr>
<tr>
<td>Summed Adams</td>
<td>Multi</td>
<td>Fixed</td>
<td>Summed</td>
<td>Single</td>
</tr>
<tr>
<td>Shampine-Gordon</td>
<td>Multi</td>
<td>Variable</td>
<td>Non-Summed</td>
<td>Single</td>
</tr>
<tr>
<td>Stormer-Cowell</td>
<td>Multi</td>
<td>Fixed</td>
<td>Non-Summed</td>
<td>Double</td>
</tr>
<tr>
<td>Gauss-Jackson</td>
<td>Multi</td>
<td>Fixed</td>
<td>Summed</td>
<td>Double</td>
</tr>
<tr>
<td>Proposed</td>
<td>Multi</td>
<td>Variable</td>
<td>Summed</td>
<td>Double</td>
</tr>
</tbody>
</table>
Variable-Step Integration

- Fixed-step integrators take more steps than needed at apogee.
- Variable-step integrators change the step size to control local error.
- An alternative to variable-step integration is to change the independent variable ($s$-integration)
  - Still a fixed-step method - no local error control.
  - Must integrate to find time - leads to in-track error.
- Test benefit of variable step by timing integrations of equivalent accuracy.
Speed Ratios at 400 km Perigee

![Graph showing speed ratios at 400 km perigee with eccentricity on the x-axis and speed ratio on the y-axis. The graph includes two lines: one for s-integration and another for Shampine-Gordon.](image)
Single / Double Integration

- Compare Adams and Störmer-Cowell
- Both use 30 sec step, 2 evaluations per step.
- Test by defining an error ratio:

\[ \rho_r = \frac{1}{r_A N_{\text{orbits}}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\Delta r_i)^2} \]

where \( \Delta r = |r_{\text{computed}} - r_{\text{ref}}| \).

- Comparisons are over 3 days.
- Reference is analytic solution (two-body).
## Double vs. Single (Two Body)

<table>
<thead>
<tr>
<th>Height (km)</th>
<th>Eccentricity</th>
<th>Störmer-Cowell</th>
<th>Adams</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>0.00</td>
<td>$2.47 \times 10^{-13}$</td>
<td>$2.66 \times 10^{-12}$</td>
</tr>
<tr>
<td>300</td>
<td>0.25</td>
<td>$3.05 \times 10^{-12}$</td>
<td>$7.90 \times 10^{-12}$</td>
</tr>
<tr>
<td>300</td>
<td>0.75</td>
<td>$4.01 \times 10^{-11}$</td>
<td>$2.66 \times 10^{-10}$</td>
</tr>
<tr>
<td>500</td>
<td>0.00</td>
<td>$3.49 \times 10^{-13}$</td>
<td>$7.90 \times 10^{-13}$</td>
</tr>
<tr>
<td>500</td>
<td>0.25</td>
<td>$2.87 \times 10^{-12}$</td>
<td>$9.21 \times 10^{-12}$</td>
</tr>
<tr>
<td>500</td>
<td>0.75</td>
<td>$2.21 \times 10^{-11}$</td>
<td>$1.69 \times 10^{-10}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.00</td>
<td>$9.63 \times 10^{-14}$</td>
<td>$4.78 \times 10^{-12}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.25</td>
<td>$3.53 \times 10^{-13}$</td>
<td>$9.58 \times 10^{-12}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.75</td>
<td>$9.70 \times 10^{-12}$</td>
<td>$7.03 \times 10^{-11}$</td>
</tr>
</tbody>
</table>
Double vs. Single

- Similar results with perturbations.
- Without second evaluation, Adams is unstable.
- Störmer-Cowell is stable with one evaluation per step.
- Variable-step double-integration only needs one evaluation per step.
- Significant advantage over Shampine-Gordon.
Shampine-Gordon

- Solve the differential equation
  \[ y' = f(x, y) \]
  by approximating \( f(x, y) \) with a polynomial \( P(x) \) interpolating through the backpoints.
- \( P(x) \) is written in \textbf{Divided Difference} form so the backpoints do not have to be equally spaced.
### Divided Differences

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$f[x_n]$</th>
<th>$f[x_n, x_{n-1}]$</th>
<th>$f[x_n, x_{n-1}, x_{n-2}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>9</td>
<td>4</td>
<td>$2/3$</td>
</tr>
</tbody>
</table>

$$P(x) = 9 + (x - 4)(4) + (x - 4)(x - 3)(2/3)$$
Shampine-Gordon Predictor

- Integrating the polynomial:

\[ p_{n+1} = y_n + \int_{x_n}^{x_{n+1}} P(x) \, dx \]

gives a predictor formula:

\[ p_{n+1} = y_n + h_{n+1} \sum_{i=1}^{k} g_{i,1} \phi_i^*(n) \]

- The \( g_{i,1} \) are integration coefficients.
- Coefficients must be calculated at each step.
- The \( \phi_i^*(n) \) are modified divided differences.
Shampine-Gordon

- After the predictor an evaluation is performed.
- The corrector is derived using a polynomial that integrates through the backpoints plus the predicted value.
- A second evaluation follows the corrector.
- Step size is modified based on local error estimate:

\[ r = \left( \frac{\epsilon}{\text{Error}} \right)^{\frac{1}{k+1}} \]

- \( r \) is bounded between 0.5 and 2, and not allowed to be between 0.9 and 2.
Double Integration - Predictor

- Solve the second order ODE
  \[ y'' = f(x, y, y') \]

- Replace \( f \) with \( P(x) \) and integrate both sides twice:
  \[ p_{n+1} = y_n + h_{n+1} y'_n + \int_{x_n}^{x_{n+1}} \int_{x_n}^{\tilde{x}} P(x) \, dx \, d\tilde{x} \]

- To get rid of \( y' \) term, integrate backwards too:
  \[ p_{n+1} = \left( 1 + \frac{h_{n+1}}{h_n} \right) y_n - \frac{h_{n+1}}{h_n} y_{n-1} + \int_{x_n}^{x_{n+1}} \int_{x_n}^{\tilde{x}} P(x) \, dx \, d\tilde{x} + \frac{h_{n+1}}{h_n} \int_{x_n}^{x_{n-1}} \int_{x_n}^{\tilde{x}} P(x) \, dx \, d\tilde{x} \]
Double Integration

- The coefficients $g_{i,2}$ from Shampine-Gordon can be used to find
  \[ \int_{x_n}^{x_{n+1}} \int_{x_n}^{\tilde{x}} P(x) \, dx \, d\tilde{x} \]
- New set of coefficients $g'_{i,2}$ needed for second integral.
- Predictor formula:
  \[
p_{n+1} = \left(1 + \frac{h_{n+1}}{h_n}\right) y_n - \frac{h_{n+1}}{h_n} y_{n-1} \\
  + h_{n+1}^2 \sum_{i=1}^{k} \left( g_{i,2} + \frac{h_{n+1}}{h_n} g'_{i,2} \right) \phi_i^*(n)
  \]
Double Integration - Implementation

• Predictor is followed by an evaluation, and then the corrector.

• A second evaluation is **Not** performed.

• The factor $r$ to change the step is calculated:

$$ r = \left( \frac{0.5\varepsilon}{\text{Error}} \right)^{\frac{1}{k+2}} $$

and bounded between 0.5 and 2.
Results

- Two implementations are tested, Matlab and Fortran.
- Implementations use 9 backpoints.
- Runge-Kutta used to start the integrator.
- Matlab test on $y'' = -y$
  Solution: $y = \sin(x)$
- Fortran test on two-body orbit propagation.
  - Implements single integration for velocity, double integration for position.
## Fortran Results

<table>
<thead>
<tr>
<th>Height (km)</th>
<th>Eccentricity</th>
<th>Error Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>0.00</td>
<td>$6.41 \times 10^{-10}$</td>
</tr>
<tr>
<td>300</td>
<td>0.25</td>
<td>$7.49 \times 10^{-11}$</td>
</tr>
<tr>
<td>300</td>
<td>0.75</td>
<td>$1.98 \times 10^{-11}$</td>
</tr>
<tr>
<td>500</td>
<td>0.00</td>
<td>$6.23 \times 10^{-10}$</td>
</tr>
<tr>
<td>500</td>
<td>0.25</td>
<td>$5.99 \times 10^{-11}$</td>
</tr>
<tr>
<td>500</td>
<td>0.75</td>
<td>$2.04 \times 10^{-11}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.00</td>
<td>$5.81 \times 10^{-10}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.25</td>
<td>$5.97 \times 10^{-11}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.75</td>
<td>$2.31 \times 10^{-11}$</td>
</tr>
</tbody>
</table>
Future Work

- Accuracy / Speed tests against other integrators.
- Start-up with variable-order implementation.
- Interpolation to get requested values.
- Choosing the best factor $r$ from the two available: single and double-integration step-size control algorithms.