ABSTRACT

Title of dissertation: ON SUPERSPACE DIMENSIONAL REDUCTION

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We describe manifestly globally supersymmetric theories in five dimensions and their dimensional reduction in superspace. This proceeds by a reduction from harmonic superspace, the universal superspace for theories in dimension six or fewer with eight supercharges, through projective superspace to simple superspace. That is, all symmetries of the original theory are retained but only a proper subgroup of the original Lorentz supergroup is realized linearly. The latter supergroup is precisely the four-dimensional Lorentz supergroup with four supercharges, making this setting ideal for the study of particle-field theories in five or six dimensions with flat four-dimensional subspaces.

ON SUPERSPACE DIMENSIONAL REDUCTION

by

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Chapter 1

Introduction

One of the most amazing observations about our universe is the fact that there exist two and only two types of particles. Bosons have the property that they like to "condense" into the same state as other identical bosons making possible, among other things, the stimulated emission of radiation, superfluidity, and superconductivity. Fermions, on the other hand, obey the *Pauli exclusion principle* meaning that two fermions can never be in the same state. This is the basis of the stability of matter; the periodic table of the elements and therefore all of chemistry (not to mention biology) would not exist without it. Indeed, the forces of nature all seem to be carried by bosons while all of fundamental matter (excluding the elusive Higgs boson) is fermionic.

Supersymmetry is a symmetry which treats bosons and fermions on an equal footing. In a theory possessing the simplest of such symmetries, every bosonic degree of freedom has a fermionic partner and *vice versa*. Considering how different these two types of particles are, it is quite surprising that such a symmetry exists. With respect to how forces are carried by bosons and matter is made up of fermions it is perhaps even more surprising that we would consider such a symmetry relevant to a description of our universe.

In fact, supersymmetry is the hallmark of a variety of interesting theories from the purely mathematical to the phenomenological. Besides the æsthetically pleasing fact that it is the "ultimate" symmetry, it is the most powerful tool available for the study of the strong coupling limit of gauge theories, has the potential to cure the standard model[1], and seems to be an essential ingredient in the ultraviolet completion of effective theories of various types, including gravity.

The standard approach to such theories is the component approach or "tensor calculus". In such an approach supersymmetry is not manifest. This is not usually considered to be a problem since many of the virtues of supersymmetric theories are theorems about renormalizability which hold whether the symmetry is manifest or not. (In fact, it is common practice in the literature to write only the bosonic part of the theory under consideration and simply announce that it is supersymmetric, by which is meant that it has a supersymmetric completion.) One advantage of this approach is that everything is written in a manner consistent with the usual

[1] That this symmetry is not in direct conflict with observation is because of the possibility that it is *spontaneously broken*, that is, it is a symmetry of the action but not of the ground state of the system. The situation can then be arranged so that the missing superpartners gain a correction to their masses which puts them above the scale of current measurements.

notation used when one learns about the quantum theory of particles and fields. This is therefore the most popular approach to the subject for beginners. It also follows that this formulation is amenable to standard field-theoretic manipulations.

An alternative approach is to use the superfield or superspace formalism. In this notation supersymmetry is manifest in the same sense that Lorentz invariance is manifest in 4-vector notation. Besides the usual advantages of having a symmetry manifest, the superspace approach is indispensable for the quantization of complicated theories such as General Relativity. The disadvantages of this approach, however, are manifold and non-negligible. Firstly, the amount of formalism one needs to learn is significant: Two of the three standard references [1, 2] on the subject are over 500 pages long. The third [3] is around 250 pages long but is almost entirely a long list of big equations. Secondly, the formalism for a given theory is not always known. For example, it is not known what the off-shell formulation is for the "maximally supersymmetric" super-Yang-Mills (sYM) theory in four dimensions. Thirdly, when the theory is known, it is often quite complicated *videlicet* four-dimensional $\mathcal{N} = 2$ supergravity. Finally, certain manipulations of a theory which one may like to perform are more difficult than in the component approach. For example, one would often like to truncate a theory defined in d dimensions to $d-1$ dimensions or fewer. The physics resulting from this reduction is easy to understand in components but not in superspace.

In the end, the fact remains that some calculations are impossible to perform in components such as the quantization of theories in the presence of gravity. Other calculations can be performed this way at the expense of great effort but become almost trivial when formulated in superspace. In yet other theories, superspace allows one to make progress on pressing questions which have resisted analysis in the component formulation. An example of this is the problem of the quantization of strings in Ramond-Ramond backgrounds. Finally, the ills of the superspace approach can be argued to be curable, in which case superspace becomes a more attractive option. Alternatively, we could use the parts we do understand well to attack the ones we do not in non-conventional ways.

In this thesis I describe what I see as evidence of these last two assertions. Specifically, we will analyze the special case of flat five[2]-dimensional superspace with

[2]Although we will consider only five dimensional superspace with the minimal amount of supersymmetry, the formalism can be extended almost trivially to six dimensions. In the formulation in which supersymmetry is manifest, the extension can be achieved by interpreting the central charge in the algebra $\triangle$ as the derivative in the "6-direction". In the projective superspace approach (and therefore also the simple superspace approach) in which we will drop the central charge, the required extension is effected by the substitution $\partial_5 \to \partial_5 \pm i\partial_6$. Some of the theories we consider in this thesis cannot be obtained in this way while keeping the fields off their mass shells.

the minimal amount of supersymmetry. We will perform, in a relatively simple way, the reduction of theories in this space to four-dimensional superspace notation. Although the resulting theory is still five-dimensional, only the four-dimentional part of the Poincaré symmetry will be manifest as will only one of its two[3] supersymmetries. Schematically, this reduction proceeds as follows:

$$\text{Harmonic} \xrightarrow{\delta_2} \text{Projective} \xrightarrow{\delta_1} \text{Simple} \ . \tag{1.0.1}$$

Here "Harmonic" refers to the five-dimensional superspace with manifest minimal supersymmetry, "Projective" refers to the five-dimensional superspace with half of the minimal supersymmetry manifest and "Simple" is the five-dimensional superspace obtained by integrating out an infinite number of auxiliary (non-dynamical)

[3]The counting of supersymmetries will be unavoidably ambiguous for historical reasons. Five-dimensional theories with minimal supersymmetry (eight real supercharges) are properly said to have $\mathcal{N} = 1$ supersymmetry. The truncation of such a theory to four dimensions gives twice the amount of minimal (*id est* four real supercharges) supersymmetry allowable in this dimension and is properly called an $\mathcal{N} = 2$ theory in four dimensions. The relevant superspace is then called conventional $\mathcal{N} = 2$ superspace. For the purpose of describing five-dimensional theories, however, this superspace (together with one extra bosonic coordinate) has the minimal amount of supersymmetry in five dimensions and is therefore $\mathcal{N} = 1$ in the five-dimensional sense. Writing a five-dimensional minimally supersymmetric theory with eight real supercharges in a superspace with only four real supercharges keeps manifest only one-half of the five-dimensional supersymmetries. However, one often describes this by saying that we keep manifest only one of the two supersymmetries.

superfields which are the hallmark of linearly realized symmetries with more than four real supercharges. The projection δ_1 is the map which implements this procedure while δ_2 is the projection developed by Sergei M. Kuzenko [4] which removes two complex co-dimension-one hyperplanes from the harmonic superspace and is the subject of section 5.

This report is organized as follows: In chapter 2 we review, very briefly, four-dimensional $\mathcal{N}=0$ and $\mathcal{N}=1$ superspaces. The purpose of section 2.1 is simply to provide a layout of the presentation of the subsequent sections in a familiar setting. Section 2.2 provides a review of field theory with manifest global supersymmetry and serves to set the notation of four-dimensional superspace.

Chapter 3 reviews the construction of five-dimensional theories with global supersymmetry in simple superspace. The sYM theory in this section is a truncation of a much more non-trivial result obtained in [5] for ten dimensions. This result was later re-derived in [6] and applied to phenomenological problems. These two papers gave the impetus for the study of a gravitational analogue which was developed in [7] and started the line of research [8, 9, 10] which eventually culminated in this thesis.

Chapter 4 is a review and extension of the harmonic superspace based on the SU(2)/U(1) coset developed in [11]. All details of this theory can be found in the textbook [12]. We restrict our attention to the case of global supersymmetry.

Chapter 5 covers the main point of this thesis: All results in five- and six-dimensional globally supersymmetric theories can be reduced to four dimensions by a simple reduction procedure first described in [4]. This reference provided a detailed embedding of projective superspace into harmonic superspace. This result was re-interpreted as a dimensional reduction in [10].

Chapter 6 provides some closing thoughts about the work presented in this thesis and is, perhaps, the most interesting chapter. Here I describe the relation of projective superspace to string theory on Calabi-Yau 3-folds and speculate on the relation of harmonic superspaces in general to the various existing formalisms for the covariant quatization of the critical superstring. In particular, I consider the relationship between the existence of generalizations of the chiral and analytic subspaces of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superspaces and the measure for the open string field theory path integral.

Chapter 2

Four-Dimensional Simple Superspace

In this section we will review some basics of simple superspace. We will not attempt to give a comprehensive presentation of this subject as it is done completely in the excellent references [1, 2] and [3] and would take far too long to repeat. Instead, we collect here some highlights. We will be adhering to the conventions of reference [2] throughout.

2.1 Minkowski $N = 0$ Superspace

The simplest of non-trivial example of a four-dimensional superspace is flat spacetime $\mathbb{R}^{1,3}$, or Minkowski space. It consists of commuting coordinates only $\{x^a\}_{a=0}^3$,

$$[x^a, x^b] = x^a x^b - x^b x^a = 0 \ , \tag{2.1.1}$$

transforming in the spin-1 representation of the Lorentz group. In particular one coordinate, $\{x^0\}$, is time-like while the other three, $\{x^i\}_{i=1}^3$, are space-like.

For future reference we point out that this space is isometric to the quotient of the Poincaré, or in-homogeneous Lorentz group, $\mathrm{ISO}(1,3)$ by the homogeneous part

SO(1, 3). That is

$$\mathbb{R}^{1,3} \approx \mathrm{ISO}(1,3)\Big/\mathrm{SO}(1,3)\ , \tag{2.1.2}$$

A representation

$$\begin{aligned} \varphi : \mathbb{R}^{1,3} &\rightarrow \mathfrak{a} \\ x &\mapsto \varphi(x)\ , \end{aligned} \tag{2.1.3}$$

defined on this space and taking values in an algebra $\mathfrak{a}$ is called *a(n $\mathfrak{a}$-valued classical) field*. The generator of translations P_a is realized on this field as a derivative $P_a = -\mathrm{i}\partial_a$.

In order to define the classical theory of an $\mathfrak{a}$-valued field $\varphi(x)$, we often define an *action* functional S as a smooth map from the space of fields and its derivatives to the real numbers. Traditionally, we further require that the action can be written as the integral of a density:

$$S[\varphi] = \int \mathrm{d}^4x \mathrm{tr} L\ . \tag{2.1.4}$$

Here $L = L[\varphi(x), \partial\varphi(x), \partial^2\varphi(x), \ldots]$ is the Lagrangian density which, in turn, is required to be a local functional of the field and its derivatives. We usually[1] require the action S to be a singlet of the Lorentz group SO(1, 3) as well as the algebra $\mathfrak{a}$.

[1]This requirement cannot always be met in the quantum theory where S might shift under a transformation. It is then only required that $S \rightarrow S + 2\pi k$ for $k \in \mathbb{Z}$ an integer.

We denote the latter projection by tr. The former condition will be satisfied if L is itself a Lorentz singlet. Alternatively, with rapidly decaying boundary conditions on the fields, the Lagrangian may transform into a total derivative, a condition we will assume throughout this work. We will always assume such conditions.

The stationary points of the action functional are the classical equations of motion[2]

$$E[\varphi] = \frac{\delta}{\delta\varphi(x)} S[\varphi] = 0 \ . \tag{2.1.5}$$

In what follows we will be studying the analogues of the action for a free complex scalar field φ given by

$$-\int \mathrm{d}^4x \partial_a \bar{\varphi} \partial^a \varphi \ , \tag{2.1.6}$$

and that of a connection 1-form A_a given by

$$-\frac{1}{4}\int \mathrm{d}^4x F_{ab}F^{ab} \quad F_{ab} = \partial_a A_b - \partial_b A_a \ . \tag{2.1.7}$$

[2]When $E = 0$ we say that the theory is *on-shell* or that the equation of motion is satisfied. Conversely, if none of the equations of motion are imposed, we say that the theory is *off-shell*.

2.2 Simple $N = 1$ Superspace

The next-to-simplest superspace is parameterized by the coordinates of Minkowski space together with a set of anti-commuting parameters $\hat{\theta}^{\hat{\alpha}}$

$$\{\hat{\theta}^{\hat{\alpha}}, \hat{\theta}^{\hat{\beta}}\} = \hat{\theta}^{\hat{\alpha}}\hat{\theta}^{\hat{\beta}} + \hat{\theta}^{\hat{\beta}}\hat{\theta}^{\hat{\alpha}} = 0 \ , \tag{2.2.8}$$

where hatted greek letters take the values $\hat{\alpha} = 1, 2, \dot{1}, \dot{2}$. These parameters are defined to transform in the spin-$\frac{1}{2}$ representation of the Lorentz group. Actually, since the algebra $\mathfrak{spin}(1,3) \approx \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ splits, these fermionic parameters split as

$$(\theta_{\hat{\alpha}}) \quad = \quad \begin{pmatrix} \theta_\alpha \\ \\ \bar{\theta}^{\dot{\alpha}} \end{pmatrix} \ , \tag{2.2.9}$$

where now $\alpha = 1, 2$ *et cetera*. θ and $\bar{\theta}$ are called *Weyl* spinors. We take $\bar{\theta}$ to be the hermitian conjugate of θ making the original $\theta^{\hat{\alpha}}$ a *Majorana* spinor. Spinorial indices are raised and lowered using the SL$(2,\mathbb{C})$-invariant tensors $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$ and their inverses.[3] An enormously helpful fact concerning this split is that spin-tensors with antisymmetric dotted or un-dotted indices are trivial representations of the Lorentz

[3] A note of caution: If a spinor ψ_α is originally defined with its index down, then $\psi^\alpha := \varepsilon^{\alpha\beta}\psi_\beta$ is the definition for this object with its index raised and analogous statements hold for general spin-tensors. There are a few exceptions to this rule. One is $\varepsilon_{\alpha\beta}$ itself. As one can easily show, $\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\varepsilon_{\gamma\delta} = -\varepsilon^{\alpha\beta}$ if, as it is defined in these conventions, $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta_\gamma^\alpha$.

group. For example,

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^\gamma\theta_\gamma = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2 \ . \tag{2.2.10}$$

Translations in these fermionic directions are generated by the supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$ and are defined to close algebraically on translations P_a in the graded sense, *id est*

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^a)_{\alpha\dot{\alpha}}P_a \ , \tag{2.2.11}$$

with all other graded commutators vanishing.[4] The set $\{Q, \bar{Q}, P, M\}$, with M denoting the Lorentz generator, therefore closes under graded commutation and is called the super-Poincaré algebra. Exponentiation of this algebra gives the (connected part of the) super-Poincaré group.

Simple superspace is now defined as the (left) quotient of this group by its Lorentz subgroup. Classical fields on this space are as before but now depend on the fermionic coordinates θ and $\bar{\theta}$. They are called *superfields*. Since the fermionic coordinates are anti-commuting, a McLauren expansion in them terminates at a finite order. The coefficients of this expansion are ordinary $N = 0$ fields called component fields in various representations of the Lorentz group. Using equation (2.2.10) we

[4]Here $\sigma_a = (\mathbf{1}, \vec{\sigma})$ is the four-dimensional extension of the standard Pauli matrices. They are constant tensors of the Lorentz group.

find that for a real scalar superfield $f(x,\theta,\bar{\theta})$,

$$\begin{aligned} f(x,\theta,\bar{\theta}) &= \varphi(x) + (\theta\psi(x) + \text{h.c.}) + (\theta^2 F(x) + \text{h.c.}) \\ &\quad + \mathrm{i}(\theta\sigma^a\bar{\theta})A_a(x) + (\bar{\theta}^2\theta\lambda(x) + \text{h.c.}) + \theta^2\bar{\theta}^2 D(x) \ . \end{aligned} \tag{2.2.12}$$

Here φ and D are real scalar fields, F is a complex scalar field and ψ and λ are Weyl fermion fields. The letters used for the component fields are not standard except F and D. These parts of a superfield are even called the F-term and D-term respectively and are important as we will see below.

The supercharges are realized on superfields as fermionic derivations

$$\begin{aligned} Q_\alpha &:= \mathrm{i}\partial_\alpha + \bar{\theta}^{\dot{\alpha}}(\sigma^a)_{\alpha\dot{\alpha}}\partial_a \\ \bar{Q}_{\dot{\alpha}} &:= -\mathrm{i}\bar{\partial}_{\dot{\alpha}} - \theta^{\dot{\alpha}}(\sigma^a)_{\alpha\dot{\alpha}}\partial_a \ , \end{aligned} \tag{2.2.13}$$

where the fermionic partial derivatives are defined to act as $\partial^\alpha\theta_\beta := \delta^\alpha_\beta$ and $\bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}}$ and annihilate everything else.[5] In order to preserve the reality of a superfield, the supersymmetry transformation with infinitesimal Weyl parameter ϵ is given by

$$\delta = \mathrm{i}(\epsilon Q + \bar{\epsilon}\bar{Q}) \ . \tag{2.2.14}$$

We now come to the most important part of the $N = 1$ superspace formalism. We want to construct covariant derivatives – derivations which commute with this

[5] We will not elaborate on this point here, but these partial derivatives are another exception the the raising/lowering rule. In fact, in the conventions of [2], $\partial^\alpha = -\varepsilon^{\alpha\beta}$ and $\bar{\partial}_{\dot{\alpha}} = -\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\beta}}$. Note also the fact that with these definitions $\bar{Q} = -(Q)^*$.

supersymmetry transformation. That these derivatives have to exist is a consequence of the fact that our superspace is a left coset and the supersymmetry transformations are generated by a left action. However, we could also consider motions generated by a right action. Since acting on the left commutes with acting on the right, the derivations effecting these actions must commute. Alternatively, we could just compute explicitly and we find that

$$
\begin{aligned}
& \partial_a \quad , \\
D_\alpha &:= \partial_\alpha + \mathrm{i}\bar{\theta}^{\dot{\alpha}}\partial_{\underline{a}} \quad , \\
\bar{D}_{\dot{\alpha}} &:= -\bar{\partial}_{\dot{\alpha}} - \mathrm{i}\theta^{\alpha}\partial_{\underline{a}} \quad ,
\end{aligned} \tag{2.2.15}
$$

all commute with the Qs. (With this normalization, $\bar{D}$ is the complex conjugate of D.)

The existence of the fermionic derivatives suggests the following alternative to the component expansion (2.2.12). The fields defined in this expansion can be extracted by acting with fermionic partial derivatives and setting to zero the explicit θ dependence in the result. For example $\psi_\alpha(x) = -\partial_\alpha f(x, \theta, \bar{\theta})|$. (The $|$ notation is a standard notation for the $\theta, \bar{\theta} \to 0$ limit.) The drawback of this is that the resulting components do not transform covariantly under supersymmetry transformations since $\{\partial_\alpha, \bar{Q}_{\dot{\alpha}}\} \neq 0$. It is easy to convince oneself that the covariant set of components defined instead by acting with D_α and $\bar{D}_{\dot{\alpha}}$ is equivalent to the this non-covariant set, the latter differing from the former by a field redefinition. Therefore, we will henceforth use the covariant component definitions exclusively. Expressions such a the component expansion (2.2.12) are to be interpreted as schematic shorthand for a covariant expansion and not literally as an expansion in the θ-variables. Once this is agreed, we are in a position to abandon the superspace translation operators altogether in favor of the covariant derivatives.

The derivatives (2.2.15) obey the algebra[6]

$$\{D_\alpha, D_\beta\} = 0 \quad , \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \ , \tag{2.2.16}$$

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} \ = \ -2\mathrm{i}\partial_{\underline{a}} \ , \tag{2.2.17}$$

together with trivial commutators with the bosonic partial derivatives. Important formulæ which follow from this algebra are

$$D_\alpha D_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}D^2 \quad ; \quad \bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}} = -\frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{D}^2 \tag{2.2.18}$$

$$[D^2, \bar{D}_{\dot{\alpha}}] = -4i\partial_{\underline{a}}D^\alpha \quad ; \quad [\bar{D}^2, D_\alpha] = 4i\partial_{\underline{a}}\bar{D}^{\dot{\alpha}} \tag{2.2.19}$$

$$D^\alpha \bar{D}^2 D_\alpha \ = \ \bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} \tag{2.2.20}$$

[6]The space of 4-vectors is isomorphic to the space $\mathfrak{su}(2)$ of hermitian 2×2 matrices. The isomorphism is given by the contraction with the Pauli matrices. It proves to be very useful, therefore, to employ a notation in which all indices are spinorial. Many references write expressions like $v_{\alpha\dot{\alpha}} := (\sigma^a)_{\alpha\dot{\alpha}} v_a$, a notation we will sometimes use. In addition, however, we will also define $v_{\underline{a}} := v_{\alpha\dot{\alpha}}$.

$$\{D^2, \bar{D}^2\} - 2D^\alpha \bar{D}^2 D_\alpha = 16\Box \quad (2.2.21)$$

$$[D^2, \bar{D}^2] = -4i\partial^{\underline{a}}[D_\alpha, \bar{D}_{\dot{\alpha}}] \quad (2.2.22)$$

The relation (2.2.16) implies the existence of non-trivial solutions to the equation

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad (2.2.23)$$

Indeed, since $\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}} \equiv 0$, the solution is given by $\Phi = -\frac{1}{4}\bar{D}^2\varphi$ for a completely arbitrary complex superfield φ. Superfields obeying equation (2.2.23) are called *chiral* superfields and are of fundamental importance. The subspace of the full superspace is called the chiral subspace and it is invariant under translations and supersymmetry transformations. Plugging the complex[7] version of the component expansion (2.2.12) into the chirality condition (2.2.23) and using the representation (2.2.15) of the covariant derivatives, one finds that only the (complex!) fields analogous to φ, ψ_α and F survive. Indeed, from the covariant projection

$$\begin{aligned} \varphi(x) &:= \Phi| \\ \psi_\alpha(x) &:= D_\alpha\Phi| \\ F(x) &:= -\frac{1}{4}D^2\Phi| \end{aligned} \quad (2.2.24)$$

[7] Real chiral fields are necessarily constant as follows directly from the algebra upon imposing the constraint.

it is immediate that these are the only components since any expression with a $\bar{D}$ in it can be reduced, using the algebra and the constraint, to zero or some combination of spacetime derivatives acting on these components.

2.3 Classical Field Theories in Four-Dimensional Simple Superspace

In order to write classical field theories in superspace, it is nice to know how to write actions in terms of Lagrangian super-densities. To this end, we need a projection from superspace to the real numbers. A general superfield $L = L(x, \theta, \bar{\theta})$ has the property that it's D-term D_L transforms into a spacetime derivative under a supersymmetry transformation. It follows that the "integral" $\int \mathrm{d}^4\theta L := D_L$ transforms into a total derivative. This term, as described above, can be isolated by covariant projection $D_L = \frac{1}{16}\bar{D}^2 D^2 L|$. We therefore define the action of L by

$$\int \mathrm{d}^4 x \int \mathrm{d}^4\theta L = \frac{1}{16} \int \mathrm{d}^4 x \bar{D}^2 D^2 L \Big| \ . \tag{2.3.25}$$

Suppose now that W is a chiral superfield. It follows immediately that the action above vanishes if we were to replace $L \to W$. However, for chiral superfields, the supersymmetry transformation of the F-term F_W is a total derivative. We therefore define the integration over the chiral subspace of the full superspace as

$\int \mathrm{d}^2\theta W := F_W$ and, consequently, the action of this so-called "superpotential" as

$$\int \mathrm{d}^2\theta W + \text{h.c.} = -\frac{1}{4}D^2 W\Big| + \text{h.c.} \tag{2.3.26}$$

Let us now turn to some applications of these observations.

2.3.1 The Scalar Multiplet I: Chiral Multiplet

The standard scalar multiplet is described in superspace by a complex field Φ satisfying the chirality condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \ . \tag{2.3.27}$$

This condition is entirely equivalent to the statement that $\Phi = -\frac{1}{4}\bar{D}^2\bar{\varphi}$ provided that φ is an unconstrained complex superfield. The mass dimension of Φ is taken to be 1 and the Lagrangian density for this theory is the well known Wess-Zumino kinetic term

$$\int d^4\theta\bar{\Phi}\Phi \ . \tag{2.3.28}$$

The equation of motion is obtained by the chiral differentiation rule or simply by varying with respect to the complex prepotential:

$$E[\varphi] = -\frac{1}{4}\bar{D}^2\bar{\Phi} \ . \tag{2.3.29}$$

It follows from this equation of motion and the chirality of Φ that:

1. The superfield Φ obeys the correct mass-shell condition $\Box\Phi = 0$.

2. The component spectrum of *on-shell* degrees of freedom is that of a complex scalar $z \sim \Phi|$ and a Weyl fermion $\lambda_\alpha \sim D_\alpha\Phi|$ which obey the Klein-Gordon equation by virtue of consequence 1.[8] In particular the *F-term* of Φ is an auxiliary field.

2.3.2 The Scalar Multiplet II: Linear Multiplet

A second description of a scalar multiplet is obtained by replacing the chiral field with a complex field Γ obeying the linear constraint

$$\bar{D}^2\Gamma = 0 \ . \qquad (2.3.30)$$

This Bianchi identity is solved in terms of an unconstrained spinor superfield ψ_α as $\Gamma = \bar{D}_{\dot{\alpha}}\psi^{\dot{\alpha}}$. This prepotential transforms as $\delta\psi_\alpha = D^\beta\tau_{\alpha\beta}$ for symmetric $\tau_{\alpha\beta}$ leaving Γ invariant.

The mass dimension of Γ is taken to be one and the Lagrangian density for

[8] In fact, the fermion obeys the Weyl equation as derived from (2.3.29) by $0 = \bar{D}_{\dot{\alpha}}\bar{E}[\varphi]| \sim \partial_{\underline{a}}D^\alpha\Phi|$. In the sequel, we will only explicitly refer to the Klein-Gordon equation. Implicitly, the component fields are obeying the correct mass-shell conditions.

this theory is simply[9]

$$-\int d^4\theta \bar{\Gamma}\Gamma \ . \tag{2.3.31}$$

This time the equation of motion is obtained by varying with respect to the spinor prepotential:

$$E[\bar{\psi}] = -\bar{D}_{\dot{\alpha}}\Gamma \ . \tag{2.3.32}$$

It follows from this equation of motion and the Bianchi identity that

1. the superfield Γ obeys the Klein-Gordon equation $\Box\Gamma = 0$,
2. the component spectrum is that of a complex scalar $z \sim \Gamma|$ and a Weyl fermion $\lambda_\alpha \sim D_\alpha\Gamma|$,
3. this theory is dual to the chiral scalar theory described above (compare equations (2.3.27↔2.3.32) and (2.3.29↔2.3.30)) since the first was off-shell chiral but on-shell linear while in this case it is the other way around.

[9]The sign preceding this density is not a mistake. Although comparing (2.3.31) to (2.3.28) suggests this sign gives ghostlike kinetic terms to the scalars, it is in fact necessary to get the correct sign as is easy to check explicitly. Alternatively, one can perform a duality transformation from the chiral field Lagrangian of the previous section to the complex linear Γ multiplet which produces a sign of the type in (2.3.30).

2.3.3 The Vector Multiplet

The chiral multiplet action (2.3.28) has a global U(1) symmetry taking $\Phi \mapsto \mathrm{e}^{\mathrm{i}a}\Phi$ with $a \in \mathbb{R}$. Gauging this symmetry while preserving the the chirality of Φ requires the replacement $a \to \Lambda$ with chiral (non-constant) superfield Λ. Note that this complexifies the $\mathrm{U}(1) \hookrightarrow \mathbb{C}^*$ in addition to making it local. The invariance of the action can now be restored by introducing a real superfield-valued connection e^V transforming under the $\mathbb{C}^*$ action as $(\mathrm{e}^V)' = \mathrm{e}^{\mathrm{i}\bar{\Lambda}}\mathrm{e}^V\mathrm{e}^{-\mathrm{i}\Lambda}$ and replacing the Lagrange super-density $\bar{\Phi}\Phi \to \bar{\Phi}\mathrm{e}^V\Phi$. In the abelian case, the field strength of the gauge pre-potential can be deduced from the linearized transformation law $\delta V = \frac{1}{2\mathrm{i}}(\Lambda - \bar{\Lambda})$. Since the operator $\bar{D}^2 D_\alpha$ annihilates both chiral and anti-chiral fields, $W_\alpha = \frac{1}{8}\bar{D}^2 D_\alpha V$ is an invariant field strength. We note immediately that this field strength superfield obeys the following constraints identically[10]

$$\bar{D}_{\dot{\alpha}} W_\alpha = 0 \quad ; \quad D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = 0 \ . \tag{2.3.33}$$

From these equations it follows that the action constructed from the Lagrangian super-density

$$\frac{1}{2}\int \mathrm{d}^2\theta W^\alpha W_\alpha \ , \tag{2.3.34}$$

[10] The second of these follows from the identitiy (2.2.20) and the reality of V.

is both real and manifestly gauge invariant. The equation of motion which follows from this is

$$E[V] = D^{\alpha} W_{\alpha} + \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \ . \tag{2.3.35}$$

Consequently,

1. The superfield strength W_{α} obeys the Dirac equation.

2. The component spectrum is that of a Weyl fermion $\lambda_{\alpha} \sim W_{\alpha}|$ and a symmetric bi-spinor field strength $F_{\alpha\beta} \sim D_{(\alpha} W_{\beta)}|$ and their conjugates. Such a component field strength is equivalent to a 2-form $F_{\underline{ab}} \sim \varepsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta} + \varepsilon_{\alpha\beta} \bar{F}_{\dot{\alpha}\dot{\beta}}$ which is the field strength of a gauge 1-form $F_{ab} \sim \partial_{[a} A_{b]}$.

3. The dual of this theory is a theory of the same type. More precisely, defining $U = \mathrm{i}V$ and writing the theory in terms of U we switch the second constraint and the equation of motion and multiply the action by a factor of -1.

An alternative approach to obtaining this theory is by interpreting it as a super-differential geometric problem. In such an approach we introduce gauge connections for all of the superspace covariant derivatives $D_A \to \mathcal{D}_A = D_A + \mathrm{i}\Gamma_A$ so that the resulting gauge covariant derivatives transform covariantly under the $\mathbb{C}^*$ action. This collection of gauge superfields is absurdly redundant. To remove all of the redundant fields, we notice that generically

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C \mathcal{D}_C + \mathrm{i} F_{AB} \ , \tag{2.3.36}$$

for some collection of superfields $T_{AB}{}^C$ and F_{AB} called *torsions* and *field strengths*, respectively. The strategy now is to find a proper non-trivial subset of these torsions and field strengths which satisfy the Bianchi identity

$$[\mathcal{D}_A, [\mathcal{D}_B, \mathcal{D}_C\}\} + \text{graded cyclic permutations} = 0 \ . \tag{2.3.37}$$

The case in which the result implies that the minimal number of independent on-shell degrees of freedom remain is called “irreducible”. It is irreducible in the sense that the resulting multiplet forms an irreducible (field) representation of supersymmetry. If, furthermore, this representation contains the minimal number of independent off-shell degrees of freedom, it is called “minimal”.

The minimal $\mathcal{N} = 1$ sYM algebra is given by [1, 2]

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0 = \left\{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\right\} \quad ; \quad \left\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\right\} = -2\mathrm{i}\mathcal{D}_{\underline{a}} \tag{2.3.38}$$

$$\left[\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\underline{b}}\right] = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta} W_\beta \quad ; \quad \left[\mathcal{D}_\alpha, \mathcal{D}_{\underline{b}}\right] = 2\mathrm{i}\varepsilon_{\alpha\beta} \bar{W}_{\dot\beta} \tag{2.3.39}$$

$$\left[\mathcal{D}_{\underline{a}}, \mathcal{D}_{\underline{b}}\right] = -\varepsilon_{\dot\alpha\dot\beta} D_\alpha W_\beta - \varepsilon_{\alpha\beta} \bar{D}_{\dot\alpha} \bar{W}_{\dot\beta} \tag{2.3.40}$$

The constraints (2.3.38) on the first line are called conventional constraints and are given by setting torsions to constant values. One representation of the unique solution to these constraints can be shown to be given by (Frohbenius' Theorem)

$$\bar{\mathcal{D}}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \quad ; \quad \mathcal{D}_\alpha = \mathrm{e}^{-2V} D_\alpha \mathrm{e}^{2V} \quad ; \quad \mathcal{D}_{\underline{a}} = \frac{\mathrm{i}}{2} \left\{ \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}} \right\} \ . \tag{2.3.41}$$

This is called the chiral representation since in this representation $\bar{\mathcal{D}}_{\dot{\alpha}}$ has no connection from which it follows that if Φ is chiral in the background where the YM field is switched off, then it is automatically gauge-covariantly chiral.

The solutions to the constraints (2.3.39) in this representation become

$$W_\alpha = -\frac{1}{8} \bar{D}^2 \left(\mathrm{e}^{-2V} D_\alpha \mathrm{e}^{2V} \right) \quad ; \quad \bar{W}_{\dot{\alpha}} = \frac{1}{8} \mathrm{e}^{-2V} \bar{D}^2 \left(\mathrm{e}^{2V} D_\alpha \mathrm{e}^{-2V} \right) \mathrm{e}^{2V} \ . \tag{2.3.42}$$

In the abelian case, these reduce to $W_\alpha = \frac{1}{8} \bar{D}^2 D_\alpha V$ and its conjugate.

Chapter 3

Five-Dimensional Simple Superspace

The superspace we are calling five-dimensional simple superspace is parameterized by the local co-ordinates of 4D, $\mathcal{N}=1$ superspace $\{x^a, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}\}$ together with one additional bosonic coordinate x^5 which parameterizes the fifth direction. The basis tangent vector in this direction is denoted as ∂_5 and is *central* in the sense that it commutes with every other tangent space generator. Contrary to the usual case, however, the 4D, $\mathcal{N}=1$ superspace generators form a closed algebra without it. In the jargon: the central charge is not gauged.

3.1 Classical Field Theories in Five-Dimensional Simple Superspace

Since we are simply extending the superspace of the previous section by an extra bosonic direction, we will find the following for the analogues of the theories above: We consider, of course, irreducible representations of the five-dimensional super-Lorentz group. However, since we are keeping manifest only the "four-dimensional" part, they will be composed of two multiplets of the previous section. In this sense the representation is reducible. However, it clearly cannot be reducible since the two “four-dimensional” superfields must transform into each other under a “second” supersymmetry (in addition to five-dimensional Lorentz transformations).

Since we are still in simple superspace, there is no new fermionic measure to define in order to write actions. Instead, we find the generic structure $S = S_0 + cS'$ for S_0 and S' actions of the previous section for some constant c. If the theory were truly four-dimensional with only $\mathcal{N} = 1$ supersymmetry, c would be an arbitrary real number. To find the precise number c one can either

1. project down to the components, assemble the components into irreducible five-dimensional multiplets, and read off the resulting value of c (not recommended),

2. take the equations of motion resulting from S, hit with appropriate combinations of covariant derivatives, use the constraints, and write the Klein-Gordon or Dirac equations for the resulting superfield strengths, thereby fixing the value of c by purely superspace methods.

The former approach was used in [7] to derive a linearized form of five-dimensional supergravity. The latter approach was described in [9] and used to show the same thing by more direct methods.

Let us now turn to the description of some five-dimensional matter multiplets in terms of this superspace.

3.1.1 The Scalar Multiplet I: The Chiral Non-Minimal Case

The closest analogue to the 4D scalar multiplet is most naturally described using a chiral field $\Phi = -\frac{1}{4}\bar{D}^2\bar{\varphi}$ and a complex linear field $\Gamma = D^\alpha\psi_\alpha - \partial_5\varphi$. They obey the Bianchi identities

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \ , \ \ -\frac{1}{4}D^2\Gamma + \partial_5\bar{\Phi} = 0 \tag{3.1.1}$$

and are both dimension one field strengths. Their Lagrangian is given by

$$\int d^4\theta \left(\bar{\Phi}\Phi - \bar{\Gamma}\Gamma\right) \ , \tag{3.1.2}$$

and the resulting equations of motion are

$$E[\Phi] = -\frac{1}{4}\bar{D}^2\bar{\Phi} - \partial_5\Gamma \ , \ E[\psi] = -D_\alpha\bar{\Gamma} \tag{3.1.3}$$

Taking $-\frac{1}{4}\bar{D}^2\bar{E}[\Phi] = 0$ and using both Bianchi identities (3.1.1), we find on-shell that $(\Box + \partial_5^2)\Phi = 0$.[1] Conversely, taking $-\frac{1}{4}\bar{D}^2$ on the second Bianchi identityin (3.1.1) and substituting the equations of motion (3.1.3), we find $(\Box + \partial_5^2)\Gamma = 0$. Hence, this theory describes the dynamics of the component fields

$$a + ib \sim \Phi| \ , \ \lambda_\alpha^{(+)} \sim D_\alpha\Phi| \ , \ x + iy \sim \Gamma| \ , \ \lambda_\alpha^{(-)} \sim D_\alpha\Gamma| \ . \tag{3.1.4}$$

[1] In fact, this how the relative coefficient in the Lagrangian (3.1.2) was found. Notice that it is unnecessary to project to component fields in order to fix this coefficient.

This multiplet is self-dual. In the limit $\partial_5 \to 0$, this multiplet becomes a 4D, $N = 2$ scalar multiplet discovered in reference [13] dubbed the "chiral non-minimal" (CNM) multiplet.

3.1.2 The Scalar Multiplet II: The Fayet-Sohnius Case

An alternative description of the five-dimensional scalar is necessarily on-shell in the case in which we switch off the central charge.[2] It is related by a Lorentz violating duality to the CNM scalar above. In particular, we switch the equation of motion and constraint for the complex linear field Γ only. This gives a description in terms of two chiral fields $\Phi^{\underline{1}} = \Phi_{\underline{2}}$ and $\Phi^{\underline{2}} = -\Phi_{\underline{1}}$.[3] In writing an action for this representation, one cannot avoid introducing naked ∂_5 derivatives and the Lagrangian is given by:

$$\sum_{i=1,2} \left\{ \int \mathrm{d}^4\theta \bar{\Phi}_i \Phi^i + \frac{1}{2} \int \mathrm{d}^2\theta \Phi^i \partial_5 \Phi_i + \frac{1}{2} \int \mathrm{d}^2\bar{\theta} \bar{\Phi}_i \partial_5 \bar{\Phi}^i \right\} . \tag{3.1.5}$$

Here $\bar{\Phi}_i = (\Phi^i)^\dagger$. The resulting equations of motion are

$$E[\Phi^{\underline{1}}] = -\frac{1}{4}\bar{D}^2\bar{\Phi}_{\underline{1}} + \partial_5\Phi_{\underline{1}} \quad ; \quad E[\Phi^{\underline{2}}] = -\frac{1}{4}\bar{D}^2\bar{\Phi}_{\underline{2}} - \partial_5\Phi_{\underline{2}} . \tag{3.1.6}$$

[2]By the way in which the central charge is related to the translation generator in six dimensions, it follows that we cannot use this multiplet to write an off-shell theory for the six-dimensional scalar.

[3]Spinor and iso-spinor indices are raised and lowered using the anti-symmetric symbol ε which is normalized such that $\varepsilon^{12} = +1$.

Together with the chirality constraints these equations imply that the Φ^i obey the Klein-Gordon equation and that at the component level they give the same spectrum as the CNM theory (3.1.4)

$$a + ib \sim \Phi^{\underline{1}}| \ , \ \lambda_\alpha^{(+)} \sim D_\alpha \Phi^{\underline{1}}| \ , \ x + iy \sim \Phi^{\underline{2}}| \ , \ \lambda_\alpha^{(-)} \sim D_\alpha \Phi^{\underline{2}}| \ . \tag{3.1.7}$$

In the centrally extended algebra, the chiral fields Φ^i are constrained by the central charge to satisfy

$$\mathrm{i}\triangle\Phi_{\underline{1}} = E[\Phi^{\underline{1}}] \quad , \quad \mathrm{i}\triangle\Phi_{\underline{2}} = E[\Phi^{\underline{1}}] \ . \tag{3.1.8}$$

This fact is a manifestation of the statement that without central charges in the algebra, the Fayet-Sohnius hypermultiplet is on-shell.

3.1.3 The Vector Multiplet

The simple superspace representation for the vector multiplet is given by a real scalar field V and a chiral scalar Φ. The real part of the latter field carries the fifth component of the gauge field sitting in V at the $\bar{\theta}\theta$ level. The gauge transformations of these prepotentials are of the form

$$\begin{aligned} \delta V &= \frac{1}{2\mathrm{i}}\left(\Lambda - \bar{\Lambda}\right) \ , \\ \delta\Phi &= \frac{1}{2\mathrm{i}}\partial_5\Lambda \ . \end{aligned} \tag{3.1.9}$$

The following field strengths are invariant under these transformations

$$
\begin{aligned}
W_\alpha &= \frac{1}{8}\bar{D}^2 D_\alpha V \ , \\
F &= \frac{1}{2}\left(\Phi + \bar{\Phi} + \partial_5 V\right) \ ,
\end{aligned}
\tag{3.1.10}
$$

and satisfy the Bianchi identities

$$
\begin{aligned}
D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} = 0 \ , \\
-\frac{1}{4}\bar{D}^2 D_\alpha F + \partial_5 W_\alpha = 0 \ .
\end{aligned}
\tag{3.1.11}
$$

The Lagrangian is given by

$$
\frac{1}{4}\left\{\int d^2\theta W^\alpha W_\alpha + \text{h.c.}\right\} + \int d^4\theta F^2 \ , \tag{3.1.12}
$$

and the resulting equations of motion are

$$
\begin{aligned}
E[V] &= D^\alpha W_\alpha + \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} - 2\partial_5 F \ , \\
E[\varphi] &= -\frac{1}{4} D^2 F \ .
\end{aligned}
\tag{3.1.13}
$$

Both field strengths obey the Klein-Gordon equation and the component content of this theory is

$$
\begin{aligned}
\lambda_\alpha^{(+)} \sim W_\alpha| \ , \ f_{\alpha\beta} \sim D_{(\alpha} W_{\beta)}| \ , \\
\varphi \sim F| \ , \ \lambda_\alpha^{(-)} \sim D_\alpha F| \ , \ F_{\underline{a}5} \sim [D_\alpha, \bar{D}_{\dot{\alpha}}]F| \ .
\end{aligned}
\tag{3.1.14}
$$

As before, it is possible to obtain these results by superspace geometrical methods. This was done explicitly in [10].

3.1.4 Abelian Chern-Simons Theory

Using the results from the previous section, we can write down the action for abelian Chern-Simons theory. The characteristic form (potential) × (field strength) × (field strength) implies that there must be a term of the form $\int d^2\theta \Phi W^\alpha W_\alpha$. Making this covariant under the gauge transformations (3.1.9) we find (up to normalization which we choose to agree with that in the literature)

$$12g^2 L_{\text{CS}} = \int d^2\theta \Phi W^\alpha W_\alpha \quad + \quad \int d^4\theta V \left[F D^\alpha W_\alpha + 2(D^\alpha F) W_\alpha \right] \\ + \quad \text{c.c.} + \ldots \tag{3.1.15}$$

Here the ellipsis stands for terms which may be separately invariant. The only possibility, on dimensional grounds, is

$$\ldots = c \int d^4\theta F^3 \ , \tag{3.1.16}$$

for a real constant $c \in \mathbb{R}$. Although we can use the same reasoning as before to fix c, we will find this value in section 5.4.4 below by superspace dimensional reduction. (The answer will be $c = 4$.)

Chapter 4

Four- and Five-Dimensional Harmonic Superspace

The traditional harmonic superspace [11, 12] is a universal framework for

- 4D, $\mathcal{N} = 2$,
- 5D, $\mathcal{N} = 1$, and
- 6D, $\mathcal{N} = (1, 0)$

supersymmetry. In this section we consider only the first two possibilities explicitly but we do so in a uniform notation. Hatted bosonic indices run over five values $\hat{a} \in \{0, 1, 2, 3, 5\}$ while unhatted ones run over the 4D subspace $a \in \{0, 1, 2, 3\}$. Similarly, hatted spinor indices run over two undotted and two dotted values. Our conventions for the five-dimensional extension of harmonic superspace were developed recently in [10] and are collected in Appendix A.[1]

Using this convention, we can treat 5D and 4D simultaneously. For example, conventional 4D, $\mathcal{N} = 2$ superspace is described by the centrally extended covariant

[1]These conventions differ from those established previously by Zupnik [14] but are more closely related to the conventions of [2] to which we adhered in the previous sections.

derivative algebra

$$\left\{D^i_{\hat\alpha}, D^j_{\hat\beta}\right\} = -2\mathrm{i}\varepsilon^{ij}\left[(\gamma^{\hat c})_{\hat\alpha\hat\beta}\partial_{\hat c} + \varepsilon_{\hat\alpha\hat\beta}\triangle\right] \ . \tag{4.0.1}$$

If we want to restrict from 5D to 4D, we need only drop the hat on the vector indices and decompose hatted spinor indices. This results in the algebra (A.22).

To pass to harmonic superspace, we introduce the auxiliary variables $(u_i^\pm) \in$ SU(2) subject to the condition $u^{+i}u_i^- = 1$. This means that superfields are now defined over the conventional superspace multiplied with SU(2). This is not quite what we want. We further restrict all fields to have integer U(1) charge under the transformation $u_i^\pm \to \mathrm{e}^{\mathrm{i}\varphi}u_i^\pm$. Then the extended superspace becomes $\mathbb{R}^{1,4|4} \times S^2$ where $S^2 \approx \mathrm{SU}(2)/\mathrm{U}(1)$.

Contracting the covariant derivatives with $u_i^\pm$ gives

$$\begin{gathered}\left\{D^+_{\hat\alpha}, D^+_{\hat\beta}\right\} = 0 \quad ; \quad \left\{D^-_{\hat\alpha}, D^-_{\hat\beta}\right\} = 0 \\ \left\{D^+_{\hat\alpha}, D^-_{\hat\beta}\right\} = 2\mathrm{i}\left[(\partial\!\!\!/)_{\hat\alpha\hat\beta} + \varepsilon_{\hat\alpha\hat\beta}\triangle\right]\end{gathered} \tag{4.0.2}$$

Note the resulting similarity with the simple covariant derivative algebra (2.2.16). The first two equations imply the existence of subspaces analogous to the chiral subspaces of section 2.2. We will call a superfield $\Phi(x, \theta^\pm_{\hat\alpha}, u_i^\pm)$ *analytic* if it obeys

$$\mathcal{D}^+_{\hat\alpha}\Phi = 0 \ . \tag{4.0.3}$$

This definition is to be compared with that of a chiral field (2.2.23). There is one crucial difference with the chiral subspace of simple superspace. This has to do with the fact that there exists on harmonic superspace an anti-involution called smile-conjugation which fixes the analytic subspace. The *smile conjugation* operation is the natural involution on harmonic superspace acting as [12]

$$(u^{+i})\breve{} = -u_i^+ \quad ; \quad (u_i^-)\breve{} = u^{-i} \tag{4.0.4}$$

on the harmonics and as complex conjugation on numbers.

What about the calculus on the harmonic sphere? Introducing the harmonic derivatives [11][2]

$$\begin{aligned} D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} \ , \quad D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}} \ , \quad D^0 &= u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} \ , \\ [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm} \ , \qquad [D^{++}, D^{--}] &= D^0 \ , \end{aligned} \tag{4.0.5}$$

one can see that D^0 is the operator of harmonic U(1) charge, $D^0\Psi^{(p)}(z,u) = p\Psi^{(p)}(z,u)$. Note that in addition to the algebra (4.1.17), we further have

$$[D^{++}, D^-_{\hat{\alpha}}] = D^+_{\hat{\alpha}} \quad \text{and} \quad [D^{--}, D^+_{\hat{\alpha}}] = D^-_{\hat{\alpha}} \ . \tag{4.0.6}$$

Notice, that the D^{++} operator fixes the analytic subspace defined by all solutions to (4.0.3). Therefore, we can consider the subspace kerD^{++} of the analytic

[2]These representations are given in the so-called central basis and are short representations. In the analytic basis, it is the $D^+_{\hat{\alpha}}$ operator which becomes short and the harmonic derivatives acquire connections.

subspace *id est* analytic solutions to the equation

$$D^{++}\Phi = 0 \ . \tag{4.0.7}$$

These functions form a ring. We will refer to solutions of (4.0.7) as holomorphic (harmonic) superfields. This nomenclature is appropriate because by the first equation in (4.0.5) fields annihilated by D^{++} depend only on u^+.

Fields defined on in harmonic superspace depend on the S^2 harmonics u. As before, we want a projection from this space to the real numbers so that we can define actions and, as before, we want the result to be a singlet. This is easily done for the harmonic sector by averaging over the 2-sphere. We normalize the integral in the standard way

$$\int \mathrm{d}u 1 = 1 \quad , \quad \int \mathrm{d}u(\mathrm{non}-\mathrm{singlet}) = 0 \tag{4.0.8}$$

Harmonic superspace has, in addition many fermionic dimensions. Let us define the operators

$$(D^{\pm})^2 = D^{\pm\hat{\alpha}} D^{\pm}_{\alpha} = D^{\pm\alpha} D^{\pm}_{\alpha} - D^{\pm}_{\dot{\alpha}} D^{\pm\dot{\alpha}} \ , \tag{4.0.9}$$

and

$$(D^{\pm})^4 = -\frac{1}{32}(\hat{D}^{\pm})^2(\hat{D}^{\pm})^2 \ . \tag{4.0.10}$$

In terms of these, the full fermionic measure is given similarly to the simple superspace case as

$$\int \mathrm{d}u \int \mathrm{d}^8\zeta L = \int \mathrm{d}u (\hat{D}^-)^4 (\hat{D}^+)^4 L \Big| \ , \tag{4.0.11}$$

for an arbitrary (unconstrained) harmonic superfield L.

More useful to us will be the measure on the analytic subspace. Let $L^{(+4)}$ be a U(1)-charge 2 *analytic* superfield. Then the following integral is manifestly supersymmetric

$$\int \mathrm{d}u \int \mathrm{d}\zeta^{(-4)} L^{(+4)} = \int \mathrm{d}u (\hat{D}^-)^4 L^{(+4)} \Big| \ . \tag{4.0.12}$$

Finally, suppose that, in addition to being harmonic, L^{++} is *holomorphic* in u^+, that is, it satisfies both (4.0.3) and (4.0.7). Then, as was shown in [15], the integral

$$\int \mathrm{d}u \int \mathrm{d}\zeta^{(-2)} L^{++} = \int \mathrm{d}u (\hat{D}^-)^2 L^{++} \Big| \quad : \quad D^{++} L^{++} = 0 \ , \tag{4.0.13}$$

is supersymmetric.

4.1 Classical Field Theories in Harmonic Superspace

We are now in a position to write down the harmonic superspace analogues of the theories considered in the previous sections.

4.1.1 The Scalar Multiplet I: The q^+-Hypermultiplet

Consider the harmonic five-dimensional algebra without central charge. Then the off-shell analogue of the CMN multiplet of section 3.1.1 is given by a charge $+1$ analytic superfield q^+. Since the operator D^{++} takes the analytic subspace into itself, $D^{++}q^+$ is again an analytic superfield. Therefore, we can make the smile-real action

$$\int \mathrm{d}u\mathrm{d}\zeta^{(-4)}\breve{q}^+ D^{++} q^+ \ . \tag{4.1.14}$$

The equation of motion which follows from this action is

$$E[\breve{q}^+] = D^{++}q^+ \ , \tag{4.1.15}$$

so that q^+ is holomorphic on-shell. The unique solution to this condition is $q^+ = u^+_{\underline{1}} q^{\underline{1}} + u^+_{\underline{2}} q^{\underline{2}}$.

4.1.2 The Scalar Multiplet II: The Fayet-Sohnius Hypermultiplet

Now consider the five-dimensional algebra with central charge $\triangle$. Consider, again, a charge $+1$ analytic superfield q^+ but this time impose the holomorphicity condition $D^{++}q^+ = 0$ off-shell, *id est* as a constraint. Then the action considered in the previous section (4.1.14) vanishes identically. However, since we now have a central charge in the algebra, we may consider the charge $+2$ analytic superfield

$L^{++} = \frac{1}{2}\breve{q}^{+} \overset{\leftrightarrow}{\triangle} q^{+}$. By the constraint imposed on the field, we see that L^{++} is in addition holomorphic, that is $D^{++}L^{++} = 0$. Therefore, the action

$$\frac{1}{2}\int \mathrm{d}u \int \mathrm{d}\zeta^{(-2)} \breve{q}^{+} \overset{\leftrightarrow}{\triangle} q^{+} \tag{4.1.16}$$

is supersymmetric.

4.1.3 The Vector Multiplet

The analogue of the constraints (2.3.38-2.3.40) in harmonic superspace are [11, 12]

$$\begin{aligned} \{\mathcal{D}^{+}_{\hat\alpha}, \mathcal{D}^{+}_{\hat\beta}\} &= 0 \ , \qquad [\mathcal{D}^{++}, \mathcal{D}^{+}_{\hat\alpha}] = 0 \ , \\ \{\mathcal{D}^{+}_{\hat\alpha}, \mathcal{D}^{-}_{\hat\beta}\} &= 2\mathrm{i}\Big(\mathcal{D}_{\hat\alpha\hat\beta} + \varepsilon_{\hat\alpha\hat\beta}(\Delta + \mathrm{i}\mathcal{W})\Big) \ , \\ [\mathcal{D}^{++}, \mathcal{D}^{-}_{\hat\alpha}] &= \mathcal{D}^{+}_{\hat\alpha} \ , \qquad [\mathcal{D}^{--}, \mathcal{D}^{+}_{\hat\alpha}] = \mathcal{D}^{-}_{\hat\alpha} \ . \end{aligned} \tag{4.1.17}$$

In the so-called λ-frame, the $\mathcal{D}^{+}$ derivatives are short while the harmonic derivatives acquire connections

$$\mathcal{D}^{++} = D^{++} + \mathrm{i}V^{++} \ . \tag{4.1.18}$$

The connection V^{++} is suffers the gauge transformation

$$\delta V^{++} = -\mathcal{D}^{++}\lambda = -D^{++}\lambda - \mathrm{i}\left[V^{++}, \lambda\right] \ . \tag{4.1.19}$$

The smile-real connection V^{++} is seen to be an analytic superfield, $D^+_{\hat\alpha} V^{++} = 0$, of harmonic U(1) charge plus two, $D^0 V^{++} = 2V^{++}$ by virtue of the algebra (4.1.17). The other harmonic connection V^{--} turns out to be uniquely determined in terms of V^{++} using the zero-curvature condition

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = D^0 \quad \Leftrightarrow \quad D^{++}V^{--} - D^{--}V^{++} + \mathrm{i}[V^{++}, V^{--}] = 0 \ , \tag{4.1.20}$$

as demonstrated in [16]. The result is

$$V^{--}(z,u) = \sum_{n=1}^{\infty} (-\mathrm{i})^{n+1} \int \mathrm{d}u_1 \ldots \mathrm{d}u_n \frac{V^{++}(z,u_1)V^{++}(z,u_2)\cdots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_n^+u^+)} \ , \tag{4.1.21}$$

with $(u_1^+u_2^+) = u_1^{+i} u^+_{2\,i}$, and the harmonic distributions on the right of (4.1.21) defined *exempli gratia* in [12].

As far as the connections $V^-_{\hat\alpha}$ and $V_{\hat a}$ are concerned, they can be expressed in terms of V^{--} with the aid of the (anti-)commutation relations (4.1.17). In particular, one obtains

$$\mathcal{W} = \frac{\mathrm{i}}{8}(\hat{D}^+)^2 V^{--} \ . \tag{4.1.22}$$

Therefore, V^{++} is the single unconstrained analytic prepotential of the theory. With the aid of (4.1.20) one can obtain the following useful expression

$$\mathcal{W} = \frac{\mathrm{i}}{8}\int \mathrm{d}u (\hat{D}^-)^2 V^{++} \ + \ O\Big((V^{++})^2\Big) \ . \tag{4.1.23}$$

In the Abelian case, only the first term on the right survives.

The field strength $\mathcal{W}$ satisfies the constraint

$$\mathcal{D}^+_{\hat\alpha}\mathcal{D}^+_{\hat\beta}\mathcal{W} = \frac{1}{4}\varepsilon_{\hat\alpha\hat\beta}(\hat{\mathcal{D}}^+)^2\mathcal{W} \quad \Rightarrow \quad \mathcal{D}^+_{\hat\alpha}\mathcal{D}^+_{\hat\beta}\mathcal{D}^+_{\hat\gamma}\mathcal{W} = 0 \ , \tag{4.1.24}$$

as a result of the algebra (4.1.17). Using this one can readily construct a covariantly analytic descendant of $\mathcal{W}$

$$-\,\mathrm{i}G^{++} = \mathcal{D}^{+\hat\alpha}\mathcal{W}\mathcal{D}^+_{\hat\alpha}\mathcal{W} + \frac{1}{4}\{\mathcal{W}, (\hat{\mathcal{D}}^+)^2\mathcal{W}\} \ , \tag{4.1.25}$$

which is analytic $\mathcal{D}^+_{\hat\alpha}G^{++} = 0$ and holomorphic $\mathcal{D}^{++}G^{++} = 0$. It follows that we may take $L^{++}_{\mathrm{YM}} = \frac{1}{4}\mathrm{tr}G^{++}$ and construct the super-Yang-Mills action in harmonic superspace as

$$\frac{1}{4}\int \mathrm{d}u \int \mathrm{d}\zeta^{(-2)} G^{++} \ . \tag{4.1.26}$$

Henceforth, we will consider for simplicity the case of abelian $\mathcal{W}$. Then the equation of motion resulting from the action (4.1.26) is given by

$$E[V^{--}] = (\hat{D}^+)^2\mathcal{W} \ . \tag{4.1.27}$$

It follows from this that the component content of $\mathcal{W}$ is given by the set

$$\varphi \sim \mathcal{W}| \quad , \quad \Psi^\pm_{\hat\alpha} \sim \mathcal{D}^\pm_{\hat\alpha}\mathcal{W}| \quad , \quad F_{ab} \sim (D^+\Sigma_{ab}D^-)\mathcal{W}| \ . \tag{4.1.28}$$

4.1.4 Abelian Chern-Simons Theory

From the analytic descendent G^{++} (4.1.25) we can immediately propose a candidate action form Chern-Simons theory:

$$\frac{1}{12}\int \mathrm{d}u \int \mathrm{d}\zeta^{(-4)} V^{++} G^{++} \ . \tag{4.1.29}$$

This is supersymmetric since V^{++} is also analytic and gauge invariant under the abelian form of (4.1.19) because G^{++} is holomorphic.

Chapter 5

Projective Superspace and Dimensional Reduction

In the previous chapter, harmonic superspace was introduced to solve the constraints (4.0.1). As shown in reference [4], this formalism may be reduced to the so-called "projective" superspace [17] by allowing the harmonic superfields to acquire isolated singularities on the sphere. In this section we review this construction.

5.1 Singular Harmonic Superfields

The harmonic sphere is more properly thought of as the one-complex-dimensional projective space $\mathbb{C}P^1 \subset \mathbb{C}^2$. Its simplest atlas consists of two co-ordinate charts to which we will refer as the northern and southern patches in which, respectively, $u^{+\underline{2}} \neq 0$ and $u^{+\underline{1}} \neq 0$. Define on these patches the standard stereographic co-ordinates $z = u^{+\underline{1}}/u^{+\underline{2}}$ and $w = u^{+\underline{2}}/u^{+\underline{1}}$ such that z is a global co-ordinate on the northern patch, w is a global co-ordinate on the southern patch and the two are related on the intersection of these patches by $z = 1/w$ (see figure 5.1).

A smooth function f on the harmonic sphere has, by definition, a well-defined

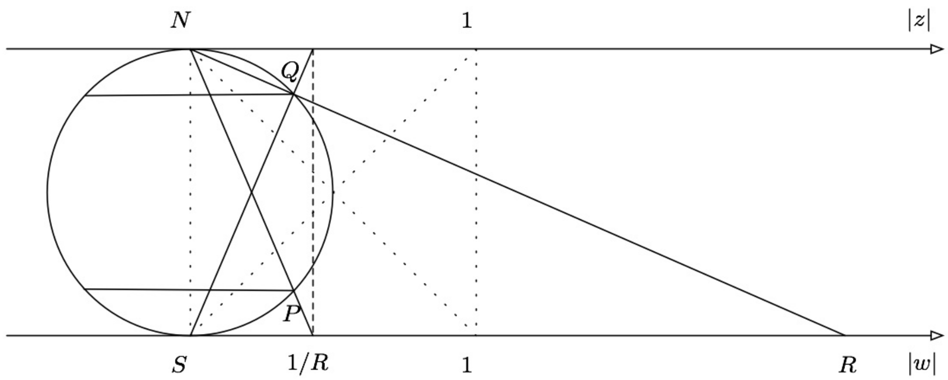

Figure 5.1: (a) The Projective Co-ordinates of $\mathbb{C}P^1$: The sphere is chosen to have radius 1/2 such that the equator is described by the same equation in both charts, namely $|z| = 1 = |w|$. The dotted lines serve to illustrate this. The circle of latitude passing throug the point Q is described in the southern chart by the equation $|w| = R$ and in the northern chart by $|z| = 1/R$ as the hyphenated line is meant to illustrate. (b) The band between the circles of latitude through the points P and Q is the region to which our cut-off functions restrict the original function.

U(1) charge q and a convergent harmonic expansion

$$f(u) \;=\; \sum_{n=0}^{\infty} f^{(i_1 \ldots i_q j_1 k_1 \ldots j_n k_n)} u^{\pm}_{i_1} \ldots u^{\pm}_{i_q} (u^{+}_{j_1} u^{-}_{k_1}) \ldots (u^{+}_{j_1} u^{-}_{k_n}) \qquad (5.1.1)$$

where the upper (lower) signs are chosen for $q > 0$ ($q < 0$). Now suppose that $F(z, u)$ is a harmonic superfield with, for definiteness, non-negative charge q. In local co-ordinates $F(z, u) = (u^{+\underline{1}})^q f(z, w, \bar{w})$. Suppose further that $\mathcal{D}^{++} F = 0$. Then it follows that $\frac{\partial}{\partial \bar{w}} f(z, w, \bar{w}) = 0$ and $f(z, w)$ is a polynomial of order q in w

with superfield-valued coefficients. Indeed, it is easy to check explicitly that in local coordinates this operator is given by

$$\mathcal{D}^{++} \;=\; (u^{+\underline{1}})^2(1+\bar{w}w)^2\frac{\partial}{\partial\bar{w}}\ . \tag{5.1.2}$$

That we get such a finite polynomial has to do with the fact that we assumed F to be globally defined on the sphere.[1] If we relax this assumption and allow F to become singular at the north pole $z = 0 \leftrightarrow w = \infty$ then we would find the more general solution $\mathbf{f}(z,w)$ for $f(z,w)$ which is an infinite power series in w

$$\mathbf{f}(z,w) \;=\; \sum_{n=-\infty}^{\infty} f_n(z)w^n \tag{5.1.3}$$

Functions of this form are well-defined on the punctured complex plane $\mathbb{C}^*$. We will refer to this space as the (doubly-)punctured sphere $S^2\backslash\{N\cup S\}$ since this space is the intersection of the northern and southern patches.

The so-called projective covariant derivative $\nabla_{\hat{\alpha}}$ is defined [17] such that $\nabla_\alpha := w\mathcal{D}^{\underline{1}}_\alpha - \mathcal{D}^{\underline{2}}_\alpha$ and $\nabla_{\dot{\alpha}} := \bar{\mathcal{D}}_{\underline{1}\dot{\alpha}} + w\bar{\mathcal{D}}_{\underline{2}\dot{\alpha}}$.[2] A holomorphic superfield $\mathbf{f}(z,w)$ with a well-defined power series expansion (5.1.3) which satisfies $\nabla_{\hat{\alpha}}\mathbf{f} = 0$ is called a *projective*

[1] The statement that a harmonic superfield $F(z,u)$ with non-negative U(1) charge q is globally defined is equivalent to the statement that in local co-ordinates $\lim_{|w|\to\infty} w^{-q}F(z,w,\bar{w})$ is smooth.

[2] These objects are usually defined in the context of 4D, $\mathcal{N}=2$ superspace. Here we consider the extension to 5D superspace but will continue to use the 4D nomenclature.

superfield. It follows from the relations

$$\mathcal{D}^+_\alpha = -u^{+\underline{1}}\nabla_\alpha \quad ; \quad \mathcal{D}^+_{\dot\alpha} = -u^{+\underline{1}}\bar\nabla_{\dot\alpha} \tag{5.1.4}$$

that if $F(z,u)$ is an analytic superfield in the harmonic sense satisfying $\mathcal{D}^{++}F = 0$, then its associated holomorphic (possibly singular) counterpart $\mathbf{f}(z,w)$ is projective.

Let us pause to consider some specific examples of such fields. The projective analogue of the q^+-hypermultiplet of section 4.1.1 is the so-called *polar* mutiplet $(\Upsilon, \breve\Upsilon)$ with

$$\Upsilon(z,w) = \sum_{n=0}^{\infty} \Upsilon_n(z) w^n \quad \text{and} \quad \breve\Upsilon(z,w) = \sum_{n=0}^{\infty} (-)^n \bar\Upsilon_n(z) \frac{1}{w^n} \tag{5.1.5}$$

These superfields are also known as *arctic* and *antarctic* superfields respectively. If the series terminates at some finite order k in w, the multiplet is referred to as a *complex* O(k) *multiplet*. Note from the definition of the co-ordinate w that the smile-conjugation (4.0.4) induces an involution on the punctured sphere acting as $\breve w = -1/w$. That this multiplet arises as the projective reduction of the q^+ hypermultiplet of section 4.1.1 can be seen as follows. The general solution to the holomorphic iso-spinor equation $\mathcal{D}^{++}q^+ = 0$ on the harmonic sphere is $q^+(z,u) = u^{+\underline{1}}[\Phi(z) + w\Gamma(z)]$. However, allowing for the more general solution singular at the south pole S, we find

$$\mathbf{q}^+ \;=\; u^{+\underline{1}}\Upsilon(z,w) \tag{5.1.6}$$

where $\Upsilon(z,w)$ is an arctic superfield. Note that the smile conugate of $\mathbf{q}^+$ is described by an antarctic superfield which is the smile conjugate of Υ. Therefore, although the superfields are well-defined on all of $\mathbb{C}$, the multiplet $(\mathbf{q}^+, \breve{\mathbf{q}}^+)$ is defined only on the doubly punctured sphere.

The next example is that of a *tropical* multiplet $V(z,w)$. It is of the form

$$V(z,w) = \sum_{n=-\infty}^{\infty} V_n(z) w^n \quad \text{with} \quad V_{-n} = (-)^n \bar{V}_n \tag{5.1.7}$$

Note that it is real with respect to smile conjugation forcing it to have antipodal singularities on the sphere. Therefore, this multiplet is also defined only on $\mathbb{C}^*$. When the series terminates at the k^{th} order in w and $1/w$ the multiplet is called a *real* O$(2k)$ *multiplet*. The tropical multiplet describes the five-dimensional Yang-Mills superfield provided it is defined up to the gauge symmetry

$$\delta V = \frac{\Lambda - \breve{\Lambda}}{2\mathrm{i}} \tag{5.1.8}$$

with $(\Lambda, \breve{\Lambda})$ a polar multiplet. It arises from the smile-real analytic superfield V^{++} of section 4.1.3 similarly to the case considered above: The solution to the constraint $\mathcal{D}^{++}V^{++} = 0$ reads $V^{++} = (u^{+\underline{1}})^2 v(z,w)$ for v again a finite polynomial in w. Since V^{++} is real with respect to smile conjugation, as is the combination $(\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})$, we instead define (again, passing to a singular version of this field)

$$\mathbf{V}^{++} \quad = \quad (\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})V(z,w) \tag{5.1.9}$$

It is easy to check that the resulting superfield $V(z,w)$ is the tropical multiplet.

5.2 The Regularization of the Singular Theories

To complete the reduction of theories described by these multiplets it remains to consider the reduction of the Lagrangian description of their dynamics. As we are allowing singular fields on the harmonic sphere, we require a regularization procedure in order to make sense of these Lagrangians. We will perform this regularization with a smooth cutoff function $F_{R,\epsilon}(x)$ sketched in figure 5.2. This function may be explicitly constructed by defining the auxiliary function

$$f_{R,\epsilon}(x) := \begin{cases} \exp\left(\frac{1}{x-(R+\epsilon)} - \frac{1}{x-R}\right) & \text{for } x \in [R, R+\epsilon], \\ 0 & \text{for } x \in [0, R] \cup [R+\epsilon, \infty) \ , \end{cases} \tag{5.2.10}$$

and making the combination

$$F_{R,\epsilon}(x) = \int_x^{R+\epsilon} \mathrm{d}t \ f_{R,\epsilon}(t) \Big/ \int_R^{R+\epsilon} \mathrm{d}t \ f_{R,\epsilon}(t) \ . \tag{5.2.11}$$

This function extrapolates smoothly from unit magnitude to zero in a small region between R, which is assumed to be large number, and $R+\epsilon$ where ϵ is small. The derivative of this function, which will be important later, localizes whatever it multiplies to this region and is normalized so that in the limit

$$\lim_{\epsilon \to 0} F_{R,\epsilon}(x) = -\delta(x-R) \tag{5.2.12}$$

as a distribution and has the property that

$$[F_{R,\epsilon}(x)]^2 = F_{R,\epsilon}(x) \ . \tag{5.2.13}$$

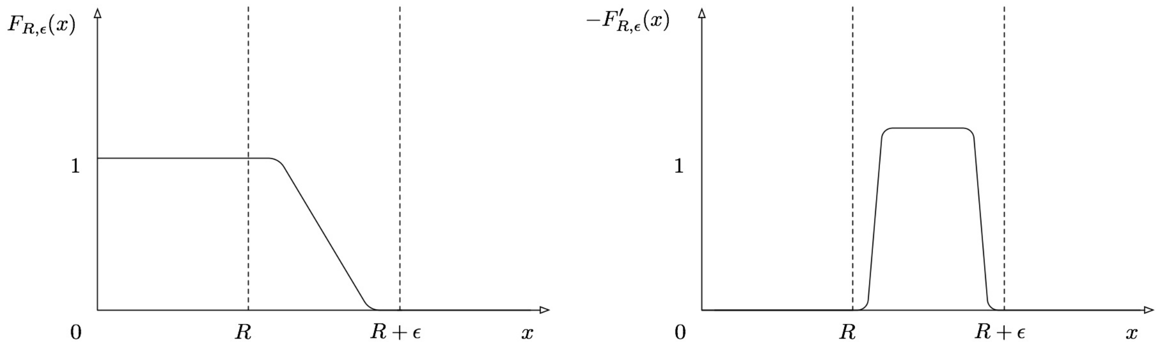

Figure 5.2: The function $F_{R,\epsilon}(x)$ smoothy interpolates from 1 to 0 in the region of width ϵ starting at R. It's derivative is a bump function with support $[R, R+\epsilon]$ and unit area.

A general singular superfield $\Phi(z,u)$ with a singularity at the south pole $z = 0 \leftrightarrow w = \infty$ can be regularized by multiplying with $F_{R,\epsilon}$ to make

$$\Phi(z,u) \to \Phi_{R,\epsilon}(z,w,\bar{w}) = F_{R,\epsilon}(|w|)\Phi(z,w,\bar{w}) \tag{5.2.14}$$

The smile conjugate of this is

$$\breve{\Phi}(z,u) \to \breve{\Phi}_{R,\epsilon}(z,w,\bar{w}) = F_{R,\epsilon}(|w|^{-1})\breve{\Phi}(z,w,\bar{w}) \tag{5.2.15}$$

so that now the singularity of $\breve{\Phi}$ at the north pole is cut off. Applying this to (the singular extension of) the q^+-multiplet, we have

$$\begin{aligned} \mathbf{q}^+(z,u) &\to q^+_{R,\epsilon}(z,w,\bar{w}) = (u^{+\underline{1}})F_{R,\epsilon}(|w|)\Upsilon(z,w) \\ \breve{\mathbf{q}}^+(z,u) &\to \breve{q}^+_{R,\epsilon}(z,w,\bar{w}) = (u^{+\underline{2}})F_{R,\epsilon}(|w|^{-1})\breve{\Upsilon}(z,w) \end{aligned} \tag{5.2.16}$$

If $\Phi(z,u)$ is real with respect to smile conjuation, then it necessarily has singularities at both poles and its regularization is given by replacing

$$\Phi(z,u) \to \Phi_{R,\epsilon}(z,w,\bar{w}) = F_{R,\epsilon}(|w|^{-1})\Phi(z,w)F_{R,\epsilon}(|w|) \tag{5.2.17}$$

The regularized form of the singular tropical superfield, then, is

$$\mathbf{V}^{++} \quad \to \quad V^{++}_{R,\epsilon} = (\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})F_{R,\epsilon}(|w|^{-1})V(z,w)F_{R,\epsilon}(|w|) \tag{5.2.18}$$

It is important to note that the projective superfields on the RHS of equations (5.2.16) and (5.2.18) are holomorphic in w. This is a consequence of the condition that the harmonic superfields from which they originated were annihilated by the $\mathcal{D}^{++}$ operator (recall equation (5.1.2) for example).

Let us pause to clarify the geometry of this construction. We have attempted to illustrate the situation in figure 5.1. The original function $g(u)$ is defined on the whole sphere S^2. It's projective counterpart is singular at the north pole N and/or the south pole S. Fix some large $R \gg 1$ and some small $\epsilon \ll 1$. Consider the circle of latitude passing through the point marked Q. In stereographic coordinates this circle is defined in the southern chart by the equation $|w| = R$ and in the northern chart by $|z| = 1/R$. Multiplying a function $g(u)$ by $F_{R,\epsilon}(|w|)$ restricts it to the region south of this circle. (In this picture we ignore the tiny difference between circles differing by ϵ in latitude.) Multiplying it instead with $F_{R,\epsilon}(|w|)$ restricts it to the region north of the circle of latitude passing through P. Multiplying by both, as will be the case for the Lagrangians considered below, restricts to the band between the circles passing through P and Q. The regulator is removed by taking first $\epsilon \to 0$ and finally $R \to \infty$. From the figure we see that the latter limit extends the band to all of S^2 except the points N and S.

5.3 The Reduced Actions

Before we can reduce the actions for the singular harmonic superfields described in chapter 4, we need to be able to reduce the analytic measure. In particular, we need the $(\hat{\mathcal{D}}^-)^2$ operator in local coordinates. Let us consider the form of this operator acting on analytic superfields $\Phi(z, u)$. The analyticity allows us to move all $D^{\underline{2}}_\alpha$ and $\bar{D}_{\underline{2}\dot{\alpha}}$ derivatives onto Φ and rewrite them in terms of $D_\alpha := D^{\underline{1}}_\alpha$ and $\bar{D}_{\dot{\alpha}} := \bar{D}_{\underline{2}\dot{\alpha}}$. When this is done, we find in local coordinates for an analytic superfield of arbitrary charge

$$(\hat{\mathcal{D}}^-)^2\Phi \quad = \quad -4(\overline{u^{+\underline{1}}})^2 \frac{(1+\bar{w}w)^2}{w} \Diamond \Phi \tag{5.3.19}$$

where the projective operator

$$\Diamond \quad := \quad \left[-\frac{1}{w}\left(-\frac{1}{4}\bar{D}^2\right) + \partial_5 + w\left(-\frac{1}{4}D^2\right)\right] \ . \tag{5.3.20}$$

Here we have repeatedly used the algebra of covariant derivatives (A.22) and the fact that $u^-_{\underline{1}} = \overline{u^{+\underline{1}}}$, $u^-_{\underline{2}} = \overline{u^{+\underline{1}}}\bar{w}$ and $|u^{+\underline{1}}|^2 = (1+\bar{w}w)^{-1}$. Recall that the operator

(5.3.19) is related to the analytic measure defined in (4.0.10) and (4.0.12) by

$$(\hat{D}^-)^4 \;=\; -\frac{1}{32}(\hat{\mathcal{D}}^-)^2(\hat{\bar{\mathcal{D}}}^-)^2 \; . \tag{5.3.21}$$

Using the analyticity of Φ again, it is easy to show that up to total derivatives

$$(\hat{D}^-)^4\Phi \;=\; (\overline{u^{+\underline{1}}})^4\frac{(1+\bar{w}w)^4}{w^2}D^4\Phi \tag{5.3.22}$$

with

$$D^4 \;:=\; \left(-\frac{1}{4}\bar{D}^2\right)\left(-\frac{1}{4}D^2\right) \tag{5.3.23}$$

We are now in a position to reduce the actions from chapter 4. The general action on the analytic subspace given in equation (4.0.12) is

$$S = \int \mathrm{d}^5x \int \mathrm{d}u(\hat{D}^-)^4 L^{(+4)}\Big| \tag{5.3.24}$$

where the harmonic integral is carried out over the entire S^2. Restricting to the doubly punctured sphere amounts to allowing the fields in $L^{(+4)}$ to become singular and regulating the integral as described above. At this point the Lagrangian is replaced with

$$L^{(+4)} \;\;\to\;\; L^{(+4)}_{R,\epsilon} = (\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})^2 F_{R,\epsilon}(|w|^{-1})L(z,w)F_{R,\epsilon}(|w|) \tag{5.3.25}$$

for a tropical superfield $L(z,w)$, the details of which depend on the theory under consideration. In the final stages we will remove the regulator by taking first $\epsilon \to 0$ and then $R \to \infty$.

In the projective coordinates the harmonic integral of a smooth function $f(u)$ of vanishing U(1) charge takes the form

$$\int \mathrm{d}u \mathcal{L}(u) = \frac{1}{\pi} \int \frac{\mathrm{d}\bar{w}\mathrm{d}w}{(1+\bar{w}w)^2} \mathcal{L}(w,\bar{w}) = \frac{1}{\pi} \int \frac{\mathrm{d}\bar{w}\mathrm{d}w}{w(1+\bar{w}w)} \frac{\partial \mathcal{L}}{\partial \bar{w}} \tag{5.3.26}$$

Taking this function to be $\mathcal{L} = (\hat{D}^{-})^4 L^{(+4)}|$ and using the explicit forms (5.3.23) and (5.3.25) we find, up to total derivatives

$$\mathcal{L}(w,\bar{w}) = F_{R,\epsilon}(|w|^{-1}) F_{R,\epsilon}(|w|) \left[D^4 L(z,w) \right] \Big| \tag{5.3.27}$$

and the integral becomes

$$\int \mathrm{d}u \mathcal{L}(u) = \frac{1}{\pi} \int \frac{\mathrm{d}\bar{w}\mathrm{d}w}{w(1+\bar{w}w)} \frac{\partial}{\partial \bar{w}} \left[F_{R,\epsilon}(|w|^{-1}) F(|w|) \right] D^4 L(z,w) \Big| \tag{5.3.28}$$

Note that it is crucial for this step that $L(z,w)$ is holomorphic. Switching to polar coordinates $w = \rho \mathrm{e}^{\mathrm{i}\varphi}$ and taking the $\epsilon \to 0$ limit, we find for this expression

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\infty} \frac{\mathrm{d}\rho}{1+\rho^2} \left[\frac{1}{\rho} \delta(\rho - 1/R) F(\rho) - F(\rho^{-1}) \delta(\rho - R) \right] & \\ \times D^4 L(\rho,\varphi) \Big| & \end{aligned} \tag{5.3.29}$$

We can now perform the ρ integral to obtain

$$\frac{1}{2\pi} \frac{R^2-1}{R^2+1} \int_0^{2\pi} \mathrm{d}\varphi D^4 L(R,\varphi) \Big| \tag{5.3.30}$$

The limit $R \to \infty$ is well defined precisely when the Lagrangian we started with is well-defined on the harmonic sphere. Switching to the holomorphic coordinate

$w = R\mathrm{e}^{\mathrm{i}\varphi}$ and taking this limit we find that for the action (5.3.24) with singular fields on the harmonic sphere

$$S = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^5 x \oint \frac{\mathrm{d}w}{w} D^4 L(z,w)\Big| \qquad (5.3.31)$$

in terms of the tropical superfield Lagrangian $L(z,w)$ defined by equation (5.3.25).

This then is the general form of the reduced action. For many purposes, however, it is convenient to have such an equation not for the full analytic superspace measure but rather for the special case considered in ref [15] and used in sections 4.1.2 and 4.1.3 for the Fayet-Sohnius hypermultiplet and the Yang-Mills gauge mutliplet respectively. In this case the Lagrangian is reduced to

$$S = \frac{\mathrm{i}}{4} \int \mathrm{d}^5 x \int \mathrm{d}u (\hat{\mathcal{D}}^-)^2 L^{++} \qquad (5.3.32)$$

with L^{++} satisfying

$$\mathcal{D}^+_{\hat{\alpha}} L^{++} = 0 \quad \text{and} \quad \mathcal{D}^{++} L^{++} = 0 \qquad (5.3.33)$$

Using the explicit form (5.3.19) and the obvious analogue of (5.1.9) for L^{++} we find for such theories, in complete analogy with the analysis above,

$$S = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^5 x \oint \frac{\mathrm{d}w}{w} \left[-\frac{1}{w}\left(-\frac{1}{4}\bar{D}^2\right) + w\left(-\frac{1}{4}D^2\right)\right] L(z,w)\Big| \qquad (5.3.34)$$

Let us give some explicit examples of these results from chapter 4.

5.4 Explicit Examples

5.4.1 Reduction of the q^+-Hypermultiplet

For clarity of exposition, we work with the free q^+-hypermultiplet of section 4.1.1. The action (4.1.14) was of the form $\int \breve{q}^+ D^{++} q^+$. Restricting the domain of the harmonic integral to the punctured sphere is implemented by replacing $(q^+, \breve{q}^+)$ with its regularized version (5.2.16). In the resulting overlap of the northern and southern patches, the D^{++} operator takes the form (5.1.2). Since this operator appears explicitly, the reduction of the harmonic integral does not happen automatically as we will be the case in all other examples. Nevertheless, the resulting integral

$$-\frac{1}{\pi}\int \mathrm{d}^5x \mathrm{d}w \mathrm{d}\bar{w} (|u^{+\underline{1}}|^2)^4 \frac{(1+|w|^2)^4}{w} F_{R,e}\left(\frac{1}{|w|}\right) \frac{\partial}{\partial \bar{w}} F_{R,\epsilon}(|w|) D^4 \left[\breve{\Upsilon}\Upsilon\right] \quad (5.4.35)$$

is trivial to do explicitly. Integrating the anti-holomorphic derivative by parts and removing the regulators, we obtain for this integral the beautiful projective superspace result

$$\frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^5 x \oint \frac{\mathrm{d}w}{w} D^4 \breve{\Upsilon}\Upsilon\Big| \,. \quad (5.4.36)$$

Expanding further $\Upsilon = \Phi + w\Gamma + \ldots$ and integrating out the infinite number of decoupled auxiliary fields gives precisely the action (3.1.2). Since Υ is a projective field, it follows that the the dynamical component fields Φ and Γ satisfy the constraints

(3.1.1). Therefore, all conclusions of section 3.1.1 hold for the dimensional reduction of the q^+-hypermultiplet.

5.4.2 Reduction of the Fayet-Sohnius Hypermultiplet

The Fayet-Sohnuis multiplet, although superficially resembling the q^+-hypermultiplet, is equivalent to the latter only on-shell. It satisfies $D^{++}q^+$ as a constraint and can therefore be written as a complex O(2) multiplet $q^+ = -u^{+\underline{1}}(\Phi_+ + w\bar{\Phi}_-^{\mathrm{T}})$ with $\Phi_\pm$ an SU(2) doublet of chiral fields. The Lagrangian $L^{++} = \frac{1}{2}\breve{q}^+ \overset{\leftrightarrow}{\triangle} q^+$ can be represented by a real O(2) multiplet $L^{++} = \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}(-\frac{1}{w}W + K + w\bar{W})$, for chiral W and real K. Explicitly, $W = \mathrm{i}L^{\underline{22}}$ and $K = -2\mathrm{i}L^{\underline{12}}$. Since this Lagrangian is also analytic, the action reduces to the form (5.3.34) for a homomorphic-analytic action in projective superspace which, in turn, can be integrated to give

$$\frac{1}{2}\int \mathrm{d}^2\theta W + \mathrm{h.c.} \tag{5.4.37}$$

The explicit form of W

$$W = -\frac{1}{4}\bar{D}^2\left(\bar{\Phi}_+\Phi_+ + \bar{\Phi}_-\Phi_-\right) - \Phi_-^{\mathrm{T}} \overset{\leftrightarrow}{\partial}_5 \Phi_+ \ , \tag{5.4.38}$$

follows from the constraints (3.1.8). The resulting Lagrangian is precisely (3.1.5).

5.4.3 Reduction of Yang-Mills Theory

As in the previous section, the Lagrangian of super-Yang-Mills theory, equation (4.1.26), satisfies $D^{++}L^{++} = 0$ and can be written as $L^{++} = \frac{1}{4}(\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})G(z,w)$ with $G(z,w)$ an O(2) multiplet

$$G(z,w) \;=\; -\frac{1}{w}W + K + w\bar{W} \tag{5.4.39}$$

where $W := \mathrm{i}L^{\underline{22}}$ is chiral in the four-dimensional sense and $K := -2\mathrm{i}L^{\underline{12}}$ is real. In terms of the field strengths defined in section 3.1.3,

$$\begin{aligned} W &= -W^{\alpha}W_{\alpha} + \frac{1}{2}\bar{D}^2(F^2) \\ K &= \{(FD^{\gamma} + 2D^{\gamma}F)W_{\gamma} + \text{h.c.}\} + 2\partial_5(F^2)\ , \end{aligned} \tag{5.4.40}$$

as can be checked explicitly. The action for this theory is defined using the reduced measure (5.3.32). The result (5.3.34) for this measure immediately gives

$$S_{\mathrm{YM}} \;=\; \int \mathrm{d}^5x \left\{ \frac{1}{4}\int \mathrm{d}^2\theta\ W^{\alpha}W_{\alpha} + \frac{1}{4}\int \mathrm{d}^2\bar{\theta}\ \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} + \int \mathrm{d}^4\theta\ F^2 \right\} \tag{5.4.41}$$

The relative factor of $\frac{1}{4}$ is not a convention and is necessary for five-dimensional Lorentz invariance (compare section 3.1.3).

5.4.4 Reduction of Chern-Simons Theory

The Chern-Simons Lagrangian (4.1.29) is of the type defined over the full analytic subspace (5.3.24). The singular charge-2 harmonic field gives rise *via* equation

(5.1.9) to a tropical field

$$V(z,w) \;=\; \ldots - \frac{1}{w}\varphi(z) + V(z) + w\bar{\varphi}(z) + \ldots \tag{5.4.42}$$

This field suffers the gauge transformation

$$\delta V(z,w) \;=\; \frac{1}{2\mathrm{i}}\Big(\Lambda(z,w) - \breve{\Lambda}(z,w)\Big) \tag{5.4.43}$$

with $\Lambda(z,w) = \Lambda(z) + w\Theta(z) + \ldots$ a polar field. This induces on the simple superfield components the transformations

$$\begin{aligned} \delta V(z) &= \frac{1}{2\mathrm{i}}\left(\Lambda(z) - \bar{\Lambda}(z)\right) \\ \delta\varphi(z) &= \frac{1}{2\mathrm{i}}\Theta(z) \; . \end{aligned} \tag{5.4.44}$$

In the reduced form, the theory will not depend on the field $\varphi(z)$ but rather it will be given in terms of its chiral projection. For this reason we introduce the simple chiral field

$$\Phi(z) \;=\; -\frac{1}{4}\bar{D}^2\bar{\varphi}(z) \; . \tag{5.4.45}$$

Using the result for the 5D analogue of the chiral non-minimal multiplet (3.1.1) above, it easily follows that Φ suffers the gauge transformation

$$\delta\Phi(x) = \frac{1}{2\mathrm{i}}\partial_5\Lambda(z) \; . \tag{5.4.46}$$

We conclude from this that the field strength $F(z)$ is given by

$$F \;=\; \frac{1}{2}\left(\Phi + \bar{\Phi} + \partial_5 V\right) \; , \tag{5.4.47}$$

up to a constant of proportionality. The constant of proportionality in this expression is found by normalizing

$$W_\alpha \;=\; \frac{1}{8}\bar{D}^2 D_\alpha V \;, \tag{5.4.48}$$

and satisfying the constraint (3.1.11) relating F and W_α.

Another "more honest" derivation is as follows. In the Abelian case, the gauge transformation (4.1.19) simplifies

$$\delta V^{++} = -D^{++}\lambda \;, \qquad D^+_{\hat\alpha}\lambda = 0 \;, \qquad \breve{\lambda} = \lambda \;. \tag{5.4.49}$$

The field strength (4.1.23) also simplifies

$$\mathcal{W} = \frac{\mathrm{i}}{8}\int \mathrm{d}u(\hat{D}^-)^2 V^{++} \;. \tag{5.4.50}$$

It is easy to see that $\mathcal{W}$ is gauge invariant.

The gauge freedom (5.4.49) can be used to choose the supersymmetric Lorentz gauge [11]

$$D^{++}V^{++} = 0 \;. \tag{5.4.51}$$

In other words, in this gauge V^{++} becomes a real O(2) multiplet,

$$V^{++} = \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}V(z,w) \;, \qquad V(z,w) = \frac{1}{w}\varphi(z) + V(z) - w\bar{\varphi}(z) \;. \tag{5.4.52}$$

Since $\mathcal{W}$ is gauge invariant, for its evaluation one can use any potential V^{++} from the same gauge orbit, in particular the one obeying the gauge condition (5.4.51). This

Lorentz gauge is particularly useful for our consideration. Using the result (5.3.19) for the holomorphic-analytic measure and recalling that $|u^{+\underline{1}}|^2 = (1+w\bar{w})^{-1}$, we can rewrite $\mathcal{W}$ in the form

$$\mathcal{W} = \frac{1}{2}\int \mathrm{d}u \diamond V(z,w) \ . \tag{5.4.53}$$

This can be further transformed to

$$\mathcal{W} = \frac{1}{4\pi \mathrm{i}} \oint \frac{\mathrm{d}w}{w} \diamond V(z,w) \ . \tag{5.4.54}$$

Indeed, the punctuation procedure of the foregoing sections 5.1-5.3 justifies the following identity

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int \mathrm{d}u\phi_{R,\epsilon}(u) = \frac{1}{2\pi\mathrm{i}}\oint \frac{\mathrm{d}w}{w}\phi(w) \ , \tag{5.4.55}$$

with the regularization $\phi_{R,\epsilon}(u) = \phi_{R,\epsilon}(w,\bar{w})$ of a holomorphic function $\phi(w)$ on $\mathbb{C}^*$ defined according to (5.2.17). Since the integrand on the right of (5.4.53) is, by construction, a smooth scalar field on S^2, we have

$$\int \mathrm{d}u \diamond V(z,w) = \lim_{R\to\infty}\lim_{\epsilon\to 0}\int \mathrm{d}u \diamond V_{R,\epsilon}(z,u) \ . \tag{5.4.56}$$

Now, we are in a position to evaluate the $\mathcal{N}=1$ field strengths (3.1.10) in terms of the prepotentials V_n. It follows from (5.3.19) that

$$F = \mathcal{W}| = \frac{1}{4\pi\mathrm{i}}\oint \frac{\mathrm{d}w}{w}\diamond V(w)\Big| = \frac{1}{2}\Big(\Phi + \bar{\Phi} + \partial_5 V\Big) \ , \tag{5.4.57}$$

where we have defined

$$\Phi = \frac{1}{4}\bar{D}^2 V_1| \ , \qquad V = V_0| \ . \tag{5.4.58}$$

The spinor field strength W_α is given by

$$W_\alpha(z) = D^2_{\alpha}\mathcal{W}| = \frac{1}{4\pi \mathrm{i}}\oint \frac{\mathrm{d}w}{w}\left([D^2_{\alpha}, \diamond] + \diamond D^2_{\alpha}\right) V(w)\Big| \ . \tag{5.4.59}$$

However, as $[D^2_{\alpha}, \diamond] = w\partial_5 D^1_{\alpha}$ and given that for any projective superfield $\phi(w)$ we have $D^2_{\alpha}\phi(w) = wD^1_{\alpha}\phi(w)$, this expression reduces to

$$W_\alpha = \frac{1}{4\pi \mathrm{i}}\oint \frac{\mathrm{d}w}{w}\left(\frac{1}{4}\bar{D}^2 D_\alpha\right) V(w)\Big| = \frac{1}{8}\bar{D}^2 D_\alpha V \ . \tag{5.4.60}$$

It can be seen that the gauge transformation (5.4.49) acts on the superfields in (5.4.58) as follows:

$$\delta V = \frac{1}{2\mathrm{i}}(\Lambda - \bar{\Lambda}) \ , \ \delta\Phi = \frac{1}{2\mathrm{i}}\partial_5\Lambda \ . \tag{5.4.61}$$

The approach presented in this section can be applied to reduce the supersymmetric Chern-Simons theory (4.1.29) to projective superspace. With G^{++} defined in (4.1.25) and the real O(2) multiplet $G(z, w)$ defined in (5.4.39) the Chern-Simons action becomes

$$12g^2 S_{\mathrm{CS}} = -\frac{1}{2\pi \mathrm{i}}\int \mathrm{d}^5x\mathrm{d}^4\theta \oint \frac{\mathrm{d}w}{w} V(w)G(w)\Big| \ . \tag{5.4.62}$$

It is then trivial to derive that the Lagrangian becomes

$$\int \mathrm{d}^2\theta \Phi W^\alpha W_\alpha + \int \mathrm{d}^4\theta V\left[F D^\alpha W_\alpha + 2(D^\alpha F)W_\alpha\right] + \mathrm{c.c.} + 4\int \mathrm{d}^4\theta F^3 \ . \tag{5.4.63}$$

This result is to be compared with that obtained in section 3.1.4.

For completeness, we also present here projective superspace extensions of the vector multiplet mass term and the Fayet-Iliopoulos term (their harmonic superspace form is given in [11]). The vector multiplet mass term is

$$-m^2 \int \mathrm{d}\zeta^{(-4)}(V^{++})^2 \longrightarrow \frac{m^2}{2\pi \mathrm{i}} \int \mathrm{d}^5x\mathrm{d}^4\theta \oint \frac{\mathrm{d}w}{w} V^2(w)\Big| \ . \tag{5.4.64}$$

The gauge invariant Fayet-Iliopoulos term is

$$\int \mathrm{d}\zeta^{(-4)} c^{++}V^{++} \longrightarrow -\frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^5x\mathrm{d}^4\theta \oint \frac{\mathrm{d}w}{w} c(w)V(w)\Big| \ , \tag{5.4.65}$$

where $c^{++} = c^{ij}u_i^+u_j^+$, with a constant real iso-vector c^{ij}. Defining $c^{++} = \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}c(w)$, with $c(w) = w^{-1}\bar{\xi}_{\mathbb{C}} + \xi_{\mathbb{R}} - w\xi_{\mathbb{C}}$, the FI action then reduces to

$$\xi_{\mathbb{R}} \int \mathrm{d}^5x\mathrm{d}^4\theta V + 2\mathrm{Re}\Big(\xi_{\mathbb{C}} \int \mathrm{d}^5x\mathrm{d}^2\theta\Phi\Big) \ . \tag{5.4.66}$$

So far the considerations in this section have been restricted to the Abelian case. It is necessary to mention that the projective superspace approach [17] can be generalized to provide an elegant description of 5D super Yang-Mills theories, which is very similar to the well-known description of 4D, $\mathcal{N} = 1$ supersymmetric theories. In particular, the Yang-Mills supermultiplet is described by a real Lie-algebra-valued tropical superfield $V(z, w)$ with the gauge transformation

$$\mathrm{e}^{V(w)} \ \rightarrow \ \mathrm{e}^{\mathrm{i}\breve{\Lambda}(w)}\mathrm{e}^{V(w)}\mathrm{e}^{-\mathrm{i}\Lambda(w)} \ , \tag{5.4.67}$$

which is the non-linear generalization of the Abelian gauge transformation (5.4.49). The hypermultiplet sector is described by an arctic superfield $\Upsilon(z, w)$ and its conjugate, with the gauge transformation

$$\Upsilon(w) \;\to\; \mathrm{e}^{\mathrm{i}\Lambda(w)}\Upsilon(w) \; . \tag{5.4.68}$$

The hypermultiplet gauge–invariant action is

$$S[\Upsilon, \breve{\Upsilon}, V] = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}w}{w} \int \mathrm{d}^5x \mathrm{d}^4\theta \breve{\Upsilon}(w) \mathrm{e}^{V(w)} \Upsilon(w) \quad . \tag{5.4.69}$$

Chapter 6

Closing Remarks

In this work, we have argued in favor of the use of harmonic superspace when dealing with (quantum) field theories in five and six dimensions. This formalism can be reduced to a five-dimensional version of projective superspace and therefore to simple superfields. The dimensional truncation to $\mathcal{N} = 2$ four-dimensional theories is then trivial to implement. The elucidation of this procedure was the goal of this work. The main motivation to study this formalism was an effort to extend the usefulness of the superspace formalism to include cases in which we would like to lower the number of dimensions from five (or six) to four or, alternatively, to embed a four-dimensional theory into a higher dimensional one.

In fact, harmonic superspaces of a more general type than the one considered here exist and have been employed in the literature. Some of these can be embedded in others as we have seen to be possible in the case of simple, projective and (the original) harmonic superspace. One example of such a harmonic superspace is that of 4D, $\mathcal{N} = 3$ superspace extended by the coset $\mathrm{SU}(3)/[\mathrm{U}(1) \times \mathrm{U}(1)]$. As explained in references [18] and [19], this harmonic superspace (among others) can be embedded in a bigger harmonic superspace in which the harmonic coordinates are formed by a so-called "pure spinor", usually denoted by λ^α, which is a twistor-like complex commuting Weyl spinor of SO(9, 1) subject to the constraint

$$\lambda\gamma^m\lambda = 0 \ . \tag{6.1}$$

This pure spinor is the main ingredient in the only method known to date of quantizing the superstring with manifest supersymmetry [18].[1] Indeed, the authors of [19] claim to derive harmonic superspaces for four-dimensional superspaces with $\mathcal{N} = 2$, 3, and 4 from the pure spinor superstring. This philosophy is very close to the one advocated here. However, there is still much work to be done in this area. A major open problem in the pure spinor superstring theory is our lack of understanding of the superspace measure. This measure is normalized to $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$ which can be interpreted as a partial cancellation of the harmonic and fermionic measures, the idea being that the 11 degrees of freedom carried by the pure spinor have cancelled 11 fermionic zero-modes leaving only five of the original 16. However, this measure is not manifestly supersymmetric since it involves integration over a proper subspace of the full superspace which is not obviously analytic in any gen-

[1] It is sometimes said that this quantization is not manifestly supersymmetric because it is necessary to solve the pure spinor constraint in terms of spinors of the U(5) subgroup of SO(9, 1) in order to compute the operator product of Lorentz currents. While this is true, it is also true that the result of this calculation is manifestly SO(9, 1) covariant.

eralized sense. As a consequence of this we cannot, for example, use it to provide a manifestly supersymmetric action for ten-dimensional supersymmetric Yang-Mills theory.[2] On the other hand, it is clear that there must be *some* proper subspace of the full superspace analogous to the chiral and analytic subspaces of 4D, $\mathcal{N} = 1$, 2, and 3 because otherwise the action would be forced to be of higher derivative type.[3]

Finding this subspace gives a clue about (and may even be equivalent to) finding the BRST operator for the ten-dimensional superstring. A study of the embedding of various superspaces into one another along the lines made explicit herein has the potential to shed light on these subspaces. Consider, for example, the case of five dimensions constructed above. Naively, the fermionic measure has eight real zero modes in it. However, due to the harmonics, we can find an analytic subspace of half this size. This subspace is the full superspace of four-dimensional theories with the minimal amount of supersymmetry as the equation for $(\hat{D}^{-})^4$ in analytic basis (5.3.22) shows explicitly. Another example of this is much more trivial: The chiral subspace of 4D, $\mathcal{N} = 1$ superspace is again the full 2D, $\mathcal{N} = (1,1)$ superspace.

[2]Notice that the question of the proper measure for a pure spinor superstring is equivalent to finding a formalism in which we can write a manifestly supersymmetric lagrangian description for 10D sYM.

[3]The analogue of this statement in Ectoplasmic Integration Theory [20, 21] is that the density projector for these theories must have "enough" terms in it which are independent of the spinor covariant derivatives.

The present situation with the higher harmonic superspaces is that they are not sufficient to describe off-shell theories with 16 or 32 real supercharges. Perhaps, turning the above reasoning around, we might try to build the higher superspaces by starting with the lower superspaces which would be the generalization of the analytic subspace.

Besides the pure spinor superstring, there exists another formulation for superstrings which keeps manifest part of the super-Poincaré symmetry equivalent to having manifest supersymmetry in compactifications to four or six dimensions [22]. This approach starts with superstrings in the RNS formalism and makes a (complicated) redefinition of the world sheet fields which splits the original RNS variables into a set of GS-like variables and a new set of RNS-like variables. The restriction to six or four dimensions comes from requiring that this split lines up with the split $\mathbb{R}^{9,1} \to \mathbb{R}^{3,1} \times \mathbb{R}^{6}$. This description of the critical superstring is appropriately called the "hybrid" formalism. The four-dimensional version seems to be very closely related to the pure spinor superstring. The more precise statement [22] is that it is related to a twistor-like GS description discovered by Sorokin, Tkach, Volkov, and Zheltukhin [23] in which the pure spinor constraint is replaced with the twistor constraint

$$\lambda\gamma^m\lambda = \Pi^m \ , \tag{6.2}$$

where Π^m is the GS world-sheet momentum, by a partial gauge fixing and additional field redefinitions. Furthermore, the pure spinor BRST operator $Q = \oint \lambda^{\hat{\alpha}} d_{\hat{\alpha}}$ decomposes as $Q = \oint (\lambda^\alpha d_\alpha - \bar{\lambda}_{\dot{\alpha}} \bar{d}^{\dot{\alpha}} + \ldots)$ in four-dimensional notation while the relation to the twistor-like string involves the gauge fixing $\lambda_\alpha = \mathrm{e}^{\mathrm{i}\rho} d_\alpha$ and $\bar{\lambda}_{\dot{\alpha}} = \mathrm{e}^{-\mathrm{i}\rho} \bar{d}_{\dot{\alpha}}$. In these formulæ, ρ is a chiral boson of the hybrid formalism and d_α is the world-sheet operator conjugate to θ^α which becomes the spinor covariant derivative in the target theory. Therefore, in this gauge $Q \sim w D^2 - \frac{1}{w} \bar{D}^2 + \ldots$ where we have written $w = \mathrm{e}^{\mathrm{i}\rho}$. Comparing with the formula (5.3.20) we see that this is *very* close to the form of the projective reduction of the $(\hat{D}^-)^2$ operator we have been calling $\diamond$, the discrepancy apparently being a rescaling of the d and $\bar{d}$ operators which undoes part of the gauge fixing:[4]

$$Q \leftrightarrow \diamond \tag{6.3}$$

In hindsight, it is should not come as a surprise that this operator is related to the BRST operator. The reason for this is that the particular operator Q with which we started is formally related to the pure spinor BRST operator of the ten-dimensional superparticle. Standard methods then tell us how to normalize the inner

[4]In fact, in the so-called $\mathcal{N} = 4$ topological method [24] this analogy is even more precise because there is another chiral boson in the theory which enters the generators $\tilde{G}^+ = \mathrm{e}^{\mathrm{i}(-2\rho + H_C)} \bar{d}^2 + \ldots$ and $\tilde{G}^- = \mathrm{e}^{\mathrm{i}(2\rho - H_C)} \bar{d}^2 + \ldots$

product of the Hilbert space (in this case the square integrable harmonic superfields), but this is none other than the harmonic superspace measure. Actually, this statement is too glib because for it to hold we need to be able to perform the map from the formalism we are using to the RNS formalism where we know how to do this normalization. Therefore, we should, in principle, be able to to this for the twistor-like superstring or the hybrid (which is where we got the $\diamond \leftrightarrow Q$ statement in the first place). The pure spinor is a different story because we do not know this map. This, again, is related to the discussion above about the measure for the pure spinor string and lends credence to the claim that string (field) theory is intimately connected to embeddings of harmonic superspaces into one another.

On the other hand, the statement (6.3) cannot be entirely trivial. For starters, the statistics seem to differ between the LHS and RHS. This mismatch comes from the identification $w \leftrightarrow \mathrm{e}^{\mathrm{i}\rho}$ and the fact that ρ is a *chiral* boson. Furthermore, such an identification would imply other relations between the hybrid formalism and projective superspace. For example, it seems superficially that the hermitian structure of the hybrid is closely related to the smile conjugation. In particular, the hermitian conjugation does not preserve the conformal weight of the world-sheet fields. On the projective superspace side there is the fact that an arctic field gets mapped to an antarctic field under smile conjugation.

Thus, although our considerations tempt us to speculate that superspace harmonics correspond to world-sheet ghosts of some superstring, there is clearly more work to be done to make this precise. A first obstacle to implementing this program is the somewhat confusing relation between the hybrid and pure spinor superstrings. In particular, it is currently not known how to embed the hybrid into the pure spinor. The complication is easy to see when we consider that the hybrid is equivalent to the twistor-like string with constraint (6.2). In order to make the correspondence precise, one of the things we would have to explain is how this is consistent with (6.1). Actually, the authors of [23] claim that the pure spinor superstring is just a further gauge fixing of the twistor superstring but for the moment it seems that the chiral boson necessary for the correct statistics of the RNS world-sheet spinor $\Psi^m \sim \mathrm{e}^{\mathrm{i}\rho}\lambda^m$ is missing [25] in the pure spinor. The resolution to this puzzle is currently being investigated.

Coming back, then, to the question of harmonic superspaces, we immediately run into the problem that the naive harmonic superspaces do not describe off-shell theories in dimensions higher than six. In the textbook [12] this is explained heuristically by considering the $\mathcal{N} = 3$ theory and showing that three facts about its harmonic superspace with harmonics in $\mathrm{SU}(3)/[\mathrm{U}(1)\times\mathrm{U}(1)]$ conspire to give a "miracle" which does not occur for the maximally supersymmetric sYM theory which harmonic factor SU(4)/S[U(2) $\times$ U(2)]. Perhaps a generalized notion of harmonic spaces is needed in which the maximally supersymmetric four-dimensional superspace is build on its analytic subspace which in turn is taken to be the full $\mathcal{N} = 2$ harmonic superspace.

Appendix A

5D notation and conventions

The 5D gamma-matrices $\Gamma_{\hat{m}} = (\Gamma_m, \Gamma_5)$, with $m = 0, 1, 2, 3$, defined by

$$\{\Gamma_{\hat{m}}, \Gamma_{\hat{n}}\} = -2\eta_{\hat{m}\hat{n}}\mathbf{1} \ , \qquad (\Gamma_{\hat{m}})^{\dagger} = \Gamma_0 \Gamma_{\hat{m}} \Gamma_0 \tag{A.1}$$

are chosen in accordance with [2, 3]

$$(\Gamma_m)_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} 0 & (\sigma_m)_{\alpha\dot{\beta}} \\ (\tilde{\sigma}_m)^{\dot{\alpha}\beta} & 0 \end{pmatrix} , \qquad (\Gamma_5)_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} -\mathrm{i}\delta_{\alpha}{}^{\beta} & 0 \\ 0 & \mathrm{i}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} , \tag{A.2}$$

such that $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5 = \mathbf{1}$. The charge conjugation matrix, $C = (\varepsilon^{\hat{\alpha}\hat{\beta}})$, and its inverse, $C^{-1} = C^{\dagger} = (\varepsilon_{\hat{\alpha}\hat{\beta}})$ are defined by

$$C\Gamma_{\hat{m}}C^{-1} = (\Gamma_{\hat{m}})^{\mathrm{T}} \ , \qquad \varepsilon^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} , \varepsilon_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} . \tag{A.3}$$

The antisymmetric matrices $\varepsilon^{\hat{\alpha}\hat{\beta}}$ and $\varepsilon_{\hat{\alpha}\hat{\beta}}$ are used to raise and lower the four-component spinor indices.

A Dirac spinor, $\Psi = (\Psi_{\hat{\alpha}})$, and its Dirac conjugate, $\Psi = (\bar{\Psi}^{\hat{\alpha}}) = \Psi^{\dagger}\Gamma_0$, look like

$$\Psi_{\hat{\alpha}} = \begin{pmatrix} \psi_{\alpha} \\ \bar{\phi}^{\dot{\alpha}} \end{pmatrix} , \qquad \bar{\Psi}^{\hat{\alpha}} = (\phi^{\alpha}, \bar{\psi}_{\dot{\alpha}}) \ . \tag{A.4}$$

One can now combine $\bar{\Psi}^{\hat{\alpha}} = (\phi^\alpha, \psi_{\dot{\alpha}})$ and $\Psi^{\hat{\alpha}} = \varepsilon^{\hat{\alpha}\hat{\beta}}\Psi_{\hat{\beta}} = (\psi^\alpha, -\bar{\phi}_{\dot{\alpha}})$ into a SU(2) doublet,

$$\Psi_i^{\hat{\alpha}} = (\Psi_i^\alpha, -\bar{\Psi}_{\dot{\alpha}i}) , \qquad (\Psi_i^\alpha)^* = \bar{\Psi}^{\dot{\alpha}i} , \qquad i = \underline{1}, \underline{2} , \tag{A.5}$$

with $\Psi_{\underline{1}}^\alpha = \phi^\alpha$ and $\Psi_{\underline{2}}^\alpha = \psi^\alpha$. It is understood that the SU(2) indices are raised and lowered by ε^{ij} and ε_{ij}, $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$, in the standard fashion: $\Psi^{\hat{\alpha}i} = \varepsilon^{ij}\Psi_j^{\hat{\alpha}}$. The Dirac spinor $\Psi^i = (\Psi_{\hat{\alpha}}^i)$ satisfies the pseudo-Majorana condition $\bar{\Psi}_i{}^{\mathrm{T}} = C\Psi_i$. This will be concisely represented as

$$(\Psi_{\hat{\alpha}}^i)^* = \Psi_i^{\hat{\alpha}} . \tag{A.6}$$

With the definition $\Sigma_{\hat{m}\hat{n}} = -\Sigma_{\hat{n}\hat{m}} = -\frac{1}{4}[\Gamma_{\hat{m}}, \Gamma_{\hat{n}}]$, the matrices $\{\mathbf{1}, \Gamma_{\hat{m}}, \Sigma_{\hat{m}\hat{n}}\}$ form a basis in the space of 4×4 matrices. The matrices $\varepsilon_{\hat{\alpha}\hat{\beta}}$ and $(\Gamma_{\hat{m}})_{\hat{\alpha}\hat{\beta}}$ are antisymmetric, $\varepsilon^{\hat{\alpha}\hat{\beta}}(\Gamma_{\hat{m}})_{\hat{\alpha}\hat{\beta}} = 0$, while the matrices $(\Sigma_{\hat{m}\hat{n}})_{\hat{\alpha}\hat{\beta}}$ are symmetric. Given a 5-vector $V^{\hat{m}}$ and an antisymmetric tensor $F^{\hat{m}\hat{n}} = -F^{\hat{n}\hat{m}}$, we can equivalently represent them as the bi-spinors $V = V^{\hat{m}}\Gamma_{\hat{m}}$ and $F = \frac{1}{2}F^{\hat{m}\hat{n}}\Sigma_{\hat{m}\hat{n}}$ with the following symmetry properties

$$V_{\hat{\alpha}\hat{\beta}} = -V_{\hat{\beta}\hat{\alpha}} , \varepsilon^{\hat{\alpha}\hat{\beta}}V_{\hat{\alpha}\hat{\beta}} = 0 , \qquad F_{\hat{\alpha}\hat{\beta}} = F_{\hat{\beta}\hat{\alpha}} . \tag{A.7}$$

The two equivalent descriptions $V_{\hat{m}} \leftrightarrow V_{\hat{\alpha}\hat{\beta}}$ and and $F_{\hat{m}\hat{n}} \leftrightarrow F_{\hat{\alpha}\hat{\beta}}$ are explicitly described as follows:

$$V_{\hat{\alpha}\hat{\beta}} = V^{\hat{m}}(\Gamma_{\hat{m}})_{\hat{\alpha}\hat{\beta}} , \qquad V_{\hat{m}} = -\frac{1}{4}(\Gamma_{\hat{m}})^{\hat{\alpha}\hat{\beta}}V_{\hat{\alpha}\hat{\beta}} , \tag{A.8}$$

$$F_{\hat{\alpha}\hat{\beta}} = \frac{1}{2} F^{\hat{m}\hat{n}} (\Sigma_{\hat{m}\hat{n}})_{\hat{\alpha}\hat{\beta}} \ , \qquad F_{\hat{m}\hat{n}} = (\Sigma_{\hat{m}\hat{n}})^{\hat{\alpha}\hat{\beta}} F_{\hat{\alpha}\hat{\beta}} \ . \tag{A.9}$$

These results can be easily checked using the identities (see e.g. [**?**]):

$$\varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon_{\hat{\gamma}\hat{\delta}} + \varepsilon_{\hat{\alpha}\hat{\gamma}}\varepsilon_{\hat{\delta}\hat{\beta}} + \varepsilon_{\hat{\alpha}\hat{\delta}}\varepsilon_{\hat{\beta}\hat{\gamma}} \ , \tag{A.10}$$

$$\varepsilon_{\hat{\alpha}\hat{\gamma}}\varepsilon_{\hat{\beta}\hat{\delta}} - \varepsilon_{\hat{\alpha}\hat{\delta}}\varepsilon_{\hat{\beta}\hat{\gamma}} = -\frac{1}{2}(\Gamma^{\hat{m}})_{\hat{\alpha}\hat{\beta}}(\Gamma_{\hat{m}})_{\hat{\gamma}\hat{\delta}} + \frac{1}{2}\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon_{\hat{\gamma}\hat{\delta}} \ , \tag{A.11}$$

and therefore

$$\varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \frac{1}{2}(\Gamma^{\hat{m}})_{\hat{\alpha}\hat{\beta}}(\Gamma_{\hat{m}})_{\hat{\gamma}\hat{\delta}} + \frac{1}{2}\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon_{\hat{\gamma}\hat{\delta}} \ , \tag{A.12}$$

with $\varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ the completely antisymmetric fourth-rank tensor.

Complex conjugation gives

$$(\varepsilon_{\hat{\alpha}\hat{\beta}})^* = -\varepsilon^{\hat{\alpha}\hat{\beta}} \ , \qquad (V_{\hat{\alpha}\hat{\beta}})^* = V^{\hat{\alpha}\hat{\beta}} \ , \qquad (F_{\hat{\alpha}\hat{\beta}})^* = F^{\hat{\alpha}\hat{\beta}} \ , \tag{A.13}$$

provided $V^{\hat{m}}$ and $F^{\hat{m}\hat{n}}$ are real.

The conventional 5D, $\mathcal{N}=1$ superspace $\mathbb{R}^{5|8}$ is parametrized by coordinates $z^{\hat{A}} = (x^{\hat{a}}, \theta^{\hat{\alpha}}_i)$. Then, a hypersurface $x^5 = \text{const}$ in $\mathbb{R}^{5|8}$ can be identified with the 4D, $\mathcal{N}=2$ superspace $\mathbb{R}^{4|8}$ parametrized by

$$z^A = (x^a, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}}) \ , \qquad (\theta^\alpha_i)^* = \bar{\theta}^{\dot{\alpha}i} \ . \tag{A.14}$$

The Grassmann coordinates of $\mathbb{R}^{5|8}$ and $\mathbb{R}^{4|8}$ are related to each other as follows:

$$\theta^{\hat{\alpha}}_i = (\theta^\alpha_i, -\bar{\theta}_{\dot{\alpha}i}) \ , \qquad \theta^i_{\hat{\alpha}} = \begin{pmatrix} \theta^i_\alpha \\ \bar{\theta}^{\dot{\alpha}i} \end{pmatrix} \ . \tag{A.15}$$

Interpreting x^5 as a central charge variable, one can view $\mathbb{R}^{5|8}$ as a 4D, $\mathcal{N}=2$ central charge superspace, see below.

The flat covariant derivatives $D_{\hat{A}} = (\partial_{\hat{a}}, D^i_{\hat{\alpha}})$ obey the algebra

$$\{D^i_{\hat{\alpha}}, D^j_{\hat{\beta}}\} = -2\mathrm{i}\varepsilon^{ij}\Big((\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\partial_{\hat{c}} + \varepsilon_{\hat{\alpha}\hat{\beta}}\Delta\Big) , \qquad [D^i_{\hat{\alpha}}, \partial_{\hat{b}}] = [D^i_{\hat{\alpha}}, \Delta] = 0 , \tag{A.16}$$

or equivalently

$$[D_{\hat{A}}, D_{\hat{B}}\} = T_{\hat{A}\hat{B}}{}^{\hat{C}} D_{\hat{C}} + C_{\hat{A}\hat{B}}\Delta , \tag{A.17}$$

with Δ the central charge. The spinor covariant derivatives are

$$D^i_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{\hat{\alpha}}_i} - \mathrm{i}(\Gamma^{\hat{b}})_{\hat{\alpha}\hat{\beta}}\theta^{\hat{\beta}i}\partial_{\hat{b}} - \mathrm{i}\theta^i_{\hat{\alpha}}\Delta . \tag{A.18}$$

One can relate the operators

$$D^i \equiv (D^i_{\hat{\alpha}}) = \begin{pmatrix} D^i_\alpha \\ \bar{D}^{\dot{\alpha}i} \end{pmatrix} , \qquad \bar{D}_i \equiv (D^{\hat{\alpha}}_i) = (D^\alpha_i, -\bar{D}_{\dot{\alpha}i}) \tag{A.19}$$

to the 4D $\mathcal{N}=2$ covariant derivatives $D_A = (\partial_a, D^i_\alpha, \bar{D}^{\dot{\alpha}}_i)$ where

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + \mathrm{i}(\sigma^b)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}i}\partial_b - \mathrm{i}\theta^i_\alpha(\Delta + \mathrm{i}\partial_5) , \tag{A.20}$$

$$\bar{D}_{\dot{\alpha}i} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} - \mathrm{i}\theta^\beta_i(\sigma^b)_{\beta\dot{\alpha}}\partial_b - \mathrm{i}\bar{\theta}_{\dot{\alpha}i}(\Delta - \mathrm{i}\partial_5) . \tag{A.21}$$

These operators obey the anti-commutation relations

$$\{D^i_\alpha, D^j_\beta\} = -2\mathrm{i}\varepsilon^{ij}\varepsilon_{\alpha\beta}(\Delta + \mathrm{i}\partial_5) ,$$

$$\begin{aligned} \{\bar{D}_{\dot{\alpha} i}, \bar{D}_{\dot{\beta} j}\} &= 2\mathrm{i}\varepsilon_{ij}\varepsilon_{\dot{\alpha}\dot{\beta}}(\Delta - \mathrm{i}\partial_5) \ , \\ \{D^i_\alpha, \bar{D}_{j\dot{\beta}}\} &= -2\mathrm{i}\delta^i_j(\sigma^c)_{\alpha\dot{\beta}}\partial_c \ , \end{aligned} \tag{A.22}$$

which correspond to the 4D, $\mathcal{N} = 2$ supersymmetry with a complex central charge, see also [26].

In terms of the operators (A.19), the operation of complex conjugation acts as follows

$$(D^i F)^\dagger \Gamma_0 = -(-1)^{\epsilon(F)} \bar{D}_i F^* \ , \tag{A.23}$$

with F an arbitrary superfield and $\epsilon(F)$ its Grassmann parity. This can be concisely represented as

$$(D^i_{\hat{\alpha}} F)^* = -(-1)^{\epsilon(F)} D^{\hat{\alpha}}_i F^* \ . \tag{A.24}$$

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