SUBMAXIMAL FUNCTION ALGEBRAS

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ABSTRACT

Title of Thesis: Submaximal Function Algebras

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Let X be a compact Hausdorff space. A function algebra on X is a complex Banach subalgebra of $C(X)$ which separates the points of X and contains the constants. Moreover, a function algebra on X is maximal if it is contained properly in no proper subalgebra of $C(X)$. We mention that maximal function algebras are large enough to have a goodly amount of structure. In order that we be able to state the ideas and results simply let us assume that for each algebra A the underlying space X is so adjusted that A contains no non-trivial ideals of $C(X)$.

Generally if A is a maximal function algebra on X, then the topological dimension of X is at most one. The idea of this thesis is to extend the notion of maximal function algebra so that on the one hand features of maximal algebras would be retained, while on the other hand the topological dimension of the underlying space could be forced to be arbitrarily large. Thus our introduction of the notion of submaximal function algebra.

We prove that all maximal algebras are submaximal. A submaximal, non-maximal algebra is $A(T^n)$, the completion of the polynomials in n-complex variables on the unit n-torus in

C^n. However, if A is submaximal on X, then each proper function algebra between A and $C(X)$ is contained in a proper maximal function algebra on X. Moreover, we show by example that the converse to this last statement is false.

If A is a submaximal function algebra on X, then every point in X has a compact neighborhood in X such that the algebra of restrictions of functions in A is dense in the continuous functions on that neighborhood. This is the (natural) analogue of the "pervasive" property of maximal function algebras.

It turns out that maximal function algebras are antisymmetric, which means that they contain no non-constant real-valued functions. This is not true in general for submaximal function algebras. However, if we render the antisymmetric property in the following way, then it holds true for submaximal algebras: if the real-valued continuous functions $f_1,\cdots,f_n$ on X along with A together generate a dense subalgebra of $C(X)$, then the continuous real-valued functions $h_1,\cdots,h_n$ on X and A together generate a dense subalgebra of $C(X)$, provided only that each h_j is sufficiently close to f_j. In addition, we show that if A is submaximal on X, then there always exist finitely many real-valued continuous functions on X which together with A generate a dense subalgebra of $C(X)$.

Finally we discuss tensor products of submaximal algebras. In particular, we prove that under certain restrictions, the tensor product of two submaximal algebras is submaximal.

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CHAPTER I

INTRODUCTION

Section 1.

The notion of maximal function algebra gained attention with Wermer's theorem [23] which says that the algebra $A(T)$--completion of the polynomials on the unit circle T--is maximal in $C(T)$. Other examples of maximal function algebras A in $C(X)$ are in papers by Wermer [24], [25], [26], [27], [28], [29]; by Hoffman and Singer [14]; and by Gamelin and Rossi [7]. All these share a common property: the topological dimension (see (1.12) in Chapter I) of the underlying space X is at most one.

The starting off point for this thesis was a desire to extend the notion of maximal function algebra in such a way that on the one hand features of maximal function algebras were retained, and on the other hand, the topological dimension of the underlying space X had to be allowed to be arbitrarily large. Thus our introduction of the concept of submaximal function algebra.

In Chapter II we first give the definition of submaximality and some equivalent formulations of the notion of submaximality. Then we show that submaximal algebras share a number of properties with maximal algebras. These latter properties chiefly involve notions of local and global approximation to $C(X)$. Along the way we prove some propositions which help us tell when a given algebra is submaximal. Especially we show that $A(T^n)$--completion of the polynomials in n-complex variables on the unit n-torus

T^n in $\mathbb{C}^n$--is submaximal and not maximal.

In Chapter III we show that under certain conditions, the tensor product of two submaximal algebras is again submaximal. Since, in general, taking tensor products increases the topological dimension of the underlying space, we obtain a number of submaximal algebras whose underlying space has large topological dimension. Thus this portion of the requirement to our extension of the notion of maximal algebras is fulfilled.

Section 2.

In this section we give preliminary notions and conventions used relevant to this thesis.

(1.1) Let X be a compact Hausdorff space, and let $C(X)$ denote the collection of all continuous complex-valued functions f on X, under the norm $\|f\| = \sup_{x \in X} |f(x)|$ and under pointwise addition and multiplication. A closed subalgebra A of $C(X)$ is called a function algebra on X provided that A contains the constant functions and separates the points of X. $C_{\mathbb{R}}(X)$ is the closed subset of $C(X)$ consisting of all real-valued functions in $C(X)$. $C_{\mathbb{R}}(X)$ is endowed with the relative topology inherited from $C(X)$. If f is in $C(X)$, then $Re(f)$ is the real part of f, and $Im(f)$ is the imaginary part of f, i.e., f can be written uniquely as a sum $u + iv$ where u and v are in $C_{\mathbb{R}}(X)$ and thus $Re(f) = u$ and $Im(f) = v$.

$$\mathcal{U}(X) = \{f \in C(X): |f(x)| = 1 \text{ for each } x \text{ in } X\}.$$

If A is a function algebra on X, then

(a) $\overline{A} = \{f: \overline{f} \in A\}$,

(b) $Re(A) = \{Re(f): f \in A\}$,

(c) $|A| = \{|f| : f \in A\}$,

(d) $\mathcal{U}(A) = \{f \in C(X)$: there are functions r, θ in $C_{\mathbb{R}}(X)$ with $re^{i\theta}$ in A and $f = e^{i\theta}\}$.

(1.2) If B is a (complex) normed space, then B^* is the collection of all continuous, complex-valued, complex-linear functionals on B.

For the remainder of this section, A is a function algebra on X.

(1.3) For the following discussion we use Gamelin [6] Chapter I and page 25 in Chapter II for a reference. The <u>maximal ideal space</u> of A, denoted by M_A, is the collection of all non-zero continuous, complex-valued, multiplicative linear functionals on A. The topology on M_A is the weak* topology on A^* restricted to M_A. In this topology, M_A is a compact Hausdorff space since 1 is in A. One has a homomorphism Φ from A into $C(M_A)$ given by $\Phi(f)(\phi) = \phi(f)$, for all f in A and all ϕ in M_A. The homomorphism Φ is called the <u>Gelfand transform</u> and $\Phi(f)$ is usually denoted by $\hat{f}$. Because A is a function algebra, Φ is an isometric isomorphism.

A closed subset Y of M_A is called a <u>boundary</u> for A if $\sup_{\phi \in Y} |\hat{f}(\phi)| = \sup_{\phi \in M_A} |\hat{f}(\phi)|$ for each f in A. There is a unique smallest boundary for A, denoted by ∂_A, called the <u>Shilov boundary</u> of A.

We can embed X into M_A by the map $\Phi_*(x) = \phi_x$, where $\hat{f}(\phi_x) = f(x)$ for each f in A. Φ_* is a homeomorphism of X onto its image $\Phi_*(X)$ in M_A. Of

course $\Phi_*(X)$ is a boundary for A. For this reason, we identify X with $\Phi_*(X)$ whenever convenient and usually we do not distinguish between f and $\hat{f}$. Thus if A is a function algebra on X we usually assume that X is a compact subset of M_A and as such X contains ∂_A. With this identification each function f in A has a unique continuous extension to M_A given by $\hat{f}$.

(1.4) If F is a compact subset of the maximal ideal space M_A of A, then the A-<u>convex hull</u> of F, denoted by $\mathrm{Hull}_A(F)$, is the set

$$\{\phi \in M_A: |\phi(f)| \leq \sup_{\psi \in F} |f(\psi)| \text{ for all } f \text{ in } A\}.$$

F is always contained in $\mathrm{Hull}_A(F)$ and F is called A-<u>convex</u> if $F = \mathrm{Hull}_A(F)$. We mention that $\mathrm{Hull}_A(F)$ is always A-convex. See Gamelin [6], p. 39.

(1.5) Let F be a compact subset of M_A. Then $A|F$ denotes the algebra of restrictions of functions in A to F. In addition A_F denotes the closure of $A|F$ in $C(F)$. The maximal ideal space M_{A_F} of A_F is $\mathrm{Hull}_A(F)$, Rickart [20], pp. 362-363. If f is in A, then $f|F$ is the restriction of f to F and $|f|_F = \sup_{x \in F} |f(x)|$.

(1.6) A compact subset F of X is called a <u>peak set</u> for A if there is a function f in A such that $F = \{x \in X: f(x) = \|f\|\}$. F is called a <u>weak peak set</u> or a p-<u>set</u> if F is an intersection of peak sets. See Gamelin [6], p. 56. If F is a weak peak set then $A|F = A_F$, i.e.,

$A|F$ is already closed in $C(F)$. See Gamelin [6], Lemma 12.3 on page 57 and Theorem 12.7 on page 58. We remark that if B is a function algebra on X and $A \subseteq B$, then any weak peak set for A is also a weak peak set for B.

(1.7) We call A <u>essential</u> if A contains no non-zero ideal of $C(X)$. If A is proper in $C(X)$, there is a unique smallest non-empty compact subset E of X, called the <u>essential set</u> of A, such that A contains every continuous function on X that vanishes on E (Bear [1], Corollary 1). If E is the essential set of a proper function subalgebra A of $C(X)$ then $A|E$ is closed in $C(E)$ and $A|E$ is an essential algebra ([1], Theorem 2). If $A = C(X)$, then we define the essential set to be the empty set. Thus the essential set of A is not empty if and only if A is a proper (function) subalgebra of $C(X)$. If B is a function algebra on X with $A \subseteq B$, then the essential set of B is contained in the essential set of A ([1], Corollary 2).

(1.8) A is called <u>antisymmetric</u> if A contains no non-constant real-valued functions; while a set $E \subseteq X$ is called a set of <u>antisymmetry</u> for A if every function in A that is real-valued on E is constant on E. If A is antisymmetric then X is a set of antisymmetry. We mention that if B is a function algebra on X with $A \subseteq B$, then each set of antisymmetry for B is also a set of antisymmetry for A. For more on sets of antisymmetry see Glicksberg [11].

(1.9) Theorem (Bishop Antisymmetric Decomposition Theorem).

> Let A be a function algebra on X. Let $\{E_\alpha\}$ be the family of all maximal sets of antisymmetry of A. The E_α are closed disjoint subsets of X whose union is X. Each E_α is a weak peak set. If f is in $C(X)$ and $f|E_\alpha$ is in $A|E_\alpha$ for all E_α, then f is in A.

This Theorem is quoted from Gamelin [6], p. 60. In particular we see that $A = C(X)$ if and only if each E_α is a single point.

(1.10) A function algebra A on X is <u>maximal</u> (or <u>maximal on</u> X) if whenever B is a function algebra on X with $A \subseteq B \subseteq C(X)$ then $B = A$ or $B = C(X)$.

(1.11) We always use T to denote the unit circle in the complex plane and T^n to denote the n-fold cartesian product of T. Thus $T^n = \{(z_1,\cdots,z_n) \in \mathbb{C}^n: |z_j| = 1$ for $j = 1,\cdots,n\}$. $A(T^n)$ is the closure in $C(T^n)$ of the restrictions to T^n of the polynomials in the n-complex variables $z_1,\cdots,z_n$. When $n = 1$, then $A(T) = A(T^1)$ is called the <u>disc algebra</u>.

(1.12) By the <u>topological dimension</u> of X we mean the analytical dimension of $C_{\mathbb{R}}(X)$, [18; p. 117]. We give here the definition of analytical dimension of $C_{\mathbb{R}}(X)$.

<u>Definition</u> (M. Katěkov). Let X be a compact Hausdorff space. A closed subalgebra R of $C_{\mathbb{R}}(X)$ is called <u>analytically closed</u> if whenever f is in $C_{\mathbb{R}}(X)$ and f

satisfies $f^n + a_1 f^{n-1} + \cdots + a_n = 0$ where a_j are in R for $j = 1,2,\cdots,n$, then f is in R.

A subset D of $C_{\mathbb{R}}(X)$ is called an <u>analytical base</u> of $C_{\mathbb{R}}(X)$ if every analytically closed subalgebra containing D coincides with $C_{\mathbb{R}}(X)$.

We call the least cardinal number of an analytical base of $C_{\mathbb{R}}(X)$ the <u>analytical dimension</u> of $C_{\mathbb{R}}(X)$.

CHAPTER II

SUBMAXIMAL FUNCTION ALGEBRAS

Section 1.

The disc algebra $A(T)$ might be considered the original maximal function algebra. Using the methods of complex analysis, one can prove the following facts about the disc algebra $A(T)$: 1) $A(T)$ is analytic on T (i.e., if f is in $A(T)$ and f vanishes on a non-empty open subset of T, then f is identically zero), 2) $A(T)$ is an integral domain, 3) $A(T)$ is antisymmetric, and 4) $A(T)$ is essential. Helson and Quigley [13] have shown that the first three statements hold for any essential maximal algebra. On the other hand, Hoffman and Singer [14; p. 221] have observed that if A is a maximal algebra on X then these four statements are (pairwise) equivalent, i.e., A is analytic on X if and only if A is an integral domain if and only if A is antisymmetric if and only if A is essential. This is why essential maximal algebras are so much "nicer" than merely maximal algebras.

As we have mentioned, the disc algebra was the first maximal algebra and is a very nice maximal algebra at that. Therefore we decided to examine the algebra $A(T^n)$, for $n \geq 2$, looking for a reasonable behavior similar to that of maximality. Here, two comments are in order. First of all, note that the topological dimension of T^n is n. Second, for $n \geq 2$, $A(T^n)$ is not a maximal subalgebra of $C(T^n)$.

In fact, we shall produce presently an uncountable linearly ordered chain (B_α) of distinct function algebras on T^n such that $A(T^n) \subsetneqq B_\alpha \subsetneqq B_\beta \subsetneqq C(T^n)$ whenever $\alpha < \beta$. Moreover, $\overline{\bigcup_\alpha B_\alpha} \subsetneqq C(T^n)$. This means $A(T^n)$ is quite far from being maximal. Fixing $n \geq 2$ and writing A for $A(T^n)$, we obtain the B_α as follows: for each real number α with $0 < \alpha \leq 2\pi$ let $X_\alpha = \{(e^{i\theta}, z_2, \cdots, z_n) \in T^n : \alpha \leq \theta \leq 2\pi\}$. Of course X_α is a compact subset of T^n and geometrically X_α is the cartesian product of an arc (in the unit circle) with the $(n-1)$-torus T^{n-1}. For each α with $0 < \alpha \leq 2\pi$ let $B_\alpha = \{f \in C(T^n) : f|X_\alpha \in A_{X_\alpha}\}$. Now then, because $X_\alpha \supseteq X_\beta$ whenever $0 < \alpha < \beta \leq 2\pi$, it follows that $B_\alpha \subseteq B_\beta$ whenever $0 < \alpha < \beta \leq 2\pi$. What we want to show is that $B_\alpha \neq B_\beta$ unless $\alpha = \beta$. We will show this by producing, for each pair (α, β) with $0 < \alpha < \beta \leq 2\pi$, a function $F_{\alpha,\beta}$ with $F_{\alpha,\beta}$ in B_β but $F_{\alpha,\beta}$ not in B_α. For each pair (α, β) of real numbers with $0 < \alpha < \beta \leq 2\pi$, let $f_{\alpha,\beta}$ be a continuous real-valued function on T such that $f_{\alpha,\beta}(e^{i\alpha}) = 1$ while $f(e^{i\theta}) = 0$ for each θ, $\beta \leq \theta \leq 2\pi$. Define $F_{\alpha,\beta}$ on T^n by $F_{\alpha,\beta}(z_1, \cdots, z_n) = f_{\alpha,\beta}(z_1) f_{\alpha,\beta}(z_2)$. It follows that $F_{\alpha,\beta}$ is in B_β since $F_{\alpha,\beta}$ restricted to X_β is identically zero. To show $F_{\alpha,\beta}$ is not in B_α we must do a little work. Let $Y = \{(e^{i\alpha}, z_2, 1, \cdots, 1) : |z_2| = 1\}$ and note that Y is a peak set for A because the polynomial $p(z_1, \cdots, z_n) = \left(\frac{e^{-i\alpha} z_1 + 1}{2}\right)\left(\frac{z_3+1}{2}\right)\left(\frac{z_4+1}{2}\right)\cdots\left(\frac{z_n+1}{2}\right)$ peaks on Y. Thus $A|Y = A_Y$

(see item 1.6 in Chapter 1), which means that $F_{\alpha,\beta}|Y$ is in A_Y if and only if there is a function G in A with $G|Y = F_{\alpha,\beta}|Y$. But $A|Y$ is isometrically isomorphic to the disc algebra. The disc algebra, and therefore $A|Y$, contains no non-constant real-valued functions. The function $F_{\alpha,\beta}$ is real-valued on Y and non-constant on Y (for example $F_{\alpha,\beta}(e^{i\alpha},e^{i\alpha},1,\cdots,1) = f_{\alpha,\beta}(e^{i\alpha})f_{\alpha,\beta}(e^{i\alpha}) = 1$ while $F_{\alpha,\beta}(e^{i\alpha},e^{i\beta},1,\cdots,1) = f_{\alpha,\beta}(e^{i\alpha})f_{\alpha,\beta}(e^{i\beta}) = 0$). Thus $F_{\alpha,\beta}|Y$ is not in $A|Y$. On the other hand, if $F_{\alpha,\beta}$ were in B_α, then there would be a sequence of functions (g_k) in A such that $|g_k - F_{\alpha,\beta}|_{X_\alpha} \longrightarrow 0$ as $k \longrightarrow \infty$. Because Y is contained in X_α, $|g_k - F_{\alpha,\beta}|_Y \longrightarrow 0$ as $k \longrightarrow \infty$. Thus $F_{\alpha,\beta}$ would be in $A_Y = A|Y$. This contradiction shows $F_{\alpha,\beta}$ is not in B_α. Therefore for each pair (α,β) with $0 < \alpha < \beta \leq 2\pi$ we have $A \subsetneqq B_\alpha \subsetneqq B_\beta \subsetneqq C(T^n)$. To see that $\overline{\bigcup B_\alpha} \neq C(T^n)$ see Example 2.19 and Definition 2.1.

After some investigation of $A(T^n)$ we discovered the following fact: if $(B_k)_{k\geq 1}$ is a sequence of function algebras on T^n with $A(T^n) \subseteq B_k \subseteq B_{k+1} \subseteq C(T^n)$ for $k \geq 1$, then $\bigcup_{k\geq 1} B_k$ is dense in $C(T^n)$ if and only if $B_k = C(T^n)$ for some $k \geq 1$. This fact, which is proved later (Example 2.19) forms the basis of our definition of submaximality. In order to consider compact spaces which are not necessarily metrizable our definition of submaximality involves the use of nets of function algebras instead of sequences of function algebras. After giving the definition

of submaximality, we will compare the use of nets to the use of sequences in the submaximality definition.

Definition 2.1. Let A be a function algebra on X. Then A is called a submaximal function algebra on X provided that whenever $(B_\gamma)_{\gamma\in\Gamma}$ is a net of function algebras on X such that the three properties

i) $A \subseteq B_\gamma$ for each γ in Γ

ii) $B_\gamma \subseteq B_\eta$ whenever $\gamma \leq \eta$ in Γ

iii) $\bigcup_{\gamma\in\Gamma} B_\gamma$ is dense in $C(X)$

imply $B_\gamma = C(X)$ for some γ in Γ.

Remark 2.2. 1) Maximal algebras are submaximal function algebras on their respective Shilov boundaries.

2) It is quite possible that A is a submaximal function algebra on some boundary X, while A is not a submaximal function algebra on some other boundary Y. Evidently $A(T)$, being maximal in $C(T)$, is a submaximal function algebra on T. On the other hand $A(T)$ is also a function algebra on the closed unit disc D in the complex plane. We can show that $A(T)$ is not a submaximal function algebra on D. In fact, write A for $A(T)$ and let D_n be the closed disc with center at the origin and with radius $\frac{1}{n}$. Let $B_n = \{f \in C(D): f|D_n \in A_{D_n}\}$. Each B_n is a proper function algebra on D and for each $n \geq 1$, $A \subseteq B_n \subseteq B_{n+1} \subsetneq C(D)$. Moreover, $\bigcup_{n\geq 1} B_n$ is dense in $C(D)$.

3) If A is a submaximal function algebra on X, then we may just say A is submaximal on X; or, if the space X in question is clearly understood, then we will sometimes merely say A is submaximal.

4) We mention that one can generalize the definition of submaximality. On the one hand, we can say that A is submaximal in B if A and B are function algebras on X with $A \subseteq B$ and if an appropriate statement holds for nets of function algebras between A and B. Having done this we can then remove the requirement that A and B be function algebras. For example B might be a topological algebra and A a closed subalgebra of B. Then A is submaximal in B if an appropriate statement holds for nets of closed subalgebras between A and B.

Before relating the use of nets and the use of sequences in the definition of submaximality, we need to prove the following proposition.

<u>Proposition 2.3</u>. Let A be a function algebra on X. Then A is submaximal on X if and only if whenever $\mathcal{D}$ is a subset of $C(X)$ such that the algebra generated by A and $\mathcal{D}$ is dense in $C(X)$ then there is a finite subset D of $\mathcal{D}$ such that the algebra generated by A and D is dense in $C(X)$.

<u>Proof</u>. ($\Longrightarrow$) Let Γ be the set of all finite subsets of $\mathcal{D}$.

Under set inclusion Γ is a directed set. For each γ in Γ, let A_γ be the function algebra generated on X by A and γ. Then $(A_\gamma)_{\gamma\in\Gamma}$ is a net of function algebras on X such that

i) $A \subseteq A_\gamma$ for each γ in Γ.

ii) $A_\gamma \subseteq A_\eta$ whenever $\gamma \leq \eta$ in Γ.

iii) $\bigcup_{\gamma\in\Gamma} A_\gamma$ contains the algebra generated by A and $\mathcal{D}$ and is therefore dense in $C(X)$.

Because A is submaximal on X, there is a γ in Γ such that $A_\gamma = C(X)$; i.e., the algebra generated by A and the finite subset γ of $\mathcal{D}$ is dense in $C(X)$. The proof is finished.

($\Longleftarrow$) Let $(B_\gamma)_{\gamma\in\Gamma}$ be a net of function algebras on X such that

i) $A \subseteq B_\gamma$ for each γ in Γ.

ii) $B_\gamma \subseteq B_\eta$ whenever $\gamma \leq \eta$ in Γ.

iii) $\bigcup_{\gamma\in\Gamma} B_\gamma$ is dense in $C(X)$.

Let $\mathcal{D} = \bigcup_{\gamma\in\Gamma} B_\gamma$. Then the algebra generated by A and $\mathcal{D}$ contains $\bigcup_{\gamma\in\Gamma} B_\gamma$ and is therefore dense in $C(X)$. Hence there are finitely many functions $f_1,\cdots,f_n$ in $\mathcal{D}$ such that the algebra generated by A and $f_1,\cdots,f_n$ is dense in $C(X)$. Because of *ii)* there is an index γ in Γ such that the functions $f_1,\cdots,f_n$ are all in B_γ. Thus the function algebra B_γ contains a dense subalgebra of $C(X)$. It follows that $B_\gamma = C(X)$, as desired.

What types of $\mathcal{D}$'s allow us to determine if A is submaximal? Corollary 2.4 provides an answer, where we utilize the definitions from (1.1) in Section 2 of Chapter I.

Corollary 2.4. Let A be submaximal on X. Let $\mathcal{D}$ be any one of the following sets: $C(X)$, $C_{\mathbb{R}}(X)$, $\mathcal{U}(X)$, $\overline{A}$, $\mathrm{Re}(A)$, $|A|$, $\mathcal{U}(A)$. Then there is a finite subset D of $\mathcal{D}$ such that the algebra generated by A and D is dense in $C(X)$.

Proof. Merely observe that if $\mathcal{D}$ is any of these aforementioned sets, then the algebra generated by A and $\mathcal{D}$ contains a conjugate closed, point separating subalgebra and is therefore dense in $C(X)$ by the Stone-Weierstrass Theorem. Apply Proposition 2.3 to obtain the desired result.

Proposition 2.3 and Corollary 2.4 give rise to a question whose statement is facilitated by the following definition:

Definition 2.5. Let A be a function algebra on X. Let $\mathcal{D}$ be a subset of $C(X)$ such that the algebra generated by A and $\mathcal{D}$ is dense in $C(X)$. Then $\delta_A(\mathcal{D}) = \inf\{\mathrm{card}(D): D \subseteq \mathcal{D}$ and the algebra generated by A and D is dense in $C(X)\}$.

Remark 2.6. When the algebra under consideration is clearly understood, we may sometimes write $\delta(\mathcal{D})$ for $\delta_A(\mathcal{D})$.

Our question is:

Question 2.7. Let A be submaximal on X. What (necessarily)

finite number is $\delta_A(\mathcal{D})$ when $\mathcal{D}$ is one of the following sets: $C(X)$, $C_{\mathbb{R}}(X)$, $\mathcal{U}(X)$, $\overline{A}$, $Re(A)$, $|A|$, $\mathcal{U}(A)$?

Remark 2.8. If A is a proper maximal algebra on X and $\mathcal{D} \subseteq C(X)$ is such that the algebra generated by A and $\mathcal{D}$ is dense in $C(X)$, then by definition of maximality $\delta(\mathcal{D}) = 1$.

Conjecture 2.9. If A is a proper submaximal algebra on X, then $\delta_A(C(X)) = 1$, i.e., there is a single function f in $C(X)$ such that the algebra generated by A and f is dense in $C(X)$.

We can prove that $\delta_{A(T^n)}(Re(A(T^n))) = 1$. However, first we need the following:

Lemma 2.10. Let A be any function algebra on X. Let Λ be a non-empty index set. Let $\{f_\lambda\}_{\lambda\in\Lambda} \subseteq Re(A)$ and let $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq \overline{A}$. Suppose that $f_\lambda = Re(g_\lambda)$ for each λ in Λ. Finally let B_1 be the closed subalgebra of $C(X)$ generated by A and $\{f_\lambda\}_{\lambda\in\Lambda}$ and let B_2 be the closed subalgebra of $C(X)$ generated by A and $\{g_\lambda\}_{\lambda\in\Lambda}$. Then $B_1 = B_2$.

Proof. It is sufficient to show that $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq B_1$ and that $\{f_\lambda\}_{\lambda\in\Lambda} \subseteq B_2$. Let λ be in Λ. Then $f_\lambda = \frac{1}{2}(g_\lambda + \overline{g}_\lambda) \in B_2$ since $g_\lambda \in B_2$ and $\overline{g}_\lambda \in A \subseteq B_2$. Moreover,

$$g_\lambda = Re(g_\lambda) + i\, Im(g_\lambda) = f_\lambda - i\, Im(\overline{g}_\lambda) = f_\lambda - (\overline{g}_\lambda - f_\lambda).$$

But $f_\lambda - (\overline{g}_\lambda - f_\lambda) = 2f_\lambda - \overline{g}_\lambda \in B_1$ since $2f_\lambda \in B_1$ and $\overline{g}_\lambda \in A \subseteq B_1$. The proof is finished.

Corollary 2.11. For any function algebra A on X, $\delta_A(\mathrm{Re}(A)) = \delta_A(\overline{A})$.

Example 2.12. $\delta_{A(T^n)}(\mathrm{Re}(A(T^n))) = 1$.

Proof. Let $f(z_1,\cdots,z_n) = \overline{z}_1 \cdots \overline{z}_n$. Then $f \in \overline{A}$. The function algebra on T^n generated by A and f contains the functions $\overline{z}_1,\cdots,\overline{z}_n$. By the Stone-Weierstrass Theorem the algebra generated by A and f is dense in $C(X)$. Thus $\delta(\overline{A(T^n)}) = 1$. By Corollary 2.11 $\delta(\mathrm{Re}(A(T^n))) = 1$. By Lemma 2.10, the algebra generated by $A(T^n)$ and $\mathrm{Re}(\overline{z}_1\cdots\overline{z}_n)$ is dense in $C(T^n)$. Notice this proves Conjecture 2.9 for $A = A(T^n)$ since the above function, f, shows $\delta_{A(T^n)}(C(T^n)) = 1$.

Professor Gulick has brought to our attention the question of the possibility of a converse to Corollary 2.3. Note that the natural converse--if the algebra generated by some function f in $C(X)$ and the algebra A is dense in $C(X)$ then A is submaximal--is false. We have seen that $A(T)$ is not submaximal on the unit disc (see the second item in Remark 2.2). However the algebra generated by $f(z) = \overline{z}$ and $A(T)$ is dense in $C(D)$ by the Stone-Weierstrass Theorem. Perhaps the following is the "right" converse-like question to ask:

Question 2.13. Let A be a function algebra on X. Let $\mathcal{D}$ be a (fixed) subset of $C(X)$ such that the algebra generated

by A and $\mathcal{D}$ is dense in $C(X)$. Then A is called an algebra of <u>type-$\mathcal{D}$</u> if whenever $\mathcal{E}$ is a subset of $\mathcal{D}$ such that the algebra generated by A and $\mathcal{E}$ is dense in $C(X)$ then there is a finite subset D of $\mathcal{E}$ such that the algebra generated by A and D is dense in $C(X)$. For which $\mathcal{D}$ is it true that A is a submaximal algebra whenever A is of type $\mathcal{D}$?

We have not been able to answer Question 2.13. However, note that Lemma 2.10 says that A is of type-$\overline{A}$ if and only if A is type-Re(A). In fact, it was Question 2.13 that led us to Lemma 2.10. We hope to return to this question in the future. We believe that if A is an algebra of type-$\mathcal{D}$ with $\mathcal{D}$ one of the sets Re(A), $\overline{A}$, $C_{\mathbb{R}}(X)$, etc., then A must at least have a good number of the properties which submaximal algebras enjoy.

We are finally ready to relate the notions of sequences of function algebras and nets of function algebras with respect to submaximality, and we obtain as a corollary the equivalence of the two notions in the event that X is metrizable.

<u>Theorem 2.14</u>. Let A be a function algebra on X. Then the following are equivalent:

a) A is submaximal on X.

b) There is a countable (finite or infinite) subset $\mathcal{D}$ of $C(X)$ such that the algebra generated by A and $\mathcal{D}$ is

dense in $C(X)$ <u>and</u> whenever $(B_n)_{n\geq 1}$ is a sequence of function algebras on X with

1) $A \subseteq B_n \subseteq B_{n+1}$ for $n \geq 1$,

2) $\bigcup_{n\geq 1} B_n$ is dense in $C(X)$,

then $B_n = C(X)$ for some integer $n \geq 1$.

<u>Proof.</u> ($a \Longrightarrow b$) The first part of b) follows from a) and Corollary 2.4. The second part of b) follows from a) directly from Definition 2.1 in the special case $\Gamma = \mathbb{N}$ = the set of positive integers with the usual ordering.

($b \Longrightarrow a$) Let $f_1, f_2, \cdots$ be an enumeration of the countable set $\mathcal{D}$; if $\mathcal{D}$ is finite with k distinct elements, say $\mathcal{D} = \{f_1, \cdots, f_k\}$, then let $f_n = f_k$ for $n \geq k$. Now let $(B_\gamma)_{\gamma\in\Gamma}$ be a net of function algebras on X such that

i) $A \subseteq B_\gamma$ for each γ in Γ,

ii) $B_\gamma \subseteq B_\eta$ whenever $\gamma \leq \eta$ in Γ,

iii) $\bigcup_{\gamma\in\Gamma} B_\gamma$ is dense in $C(X)$.

We must show that $B_\gamma = C(X)$ for some γ in Γ. The idea of the proof is to extract from Γ a sequence γ_n such that $\gamma_n \leq \gamma_{n+1}$ for $n \geq 1$ and such that $\bigcup_{n\geq 1} B_{\gamma_n}$ is dense in $C(X)$. Observe that, because $\bigcup_{\gamma\in\Gamma} B_\gamma$ is dense in $C(X)$ and because $B_\gamma \subseteq B_\eta$ whenever $\gamma \leq \eta$ in Γ, given any $\varepsilon > 0$ and any finite set $\{h_1, \cdots, h_n\}$ in $C(X)$ there is a γ in Γ such that

$$\inf_{g\in B_\gamma} \|g-h_j\| \leq \varepsilon \quad \text{for } j = 1,2,\cdots,n.$$

Let γ_1 be any index in Γ such that $\inf_{g\in B_{\gamma_1}} \|g-f_1\| \leq 1$. By induction, let γ_n be any index in Γ such that $\gamma_n \geq \gamma_{n-1}$ and such that $\inf_{g\in B_{\gamma_n}} \|g-f_j\| \leq \frac{1}{n}$ for $j = 1,2,\cdots,n$. We claim $\bigcup_{n\geq 1} B_{\gamma_n}$ is dense in $C(X)$. First of all note that $B_{\gamma_n} \subseteq B_{\gamma_{n+1}}$ for each $n \geq 1$ and therefore $\bigcup_{n\geq 1} B_{\gamma_n}$ is a (not necessarily closed) subalgebra of $C(X)$. Let B be the closure of $\bigcup_{n\geq 1} B_{\gamma_n}$ in $C(X)$. Second, note that for each f_k, there is a sequence $(g_{k,\ell})_{\ell\geq 1}$ with $g_{k,\ell}$ in B_{γ_ℓ} and with $\|g_{k,\ell}-f_k\| \longrightarrow 0$ as $\ell \longrightarrow \infty$. Thus because B is closed, B contains $\mathcal{D} = \{f_1, f_2, \cdots\}$. Finally, note $A \subseteq \bigcup_{\ell\geq 1} B_{\gamma_\ell} \subseteq B$ so that B contains the algebra generated by A and $\mathcal{D}$. This latter algebra is dense in $C(X)$. Thus $B = C(X)$. By the second part of b) there is an integer $n \geq 1$ for which $B_{\gamma_n} = C(X)$. The proof is finished.

Remark 2.15. We do not know whether or not the preceding theorem remains true if we delete from b) the requirement that there be a countable subset $\mathcal{D}$ of $C(X)$ such that the algebra generated by A and $\mathcal{D}$ is dense in $C(X)$.

Corollary 2.16. Let A be a function algebra on X and suppose X is metrizable. Then the following are equivalent:

1) A is submaximal on X,

2) whenever $(B_n)_{n\geq 1}$ is a sequence of function algebras on X with $A \subseteq B_n \subseteq B_{n+1}$ for $n \geq 1$ and $\bigcup_{n\geq 1} B_n$ dense in $C(X)$, then $B_n = C(X)$ for some $n \geq 1$.

Proof. Because X is metrizable, $C(X)$ is separable. Let $\mathcal{D}$ be a countable dense subset of $C(X)$. It follows that the algebra generated by A and $\mathcal{D}$ is dense in $C(X)$. The corollary now follows from Theorem 2.14.

Remark 2.17. We do not know if this corollary is true without the assumption of metrizability on X.

Section 2.

In this section we prove some propositions that allow us to exhibit examples of submaximal algebras.

Proposition 2.18. Let A be a function algebra on X. Suppose that there are finitely many functions $f_1, \cdots, f_n$ in A such that $f_1, \cdots, f_n$ separate the points of X and such that $|f_1(x)| = \cdots = |f_n(x)| = 1$ for each x in X. Then A is submaximal on X.

Proof. On X, $\frac{1}{f_j} = \overline{f}_j$ for $j = 1, 2, \cdots, n$. It follows from the Stone-Weierstrass Theorem that the algebra generated by A and $\overline{f}_1, \cdots, \overline{f}_n$ is dense in $C(X)$. Thus by Theorem 2.14 we can use sequences of function algebras to test for submaximality of A. Suppose, therefore, that $(B_k)_{k \geq 1}$ is a sequence of function algebras on X with $A \subseteq B_k \subseteq B_{k+1}$ for $k \geq 1$ and with $\bigcup_{k \geq 1} B_k$ dense in $C(X)$. If for each j, $1 \leq j \leq n$, there is an integer k_j with f_j invertible in B_{k_j}, then there is an integer m such that each f_j, $1 \leq j \leq n$, is invertible in B_m. As noted above, the Stone-

Weierstrass Theorem then says that $B_m = C(X)$. Hence if $B_k \neq C(X)$ for all $k \geq 1$, there is an integer j, $1 \leq j \leq n$, such that f_j is not invertible in B_k for each $k \geq 1$. This means that for each $k \geq 1$ there is a non-zero homomorphism ϕ_k on B_k such that $\phi_k(f_j) = 0$. Denote by Φ_k a norm preserving extension of ϕ_k to $C(X)$. Thus $(\Phi_k)_{k\geq 1}$ is a sequence of linear functions in the unit ball of $C^*(X)$. Therefore there is a subnet (Φ_{k_λ}) of (Φ_k) and a functional Φ in $C^*(X)$ such that $\Phi_{k_\lambda} \longrightarrow \Phi$ as $\lambda \longrightarrow \infty$ in the weak* topology. Now let f and g be in $\bigcup_{k\geq 1} B_k$. Then for some $\ell \geq 1$, f and g are in B_ℓ. Because (Φ_{k_λ}) is a subnet of (Φ_k) there is a λ_0 such that $k_\lambda \geq \ell$ for all $\lambda \geq \lambda_0$. But if $k_\lambda \geq \ell$ then f and g are in B_{k_λ} and we have

$$\Phi_{k_\lambda}(fg) = \phi_{k_\lambda}(fg) = \phi_{k_\lambda}(f)\phi_{k_\lambda}(g) = \Phi_{k_\lambda}(f)\Phi_{k_\lambda}(g).$$

Thus

$$\begin{aligned}
\Phi(fg) &= \lim_{\lambda\to\infty} \Phi_{k_\lambda}(fg) \\
&= \lim_{\substack{\lambda\to\infty \\ \lambda\geq\lambda_0}} \Phi_{k_\lambda}(fg) \\
&= \lim_{\substack{\lambda\to\infty \\ \lambda\geq\lambda_0}} \Phi_{k_\lambda}(f)\Phi_{k_\lambda}(g) \\
&= [\lim_{\lambda\to\infty} \Phi_{k_\lambda}(f)]\cdot[\lim_{\lambda\to\infty} \Phi_{k_\lambda}(g)] \\
&= \Phi(f)\Phi(g).
\end{aligned}$$

We see that Φ is multiplicative on $\bigcup_{k\geq 1} B_k$. But Φ is already continuous on $C(X)$. Thus Φ is continuous and

multiplicative on the closure in $C(X)$ of $\bigcup_{k\geq 1} B_k$, which closure is $C(X)$. We have consequently shown that Φ is a continuous homomorphism of $C(X)$. Also

$$\begin{aligned}\Phi(1) &= \lim_{\lambda\to\infty} \Phi_{k_\lambda}(1)\\ &= \lim_{\lambda\to\infty} \phi_{k_\lambda}(1)\\ &= 1.\end{aligned}$$

Thus Φ is a non-zero homomorphism of $C(X)$. Therefore there is a point x in X such that $\Phi(f) = f(x)$ for all f in $C(X)$. In particular $|\Phi(f_j)| = |f_j(x)| = 1$. On the other hand $\Phi(f_j) = \lim_{\lambda\to\infty} \Phi_{k_\lambda}(f_j) = \lim_{\lambda\to\infty} \phi_{k_\lambda}(f_j) = 0$. This contradiction shows that for some $m \geq 1$ $B_m = C(X)$. The proof is finished.

Example 2.19. The function algebra $A(T^n)$ is submaximal on T^n. We note that the coordinate functions $z_1,\cdots,z_n$ are in $A(T^n)$, separate the points of T^n, and are each of constant modulus 1 on T^n so that Proposition 2.18 applies.

Let us discuss two inheritance properties of submaximal algebras.

Proposition 2.20. Let A be submaximal on X. Let F be a nonempty compact subset of X. Then A_F is submaximal on F.

Proof. Let $(B_\gamma)_{\gamma\in\Gamma}$ be a net of function algebras on F and suppose $(B_\gamma)_{\gamma\in\Gamma}$ satisfies conditions *i*), *ii*) and *iii*) of the definition of submaximality with respect to A_F. For each

γ in Γ let $D_\gamma = \{f \in C(X): f|_F \in B_\gamma\}$. Then $(D_\gamma)_{\gamma\in\Gamma}$ satisfies conditions *i*), *ii*) and *iii*) of the definition of submaximality with respect to A. Thus because A is submaximal on X, $D_\gamma = C(X)$ for some γ in Γ and therefore $B_\gamma = C(X)$. The proof is finished.

<u>Proposition 2.21</u>. Suppose that A is a function algebra on X. Suppose that $X = X_1 \cup \cdots \cup X_n$ where the X_j are pairwise disjoint non-empty compact subsets of X. Then A_{X_j} is submaximal on X_j for $j = 1,2,\cdots,n$ if and only if A is submaximal on X.

<u>Proof</u>. ($\Longrightarrow$) We can use induction, so we may as well assume $n = 2$. Let $(B_\gamma)_{\gamma\in\Gamma}$ be a net of function algebras satisfying conditions *i*), *ii*) and *iii*) in the definition of submaximality. Observe that χ_1 and χ_2, the characteristic functions of X_1 and X_2 respectively, are in $C(X)$. By condition *iii*) there is a γ in Γ such that in B_γ there are functions f_1 and f_2 with $\|\chi_1 - f_1\| < \frac{1}{2}$ and $\|\chi_2 - f_2\| < \frac{1}{2}$. It follows that the maximal ideal space $M_{B_\gamma} = M_1 \cup M_2$ where M_1 and M_2 are disjoint compact subsets of M_{B_γ} with $X_1 \subseteq M_1$ and $X_2 \subseteq M_2$. In fact $M_1 = f_1^{-1}(\{z: |1-z| < \frac{1}{2}\})$ and $M_2 = f_1^{-1}(\{z: |z| < \frac{1}{2}\})$. It follows from the Shilov Idempotent Theorem that χ_1' and χ_2', the characteristic functions of M_1 and M_2 respectively, are in B_γ. But B_γ is a function algebra on X and this means that $\chi_1'|_X$ and $\chi_2'|_X$ are in B_γ, i.e., χ_1 and χ_2 are in B_γ. Thus X_1 and X_2 are

peak sets for B_γ and hence peak sets for B_η for $\eta \geq \gamma$ (we are using the fact that $B_\eta \supseteq B_\gamma$ whenever $\eta \geq \gamma$ in Γ). Because X_1 and X_2 are peak sets for B_η with $\eta \geq \gamma$, $B_\eta|X_j = B_{\eta_{X_j}}$ for $j = 1,2$. But now the nets $(B_\eta|X_j)_{\eta\geq\gamma}$ $j = 1,2$ satisfy conditions *i*), *ii*), *iii*) with respect to A_{X_j} $j = 1,2$ in the definition of submaximality. Thus there is an index $\eta \geq \gamma$ such that $B_\eta|X_j = C(X_j)$ for $j = 1,2$, whereupon $B_\eta = C(X)$.

($\Longleftarrow$) This is just Proposition 2.20.

Remark 2.22. This is a proof suggested to us by Professor Gulick and this proof replaces our earlier proof.

We now have a corollary which generalizes Proposition 2.18.

Corollary 2.23. Let A be a function algebra on X. Suppose there are finitely many functions $f_1,\cdots,f_n$ in A such that

1) the functions $f_1,\cdots,f_n$ separate the points of X.
2) the functions $|f_1|,\cdots,|f_n|$ are locally constant on X.

Then A is submaximal on X.

Proof. Consider the mapping $F: X \longrightarrow \mathbb{R}^n$ defined by $F(x) = (|f_1(x)|,\cdots,|f_n(x)|)$. Because each $|f_j|$ is locally constant on X, each $|f_j|$ takes on only finitely many

values. Thus $F(X)$ is a finite point set of $\mathbb{R}^n$, say $F(X) = \{r_1, \cdots, r_k\}$. Let $X_\ell = F^{-1}(r_\ell)$ for each ℓ, $1 \leq \ell \leq k$. Each function $|f_j|$ is constant on X_ℓ, $1 \leq \ell \leq k$. On X_ℓ, the functions separate the points. Thus by Proposition 2.18, A_{X_ℓ} is submaximal on X_ℓ for $\ell = 1, 2, \cdots, k$. Of course the sets $X_1, \cdots, X_k$ are pairwise disjoint non-empty compact subsets of X with $X = X_1 \cup \cdots \cup X_k$. Therefore by Proposition 2.21, A is submaximal on X.

<u>Example 2.24</u>. Let $X = \{z: |z| = \frac{1}{2}$ or $|z| = 1\}$. Let $\mathcal{P}(X)$ be the closure in $C(X)$ of the polynomials in the complex variable z. Then $p(z) = z$ separates the points of X is locally of constant modulus on X. Thus $\mathcal{P}(X)$ is submaximal on X by Corollary 2.23. Note $\mathcal{P}(X)$ is not maximal on X. In fact if we let $B = \{f \in C(X): f|T \in \mathcal{P}(X)|T\}$, then B is a function algebra on X because T is the Shilov boundary for $\mathcal{P}(X)$ and thus $\mathcal{P}(X)|T$ is closed in $C(T)$. Moreover $\mathcal{P}(X) \subsetneqq B \subsetneqq C(X)$.

<u>Example 2.25</u>. Let $Y = \{(z_1, \cdots, z_n) \in \mathbb{C}^n: \frac{1}{2} \leq |z_j| \leq 1$ for $1 \leq j \leq n\}$. Let $\mathcal{R}(Y)$ be the closure in $C(Y)$ of the rational functions in the complex variables $z_1, \cdots, z_n$ with poles off Y. Let $X = \{(z_1, \cdots, z_n) \in \mathbb{C}^n: |z_j| = 1$ or $|z_j| = \frac{1}{2}$ for $1 \leq j \leq n\}$. Then by the maximum modulus principle, $\mathcal{R}(Y)$ is a function algebra on X. Since on X the functions $z_1, \cdots, z_n$ are locally of constant modulus

and since they separate the points of X, $\mathcal{R}(Y)$ is submaximal on X. Of course $\mathcal{R}(Y)$ is not submaximal on Y.

Section 3.

The local boundary behavior of an essential maximal algebra is as stable as we could desire. By this statement, we mean that the following theorem, which may be found in Hoffman and Singer [14; p. 220] and implicitly in Helson and Quigley [13], is available:

Theorem 2.26. Let A be an essential maximal algebra on X. Then $A_F = C(F)$ for each proper compact subset F of X.

Proof. A proper compact subset F of X is a non-empty compact set different from X. Let $B = \{f \in C(X): f|F \in A_F\}$. Then $A \subseteq B \subseteq C(X)$. However, $B \neq A$ since B is not essential. Thus $B = C(X)$ and $A_F = C(F)$.

The conclusion of this theorem is not generally true for submaximal function algebras. See for example the proof (at the beginning of this chapter) of the fact that $A(T^2)$ is not maximal on T^2 and observe that any of the algebras B_α provides a counterexample to the above theorem for the essential submaximal algebra $A(T^2)$. Rather, the above theorem is locally true for submaximal algebras.

Proposition 2.27. Let A be submaximal on X. Let F be a non-empty compact subset of X such that $A_F = C(F)$. Then there is a compact neighborhood G of F such that $A_G = C(G)$.

Proof. Let $\mathcal{G}$ be the collection of compact neighborhoods G of F in X. Order $\mathcal{G}$ by: $G_1 \leq G_2$ if and only if $G_2 \subseteq G_1$. Then $\mathcal{G}$ is a directed set. For each G in $\mathcal{G}$, let $A(G) = \{f \in C(X): f|G \in A_G\}$. Observe that $\{A(G)\}_{G\in\mathcal{G}}$ is a net of function algebras on X such that

i) $A \subseteq A(G)$ for each G in $\mathcal{G}$,

ii) $A(G_1) \subseteq A(G_2)$ whenever $G_1 \leq G_2$ in $\mathcal{G}$,

iii) $\bigcup_{G\in\mathcal{G}} A(G)$ is dense in $C(X)$.

To see that *iii)* holds, let B be the closure of $\bigcup_{G\in\mathcal{G}} A(G)$ in $C(X)$. Let E be the essential set of B. Now $E \subseteq G$ for each G in $\mathcal{G}$ since $A(G)$ and therefore B contains every function in $C(X)$ that vanishes on G. Thus $E \subseteq \bigcap_{G\in\mathcal{G}} G = F$. Because $A \subseteq B \subseteq C(X)$, $C(E) = A_E \subseteq B_E = B|E \subseteq C(E)$. But $B = C(X)$ since if $B \neq C(X)$ then $E \neq \phi$ and thus $B|E \subsetneq C(E)$. This means *iii)* holds.

Thus by submaximality, $A(G) = C(X)$ for some G in $\mathcal{G}$, i.e., $A_G = C(G)$. The proof is finished.

Corollary 2.28. Each point x in X has a compact neighborhood G in X such that $A_G = C(G)$. In particular x has an A-convex neighborhood in X.

Proof. Note that $A|\{x\} = C(\{x\})$ and apply Proposition 2.27. By item 1.5 in Chapter I $M_{A_G} = \mathrm{Hull}_A(G)$. But $A_G = C(G)$ so that $G = M_{A_G}$.

Proposition 2.27 has a very useful interpretation. Let

A be submaximal on X. Let $\mathcal{X}$ be the collection of non-empty compact subsets of X. Let $\mathcal{E} = \{E \in \mathcal{X} : A_E \neq C(E)\}$. Let $\mathcal{G} = \{G \in \mathcal{X} : A_G = C(G)\}$. Then $\mathcal{E} \cap \mathcal{G} = \phi$ and $\mathcal{E} \cup \mathcal{G} = \mathcal{X}$. What we would like to say is that $\mathcal{E}$ is a compact subset of $\mathcal{X}$ and that $\mathcal{G}$ is open in $\mathcal{X}$.

Our duty is clear: we must find a compact Hausdorff topology for $\mathcal{X}$. Such a topology is available, is called the <u>finite topology</u>, and is due to L. Vietoris [30].

<u>Definition 2.29</u> (Victoris [30; p. 258]). Let X be a (non-empty) compact Hausdorff space. Let $\mathcal{X}$ be the family of all non-empty compact subsets of X. For each finite collection of non-empty open sets $V_1, \cdots, V_n$ in X, let

$$\mathcal{O}(V_1, \cdots, V_n) = \{F \in \mathcal{X} : F \subseteq \bigcup_{j=1}^{n} V_j \text{ and } F \cap V_j \neq \phi \text{ for } j = 1, 2, \cdots, n\}.$$

<u>Theorem 2.30</u> (Vietoris [30; #12]). Let X be a compact Hausdorff space. Then the set $\{\mathcal{O}(V_1, \cdots, V_n) : n \geq 1$, and $V_1, \cdots, V_n$ open in $X\}$ is a basis of open sets for a compact Hausdorff topology on $\mathcal{X}$.

<u>Proposition 2.31</u>. Let A be submaximal on X. Let $(E_\gamma)_{\gamma \in \Gamma}$ be a net of non-empty compact subsets of X. Let E be a non-empty compact subset of X such that in the finite topology $E_\lambda \longrightarrow E$ and such that $A_E = C(E)$. Then there is a γ in Γ such that $A_{E_\eta} = C(E_\eta)$ for each $\eta \geq \gamma$.

<u>Proof</u>. By Proposition 2.27, there is a compact neighborhood

G of E such that $A_G = C(G)$. Let V be the interior of G. Then $E \subset V$, and $\mathcal{O}(V)$ is a neighborhood of E in the finite topology. Hence there is a γ in Γ such that $E_\eta \in \mathcal{O}(V)$ for each $\eta \geq \gamma$, i.e., $E_\eta \subseteq V$ for each $\eta \geq \gamma$. Thus $E_\eta \subseteq G$, and since $A_G = C(G)$ it follows that $A_{E_\eta} = C(E_\eta)$ for each $\eta \geq \gamma$.

<u>Corollary 2.32</u>. $\mathcal{E} = \{E \in \mathcal{X} : A_E \neq C(E)\}$ is compact in $\mathcal{X}$ and $\mathcal{G} = \{G \in \mathcal{X} : A_G = C(G)\}$ is open and dense in $\mathcal{X}$.

<u>Proof</u>. Let $(E_\gamma)_{\gamma \in \Gamma}$ be a net in $\mathcal{E}$ and suppose that $E_\gamma \longrightarrow E$ in $\mathcal{X}$. Then by the previous proposition E is in $\mathcal{E}$. Thus $\mathcal{E}$ is a closed subset of the compact Hausdorff space $\mathcal{X}$ and is therefore compact. Since $\mathcal{G} = \mathcal{X} \setminus \mathcal{E}$, $\mathcal{G}$ is open in $\mathcal{X}$. If the open set $\mathcal{O}(V_1, \cdots, V_n)$ is contained in $\mathcal{X} \setminus \bar{\mathcal{G}}$, then $\mathcal{O}(V_1, \cdots, V_n) \subseteq \mathcal{E}$. Let $x_j \in V_j$ for $j = 1, 2, \cdots, n$. Then $\{x_1, \cdots, x_n\} \in \mathcal{O}(V_1, \cdots, V_n) \subseteq \mathcal{E}$. Therefore $A_{\{x_1, \cdots, x_n\}} \neq C(\{x_1, \cdots, x_n\})$, which is absurd.

<u>Section 4</u>.

As we have noted before, every essential maximal algebra is antisymmetric, i.e., contains no non-constant real functions. The corresponding statement for essential submaximal algebras is not true. For an example, see Example 3.13 in the next chapter. Nevertheless, there is an analogue to this antisymmetry property for all submaximal algebras. Observe that for an essential maximal algebra A on X the antisymmetry property can be stated as follows: the set

$G^1_{\mathbb{R}} = \{f \in C_{\mathbb{R}}(X)$: the algebra generated by A and f is dense in $C(X)\}$ is open in $C_{\mathbb{R}}(X)$. After all, $G^1_{\mathbb{R}} = C_{\mathbb{R}}(X) \setminus \{f \in C_{\mathbb{R}}(X)$: f is constant$\}$. This is the general form our theorem takes. But before we state and prove it we need some preliminary groundwork.

<u>Proposition 2.33</u>. Let A be submaximal on X. Then for each proper function algebra R on X with $R \supseteq A$, there is a proper maximal algebra S on X with $S \supseteq R$.

<u>Proof</u>. If $A = C(X)$, there is nothing to prove. Thus we may assume $A \neq C(X)$. Let R be any proper function algebra on X with $R \supseteq A$. Let

$\mathfrak{S} = \{S$: S is a proper function algebra on X and $S \supseteq R\}$.

Let $(S_\gamma)_{\gamma\in\Gamma}$ be a linearly ordered chain in $\mathfrak{S}$. Let S be the closure of $\bigcup_{\gamma\in\Gamma} S_\gamma$ in $C(X)$. Then S is a function algebra on X and $S \supseteq R$. Moreover, since A is submaximal on X and since $A \subseteq S_\gamma$ for each γ in Γ and $S_\gamma \subseteq S_\eta$ whenever $\gamma \leq \eta$ in Γ, it follows S is a proper function algebra on X, i.e., $(S_\gamma)_{\gamma\in\Gamma}$ has properties *i*) and *ii*) of the definition of submaximality, so if each $S_\gamma \neq C(X)$ then $(S_\gamma)_{\gamma\in\Gamma}$ must not satisfy property *iii*). Thus $S = \overline{\bigcup_{\gamma\in\Gamma} S_\gamma} \neq C(X)$. Thus S is in $\mathfrak{S}$ and the proposition follows by Zorn's lemma.

The converse to Proposition 2.33 is not true as the following example shows.

Example 2.34 (The Cone Algebra). Let

$$X = \{(z,w) \in \mathbb{C}^2 : w \text{ is real}, \ 0 \leq w \leq 1 \text{ and } |z| = w\}.$$

X is just a bounded right circular cone. Let A be the closure in $C(X)$ of the restrictions to X of the polynomials in the complex variables z, w. Let U be a neighborhood in X of the point $(0,0)$. Then for w_0 sufficiently small but positive the circle $\{(z,w_0): |z| = w_0\}$ is contained in U. Thus $\mathrm{Hull}_A(U)$ contains $\mathrm{Hull}_A\{(z,w_0): |z| = w_0\}$. But $(0,w_0)$ is in the $\mathrm{Hull}_A\{(z,w_0): |z| = w_0\}$ whereas $(0,w_0)$ is not in X let alone U. Hence $\mathrm{Hull}_A(U) \neq U$ and the point $(0,0)$ has no A-convex neighborhood in X. In particular by Corollary 2.28 A is not submaximal. Let B be a proper function algebra on X with $B \supseteq A$. We show there is a proper maximal function algebra S on X with $S \supseteq B$. Let E be a maximal set of antisymmetry for B such that $B|E$ is closed and proper in $C(E)$. Such an E exists by the Bishop Antisymmetric Decomposition Theorem since B is a proper subalgebra of $C(X)$. Now the function $F(z,w) = w$ is in A and is real-valued by the definition of X. Thus $E \subseteq F^{-1}(w_0)$ for some w_0. But $F^{-1}(w_0) = \{(z,w_0): |z| = w_0\}$. It follows that $w_0 \neq 0$ since otherwise $E = \{(0,0)\}$ and this would mean $B|E = C(E)$. Now $A|F^{-1}(w_0) = B|F^{-1}(w_0)$ since each is closed in $C(F^{-1}(w_0))$ and $A|F^{-1}(w_0)$ is maximal on $F^{-1}(w_0)$. ($A|F^{-1}(w_0)$ is the disc algebra of radius w_0.) It follows $E = F^{-1}(w_0)$ since E is a set of antisymmetry for $A|F^{-1}(w_0)$.

If we let $S = \{f \in C(X): f|E \in A|E = B|E\}$ then S is a proper maximal algebra on X and $S \supseteq B$.

<u>Lemma 2.35</u>. Let A be a proper submaximal algebra on X. Let $f_1,\cdots,f_n$ be functions in $C(X)$ such that the algebra generated by A and $f_1,\cdots,f_n$ is dense in $C(X)$. Let $\mathfrak{B}$ be the family of all proper maximal subalgebras B of $C(X)$ with $B \supseteq A$. Let E_B be the essential set of B for each B in $\mathfrak{B}$. Then $\inf_{B\in\mathfrak{B}} \max_{1\le j\le n} \sup_{x,y\in E_B} |f_j(x)-f_j(y)| > 0$.

<u>Proof</u>. Since $A \neq C(X)$, $\mathfrak{B}$ is not empty, by the previous proposition. Observe that if for some B in $\mathfrak{B}$, $\sup_{x,y\in E_B} |f_j(x)-f_j(y)| = 0$ for $j = 1,2,\cdots,n$, then each f_j is constant on E_B. Thus $A_{E_B} = C(E_B)$ since on E_B the algebra generated by A, and the algebra generated by A and the functions $f_1,\cdots,f_n$ are the same because of the fact that each f_j is constant on E_B. By hypothesis the algebra generated by A and $f_1,\cdots,f_n$ is dense in $C(X)$ so the algebra generated by A and $f_1,\cdots,f_n$ on E_B is dense in $C(E_B)$. But $A_{E_B} \subseteq B_{E_B} = B|E_B \subsetneq C(E_B)$. This contradiction shows that for each B in $\mathfrak{B}$,

$$\sup_{x,y\in E_B} |f_j(x)-f_j(y)| > 0 \quad \text{for some } j_B,\ 1 \le j_B \le n.$$

Hence for B in $\mathfrak{B}$ we have

$$\max_{1\le j\le n} \sup_{x,y\in E_B} |f_j(x)-f_j(y)| > 0.$$

Now suppose that for some net $(B_\gamma)_{\gamma\in\Gamma}$ in $\mathfrak{B}$

$$\lim_{\gamma\to\infty} \max_{1\le j\le n} \sup_{x,y\in E_B} |f_j(x)-f_j(y)| = 0.$$

Recall by Theorem 2.30, the space $\mathcal{X}$ of all non-empty compact subsets of X is compact in the finite topology. Therefore there is a subnet of $(E_{B_\gamma})_{\gamma\in\Gamma}$ which converges to some non-empty compact subset E of X. For notational convenience we denote this subnet by $(E_{B_\gamma})_{\gamma\in\Gamma}$. Now we claim that $\sup_{x,y\in E} |f_j(x)-f_j(y)| = 0$ for $j = 1,2,\cdots,n$, i.e., each f_j is constant on E. For suppose to the contrary that for some j, $1 \le j \le n$, $\sup_{x,y\in E} |f_j(x)-f_j(y)| = \varepsilon > 0$. Then for some x_0 and y_0 in E, $|f_j(x_0)-f_j(y_0)| = \varepsilon$. By continuity of the f_j, there are open sets U and V in X with x_0 in U and y_0 in V such that

$$|f_j(x)-f_j(y)| > \frac{\varepsilon}{2} \quad \text{for each } (x,y) \text{ in } U\times V.$$

If $\mathcal{W} = \{F\in\mathcal{X} : F\cap U \ne \phi \text{ and } F\cap V \ne \phi\}$, then $\mathcal{W}$ is an open neighborhood of E in $\mathcal{X}$ with the finite topology. ($\mathcal{W} = \mathcal{O}(U,V,X)$). Thus there is a γ_0 in Γ such that for all $\gamma \ge \gamma_0$ in Γ, $E_{B_\gamma}\cap U \ne \phi$ and $E_{B_\gamma}\cap V \ne \phi$. Let x_γ be in $E_{B_\gamma}\cap U$ and y_γ be in $E_{B_\gamma}\cap V$ for each $\gamma \ge \gamma_0$. We see that

$$\sup_{x,y\in E_{B_\gamma}} |f_j(x)-f_j(y)| \ge |f_j(x_\gamma)-f_j(y_\gamma)| \ge \frac{\varepsilon}{2}.$$

So $\sup_{x,y\in E_{B_\gamma}} |f_j(x)-f_j(y)|$ does not converge to zero as $\gamma\longrightarrow\infty$. Thus

$$\lim_{\gamma\to\infty} \max_{1\le j\le n} \sup_{x,y\in E_{B_\gamma}} |f_j(x)-f_j(y)| \ne 0$$

as assumed. This contradiction establishes the claim: $\sup_{x,y\in E} |f_j(x)-f_j(y)| = 0$ for $j = 1,2,\cdots,n$. As before, $A_E = C(E)$ since each f_j is contant on E. By Proposition 2.27, there is a compact neighborhood G of E such that $A_G = C(G)$. Let V be the interior of G. Then $E \subset V$ and $\mathcal{U} = \{F \in \mathcal{X}: F \cap V \neq \phi \text{ and } F \subseteq V\} \equiv \mathcal{O}(V)$ is an open neighborhood of E in $\mathcal{X}$. Since E_{B_γ} converges to E in $\mathcal{X}$, there is an index γ_0 in Γ such that for all $\gamma \geq \gamma_0$ $E_{B_\gamma} \subseteq V$. Thus $A_{E_{B_\gamma}} = C(E_{B_\gamma})$ for all $\gamma \geq \gamma_0$, while $A_{E_{B_\gamma}} \subseteq B_{E_{B_\gamma}} = B|_{E_{B_\gamma}} \subsetneq C(E_{B_\gamma})$ for all $\gamma \geq \gamma_0$. This contradiction establishes the lemma.

In order to state the principle theorem in this section, it is convenient to adopt the following notation: if A is a function algebra on X, then $\prod_{j=1}^n C_{\mathbb{R}}(X) \equiv$ n-fold cartesian product of $C_{\mathbb{R}}(X)$, and $\prod_{j=1}^n C_{\mathbb{R}}(X)$ is endowed with the product topology.

$$G_{\mathbb{R}}^n = \{(f_1,\cdots,f_n) \in \prod_{j=1}^n C_{\mathbb{R}}(X): \text{the algebra generated by } A \text{ and } f_1,\cdots,f_n \text{ is dense in } C(X)\}.$$

Our theorem is:

<u>Theorem 2.36</u>. Let A be submaximal on X. Then

1) $G_{\mathbb{R}}^n$ is open in $\prod_{j=1}^n C_{\mathbb{R}}(X)$ for each $n \geq 1$.

2) $G_{\mathbb{R}}^k$ is not empty for some $k \geq 1$.

<u>Proof</u>. Of course 2) is just part of Corollary 2.4, (where $\mathcal{D} = C_{\mathbb{R}}(X)$). The proof of 1) follows from the previous lemma

in the following way: just observe that if $A = C(X)$, then $G^n_{\mathbb{R}} = \prod_{j=1}^n C_{\mathbb{R}}(X)$ for every $n \geq 1$ and we are finished. Thus we may assume $A \neq C(X)$ so that Lemma 2.35 applies if $(h_1, \cdots, h_n)$ is in $G^n_{\mathbb{R}}$, i.e.,

$$0 < \varepsilon = \inf_{B \in \mathfrak{B}} \max_{1 \leq j \leq n} \sup_{x,y \in E_B} |h_j(x) - h_j(y)|.$$

Then for each B in $\mathfrak{B}$, there is an integer j_B, $1 \leq j_B \leq n$, such that $\sup_{x,y \in E_B} |h_{j_B}(x) - h_{j_B}(y)| > \varepsilon$. For each B in $\mathfrak{B}$, select x_B and y_B in E_B so that $|h_{j_B}(x_B) - h_{j_B}(y_B)| \geq \varepsilon$. Let $(f_1, \cdots, f_n)$ be in $\prod_{j=1}^n C_{\mathbb{R}}(X)$ and suppose that $\|f_j - h_j\| < \frac{\varepsilon}{4}$ for $j = 1, 2, \cdots, n$. We want to show that $(f_1, \cdots, f_n)$ is in $G^n_{\mathbb{R}}$. We have

$$|f_{j_B}(x_B) - h_{j_B}(x_B) + h_{j_B}(y_B) - f_{j_B}(y_B)| \leq 2\,\|h_{j_B} - f_{j_B}\| \leq \frac{\varepsilon}{2}.$$

Thus

$$\sup_{x,y \in E_B} |f_{j_B}(x) - f_{j_B}(y)|$$

$$= \sup_{x,y \in E_B} |f_{j_B}(x) - h_{j_B}(x) + h_{j_B}(x) - h_{j_B}(y) + h_{j_B}(y) - f_{j_B}(y)|$$

$$\geq |h_{j_B}(x) - h_{j_B}(y) + f_{j_B}(x) - h_{j_B}(x) - f_{j_B}(y) + h_{j_B}(y)|$$

for all x, y in E_B. In particular

$$\sup_{x,y \in E_B} |f_{j_B}(x) - f_{j_B}(y)|$$

$$\geq |h_{j_B}(x_B) - h_{j_B}(y_B) + f_{j_B}(x_B) - h_{j_B}(x_B) + h_{j_B}(y_B) - f_{j_B}(y_B)|$$

$$\geq |h_{j_B}(x_B) - h_{j_B}(y_B)| - |f_{j_B}(x_B) - h_{j_B}(x_B) + h_{j_B}(y_B) - f_{j_B}(y_B)|$$

$\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0.$

Thus for each B in $\mathfrak{B}$ there is an integer j_B, $1 \leq j \leq n$, such that f_{j_B} is not constant on E_B. Let D be the function algebra on X generated by A and $f_1, \cdots, f_n$. We finally show that $D = C(X)$ since otherwise there is a B in $\mathfrak{B}$ with $D \subseteq B$ by Proposition 2.33. Thus $D_{E_B} \subseteq B_{E_B} = B|E_B \subsetneq C(E_B)$. But f_{j_B} is not constant on E_B and $f|E_B$ is in D_{E_B} hence $f|E_B$ is in B_{E_B}. Thus the essential maximal algebra B_{E_B} contains a non-constant real-valued function. This impossibility shows $D = C(X)$. The proof is finished.

CHAPTER III

TENSOR PRODUCTS

Section 1.

The primary question in this chapter is the tensor product question, namely: if A and B are submaximal on X and Y respectively, then is $A \otimes_\lambda B$ submaximal on $X \times Y$? We give here the definition of $A \otimes_\lambda B$:

Definition 3.1. Let A and B be function algebras on X and Y respectively. Then $A \otimes_\lambda B$ is the closure in $C(X \times Y)$ of the algebra

$$\{ \sum_{j=1}^{n} f_j g_j : \ n \geq 1; \ f_j \in A, \ g_j \in B \text{ for } 1 \leq j \leq n \}.$$

If f is in A and g is in B then the function fg is in $A \otimes_\lambda B$ by the convention $fg(x,y) = f(x)g(y)$.

For more information on tensor products see Gelbaum [9] or Gil de Lamadrid [10].

As it turns out, there is a possibly stronger notion which is required for an affirmative answer to the tensor product question. This notion is that of continuous submaximality.

Definition 3.2. Let A be a function algebra on X. A net $(B_\gamma)_{\gamma \in \Gamma}$ of function algebras on X is said to <u>approximate</u> $C(X)$ <u>with respect to</u> A if

i) $A \subseteq B_\gamma$ for each γ in Γ,

ii) for each f in $C(X)$ there is a net $(f_\gamma)_{\gamma\in\Gamma}$ in $C(X)$ with f_γ in B_γ for every γ in Γ such that $\|f-f_\gamma\| \longrightarrow 0$ as $\gamma \longrightarrow \infty$.

A function algebra A on X is called <u>continuous submaximal on</u> X if whenever $(B_\gamma)_{\gamma\in\Gamma}$ is a net of function algebras on X that approximates $C(X)$ with respect to A there is a γ in Γ so that $B_\eta = C(X)$ for [illegible] $\eta \geq \gamma$.

<u>Remark 3.3</u>. We shall quite often shorten the terminology when there should arise no confusion. For instance we might say the net $(B_\gamma)_{\gamma\in\Gamma}$ approximates $C(X)$ instead of saying the net $(B_\gamma)_{\gamma\in\Gamma}$ of function algebras on X approximates $C(X)$ with respect to A. Also we sometimes say A is continuous submaximal instead of A is continuous submaximal on X.

The principal theorems in this section are: first, A is continuous submaximal on X if and only if $A \otimes_\lambda C(Y)$ is submaximal on $X \times Y$ for every compact space Y; and second, A is continuous submaximal on X if and only if $A \otimes_\lambda C(Y)$ is continuous submaximal on $X \times Y$ for every compact space Y. We begin our travel to that destination with the following lemma, whose shortened proof was given to us by Professor Gulick.

<u>Lemma 3.4</u>. Let A be a function algebra on X. Let $(B_\gamma)_{\gamma\in\Gamma}$ be a net of function algebras on X such that

i) $A \subseteq B_\gamma$ for each γ in Γ,

ii) $B_\gamma \subseteq B_\eta$ whenever $\gamma \leq \eta$ in Γ,

iii) $\bigcup_{\gamma\in\Gamma} B_\gamma$ is dense in $C(X)$.

Then $(B_\gamma)_{\gamma\in\Gamma}$ approximates $C(X)$ with respect to A.

<u>Proof</u>. Let f be in $C(X)$ and $\varepsilon_\gamma = \inf_{g\in B_\gamma} \|g-f\|$ for each γ in Γ. Since $\gamma \leq \eta$ in Γ implies $B_\gamma \subseteq B_\eta$ we know $\varepsilon_\gamma \geq \varepsilon_\eta$ whenever $\gamma \leq \eta$ in Γ. On the other hand, $\bigcup_\gamma B_\gamma$ dense in $C(X)$ implies $\inf_\gamma \varepsilon_\gamma = 0$. Thus $\lim_{\gamma\to\infty} \varepsilon_\gamma = 0$, and if we pick f_γ in B_γ such that $\|f-f_\gamma\| \leq 2\varepsilon_\gamma$, then $\lim_{\gamma\to\infty} \|f_\gamma - f\| = 0$. The proof is finished.

<u>Proposition 3.5</u>. If A is continuous submaximal on X, then A is submaximal on X.

<u>Proof</u>. Let $(B_\gamma)_{\gamma\in\Gamma}$ be a net of function algebras on X such that

i) $A \subseteq B_\gamma$ for each γ in Γ,

ii) $B_\gamma \subseteq B_\eta$ whenever $\gamma \leq \eta$,

iii) $\bigcup_\gamma B_\gamma$ is dense in $C(X)$.

By Lemma 3.4, $(B_\gamma)_{\gamma\in\Gamma}$ approximates $C(X)$. Since A is continuous submaximal on X, there is a γ in Γ such that $B_\gamma = C(X)$. Thus A is submaximal.

Question 3.6. Does submaximality imply continuous submaximality?

We have spent some time on this question but as yet have been unable to answer it. We do know (Corollary 3.12) however that sometimes the answer is yes; there are some compact spaces Y such that for any function algebra A on X, $A \otimes_\lambda C(Y)$ is submaximal on $X \times Y$ if and only if $A \otimes_\lambda C(Y)$ is continuous submaximal.

Before we can state and prove the principal theorems of this section, we need a proposition relating the use of sequences and nets of function algebras with respect to continuous submaximality. The proof of this latter proposition uses the following lemma.

Lemma 3.7. Let A be a function algebra on X. Let $(B_\gamma)_{\gamma\in\Gamma}$ a net of function algebras on X with $A \subseteq B_\gamma$ for each γ in Γ. Let $B = \{f \in C(X)$: there is a net of functions $(f_\gamma)_{\gamma\in\Gamma}$ in $C(X)$ such that f_γ is in B_γ for each γ in Γ and such that $\lim_{\gamma\to\infty} \|f_\gamma - f\| = 0\}$. Then B is a function algebra on X and $B \supseteq A$.

Proof. If α and β are complex numbers, if f and g are in B and if $(f_\gamma)_{\gamma\in\Gamma}$ and $(g_\gamma)_{\gamma\in\Gamma}$ are nets of functions in $C(X)$ such that f_γ and g_γ are in B_γ for each γ in Γ and such that $\lim_{\gamma\to\infty} \|g_\gamma - g\| = \lim_{\gamma\to\infty} \|f_\gamma - f\| = 0$, then $\alpha f_\gamma + \beta g_\gamma$ is in B_γ for each γ in Γ and $\lim_{\gamma\to\infty} \|\alpha f_\gamma + \beta g_\gamma - (\alpha f + \beta g)\| = 0$. Thus $\alpha f + \beta g$ is in B. Moreover, $f_\gamma g_\gamma$ is in B_γ

for each γ in Γ. In addition from the inequalities

$$\|f_\gamma g_\gamma - fg\| \leq \|f_\gamma g_\gamma - f_\gamma g\| + \|f_\gamma g - fg\|$$

$$\leq \|f_\gamma\| \cdot \|g_\gamma - g\| + \|f_\gamma - f\| \cdot \|g\|$$

and from the equalities $\lim_{\gamma\to\infty} \|f_\gamma - f\| = \lim_{\gamma\to\infty} \|g_\gamma - g\| = 0$ we obtain $\lim_{\gamma\to\infty} \|f_\gamma g_\gamma - fg\| = 0$. Thus B is an algebra.

Suppose that $(f_n)_{n\geq 1}$ is a sequence in B such that $\lim_{n\to\infty} \|f_n - f\| = 0$ for some f in $C(X)$. For each γ in Γ let $\varepsilon_\gamma = \inf_{g\in B_\gamma} \|g-f\|$. If $\varepsilon_\gamma \not\longrightarrow 0$ as $\gamma \longrightarrow \infty$, then $\varepsilon_\gamma \geq \varepsilon > 0$ for some $\varepsilon > 0$ and all γ in some subnet Γ_0 of Γ. Let k be an integer such that $\|f_k - f\| \leq \frac{\varepsilon}{3}$. Select an index γ in Γ_0 and a function h_γ in B_γ such that $\|h_\gamma - f_k\| < \frac{\varepsilon}{3}$. Thus $\|h_\gamma - f\| \leq \frac{2\varepsilon}{3}$ and so $\varepsilon_\gamma \leq \frac{2\varepsilon}{3} < \varepsilon \leq \varepsilon_\gamma$. This contradiction shows $\varepsilon_\gamma \longrightarrow 0$ as $\gamma \longrightarrow \infty$ and if we pick h_γ in B_γ such that $\|h_\gamma - f\| \leq 2\varepsilon_\gamma$ then $\lim_{\gamma\to\infty} \|h_\gamma - f\| = 0$. Thus f is in B and B is closed in $C(X)$. It is clear from the definition of B that $A \subseteq B$. Hence B is a function algebra on X because A is.

<u>Proposition 3.8.</u> Let A be a function algebra on X. Then the following are equivalent:

a) A is continuous submaximal on X.

b) There is a countable subset $\mathcal{D} \subseteq C(X)$ such that the algebra generated by A and $\mathcal{D}$ is dense in $C(X)$ <u>and</u> whenever $(B_n)_{n\geq 1}$ is a sequence of function

algebras on X that approximates $C(X)$ with respect to A there is an integer $n \geq 1$ such that $B_n = C(X)$.

<u>Proof</u>. a)$\Longrightarrow$b). The first part of b) follows from a) and the fact (Proposition 3.5) that A must also be submaximal. Because A is submaximal, Corollary 2.4 implies that we can find such a set $\mathcal{D}$. The second part of b) follows directly from the definition of continuous submaximality in the special case that $\Gamma = \mathbb{N} =$ The of positive integers with the usual ordering.

b)$\Longrightarrow$a). Let $(B_\gamma)_{\gamma\in\Gamma}$ approximate $C(X)$. There are two possibilities: 1) there is a γ in Γ such that $B_\eta = C(X)$ for each $\eta \geq \gamma$ in Γ; or else 2) for γ in Γ there is an index θ_γ in Γ such that $\theta_\gamma \geq \gamma$ and $B_{\theta_\gamma} \neq C(X)$. To show the former holds assume 2) and obtain a contradiction. To this end we note that by assumption 2) $\{\theta_\gamma : \gamma \in \Gamma\}$ is cofinal in Γ, and therefore, the net $(B_{\theta_\gamma})_{\gamma\in\Gamma}$ also approximates $C(X)$. For notational convenience, assume $B_\gamma \neq C(X)$ for all γ in Γ. Let $f_1, f_2, \cdots$ be an enumeration of $\mathcal{D}$. For each integer $n \geq 1$, select an index γ_n such that $\inf_{g\in B_{\gamma_n}} \|g-f_j\| \leq \frac{1}{n}$ for $j = 1,2,\cdots,n$. This can be done because $(B_\gamma)_{\gamma\in\Gamma}$ approximates $C(X)$. Let us consider $(B_{\gamma_n})_{n\geq 1}$ as a sequence of function algebras. We claim that $(B_{\gamma_n})_{n\geq 1}$ approximates $C(X)$. Let $B = \{f \in C(X)$: there is a sequnce $(g_n)_{n\geq 1}$ in $C(X)$ such that g_n is in B_{γ_n} for each $n \geq 1$ and such that $\lim_{n\to\infty} \|g_n - g\| = 0\}$. Note

that B contains $\mathcal{D}$ and by Lemma 3.7, B is a function algebra on X and $B \supseteq A$. Since by hypothesis, the algebra generated by A and $\mathcal{D}$ is dense in $C(X)$ it follows $B = C(X)$, i.e., $(B_{\gamma_n})_{n \geq 1}$ approximates $C(X)$. By hypothesis b), there is an integer $n \geq 1$ such that $B_{\gamma_n} = C(X)$. This contradicts the fact that is B_γ is a proper subalgebra of $C(X)$. The proof is finished.

We are finally ready for the principal theorems of this section. In the statement of the following theorems we identify the space c of all convergent complex-valued sequences with the space $C(M)$ where $M = \{\frac{1}{n}: n \text{ is a positive integer}\} \cup \{0\}$ with the relative topology as a subset of the reals with the usual topology. This identification is made so that we can talk about the tensor product of a function algebra with c as tensor products are defined in Definition 3.1.

<u>Theorem 3.9</u>. Let A be a function algebra on X. Then the following are equivalent:

1. A is continuous submaximal on X.
2. $A \otimes_\lambda C(Y)$ is submaximal on $X \times Y$ for every compact Hausdorff space Y.
3. $A \otimes_\lambda c$ is submaximal on $X \times M$.

<u>Proof</u>. $(1 \Longrightarrow 2)$ Let $(D_\gamma)_{\gamma \in \Gamma}$ be a net of function algebras on $X \times Y$ such that

i) $A \otimes_\lambda C(Y) \subseteq D_\gamma$ for each γ in Γ,

ii) $D_\gamma \subseteq D_\eta$ whenever $\gamma \leq \eta$ in Γ,

iii) $\bigcup_\gamma D_\gamma$ is dense in $C(X \times Y)$.

Suppose that $D_\gamma \neq C(X \times Y)$ for each γ in Γ. Thus by the Bishop Antisymmetric Decomposition Theorem there is a maximal set of antisymmetry E_γ of D_γ such that $D|E_\gamma$ is closed and proper in $C(E_\gamma)$. Now we claim $E_\gamma \subseteq X \times \{y_\gamma\}$ for some y_γ in Y. Indeed [illegible] in Y, then there is a continuous real-valued function g on Y such that $g(z) \neq g(t)$. Therefore $G(x,y) = g(y)$ is a continuous real-valued function on $X \times Y$ and G is in D_γ for any γ in Γ by the definition of the tensor product. This means that any set of antisymmetry of D_γ cannot have non-empty intersection with the distinct slices $X \times \{z\}$ and $X \times \{t\}$. Thus the claim is established. Identify E_γ with the compact subset E'_γ of X, where $E_\gamma = E'_\gamma \times \{y_\gamma\}$ and write $E_\gamma = E'_\gamma$. Let $B_\gamma = \{f \in C(X): f|E_\gamma \in D_\gamma|E_\gamma\}$. $(B_\gamma)_{\gamma\in\Gamma}$ is a net of function algebras because $D_\gamma|E_\gamma$ is closed in $C(E_\gamma)$. We claim B_γ approximates $C(X)$. Let f be in $C(X)$ and consider f as a function in $C(X \times Y)$ by $f(x,y) = f(x)$. By Lemma 3.4, $(D_\gamma)_{\gamma\in\Gamma}$ approximates $C(X \times Y)$ so that there exists a net $(f_\gamma)_{\gamma\in\Gamma}$ with $\lim_{\gamma\to\infty} \|f_\gamma - f\| = 0$ and with f_γ in D_γ. Thus $f_\gamma|X \times \{y_\gamma\}$ is in B_γ and $|f-f_\gamma|_{X\times\{y_\gamma\}} \longrightarrow 0$ as $\gamma \longrightarrow \infty$. Since A is continuous submaximal, there is an index γ in Γ with $B_\gamma = C(X_\gamma)$. However this is impossible

since $B_\gamma|E_\gamma = D|E_\gamma \subsetneqq C(E_\gamma)$. Thus $A \otimes_\lambda C(Y)$ is submaximal.

$(2 \Longrightarrow 3)$. 3 is just a special case of 2.

$(3 \Longrightarrow 1)$. By Proposition 2.20 and Corollary 2.4 and the fact that $A = A \otimes_\lambda c|X \times \{1\}$ there are finitely many functions $f_1, \cdots, f_k$ in $C(X)$ such that the algebra generated by A and $f_1, \cdots, f_k$ is dense in $C(X)$. Thus by Proposition 3.8 we can use sequences of function algebras to see whether or not A is continuous submaximal. Suppose that $(B_n)_{n \geq 1}$ is a sequence of function algebras that approximates $C(X)$. Let $D_n = \{f \in C(X \times M): f|X \times \{\frac{1}{n}\} \in B_n\}$ and set $E_n = \bigcap_{\ell \geq n} D_\ell$. Then $(E_n)_{n \geq 1}$ is a sequence of function algebras on $X \times M$ such that $A \otimes_\lambda c \subseteq E_n \subseteq E_{n+1}$ for $n \geq 1$. We presently show $\bigcup_{n \geq 1} E_n$ is dense in $C(X \times M)$. Let F be in $C(X \times M)$ and let $\varepsilon > 0$ be given. Put $f = F|X \times \{0\}$. Thus there is a sequence $(f_n)_{n \geq 1}$ of functions on X with f_n in B_n and $\lim_{n \to \infty} \|f_n - f\| = 0$. Let m be an integer such that $\|f_n - f\| < \varepsilon$ for all $n \geq m$. Define G on $X \times M$ by

$$G(x,z) = \begin{cases} f(x,0) \equiv f(x) & \text{if } z = 0 \\ F(x,\frac{1}{n}) & \text{if } z = \frac{1}{n} \text{ and } 1 \leq n \leq m \\ f_n(x) & \text{if } z = \frac{1}{n} \text{ and } m < n. \end{cases}$$

Then G is in $C(X \times M)$. Observe that G is in D_n for all $n \geq m$. Hence G is in E_m. Furthermore $\|G - F\| < \varepsilon$ by construction. Thus $\bigcup_{n \geq 1} E_n$ is dense in $C(X \times M)$. Because $A \otimes_\lambda c$ is submaximal there is an integer $p \geq 1$

such that $E_p \subset C(X \times M)$. Consequently $B_n = C(X)$ for all $n \geq p$. The proof is complete.

<u>Remark 3.10.</u> The upshot of Theorem 3.9 is that if we expect $A \otimes_\lambda B$ to be submaximal, then except perhaps for trivial cases, both A and B must already be continuous submaximal. After all, if B is a function algebra on Y and if Y has a convergent sequence say $(y_n)_{n \geq 1} \subset Y$ with $y_n \to y_0$ and with $y_n \neq y_0$ there we can argue as follows: If $A \otimes_\lambda B$ is submaximal, then so is $A \otimes_\lambda C(Y)$ since $A \otimes_\lambda C(Y) \supseteq A \otimes_\lambda B$. Now $Y_0 = \{y \in Y: y = y_n$ for some $n \geq 0\}$ is a compact subset of Y and also a weak peak set. Thus $X \times Y_0$ is a is a weak peak set for $A \otimes_\lambda C(Y)$ and so $A \otimes_\lambda C(Y)|X \times Y_0$ is closed in $C(X \times Y_0)$. In fact $A \otimes_\lambda C(Y)|X \times Y_0 = A \otimes_\lambda C(Y_0)$. Therefore on the one hand by Proposition 2.20 $A \otimes_\lambda C(Y)|X \times Y_0$ is submaximal while on the other hand since $C(Y_0) = C$ it follows from the preceding theorem that A is continuous submaximal.

We can actually put in the words "continuous submaximal" for the word "submaximal" in Theorem 3.9 and we do this forthwith.

<u>Theorem 3.11.</u> Let A be a function algebra on X. Then the following are equivalent:

1. A is continuous submaximal on X,
2. $A \otimes_\lambda C(Y)$ is continuous submaximal on $X \times Y$ for every compact Hausdorff space Y.

3. $A \otimes_\lambda C$ is continuous submaximal on $X \times M$.

<u>Proof</u>. (1 $\Longrightarrow$ 2) Observe as in the proof of Theorem 3.9 if B is any function algebra on $X \times Y$ with $B \supseteq A \otimes_\lambda C(Y)$ then every maximal set of antisymmetry E of B is of the form $E = E' \times \{y\}$ for some compact set $E' \subseteq X$ and some y in Y. Let $(B_\gamma)_{\gamma \in \Gamma}$ approximate $C(X \times Y)$. Suppose that $(B_\gamma)_{\gamma \in \Gamma}$ has a subnet $(B_\gamma)_{\gamma \in \Gamma_0}$ such that $B_\gamma \neq C(X \times Y)$ for every γ in Γ_0. Then for each γ in Γ_0 there is a maximal set of antisymmetry $E_\gamma = E'_\gamma \times \{y_\gamma\}$ such that $B_\gamma | E_\gamma$ is closed and proper in $C(E_\gamma)$. Let $D_\gamma = \{f \in C(X): f|E_\gamma \in B_\gamma|E_\gamma\}$. Then $(D_\gamma)_{\gamma \in \Gamma_0}$ is a net of function algebras on X. Moreover $(D_\gamma)_{\gamma \in \Gamma_0}$ approximates $C(X)$. In fact, if f is in $C(X)$ consider f as a function in $C(X \times Y)$ by $f(x,y) = f(x)$. Then there is a net $(h_\gamma)_{\gamma \in \Gamma}$ with h_γ in B_γ and with $\lim_{\gamma \to \infty} \|h_\gamma - f\| = 0$. It follows that $h_\gamma | X \times \{y_\gamma\}$ is in D_γ for each γ in Γ_0 and that $\lim_{\substack{\gamma \to \infty \\ \gamma \text{ in } \Gamma_0}} \|h_\gamma | X \times \{y_\gamma\} - f\| = 0$. Because A is continuous submaximal, $D_\gamma = C(X)$ for some γ in Γ_0. This means that $B_\gamma = C(E_\gamma)$ for this same γ. This contradiction finishes the proof.

(2 $\Longrightarrow$ 3). This is immediate.

(3 $\Longrightarrow$ 1). This follows from Proposition 3.5 and Theorem 3.9.

<u>Corollary 3.12</u>. Let A be a function algebra on X. Then $A \otimes_\lambda C$ is submaximal if and only if $A \otimes_\lambda C$ is continuous submaximal.

Example 3.13. $A = A(T) \otimes_\lambda C[0,1]$ is continuous submaximal on $T \times [0,1]$. A is not antisymmetric since A contains the non-constant real-valued function $F(z,y) = y$ where z is in T and y is in $[0,1]$. A is essential since each maximal set of sntisymmetry is of the form $T \times \{y\}$ for some y in $[0,1]$. The function F shows that no set of antisymmetry is larger than $T \times \{y\}$ while $T \times \{y\}$ is a peak set for A since $\{y\}$ is a peak set for $C[0,1]$ for each y in $[0,1]$. Thus $A|_{T \times \{y\}} = A(T)$ and T is a set of antisymmetry for $A(T)$.

Section 2.

The objectives of this section are two-fold: first to develop the machinery so that we can more easily tell if a given function algebra is continuous submaximal and second to show that sometimes the tensor product of continuous submaximal function algebras is continuous submaximal.

The most interesting description of continuous submaximality is given by the next theorem. To motivate the theorem we first discuss a property of maximal algebras. Suppose therfore that A is a maximal algebra on X. Let $G_1 = \{f \in C(X)$: the algebra generated by A and f is dense in $C(X)\}$. Observe G_1 is an open subset of $C(X)$. After all, $G_1 = C(X) \setminus A$ by the definition of maximality and A is closed in $C(X)$. There is a corresponding statement for continuous submaximal algebras. If we let $\prod_{j=1}^{n} C(X)$ be the n-fold cartesian product of $C(X)$ and endow $\prod_{j=1}^{n} C(X)$ with the

product topology, then we shall show that the set $G_n = \{(f_1,\cdots,f_n) \in \prod_{j=1}^n C(X)$: the algebra generated by A and $f_1,\cdots,f_n$ is dense in $C(X)\}$ is open in $\prod_{j=1}^n C(X)$ for every $n \geq 1$, provided A is continuous submaximal on X. Since every continuous submaximal algebra is submaximal (Proposition 3.5), it will therefore follow that G_n is even non-empty (Corollary 2.4) for some $n \geq 1$. Moreover, the converse is true: namely, if G_n is open and non-empty for some $n \geq 1$, then A is continuous submaximal on X. That is the content of our ensuing theorem.

<u>Theorem 3.14</u>. Let A be a function algebra on X. Then the following are equivalent:

1) A is continuous submaximal on X,

2) G_n is open in $\prod_{j=1}^n C(X)$ for every $n \geq 1$, and G_n is not empty for some $n \geq 1$.

3) $\mathrm{int}(G_n)$ is not empty for some $n \geq 1$.

<u>Proof</u>. (1$\Longrightarrow$2). We have shown (Proposition 3.5) that A is submaximal. By Corollary 2.4, G_n is not empty for some $n \geq 1$. Let $(f_1,\cdots,f_n)$ be in G_n and assume that $(f_1,\cdots,f_n)$ is not in the interior of G_n. Then for each integer $m \geq 1$, there is a proper function algebra B_m on X such that $B_m \supseteq A$ and $\inf_{g\in B_m} \|g-f_j\| \leq \frac{1}{m}$ for $j = 1,\cdots,n$. We let $B = \{f \in C(X)$: there is a sequence $(g_m)_{m\geq 1}$ in $C(X)$ such that g_m is in B_m and $\lim_{m\to\infty} \|g-g_m\| = 0\}$. Then

$B \supseteq A$ and B is a closed subalgebra of $C(X)$ by Lemma 3.6. By design $\{f_1,\cdots,f_n\} \subseteq B$. But $(f_1,\cdots,f_n)$ is in G_n, so $B = C(X)$ which means that $(B_m)_{m\geq 1}$ approximates $C(X)$. Consequently 1) implies $B_k = C(X)$ for some $k \geq 1$. This contradiction shows $1 \Longrightarrow 2$.

$(2 \Longrightarrow 3)$. This is immediate.

$(3 \Longrightarrow 1)$. Suppose $\mathrm{int}(G_n)$ is not empty. Let $(f_1,\cdots,f_n)$ be in $\mathrm{int}(G_n)$ and let $\varepsilon > 0$ be such that whenever $(g_1,\cdots,g_n)$ are in $\prod_{j=1}^{n} C(X)$ and $\|f_j-g_j\| \leq \varepsilon$ for $j = 1,2,\cdots,n$, then $(g_1,\cdots,g_n)$ is in G_n. Let $(B_\gamma)_{\gamma\in\Gamma}$ approximate $C(X)$. Then there is a γ in Γ so that $\inf_{g\in B_\eta} \|g-f_j\| \leq \frac{\varepsilon}{2}$ for $j = 1,2,\cdots,n$, and for $\eta \geq \gamma$. If for each $\eta \geq \gamma$ we select g_j^η in B_η so that $\|g_j^\eta-f_j\| \leq \varepsilon$ for $j = 1,\cdots,n$ then $(g_1^\eta,\cdots,g_n^\eta)$ is in G_n. for all $\eta \geq \gamma$ and it follows $B_\eta = C(X)$ for every $\eta \geq \gamma$. The proof of the theorem is complete.

In the remaining portion of this section, the theorem we want to prove is that if A is any continuous submaximal function algebra on X and if B is a function algebra on Y and if B contains some very nice functions, then $A \otimes_\lambda B$ is continuous submaximal. By "very nice functions" we mean that B contains finitely many functions $f_1,\cdots,f_n$ such that each f_j vanishes at no point of Y and such that the algebra generated on Y by B and $\frac{1}{f_1},\cdots,\frac{1}{f_n}$ is dense in $C(Y)$. For example, let $Y = T^n$, let $B = A(T^n)$ and

let $f_j(z_1,\cdots,z_n) = z_j$ for $j = 1,2,\cdots,n$. Before we can give this theorem however, we need some technical information about the behavior of approximating nets with respect to compact subsets of $M_A \setminus X$, where A is a function algebra on X. As a consequence of this latter information, we obtain the fact that $A(T^n)$ is continuous submaximal for all $n \geq 1$.

<u>Lemma 3.15</u>. Let B be a Banach space. Suppose $(x_\gamma^*)_{\gamma\in\Gamma}$ is a net in B^* with $\sup_\gamma \|x_\gamma^*\| <$ and suppose that $x^* = \lim_{\gamma\to\infty} x_\gamma^*$ in the weak* topology in B^*. Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in B and suppose that $\lim_{\gamma\to\infty} \|x-x_\gamma\| = 0$ for some x in B. Then $\lim_{\gamma\to\infty} x_\gamma^*(x_\gamma) = x^*(x)$.

<u>Proof</u>. Let $\varepsilon > 0$ be given and set $M = \sup_\gamma \|x_\gamma^*\|$. There is a γ_0 in Γ such that $\|x-x_\gamma\| < \frac{\varepsilon}{2M}$ for all $\gamma \geq \gamma_0$ and such that $|x^*(x)-x_\gamma^*(x)| < \frac{\varepsilon}{2}$ for all $\gamma \geq \gamma_0$. Thus

$$\begin{aligned} |x^*(x)-x_\gamma^*(x_\gamma)| &\leq |x^*(x)-x_\gamma^*(x)| + |x_\gamma^*(x-x_\gamma)| \\ &\leq |x^*(x)-x_\gamma^*(x)| + M\cdot\|x-x_\gamma\| \\ &\leq \frac{\varepsilon}{2} + M\cdot\frac{\varepsilon}{2M} = \varepsilon \quad \text{for all } \gamma \geq \gamma_0. \end{aligned}$$

<u>Lemma 3.16</u>. Let A be a function algebra on X. Let $(B_\gamma)_{\gamma\in\Gamma}$ approximate $C(X)$. Let ϕ_γ be in M_{B_γ} and let Φ_γ be a norm preserving extension of ϕ_γ to $C(X)$. Assume $\Phi = \lim_{\gamma\to\infty} \Phi_\gamma$ in the weak* topology in $C^*(X)$ for some Φ in $C^*(X)$. Then Φ is a non-zero homomorphism of $C(X)$.

<u>Proof</u>. By hypothesis $1 \in B_\gamma$ for all γ in Γ so $\Phi(1) = \lim_\gamma \Phi_\gamma(1) = \lim_\gamma \phi_\gamma(1) = 1$ and Φ is not identically zero. It remains to show Φ is multiplicative. Let f and g be in $C(X)$. Since $(B_\gamma)_{\gamma\in\Gamma}$ approximates $C(X)$, there are nets $(f_\gamma)_{\gamma\in\Gamma}$ and $(g_\gamma)_{\gamma\in\Gamma}$ in $C(X)$ with f_γ and g_γ in B_γ for each γ in Γ such that $\lim_\gamma \|g_\gamma - g\| = \lim_\gamma \|f_\gamma - f\| = 0$. It follows that $f_\gamma g_\gamma$ is in B_γ for each γ in Γ and $\lim_\gamma \|f_\gamma g_\gamma - fg\| = 0$. Consequently, by Lemma 3.15, $\Phi(fg) = \lim_\gamma \Phi_\gamma(f_\gamma g_\gamma) = \lim_\gamma \phi_\gamma(f_\gamma g_\gamma) = \lim_\gamma \phi_\gamma(f_\gamma)\phi_\gamma(g_\gamma) = \lim_\gamma \Phi_\gamma(f_\gamma)\Phi_\gamma(g_\gamma) = [\lim_\gamma \Phi_\gamma(f_\gamma)][\lim_\gamma \Phi_\gamma(g_\gamma)]$ if these latter limits exist. But $[\lim_\gamma \Phi_\gamma(f_\gamma)][\lim_\gamma \Phi_\gamma(g_\gamma)] = \Phi(f)\Phi(g)$ by Lemma 3.15. The proof is finished.

For the next proposition we have the following situation: A is a function algebra on X and $(B_\gamma)_{\gamma\in\Gamma}$ is a net of function algebras on X such that $A \subseteq B_\gamma$ for each γ in Γ. We define $\pi_\gamma : M_{B_\gamma} \longrightarrow M_A$ by the restriction of a homomorphism on B_γ to A. Having made this definition, we then agree to omit the subscript γ on π_γ and write π for any of the maps π_γ. We mention π is continuous, [5; p. 478]. The following proposition says, with the above notation, that if K is a compact subset of $M_A \setminus X$ and $(B_\gamma)_{\gamma\in\Gamma}$ approximates $C(X)$, then $\pi(M_{B_\gamma})$ eventually leaves K and never returns to K.

<u>Proposition 3.17</u>. Let A be a function algebra on X. Let K be a compact subset of $M_A \setminus X$. Let $(B_\gamma)_{\gamma\in\Gamma}$ approximate

$C(X)$. Then there is a γ in Γ such that for all $\eta \geq \gamma$, $\pi(M_{B_\eta}) \cap K = \phi$, i.e., for all $\eta \geq \gamma$ no homomorphism in K extends to a homomorphism on B_η.

<u>Proof</u>. Suppose otherwise. Then without loss of generality we may suppose $\pi(M_{B_\gamma}) \cap K \neq \phi$ for every γ in Γ. Let ϕ_γ be in $\pi(M_{B_\gamma}) \cap K$ for each γ in Γ and denote by Φ_γ a norm preserving extension of ϕ_γ to $C(X)$. Thus $\|\Phi_\gamma\| \leq 1$ for all γ in Γ. Because the unit ball of $C^*(X)$ is weak* compact, we may for convenience suppose that $\Phi_\gamma \longrightarrow \Phi$ in the weak* topology of $C(X)$ for some Φ in $C^*(X)$. By Lemma 3.16, Φ is in $M_{C(X)} - X$. Thus there is an x in X such that $\Phi(f) = f(x)$ for all f in $C(X)$. On the one hand $\pi(\Phi)$ is in K, since $\pi(\Phi) = \lim_\gamma \pi(\Phi_\gamma)$ and $\pi(\Phi_\gamma)$ is in the compact set K for each γ. On the other hand $K \subset M_A \setminus X$. This is impossible. The proof is finished.

We are nearly ready for our tensor product theorem. However we give here a proposition, two corollaries and as an example we show $A(T^n)$ is continuous submaximal.

<u>Proposition 3.18</u>. Let A be a function algebra on X and let K be a compact subset of $M_A \setminus X$. Suppose that for each function algebra B on X either $B = C(X)$ or $\pi(M_B) \cap K \neq \phi$. Then A is continuous submaximal on X.

<u>Proof</u>. Let $(B_\gamma)_{\gamma \in \Gamma}$ approximate $C(X)$. By Proposition 3.17 there is a γ in Γ so that $\pi(B_\eta) \cap K = \phi$ for each $\eta \geq \gamma$. By hypothesis, $B_\eta = C(X)$ for $\eta \geq \gamma$.

Corollary 3.19. Let A be a function algebra on X. Suppose there are finitely many functions $f_1, \cdots, f_n$ in A such that each f_j vanishes at no point of X and the algebra generated by A and $\frac{1}{f_1}, \cdots, \frac{1}{f_n}$ is dense in $C(X)$. Then A is continuous submaximal on X.

Proof. Let $K = \{\phi \in M_A : \phi(f_j) = 0$ for some j, $1 \leq j \leq n\}$. Then K is a compact subset of $M_A \setminus X$. If B is a function algebra on X and $B \supseteq A$ then $B = C(X)$ if and only if $\frac{1}{f_1}, \cdots, \frac{1}{f_n}$ are in B and this happens if and only if $\pi(M_B) \cap K = \phi$. By Proposition 3.18, A is continuous submaximal.

Corollary 3.20. Let A be a function algebra on X. Suppose there are finitely many functions $f_1, \cdots, f_n$ in A such that $|f_1(x)| + \cdots + |f_n(x)| = 1$ for all x in X and such that the functions separate the points of X. Then A is continuous submaximal.

Proof. By the Stone-Weierstrass Theorem the algebra generated by A and $\frac{1}{f_1}, \cdots, \frac{1}{f_n}$ is dense in $C(X)$. Apply Corollary 3.19 to finish the proof.

Example 3.21. $A(T^n)$ is continuous submaximal on T^n for all $n \geq 1$.

We proceed with our tensor product theorem.

Theorem 3.22. Let A and B be function algebras on X and Y respectively. Suppose that $g_1, \cdots, g_n$ are in B and suppose each g_j vanishes at no point of Y while the algebra generated by B and $\frac{1}{g_1}, \cdots, \frac{1}{g_n}$ is dense in $C(Y)$. Let A be continuous submaximal on X. Then $A \otimes_\lambda B$ is continuous submaximal on $X \times Y$.

Proof. From [6; p. 96] we have $M_{A \otimes_\lambda B} = M_A \times M_B$. Let $K = \{\phi \in M_B : \phi(g_j) = 0$ for some j, $1 \leq j \leq n\}$. By hypothesis K is a compact subset of $M_B \setminus X$ so that $M_A \times K$ is a compact subset of $M_A \times M_B \setminus X \times Y$. Since $M_{A \otimes_\lambda B} = M_A \times M_B$, $M_A \times K = \{\phi \in M_{A \otimes_\lambda B} : \phi(g_j) = 0$ for some j, $1 \leq j \leq n\}$. Let $(D_\gamma)_{\gamma \in \Gamma}$ approximate $C(X \times Y)$ with respect to $A \otimes_\lambda B$. By Proposition 3.17, there is a γ_0 in Γ such that $\pi(M_{D_\gamma}) \cap K = \phi$ for each $\gamma \geq \gamma_0$. In particular, $g_1, \cdots, g_n$ are invertible in D_γ for all $\gamma \geq \gamma_0$. Thus for $\gamma \geq \gamma_0$, $D_\gamma \supseteq A \otimes_\lambda C(Y)$. By Theorem 3.11, $A \otimes_\lambda C(Y)$ is continuous submaximal. Thus there is a γ_1 in Γ with $\gamma_1 \geq \gamma_0$ and with $D_\gamma = C(X \times Y)$ for each $\gamma \geq \gamma_1$. The proof is finished.

Corollary 3.23. If A and B are maximal algebras on X and Y respectively and if B contains a function that vanishes at some point of $M_B \setminus Y$ but never vanishes at any point of Y, then $A \otimes_\lambda B$ is continuous submaximal on $X \times Y$.

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