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Abstract

In this paper the fixed step Gauss-Jackson method is compared to two variable step integrators. The first is the variable step, variable order Shampine-Gordon method. The second is s-integration, which may be considered an analytical step regulation. Speed tests are performed for orbit propagation with the integrators set to give equivalent accuracy. The integrators are also tested for orbit determination, to determine the speed benefit of the variable step methods. The tests give an indication of the types of orbits where variable step methods are more efficient than fixed step methods.

INTRODUCTION

Performing orbit propagation and orbit determination with a numerical integrator gives a significant accuracy improvement over analytic techniques. Numerical methods take a great deal more computation time than analytic methods, but are becoming increasingly popular as computer speeds increase. Many different numerical integration methods exist, and the choice of integration method is made either to give the quickest calculation for a given accuracy requirement, or the most accurate calculation for an alloted computation time. The best method may be dependent on the type of orbit, so applications which process a wide variety of orbit types need to have the flexibility to use different integrators for different orbits.

In this paper, we analyze the speed and accuracy effect of variable step integration. The fixed step Gauss-Jackson method is compared to two variable step methods. The first is the variable step, variable order Shampine-Gordon method(Ref. 1). The Shampine-Gordon integrator uses local error control to adjust the step size and the order of the method. The second is an implementation of a generalized Sundman transformation, or s-integration (Ref. 2), which may be considered an analytical step regulation. In s-integration the independent variable is changed from time, t, to an angle s, which spreads the integration steps more evenly about an elliptical orbit.

Speed and accuracy tests are performed for both orbit propagation and orbit determination. Comparing these three integration methods for orbit determination and propagation for a wide variety of orbits indicates when variable step methods are preferable to fixed step methods.

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TEST CASES

In this paper the integrators are tested for orbit propagation on test case orbits with varying eccentricity and perigee heights. The test cases all have an inclination of 40° and a ballistic coefficient of $0.01\,{}^{\mathrm{m}^2}\!/_{\mathrm{kg}}$. The epoch of the test cases is 1999-10-01 00:00:00 UTC. At epoch the test cases are at perigee, which is on the X axis. When perturbations are considered, they are lunar and solar forces, the Jacchia 70 drag model, and the 36×36 WGS-84 geopotential. In the orbit determination tests later in this paper, real satellite data is used. The orbit propagation and determination tests are performed using the SPeCIAL-K software suite (Ref. 3), developed by the Naval Research Laboratory, which is used operationally by Naval Network and Space Operations Command (NNSOC, formerly Naval Space Command). The version of SPeCIAL-K used in this paper has been modified to use additional numerical integrators.

EVALUATIONS

When comparing the computation time of integrators, the main factor determining the execution time is the number of force model evaluations performed. For instance, a 400km circular orbit integrated with an eighth order Gauss-Jackson with a 30 second step size for 30 days with perturbations takes 35.0 seconds of user time on an SGI Origin 200. Without perturbations, the computation time is 3.33 seconds. So 90.5% of the run-time is spent evaluating the perturbations. For one integrator to be advantageous over another, it must have fewer evaluations per orbit, either by taking fewer steps per orbit, or by having fewer evaluations per step.

GAUSS-JACKSON INTEGRATION

The eighth order Gauss-Jackson (Ref. 4) is an integration method commonly used in space surveillance. Gauss-Jackson is a fixed-step predictor-corrector multistep method, with no control over the local error. In Ref. 4, a procedure for Gauss-Jackson integration is given in which the integrator predicts the state (P), evaluates at the predicted state (E), and corrects the state (C). The integrator then tests for convergence of the predicted and corrected state, and performs more evaluate and correct cycles (EC...), if the convergence criteria is not met. This is a PECEC...cycle, in which there are normally two or more evaluations per step.

Herrick (Ref. 5) claims that the corrector is not usually required in Gauss-Jackson integration. This saves computation time, because only one evaluation is required (PE) per step. A slight improvement over the PE method can be made by applying the corrector once (PEC), which does not significantly affect the computation time. Table 1 shows the maximum position difference over three days between using Gauss-Jackson with the PEC method and the PECEC...method for various orbits. Both integrations use a 30 second step size, and perturbations are included. Though the PECEC...method takes at least twice as long as the PEC method, it gives an improvement of less than a meter, and less than a centimeter for most cases.

Table 2 shows the maximum position difference over 3 days between Gauss-Jackson with a 30 second step size, and Gauss-Jackson with a 15 second step size. Both use the PEC method. Comparing Table 1 to Table 2, we see that a much larger improvement is given by cutting the step size than by performing additional evaluations, though both double the run time. This shows that the additional evaluations are not necessary; it is better to decrease the step size. The results presented later in this paper use the PEC method for Gauss-Jackson.

GENERALIZED SUNDMAN TRANSFORMATION

For elliptical orbits, fixed step integrators are less efficient than variable step methods, because many steps have to be taken at apogee in order to get the desired accuracy at perigee. One way

Table 1: Effect of additional evaluations in Gauss-Jackson.

perigee height (km)	e	position difference (mm)
300	0.0	8.8
300	0.25	26
300	0.75	160
500	0.0	1.8
500	0.25	5.9
500	0.75	38
1000	0.0	0.1
1000	0.25	0.3
1000	0.75	4.6

Table 2: Effect of step-size halving in Gauss-Jackson.

perigee height (km)	e	position difference (mm)
300	0.0	1800
300	0.25	1200
300	0.75	1400
500	0.0	110
500	0.25	63
500	0.75	94
1000	0.0	0.3
1000	0.25	2.2
1000	0.75	45

of increasing the efficiency of numerically integrating elliptical orbits is to change the independent variable using a generalized Sundman transformation (Ref. 2). This is known as s-integration, while conventional integration with time as the independent variable is known as t-integration.

Sundman (Ref. 6) and Levi-Civita (Ref. 7), in attempting to solve the restricted problem of three bodies, introduced the transformation of the independent variable

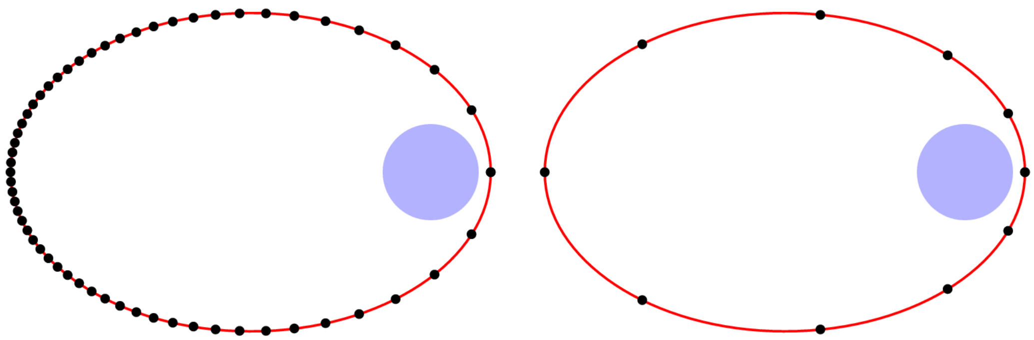
$$dt = cr\,ds, \tag{1}$$

with c constant for the two-body orbit, because this transformation regularizes, and in fact linearizes, the equations of motion. Later investigators raised r to different powers in the transformation,

$$dt = cr^n\,ds, \tag{2}$$

known as the generalized Sundman transformation. If $n = 1$ and $c = \sqrt{a/\mu}$, s is the eccentric anomaly. This was Sundman's original transformation. If $n = 2$ and $c = [\mu a(1 - e^2)]^{-1/2}$, s is the true anomaly. Merson (Ref. 8) gave analysis showing that $n = 3/2$ equally distributes the integration error around an orbit, even with high eccentricity. Nacozy (Ref. 9), expressed s in terms of an elliptic integral of the true anomaly for $n = 3/2$ and $c = 1/\sqrt{\mu}$, and dubbed this angle the *intermediate anomaly*. Our implementation of s-integration is in the same form as Merson and Nacozy, with $n = 3/2$ and $c = 1/\sqrt{\mu}$.

Integration steps equally spaced by s are spread more evenly about an elliptical orbit than steps equally spaced by time, so fewer steps are needed about the orbit to achieve the same integration accuracy as t-integration at perigee. As an example, Figure 1 shows the distribution of integration points for a 0.75 eccentricity orbit using both t- and s-integration. Both have the same step size at perigee, of $\Delta\nu \approx 1.0$.



(a) t-integration (n=0) with 58 steps. (b) s-integration (n=3/2) with 10 steps.

Figure 1: Distribution of integration points for t- and s-integration, $e = 0.75$.

One disadvantage of s-integration is that a seventh differential equation (2) must be solved to find time. This makes time subject to integration error, which can significantly contribute to in-track error. In Ref. 2, we observed that s-integration seemed to be affected by round-off error. An improvement was made to s-integration to reduce the round-off error in the time equation. In the previous paper, we were storing time in units of minutes since 1950. We have changed the code to store time as minutes from epoch, so that the round-off error in the integration of time is at a less significant digit. The s-integration results given in this paper are improved over Ref. 2 due to this code modification.

Another disadvantage of s-integration is that there is still no control over the local error. Though s-integration provides analytic step regulation by varying the steps through the orbit as shown in Figure 1, it is still a fixed step method. The steps are equally spaced by s, instead of equally spaced by time, t. This means that there is still a user defined step size, Δs, measured in radians. Because s is not a physical characteristic of the orbit, we choose a step size in time at perigee, and convert that to Δs,

$$\Delta s = \sqrt{\mu} r_p^{\frac{3}{2}} \Delta t. \tag{3}$$

In this paper we refer to step sizes for s-integration in units of time; these are actually the step sizes at perigee which must be converted to s-steps with (3).

When s-integration is implemented as a predict, evaluate, correct cycle, the integration becomes unstable. Additional evaluations make the integration stable, for instance a PECEC implementation is stable. This doubles the computation time, reducing the benefit of s-integration over t-integration. However, it is not necessary for the second evaluation to be a full evaluation to maintain stability. Because the difference between the predicted and corrected states is small, the difference in perturbation forces at the two states is small. This means that computation time can be saved by simply re-evaluating the two-body force during the second evaluation, and adding it to the perturbation force from the first evaluation. We call this second evaluation a *pseudo-evaluation* and denote it $\tilde{\mathrm{E}}$, so it is a PEC$\tilde{\mathrm{E}}$C method.

Table 3 shows the maximum position difference over three days between using s-integration with the PECEC method and the PEC$\tilde{\mathrm{E}}$C method for a variety of orbits. Each integration was performed with perturbations, with a 30 second step size at perigee. Though the PECEC method takes twice as long, it changes the result by less than 10 mm. The s-integration results presented in this paper are obtained using the PEC$\tilde{\mathrm{E}}$C method.

Table 3: Effect of pseudo-evaluation in s-integration.

perigee height (km)	e	position difference (mm)
300	0.25	1.96
300	0.50	2.12
300	0.75	2.30
500	0.25	1.91
500	0.50	2.17
500	0.75	2.27
1000	0.25	1.85
1000	0.50	2.05
1000	0.75	8.79

SHAMPINE-GORDON INTEGRATION

Another way to more efficiently integrate elliptical orbits is to use a variable step integrator. The Shampine-Gordon integrator is a variable order, variable step multistep integrator (Ref. 1). It is based on the Adams-Bashforth and Adams-Moulton multistep integrators, and uses a PECE scheme. The main advantage of the Shampine-Gordon method is that it provides control over the local error, and adjusts the step size and the order of the method appropriately if the local error is outside user-defined bounds. Because the step size needed to meet a given accuracy is dependent on the orbit type, local error control is particularly useful in applications which process a wide variety of orbit types, since the integrator automatically finds the best step size for each orbit instead of the user providing step size values. Local error control is also useful for elliptical orbits because it allows larger steps to be taken at apogee.

The Shampine-Gordon integrator allows the user to specify a relative error tolerance, ϵ_r and an absolute error tolerance, ϵ_a, for local error, ξ. The code performs a test at each step, on each state variable X to see that the local error is below the tolerance,

$$\xi \leq \epsilon_r X + \epsilon_a. \tag{4}$$

In the orbit propagation and determination tests the absolute tolerance is $\epsilon_a = 1 \times 10^{-31}$, while the relative tolerance ϵ_r is set to meet a specified accuracy requirement.

The main disadvantage of Shampine-Gordon integration is that there are two evaluations per step. In order to have an advantage over the fixed step method, Shampine-Gordon needs to have half or fewer the number of steps per orbit.

ORBIT PROPAGATION TESTS

Our goal is to identify when variable step methods give a speed advantage over fixed step methods, given that the two methods are of equivalent accuracy. To do this we perform speed tests for orbit propagation with the integrators set to give approximately the same accuracy. A metric for integration accuracy is defined using an error ratio defined in terms of the RMS error of the integration (Ref. 8). First define position errors as

$$\Delta r = |r_{\text{computed}} - r_{\text{reference}}|, \tag{5}$$

$$\tag{6}$$

The RMS position error can be calculated,

$$\Delta r_{\text{RMS}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\Delta r_i)^2}. \tag{7}$$

The RMS position error is normalized by the apogee distance and the number of orbits to find the position error ratio,

$$\rho_r = \frac{\Delta r_{\mathrm{RMS}}}{r_A N_{\mathrm{orbits}}}. \tag{8}$$

In Ref. 10, several techniques are described to assess the accuracy of numerical integrators. In each of these techniques a different reference value is used in (5). In the step-size halving method, the reference values are found by integrating with the same integrator, but with half the step size. This method gives a good indication of the accuracy of a numerical integrator. Another method is the two-body test, in which the numerical integrator being tested is used with only the two-body force, and compared to the analytic solution from Kepler's equation. This method gives an exact indication of integration error when no perturbations are present, but does not indicate how well an integrator handles perturbations. Some integrators can integrate the two-body force very accurately, but are less accurate in the presence of perturbations, especially drag.

For Gauss-Jackson, the step size is found which gives an error ratio of approximately 1×10^{-9} in the step-size halving test, for both t- and s-integration. For Shampine-Gordon, the step-size halving test cannot be used, because it is a variable step method. Instead, the relative error tolerance is found which gives an error ratio of 1×10^{-9} in the two-body test. By using the two-body test, we are assuming that a tolerance that gives a certain error ratio without perturbations gives a similar error ratio in the presence of perturbations. Because the tolerance is a control over the local error, this is a reasonable assumption. Table 4 shows the error ratio over 3 days between t-integration and Shampine-Gordon with perturbations using the step-size or tolerance specified by the tests for a sample of orbits. These error ratios are all within range of 1×10^{-9}, indicating that tolerance specified by the two-body test for Shampine-Gordon is also valid with perturbations. To compare computation time, the time to propagate 30 days with the step size or tolerance specified by the tests is found.

Table 4: Error ratio between t-integration and Shampine-Gordon.

Perigee Height (km)	e	Error Ratio
300	0.0	3.6×10^{-9}
300	0.4	6.8×10^{-10}
400	0.7	7.9×10^{-10}
400	0.9	1.0×10^{-9}
500	0.0	2.9×10^{-10}
500	0.5	7.1×10^{-10}

Tables 5, 6, 7, and 8 show the results of the propagation tests at perigee heights of 300km, 400km, 500km, and 1000km, respectively. The tables show the step sizes needed for Gauss-Jackson with t-integration and s-integration and the relative error tolerance needed for Shampine-Gordon to achieve an error ratio of approximately 1×10^{-9}. The tables also show the user time to propagate 30 days with perturbations on an SGI Origin 200 with the listed step size or tolerance. The speed ratio for the variable step methods, which is the time of the variable step methods divided by the time of the fixed step t-integration, is also shown.

Figures 2, 3, 4, and 5 show plots of the speed ratio vs. eccentricity for perigee heights of 300km, 400km, 500km, and 1000km, respectively. Plots for both s-integration and Shampine-Gordon are shown on each figure. The horizontal line on each figure represents an speed ratio of 1. The variable step methods are more efficient than the fixed step method when the plots are above this line. The figures show that s-integration is more efficient that t-integration at an eccentricity of approximately 0.15, and that Shampine-Gordon is more efficient than Gauss-Jackson with t-integration at an eccentricity of approximately 0.60. The eccentricity where the variable step methods are more efficient is independent of the perigee height. The plots also show that s-integration is always more efficient

Table 5: Comparisons for perigee height of 300 km.

	Step Size / Tolerance			Time for 30 Day Run (sec)			Speed Ratio to t	
e	t	s	SG	t	s	SG	s	SG
0	12	8	3×10^{-11}	57.6	89.6	107	0.64	0.54
0.10	30	20	2×10^{-11}	34.6	47.4	106	0.73	0.33
0.15	30	26	4×10^{-11}	34.5	33.7	81.3	1.0	0.42
0.20	20	15	2×10^{-11}	50.8	50.8	85.3	1.0	0.60
0.30	36	31	1×10^{-11}	27.9	21.2	69.6	1.3	0.40
0.40	37	20	1×10^{-11}	27.1	25.6	56.1	1.1	0.48
0.50	34	24	1×10^{-11}	29.3	16.7	45.1	1.8	0.65
0.60	30	26	1×10^{-11}	33.1	11.5	34.3	2.9	0.97
0.65	26	24	3×10^{-11}	38.2	10.3	25.1	3.7	1.5
0.70	28	24	1×10^{-11}	35.4	8.35	23.7	4.2	1.5
0.80	26	23	2×10^{-11}	38.1	5.11	13.7	7.5	2.8
0.90	24	19	1×10^{-10}	41.2	2.52	5.11	16	8.1
0.95	24	19	2×10^{-10}	41.2	1.06	2.10	39	20

Table 6: Comparisons for perigee height of 400 km.

	Step Size / Tolerance			Time for 30 Day Run (sec)			Speed Ratio to t	
e	t	s	SG	t	s	SG	s	SG
0	32	16	5×10^{-11}	32.5	69.3	141	0.47	0.23
0.10	45	40	3×10^{-11}	23.1	25.2	90.2	0.92	0.26
0.15	40	36	6×10^{-11}	25.7	25.1	69.4	1.0	0.37
0.20	38	34	2×10^{-11}	26.8	23.7	70.8	1.1	0.38
0.30	39	34	1×10^{-11}	25.8	19.6	63.6	1.3	0.41
0.40	35	32	2×10^{-11}	28.6	16.6	47.2	1.7	0.61
0.50	33	30	1×10^{-11}	30.2	13.6	41.6	2.2	0.73
0.55	35	26	2×10^{-11}	28.4	13.4	33.5	2.1	0.85
0.60	31	28	3×10^{-11}	32.1	10.7	27.7	3.0	1.2
0.70	29	26	5×10^{-11}	34.2	7.76	18.6	4.4	1.8
0.80	27	24	3×10^{-11}	37.4	4.93	12.4	7.6	3.0
0.90	25	22	1×10^{-10}	39.5	2.15	4.86	18	7.9
0.95	25	22	2×10^{-10}	39.5	0.95	2.01	42	20

Table 7: Comparisons for perigee height of 500 km.

	Step Size / Tolerance			Time for 30 Day Run (sec)			Speed Ratio to t	
e	t	s	SG	t	s	SG	s	SG
0	45	40	5×10^{-11}	23.1	29.4	131	0.79	0.18
0.10	44	39	2×10^{-11}	23.6	25.5	88.1	0.93	0.27
0.15	42	37	6×10^{-11}	24.4	24.1	64.6	1.0	0.38
0.20	42	38	2×10^{-11}	24.2	21.5	65.9	1.1	0.37
0.30	40	35	7×10^{-12}	25.2	18.9	63.2	1.3	0.40
0.50	36	32	2×10^{-11}	27.7	12.8	36.0	2.2	0.77
0.55	34	30	1×10^{-11}	29.3	11.8	35.2	2.5	0.83
0.60	34	30	2×10^{-11}	29.3	10.1	27.7	2.9	1.1
0.70	32	28	1×10^{-11}	31.0	7.24	21.3	4.3	1.5
0.80	29	26	1×10^{-11}	34.2	4.55	13.7	7.5	2.5
0.90	27	24	1×10^{-10}	36.6	2.00	4.65	18	7.9
0.95	27	24	1×10^{-10}	36.6	0.89	2.17	41	17

Table 8: Comparisons for perigee height of 1000 km.

	Step Size / Tolerance			Time for 30 Day Run (sec)			Speed Ratio to t	
e	t	s	SG	t	s	SG	s	SG
0	70	60	4×10^{-11}	15.0	19.6	94.0	0.77	0.16
0.10	65	56	2×10^{-11}	15.9	17.7	62.9	0.90	0.25
0.15	62	55	5×10^{-11}	16.5	16.3	48.8	1.0	0.34
0.20	60	53	2×10^{-11}	16.9	15.3	50.1	1.1	0.34
0.30	57	50	1×10^{-11}	17.6	13.3	46.9	1.3	0.38
0.50	52	46	1×10^{-11}	19.2	9.00	32.3	2.1	0.59
0.60	50	44	5×10^{-11}	20.0	6.95	20.6	2.9	0.97
0.70	46	41	4×10^{-11}	21.6	5.05	15.2	4.3	1.4
0.90	38	34	1×10^{-10}	26.1	1.50	4.20	17	6.2

than Shampine-Gordon, most likely due to the extra evaluation required by Shampine-Gordon.

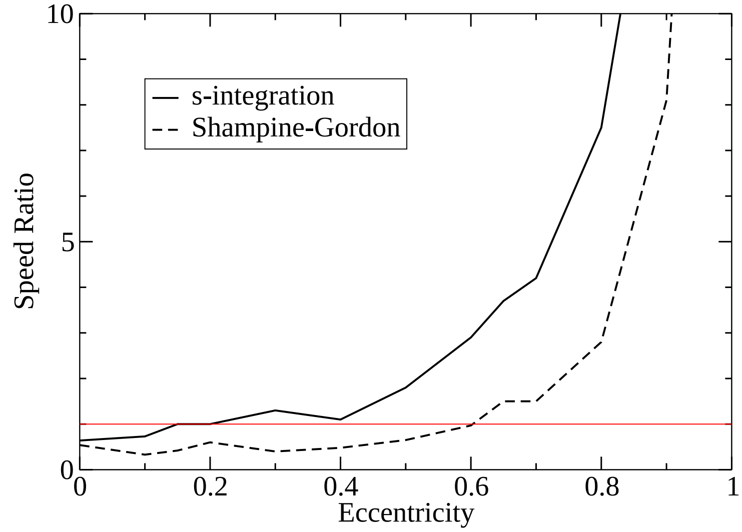


Figure 2: Speed ratios to t integration at 300 km perigee.

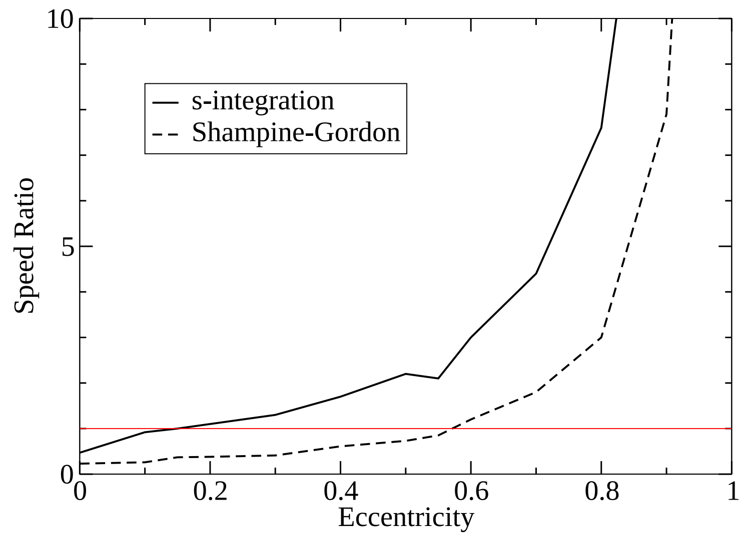


Figure 3: Speed ratios to t integration at 400 km perigee.

ORBIT DETERMINATION TESTS

For orbit determination, testing is performed on a test set of cataloged satellites for 1999-09-29. There are 8003 objects in the catalog, of which 1000 are randomly selected for testing. The goal of

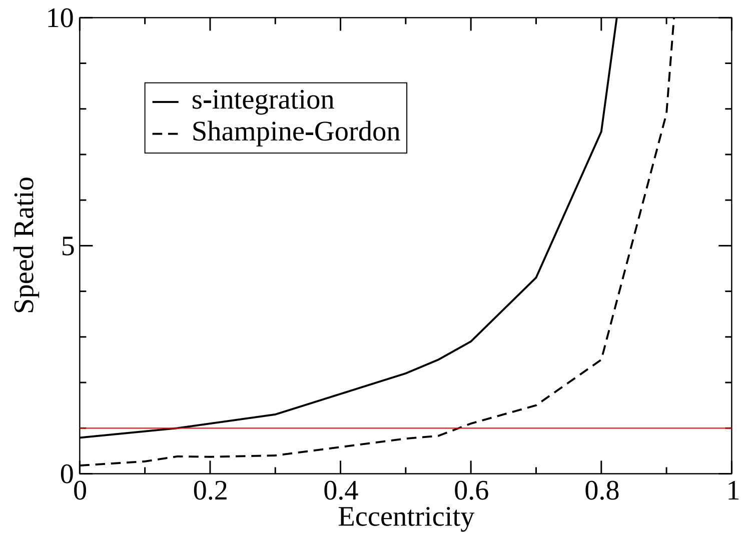


Figure 4: Speed ratios to t integration at 500 km perigee.

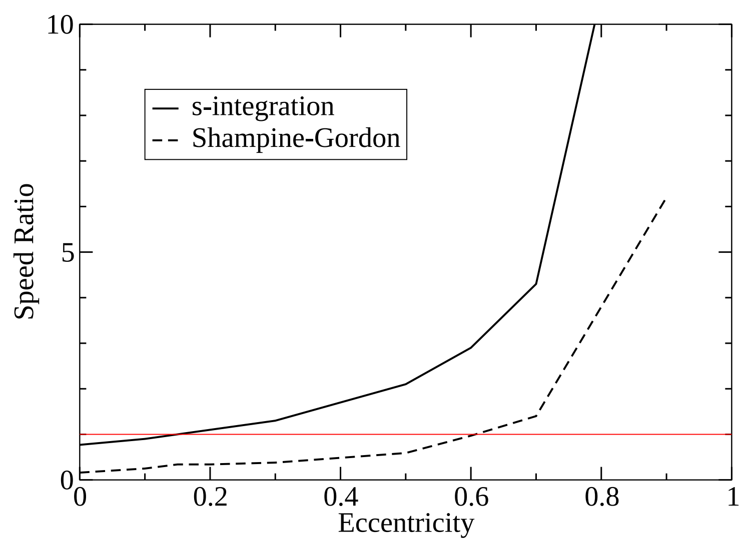


Figure 5: Speed ratios to t integration at 1000 km perigee.

the test is to find the improvement in computation time by using the variable step methods where the orbit propagation tests show they are more efficient, as well to validate that the variable step methods give comparable results to the fixed step Gauss-Jackson.

The initial vectors from the catalog are fit using differential correction with a fitspan between 1.5 and 10 days. The fitspan is determined by operational algorithms and depends on the mean motion and rate of change of mean motion. The fit includes observations up to 1999-10-01 00:00:00, going back through the length of the fitspan. Before the fit, the initial vector is propagated forward to the time of the last observation. The fit solves for position and velocity at the time of the last observation, and also solves for the ballistic coefficient when the perigee height is below 1200km.

To get a baseline for computation time, the time is found to fit the 1000 test objects using Gauss-Jackson with t-integration. The test is performed on a 450 MHz Pentium II machine running Linux. The total user time is 11.0 hours. Of the 1000 objects, 915 update, while 85 do not update because they fail some criteria, such as a final RMS that is too large, or there are not enough observations.

To test s-integration, the objects with eccentricities above 0.15 are fit with Gauss-Jackson using both t-integration and s-integration. The time using t-integration is 2.24 hrs, and the time using s-integration is 0.65 hrs. There are 135 objects with eccentricities above 0.15, of which 103 update and 32 do not. Comparing the final states given by t- and s-integration, 82 of the objects have a final position difference of less than 1m. The remaining 21 objects are shown in Table 9, which gives the final position difference in meters, and the difference in final weighted RMS. A negative value for RMS difference in the table indicates that the s-integration has a lower final weighted RMS.

Table 9: Orbit Determination Differences for t- vs. s-integration.

Satellite Number	Position Difference (m)	RMS Difference
3827	112.8	0.0007
10960	2.7	0.0000
13970	2.5	-0.0001
19622	378.8	-0.0912
19884	1.1	0.0000
19994	7.1	-0.0007
19998	1.1	-0.0004
21589	264.5	-0.0555
22020	14.0	0.0000
22238	7.4	0.0000
22274	1.0	-0.4953
22997	1.7	0.0007
23174	6.4	0.0000
23332	15.0	0.0000
23420	5.8	0.0000
23824	2.9	0.0002
24211	10.3	-0.0001
24655	14.9	0.0000
24764	17.3	-0.0001
25542	18.3	0.0000
25552	2.9	0.0000

The objects with the largest position difference in Table 9, 19622 and 21589, have a lower weighted RMS with s-integration, indicating that s-integration gives a better fit. The object with the next highest position difference, 3827, converged on a different iteration in the differential correction, and accepted a different number of observations. Though this position difference is relatively large, it is still within the accuracy of the observations and the force model. The remaining position differences are relatively minor, and well within the accuracy of the observations.

Using s-integration to perform the fits saves 1.59 hrs over t-integration. In the entire set of 1000 objects, if t-integration is used to fit the objects with eccentricities below 0.15 and s-integration is used to fit objects with eccentricities above 0.15, the total computation time would be 9.41 hrs, which is a 14.5% savings over using only t-integration.

To test Shampine-Gordon, the objects with eccentricities above 0.6 are fit with both Gauss-Jackson using t-integration and Shampine-Gordon. Gauss-Jackson takes 1.58 hrs to process the 86 objects, while Shampine-Gordon takes 0.81 hrs. Both integrators update 68 of the objects. The final position differences after the fit between the two integrators is less than 1 meter for 55 of the objects, the remaining 13 are shown in Table 10. A negative RMS difference in the table indicates that the weighted RMS is lower with Shampine-Gordon.

Table 10: Orbit Determination Differences for t-integration vs. Shampine-Gordon.

Satellite Number	Position Difference (m)	RMS Difference
8195	1.8	-0.0001
9911	2.5	0.0000
12992	47.3	0.0001
17078	1.9	0.0000
19622	1.7	0.0003
19807	1.1	0.0000
19884	1.3	-0.0002
20649	1.8	0.0004
21589	259.7	-0.0545
22068	1.2	-0.0003
22274	11.8	-0.6430
22997	2.5	0.0014
24655	36.1	0.5816

Again the object with the largest position difference, 21589, has a lower weighted RMS with Shampine-Gordon. The remaining objects have position differences that are relatively small, and well within the accuracy of the observations.

Using Shampine-Gordon on eccentricities over 0.60 saves 0.77 hrs over t-integration. If the entire set of 1000 objects is fit with Gauss-Jackson using t-integration for eccentricities below 0.60 and with Shampine-Gordon for eccentricities above 0.60, the total computation time would be 10.23 hrs, which is a 7.0% savings over using only Gauss-Jackson with t-integration.

CONCLUSION

In this paper we compare the fixed step Gauss-Jackson integrator using t-integration to the Gauss-Jackson integrator using s-integration, as well as to the variable order variable step Shampine-Gordon integrator. The Gauss-Jackson integrator with t-integration performs one force model evaluation per step. With s-integration, the integrator performs one evaluation and one pseudo-evaluation, which saves computation time. The Shampine-Gordon integrator performs two evaluations per step.

Orbit propagation tests in which the integrators are set to give equivalent accuracy show that s-integration has an advantage over t-integration above an eccentricity of 0.15, and Shampine-Gordon has an advantage over t-integration above an eccentricity of 0.60. These eccentricities are independent of the perigee height. The Shampine-Gordon integrator is never more efficient than s-integration, most likely because Shampine-Gordon performs two evaluations per step while s-integration performs only one. It may be possible to make Shampine-Gordon faster by performing only one evaluation per step, or by using the pseudo-evaluation technique. This is a topic of further study.

Orbit determination tests show that s-integration and Shampine-Gordon give comparable results to t-integration when used at the eccentricities specified by the orbit propagation tests. A time improvement of 14.5% is achieved by using s-integration at eccentricities above 0.15, and a time improvement of 7.0% is achieved by using Shampine-Gordon at eccentricities above 0.60.

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