ABSTRACT

Title of dissertation: Estimating Common Odds Ratio with Missing Data

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We derive estimates of expected cell counts for $I \times J \times K$ contingency tables where the stratum variable C is always observed but the column variable B and row variable A might be missing. In particular, we investigate cases where only row variable A might be missing, either randomly or informatively. For $2 \times 2 \times K$ tables, we use Taylor expansion to study the biases and variances of the Mantel-Haenszel estimator and modified Mantel-Haenszel estimators of the common odds ratio using one pair of pseudotables for data without missing values and for data with missing values, based either on the completely observed subsample or on estimated cell means when both stratum and column variables are always observed. We examine both large table and sparse table asymptotics.

Analytic studies and simulation results show that the Mantel-Haenszel estimators overestimate the common odds ratio but adding one pair of pseudotables reduces bias and variance. Mantel-Haenszel estimators with jackknifing also reduces the biases and variances. Estimates using only the complete subsample seem to have larger bias than those based on full data, but when the total number of observations gets large, the bias is reduced. Estimators based on estimated cell means seem to have larger biases and variances than those based only on complete subsample with randomly missing data. With informative missingness, estimators based on the estimated cell means do not converge to the correct common odds ratio under sparse asymptotics, and converge slowly for the large table asymptotics.

The Mantel-Haenszel estimators based on incorrectly estimated cell means when the variable A is informatively missing behave similarly to those based on the only complete subsamples. The asymptotic variance formula of the ratio estimators had smaller biases and variances than those based on jackknifing or bootstrapping. Bootstrapping may produce zero divisors and unstable estimates, but adding one pair of pseudotables eliminates these problems and reduces the variability.

Estimating Common Odds Ratio With Missing Data

by

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To my families.

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Chapter 1

Introduction

In real world studies where sequences of 2×2 contingency table data are collected, frequently some tables are incomplete. Often investigators simply analyze the subsample of complete observations, ignoring possible effects of missing data. While this may be reasonable when only a small proportion of observations are incomplete, it could possibly cause incorrect estimates and variability when large amounts of data are missing. Another approach, imputation, is not always valid. When a large proportion of data is missing, incorrect estimation of variability is possible even if the imputation scheme does not produce bias.

Statistical studies of the effects of missing data are summarized by Little and Rubin [30]. They show that, in multivariate data with missing observations, estimates using only complete observations do lead to biased estimates of means and variances. They formulate mathematical models for missing data and suggest better techniques for dealing with missing data.

In this dissertation, we study the effects of estimating the common odds ratio when the row variable is missing at random with missingness probability depending on the stratum and the column variables or when it is missing informatively with missingness probability depending on the variable itself and the stratum variable. We assume the column variable and the stratum variable are always observed, as in case control studies. We investigate estimators based on only the complete observations and in the closed-form imputation introduced by Baker, Rosenberger and Dersimonian [5].

We use both mathematical proofs and simulations to study the effect of ignoring missing data, imputing missing data, and misspecifying the missing data mechanism.

The dissertation is organized as follows. In Chapter 2, we review the literature about missing data analysis, particular using the closed-form imputation of the cell counts for contingency tables and the common odds ratio estimation. In Chapter 3, we will show the three-way contingency tables closed-from. In Chapter 4, we study the biases for Mantel-Haenszel estimator of the common odds ratio and its adjustments, both in complete and incomplete contingency tables. for incomplete contingency tables. The results of the simulation are in Chapter 5. Chapter 6 states our conclusions and discusses some possible directions of further research.

Chapter 2

Literature Review

2.1 Missing Data

Let $\mathbf{V}$ be a matrix of random variables with n independent rows and p columns. Each row represents a different observation and each column represents a different categorical variable $\mathbf{Y} = (Y_1, Y_2, ..., Y_p)$. One could rearrange $\mathbf{V}$ as a p-dimensional contingency table with W cells defined by the joint levels of the variables. The entries in the table are counts $\{z_{ijk...t}\}$, where $z_{ijk...t}$ is the number of sampled cases in the cell with $Y_1 = i, Y_2 = j, ..., Y_p = t$.

If the data matrix $\mathbf{V}$ has missing items, that is one or more column variables are not identifiable in some rows, we will convert the data matrix to a contingency table with p extra variables of $(R_1, ..., R_p) = \mathbf{R}$ which indicate whether $Y_1, ...Y_p$ were observed. We write $R_r = 1$ when the rth variable is observed and $R_r = 0$ otherwise.

If the event that Y_i is missing does not depend on Y_i itself or any other Y_k, then we say Y_i is Missing Completely at Random (MCAR). That is, $P(R_i|\mathbf{Y}, \phi) = P(R_i|\phi) = \pi$ where ϕ denotes unknown parameters. If the event that Y_i is missing does not depend on Y_i itself but does depend on some other observed variables $Y_k, k \neq i$, then we say Y_i is Missing at Random (MAR). That is, $P(R_i|\mathbf{Y}, \phi) = P(R_i|\{Y_k, R_k = 1\}, \phi)$ (Little and Rubin [30]). In this situation, we will use the notation Y_i is MAR$(Y_k, R_k = 1)$, or more simply MAR(Y_k). Nonignorable missingness or informative missingness means that the missingness of Y_i depends on either Y_i itself or some unobserved Y_k. That is Y_i is not MAR.

Standard statistical methods were not designed to analyze missing data, so when some data are missing, we should use other methods. Four groups of methods to analyze a data set with missing values were presented by Little and Rubin [30].

(a) *Procedures based on complete recorded units:* When some variables are missing for some observations, a simple method is to analyze only the units with complete data. In general, this method can lead to serious biases and it may not be very efficient. However, under MCAR, the complete observations are a random sample of the full data set, so no bias occurs.

(b) *Imputation-based procedures:* One fills in values for the missing data so that the data set become a completed data set, and then one simply uses standard methods to analyze the data. There are several imputation methods including *hot deck* imputation, *mean* imputation and *regression* imputation. The idea is somehow to find $\hat{Z}_{i,j}$, an estimate of the expected value of a missing $Z_{i,j}$ given the observed data, and substitute $\hat{Z}_{i,j}$ in the sample.

(c) *Weighting procedures:* In a complex survey, observations are given weights π_i which are inversely proportional to the probability of selection. For instance, let x_i be the ith value of a variable X. Then the population mean of X is often estimated by

$$\sum_i \pi_i^{-1} x_i / \sum_i \pi_i^{-1}, \tag{2.1}$$

where π_i is the probability of that unit i is observed and π_i^{-1} is the design weight for observation i. Weighting procedures modify the weights in an attempt to adjust for nonresponse. The estimator (2.1) is replaced by $\sum (\pi_i \hat{p}_i)^{-1} x_i / \sum (\pi_i \hat{p}_i)^{-1}$, where the sums are over data units where X is observed, and $\hat{p}_i$ is an estimate of the probability that X is observed for unit i. If the design weights are constant in subclasses of the sample, then the mean imputation and weighting lead to the same estimates of population means, although not the same estimates of sampling variances unless we make adjustments to the data with mean imputation.

(d) *Model-based procedures:* The analyst defines a model for the missing data mechanism

Table 2.1: Two-way contingency table with missing data

		$B=1$		$B=2$	
		$R_b=1$	$R_b=0$	$R_b=1$	$R_b=0$
$A=1$	$R_a=1$	z_{1111}	z_{1110}	z_{1211}	z_{1210}
	$R_a=0$	z_{1101}	z_{1100}	z_{1201}	z_{1200}
$A=2$	$R_a=1$	z_{2111}	z_{2110}	z_{2211}	z_{2210}
	$R_a=0$	z_{2101}	z_{2100}	z_{2201}	z_{2200}

and bases inferences on the likelihood under that model. The advantages of this procedure are flexibility. However, there is a possibility of introducing bias if the model for missingness is misspecified.

Methods (b), (v), and (d) all involve the use of a model, whether implicit or explicit. They are therefore potentially subject to biases.

2.2 Closed Form Estimates

Let A and B are two categorical variables with 2 levels each and let R_a and R_b be the indicator variables we discussed in section 2.1. The 2×2 contingency table with missing data is as Table 2.1

Let μ_{ijkl} denote the expected cell counts of the (i,j) cells in a 2×2 contingency table with $R_a = k$ and $R_b = l$. The log-linear model for two partially observed categorical variables with no three- or four-way interactions is

$$\log\left(\mu_{ijkl}\right) = \mu + \alpha_i^A + \alpha_j^B + \alpha_{ij}^{AB} + \beta_k^{R_a} + \beta_l^{R_b} + \beta_{kl}^{R_a R_b} + \gamma_{ik}^{AR_a} + \gamma_{il}^{AR_b} + \gamma_{jk}^{BR_a} + \gamma_{jl}^{BR_b}$$

Baker, Rosenberger and Dersimonian[5] introduced the following parameterization of the model and led to closed-form ML estimates:

$$
\begin{aligned}
m_{ij} &= NPr(A=i,B=j,R_a=1,R_b=1) \\
&= \exp\{\mu+\alpha_i^A+\alpha_j^B+\alpha_{ij}^{AB}+\beta_1^{R_a}+\beta_1^{R_b}+\beta_{11}^{R_aR_b}+\gamma_{i1}^{AR_a}+\gamma_{i1}^{AR_b}+\gamma_{j1}^{BR_a}+\gamma_{J1}^{BR_b}\}, \\
a_{ij} &= \frac{Pr(R_a=0,R_b=1|A=i,B=j)}{Pr(R_a=1,R_b=1|A=i,B=j)} \\
&= \exp\{-2[\beta_1^{R_a}+\beta_{11}^{R_aR_b}+\gamma_{i1}^{AR_a}+\gamma_{j1}^{BR_a}]\} \\
b_{ij} &= \frac{Pr(R_a=1,R_b=0|A=i,B=j)}{Pr(R_a=1,R_b=1|A=i,B=j)}, \\
&= \exp\{-2[\beta_1^{R_b}+\beta_{11}^{R_aR_b}+\gamma_{i1}^{AR_b}+\gamma_{j1}^{BR_b}]\} \\
g &= \frac{Pr(R_a=1,R_b=1|A=i,B=j)Pr(R_a=0,R_b=0|A=i,B=j)}{Pr(R_a=1,R_b=0|A=i,B=j)Pr(R_a=0,R_b=1|A=i,B=j)}. \\
&= \exp\{4\beta_{11}^{R_aR_b}\},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\mu_{ij11}=m_{ij}, \qquad \mu_{ij10}=m_{ij}b_{ij}, \\
&\mu_{ij01}=m_{ij}a_{ij}, \qquad \mu_{ij00}=m_{ij}a_{ij}b_{ij}g, \\
&m_{ij}\ge 0, \quad a_{ij}\ge 0, \quad b_{ij}\ge 0, \quad g\ge 0, \\
&\textstyle\sum_i\sum_j m_{ij}(1+a_{ij}+b_{ij}+a_{ij}b_{ij}g)=N.
\end{aligned}
$$

The cell probabilities are

$$
\pi_{ij..}=Pr(A=i,B=j) \quad = \quad m_{ij}(1+a_{ij}+b_{ij}+a_{ij}b_{ij}g),
$$

and the log-likelihood is

$$
L=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=0}^{1}\sum_{l=0}^{1}\{z_{ij11}\log(\mu_{ij11})\Delta_{11}+z_{i+10}\log(\mu_{i+10})\Delta_{10} \\
+z_{+j01}\log(\mu_{+j01})\Delta_{01}+z_{++00}\log(\mu_{++00})\Delta_{00}\}-\mu_{++++}
$$

where $\Delta_{cd}=1$, if $k=c$ and $l=d$, and 0, otherwise.

To compute maximum likelihood estimator, one solves the system of equations $\partial L/\partial\theta_{ijkl}=0$, where θ_{ijkl} includes all of μ, α's, β's and γ's. Closed-form solutions might be available, and boundary conditions might need to be used if any of the solutions to the likelihood equations falls outside the parameter space.

Table 2.2: One Pair of Pseudotables

1	0		0	1
0	1		1	0

2.3 Common Odds Ratio Estimators

2.3.1 Mantel-Haenszel Estimator

Suppose there are K 2×2 tables, let a_k, b_k, c_k and d_k be the data of the kth table and let n_k be the sum of the data. The Mantel-Haenszel estimator [27] for the common odds ratio is

$$\hat{\theta}_{MH} = \frac{\sum_1^K a_k d_k/n_k}{\sum_1^K b_k c_k/n_k}. \tag{2.2}$$

Agresti [1] and Santner and Duffy [42] discussed Mantel-Haenszel estimator. The Mantel-Haenszel estimator tends to overestimate the common odds ratio Breslow [7] studied for the sparse data and Hauck, Anderson and Leahy [21] studied for the large strata cases. They modified the Mantel-Haenszel estimator by adding pseudocounts to each cells to reduce the bias and the common odds ratio estimator for adding s observations to each table is

$$\hat{\theta}_{PMHO} = \frac{\sum_1^K (a_k + s/4)(d_s + s/4)}{\sum_1^K (b_k + s/4)(c_k + s/4)}.$$

In their study, the best choice of s is 0.25.

2.3.2 Mantel-Haenszel with Pseudotable Methods

To reduce the bias of the Mantel-Haenszel estimator, Wypij and Santner ([48]) introduced the Pseudodata methods by adding one or more pairs of pseudotables as in Table 2.2. They justified the pseudotable method using a Bayesian argument. The common odds ratio estimator for adding r pair of the pseudotables is

$$\hat{\theta}_{PMH} = \frac{\sum_1^K a_k d_k/n_k + r/2}{\sum_1^K b_k c_k/n_k + r/2}$$

2.3.3 Jackknifing Mantel-Haenszel Method

Pigeot and Strugholtz [35] used the idea of Quenouille [39] of using "pseudo-values" by calculating the same type of the estimators based on a reduced sample and investigated two jackknife techniques applied to the Mantel-Haenszel estimator by dropping one table every time or one observation at a time.

When dropping table technique is in use, they calculated the ith pseudo-value J_i^I which can be determined as

$$\begin{aligned} J_i^I &= K\hat{\theta}_{MH} - (K-1)\hat{\theta}_{MH,i} \\ \hat{\theta}_{MH} &= \frac{\sum_1^K a_k d_k/n_k}{\sum_1^K b_k c_k/n_k} \\ \hat{\theta}_{MH,i} &= \frac{\sum_{k=1,k\neq i}^K a_k d_k/n_k}{\sum_{k=1,k\neq i}^K b_k c_k/n_k} \\ \hat{\theta}_{JMH} &= \sum_{k=1}^K \theta_{MH,i}/K \end{aligned}$$

2.4 Variance Estimation Formula

The Mantel-Haenszel estimator is a ratio of sums of random terms which are not identically distributed. Using Taylor expansions on expressions of the form

$$\hat{\theta}_{MH} = \frac{\sum_1^K y_i}{\sum_1^K x_i}$$

centered at $\theta = \sum_1^K E[y_i]/\sum_1^K E[x_i]$, one can write

$$\hat{\theta}_{MH} - \theta \doteq \frac{1}{\sum_1^K E[x_i]} E[\sum_1^K y_i - \theta \sum_1^K x_i].$$

This suggests that approximate variance formula

$$\text{Var}(\hat{\theta}_{MH}) \doteq E[(\hat{\theta}_{MH} - \theta)^2] \doteq \left[\frac{1}{\sum_1^K E[x_i]}\right]^2 \sum_1^K \text{Var}(y_i - \theta x_i).$$

This approach is detailed in Cochran [12] for ratio estimator and was applied to the Mantel-Haenszel estimator by Breslow [7].

Chapter 3

Closed-Form Estimators for $I \times J \times K$ Tables

Consider a three-way contingency table in which some cells are only partially observed. The $k^{\underline{th}}$ table is arranged as in Table 3.1, where z_{ijklmn} denotes the data in the kth table with the level of variable $A = i$ and variable $B = j$. Here $l = 1$ indicates that variable A is observed (i is known) and $l = 0$ indicates that the level of variable A is unobserved ($i = .$ denotes unknown). The meaning of m and n are similar. The goal is to estimate the cell means μ^*_{ijk} from a fully observed $I \times J \times K$ table.

Let $R_a = 1$ indicate that variable A is observed and 0 if not. The meanings of R_b and R_c are similar. Assume there are no four-way or higher interactions for the categorical variables A, B, C, R_a, R_b and R_c. The log-linear model is

$$\begin{aligned}
\log(\mu_{ijklmn}) &= \mu + \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{ik}^{AC} + \alpha_{jk}^{BC} + \alpha_{ijk}^{ABC} \\
&\quad + \beta_l^{R_a} + \beta_m^{R_b} + \beta_n^{R_c} + \beta_{lm}^{R_aR_b} + \beta_{ln}^{R_aR_b} + \beta_{mn}^{R_bR_c} + \beta_{lmn}^{R_aR_bR_c}, \\
&\quad + \gamma_{il}^{AR_a} + \gamma_{im}^{AR_b} + \gamma_{in}^{AR_c} + \gamma_{jl}^{BR_a} + \gamma_{jm}^{BR_b} + \gamma_{jn}^{BR_c} + \gamma_{kl}^{CR_a} + \gamma_{km}^{CR_b} + +\gamma_{kn}^{CR_c} \\
&\quad + \gamma_{ijl}^{ABR_a} + \gamma_{ijm}^{ABR_b} + \gamma_{ijn}^{ABR_c} + \gamma_{ikl}^{ACR_a} + \gamma_{ikm}^{ACR_b} + \gamma_{ikn}^{ACR_c} + \gamma_{jkl}^{BCR_a} + \gamma_{jkm}^{BCR_b} \\
&\quad + \gamma_{jkn}^{BCR_c} + \gamma_{ilm}^{AR_aR_b} + \gamma_{iln}^{AR_aR_c} + \gamma_{imn}^{AR_bR_c} + \gamma_{jlm}^{BR_aR_b} + \gamma_{jln}^{BR_aR_c} + \gamma_{jmn}^{BR_bR_c} \\
&\quad + \gamma_{klm}^{CR_aR_b} + \gamma_{kln}^{CR_aR_c} + \gamma_{kmn}^{CR_bR_c}.
\end{aligned}$$

We extend Baker, Rosenberger and Dersimonian's [5] method to reparameterize as

$$\begin{aligned}
m_{ijk} &= NP(A = i, B = j, C = k, R_a = 1, R_b = 1, R_c = 1) \\
&= \exp(\mu + \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{ik}^{AC} + \alpha_{jk}^{BC} + \alpha_{ijk}^{ABC} \\
&\quad + \beta_1^{R_a} + \beta_1^{R_b} + \beta_1^{R_c} + \beta_{11}^{R_aR_b} + \beta_{11}^{R_aR_c} + \beta_{11}^{R_bR_c} + \beta_{111}^{R_aR_bR_c})
\end{aligned}$$

Table 3.1: the kth table

	$B=1$	$B=2$	...	$B=J$	$B=.$
$A=1$	z_{11k111}	z_{12k111}	...	z_{1Jk111}	z_{1+k101}
$A=2$	z_{21k111}	z_{22k111}	...	z_{2Jk111}	z_{2+k101}
.	...	...	...	...	...
.	...	...	...	...	...
.	...	...	...	...	...
$A=I$	z_{I1k111}	z_{I2k111}	...	z_{IJk111}	z_{I+k101}
$A=.$	z_{+1k011}	z_{+2k011}	...	z_{+JK011}	z_{++k001}

$$\begin{aligned}
a_{ijk} &= \frac{P(R_a=0,R_b=1,R_c=1|A=i,B=j,C=k)}{P(R_a=1,R_b=1,R_c=1|A=i,B=j,C=k)} \\
&= \exp(\mu+\alpha_i^A+\alpha_j^B+\alpha_k^C+\alpha_{ij}^{AB}+\alpha_{ik}^{AC}+\alpha_{jk}^{BC}+\alpha_{ijk}^{ABC} \\
&\quad +\beta_0^{R_a}+\beta_1^{R_b}+\beta_1^{R_c}+\beta_{01}^{R_aR_b}+\beta_{01}^{R_aR_c}+\beta_{11}^{R_bRc}+\beta_{011}^{R_aR_bR_c} \\
&\quad +\gamma_{i0}^{AR_a}+\gamma_{i1}^{AR_b}+\gamma_{i1}^{AR_c}+\gamma_{j0}^{BR_a}+\gamma_{j1}^{BR_b}+\gamma_{j1}^{BR_c}+\gamma_{k0}^{CR_a}+\gamma_{k1}^{CR_b}+\gamma_{k1}^{CR_c} \\
&\quad +\gamma_{ij0}^{ABR_a}+\gamma_{ji1}^{ABR_b}+\gamma_{ji1}^{ABR_c}+\gamma_{ik0}^{ACR_a}+\gamma_{ik1}^{ACR_b}+\gamma_{ik1}^{ACR_c}+\gamma_{jk0}^{BCR_a}+\gamma_{jk1}^{BCR_b} \\
&\quad +\gamma_{jk1}^{BCR_c}+\gamma_{i01}^{AR_aR_b}+\gamma_{i01}^{AR_aR_c}+\gamma_{i11}^{AR_bR_c}+\gamma_{j01}^{BR_aR_b}+\gamma_{j01}^{BR_aR_c}+\gamma_{j11}^{BR_bR_c} \\
&\quad \gamma_{k01}^{CR_aR_b}+\gamma_{k01}^{CR_aR_c}+\gamma_{k11}^{CR_bR_c}) \\
&\quad /\exp(\mu+\alpha_i^A+\alpha_j^B+\alpha_k^c+\alpha_{ij}^{AB}+\alpha_{ik}^{AC}+\alpha_{jk}^{BC}+\alpha_{ijk}^{ABC} \\
&\quad +\beta_1^{R_a}+\beta_1^{R_b}+\beta_1^{R_c}+\beta_{11}^{R_aR_b}+\beta_{11}^{R_aR_c}+\beta_{11}^{R_bRc}+\beta_{111}^{R_aR_bR_c} \\
&\quad +\gamma_{i1}^{AR_a}+\gamma_{i1}^{AR_b}+\gamma_{i1}^{AR_c}+\gamma_{j1}^{BR_a}+\gamma_{j1}^{BR_b}+\gamma_{j1}^{BR_c}+\gamma_{k1}^{CR_a}+\gamma_{k1}^{CR_b}+\gamma_{k1}^{CR_c} \\
&\quad +\gamma_{ij1}^{ABR_a}+\gamma_{ji1}^{ABR_b}+\gamma_{ji1}^{ABR_c}+\gamma_{ik1}^{ACR_a}+\gamma_{ik1}^{ACR_b}+\gamma_{ik1}^{ACR_c}+\gamma_{jk1}^{BCR_a}+\gamma_{jk1}^{BCR_b} \\
&\quad +\gamma_{jk1}^{BCR_c}+\gamma_{i11}^{AR_aR_b}+\gamma_{i11}^{AR_aR_c}+\gamma_{i11}^{AR_bR_c}+\gamma_{j11}^{BR_aR_b}+\gamma_{j11}^{BR_aR_c}+\gamma_{j11}^{BR_bR_c} \\
&\quad \gamma_{k11}^{CR_aR_b}+\gamma_{k11}^{CR_aR_c}+\gamma_{k11}^{CR_bR_c})
\end{aligned}$$

$$
\begin{aligned}
&= \exp(\beta_0^{R_a} + \beta_{01}^{R_aR_b} + \beta_{01}^{R_aR_c} + \beta_{011}^{R_aR_bR_c} + \gamma_{i0}^{AR_a} + \gamma_{j0}^{BR_a} + \gamma_{k0}^{CR_a} + \gamma_{ij0}^{ABR_a} \\
&\quad + \gamma_{ik0}^{ACR_a} + \gamma_{jk0}^{BCR_a} + \gamma_{i01}^{AR_aR_b} + \gamma_{i01}^{AR_aR_c} + \gamma_{j01}^{BR_aR_b} + \gamma_{j01}^{BR_aR_c} + \gamma_{k01}^{CR_aR_b} + \gamma_{k01}^{CR_aR_c} \\
&\quad - \beta_1^{R_a} - \beta_{11}^{R_aR_b} - \beta_{11}^{R_aR_c} - \beta_{111}^{R_aR_bR_c} - \gamma_{i1}^{AR_a} - \gamma_{j1}^{BR_a} - \gamma_{k1}^{CR_a} - \gamma_{ij1}^{ABR_a} \\
&\quad - \gamma_{ik1}^{ACR_a} - \gamma_{jk1}^{BCR_a} - \gamma_{i11}^{AR_aR_b} - \gamma_{i11}^{AR_aR_c} - \gamma_{j11}^{BR_aR_b} - \gamma_{j11}^{BR_aR_c} - \gamma_{k11}^{CR_aR_b} - \gamma_{k11}^{CR_aR_c}) \\
&= \exp[-2(\beta_1^{R_a} + \beta_{11}^{R_aR_b} + \beta_{11}^{R_aR_c} + \beta_{111}^{R_aR_bR_c} + \gamma_{i1}^{AR_a} + \gamma_{j1}^{BR_a} + \gamma_{k1}^{CR_a} + \gamma_{ij1}^{ABR_a} + \gamma_{ik1}^{ACR_a} \\
&\quad + \gamma_{jk1}^{BCR_a} + \gamma_{i11}^{AR_aR_b} + \gamma_{j11}^{BR_aR_b} + \gamma_{k11}^{CR_aR_b})] \\
b_{ijk} &= \frac{P(R_a=1, R_b=0, R_c=1|A=i, B=j, C=k)}{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)} \\
&= \exp[-2(\beta_1^{R_b} + \beta_{11}^{R_aR_b} + \beta_{11}^{R_bR_c} + \beta_{111}^{R_aR_bR_c} + \gamma_{i1}^{AR_b} + \gamma_{j1}^{BR_b} + \gamma_{k1}^{CR_b} + \gamma_{ij1}^{ABR_b} + \gamma_{ik1}^{ACR_b} \\
&\quad + \gamma_{jk1}^{BCR_b} + \gamma_{i11}^{AR_aR_c} + \gamma_{j11}^{BR_aR_c} + \gamma_{k11}^{CR_aR_c})] \\
c_{ijk} &= \frac{P(R_a=1, R_b=1, R_c=0|A=i, B=j, C=k)}{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)} \\
&= \exp[-2(\beta_1^{R_c} + \beta_{11}^{R_aR_c} + \beta_{11}^{R_bR_c} + \beta_{111}^{R_aR_bR_c} + \gamma_{i1}^{AR_C} + \gamma_{j1}^{BR_c} + \gamma_{k1}^{CR_c} + \gamma_{ij1}^{ABR_c} + \gamma_{ik1}^{ACR_c} \\
&\quad + \gamma_{jk1}^{BCR_c} + \gamma_{i11}^{AR_bR_c} + \gamma_{j11}^{BR_BR_c} + \gamma_{k11}^{CR_bR_c})] \\
d_{ijk} &= \frac{P(R_a=0, R_b=0, R_c=1|A=i, B=j, C=k)}{P(R_a=0, R_b=1, R_c=1|A=i, B=j, C=k)} \\
&\quad \times \frac{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)}{P(R_a=1, R_b=0, R_c=1|A=i, B=j, C=k)} \\
&= \exp[-2(\beta_1^{R_b} + \beta_{01}^{R_aR_b} + \beta_{11}^{R_bR_c} + \beta_{011}^{R_aR_bR_c} + \gamma_{i1}^{AR_b} + \gamma_{j1}^{BR_b} + \gamma_{k1}^{CR_b} + \gamma_{ij1}^{ABR_b} \\
&\quad + \gamma_{ik1}^{ACR_b} + \gamma_{jk1}^{BCR_b} + \gamma_{i01}^{AR_aR_b} + \gamma_{i11}^{AR_bR_c} + \gamma_{j01}^{BR_aR_b} + \gamma_{j11}^{BR_bR_c} + \gamma_{k01}^{CR_aR_b} + \gamma_{k11}^{CR_bR_c})] \\
&\quad \times \exp[2(\beta_1^{R_b} + \beta_{11}^{R_aR_b} + \beta_{11}^{R_bR_c} + \beta_{111}^{R_aR_bR_c} + \gamma_{i1}^{AR_b} + \gamma_{j1}^{BR_b} + \gamma_{k1}^{CR_b} + \gamma_{ij1}^{ABR_b} \\
&\quad + \gamma_{ik1}^{ACR_b} + \gamma_{jk1}^{BCR_b} + \gamma_{i11}^{AR_aR_b} + \gamma_{i11}^{AR_bR_c} + \gamma_{j11}^{BR_aR_b} + \gamma_{j11}^{BR_bR_c} + \gamma_{k11}^{CR_aR_b} + \gamma_{k11}^{CR_bR_c})] \\
&= \exp[4(\beta_{11}^{R_aR_b} + \beta_{111}^{R_aR_bR_c} + \gamma_{i11}^{AR_aR_b} + \gamma_{j11}^{BR_aR_b} + \gamma_{k11}^{CR_aR_b})] \\
e_{ijk} &= \frac{P(R_a=0, R_b=1, R_c=0|A=i, B=j, C=k)}{P(R_a=0, R_b=1, R_c=1|A=i, B=j, C=k)} \\
&\quad \times \frac{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)}{P(R_a=1, R_b=1, R_c=0|A=i, B=j, C=k)} \\
&= \exp[4(\beta_{11}^{R_aR_c} + \beta_{111}^{R_aR_bR_c} + \gamma_{i11}^{AR_aR_c} + \gamma_{j11}^{BR_aR_c} + \gamma_{k11}^{CR_aR_c})] \\
f_{ijk} &= \frac{P(R_a=1, R_b=0, R_c=0|A=i, B=j, C=k)}{P(R_a=1, R_b=0, R_c=1|A=i, B=j, C=k)} \\
&\quad \times \frac{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)}{P(R_a=1, R_b=1, R_c=0|A=i, B=j, C=k)}
\end{aligned}
$$

$$= \exp[4(\beta_{11}^{R_bR_c} + \beta_{111}^{R_aR_bR_c} + \gamma_{i11}^{AR_bR_c} + \gamma_{j11}^{BR_bR_c} + \gamma_{k11}^{CR_bR_c})]$$

$$\begin{aligned} g_{ijk} &= \frac{P(R_a=0,R_b=0,R_c=0|A=i,B=j,C=k)}{P(R_a=0,R_b=0,R_c=1|A=i,B=j,C=k)} \\ &\quad \times \frac{P(R_a=0,R_b=1,R_c=1|A=i,B=j,C=k)}{P(R_a=0,R_b=1,R_c=0|A=i,B=j,C=k)} \\ &\quad \times \frac{P(R_a=1,R_b=0,R_c=1|A=i,B=j,C=k)}{P(R_a=1,R_b=0,R_c=0|A=i,B=j,C=k)} \\ &\quad \times \frac{P(R_a=1,R_b=1,R_c=0|A=i,B=j,C=k)}{P(R_a=1,R_b=1,R_c=1|A=i,B=j,C=k)} \\ &= \exp[-2(\beta_1^{R_c} + \beta_{01}^{R_aR_c} + \beta_{01}^{R_bR_c} + \beta_{001}^{R_aR_bR_c} + \gamma_{i0}^{AR_c} + \gamma_{j0}^{BR_c} + \gamma_{k0}^{CR_c} + \gamma_{ij0}^{ABR_c} \\ &\quad + \gamma_{ik0}^{ACR_c} + \gamma_{jk0}^{BCR_c} + \gamma_{i01}^{AR_aR_c} + \gamma_{i01}^{AR_bR_c} + \gamma_{j01}^{BR_aR_c} + \gamma_{j01}^{BR_bR_c} + \gamma_{k01}^{CR_aR_c} + \gamma_{k01}^{CR_bR_c})] \\ &\quad \times \exp[2(\beta_1^{R_c} + \beta_{01}^{R_aR_c} + \beta_{11}^{R_bR_c} + \beta_{011}^{R_aR_bR_c} + \gamma_{i1}^{AR_c} + \gamma_{j1}^{BR_c} + \gamma_{k1}^{CR_c} + \gamma_{ij1}^{ABR_c} + \gamma_{ik1}^{ACR_c} \\ &\quad + \gamma_{jk1}^{BCR_c} + \gamma_{i01}^{AR_aR_c} + \gamma_{i11}^{AR_bR_c} + \gamma_{j01}^{BR_aR_c} + \gamma_{j11}^{BR_bR_c} + \gamma_{k01}^{CR_aR_c} + \gamma_{k11}^{CR_bRc})] \\ &\quad \times \exp[2(\beta_1^{R_c} + \beta_{11}^{R_aR_c} + \beta_{01}^{R_bR_c} + \beta_{101}^{R_aR_bR_c} + \gamma_{i1}^{AR_c} + \gamma_{j1}^{BR_c} + \gamma_{k1}^{CR_c} + \gamma_{ij1}^{ABR_c} + \gamma_{ik1}^{ACR_c} \\ &\quad + \gamma_{jk1}^{BCR_c} + \gamma_{i11}^{AR_aR_c} + \gamma_{i01}^{AR_bR_c} + \gamma_{j11}^{BR_aR_c} + \gamma_{j01}^{BR_bR_c} + \gamma_{k11}^{CR_aR_c} + \gamma_{j01}^{CR_bR_c})] \\ &\quad \times \exp[-2(\beta_1^{R_c} + \beta_{11}^{R_aR_c} + \beta_{11}^{R_bR_c} + \beta_{111}^{R_aR_bR_c} + \gamma_{i1}^{AR_c} + \gamma_{j1}^{BR_c} + \gamma_{k1}^{CR_c} + \gamma_{ij1}^{ABR_c} + \gamma_{ik1}^{ACR_c} \\ &\quad + \gamma_{jk1}^{BCR_c} + \gamma_{i11}^{AR_aR_c} + \gamma_{i11}^{AR_bR_c} + \gamma_{j11}^{BR_aR_c} + \gamma_{j11}^{BR_bR_c} + \gamma_{k11}^{CR_aR_c} + \gamma_{k11}^{CR_bR_c})] \\ &= \exp[-8\beta_{111}^{R_aR_bR_c}] \\ &= g \text{ independent of } i,j,k. \end{aligned}$$

Therefore,

$$\begin{aligned} &\mu_{ijk111} = m_{ijk}, && \mu_{ijk011} = m_{ijk}a_{ijk}, \\ &\mu_{ijk101} = m_{ijk}b_{ijk}, && \mu_{ijk110} = m_{ijk}c_{ijk}, \\ &\mu_{ijk001} = m_{ijk}a_{ijk}b_{ijk}d_{ijk}, && \mu_{ijk010} = m_{ijk}a_{ijk}c_{ijk}e_{ijk}, \\ &\mu_{ijk100} = m_{ijk}b_{ijk}c_{ijk}f_{ijk}, && \mu_{ijk000} = m_{ijk}a_{ijk}b_{ijk}c_{ijk}d_{ijk}e_{ijk}f_{ijk}g, \\ &m_{ijk} \geq 0, \quad a_{ijk} \geq 0, && b_{ijk} \geq 0, \\ &d_{ijk} \geq 0, \quad e_{ijk} \geq 0, && f_{ijk} \geq 0, \quad g \geq 0 \end{aligned}$$

3.1 Variable C always observed

We examine the simpler case where the stratum variable C is always observed, but the row variable A and column variable B may be missing. As we assumed before, there are no four-way or higher interactions, so that, after simplifying the subscript notation

$$\begin{aligned}\log(\mu_{ijklm}) &= \mu + \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{ik}^{AC} + \alpha_{jk}^{BC} + \alpha_{ijk}^{ABC} \\ &+\beta_l^{R_a} + \beta_m^{R_b} + \beta_{lm}^{R_aR_b} \\ &+\gamma_{il}^{AR_a} + \gamma_{im}^{AR_b} + \gamma_{jl}^{BR_a} + \gamma_{jm}^{BR_b} + \gamma_{kl}^{CR_a} + \gamma_{km}^{CR_b} + \gamma_{ijl}^{ABR_a} + \gamma_{ijm}^{ABR_b} \\ &+\gamma_{ikl}^{ACR_a} + \gamma_{ikm}^{ACR_b} + \gamma_{ilm}^{AR_aR_b} + \gamma_{jkl}^{BCR_a} + \gamma_{jkm}^{BCR_b} + \gamma_{jlm}^{BR_aR_b} + \gamma_{klm}^{CR_aR_b}.\end{aligned}$$

Hence

$$\begin{aligned}m_{ijk} &= NP(A=i, B=j, C=k, R_a=1, R_b=1), \\ a_{ijk} &= \frac{P(R_a=0, R_b=1, R_c=1|A=i, B=j, C=k)}{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)}, \\ b_{ijk} &= \frac{P(R_a=1, R_b=0, R_c=1|A=i, B=j, C=k)}{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)}, \\ d_{ijk} &= \frac{P(R_a=0, R_b=0, R_c=1|A=i, B=j, C=k)}{P(R_a=0, R_b=1, R_c=1|A=i, B=j, C=k)}, \\ &\times\frac{P(R_a=1, R_b=1, R_c=1|A=i, B=j, C=k)}{P(R_a=1, R_b=0, R_c=1|A=i, B=j, C=k)},\end{aligned}$$

$$\mu_{ijk11} = m_{ijk}, \qquad \mu_{ijk01} = m_{ijk}a_{ijk},$$

$$\mu_{ijk10} = m_{ijk}b_{ijk}, \qquad \mu_{ijk00} = m_{ijk}a_{ijk}b_{ijk}d_{ijk},$$

$$m_{ijk} \geq 0, \quad a_{ijk} \geq 0, \quad b_{ijk} \geq 0, \quad d_{ijk} \geq 0.$$

Since R_c always is 1 in this case, we drop the sixth subscript to simplify the notation. Also note that $e_{ijk} = f_{ijk} = g_{ijk} = 0$. In this case one computes separate estimates for the K subtables of order $I \times J$. Tables 3.2, 3.3 and 3.4 list the closed form solutions for various missingness models.

Under the common odds ratio assumption, the log-linear model is

$$\begin{aligned}\log(\mu_{ijklm}) &= \mu + \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{ik}^{AC} + \alpha_{jk}^{BC} \\ &+\beta_l^{R_a} + \beta_m^{R_b} + \beta_n^{R_c} + \beta_{lm}^{R_aR_b} + \beta_{ln}^{R_aR_b} + \beta_{mn}^{R_bR_c} + \beta_{lmn}^{R_aR_bR_c} \\ &+\gamma_{il}^{AR_a} + \gamma_{im}^{AR_b} + \gamma_{in}^{AR_c} + \gamma_{jl}^{BR_a} + \gamma_{jm}^{BR_b} + \gamma_{jn}^{BR_c} + \gamma_{kl}^{CR_a} + \gamma_{km}^{CR_b} + +\gamma_{kn}^{CR_c} \\ &+\gamma_{ijl}^{ABR_a} + \gamma_{ijm}^{ABR_b} + \gamma_{ijn}^{ABR_c} + \gamma_{ikl}^{ACR_a} + \gamma_{ikm}^{ACR_b} + \gamma_{ikn}^{ACR_c} + \gamma_{jkl}^{BCR_a} + \gamma_{jkm}^{BCR_b} \\ &+\gamma_{jkn}^{BCR_c} + \gamma_{ilm}^{AR_aR_b} + \gamma_{iln}^{AR_aR_c} + \gamma_{imn}^{AR_bR_c} + \gamma_{jlm}^{BR_aR_b} + \gamma_{jln}^{BR_aR_c} + \gamma_{jmn}^{BR_bR_c} \\ &+\gamma_{klm}^{CR_aR_b} + \gamma_{kln}^{CR_aR_c} + \gamma_{kmn}^{CR_bR_c}.\end{aligned}$$

Table 3.2: Closed Form Estimates for $I \times J \times K$ Tables

(a)	Missingness of both (A, B) depends on C		
	$\hat{m}_{ijk}^{(0)}$	$=$	z_{ijk11}
	$\hat{m}_{ijk}^{(t+1)}$	$=$	$[z_{ijk11} + z_{+jk01}(\hat{m}_{ijk}^{(t)}/\hat{m}_{+jk}^{(t)}) + z_{i+k10}(\hat{m}_{ijk}^{(t)})/(\hat{m}_{i+k}^{(t)})]$
			$/(z_{++k11} + z_{++k10} + z_{++k01})$
	$\hat{a}_k$	$=$	z_{++k01}/z_{++k11}
	$\hat{b}_k$	$=$	z_{++k10}/z_{++k11}
	$\hat{d}_k$	$=$	$z_{++k11}z_{++k00}/z_{++k10}z_{++k01}$
(b1)	Missingness of A depends on C, Missingness of B depends on (A, C)		
	$\hat{m}_{ijk}$	$=$	$z_{ijk11}z_{++k11}z_{+jk+1}/z_{++k+1}z_{+jk+1}$
	$\hat{a}_k$	$=$	z_{++k01}/z_{++k11}
	$\hat{b}_{ik}$	$=$	$z_{i+k10}/\hat{m}_{i+k}$
	$\hat{d}_k$	$=$	$z_{++k11}z_{++k00}/z_{++k10}z_{++k01}$
(b2)	Missingness of A depends on (B, C) Missingness of B depends on C		
	$\hat{m}_{ijk}$	$=$	$z_{ijk11}z_{++k11}z_{i+k1+}/z_{++k1+}z_{i+k1+}$
	$\hat{a}_{jk}$	$=$	$z_{+jk01}/\hat{m}_{+jk}$
	$\hat{b}_k$	$=$	z_{++k10}/z_{++k}
	$\hat{d}_k$	$=$	$z_{++k11}z_{++k00}/z_{++k10}z_{++k01}$

Table 3.3: Closed Form Estimates for $I \times J \times K$ Tables (Continued)

(c1)	Missingness of A depends on C		
	Missingness of B depends on (B, C)		
	$\hat{m}_{ijk}$	$=$	$z_{ijk11} z_{+jk+1} z_{++k11} / z_{++k+1} z_{+jk11}$
	$\hat{a}_k$	$=$	z_{++k01} / z_{++k11}
	$\hat{b}_{jk}$	such that	$\sum_j \hat{m}_{ijk} \hat{b}_{jk} = z_{i+k10}$
	$\hat{d}_k$	$=$	$z_{++k11} z_{++k00} / z_{++k10} z_{++k01}$
(c2)	Missingness of A depends on (A, C)		
	Missingness of B depends on C		
	$\hat{m}_{ijk}$	$=$	$z_{ijk11} z_{i+k1+} z_{++k11} / z_{i+k11} z_{++k1+}$
	$\hat{a}_{ik}$	such that	$\sum_i \hat{m}_{ijk} \hat{a}_{ik} = z_{+jk01}$
	$\hat{b}_k$	$=$	z_{++k10} / z_{++k11}
	$\hat{d}_k$	$=$	$z_{++k11} z_{++k00} / z_{++k10} z_{++k01}$
(d1)	Missingness of (A, B) depends on (A, C)		
	$\hat{m}_{ijk}$	$=$	z_{ijk11}
	$\hat{a}_{ik}$	such that	$\sum_i z_{ijk11} \hat{a}_{ik} = z_{+jk01}$
	$\hat{b}_{ik}$	$=$	z_{i+k10} / z_{i+k11}
	$\hat{d}_k$	$=$	$z_{++k00} / \sum_i z_{i+k11} \hat{a}_{ik} \hat{b}_{ik}$
(d2)	Missingness of (A, B) depends on (B, C)		
	$\hat{m}_{ijk}$	$=$	z_{ijk11}
	$\hat{a}_{jk}$	$=$	z_{+jk01} / z_{+jk11}
	$\hat{b}_{jk}$	such that	$\sum_j z_{ijk11} \hat{b}_{jk} = z_{i+k10}$
	$\hat{d}_k$	$=$	$z_{++k00} / \sum_j z_{+jk11} \hat{a}_{jk} \hat{b}_{jk}$

Table 3.4: Closed Form Estimates for $I \times J \times K$ Tables (Continued)

(e)	Missingness of A depends on (A, C)		
	Missingness of B depends on (B, C)		
	$\hat{m}_{ijk}$	$=$	z_{ijk11}
	$\hat{a}_{ik}$	such that $\sum_i \hat{m}_{ijk}\hat{a}_{ik} = z_{+jk01}$	
	$\hat{b}_{jk}$	such that $\sum_j z_{ijk11}\hat{b}_{jk} = z_{i+k10}$	
	$\hat{d}_k$	$=$	$z_{++k00}/\sum_i \sum_j z_{ijk11}\hat{a}_{ik}\hat{b}_{jk}$
(f)	Missingness of A depends on (B, C)		
	Missingness of B depends on (A, C)		
	$\hat{m}_{ijk}$	$=$	z_{ijk11}
	$\hat{a}_{jk}$	$=$	z_{+jk01}/z_{+jk11}
	$\hat{b}_{ik}$	$=$	z_{i+k10}/z_{i+k11}
	$\hat{d}_k$	$=$	$z_{++k00}/\sum_i \sum_j z_{ijk11}\hat{a}_{jk}\hat{b}_{ik}$

As $\alpha_{ijk}^{ABC} = 0$ in the common odds ratio models, the closed form for $\hat{m}_{ijk}$ will be the solutions of the system of equations

$$\begin{aligned}
\sum_i m_{ijk}(1 + a_{ijk} + b_{ijk}) &= z_{+jk11} + \sum_i z_{+jk01}\frac{m_{ijk}a_{ijk}}{m_{+jk}a_{ijk}} + \sum_i z_{i+k10}\frac{m_{ijk}b_{ijk}}{m_{i+k10}b_{ijk}} m_{i+k}b_{ijk} \\
\sum_j m_{ijk}(1 + a_{ijk} + b_{ijk}) &= z_{i+k11} + \sum_j z_{+jk01}\frac{m_{ijk}a_{ijk}}{m_{+jk}a_{ijk}} + \sum_j z_{i+k10}\frac{m_{ijk}b_{ijk}}{m_{i+k10}b_{ijk}} m_{i+k}b_{ijk} \\
\sum_k m_{ijk}(1 + a_{ijk} + b_{ijk}) &= z_{ij+11} + \sum_k z_{+jk01}\frac{m_{ijk}a_{ijk}}{m_{+jk}a_{ijk}} + \sum_k z_{i+k10}\frac{m_{ijk}b_{ijk}}{m_{i+k10}b_{ijk}} m_{i+k}b_{ijk}.
\end{aligned}$$

But the $\hat{a}_{ijk}, \hat{b}_{ijk}, \hat{d}_{ijk}$ are still the same as in Tables 3.2, 3.3 and 3.4.

3.2 Variables B and C always observed

In the special cases that both variables B and C are always observed, we consider the models in which missingness of variable A depends on C and either A or B but not both. In these models there are no four-way or higher interactions. After simplifying the subscript notation, we use μ_{ijkl} denote the expected cell counts and z_{ijkl} are the observations, and the log-linear model for the partially observed categorical variables is

$$\begin{aligned}
\log(\mu_{ijkl}) &= \mu + \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{ik}^{AC} + \alpha_{jk}^{BC} + \alpha_{ijk}^{ABC} \\
&\quad + \beta_l^{R_a} + \gamma_{il}^{AR_a} + \gamma_{jl}^{BR_a} + \gamma_{kl}^{CR_a} + \gamma_{ikl}^{ACR_a} + \gamma_{jkl}^{BCR_a}.
\end{aligned} \tag{3.1}$$

When missingness of A depends on (A, C) (informative missingness), only $(\gamma_{il}^{AR_a}, \gamma_{kl}^{CR_a}, \gamma_{ikl}^{ACR_a})$ will be in the model, and when missingness of A depends on (B, C) (Missing at Random or MAR(B, C)), only $(\gamma_{jl}^{BR_a}, \gamma_{kl}^{CR_a}, \gamma_{jkl}^{BCR_a})$ will be in the model. Moreover, as both B and C are always observed, $b_{ijk} = 0$ and $g = 0$ in both models, so only m_{ijk} and a_{ijk} need to be estimated. Table 3.5 shows the closed forms for these two models.

Table 3.5: Closed Form Estimator – B and C are Always Observed

	Model$MAR(B,C)$	Missing Informative(A,C)
$\hat{m}_{ijk}$	z_{ijk1}	z_{ijk1}
$\hat{a}_{ijk}$	$\hat{a}_{.jk} = \frac{z_{+jk0}}{z_{+jk1}}$	$\hat{a}_{i.k}$, such that
		$\sum_i \hat{m}_{ijk}\hat{a}_{i.k} = z_{+jk0}$
		Closed-form solution requires I=J
		and non-negative estimates.

3.3 Negative $\hat{\alpha}_i$ – boundary problems

In models (c1), (c2), (d1) and (d2) in Table 3.3, model (e) in Table 3.4 and the informative missingness model in Table 3.5, the closed-form solution might exist when $I = J$ and the solutions of $\hat{a}_{i.k}$ or $\hat{b}_{.jk}$ are nonnegative. If any solution of $\hat{a}_{i.k}$ is negative, the ML estimate lies on the boundary of the parameter space and boundary solutions need to be investigated. In the case that $I = J = 2$, we can obtain closed-form ML boundary estimates by setting one of the $\hat{a}_{i.k} = 0$ in the likelihood equations. According to Baker, Rosenberger and Dersimonian [5], we must evaluate both boundaries $\hat{a}_{1.k}$ and $\hat{a}_{2.k}$ and calculate G^2 in (3.3) to figure out which parameter is falling on the boundary of the parameter space.

$$\begin{aligned} G^2 \quad = \quad & -2\left\{\sum_i\sum_j z_{ijk11}\log\left[\frac{\hat{m}_{ijk}}{z_{ijk11}}\right] + \sum_i z_{i+10}\log\left[\frac{\sum_j \hat{m}_{ijk}\hat{b}_{ijk}}{z_{i+k10}}\right]\right. \\ & \left. + \sum_j z_{+jk01}\log\left[\frac{\sum_i \hat{m}_{ijk}\hat{a}_{ijk}}{z_{+jk01}}\right] + z_{++k00}\log\left[\frac{\sum_i\sum_j \hat{m}_{ijk}\hat{a}_{ijk}\hat{b}_{ijk}\hat{d}}{z_{++k00}}\right]\right\} \end{aligned} \tag{3.2}$$

If $G_1^2 = G_2^2$, then both $\hat{a}_{1.k}$ and $\hat{a}_{2.k}$ are set to be zero. Also, if there are not enough independent equations to solve for $\hat{a}_{i.k}$, then the boundary solution will be used.

In the informative missingness case,

$$G^2 = (-2)\left(\sum_i\sum_j z_{ijk11}\log\left[\hat{m}_{ijk}/z_{ijk11}\right] + \sum_j z_{+jk01}\log\left[\frac{\sum_i \hat{m}_{ijk}\hat{a}_{ijk}}{z_{+jk01}}\right]\right)$$

Table 3.6: Boundary Solution for Informative(A, C) when both B and C are always observed

	$\hat{a}_{1.k} = 0$	$\hat{a}_{2.k} = 0$
$\hat{m}_{1jk}$	z_{1jk1}	$\frac{z_{1+k1}(z_{1jk1}+z_{+jk0})}{z_{1+k1}+z_{++k0}}$
$\hat{m_{2jk}}$	$\frac{z_{2+k1}(z_{2jk1}+z_{+jk0})}{z_{2+k1}+z_{++k0}}$	z_{2jk1}
$\hat{a}_{1.k}$	0	$\frac{z_{++k0}}{z_{1+k1}}$
$\hat{a}_{2.k}$	$\frac{z_{++k0}}{z_{2+k1}}$	0

The boundary estimates for $2 \times 2 \times K$ tables when both the stratum variable C and column variable B are always observed are presented in Table 3.6

Chapter 4

Common Odds Ratio Estimation

Consider K 2×2 contingency tables where each table has n_k observations and $\sum_1^K n_k = N$. When the stratum variable $C = k$, and no data are missing, we observe the table

	$B=1$	$B=2$	Total
$A=1$	z_{11k}	z_{12k}	n_{1k}
$A=2$	z_{21k}	z_{22k}	n_{2k}
			n_k

for $k = 1, \cdots, K$. We assume that the odds ratio $\pi_{11k}\pi_{22k}/\pi_{12k}\pi_{21k} = \theta$ independent of k (common odds ratio). We consider the cases where the stratum variable C and the column variable B are always observed but the row variable A might be missing for some of the n_k observations. Variable A is either missing at random depending on variables B and C (MAR(B, C)) or informatively missing depending on variables A and C (Informative(A, C)). We study estimation of the common odds ratio based on the hypothetical fully observed data set, only the completely observed subset and closed-form estimated data set of Chapter 3.

4.1 Methods

4.1.1 Mantel-Haenszel Estimator

The Mantel-Haenszel estimator for the common odds ratio is

$$\hat{\theta}_{MH} = \frac{\sum_{k=1}^K z_{11k} z_{22k}/n_k}{\sum_{k=1}^K z_{12k} z_{21k}/n_k}$$

so that

$$\hat{\theta}_{MH} - \theta = \frac{\sum_{k=1}^K (z_{11k} z_{22k}/n_k - \theta z_{12k} z_{21k}/n_k)}{\sum_{k=1}^K z_{12k} z_{21k}/n_k}$$

The quantities $(z_{11k},\, z_{12k},\, z_{21k})$ have a multinomial distribution with parameters n_k, π_{11k}, π_{12k}, and π_{21k}, where $n_k - z_{11k} - z_{12k} - z_{21k} = z_{22k}$ and $1 - \pi_{11k} - \pi_{12k} - \pi_{21k} = \pi_{22k}$. Under the common odds ratio assumption,

$$\begin{aligned}
\theta &= \frac{\pi_{11k}\pi_{22k}}{\pi_{12k}\pi_{21k}}, \\
\theta\pi_{12k}\pi_{21k} &= \pi_{11k}\pi_{22k}, \\
\theta\pi_{12k}\pi_{21k} &= \pi_{11k}(1 - \pi_{11k} - \pi_{12k} - \pi_{21k}), \\
\theta\pi_{12k}\pi_{21k} &= \pi_{11k} - \pi_{11k}^2 - \pi_{11k}\pi_{12k} - \pi_{11k}\pi_{21k}, \\
(\pi_{11k} + \pi_{12k}\theta)\pi_{21k} &= \pi_{11k}(1 - \pi_{11k} - \pi_{12k}), \\
\pi_{21k} &= \frac{\pi_{11k}(1 - \pi_{11k} - \pi_{12k})}{\pi_{11k} + \pi_{12k}\theta}, \\
\pi_{22k} &= 1 - \pi_{11k} - \pi_{12k} - \pi_{21k} \\
&= 1 - \pi_{11k} - \pi_{12k} - \frac{\pi_{11k}(1 - \pi_{11k} - \pi_{12k})}{\pi_{11k} + \pi_{12k}\theta}.
\end{aligned}$$

Using properties of the multinomial distribution

$$\begin{aligned}
E[z_{11k}z_{22k}/n_k] &= E\left[\frac{z_{11k}z_{22k}}{n_k}\right] \\
&= \frac{1}{n_k}E[z_{11k}z_{22k}] \\
&= \frac{1}{n_k}n_k(n_k - 1)\pi_{11k}\pi_{22k} \\
&= (n_k - 1)\pi_{11k}\pi_{22k},
\end{aligned}$$

and

$$E[z_{12k}z_{21k}/n_k] = (n_k - 1)\pi_{12k}\pi_{21k}.$$

Therefore

$$\begin{aligned}
E\left[\sum z_{12k}z_{21k}/n_k\right] &= \sum_{k=1}^{K}(n_k - 1)\pi_{12k}\pi_{21k} \\
&= \sum_{k=1}^{K}(n_k - 1)\pi_{11k}\pi_{22k}/\theta \\
&= \mu_x.
\end{aligned}$$

When K and θ are fixed, μ_x is $O(N)$ and when n_k, and θ are fixed and the π_{ijk} are bounded away from 0, μ_x is $O(K)$. When K and n_k are fixed, μ_x is $O(1/\theta)$.

Writing $\hat{\theta}_{MH} = y/x$ and using a Taylor expansion, $\hat{\theta}_{MH} - \theta$ can be written as

$$\begin{aligned} \frac{y-\theta x}{x} &= \frac{y-\theta x}{\mu_x}\left[1 - \frac{x-\mu_x}{\mu_x} + \cdots\right] \\ &= \frac{y-\theta x}{\mu_x} - \left(\frac{y-\theta x}{\mu_x}\right)\left(\frac{x-\mu_x}{\mu_x}\right) + \left(\frac{y-\theta x}{\mu_x}\right)\left(\frac{x-\mu_x}{\mu_x}\right)^2 - \cdots \end{aligned} \tag{4.1}$$

The expected value of the first term of (4.1) is

$$\begin{aligned} & E\left[\sum_{k=1}^{K}\left(z_{11k}z_{22k}/n_k - \theta z_{12k}z_{21k}/n_k\right)/\mu_x\right] \\ &= \frac{1}{\mu_x}\sum \frac{1}{n_k}E[z_{11k}z_{22k} - \theta z_{12k}z_{21k}] \\ &= \frac{1}{\mu_x}\sum \frac{1}{n_k}(n_k(n_k-1)\pi_{11k}\pi_{22k} - \theta n_k(n_k-1)\pi_{12k}\pi_{21k}) \\ &= \frac{1}{\mu_x}\sum (n_k-1)(\pi_{11k}\pi_{22k} - \theta\pi_{12k}\pi_{21k}) \\ &= 0. \end{aligned}$$

The expected value of the second term of (4.1) is

$$\begin{aligned} & E\left[\left(\frac{\sum_1^K (z_{11k}z_{22k}/n_k - \theta z_{12k}z_{21k}/n_k)}{\mu_x}\right)\left(\frac{\sum_1^K z_{12k}z_{21k}/n_k - \mu_x}{\mu_x}\right)\right] \\ &= E\left[\frac{1}{\mu_x^2}\sum\left(\frac{z_{11k}z_{22k} - \theta z_{12k}z_{21k}}{n_k}\right)\left(\frac{z_{12k}z_{21k}}{n_k} - E\left[\frac{z_{12k}z_{21k}}{n_k}\right]\right)\right] \\ &\quad + E\left[\frac{1}{\mu_x^2}\sum_{k\neq k'}\left(\frac{z_{11k}z_{22k} - \theta z_{12k}z_{21k}}{n_k}\right)\left(\frac{z_{12k'}z_{21k'}}{n_{k'}} - E\left[\frac{z_{12k'}z_{21k'}}{n_{k'}}\right]\right)\right] \\ &= \frac{1}{\mu_x^2}\left(\sum_{k=1}^{K} A_k + B\right). \end{aligned}$$

We have

$$\begin{aligned} A_k &= E\left[\left(\frac{z_{11k}z_{22k} - \theta z_{12k}z_{21k}}{n_k}\right)\left(\frac{z_{12k}z_{21k}}{n_k} - E\left[\frac{z_{12k}z_{21k}}{n_k}\right]\right)\right] \\ &= E\left[\frac{(z_{11k}z_{22k} - \theta z_{12k}z_{21k})(z_{12k}z_{21k})}{n_k^2}\right] \\ &\quad - E\left[\frac{z_{12k}z_{21k}}{n_k}\right]E\left[\frac{z_{11k}z_{22k} - \theta z_{12k}z_{21k}}{n_k}\right] \\ &= E\left[\frac{1}{n_k^2}z_{11k}z_{12k}z_{21k}z_{22k}\right] - \theta E\left[\frac{1}{n_k^2}z_{12k}^2 z_{21k}^2\right] \end{aligned}$$

because $E[z_{11k}z_{22k} - \theta z_{12k}z_{21k}] = 0$.

As

$$
\begin{aligned}
& E\left[\frac{1}{n_k^2} z_{11k}z_{12k}z_{21k}z_{22k}\right] \\
= \quad & \frac{1}{n_k^2} E[z_{11k}z_{12k}z_{21k}(n_k - z_{11k} - z_{12k} - z_{21k})] \\
= \quad & \frac{1}{n_k^2} E[n_k z_{11k}z_{12k}z_{21k} - z_{11k}^2 z_{12k}z_{21k} - z_{11k}z_{12k}^2 z_{21k} - z_{11k}z_{12k}z_{21k}^2] \\
= \quad & \frac{1}{n_k^2} E[n_k z_{11k}z_{12k}z_{21k} - z_{11k}(z_{11k}-1)z_{12k}z_{21k} - z_{11k}z_{12k}(z_{12k}-1)z_{21k}] \\
& \quad - \frac{1}{n_k^2} E[z_{11k}z_{12k}z_{21k}(z_{21k}-1) - 3z_{11k}z_{12k}z_{21k}] \\
= \quad & \frac{1}{n_k^2}(n_k n_k(n_k-1)(n_k-2)\pi_{11k}\pi_{12k}\pi_{21k} - n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{12k}\pi_{21k}) \\
& \quad - \frac{1}{n_k^2} n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}^2\pi_{21k} \\
& \quad - \frac{1}{n_k^2} n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}\pi_{21k}^2 \\
& \quad - \frac{1}{n_k^2} 3n_k(n_k-1)(n_k-2)\pi_{11k}\pi_{12k}\pi_{21k} \\
= \quad & \frac{n_k(n_k-1)(n_k-2)}{n_k^2}\pi_{11k}\pi_{12k}\pi_{21k}(n_k - (n_k-3)(\pi_{11k}+\pi_{12k}+\pi_{21k}) - 3) \\
= \quad & \frac{(n_k-1)(n_k-2)(n_k-3)}{n_k}\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k}
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\frac{1}{n_k^2} z_{12k}^2 z_{21k}^2\right] \\
= \quad & \frac{1}{n_k^2} E[z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1) + z_{12k}(z_{12k}-1)z_{21k}] \\
& \quad + \frac{1}{n_k^2} E[z_{12k}z_{21k}(z_{21k}-1) + z_{12k}z_{21k}] \\
= \quad & \frac{1}{n_k^2}(n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^2\pi_{21k}^2 + n_k(n_k-1)(n_k-2)\pi_{12k}^2\pi_{21k}) \\
& \quad + \frac{1}{n_k^2}(n_k(n_k-1)(n_k-2)\pi_{12k}\pi_{21k}^2 + n_k(n_k-1)\pi_{12k}\pi_{21k}) \\
= \quad & \frac{(n_k-1)(n_k-2)(n_k-3)}{n_k}\pi_{12k}\pi_{21k}(\pi_{11k}\pi_{22k}/\theta) \\
& \quad + \frac{(n_k-1)(n_k-2)}{n_k}\pi_{12k}\pi_{21k}(\pi_{12k}+\pi_{21k}) + \frac{n_k-1}{n_k}\pi_{12k}\pi_{21k}.
\end{aligned}
$$

Then

$$
E\left[\frac{1}{n_k^2} z_{11k}z_{12k}z_{21k}z_{22k}\right] - \theta E\left[\frac{1}{n_k^2} z_{12k}^2 z_{21k}^2\right]
$$

$$\begin{aligned}
&= \frac{(n_k-1)(n_k-2)(n_k-3)}{n_k}\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k} \\
&\quad - \theta\frac{(n_k-1)(n_k-2)(n_k-3)}{n_k}\pi_{12k}\pi_{21k}(\pi_{11k}\pi_{22k}/\theta) \\
&\quad - \theta\frac{(n_k-1)(n_k-2)}{n_k}\pi_{12k}\pi_{21k}(\pi_{12k}+\pi_{21k}) + \frac{n_k-1}{n_k}\pi_{12k}\pi_{21k} \\
&= -\theta\frac{(n_k-1)}{n_k}\pi_{12k}\pi_{21k}[(n_k-2)(\pi_{12k}+\pi_{21k})+1] \\
&= -\frac{n_k-1}{n_k}\pi_{11k}\pi_{22k}[(n_k-2)(1-\pi_{11k}-\pi_{22k})+1].
\end{aligned}$$

This implies

$$\begin{aligned}
A_k &= -\frac{(n_k-1)}{n_k}\pi_{11k}\pi_{22k}((n_k-2)(1-\pi_{11k}-\pi_{22k})+1) \\
B &= E\left[\frac{1}{\mu_x^2}\sum_{k\neq k'}\left(\frac{z_{11k}z_{22k}-\theta z_{12k}z_{21k}}{n_k}\right)\left(\frac{z_{12k'}z_{21k'}}{n_{k'}} - E\left[\frac{z_{12k'}z_{21k'}}{n_{k'}}\right]\right)\right] \\
&= \frac{1}{\mu_x^2}\sum_{k\neq k'}\left(\frac{1}{n_k}E[z_{11k}z_{22k}-\theta z_{12k}z_{21k}]\frac{1}{n_{k'}}E[z_{12k'}z_{21k'} - E[z_{12k'}z_{21k'}]]\right) \\
&= 0.
\end{aligned}$$

Therefore, the leading term of the bias for the Mantel-Haenszel estimator is

$$\begin{aligned}
-\frac{1}{\mu_x^2}\sum_{k=1}^{K}A_k &= -\frac{1}{\mu_x^2}\sum_{k=1}^{K}\left(-\frac{(n_k-1)}{n_k}\pi_{11k}\pi_{22k}((n_k-2)(1-\pi_{11k}-\pi_{22k})+1)\right) \\
&= \frac{1}{\mu_x^2}\sum\frac{(n_k-1)}{n_k}\pi_{11k}\pi_{22k}[(n_k-2)(1-\pi_{11k}-\pi_{22k})+1] \\
&\geq 0.
\end{aligned}$$

Because $\mu_x^{-2} = O(1/N^2)$ and the sum is $O(\sum n_k)$,

$$-\frac{1}{\mu_x^2}\sum_1^K A_k = O(1/N).$$

The expected value of the third term of (4.1) is

$$\begin{aligned}
&E\left[\left(\frac{\sum_1^K(z_{11k}z_{22k}/n_k - \theta z_{12k}z_{21k}/n_k)}{\mu_x}\right)\left(\frac{\sum_1^K(z_{12k}z_{21k}/n_k - E[z_{12k}z_{21k}/n_k])}{\mu_x}\right)^2\right] \\
&\quad = \frac{1}{\mu_x^3}\sum_1^K E\left[\frac{z_{11k}z_{22k}-\theta z_{12k}z_{21k}}{n_k}\left(\frac{z_{12k}z_{21k}}{n_k} - \frac{E[z_{12k}z_{21k}]}{n_k}\right)^2\right]. \qquad (4.2)
\end{aligned}$$

As the observations from different strata are independent, all of the expectations of cross-product terms are zeros.

The kth term of the summation in (4.2) is

$$
\begin{aligned}
& E\left[\frac{z_{11k}z_{22k}-\theta z_{12k}z_{21k}}{n_k}\left(\frac{z_{12k}z_{21k}}{n_k}-\frac{E[z_{12k}z_{21k}]}{n_k}\right)^2\right] \\
&= E\left[\frac{z_{11k}z_{12k}^2z_{21k}^2z_{22k}}{n_k^3}\right]-\theta E\left[\frac{z_{12k}^3z_{21k}^3}{n_k^3}\right] \\
&\quad -2\left(E\left[\frac{z_{11k}z_{12k}z_{21k}z_{22k}E[z_{12k}z_{21k}]}{n_k^3}\right]+\theta E\left[\frac{z_{12k}^2z_{21k}^2E[z_{12k}z_{21k}]}{n_k^3}\right]\right) \\
&\quad + E\left[\frac{z_{11k}z_{22k}-\theta z_{12k}z_{21k}}{n_k}\right]\left(E\left[\frac{z_{12k}z_{21k}}{n_k}\right]^2\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
& E\left[\frac{z_{11k}z_{12k}^2z_{21k}^2z_{22k}}{n_k^3}\right] \\
&= \frac{1}{n_k^3}E[z_{11k}z_{12k}^2z_{21k}^2z_{22k}] \\
&= \frac{1}{n_k^3}E[z_{11k}z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1)z_{22k}+z_{11k}z_{12k}(z_{12k}-1)z_{21k}z_{22k}] \\
&\quad +\frac{1}{n_k^3}E[z_{11k}z_{12k}z_{21k}(z_{21k}-1)z_{22k}+z_{11k}z_{12k}z_{21k}z_{22k}] \\
&= \frac{1}{n_k^3}n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)(n_k-5)\pi_{11k}\pi_{12k}^2\pi_{21k}^2\pi_{22k} \\
&\quad +\frac{1}{n_k^3}n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k}(\pi_{12k}+\pi_{21k}) \\
&\quad +\frac{1}{n_k^3}n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k} \\
&= \frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)(n_k-5)\pi_{11k}\pi_{12k}^2\pi_{21k}^2\pi_{22k} \\
&\quad +\frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k}(\pi_{12k}+\pi_{21k}) \\
&\quad +\frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
z_{12k}^3z_{21k}^3 &= z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1)(z_{21k}-2)+3z_{12k}^3z_{21k}^2+3z_{12k}^2z_{21k}^3 \\
&\quad -2z_{12k}^3z_{21k}-9z_{12k}^2z_{21k}^2-2z_{12k}z_{21k}^3+6z_{12k}^2z_{21k}+6z_{12k}z_{21k}^2-4z_{12k}z_{21k} \\
z_{12k}^3z_{21k}^2 &= z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1)+z_{12k}^3z_{21k}+3z_{12k}^2z_{21k}^2 \\
&\quad -3z_{12k}^2z_{21k}-2z_{12k}z_{21k}^2+2z_{12k}z_{21k} \\
z_{12k}^2z_{21k}^3 &= z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1)(z_{21k}-2)+3z_{12k}^2z_{21k}^2+z_{12k}z_{21k}^3 \\
&\quad -2z_{12k}^2z_{21k}-3z_{12k}z_{21k}^2+2z_{12k}z_{21k}
\end{aligned}
$$

$$\begin{aligned}
z_{12k}^3 z_{21k} &= z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k} + 3z_{12}^2 z_{21k} - 2z_{12k}z_{21k} \\
z_{12k}^2 z_{21k}^2 &= z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1) + z_{12k}^2 z_{21k} + z_{12k}z_{21k}^2 - z_{12k}z_{21k} \\
z_{12k} z_{21k}^3 &= z_{12k}z_{21k}(z_{21k}-1)(z_{21k}-2) + 3z_{12}z_{21k}^2 - 2z_{12k}z_{21k} \\
z_{12k}^2 z_{21k} &= z_{12k}(z_{12k}-1)z_{21k} + z_{12k}z_{21k} \\
z_{12k} z_{21k}^2 &= z_{12k}z_{21k}(z_{12k}-1) + z_{12k}z_{21k}.
\end{aligned}$$

So

$$\begin{aligned}
z_{12k}^3 z_{21k}^3 &= z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1)(z_{21k}-2) \\
&\quad + 3z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1) \\
&\quad + 3z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1)(z_{21k}-2) \\
&\quad + z_{12k}^3 z_{21k} + 9z_{12k}^2 z_{21k}^2 + z_{12k}z_{21k}^3 - 9z_{12k}^2 z_{21k} - 9z_{12k}z_{21k}^2 + 8z_{12k}z_{21k}, \\
&= z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1)(z_{21k}-2) \\
&\quad + 3z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1) \\
&\quad + 3z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1)(z_{21k}-2) \\
&\quad + z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k} + 9z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1) \\
&\quad + z_{12k}z_{21k}(z_{21k}-1)(z_{21k}-2) + 12(z_{12k}(z_{12k}-1)z_{21k} + z_{12k}z_{21k}) \\
&\quad + 12(z_{12k}z_{21k}(z_{12k}-1) + z_{12k}z_{21k}) - 23z_{12k}z_{21k}, \\
&= z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1)(z_{21k}-2) \\
&\quad + 3z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1) \\
&\quad + 3z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1)(z_{21k}-2) \\
&\quad + z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k} + 9z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1) \\
&\quad + z_{12k}z_{21k}(z_{21k}-1)(z_{21k}-2) + 12z_{12k}(z_{12k}-1)z_{21k} \\
&\quad + 12z_{12k}z_{21k}(z_{12k}-1) + z_{12k}z_{21k}.
\end{aligned}$$

We have

$$E[z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1)(z_{21k}-2)|n_k]$$

$$
\begin{aligned}
&= n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)(n_k-5)\pi_{12k}^3\pi_{21k}^3 \\
&E[z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}(z_{21k}-1)|n_k] \\
&= n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{12k}^3\pi_{21k}^2 \\
&E[z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1)(z_{21k}-2)|n_k] \\
&= n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{12k}^2\pi_{21k}^3
\end{aligned}
$$

$$
\begin{aligned}
E[z_{12k}(z_{12k}-1)(z_{12k}-2)z_{21k}] &= n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^3\pi_{21k} \\
E[z_{12k}(z_{12k}-1)z_{21k}(z_{21k}-1)] &= n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^2\pi_{21k}^2 \\
E[z_{12k}z_{21k}(z_{21k}-1)(z_{21k}-2)] &= n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}\pi_{21k}^3 \\
E[z_{12k}(z_{12k}-1)z_{21k}] &= n_k(n_k-1)(n_k-2)\pi_{12k}^2\pi_{21k} \\
E[z_{12k}z_{21k}(z_{12k}-1)] &= n_k(n_k-1)(n_k-2)\pi_{12k}\pi_{21k}^2 \\
E[z_{12k}z_{21k}] &= n_k(n_k-1)\pi_{12k}\pi_{21k}
\end{aligned}
$$

Hence

$$
\begin{aligned}
E[z_{12k}^3z_{21k}^3] = {} & n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)(n_k-5)\pi_{12k}^3\pi_{21k}^3 \\
& + 3n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{12k}^3\pi_{21k}^2 \\
& + 3n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{12k}^2\pi_{21k}^3 \\
& + n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^3\pi_{21k} \\
& + 9n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^2\pi_{21k}^2 \\
& + n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}\pi_{21k}^3 + 12n_k(n_k-1)(n_k-2)\pi_{12k}^2\pi_{21k} \\
& + 12n_k(n_k-1)(n_k-2)\pi_{12k}\pi_{21k}^2 + n_k(n_k-1)\pi_{12k}\pi_{21k}.
\end{aligned}
$$

Therefore,

$$
E\left[\frac{z_{12k}^3z_{21k}^3}{n_k^3}\right] = \frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)(n_k-5)\pi_{11k}^3\pi_{22k}^3/\theta^3)
$$

$$
\begin{aligned}
&+ \frac{3}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}^2\pi_{22k}^2/\theta^2)(\pi_{12k}+\pi_{21k}) \\
&+ \frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^2(\pi_{11k}\pi_{22k}/\theta) \\
&+ \frac{9}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2/\theta^2) \\
&+ \frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{21k}^2(\pi_{11k}\pi_{22k}/\theta) \\
&+ \frac{12}{n_k^2}(n_k-1)(n_k-2)\pi_{12k}(\pi_{11k}\pi_{22k}/\theta) \\
&+ \frac{12}{n_k^2}(n_k-1)(n_k-2)\pi_{21k}(\pi_{11k}\pi_{22k}/\theta) + \frac{n_k-1}{n_k^2}\pi_{11k}\pi_{22k}/\theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E\left[\frac{z_{11k}z_{12k}^2z_{21k}^2z_{22k}}{n_k^3}\right] - \theta E\left[\frac{z_{12k}^3z_{21k}^3}{n_k^3}\right] \\
= \;& \frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)(n_k-5)\pi_{11k}^3\pi_{22k}^3/\theta^2 \\
&+ \frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}^2\pi_{22k}^2/\theta(1-\pi_{11k}-\pi_{22k}) \\
&+ \frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2/\theta \\
&- \frac{\theta}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)(n_k-5)\pi_{11k}^3\pi_{22k}^3/\theta^3) \\
&- \frac{3\theta}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}^2\pi_{22k}^2/\theta^2(1-\pi_{11k}-\pi_{22k}) \\
&- \frac{\theta}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(\pi_{12k}^2+\pi_{21k}^2)\pi_{11k}\pi_{22k}/\theta \\
&- \frac{9\theta}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2/\theta^2 \\
&- \frac{12\theta}{n_k^2}(n_k-1)(n_k-2)(\pi_{12k}+\pi_{21k})(\pi_{11k}\pi_{22k}/\theta) \\
&- \frac{\theta}{n_k^2}(n_k-1)\pi_{11k}\pi_{22k}/\theta \\
= \;& -\frac{2}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}^2\pi_{22k}^2/\theta(1-\pi_{11k}-\pi_{22k}) \\
&- \frac{8}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2/\theta \\
&- \frac{\theta}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(\pi_{12k}^2+\pi_{21k}^2)(\pi_{11k}\pi_{22k}/\theta) \\
&- \frac{12\theta}{n_k^2}(n_k-1)(n_k-2)(\pi_{12k}+\pi_{21k})(\pi_{11k}\pi_{22k}/\theta) \\
&- \frac{\theta}{n_k^2}(n_k-1)\pi_{11k}\pi_{22k}/\theta \\
= \;& -\frac{2}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}^2\pi_{22k}^2(1-\pi_{11k}-\pi_{22k})/\theta \\
&- \frac{6}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2/\theta
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{(n_k-1)(n_k-2)(n_k-3)}{n_k^2}\pi_{11k}\pi_{22k}(\pi_{12k}+\pi_{21k})^2 \\
&- 12\frac{(n_k-1)(n_k-2)}{n_k^2}\pi_{11k}\pi_{22k}(1-\pi_{11k}-\pi_{22k}) \\
&- \frac{n_k-1}{n_k^2}\pi_{11k}\pi_{22k}
\end{aligned}
$$

and

$$
\begin{aligned}
&-2E\left[\frac{z_{11k}z_{12k}z_{21k}z_{22k}E[z_{12k}z_{21k}]}{n_k^3}\right] + 2\theta E\left[\frac{z_{12k}^2 z_{21k}^2 E[z_{12k}z_{21k}]}{n_k^3}\right] \\
&= -2E\left[\frac{1}{n_k^3}(z_{11k}z_{12k}z_{21k}z_{22k} - \theta z_{12k}^2 z_{21k}^2)E[z_{12k}z_{21k}]\right] \\
&= -2n_k(n_k-1)\pi_{12k}\pi_{21k}E\left[\frac{1}{n_k}\left(E\left[\frac{1}{n_k^2}z_{11k}z_{12k}z_{21k}z_{22k}\right] - \theta E\left[\frac{1}{n_k^2}z_{12k}^2 z_{21k}^2\right]\right)\right] \\
&= -2(n_k-1)\pi_{12k}\pi_{21k}\left(-\theta\frac{n_k-1}{n_k}\pi_{12k}\pi_{21k}[(n_k-2)(\pi_{12k}+\pi_{21k})+1]\right) \\
&= 2\frac{(n_k-1)^2}{n_k}\pi_{11k}^2\pi_{22k}^2[(n_k-2)(1-\pi_{11k}-\pi_{22k})+1]/\theta \\
&E\left[\frac{z_{11k}z_{22k}-\theta z_{12k}z_{21k}}{n_k}\right]\left(E\left[\frac{z_{12k}z_{21k}}{n_k}\right]^2\right) \\
&= 0.
\end{aligned}
$$

So

$$
\begin{aligned}
&E\left[\frac{z_{11k}z_{22k}-\theta z_{12k}z_{21k}}{n_k}\left(\frac{z_{12k}z_{21k}}{n_k} - \frac{E[z_{12k}z_{21k}]}{n_k}\right)^2\right] \\
&= -\frac{2}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{11k}^2\pi_{22k}^2(1-\pi_{11k}-\pi_{22k})/\theta \\
&\quad - \frac{6}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2/\theta \\
&\quad - \frac{(n_k-1)(n_k-2)(n_k-3)}{n_k^2}\pi_{11k}\pi_{22k}(1-\pi_{11k}-\pi_{22k})^2 \\
&\quad - 12\frac{(n_k-1)(n_k-2)}{n_k^2}\pi_{11k}\pi_{22k}(1-\pi_{11k}-\pi_{22k}) \\
&\quad - \frac{n_k-1}{n_k^2}\pi_{11k}\pi_{22k} \\
&\quad + 2\frac{(n_k-1)^2(n_k-2)}{n_k}\pi_{11k}^2\pi_{22k}^2(1-\pi_{11k}-\pi_{22k})/\theta \\
&\quad + 2\frac{(n_k-1)^2}{n_k}\pi_{11k}^2\pi_{22k}^2/\theta.
\end{aligned}
$$

We also have

$$
-\frac{2}{n_k^2}(n_k-1)(n_k-2)(n_k-3)(n_k-4) + 2\frac{(n_k-1)^2(n_k-2)}{n_k}
$$

$$
\begin{aligned}
&= 2\frac{(n_k-1)(n_k-2)}{n_k^2}(-(n_k-3)(n_k-4)+(n_k-1)n_k) \\
&= 2\frac{(n_k-1)(n_k-2)}{n_k^2}(-n_k^2+7n_k-12+n_k^2-n_k) \\
&= 2\frac{(n_k-1)(n_k-2)}{n_k^2}(6n_k-12) \\
&= 12\frac{(n_k-1)(n_k-2)^2}{n_k^2}.
\end{aligned}
$$

Writing $N = \sum_1^K n_k$, for fixed K and θ, $\mu_x = O(\sum n_k)$ and the third term of (4.1) is

$$
\begin{aligned}
& E\left[\left(\frac{\sum_1^K z_{11k}z_{22k}/n_k - \theta z_{12k}z_{21k}/n_k}{\mu_x}\right)\left(\frac{\sum_1^K z_{12k}z_{21k}/n_k - E[z_{12k}z_{21k}/n_k]}{\mu_x}\right)^2\right] \\
&= \frac{1}{\mu_x^3}\sum_1^K\Bigg(\frac{12}{n_k^2}(n_k-1)(n_k-2)^2\pi_{11k}^2\pi_{22k}^2(1-\pi_{11k}-\pi_{22k}) \\
&\qquad -\frac{6}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2/\theta \\
&\qquad -\frac{1}{n_k^2}(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{22k}(1-\pi_{11k}-\pi_{22k})^2 \\
&\qquad -\frac{12}{n_k^2}(n_k-1)(n_k-2)\pi_{11k}\pi_{22k}(1-\pi_{11k}-\pi_{22k}) \\
&\qquad -\frac{1}{n_k^2}(n_k-1)\pi_{11k}\pi_{22k}+2\frac{(n_k-1)^2}{n_k}\pi_{11k}^2\pi_{22k}^2/\theta\Bigg) \\
&= O(1/N^3)\sum_1^K O(n_k). \\
&= O(1/N^2)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{Bias}_{MH} &= -\frac{1}{\mu_x^2}\sum_1^K A_k + \text{ Remainder} \\
&= \frac{1}{\mu_x^2}\sum_{k=1}^K \frac{(n_k-1)}{n_k}\pi_{11k}\pi_{22k}[(n_k-2)(1-\pi_{11k}-\pi_{22k})+1] + O(1/N^2) \qquad (4.3) \\
&= O(1/N),
\end{aligned}
$$

as $N \to \infty$, and the dominant term is positive.

When n_k and K are fixed, we have $\mu_x = O(1/\theta)$ and $A_k = O(1)$, then the second term of (4.1) is

$$
-\frac{1}{\mu_x^2}\sum_1^K A_k = O(\theta^2)O(1) = O(\theta^2),
$$

and the third term of (4.1) is $O(\theta^3)O(1) = O(\theta^3)$.

Therefore, for fixed K and N, when $\theta \to \infty$, the bias $\to \infty$.

For a fixed θ, bounded n_k and the π_{ijk} are bounded, away from 0,

$$\begin{aligned}
\mu_x &= \sum_1^K (n_k - 1)\pi_{12k}\pi_{21k} \\
&= \sum_1^K (n_k - 1)c_k \\
&\leq \sum_1^K \max n_k c_k \\
&= O(K).
\end{aligned}$$

Then

$$\begin{aligned}
\text{Leading term of the Bias} &= -\frac{1}{\mu_x^2}\sum_1^K A_k \\
&= O(1/K^2)O(K) \\
&= O(1/K)
\end{aligned}$$

and the third term of (4.1) is $O(1/K^3)O(K) = O(1/K^2)$. So if n_k is fixed, $K \to \infty$ and $\sum_1^K \pi_{12k}\pi_{21k} \to \infty$, the leading term of the bias also converges to 0.

Next we investigate the variance. We have

$$\begin{aligned}
\text{Var }(\hat{\theta}_{MH} - \theta) &\doteq \text{Var}\left(\sum_{k=1}^K \left(\frac{z_{11k}z_{22k}/n_k - \theta z_{12k}z_{21k}/n_k}{\mu_x}\right)\right) \\
&= \frac{1}{\mu_x^2}\sum_1^K E\left[\left(\frac{z_{11k}z_{22k} - \theta z_{12k}z_{21k}}{n_k}\right)^2\right] - 0 \\
&= \frac{1}{\mu_x^2}\sum_1^K \frac{1}{n_k^2} E[z_{11k}^2 z_{22k}^2 - 2\theta z_{11k}z_{12k}z_{21k}z_{22k} + \theta^2 z_{12k}^2 z_{21k}^2] \\
&= \frac{1}{\mu_x^2}\sum_1^K \frac{1}{n_k^2}(E[z_{11k}^2 z_{22k}^2] + E[z_{12k}^2 z_{21k}^2] - 2\theta E[z_{11k}z_{12k}z_{21k}z_{22k}]).
\end{aligned}$$

We have

$$E[z_{11k}^2 z_{22k}^2] = E[z_{11k}(z_{11k}-1)z_{22k}(z_{22k}-1) + z_{11k}(z_{11k}-1)z_{22k}]$$

$$
\begin{aligned}
&+ E[z_{11k}z_{22k}(z_{22k}-1) + z_{11k}z_{22k}] \\
= \quad & n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2 \\
&+ n_k(n_k-1)(n_k-2)\pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}) + n_k(n_k-1)\pi_{11k}\pi_{22k}
\end{aligned}
$$

and

$$
\begin{aligned}
E[z_{11k}z_{12k}z_{21k}z_{22k}] &= n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k} \\
E[z_{12k}^2 z_{21k}^2] &= n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^2\pi_{21k}^2 \\
&\quad + n_k(n_k-1)(n_k-2)\pi_{12k}\pi_{21k}(\pi_{12k}+\pi_{21k}) + n_k(n_k-1)\pi_{12k}\pi_{21k}.
\end{aligned}
$$

Under the common odds ratio assumption, $\pi_{11k}\pi_{22k} = \theta\pi_{12k}\pi_{21k}$. Therefore the unconditional expected value of the kth term of the summation in the variance formula is

$$
\begin{aligned}
&E[(z_{11k}z_{22k} - \theta z_{12k}z_{21k})^2] \\
= \quad & n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}^2\pi_{22k}^2 \\
&+ n_k(n_k-1)(n_k-2)\pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}) + n_k(n_k-1)\pi_{11k}\pi_{22k} \\
&- 2\theta n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k} \\
&+ \theta^2 n_k(n_k-1)(n_k-2)(n_k-3)\pi_{12k}^2\pi_{21k}^2 \\
&+ \theta^2 n_k(n_k-1)(n_k-2)\pi_{12k}\pi_{21k}(\pi_{12k}+\pi_{21k}) + \theta^2 n_k(n_k-1)\pi_{12k}\pi_{21k} \\
= \quad & n_k(n_k-1)(n_k-2)(n_k-3)\theta\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k} \\
&+ n_k(n_k-1)(n_k-2)\pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}) + n_k(n_k-1)\pi_{11k}\pi_{22k} \\
&- 2\theta n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k} \\
&+ \theta n_k(n_k-1)(n_k-2)(n_k-3)\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k} \\
&+ \theta n_k(n_k-1)(n_k-2)\pi_{11k}\pi_{22k}(\pi_{12k}+\pi_{21k}) + \theta n_k(n_k-1)\pi_{11k}\pi_{22k} \\
= \quad & n_k(n_k-1)(n_k-2)\pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}+\theta(\pi_{12k}+\pi_{21k})) \\
&+ n_k(n_k-1)\pi_{11k}\pi_{22k}(1+\theta).
\end{aligned}
$$

So the variance is

$$
\text{Var }(\hat{\theta}_{MH} - \theta) \quad \doteq \quad \frac{1}{\mu_x^2}\sum_1^K \left(\frac{(n_k-1)(n_k-2)}{n_k}\pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}+\theta(1-\pi_{11k}-\pi_{22k})) \right.
$$

$$+\frac{(n_k-1)}{n_k}\pi_{11k}\pi_{22k}(1+\theta)\Big).$$

As μ_x is $O(N)$ when K and θ are fixed and $O(K)$ when N and θ are fixed and the π_{ijk} are bounded away from 0, $\text{Var}\,(\hat{\theta}_{MH}-\theta)$ is $O(1/N)$ for fixed K and θ and $O(1/K)$ for fixed N and θ with all cell probabilities bounded away from 0. For fixed N and K, μ_x is $O(1/\theta)$ that implies $\text{Var}\,(\hat{\theta}_{MH}-\theta)$ is $O(\theta^3)$.

Therefore, the Mantel-Haenszel method overestimates the common odds ratio, but when either the minimum table size or the number of strata goes to infinity bounded away from 0 cell probabilities, the estimator converges to the true common odds ratio. Hauck [19] and Breslow [7] derived similar results.

4.1.2 Mantel-Haenszel Estimator with pseudo-data

To reduce the bias of the Mantel-Haenszel estimator, Wypij and Santner [48] introduced the pseudotable methods by adding to the original $2 \times 2 \times K$ table one or more pairs of the following pseudotables:

1	0
0	1

0	1
1	0

Here we focus on the case where only one pair of pseudotables are added. Then

$$\begin{aligned}
\hat{\theta}_{PMH1} &= \frac{\sum_{k=1}^{K}(z_{11k}z_{22k}/n_k)+1/2}{\sum_{k=1}^{K}(z_{12k}z_{21k}/n_k)+1/2} \\
\hat{\theta}_{PMH1}-\theta &= \frac{\sum_{k=1}^{K}(z_{11k}z_{22k}/n_k)+\frac{1}{2}-\theta\left(\sum_{k=1}^{K}(z_{12k}z_{21k}/n_k)+\frac{1}{2}\right)}{\sum_k(z_{12k}z_{21k}/n_k)+\frac{1}{2}} \\
E[\text{ Denominator}] &= E\left[\sum_k \frac{z_{12k}z_{21k}}{n_k}+\frac{1}{2}\right] \\
&= \sum E\left[\frac{z_{12k}z_{21k}}{n_k}\right]+\frac{1}{2} \\
&= \sum (n_k-1)\pi_{12k}\pi_{21k}+\frac{1}{2}
\end{aligned}$$

$$= \quad \mu_x + \frac{1}{2}$$

$$= \quad \mu_{x,PMH1}.$$

As in section 4.1.1, writing $\hat{\theta}_{PMH1} = y_{PMH1}/x_{PMH1}$ and using Taylor expansion, $\hat{\theta}_{PMH1} - \theta$ can be written as

$$\begin{aligned}
\frac{y_{PMH1} - \theta x_{PMH1}}{x_{PMH1}} &= \frac{y_{PMH1} - \theta x_{PMH1}}{\mu_{x,PMH1}} \left[1 - \frac{x_{PMH1} - \mu_{x,PMH1}}{\mu_{x,PMH1}} + ...\right] \\
&= \frac{y_{PMH1} - \theta x_{PMH1}}{\mu_{x,PMH1}} - \left(\frac{y_{PMH1} - \theta x_{PMH1}}{\mu_{x,PMH1}}\right)\left(\frac{x_{PMH1} - \mu_{x,PMH1}}{\mu_{x,PMH1}}\right) \\
&\quad + \left(\frac{y_{PMH1} - \theta x_{PMH1}}{\mu_{x,PMH1}}\right)\left(\frac{x_{PMH1} - \mu_{x,PMH1}}{\mu_{x,PMH1}}\right)^2 + \cdots . \qquad (4.4)
\end{aligned}$$

The expected value of the first term of (4.4) is

$$\begin{aligned}
&E\left[\frac{y_{PMH1} - \theta x_{PMH1}}{\mu_{x,PMH1}}\right] \\
&= E\left[\frac{1}{\mu_{x,PMH1}}\left[\sum_{k=1}^{K}\left(\frac{z_{11k}z_{22k}}{n_k}\right) + \frac{1}{2} - \theta\left(\sum_{k=1}^{K}\left(\frac{z_{12k}z_{21k}}{n_k}\right) + \frac{1}{2}\right)\right]\right] \\
&= \frac{1}{\mu_{x,PMH1}}\left(\sum_{k=1}^{K} E\left[\frac{z_{11k}z_{22k}}{n_k} - \theta\left(\frac{z_{12k}z_{21k}}{n_k}\right)\right] + \frac{1-\theta}{2}\right) \\
&= \frac{1}{\mu_{x,PMH1}}\left[\sum \frac{n_k(n_k-1)}{n_k}\pi_{11k}\pi_{22k} - \theta\frac{n_k(n_k-1)}{n_k}\pi_{12k}\pi_{21k} + \frac{1-\theta}{2}\right] \\
&= \frac{1}{\mu_{x,PMH1}}\sum\left[\frac{n_k-1}{n_k}(\pi_{11k}\pi_{22k} - \theta\pi_{12k}\pi_{21k}) + \frac{1}{2}(1-\theta)\right] \\
&= \frac{1-\theta}{2\mu_{x,PMH1}}.
\end{aligned}$$

In the cases of $\theta > 1$ the expected value of the first term is negative. The expected value of the second term of (4.4) is

$$\begin{aligned}
&E\left[\frac{\sum(z_{11k}z_{22k}/n_k) + 1/2 - \theta\left(\sum(z_{12k}z_{21k}/n_k) + 1/2\right)}{\mu_{x,PMH1}}\left(\frac{\sum(z_{12k}z_{21k}/n_k) + 1/2 - \mu_{x,PMH1}}{\mu_{x,PMH1}}\right)\right] \\
&= \frac{1}{\mu^2_{x,PMH1}} E\left[\sum\frac{z_{11k}z_{22k}}{n_k} - \theta\left(\sum\frac{z_{12k}z_{21k}}{n_k}\right)\right]\left[\sum\frac{z_{12k}z_{21k}}{n_k} + \left(\frac{1}{2} - \mu_{x,PMH1}\right)\right] \\
&\quad + \frac{1}{2}\frac{(1-\theta)}{\mu^2_{x,PMH1}} E\left[\sum\frac{z_{12k}z_{21k}}{n_k} + \left(\frac{1}{2} - \mu_{x,PMH1}\right)\right] \\
&= \frac{1}{\mu^2_{x,PMH1}} E\left[\sum\left(\frac{z_{11k}z_{22k}}{n_k} - \theta\frac{z_{12k}z_{21k}}{n_k}\right)\left(\frac{z_{12k}z_{21k}}{n_k}\right)\right] \\
&\quad + \frac{1}{\mu^2_{x,PMH1}} E\left[\frac{1}{2}(1-\theta)\sum\left(\frac{z_{12k}z_{21k}}{n_k}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\mu^2_{x,PMH1}}(1-\theta)\left(\frac{1}{2}-\mu_{x,PMH1}\right)\\
= \quad & A+B+\frac{1}{\mu^2_{x,PMH1}}(1-\theta)\left(\frac{1}{2}-\mu_{x,PMH1}\right).
\end{aligned}
$$

The observations from different strata are independent, so the crossproduct terms all vanish. Similar to the calculations of section (4.1.1)

$$
\begin{aligned}
-A &= -\frac{1}{\mu^2_{x,PMH1}}\sum_{k=1}^{K} E\left[\frac{z_{11k}z_{12k}z_{21k}z_{22k}}{n_k^2}-\theta\frac{z_{12k}^2 z_{21k}^2}{n_k^2}\right]\\
&= -\frac{1}{\mu^2_{x,PMH1}}\sum_{k=1}^{K}\left(-\theta\frac{n_k-1}{n_k}\pi_{12k}\pi_{21k}[(n_k-2)(\pi_{12k}+\pi_{21k})+1]\right)\\
&= -\left(\frac{\mu_x}{\mu_{x,PMH1}}\right)^2 \text{ the expected value of the second term of 4.1)}\\
&< \text{ the expected value of the second term of (4.1)) because } \mu_{x,PMH1}>\mu_x.
\end{aligned}
$$

and

$$
\begin{aligned}
B &= \frac{1}{\mu^2_{x,PMH1}}E\left[\frac{1}{2}(1-\theta)\sum\frac{z_{12k}z_{21k}}{n_k}\right]\\
&= \frac{1}{\mu^2_{x,PMH1}}\frac{1}{2}(1-\theta)\sum(n_k-1)\pi_{12k}\pi_{21k}.\\
&\leq 0 \text{ when } \theta\geq 1.
\end{aligned}
$$

Therefore, the expected value of the second term of (4.4) is

$$
\begin{aligned}
&E\left[\frac{\sum(z_{11k}z_{22k}/n_k)+1/2-\theta\left(\sum(z_{12k}z_{21k}/n_k)+1/2\right)}{\mu_{x,PMH1}}\left(\frac{\sum(z_{12k}z_{21k}/n_k)+1/2-\mu_{x,PMH1}}{\mu_{x,PMH1}}\right)\right]\\
&= -\frac{1}{\mu^2_{x,PMH1}}\sum_{k=1}^{K}\left(-\theta\frac{n_k-1}{n_k}\pi_{12k}\pi_{21k}[(n_k-2)(\pi_{12k}+\pi_{21k})+1]\right)\\
&\quad+\frac{1}{\mu^2_{x,PMH1}}\frac{1}{2}(1-\theta)\sum(n_k-1)\pi_{12k}\pi_{21k}\\
&\quad+\frac{1}{\mu^2_{x,PMH1}}(1-\theta)\left(\frac{1}{2}-\mu_{x,PMH1}\right)\\
&= -\frac{1}{\mu^2_{x,PMH1}}\sum_{k=1}^{K}\left(-\theta\frac{n_k-1}{n_k}\pi_{12k}\pi_{21k}[(n_k-2)(\pi_{12k}+\pi_{21k})+1]\right)\\
&\quad-\frac{1}{\mu^2_{x,PMH1}}\frac{1}{2}(1-\theta)\left(\mu_{x,PMH1}-\frac{1}{2}\right).
\end{aligned}
$$

The expected value of the third term of (4.4) is

$$\frac{1}{\mu_{x,PMH1}^3} E\left[\left(\sum_1^K \frac{z_{11k}z_{22k}}{n_k} + \frac{1}{2} - \theta\left(\sum_1^K \frac{z_{12k}z_{21k}}{n_k} + \frac{1}{2}\right)\right)\left(\sum_1^K \frac{z_{12k}z_{21k}}{n_k} + \frac{1}{2} - \mu_{x,PMH1}\right)^2\right]$$

$$\begin{aligned}
&= \frac{1}{\mu_{x,PMH1}^3} E\left[\left(\sum_1^K \left(\frac{z_{11k}z_{22k}}{n_k} - \theta\frac{z_{12k}z_{21k}}{n_k}\right) + \frac{1}{2}(1-\theta)\right)\right.\\
&\qquad \left.\times\left(\sum_1^K \frac{z_{12k}z_{21k}}{n_k} + \frac{1}{2} - \left(\sum_1^K E\left[\frac{z_{12k}z_{21k}}{n_k}\right] + \frac{1}{2}\right)\right)^2\right]\\
&= \frac{1}{\mu_{x,PMH1}^3}\sum_1^K \frac{1}{n_k^3} E[(z_{11k}z_{22k} - \theta z_{12k}z_{21k})(z_{12k}z_{21k} - E[z_{12k}z_{21k}])^2]\\
&\quad + \frac{1}{\mu_{x,PMH1}^3}\frac{1}{2}(1-\theta)E\left[\left(\sum_1^K \frac{z_{12k}z_{21k}}{n_k} - \sum_1^K E\left[\frac{z_{12k}z_{21k}}{n_k}\right]\right)^2\right]\\
&= C + D.
\end{aligned}$$

Here C is $(\mu_x/\mu_{x,PMH1})^3$ times the third term of the Taylor expansion for the Mantel-Haenszel estimator. As $\mu_{x,PMH1} = \mu_x + 1/2$, $C = O(1/N^2)$ for fixed K and θ, $O(\theta^3)$ for fixed K and N, and $O(1/K^2)$ for fixed table sizes and θ and bounded away from 0 cell probabilities. Furthermore,

$$\begin{aligned}
&E\left[\left(\sum_1^K \frac{z_{12k}z_{21k}}{n_k} - \sum_1^K E\left[\frac{z_{12k}z_{21k}}{n_k}\right]\right)^2\right]\\
&= \sum_1^K \frac{1}{n_k^2} E[(z_{12k}z_{21k} - E[z_{12k}z_{21k}])^2]\\
&= \sum_1^K \frac{1}{n_k^2}(E[z_{12k}^2 z_{21k}^2] - E[z_{12k}z_{21k}]^2)\\
&= \sum_1^K \frac{(n_k-1)(n_k-2)(n_k-3)}{n_k}\frac{\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k}}{\theta} - \sum_1^K (n_k-1)^2\pi_{12k}^2\pi_{21k}^2\\
&\quad - \sum_1^K \frac{(n_k-1)(n_k-2)}{n_k}\pi_{12k}\pi_{21k}(\pi_{12k}+\pi_{21k}) - \sum_1^K \frac{n_k-1}{n_k}\pi_{12k}\pi_{21k}\\
&= \sum_1^K \frac{(n_k-1)}{n_k}\frac{\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k}}{\theta}((n_k-2)(n_k-3) - (n_k-1)n_k)\\
&\quad - \sum_1^K \frac{(n_k-1)(n_k-2)}{n_k}\pi_{12k}\pi_{21k}(\pi_{12k}+\pi_{21k}) - \sum_1^K \frac{n_k-1}{n_k}\pi_{12k}\pi_{21k}\\
&= \sum_1^K \frac{-2(n_k-1)(2n_k-3)}{n_k}\pi_{11k}^2\pi_{22k}^2/\theta^2\\
&\quad - \sum_1^K \frac{(n_k-1)(n_k-2)}{n_k}\pi_{11k}\pi_{22k}/\theta(1-\pi_{11k}-\pi_{22k}) - \sum_1^K \frac{n_k-1}{n_k}\pi_{11k}\pi_{22k}/\theta.
\end{aligned}$$

Therefore

$$\begin{aligned} D &= \frac{1-\theta}{2\mu^3_{x,PMH1}} \sum_1^K \left[\frac{-2(n_k-1)(2n_k-3)}{n_k} \frac{\pi^2_{11k}\pi^2_{22k}}{\theta^2} \right. \\ &\qquad \left. - \frac{(n_k-1)(n_k-2)}{n_k} \pi_{11k}\pi_{22k}/\theta(1-\pi_{11k}-\pi_{22k}) - \frac{n_k-1}{n_k}\pi_{11k}\pi_{22k}/\theta \right] \\ &= O(1/N^3)O(\sum n_k) \\ &= O(1/N^2). \end{aligned}$$

Therefore, for fixed K and θ

$$\begin{aligned} &\text{Bias}_{PMH1} \\ &= E[\hat{\theta}_{PMH1} - \theta] \\ &= \frac{1-\theta}{2\mu_{x,PMH1}} \\ &\quad + \left(\frac{\mu_x}{\mu_{x,PMH1}}\right)^2 \sum_{k=1}^K \left(\frac{n_k-1}{n_k} \pi_{11k}\pi_{22k}[(n_k-2)(1-\pi_{11k}-\pi_{22k})+1] \right) \\ &\quad + \frac{1}{\mu^2_{x,PMH1}} \frac{1}{2}(1-\theta)\left(\mu_{x,PMH1} - \frac{1}{2}\right) + \text{ Remainder} \\ &= \frac{1}{4}\frac{1-\theta}{\mu^2_{x,PMH1}} \\ &\quad + \left(\frac{\mu_x}{\mu_{x,PMH1}}\right)^2 \sum_{k=1}^K \left(\frac{n_k-1}{n_k} \pi_{11k}\pi_{22k}[(n_k-2)(1-\pi_{11k}-\pi_{22k})+1] \right) + O(\frac{1}{N^2}). \quad (4.5) \\ &= O(1/N). \end{aligned}$$

For $\theta = 1$,

$$\begin{aligned} \text{Bias}_{PMH1} &\doteq (\frac{\mu_x}{\mu_{x,PMH1}})^2 \text{ Bias}_{MH} + + \frac{1-\theta}{\mu_{x,PMH1}} + O(1/N^2) \\ &< \text{Bias}_{MH}. \end{aligned}$$

For $\theta > 1$, the leading bias term of Bias_{PMH1} is $(\mu_x/\mu_{x,PMH1})^2$ times the leading bias term of Bias_{MH}.

We already knew that the Mantel-Haenszel estimator overestimates the common odds ratio, but adding one pair of pseudotables reduces the bias.

For fixed table size and the common odds ratio of θ, when the cell probabilities are bounded away from 0, the bias is order of $1/K$. Therefore, when the table size or the number of strata goes to infinity, the common odds ratio estimator converges to the true value.

Preliminary simulation studies suggest that adding one set of pseudotables is the best choice. Adding more pseudotables or adding pseudo-data in each table will overcorrect the bias and pseudotable estimators will underestimate θ.

For fixed K and N, $\mu_{x,PMH1} = O(1/\theta)$ and the second term of (4.4) is $O(\theta^2)$ and the third term of (4.4) is $O(\theta^3)$, and suggesting that we incur larger bias in a bigger common odds ratio case.

Next we investigate the variance of the pseudotable estimators. We have

$$
\begin{aligned}
&Var(\hat{\theta}_{PMH1} - \theta) \\
&\doteq \quad \text{Var}\left(\frac{1}{\mu_{x,PMH1}}\sum_{k=1}^{K}\left(\frac{z_{11k}z_{22k}}{n_k}\right) + \frac{1}{2} - \theta\left(\sum_{k=1}^{K}\left(\frac{z_{12k}z_{21k}}{n_k}\right) + \frac{1}{2}\right)\right) \\
&= \quad \text{Var}\left(\frac{1}{\mu_{x,PMH1}}\sum_{k=1}^{K}\left(\frac{z_{11k}z_{22k}}{n_k} - \theta\frac{z_{12k}z_{21k}}{n_k}\right) + \frac{1}{2}(1-\theta)\right) \\
&= \quad \text{Var}\left(\frac{1}{\mu_{x,PMH1}}\sum_{k=1}^{K}\left(\frac{z_{11k}z_{22k}}{n_k} - \theta\frac{z_{12k}z_{21k}}{n_k}\right)\right) \\
&= \quad \left(\frac{\mu_x}{\mu_{x,PMH1}}\right)^2 \text{Var}\ (\hat{\theta}_{MH} - \theta) \\
&\leq \quad \text{Var}\ (\hat{\theta}_{MH} - \theta) \text{ as } \mu_{x,PMH1} = \mu_x + \frac{1}{2}.
\end{aligned}
$$

Therefore, adding one pair of pseudotables not only reduces the bias of the common odds ratio estimator, but it also reduces the variances.

4.2 Missing Data

In Section 4.1, the methods apply to a fully observed data set. Therefore, when the some of the observations are not fully observed, we must either impute the data or use the fully observed subset (complete data) and ignore the partially observed data when we compute the Mantel-Haenszel statistics.

The log-linear model when missingness of categorical variable A depends on the other two fully observed categorical variables B and C (MAR Model) is given in (4.6), and the model for missingness of A depends on A and C (Informative Model) is given in (4.7):

$$\begin{aligned}\log(\mu_{ijkl}) &= \mu + \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{ik}^{AC} + \alpha_{jk}^{BC} + \alpha_{ijk}^{ABC} + \beta_l^{R_a} \\ &\quad + \gamma_{jl}^{BR_a} + \gamma_{kl}^{CR_a} + \gamma_{jkl}^{BCR_a} \qquad (4.6)\\ \log(\mu_{ijkl}) &= \mu + \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{ik}^{AC} + \alpha_{jk}^{BC} + \alpha_{ijk}^{ABC} + \beta_l^{R_a} \\ &\quad + \gamma_{il}^{AR_a} + \gamma_{kl}^{CR_a} + \gamma_{ikl}^{ACR_a}. \qquad (4.7)\end{aligned}$$

In our simulations, we dropped out the cases that there was no fully observed data. So the actual number of strata might be different to the original number. The proportion of "bad" tables might be different from the missingness models and the value of parameters.

4.2.1 Complete Data Only–MAR Model

When some of the row information is not available, one way to estimate the common odds ratio is to use only the fully observed data.

Let z_{ijkl} denote the number of the observations for $A = i$, $B = j$, $C = k$ and $R_a = l$ where $l = 1$ if variable A is observed, 0 otherwise. The data table is in Table 3.5 and let $n_k^c = z_{11k1} + z_{12k1} + z_{21k1} + z_{22k1}$ denote the total number of fully observed data in the k table. Then as we mentioned before, conditioning on the total number of observations n_k, $(z_{11k}, z_{12k}, z_{21k})$, where $z_{ijk} = z_{ijk1} + z_{ijk0}$ have a multinomial distribution with parameters n_k, π_{11k}, π_{12k}, π_{21k}, $n_k - z_{11k} - z_{12k} - z_{21k} = z_{22k}$, $1 - \pi_{11k} - \pi_{12k} - \pi_{21k} = \pi_{22k}$ and $\pi_{11k}\pi_{22k}/(\pi_{12k}\pi_{21k}) = \theta$. In the case where variable A is MAR(B, C), we assume z_{ijk1} is binomial$(z_{ijk}, 1 - p_{jk}^{missing})$. Then the expected cell counts for the complete subtable of table k are

	$B = 1$	$B = 2$
$A = 1$	$n_k\pi_{11k}(1 - p_{1k}^{missing})$	$n_k\pi_{12k}(1 - p_{2k}^{missing})$
$A = 2$	$n_k\pi_{21k}(1 - p_{1k}^{missing})$	$n_k\pi_{22k}(1 - p_{2k}^{missing})$

Therefore

$$\begin{aligned}
\theta_k &= \frac{n_k\pi_{11k}(1-p_{1k}^{missing})n_k\pi_{22k}(1-p_{2k}^{missing})}{n_k\pi_{12k}(1-p_{2k}^{missing})n_k\pi_{21k}(1-p_{1k}^{missing})} \\
&= \frac{\pi_{11k}\pi_{22k}}{\pi_{12k}\pi_{22k}} \\
&= \theta.
\end{aligned}$$

Therefore the fully observed subtables are also satisfy the common odds ratio assumption but with fewer observations. Then the Mantel-Haenszel estimators when using the fully observed subtables would have similar behavior as described in Section 4.1 but may lose efficiency. Let the observations of the complete subtable of table k be

	$B = 1$	$B = 2$	Total
$A = 1$	z_{11k1}	z_{12k1}	n_{1k}^c
$A = 2$	z_{21k1}	z_{22k1}	n_{2k}^c
			n_k^c

The Mantel-Haenszel estimator for fully observed subtable is

$$\hat{\theta}_{MH,Comp}^{MAR} = \frac{\sum_1^K z_{11k1}z_{22k1}/n_k^c}{\sum_1^K z_{12k1}z_{21k1}/n_k^c}. \tag{4.8}$$

In the large table cases, we assume that $n_k^c/n_k \to \lambda_k$ as $n_k^c \to \infty$. In the sparse cases, we assume that $P(missing)$ are bounded away from 1. As

$$\hat{\theta}_{MH,Comp}^{MAR} - \theta = \frac{\sum_1^K (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})/n_k^c}{\sum_1^K z_{12k1}z_{21k1}/n_k^c},$$

where

$$\begin{aligned}
E[n_k^c] &= E[E[n_k^c|z_{11k}, z_{12k}, z_{21k}, z_{22k}]] \\
&= E[(z_{11k}+z_{21k})(1-p_{1k}^{missing}) + (z_{12k}+z_{21k})(1-p_{2k}^{missing})] \\
&= n_k[(\pi_{11k}+\pi_{21k})(1-p_{1k}^{missing}) + (\pi_{12k}+\pi_{22k})(1-p_{2k}^{missing})] \\
&= n_k - n_k[(\pi_{11k}+\pi_{21k})p_{1k}^{missing} + (\pi_{12k}+\pi_{22k})p_{2k}^{missing}].
\end{aligned} \tag{4.9}$$

Conditioning on the total number of fully observed data n_k^c, $(z_{11k1}, z_{12k1}, z_{21k1})$ have a multinomial distribution with parameters n_k^c, π_{11k1}^{comp}, π_{12k1}^{comp}, π_{21k1}^{comp}. Here $n_k^c - z_{11k1} - z_{12k1} - z_{21k1}$ equals to

z_{22k1}, and $1 - \pi_{11k1}^{comp} - \pi_{12k1}^{comp} - \pi_{21k1}^{comp} = \pi_{22k1}^{comp}$, where $\pi_{i1k1}^{comp} = n_k\pi_{i1k}(1 - p_{1k}^{missing})/E[n_k^c]$ and $\pi_{i2k1}^{comp} = n_k\pi_{i2k}(1 - p_{2k}^{missing})/E[n_k^c]$. The π_{ijk1}^{comp} satisfy the common odds assumption:

$$
\begin{aligned}
&\pi_{11k1}^{comp}\pi_{22k1}^{comp} - \theta\pi_{12k1}^{comp}\pi_{21k1}^{comp} \\
&= n_k\pi_{11k}(1 - p_{1k}^{misisng})n_k\pi_{22k})(1 - p_{2k}^{missing}) \\
&\quad - \theta n_k\pi_{12k}(1 - p_{2k}^{missing})n_k\pi_{21k}(1 - p_{1k}^{missing}) \\
&= n_k n_k(1 - p_{1k}^{missing})(1 - p_{2k}^{missing})(\pi_{11k}\pi_{22k} - \theta(\pi_{12k}\pi_{21k}) \\
&= 0.
\end{aligned}
$$

We have

$$
\begin{aligned}
E[z_{11k1}z_{22k1}/n_k^c] &= E[E[z_{11k1}z_{22k1}/n_k^c|n_k^c]] \\
&= E\left[\frac{1}{n_k^c}E[z_{11k1}z_{22k1}|n_k^c]\right] \\
&= E\left[\frac{1}{n_k^c}n_k^c(n_k^c - 1)\pi_{11k1}^{comp}\pi_{22k1}^{comp}\right] \\
&= E[(n_k^c - 1)\pi_{11k1}^{comp}\pi_{22k1}^{comp}] \\
&= \pi_{11k1}^{comp}\pi_{22k1}^{comp}E[n_k^c - 1] \\
&= \frac{n_k\pi_{11k}(1 - p_{1k}^{missing})}{E[n_k^c]}\frac{n_k\pi_{22k}(1 - p_{2k}^{missing})}{E[n_k^c]}(E[n_k^c] - 1) \\
&= \left(\frac{1}{E[n_k^c]} - \frac{1}{E[n_k^c]^2}\right)(n_k^2\pi_{11k}\pi_{22k}(1 - p_{1k}^{missing})(1 - p_{2k}^{missing})) \\
E[z_{12k1}z_{21k1}/n_k^c] &= E[E[z_{12k1}z_{21k1}/n_k^c|n_k^c]] \\
&= E\left[\frac{1}{n_k^c}E[z_{12k1}z_{21k1}|n_k^c]\right] \\
&= E\left[\frac{1}{n_k^c}n_k^c(n_k^c - 1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\right] \\
&= E[(n_k^c - 1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}] \\
&= \pi_{12k1}^{comp}\pi_{21k1}^{comp}E[n_k^c - 1] \\
&= \frac{n_k\pi_{12k}(1 - p_{2k}^{missing})}{E[n_k^c]}\frac{n_k\pi_{21k}(1 - p_{1k}^{missing})}{E[n_k^c]}(E[n_k^c] - 1) \\
&= \left(\frac{1}{E[n_k^c]} - \frac{1}{E[n_k^c]^2}\right)(n_k^2\pi_{12k}\pi_{21k}(1 - p_{2k}^{missing})(1 - p_{1k}^{missing})) \\
&= \frac{E[z_{11k1}z_{22k}/n_k^c]}{\theta},
\end{aligned}
$$

and therefore

$$\begin{aligned}
\mu_{x,Comp}^{MAR} &= E\left[\sum z_{12k1}z_{21k1}/n_k^c\right] \\
&= \sum_{k=1}^{K} E[z_{12k1}z_{21k1}/n_k^c] \\
&= \sum_{k=1}^{K} E[n_k^c - 1]\pi_{12k1}^{comp}\pi_{21k1}^{comp} \\
&= \sum_{1}^{k} \left[\frac{1}{E[n_k^c]} - \frac{1}{(E[n_k^c])^2}\right](n_k^2\pi_{12k}\pi_{21k}(1-p_{1k}^{missing})(1-p_{2k}^{missing})).
\end{aligned}$$

Let $N^c = \sum_{k=1}^{K} n_k^c$. Then when K and θ are fixed, $\mu_{x,Comp}^{MAR}$ is $O(E[N^c])$ and when n_k^c and θ are fixed, and the observed probabilities π_{ijk}^{comp} are bounded away from 0, $\mu_{x,Comp}^{MAR}$ is $O(K)$. When K and n_k are fixed, $\mu_{x,Comp}^{MAR}$ is $O(1/\theta)$.

As in section 4.1.1, writing $\hat{\theta}_{MH,Comp}^{MAR} = y_{Comp}^{MAR}/x_{Comp}^{MAR}$ and using Taylor expansion, $\hat{\theta}_{MH,Comp}^{MAR} - \theta$ can be written as

$$\begin{aligned}
\frac{y_{Comp}^{MAR} - \theta x_{Comp}^{MAR}}{x_{Comp}^{MAR}} &= \frac{y_{Comp}^{MAR} - \theta x_{Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\left[1 - \frac{x_{Comp}^{MAR} - \mu_{x,Comp}^{MAR}}{\mu_{x,Comp}^{MAR}} + \cdots\right] \qquad (4.10) \\
&= \frac{y_{Comp}^{MAR} - \theta x_{Comp}^{MAR}}{\mu_{x,Comp}^{MAR}} \\
&\quad - \left(\frac{y_{Comp}^{MAR} - \theta x_{Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right)\left(\frac{x_{Comp}^{MAR} - \mu_{x,Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right) \\
&\quad + \left(\frac{y_{Comp}^{MAR} - \theta x_{MH,Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right)\left(\frac{x_{Comp}^{MAR} - \mu_{x,Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right)^2 + \cdots.
\end{aligned}$$

The expected value of the first term of (4.10) is

$$\begin{aligned}
E\left[\frac{y_{Comp}^{MAR} - \theta x_{Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right] &= E\left[\frac{1}{\mu_{x,Comp}^{MAR}}\sum_{k=1}^{K}\left(\frac{z_{11k1}z_{22k1}}{n_k^c} - \theta\left(\frac{z_{12k1}z_{21k1}}{n_k^c}\right)\right)\right] \\
&= \frac{1}{\mu_{x,Comp}^{MAR}}\sum_{k=1}^{K}\left(E\left[\frac{z_{11k1}z_{22k1}}{n_k^c}\right] - \theta E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]\right) \\
&= \frac{1}{\mu_{x,Comp}^{MAR}}\sum_{k=1}^{K}\left(\frac{1}{E[n_k^c]} - \frac{1}{E[n_k^c]^2}\right)(1-p_{1k}^{missing})(1-p_{2k}^{missing}) \\
&\qquad \times\left(n_k^2\pi_{11k}\pi_{22k} - \theta n_k^2\pi_{12k}\pi_{21k}\right) \\
&= 0.
\end{aligned}$$

The expected value of the second term of (4.10) is

$$E\left[\left(\frac{y_{Comp}^{MAR} - \theta x_{Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right)\left(\frac{x_{Comp}^{MAR} - \mu_{x,Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{(\mu^{MAR}_{x,Comp})^2} E\left[\left(\sum \frac{z_{11k1}z_{22k1}}{n^c_k} - \theta\left(\sum \frac{z_{12k1}z_{21k1}}{n^c_k}\right)\right)\left(\sum \frac{z_{12k1}z_{21k1}}{n^c_k} - \mu^{MAR}_{x,Comp}\right)\right] \\
&= \frac{1}{(\mu^{MAR}_{x,Comp})^2} E\left[\left(\sum \left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n^c_k}\right)\right)\left(\sum \left(\frac{z_{12k1}z_{21k1}}{n^c_k} - E\left[\frac{z_{12k1}z_{21k1}}{n^c_k}\right]\right)\right)\right] \\
&= \frac{1}{(\mu^{MAR}_{x,Comp})^2} E\left[\sum \left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n^c_k}\right)\left(\frac{z_{12k1}z_{21k1}}{n^c_k} - E\left[\frac{z_{12k1}z_{21k1}}{n^c_k}\right]\right)\right]\right] \\
&\quad + \frac{1}{(\mu^{MAR}_{x,Comp})^2} E\left[\sum_{k\neq k'} \left\{\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n^c_k}\right)\left(\frac{z_{12k'1}z_{21k'1}}{n^c_{k'}} - E\left[\frac{z_{12k'1}z_{21k'1}}{n^c_{k'}}\right]\right)\right\}\right] \\
&= \frac{1}{(\mu^{MAR}_{x,Comp})^2}(\sum_k A_{k,comp} + B_{comp}).
\end{aligned}
$$

Here

$$
\begin{aligned}
A_{k,comp} &= E\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n^c_k}\right)\left(\frac{z_{12k1}z_{21k1}}{n^c_k} - E\left[\frac{z_{12k1}z_{21k1}}{n^c_k}\right]\right)\right] \\
&= E\left[\frac{(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})}{(n^c_k)^2}\right] \\
&\quad - E\left[\frac{z_{12k1}z_{21k1}}{n^c_k}\right] E\left[\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n^c_k}\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
&E\left[\frac{(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})}{(n^c_k)^2}\right] \\
&\quad = E\left[E\left[\frac{(z_{11k1}z_{12k1}z_{21k1}z_{22k1} - \theta z^2_{12k1}z^2_{21k1})}{(n^c_k)^2}|n^c_k\right]\right] \\
&\quad = E\left[\frac{1}{(n^c_k)^2} E[z_{11k1}z_{12k1}z_{21k1}z_{22k1} - \theta z^2_{12k1}z^2_{21k1}|n^c_k]\right].
\end{aligned}
$$

Also

$$
\begin{aligned}
&E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}|n^c_k] \\
&= E[z_{11k1}z_{12k1}z_{21k1}(n^c_k - z_{11k1} - z_{12k1} - z_{21k1})|n^c_k] \\
&= E[n^c_k z_{11k1}z_{12k1}z_{21k1} - z^2_{11k1}z_{12k1}z_{21k1} - z_{11k1}z^2_{12k1}z_{21k1} - z_{11k1}z_{12k1}z^2_{21k1}|n^c_k] \\
&= E[n^c_k z_{11k1}z_{12k1}z_{21k1} - z_{11k1}(z_{11k1}-1)z_{12k1}z_{21k1} - z_{11k1}z_{12k1}(z_{12k1}-1)z_{21k1}|n^c_k] \\
&\quad - E[z_{11k1}z_{12k1}z_{21k1}(z_{21k1}-1) - 3z_{11k1}z_{12k1}z_{21k1}|n^c_k] \\
&= n^c_k n^c_k(n^c_k-1)(n^c_k-2)\pi^{comp}_{11k1}\pi^{comp}_{12k1}\pi^{comp}_{21k1} - n^c_k(n^c_k-1)(n^c_k-2)(n^c_k-3)(\pi^{comp}_{11k1})^2\pi^{comp}_{12k1}\pi^{comp}_{21k1} \\
&\quad - n^c_k(n^c_k-1)(n^c_k-2)(n^c_k-3)\pi^{comp}_{11k1}(\pi^{comp}_{12k1})^2\pi^{comp}_{21k1} \\
&\quad - n^c_k(n^c_k-1)(n^c_k-2)(n^c_k-3)\pi^{comp}_{11k1}\pi^{comp}_{12k1}(\pi^{comp}_{21k1})^2
\end{aligned}
$$

$$
\begin{aligned}
& - 3n_k^c(n_k^c-1)(n_k^c-2)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp} \\
= \quad & n_k^c(n_k^c-1)(n_k^c-2)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}(n_k^c-3)(1-\pi_{11k1}^{comp}-\pi_{12k1}^{comp}-\pi_{21k1}^{comp}) \\
= \quad & n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp},
\end{aligned}
$$

and

$$
\begin{aligned}
& E[\theta z_{12k1}^2 z_{21k1}^2 | n_k^c] \\
= \quad & \theta E[z_{12k1}(z_{12k1}-1)z_{21k1}(z_{21k1}-1) + z_{12k1}(z_{12k1}-1)z_{21k1} + z_{12k1}z_{21k1}(z_{21k1}-1)|n_k^c] \\
& + \theta E[z_{12k1}z_{21k1}|n_k^c] \\
= \quad & \theta(n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2 + n_k^c(n_k^c-1)(n_k^c-2)(\pi_{12k1}^{comp})^2\pi_{21k1}^{comp}) \\
& + \theta(n_k^c(n_k^c-1)(n_k^c-2)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^2 + n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}) \\
= \quad & \theta n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\frac{\pi_{11k1}^{comp}\pi_{22k1}^{comp}}{\theta})) \\
& + \theta n_k^c(n_k^c-1)\pi_{12k1}\pi_{21k1}((n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1).
\end{aligned}
$$

Then

$$
\begin{aligned}
& E[(z_{11k1}z_{12k1}z_{21k1}z_{22k1} - \theta z_{12k1}^2 z_{21k1}^2)|n_k^c] \\
= \quad & n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp} \\
& - \theta n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\left(\frac{\pi_{11k1}^{comp}\pi_{22k1}^{comp}}{\theta}\right) \\
& - \theta n_k^c(n_k^c-1)\pi_{12k1}\pi_{21k1}((n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1) \\
= \quad & -\theta n_k^c(n_k^c-1)\pi_{12k1}\pi_{21k1}((n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& E\left[\frac{(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})}{(n_k^c)^2}\right] \\
= \quad & E\left[\frac{1}{(n_k^c)^2}(z_{11k1}z_{12k1}z_{21k1}z_{22k1} - \theta z_{12k1}^2 z_{21k1}^2)|n_k^c\right] \\
= \quad & -E\left[\frac{1}{(n_k^c)^2}\theta n_k^c(n_k^c-1)\pi_{12k1}\pi_{21k1}((n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1)\right] \\
= \quad & -\theta\pi_{12k1}\pi_{21k1}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right] - \theta E\left[\frac{(n_k^c-1)}{n_k^c}\right].
\end{aligned}
$$

Also,

$$\begin{aligned}
&E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right]\\
&= E\left[E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}|n_k^c\right]\right]\\
&= E\left[\frac{1}{n_k^c}E[z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}|n_k^c]\right]\\
&= E\left[\frac{1}{n_k^c}(n_k^c(n_k^c-1)\pi_{11k1}^{comp}\pi_{22k1}^{comp}-\theta n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp})\right]\\
&= 0.
\end{aligned}$$

Hence,

$$A_{k,comp} = -\theta\pi_{12k1}\pi_{21k1}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]-\theta E\left[\frac{(n_k^c-1)}{n_k^c}\right].$$

Since table k and table k' are independent, so are the complete subtables of tables k and k'. Hence

$$\begin{aligned}
B_{comp} &= E\left[\sum_{k\neq k'}\left\{\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k'1}z_{21k'1}}{n_k^c}-E\left[\left(\frac{z_{12k'1}z_{21k'1}}{n_k^c}\right)\right]\right)\right\}\right]\\
&= \sum_{k\neq k'}E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k'1}z_{21k'1}}{n_k^c}-E\left[\frac{z_{12k'1}z_{21k'1}}{n_k^c}\right]\right)\right]\\
&= \sum_{k\neq k'}E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right]E\left[\left(\frac{z_{12k'1}z_{21k'1}}{n_k^c}-E\left[\frac{z_{12k'1}z_{21k'1}}{n_k^c}\right]\right)\right]\\
&= 0.
\end{aligned}$$

So the expected value of the second term of (4.10) is

$$\begin{aligned}
&E\left[\left(\frac{y_{Comp}^{MAR}-\theta x_{Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right)\left(\frac{x_{Comp}^{MAR}-\mu_{x,Comp}^{MAR}}{\mu_{x,Comp}^{MAR}}\right)\right]\\
&= \frac{1}{(\mu_{x,Comp}^{MAR})^2}\sum_k A_{k,comp}\\
&= \frac{1}{(\mu_{x,Comp}^{MAR})^2}\sum_k\left(-\theta\pi_{12k1}\pi_{21k1}\left((\pi_{12k1}^{comp}+\pi_{21k1}^{comp})E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]-E\left[\frac{(n_k^c-1)}{n_k^c}\right]\right)\right).
\end{aligned}$$

The expected value of the third term of (4.10) is

$$\begin{aligned}
&E\left[\left(\frac{\sum_1^K(z_{11k1}z_{22k1}/n_k^c-\theta z_{12k1}z_{21k1}/n_k^c)}{\mu_{x,Comp}^{MAR}}\right)\left(\frac{\sum_1^K(z_{12k1}z_{21k1}/n_k^c-E[z_{12k1}z_{21k1}/n_k])}{\mu_{x,Comp}^{MAR}}\right)^2\right]\\
&= \frac{1}{(\mu_{x,Comp}^{MAR})^3}\sum_1^K E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right] \quad (4.11)
\end{aligned}$$

Since the observations from different strata are independent, all of the expectations of cross-product terms are zero. According to the same reasoning as in Section 4.1.1, the kth term of the summation in (4.11) is

$$\begin{aligned}
&E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right] \\
&= E\left[\frac{z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}}{(n_k^c)^3}\right]-\theta E\left[\frac{z_{12k1}^3z_{21k1}^3}{(n_k^c)^3}\right] \\
&\quad -2\left(E\left[\frac{z_{11k1}z_{12k1}z_{21k1}z_{22k1}E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]+\theta E\left[\frac{z_{12k1}^2z_{21k1}^2E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]\right) \\
&\quad +E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right]\left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]^2\right).
\end{aligned}$$

We have

$$\begin{aligned}
&E\left[\frac{z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}}{(n_k^c)^3}\right] \\
&= E\left[\frac{1}{(n_k^c)^3}E[z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}|n_k^c]\right] \\
&= E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)(n_k^c-5)\pi_{11k1}^{comp}(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2\pi_{22k1}^{comp}\right] \\
&\quad +E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\right] \\
&\quad +E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}\right],
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\frac{z_{12k1}^3z_{21k1}^3}{(n_k^c)^3}\right] \\
&= E\left[\frac{1}{(n_k^c)^3}E[z_{12k1}^3z_{21k1}^3|n_k^c]\right] \\
&= E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)(n_k^c-5)(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2(\pi_{11k1}^{comp}\pi_{22k1}^{comp}/\theta)\right] \\
&\quad +E\left[\frac{3}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\left(\frac{\pi_{11k1}^{comp}\pi_{22k1}^{comp}}{\theta}\right)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\right] \\
&\quad +E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(\pi_{12k1}^{comp})^3\pi_{21k1}^{comp}\right] \\
&\quad +E\left[\frac{9}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{11k1}^{comp}\pi_{22k1}^{comp}/\theta)\right] \\
&\quad +E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^3\right] \\
&\quad +E\left[\frac{12}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(\pi_{12k1}^{comp})^2\pi_{21k1}^{comp}\right]
\end{aligned}$$

$$+E\left[\frac{12}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^2+\frac{n_k^c-1}{(n_k^c)^2}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\right]$$

Therefore,

$$
\begin{aligned}
&E\left[\frac{z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}}{(n_k^c)^3}\right]-\theta E\left[\frac{z_{12k1}^3z_{21k1}^3}{(n_k^c)^3}\right]\\
= \quad & E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)(n_k^c-5)\pi_{11k1}^{comp}(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2\pi_{22k1}^{comp}\right]\\
&+E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\right]\\
&+E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}\right]\\
&-\theta E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)(n_k^c-5)(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2(\pi_{11k1}^{comp}\pi_{22k1}^{comp}/\theta)\right]\\
&-\theta E\left[\frac{3}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{11k1}^{comp}\pi_{22k1}^{comp}/\theta)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\right]\\
&-\theta E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(\pi_{12k1}^{comp})^3\pi_{21k1}^{comp}\right]\\
&-\theta E\left[\frac{9}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{11k1}^{comp}\pi_{22k1}^{comp}\theta)\right]\\
&-\theta E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^3\right]\\
&-\theta E\left[\frac{12}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(\pi_{12k1}^{comp})^2\pi_{21k1}^{comp}\right]\\
&-\theta E\left[\frac{12}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^2+\frac{1}{(n_k^c)^2}(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\right]\\
= \quad & -E\left[\frac{2}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\right]\\
&-E\left[\frac{\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(\pi_{12k1}^{comp})^3\pi_{21k1}^{comp}\right]\\
&-E\left[\frac{8}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}\right]\\
&-E\left[\frac{\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^3\right]\\
&-E\left[\frac{12\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(\pi_{12k1}^{comp})^2\pi_{21k1}^{comp}\right]\\
&-E\left[\frac{12\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^2+\frac{\theta}{(n_k^c)^2}(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\right]\\
= \quad & -E\left[\frac{2}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2(1-\pi_{11k1}^{comp}-\pi_{22k1}^{comp})/\theta\\
&-E\left[\frac{6}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2/\theta\\
&-E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{(n_k^c)^2}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})^2\\
&-E\left[\frac{12}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})
\end{aligned}
$$

$$-E\left[\frac{n_k^c-1}{(n_k^c)^2}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}.$$

Also

$$\begin{aligned}
&-2E\left[\frac{z_{11k1}z_{12k1}z_{21k1}z_{22k}E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]+2\theta E\left[\frac{z_{12k1}^2z_{21k1}^2E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]\\
&= \quad -2E\left[\frac{1}{(n_k^c)^3}E[(z_{11k1}z_{12k1}z_{21k1}z_{22k1}-\theta z_{12k1}^2z_{21k1}^2)E[z_{12k1}z_{21k1}]|n_k^c]\right]\\
&= \quad -2E[n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}]E\left[\frac{1}{n_k^c}E\left[\frac{1}{(n_k^c)^2}z_{11k1}z_{12k1}z_{21k1}z_{22k1}\middle| n_k^c\right]\right]\\
&\qquad -2\theta E[n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}]E\left[\frac{1}{n_k^c}E\left[\frac{1}{(n_k^c)^2}z_{12k1}^2z_{21k1}^2\middle| n_k^c\right]\right]\\
&= \quad -2E[n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}]E\left[\frac{1}{n_k^c}\left(-\theta\frac{n_k^c-1}{n_k^c}\pi_{12k1}^{comp}\pi_{21k1}^{comp}[(n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1]\right)\right]\\
&= \quad 2E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\\
&\qquad 2E[n_k^c(n_k^c-1)]E\left[\frac{n_k^c-1}{(n_k^c)^2}\right](\pi_{11k1}^{comp})(\pi_{22k1}^{comp})/\theta,
\end{aligned}$$

and

$$E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right]\left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]^2\right)=0.$$

Therefore,

$$\begin{aligned}
&E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right]\\
&= \quad -E\left[\frac{2}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2(1-\pi_{12k1}^{comp}+\pi_{21k1}^{comp})/\theta\\
&\qquad -E\left[\frac{6}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2/\theta\\
&\qquad -E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{(n_k^c)^2}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(1-\pi_{11k1}^{comp}-\pi_{22k1}^{comp})^2\\
&\qquad -E\left[\frac{12}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\\
&\qquad -E\left[\frac{n_k^c-1}{(n_k^c)^2}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}\\
&\qquad +2E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2(1-\pi_{11k1}^{comp}-\pi_{22k1}^{comp})/\theta\\
&\qquad +2E[n_k^c(n_k^c-1)]E\left[\frac{n_k^c-1}{(n_k^c)^2}\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2/\theta\\
&= \quad -2E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2(1-\pi_{11k1}^{comp}-\pi_{22k1}^{comp})/\theta\\
&\qquad +2E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2(1-\pi_{11k1}^{comp}-\pi_{22k1}^{comp})/\theta
\end{aligned}$$

$$
\begin{aligned}
& - E\left[\frac{6}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2/\theta \\
& - E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(1-\pi_{11k1}^{comp}-\pi_{22k1}^{comp})^2 \\
& - E\left[\frac{12}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(1-\pi_{11k1}^{comp}-\pi_{22k1}^{comp}) \\
& - E\left[\frac{n_k^c-1}{(n_k^c)^2}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp} + 2E[n_k^c(n_k^c-1)]E\left[\frac{n_k^c-1}{(n_k^c)^2}\right](\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2/\theta.
\end{aligned}
$$

As

$$
\begin{aligned}
& E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right] \\
&\quad = E\left[\frac{1}{(n_k^c)^2}((n_k^c)^4-10(n_k^c)^3+35(n_k^c)^2-50n_k^c+24)\right] \\
&\quad = E\left[(n_k^c)^2-10n_k^c+35-\frac{50}{n_k^c}+\frac{24}{(n_k^c)^2}\right] \\
&\quad = E[(n_k^c)^2]-10E[n_k^c]+35-50E\left[\frac{1}{n_k^c}\right]+24E\left[\frac{1}{(n_k^c)^2}\right], \\
& E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right] \\
&\quad = E[(n_k^c)^2-n_k^c]E\left[\frac{1}{(n_k^c)^2}((n_k^c)^2-3n_k^c+2)\right] \\
&\quad = (E[(n_k^c)^2]-E[n_k^c])\left(E\left[1-\frac{3}{n_k^c}+\frac{2}{(n_k^c)^2}\right]\right) \\
&\quad = (E[(n_k^c)^2]-E[n_k^c])\left(1-3E\left[\frac{1}{n_k^c}\right]+2E\left[\frac{1}{(n_k^c)^2}\right]\right) \\
&\quad = E[(n_k^c)^2]-E[n_k^c]+(E[(n_k^c)^2]-E[n_k^c])\left(-3E\left[\frac{1}{n_k^c}\right]+2E\left[\frac{1}{(n_k^c)^2}\right]\right), \\
& E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right]-E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right] \\
&\quad = E[(n_k^c)^2]-10E[n_k^c]+35-50E\left[\frac{1}{n_k^c}\right]+24E\left[\frac{1}{(n_k^c)^2}\right] \\
&\qquad -E[(n_k^c)^2]+E[n_k^c]-(E[(n_k^c)^2]-E[n_k^c])\left(-3E\left[\frac{1}{n_k^c}\right]+2E\left[\frac{1}{(n_k^c)^2}\right]\right) \\
&\quad = -9E[n_k^c]+35-50E\left[\frac{1}{n_k^c}\right]+24E\left[\frac{1}{(n_k^c)^2}\right] \\
&\qquad -(E[(n_k^c)^2]-E[n_k^c])\left(-3E\left[\frac{1}{n_k^c}\right]+2E\left[\frac{1}{(n_k^c)^2}\right]\right),
\end{aligned}
$$

then

$$
E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right] \quad = \quad O(E(n_k^c))
$$

Since $\mu_{x,Comp}^{MAR}$ is $O(E(N^c))$ when K and θ are fixed, the expected value of the third term of (4.10) is $O(1/(E(N^c))^2)$ or $O(1/K^2)$ for fixed n_k and θ and bounded away from 0 cell probabilities. So the leading term of the bias of the common odds ratio estimator is $O(1/N)$ for fixed K and θ and $O(1/K)$ for fixed n_k and θ when we use the fully observed subtable. The bias of the complete data only is negligible when the total observations gets larger, either larger tables or more strata with the cell probabilities are bounded away from 0.

Now we investigate the variance. We have

$$
\begin{aligned}
&\text{Var }(\hat{\theta}^{MAR}_{MH,Comp} - \theta) \\
\doteq\ & \text{Var }\left(\sum_{k=1}^{K}\left(\frac{z_{11k1}z_{12k1}/n_k^c - \theta z_{12k1}z_{21k1}/n_k^c}{\mu^{MAR}_{x,Comp}}\right)\right) \\
=\ & \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}\text{ Var }\left(\frac{z_{11k1}z_{12k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\right) \\
=\ & \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}\text{ Var }\left(E\left[\frac{z_{11k1}z_{12k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\middle| n_k^c\right]\right) \\
&+ \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}E\left[\text{ Var }\left(\frac{z_{11k1}z_{12k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\middle| n_k^c\right)\right] \\
=\ & \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}E\left[\text{ Var }\left(\frac{z_{11k1}z_{12k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\middle| n_k^c\right)\right] \\
=\ & \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}E\left[E\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\right)^2\middle| n_k^c\right]\right] - 0 \\
=\ & \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}E\left[E\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\right)^2\middle| n_k^c\right]\right] \\
=\ & \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}E\left[\frac{1}{(n_k^c)^2}E[z^2_{11k1}z^2_{22k1} - 2\theta z_{11k1}z_{12k1}z_{21k1}z_{22k1} + \theta^2 z^2_{12k1}z^2_{21k1}|n_k^c]\right] \\
=\ & \frac{1}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}E\left[\frac{1}{(n_k^c)^2}(E[z^2_{11k1}z^2_{22k1}|n_k^c] + \theta^2 E[z^2_{12k1}z^2_{21k1}|n_k^c])\right] \\
&- \frac{2\theta}{(\mu^{MAR}_{x,Comp})^2}\sum_{1}^{K}E\left[\frac{1}{(n_k^c)^2}E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}|n_k^c]\right].
\end{aligned}
$$

Here

$$
\begin{aligned}
E[z^2_{11k1}z^2_{22k1}|n_k^c] &= E[z_{11k1}(z_{11k1}-1)z_{22k1}(z_{22k1}-1) + z_{11k1}(z_{11k1}-1)z_{22k1}|n_k^c] \\
&\quad + E[z_{11k1}z_{22k1}(z_{22k1}-1) + z_{11k1}z_{22k1}|n_k^c] \\
&= n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)(\pi^{comp}_{11k1})^2(\pi^{comp}_{22k1})^2 \\
&\quad + n_k^c(n_k^c-1)(n_k^c-2)\pi^{comp}_{11k1}\pi^{comp}_{22k1}(\pi^{comp}_{11k1} + \pi^{comp}_{22k1}) + n_k^c(n_k^c-1)\pi^{comp}_{11k1}\pi^{comp}_{22k1},
\end{aligned}
$$

$$E[z_{12k1}^2 z_{21k1}^2|n_k^c] = n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2$$
$$+n_k^c(n_k^c-1)(n_k^c-2)\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp},$$

and

$$E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}|n_k^c] = n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}.$$

Under the common odds ratio assumption, $\pi_{11k1}^{comp}\pi_{22k1}^{comp} = \theta\pi_{12k1}^{comp}\pi_{21k1}^{comp}$. Therefore the unconditional expected value of the kth term of the summation of the variance is

$$\begin{aligned}
&E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right)^2\right] \\
&= E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{n_k^c}(\pi_{11k1}^{comp})^2(\pi_{22k1}^{comp})^2\right] \\
&\quad + E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\pi_{11k1}^{comp}\pi_{22k1}^{comp}(\pi_{11k1}^{comp}+\pi_{22k1}^{comp})+\frac{n_k^c-1}{n_k^c}\pi_{11k1}^{comp}\pi_{22k1}^{comp}\right] \\
&\quad - 2\theta E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{n_k^c}\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}\right] \\
&\quad + \theta^2 E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{n_k^c}(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2\right] \\
&\quad + \theta^2 E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+\frac{n_k^c-1}{n_k^c}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\right] \\
&= \theta E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp} \\
&\quad + E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(\pi_{11k1}^{comp}+\pi_{22k1}^{comp}) + E\left[\frac{n_k^c-1}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp} \\
&\quad - 2\theta E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp} \\
&\quad + \theta E\left[\frac{(n_k^c-1)(n_k^c-2)(n_k^c-3)}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp} \\
&\quad + \theta E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp}) + \theta E\left[\frac{n_k^c-1}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp} \\
&= E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}((\pi_{11k1}^{comp}+\pi_{22k1}^{comp})+\theta(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})) \\
&\quad + E\left[\frac{n_k^c-1}{n_k^c}\right]\pi_{11k1}^{comp}\pi_{22k1}^{comp}(1+\theta) \\
&= E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]\frac{n_k^3}{E[n_k^c]^3}\pi_{11k}(1-p_{1k}^{missing})\pi_{22k}(1-p_{2k}^{missing}) \\
&\quad \times[(\pi_{11k}+\theta\pi_{21k})(1-p_{1k}^{missing})+(\pi_{22k}+\theta\pi_{12k})(1-p_{2k}^{missing})] \\
&\quad + E\left[\frac{n_k^c-1}{n_k^c}\right]\frac{n_k^2}{E[n_k^c]^2}(1+\theta)\pi_{11k}\pi_{22k}(1-p_{1k}^{missing})(1-p_{2k}^{missing}).
\end{aligned}$$

Let $p_k^{missing} = \max(p_{1k}^{missin}, p_{2k}^{missing})$, so $1 - p_k^{missing} = \min(1 - p_{1k}^{missing}, 1 - p_{2k}^{missing})$. Then

$$\begin{aligned}
&E\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\right)^2\right] \\
&\quad > \quad E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]\frac{n_k^3(1-p_k^{missing})^3}{E[n_k^c]^3}\pi_{11k}\pi_{22k}((\pi_{11k}+\pi_{22k})+\theta(\pi_{12k}+\pi_{21k})) \\
&\qquad + E\left[\frac{n_k^c-1}{n_k^c}\right]\frac{n_k^2(1-p_k^{missing})^2}{E[n_k^c]^2}(1+\theta)\pi_{11k}\pi_{22k}.
\end{aligned}$$

Since the sample size is reduced, we expect the variance of estimating using only the complete subsample will be larger than those using the full data as the simulation results show:

$$\begin{aligned}
\text{Var}\ (\hat{\theta}_{MH,Comp}^{MAR} - \theta) \quad &> \quad \text{Var}\ (\hat{\theta}_{MH} - \theta) \\
&\doteq \quad \frac{1}{\mu_x^2}\sum_1^K\left(\frac{(n_k-1)(n_k-2)}{n_k}\pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}+\theta(\pi_{12k}+\pi_{21k}))\right. \\
&\qquad \left. + \ \frac{n_k-1}{n_k}\pi_{11k}\pi_{22k}(1+\theta)\right).
\end{aligned}$$

As $\mu_{x,Comp}^{MAR}$ is $O(E[N^c])$ when K and θ are fixed and $O(K)$ when $E[n_k^c]$ and θ are fixed and the observed cell probabilities are bounded away from 0, Var $(\hat{\theta}_{MH,Comp}^{MAR})$ is $O(1/E[N^c])$ for fixed K and θ and $O(1/K)$ for fixed n_k and θ. For fixed $E[N^c]$ and K, $\mu_{x,Comp}^{MAR}$ is $O(1/\theta)$ that implies Var $(\hat{\theta}_{MH,Comp}^{MAR})$ is $O(\theta^3)$.

4.2.2 Complete Data Only–Informative Missingness

In the case where variable A is missing informatively, we assume z_{ijk1} is binomial(z_{ijk}, $(1 - p_{ik}^{missing})$). Then the expected cell counts for the complete subtable of table k are

	$B = 1$	$B = 2$
$A = 1$	$n_k\pi_{11k}(1 - p_{1k}^{missing})$	$n_k\pi_{12k}(1 - p_{1k}^{missing})$
$A = 2$	$n_k\pi_{21k}(1 - p_{2k}^{missing})$	$n_k\pi_{22k}(1 - p_{2k}^{missing})$

Therefore

$$\begin{aligned}\theta_k &= \frac{n_k\pi_{11k}(1-p_{1k}^{missing})n_k\pi_{22k}(1-p_{2k}^{missing})}{n_k\pi_{12k}(1-p_{1k}^{missing})n_k\pi_{21k}(1-p_{1k}^{missing})} \\ &= \frac{\pi_{11k}\pi_{22k}}{\pi_{12k}\pi_{22k}} \\ &= \theta.\end{aligned}$$

Therefore the fully observed subdata tables also satisfy the common odds ratio assumption but with fewer observations. Then the Mantel-Haenszel estimators when using the fully observed subdata would have the similar behavior as we discussed in Section 4.1 but might lose. Let the observations in the complete subtable of table k be

	$B=1$	$B=2$	Total
$A=1$	z_{11k1}	z_{12k1}	n_{1k}^c
$A=2$	z_{21k1}	z_{22k1}	n_{2k}^c
			n_k^c

The Mantel-Haenszel estimator for fully observed subdata is

$$\hat{\theta}_{MH,Comp}^{Informative} = \frac{\sum_1^K z_{11k1}z_{22k1}/n_k^c}{\sum_1^K z_{12k1}z_{21k1}/n_k^c}. \tag{4.12}$$

In the large table cases, we assume that $n_k^c/n_k \to \lambda_k$ as $n_k^c \to \infty$. In the sparse cases, we assume that $P(missing)$ are bounded away from 1. Then (4.12) is the same as (4.8) except that z_{12k1} and z_{21k1} are different from the MAR model in terms of the missingness probabilities. As before

$$\hat{\theta}_{MH,Comp}^{Informative} - \theta = \frac{\sum_1^K (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})/n_k^c}{\sum_1^K z_{12k1}z_{21k1}/n_k^c},$$

and

$$\begin{aligned}E[n_k^c] &= E[E[n_k^c|z_{11k}, z_{12k}, z_{21k}, z_{22k}]] \\ &= E[(z_{11k}+z_{12k})(1-p_{1k}^{missing}) + (z_{21k}+z_{22k})(1-p_{2k}^{missing})] \\ &= n_k[(\pi_{11k}+\pi_{12k})(1-p_{1k}^{missing}) + (\pi_{21k}+\pi_{22k})(1-p_{2k}^{missing})] \\ &= n_k - n_k[(\pi_{11k}+\pi_{12k})p_{1k}^{missing} + (\pi_{21k}+\pi_{22k})p_{2k}^{missing}]] \qquad (4.13)\end{aligned}$$

Conditioning on the total number of the fully observed data n_k^c, $(z_{11k1}, z_{12k1}, z_{21k1})$ have a multinomial distribution with parameters n_k^c, π_{11k1}^{comp}, π_{12k1}^{comp}, π_{21k1}^{comp}. $z_{22k1} = n_k^c - z_{11k1} - z_{12k1} - z_{21k1}$,

$1-\pi_{11k1}^{comp}-\pi_{12k1}^{comp}-\pi_{21k1}^{comp}=\pi_{22k1}^{comp}$, $\pi_{1j1}=n_k\pi_{1jk}(1-p_{1k}^{missing})/E[n_k^c]$ and $\pi_{2jk1}=n_k\pi_{2jk}(1-p_{2k}^{missing})/E[n_k^c]$. The π_{ijk1} satisfy the common odds assumption, so that

$$\begin{aligned}
&\pi_{11k1}^{comp}\pi_{22k1}^{comp}-\theta\pi_{12k1}^{comp}\pi_{21k1}^{comp}\\
&= n_k\pi_{11k}(1-p_{1k}^{misisng})n_k\pi_{22k})(1-p_{2k}^{missing})\\
&\quad -\theta n_k\pi_{12k}(1-p_{1k}^{missing})n_k\pi_{21k}(1-p_{2k}^{missing})\\
&= n_k^2(1-p_{1k}^{missing})(1-p_{2k}^{missing})(\pi_{11k}\pi_{22k}-\theta(\pi_{12k}\pi_{21k}))\\
&= 0.
\end{aligned}$$

Furthermore

$$\begin{aligned}
E[z_{11k1}z_{22k1}/n_k^c] &= E[E[z_{11k1}z_{22k1}/n_k^c|n_k^c]]\\
&= E\left[\frac{1}{n_k^c}E[z_{11k1}z_{22k1}|n_k^c]\right]\\
&= E\left[\frac{1}{n_k^c}n_k^c(n_k^c-1)\pi_{11k1}^{comp}\pi_{22k1}^{comp}\right]\\
&= E[(n_k^c-1)\pi_{11k1}^{comp}\pi_{22k1}^{comp}]\\
&= \pi_{11k1}^{comp}\pi_{22k1}^{comp}E[n_k^c-1]\\
&= \frac{n_k\pi_{11k}(1-p_{1k}^{missing})}{E[n_k^c]}\frac{n_k\pi_{22k}(1-p_{2k}^{missing})}{E[n_k^c]}(E[n_k^c]-1)\\
&= \left(\frac{1}{E[n_k^c]}-\frac{1}{E[n_k^c]^2}\right)(n_k^2\pi_{11k}\pi_{22k}(1-p_{1k}^{missing})(1-p_{2k}^{missing}))\\
E[z_{12k1}z_{21k1}/n_k^c] &= E[E[z_{12k1}z_{21k1}/n_k^c|n_k^c]]\\
&= E\left[\frac{1}{n_k^c}E[z_{12k1}z_{21k1}|n_k^c]\right]\\
&= E\left[\frac{1}{n_k^c}n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\right]\\
&= E[(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}]\\
&= \pi_{12k1}^{comp}\pi_{21k1}^{comp}E[n_k^c-1]\\
&= \frac{n_k\pi_{12k}(1-p_{1k}^{missing})}{E[n_k^c]}\frac{n_k\pi_{21k}(1-p_{2k}^{missing})}{E[n_k^c]}(E[n_k^c]-1)\\
&= \left(\frac{1}{E[n_k^c]}-\frac{1}{E[n_k^c]^2}\right)(n_k^2\pi_{12k}\pi_{21k}(1-p_{2k}^{missing})(1-p_{1k}^{missing}))\\
&= \frac{E[z_{11k1}z_{22k}/n_k^c]}{\theta},
\end{aligned}$$

so that

$$\begin{aligned}
E\left[\sum z_{12k1}z_{21k1}/n_k^c\right] &= \sum_{k=1}^{K} E[z_{12k1}z_{21k1}/n_k^c] \\
&= \sum_{k=1}^{K} E[n_k^c - 1]\pi_{12k1}^{comp}\pi_{21k1}^{comp} \\
&= \sum_{1}^{k}\left[\frac{1}{E[n_k^c]} - \frac{1}{(E[n_k^c])^2}\right](n_k^2\pi_{12k}\pi_{21k}(1-p_{1k}^{missing})(1-p_{2k}^{missing})) \\
&= \mu_{x,Comp}^{Informative}.
\end{aligned}$$

Even though the expression for $\mu_{x,Comp}^{Informative}$ is the same as $\mu_{x,Comp}^{MAR}$, because $E[n_k^c]$ is different in MAR and Informative missingness model, in general, $\mu_{x,Comp}^{Informative}$ is not the same as $\mu_{x,Comp}^{MAR}$. As in previous sections, we write $\hat{\theta}_{MH,Comp}^{Informative} = y_{Comp}^{Informative}/x_{Comp}^{Informative}$ and using Taylor expansion, $\hat{\theta}_{MH,Comp}^{Informative} - \theta$ can be written as

$$\begin{aligned}
&\frac{y_{Comp}^{Informative} - \theta x_{Comp}^{Informative}}{x_{Comp}^{Informative}} \\
&= \frac{y_{Comp}^{Informative} - \theta x_{Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\left[1 - \frac{x_{Comp}^{Informative} - \mu_{x,Comp}^{Informative}}{\mu_{x,Comp}^{Informative}} + \cdots\right] \qquad (4.14) \\
&= \frac{y_{Comp}^{Ifnormative} - \theta x_{Comp}^{Informative}}{\mu_{x,Comp}^{Informative}} \\
&\quad - \left(\frac{y_{Comp}^{Informative} - \theta x_{Comp}^{Informative}}{\mu_{x,Comp}^{Informormaitve}}\right)\left(\frac{x_{Comp}^{Informative} - \mu_{x,Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right) \\
&\quad + \left(\frac{y_{Comp}^{Informative} - \theta x_{MH,Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right)\left(\frac{x_{Comp}^{Informative} - \mu_{x,Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right)^2 + \cdots.
\end{aligned}$$

From the calculation in section 4.2.1, we can see that the expected value of the first term of (4.14),

$$E\left[\frac{y_{Comp}^{Ifnormative} - \theta x_{Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right] = 0.$$

We can use the same techniques and arguments used to calculate the expected value of the second term of (4.10) to find the expected value of the second term of (4.14):

$$\begin{aligned}
&E\left[\left(\frac{y_{Comp}^{Informative} - \theta x_{Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right)\left(\frac{x_{Comp}^{Informative} - \mu_{x,Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right)\right] \\
&= \frac{1}{(\mu_{x,Comp}^{Informative})^2} E\left[\sum\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c} - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]\right)\right]\right] \\
&= \frac{1}{(\mu_{x,Comp}^{Informative})^2}\sum_k A_{k,comp}^{Informative}.
\end{aligned}$$

Since table k and table k' are independent, so are the complete subtables of table k and k', hence, the cross-product terms are 0.

$$
\begin{aligned}
A_{k,comp}^{Informative} &= E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]\right)\right] \\
&= E\left[\frac{1}{(n_k^c)^2}E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}-\theta z_{12k1}^2 z_{21k1}^2|n_k^c]\right] \\
&\quad - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right].
\end{aligned}
$$

Because

$$
E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}|n_k^c] = n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp},
$$

and

$$
\begin{aligned}
E[\theta z_{12k1}^2 z_{21k1}^2|n_k^c] &= \theta n_k^c(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{11k1}^{comp}\pi_{22k1}^{comp}/\theta \\
&\quad + \theta n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}((n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1),
\end{aligned}
$$

then

$$
\begin{aligned}
&E[(z_{11k1}z_{12k1}z_{21k1}z_{22k1}-\theta z_{12k1}^2 z_{21k1}^2)|n_k^c] \\
&= -\theta n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}((n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&E\left[\frac{(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})}{(n_k^c)^2}\right] \\
&= -E\left[\frac{1}{(n_k^c)^2}\theta n_k^c(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}((n_k^c-2)(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})+1)\right] \\
&= -\theta\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]-\theta\pi_{12k1}^{comp}\pi_{21k1}^{comp}E\left[\frac{(n_k^c-1)}{n_k^c}\right].
\end{aligned}
$$

Also,

$$
E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right] = 0.
$$

Hence,

$$
\begin{aligned}
A_{k,comp}^{Informative} &= -\theta\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right] \\
&\quad -\theta\pi_{12k1}^{comp}\pi_{21k1}^{comp}E\left[\frac{(n_k^c-1)}{n_k^c}\right].
\end{aligned}
$$

So the expected value of the second term of (4.14) is

$$
\begin{aligned}
&E\left[\left(\frac{y_{Comp}^{Informative}-\theta x_{Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right)\left(\frac{x_{Comp}^{Informative}-\mu_{x,Comp}^{Informative}}{\mu_{x,Comp}^{Informative}}\right)\right] \\
&= \frac{1}{(\mu_{x,Comp}^{Informative})^2}\sum_k A_{k,comp}^{Informative} \\
&= \frac{-\theta}{(\mu_{x,Comp}^{Informative})^2}\sum_k\left(\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right]\right) \\
&\quad +\frac{-\theta}{(\mu_{x,Comp}^{Informative})^2}\sum_k\left(\pi_{12k1}^{comp}\pi_{21k1}^{comp}E\left[\frac{(n_k^c-1)}{n_k^c}\right]\right)
\end{aligned}
$$

The expected value of the third term of (4.14) is

$$
\begin{aligned}
&E\left[\left(\frac{\sum_1^K(z_{11k1}z_{22k1}/n_k^c-\theta z_{12k1}z_{21k1}/n_k^c)}{\mu_{x,Comp}^{MAR}}\right)\left(\frac{\sum_1^K(z_{12k1}z_{21k1}/n_k^c-E[z_{12k1}z_{21k1}/n_k])}{\mu_{x,Comp}^{MAR}}\right)^2\right] \\
&= \frac{1}{(\mu_{x,Comp}^{MAR})^3}\sum_1^K E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right] \qquad (4.15)
\end{aligned}
$$

The kth term of the summation in (4.15) is

$$
\begin{aligned}
&E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right] \\
&= E\left[\frac{z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}}{(n_k^c)^3}\right]-\theta E\left[\frac{z_{12k1}^3z_{21k1}^3}{(n_k^c)^3}\right] \\
&\quad -2\left(E\left[\frac{z_{11k1}z_{12k1}z_{21k1}z_{22k1}E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]+\theta E\left[\frac{z_{12k1}^2z_{21k1}^2E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]\right) \\
&\quad +E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right]\left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]^2\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
&E\left[\frac{z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}}{(n_k^c)^3}\right] \\
&= E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)(n_k^c-5)\pi_{11k1}^{comp}(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2\pi_{22k1}^{comp}\right] \\
&\quad +E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\right] \\
&\quad +E\left[\frac{3}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}\right].
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[\frac{z_{12k1}^3z_{21k1}^3}{(n_k^c)^3}\right] \\
&= E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)(n_k^c-5)(\pi_{12k1}^{comp})^2(\pi_{21k1}^{comp})^2(\pi_{11k1}^{comp}\pi_{22k1}^{comp}/\theta)\right]
\end{aligned}
$$

$$+E\left[\frac{3}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\frac{\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{11k1}^{comp}\pi_{22k1}^{comp}}{\theta}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\right]$$
$$+E\left[\frac{7}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(\pi_{12k1}^{comp})^3\pi_{21k1}^{comp}\right]$$
$$+E\left[\frac{9}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{11k1}^{comp}\pi_{22k1}^{comp}\theta)\right]$$
$$+E\left[\frac{7}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^3\right]$$
$$+E\left[\frac{21}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(\pi_{12k1}^{comp})^2\pi_{21k1}^{comp}\right]$$
$$+E\left[\frac{21}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\pi_{12k1}^{comp}(\pi_{21k1}^{comp})^2+\frac{50}{(n_k^c)^2}(n_k^c-1)\pi_{12k1}^{comp}\pi_{21k1}^{comp}\right]$$

Therefore,

$$
\begin{aligned}
& E\left[\frac{z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}}{(n_k^c)^3}\right]-\theta E\left[\frac{z_{12k1}^3z_{21k1}^3}{(n_k^c)^3}\right]\\
&= -E\left[\frac{2}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\\
&\quad -E\left[\frac{6}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}\\
&\quad -E\left[\frac{7\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right]\pi_{12k1}^{comp}\pi_{21k1}^{comp}((\pi_{12k1}^{comp})^2+(\pi_{21k1}^{comp})^2)\\
&\quad -E\left[\frac{21\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\\
&\quad -E\left[\frac{50\theta}{(n_k^c)^2}(n_k^c-1)\right]\pi_{12k1}^{comp}\pi_{21k1}^{comp},\\
& -2E\left[\frac{z_{11k1}z_{12k1}z_{21k1}z_{22k}E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]+2\theta E\left[\frac{z_{12k1}^2z_{21k1}^2E[z_{12k1}z_{21k1}]}{(n_k^c)^3}\right]\\
&= 2E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\\
&\quad 2E[n_k^c(n_k^c-1)]E\left[\frac{n_k^c-1}{(n_k^c)^2}\right]\pi_{12k1}^{comp}\pi_{21k1}^{comp},\\
& \text{and } E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right]\left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}\right]^2\right)\\
&= 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right]\\
&= -2E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\\
&\quad +2E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})
\end{aligned}
$$

$$
\begin{aligned}
&- E\left[\frac{6}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right]\pi_{11k1}^{comp}\pi_{12k1}^{comp}\pi_{21k1}^{comp}\pi_{22k1}^{comp}\\
&- E\left[\frac{7\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)\right]\pi_{12k1}^{comp}\pi_{21k1}^{comp}((\pi_{12k1}^{comp})^2+(\pi_{21k1}^{comp})^2)\\
&- E\left[\frac{21\theta}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\pi_{12k1}^{comp}\pi_{21k1}^{comp}(\pi_{12k1}^{comp}+\pi_{21k1}^{comp})\\
&- \left(E\left[\frac{50\theta}{(n_k^c)^2}(n_k^c-1)\right]+2E[n_k^c(n_k^c-1)]E\left[\frac{n_k^c-1}{(n_k^c)^2}\right]\right)\pi_{12k1}^{comp}\pi_{21k1}^{comp}.
\end{aligned}
$$

Since

$$
\begin{aligned}
& E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)(n_k^c-3)(n_k^c-4)\right]-E[n_k^c(n_k^c-1)]E\left[\frac{1}{(n_k^c)^2}(n_k^c-1)(n_k^c-2)\right]\\
&= E[(n_k^c)^2]-10E[n_k^c]+35-50E\left[\frac{1}{n_k^c}\right]+24E\left[\frac{1}{(n_k^c)^2}\right]\\
&\quad - E[(n_k^c)^2]+E[n_k^c]-(E[(n_k^c)^2]-E[n_k^c])\left(-3E\left[\frac{1}{n_k^c}\right]+2E\left[\frac{1}{(n_k^c)^2}\right]\right)\\
&= -9E[n_k^c]+35-50E\left[\frac{1}{n_k^c}\right]+24E\left[\frac{1}{(n_k^c)^2}\right]\\
&\quad -(E[(n_k^c)^2]-E[n_k^c])\left(-3E\left[\frac{1}{n_k^c}\right]+2E\left[\frac{1}{(n_k^c)^2}\right]\right),
\end{aligned}
$$

then

$$
E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\left(\frac{z_{12k1}z_{21k1}}{n_k^c}-\frac{E[z_{12k1}z_{21k1}]}{n_k^c}\right)^2\right] = O(E[n_k^c])
$$

Since $\mu_{x,Comp}^{MAR}$ is $O(E[N^c])$ when K and θ are fixed, the expected value of the third term of (4.10) is $O(1/(E[N^c])^2)$ or $O(1/K^2)$ for fixed $E[n_k^c]$ and θ and bounded away cell probabilities. Therefore, the leading term of the bias is $O(1/E[N^c])$ when K and θ are fixed or $O(1/K)$ when $E[n_k^c]$ and θ are fixed and cell probabilities are bounded away from 0. Using the only fully observed subtables are incurs the same order of the bias as when using full data no matter the data is missing MAR or missing informative depending on A itself and the stratum variable C.

Now we investigate the variance. we have

$$
\begin{aligned}
&\text{Var }(\hat{\theta}_{MH,Comp}^{Informative}-\theta)\\
&\doteq \text{Var}\left(\sum_{k=1}^{K}\left(\frac{z_{11k1}z_{12k1}/n_k^c-\theta z_{12k1}z_{21k1}/n_k^c}{\mu_{x,Comp}^{Informative}}\right)\right)\\
&= \frac{1}{(\mu_{x,Comp}^{Informative})^2}\sum_{1}^{K}E\left[\frac{1}{(n_k^c)^2}(E[z_{11k1}^2Z_{22k1}^2|n_k^c]+E[z_{12k1}^2z_{21k1}^2|n_k^c])\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2\theta}{(\mu_{x,Comp}^{Informative})^2} \sum_{1}^{K} E\left[\frac{1}{(n_k^c)^2} E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}|n_k^c]\right] \\
= \quad & \frac{1}{(\mu_{x,Comp}^{Informative})^2} \left(E\left[\frac{(n_k^c-1)(n_k^c-2)}{n_k^c}\right] \frac{n_k^3}{(E[n_k^c])^3} \pi_{11k}\pi_{22k}(1-p_{1k}^{missing})(1-p_{2k}^{missing}) \right. \\
& \times((\pi_{11k}+\theta\pi_{12k})(1-p_{1k}^{misisng}) + (\pi_{22k}+\theta\pi_{21k})(1-p_{2k}^{missing})) \\
& \left. + E\left[\frac{n_k^c-1}{n_k^c}\right] \frac{n_k^2}{(E[n_k^c])^2}(1+\theta)\pi_{11k}\pi_{22k}(1-p_{1k}^{missing})(1-p_{2k}^{missing}) \right).
\end{aligned}
$$

As $\mu_{x,Comp}^{Informative}$ is $O(E[N^c])$ when K and θ are fixed and $O(K)$ when $E[n_k^c]$ and θ are fixed and the cell probabilities are bounded away from 0, Var $(\hat{\theta}_{MH,Comp}^{Informative})$ is $O(1/E[N^c])$ for fixed K and θ and $O(1/K)$ for fixed $E[n_k^c]$ and θ with bounded away from 0 cell probabilities. For fixed $E[N^c]$ and K, $\mu_{x,Comp}^{Informative}$ is $O(1/\theta)$ that implies Var $(\hat{\theta}_{MH,Comp}^{Informative})$ is $O(\theta^3)$.

4.2.3 Closed Form Estimated Data for MAR Model

As closed form estimates for cell means under the MAR(B, C) and Informative(A, C) models are found in the previous chapter, we also use the closed form estimated cell means to estimate the common odds ratio.

The kth table of the original data is a 3×2 frequency table of the form

	$B = 1$	$B = 2$	Total
$A = 1$	z_{11k1}	z_{12k1}	z_{1+k1}
$A = 2$	z_{21k1}	z_{22k1}	z_{2+k1}
A unobserved	z_{+1k0}	z_{+2k0}	z_{++k0}
			n_k

and if we combine the fully observed data of each column, we could collapse the table to

	$B = 1$	$B = 2$	Total
A is observed	z_{+1k1}	z_{+2k1}	z_{++k1}
A is unobserved	z_{+1k0}	z_{+2k0}	z_{++k0}
			n_k

Given the number of total observations n_k, $(z_{+1k1},\ z_{+2k1},\ z_{+1k0})$ have a multinomial distribution with parameters $n_k,\ \pi_{+1k1},\ \pi_{+2k1},\ \pi_{+1k0}$ where $\pi_{+1k1} = \pi_{+1k}(1 - p_{1k}^{missing})$, $\pi_{+2k1} = \pi_{+2k}(1 - p_{2k}^{missing})$, and $\pi_{+1k0} = \pi_{+1k}p_{1k}^{missing}$. Moreover, $z_{+2k0} = n_k - z_{+1k1} - z_{+2k1} - z_{+1k0}$ and $\pi_{+2k0} = 1 - \pi_{+1k1} - \pi_{+2k1} - \pi_{+1k0}$. The total number of fully observed data $n_k^c = z_{+1k1} + z_{+2k1}$ and conditioning on z_{+jk1}, z_{ijk1} is binomial$(z_{+jk1},\ \pi_{ijk}/\pi_{+jk})$.

The closed forms for MAR(B, C) model as in Equation (4.6) are $\hat{m}_{ijk} = z_{ijk1}$ and $\hat{a}_{.jk} = z_{+jk0}/z_{+jk1}$. To avoid infinite estimators, we only consider the cases where $z_{+jk1} \neq 0$. We drop out the "bad" cases in the simulations. All of the proofs assume $z_{+jk1} > 0$. Therefore, the estimated (i, j) cell frequencies are

$$\begin{aligned}\hat{\mu}_{ijk} &= \hat{m}_{ijk}(1 + \hat{a}_{.jk}) \\ &= z_{ijk1}\left(1 + \frac{z_{+jk0}}{z_{+jk1}}\right) \\ &= z_{ijk1}\frac{z_{+jk+}}{z_{+jk1}}, \end{aligned} \tag{4.16}$$

and the expected cell counts are

$$\begin{aligned}E[\hat{\mu}_{ijk}] &= E\left[z_{ijk1}\frac{z_{+jk+}}{z_{+jk1}}\right] \\ &= E\left[\frac{z_{+jk+}}{z_{+jk1}}E[z_{ijk1}|z_{+jk1}, z_{+jk0}]\right] \\ &= E\left[\frac{z_{+jk+}}{z_{+jk1}}z_{+jk1}\frac{\pi_{ijk}}{\pi_{+jk}}\right] \\ &= \frac{\pi_{ijk}}{\pi_{+jk}}E[E[z_{+jk+}|n_k]] \\ &= n_k\pi_{+jk}\frac{\pi_{ijk}}{\pi_{+jk}} \\ &= n_k\pi_{ijk}.\end{aligned}$$

Hence the estimated full counts for table k are

	$B = 1$	$B = 2$
$A = 1$	$z_{11k1}\frac{z_{+1k+}}{z_{+1k1}}$	$z_{12k1}\frac{z_{+2k+}}{z_{+2k1}}$
$A = 2$	$z_{21k1}\frac{z_{+1k+}}{z_{+1k1}}$	$z_{22k1}\frac{z_{+2k+}}{z_{+2k1}}$

the total estimate table counts are n_k and the expected full cell counts for table k are

	$B=1$	$B=2$
$A=1$	$n_k\pi_{11k}$	$n_k\pi_{12k}$
$A=2$	$n_k\pi_{21k}$	$n_k\pi_{22k}$

Therefore

$$\begin{aligned}\theta_k &= \frac{n_k\pi_{11k}n_k\pi_{22k}}{n_k\pi_{12k}n_k\pi_{21k}} \\ &= \frac{\pi_{11k}\pi_{22k}}{\pi_{12k}\pi_{21k}} \\ &= \theta\end{aligned}$$

and

$$\begin{aligned}&\frac{\hat{\mu}_{11k}\hat{\mu}_{22k}}{\hat{\mu}_{12k}\hat{\mu}_{21k}} \\ &= \frac{(z_{11k1}z_{+1k+}/z_{+1k1})(z_{22k1}z_{+2k+}/z_{+2k1})}{(z_{12k1}z_{+2k+}/z_{+2k1})(z_{21k1}z_{+1k+}/z_{+1k1})} \\ &= \frac{z_{11k1}z_{22k1}}{z_{12k1}z_{21k1}}.\end{aligned}$$

We define

$$\begin{aligned}\hat{\theta}^{MAR}_{MH,EST} &= \frac{\sum_1^K \hat{\mu}_{11k}\hat{\mu}_{22k}/n_k}{\sum_1^K \hat{\mu}_{12k}\hat{\mu}_{21k}/n_k} \\ &= \frac{\sum_1^K [(z_{11k1}z_{+1k+}/z_{+1k1})(z_{22k1}z_{+2k+}/z_{+2k1})]/n_k}{\sum_1^K [(z_{12k1}z_{+2k+}/z_{+2k1})(z_{21k1}z_{+1k+}/z_{+1k1})]/n_k} \\ &= \frac{\sum_1^K (z_{11k1}z_{22k1}/n_k^c)w_k}{\sum_1^K (z_{12k1}z_{21k1}/n_k^c)w_k},\end{aligned}$$

where

$$w_k = \left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)\left(\frac{n_k^c}{n_k}\right).$$

Then

$$\hat{\theta}^{MAR}_{MH,EST} - \theta = \frac{\sum_1^K [(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})/n_k^c]w_k}{\sum_1^K (z_{12k1}z_{21k1}/n_k^c)w_k}.$$

We have

$$\begin{aligned}&E\left[\frac{z_{11k1}z_{22k1}}{n_k^c}w_k\right] \\ &= E\left[E\left[\frac{z_{11k1}z_{22k1}}{n_k^c}\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\frac{n_k^c}{n_k}\middle| z_{+1k1}, z_{+2k1}, z_{+1k0}, z_{+2k0}\right]\right]\end{aligned}$$

$$
\begin{aligned}
&= E\left[\frac{1}{z_{+1k1}+z_{+2k1}}\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\frac{z_{+1k1}+z_{+2k1}}{n_k}E[z_{11k1}z_{22k1}|z_{+1k1},z_{+2k1},z_{+1k0},z_{+2k0}]\right] \\
&= E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}z_{+1k1}\frac{\pi_{11k}}{\pi_{+1k}}z_{+2k1}\frac{\pi_{22k}}{\pi_{+2k}}\right] \\
&= \frac{\pi_{11k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}}E\left[\frac{z_{+1k+}z_{+2k+}}{n_k}\right] \\
&= \frac{\pi_{11k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k}E[(z_{+1k1}+z_{+1k0})(z_{+2k1}+z_{+2k0})] \\
&= \frac{\pi_{11k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k}E[z_{+1k1}z_{+2k1}+z_{+1k0}z_{+2k1}+z_{+1k1}z_{+2k0}+z_{+1k0}z_{+2k0}] \\
&= \frac{\pi_{11k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}}(n_k-1)\pi_{+1k}\pi_{+2k}((1-p_{1k}^{missing})(1-p_{2k}^{missing})+p_{1k}^{missing}(1-p_{2k}^{missing})) \\
&\quad +\frac{\pi_{11k}\pi_{22k}}{\pi_{+2k}\pi_{+1k}}(n_k-1)\pi_{+1k}\pi_{+2k}((1-p_{1k}^{missing})p_{2k}^{missing}+p_{1k}^{missing}p_{2k}^{missing}) \\
&= (n_k-1)\pi_{11k}\pi_{22k},
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] \\
&= E\left[\frac{1}{z_{+1k1}+z_{+2k1}}\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\frac{z_{+1k1}+z_{+2k1}}{n_k}E[z_{12k1}z_{21k1}|z_{+1k1},z_{+2k1},z_{+1k0},z_{+2k0}]\right] \\
&= E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}z_{+1k1}\frac{\pi_{21k}}{\pi_{+1k}}\right] \\
&= \frac{\pi_{12k}\pi_{21k}}{\pi_{+2k}\pi_{+1k}}E\left[\frac{z_{+1k+}z_{+2k+}}{n_k}\right] \\
&= (n_k-1)\pi_{12k}\pi_{21k} \\
&= (n_k-1)\frac{\pi_{11k}\pi_{22k}}{\theta},
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[\sum\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] \\
&= \sum_{k=1}^{K}E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] \\
&= \sum_{k=1}^{K}(n_k-1)\pi_{12k}\pi_{21k} \\
&= \mu_{x,EST}^{MAR}=\mu_x
\end{aligned}
$$

Let $N=\sum_{k=1}^{K}n_k$. Then when K and θ are fixed, $\mu_{x,EST}^{MAR}$ is $O(N)$ and when n_k and θ are fixed, $\mu_{x,EST}^{MAR}$ is $O(K)$. When K and n_k are fixed, $\mu_{x,EST}^{MAR}$ is $O(1/\theta)$.

As in previous sections, writing $\hat{\theta}^{MAR}_{MH,EST} = y^{MAR}_{EST}/x^{MAR}_{EST}$ and using Taylor expansion, $\hat{\theta}^{MAR}_{MH,EST} - \theta$ can be written as

$$
\begin{aligned}
\frac{y^{MAR}_{EST} - \theta x^{MAR}_{EST}}{x^{MAR}_{EST}} &= \frac{y^{MAR}_{EST} - \theta x^{MAR}_{EST}}{\mu^{MAR}_{x,EST}} \left[1 - \frac{x^{MAR}_{EST} - \mu^{MAR}_{x,EST}}{\mu^{MAR}_{x,EST}} + \cdots \right] \\
&= \frac{y^{MAR}_{EST} - \theta x^{MAR}_{EST}}{\mu^{MAR}_{x,EST}} \\
&\quad - \left(\frac{y^{MAR}_{EST} - \theta x^{MAR}_{EST}}{\mu^{MAR}_{x,EST}} \right) \left(\frac{x^{MAR}_{EST} - \mu^{MAR}_{x,EST}}{\mu^{MAR}_{x,EST}} \right) \\
&\quad + \left(\frac{y^{MAR}_{EST} - \theta x^{MAR}_{MH,EST}}{\mu^{MAR}_{x,EST}} \right) \left(\frac{x^{MAR}_{EST} - \mu^{MAR}_{x,EST}}{\mu^{MAR}_{x,EST}} \right)^2 + \cdots .
\end{aligned}
\tag{4.17}
$$

The expected value of the first term of (4.17) is

$$
\begin{aligned}
E\left[\frac{y^{MAR}_{EST} - \theta x^{MAR}_{EST}}{\mu^{MAR}_{x,EST}} \right] &= E\left[\frac{1}{\mu^{MAR}_{x,EST}} \sum_{k=1}^{K} \left(\frac{z_{11k1} z_{22k1}}{n^c_k} w_k - \theta \left(\frac{z_{12k1} z_{21k1}}{n^c_k} w_k \right) \right) \right] \\
&= \frac{1}{\mu^{MAR}_{x,EST}} \sum_{k=1}^{K} \left(E\left[\frac{z_{11k1} z_{22k1}}{n^c_k} w_k \right] - \theta E\left[\frac{z_{12k1} z_{21k1}}{n^c_k} w_k \right] \right) \\
&= \frac{1}{\mu^{MAR}_{x,EST}} \sum_{k=1}^{K} \left((n_k - 1)\pi_{11k}\pi_{22k} - \theta (n_k - 1) \frac{\pi_{11k}\pi_{22k}}{\theta} \right) \\
&= 0
\end{aligned}
$$

The expected value of the second term of (4.17) is

$$
\begin{aligned}
& E\left[\left(\frac{y^{MAR}_{EST} - \theta x^{MAR}_{EST}}{\mu^{MAR}_{x,EST}} \right) \left(\frac{x^{MAR}_{EST} - \mu^{MAR}_{x,EST}}{\mu^{MAR}_{x,EST}} \right) \right] \\
&= \frac{1}{(\mu^{MAR}_{x,EST})^2} E\left[\left(\sum \frac{z_{11k1} z_{22k1}}{n^c_k} w_k - \theta \left(\sum \frac{z_{12k1} z_{21k1}}{n^c_k} w_k \right) \right) \left(\sum \frac{z_{12k1} z_{21k1}}{n^c_k} w_k - \mu^{MAR}_{x,EST} \right) \right] \\
&= \frac{1}{(\mu^{MAR}_{x,EST})^2} E\left[\left(\sum \left(\frac{z_{11k1} z_{22k1} - \theta z_{12k1} z_{21k1}}{n^c_k} w_k \right) \right) \right. \\
&\qquad \left. \times \left(\sum \left(\frac{z_{12k1} z_{21k1}}{n^c_k} w_k - E\left[\frac{z_{12k1} z_{21k1}}{n^c_k} w_k \right] \right) \right) \right] \\
&= \frac{1}{(\mu^{MAR}_{x,EST})^2} E\left[\sum \left[\left(\frac{z_{11k1} z_{22k1} - \theta z_{12k1} z_{21k1}}{n^c_k} w_k \right) \left(\frac{z_{12k1} z_{21k1}}{n^c_k} w_k - E\left[\frac{z_{12k1} z_{21k1}}{n^c_k} w_k \right] \right) \right] \right] \\
&= \frac{1}{(\mu^{MAR}_{x,EST})^2} \sum_k A^{MAR}_{k,EST},
\end{aligned}
$$

Since tables k and k' are independent, the cross-product terms are 0. We have

$$
A^{MAR}_{k,EST} = E\left[\left(\frac{z_{11k1} z_{22k1} - \theta z_{12k1} z_{21k1}}{n^c_k} w_k \right) \left(\frac{z_{12k1} z_{21k1}}{n^c_k} w_k - E\left[\frac{z_{12k1} z_{21k1}}{n^c_k} w_k \right] \right) \right]
$$

$$
\begin{aligned}
&= E\left[\frac{(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})}{(n_k^c)^2}w_k^2\right] \\
&\quad - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right] \\
&= (n_k-1)\pi_{12k}\pi_{21k}E\left[\frac{z_{11k1}z_{22k1}}{n_k^c}w_k-\theta\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] \\
&= (n_k-1)\pi_{12k}\pi_{21k}\left((n_k-1)\pi_{11k}\pi_{22k}-\theta(n_k-1)\frac{\pi_{11k}\pi_{22k}}{\theta}\right) \\
&= 0.
\end{aligned}
$$

Since

$$
w_k = \left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)\left(\frac{n_k^c}{n_k}\right)
$$

$$
\begin{aligned}
A_{k,EST}^{MAR} &= E\left[\frac{(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})w_k^2}{(n_k^c)^2}\right] \\
&= E\left[E\left[\left.\frac{(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})w_k^2}{(n_k^c)^2}\right| z_{+1k1},z_{+2k1},z_{+1k0},z_{+2k0}\right]\right] \\
&= E\left[\frac{w_k^2}{(n_k^c)^2}E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}-\theta z_{12k1}^2z_{21k1}^2|z_{+1k1},z_{+2k1},z_{+1k0},z_{+2k0}]\right] \\
&= E\left[\frac{w_k^2}{(n_k^c)^2}E[(z_{+1k1}-z_{21k1})z_{12k1}z_{21k1}(z_{+2k1}-z_{12k1})|z_{+1k1},z_{+2k1},z_{+1k0},z_{+2k0}]\right] \\
&\quad -\theta E\left[\frac{w_k^2}{(n_k^c)^2}E[z_{12k1}^2z_{21k1}^2|z_{+1k1},z_{+2k1},z_{+1k0},z_{+2k0}]\right]
\end{aligned}
$$

To simplify the notations, let $Z=(z_{+1k1},z_{+2k1},z_{+1k0},z_{+2k0})$. We have

$$
\begin{aligned}
&E[(z_{+1k1}-z_{21k1})z_{12k1}z_{21k1}(z_{+2k1}-z_{12k1})-\theta z_{12k1}^2z_{2k1}^2|Z] \\
&= E[z_{+1k1}z_{+2k1}z_{12k1}z_{21k1}-z_{+1k1}z_{21k1}z_{12k1}^2-z_{+2k1}z_{12k1}z_{21k1}^2+(1-\theta)z_{21k1}^2z_{12k1}^2|Z] \\
&= z_{+1k1}^2z_{+2k1}^2\frac{\pi_{12k}\pi_{21k}}{\pi_{+1k}\pi_{+2k}}-z_{+1k1}^2\frac{\pi_{21k}}{\pi_{+1k}}\left[z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2+z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\right] \\
&\quad -z_{+2k1}^2\frac{\pi_{12k}}{\pi_{+2k}}\left[z_{+1k1}^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2+z_{+1k1}\frac{\pi_{21k}}{\pi_{+2k}}\frac{\pi_{11k}}{\pi_{+1k}}\right] \\
&\quad +(1-\theta)\left(z_{+1k1}^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2+z_{+1k1}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{11k}}{\pi_{+1k}}\right)\left(z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2+z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\right) \\
&= z_{+1k1}^2z_{+2k1}^2\frac{\pi_{12k}\pi_{21k}}{\pi_{+2k}\pi_{+1k}}\left(1-\frac{\pi_{12k}}{\pi_{+2k}}-\frac{\pi_{21k}}{\pi_{+1k}}+\frac{\pi_{12k}\pi_{21k}}{\pi_{+1k}\pi_{+21k}}-\theta\frac{\pi_{12k}\pi_{21k}}{\pi_{+1k}\pi_{+21k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- z_{+1k1}^2 z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} - z_{+2k1}^2 z_{+1k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} \\
&+ (1-\theta) \left(z_{+1k1}^2 z_{+2k1} \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} + z_{+1k1} z_{+2k1}^2 \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \right) \\
&+ (1-\theta) z_{+1k1} z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}},
\end{aligned}
$$

where

$$
\begin{aligned}
& z_{+1k1}^2 z_{+2k1}^2 \frac{\pi_{12k}\pi_{21k}}{\pi_{+2k}\pi_{+1k}} \left(1 - \frac{\pi_{12k}}{\pi_{+2k}} - \frac{\pi_{21k}}{\pi_{+1k}} + \frac{\pi_{12k}\pi_{21k}}{\pi_{+1k}\pi_{+21k}} - \theta \frac{\pi_{12k}\pi_{21k}}{\pi_{+1k}\pi_{+21k}} \right) \\
&= z_{+1k1}^2 z_{+2k1}^2 \frac{\pi_{12k}\pi_{21k}}{\pi_{+2k}\pi_{+1k}} \left(\left(1 - \frac{\pi_{12k}}{\pi_{+2k}} \right) \left(1 - \frac{\pi_{21k}}{\pi_{+1k}} \right) - \theta \frac{\pi_{12k}\pi_{21k}}{\pi_{+2k}\pi_{+1k}} \right) \\
&= z_{+1k1}^2 z_{+2k1}^2 \frac{\pi_{12k}\pi_{21k}}{\pi_{+2k}\pi_{+1k1}} \left(\frac{1}{\pi_{+1k}\pi_{+2k}} (\pi_{11k}\pi_{22k} - \theta \pi_{12k}\pi_{21k}) \right) \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& -z_{+1k1}^2 z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} + (1-\theta) \left(z_{+1k1}^2 z_{+2k1} \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \right) \\
&= z_{+1k1}^2 z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \left(-1 + (1-\theta) \frac{\pi_{21k}}{\pi_{+1k}} \right) \\
&= -z_{+1k1}^2 z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \left(\frac{\pi_{11k}}{\pi_{+1k}} + \theta \frac{\pi_{21k}}{\pi_{+1k}} \right) \\
&= -z_{+1k1}^2 z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \left(\frac{\pi_{11k}}{\pi_{+1k}} \right) - \theta z_{+1k1}^2 z_{+2k1} \left(\frac{\pi_{21k}}{\pi_{+1l}} \right)^2 \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}}
\end{aligned}
$$

$$
\begin{aligned}
& -z_{+2k1}^2 z_{+1k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} + (1-\theta) z_{+1k1} z_{+2k1}^2 \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \\
&= z_{+2k1}^2 z_{+1k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} \left(-1 + (1-\theta) \frac{\pi_{12k}}{\pi_{+2k}} \right) \\
&= -z_{+2k1}^2 z_{+1k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} - \theta z_{+2k1}^2 z_{+1k1} \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E[(z_{+1k1} - z_{21k1}) z_{12k1} z_{21k1} (z_{+2k1} - z_{12k1}) - \theta z_{12k1}^2 z_{2k1}^2 Z] \\
&= -z_{+1k1}^2 z_{+2k1} \frac{\pi_{21k}}{\pi_{+1l}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \frac{\pi_{11k}}{\pi_{+1k}} - \theta z_{+1k1}^2 z_{+2k1} \left(\frac{\pi_{21k}}{\pi_{+1l}} \right)^2 \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \\
&\quad - z_{+2k1}^2 z_{+1k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} - \theta z_{+2k1}^2 z_{+1k1} \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{11k}}{\pi_{+1k}}
\end{aligned}
$$

$$
\begin{aligned}
& + (1-\theta) z_{+1k1} z_{+2k1} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \\
= \quad & z_{+1k1} z_{+2k1} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} (-z_{+1k1} - z_{+2k1} + 1 - \theta) \\
& - \theta z_{+1k1} z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \left(\frac{z_{+1k1}\pi_{22k}\pi_{21k} - z_{+2k1}\pi_{11k}\pi_{12k}}{\pi_{+1k}\pi_{+2k}} \right).
\end{aligned}
$$

Then

$$
\begin{aligned}
A^{MAR}_{k,EST} \quad = \quad & E\left[\frac{w_k^2}{(n_k^c)^2} z_{+1k1} z_{+2k1} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} (-z_{+1k1} - z_{+2k1} + 1 - \theta) \right] \\
& - \theta E\left[\frac{w_k^2}{(n_k^c)^2} z_{+1k1} z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \left(\frac{z_{+1k1}\pi_{22k}\pi_{21k} + z_{+2k1}\pi_{11k}\pi_{12k}}{\pi_{+1k}\pi_{+2k}} \right) \right] \\
= \quad & \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} E\left[\left(\frac{z_{+1k+} z_{+2k+}}{z_{+1k1} z_{+2k1} n_k} \right)^2 z_{+1k1} z_{+2k1} (-z_{+1k1} - z_{+2k1} + 1 - \theta) \right] \\
& - \theta \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}\pi_{21k}}{\pi_{+1k}\pi_{+2k}} E\left[\left(\frac{z_{+1k+} z_{+2k+}}{z_{+1k1} z_{+2k1} n_k} \right)^2 z_{+1k1}^2 z_{+2k1} \right] \\
& - \theta \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{11k}\pi_{12k}}{\pi_{+1k}\pi_{+2k}} E\left[\left(\frac{z_{+1k+} z_{+2k+}}{z_{+1k1} z_{+2k1} n_k} \right)^2 z_{+1k1} z_{+2k1}^2 \right].
\end{aligned}
$$

Averaging over $(z_{+1k1},\, z_{+2k1},\, z_{+1k0})$, we have

$$
\begin{aligned}
& E\left[\left(\frac{z_{+1k+} z_{+2k+} n_k^c}{z_{+1k1} z_{+2k1} n_k} \right)^2 \frac{1}{(n_k^c)^2} z_{+1k1} z_{+2k1} (-z_{+1k1} - z_{+2k1} + 1 - \theta) \right] \\
= \quad & E\left[\left(\frac{z_{+1k+} z_{+2k+} n_k^c}{z_{+1k1} z_{+2k1} n_k} \right)^2 \frac{1}{(n_k^c)^2} z_{+1k1} z_{+2k1} (-z_{+1k1} - z_{+2k1} + 1 - \theta) \right] \\
= \quad & \frac{1}{n_k^2} E\left[\left(\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1} z_{+2k1}} \right) (-z_{+1k1} - z_{+2k1} + 1 - \theta) \right] \\
= \quad & \frac{1}{n_k^2} O(n_k^3) \\
= \quad & O(n_k) \text{ with negative leading coefficient,}
\end{aligned}
$$

and

$$
\begin{aligned}
& -E\left[\left(\frac{z_{+1k+} z_{+2k+} n_k^c}{z_{+1k1} z_{+2k1} n_k} \right)^2 \frac{z_{+1k1}^2 z_{+2k1}}{(n_k^c)^2} \right] \\
= \quad & -E\left[\left(\frac{z_{+1k+} z_{+2k+} n_k^c}{z_{+1k1} z_{+2k1} n_k} \right)^2 \frac{z_{+1k1}^2 z_{+2k1}}{(n_k^c)^2} \right] \\
= \quad & -\frac{1}{n_k^2} \left[\left(\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}} \right) \right] \\
= \quad & -\frac{1}{n_k^2} O(n_k^3) \\
= \quad & O(n_k) \text{ with negative leading coefficient.}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&-E\left[\left(\frac{z_{+1k+}z_{+2k+}n_k^c}{z_{+1k1}z_{+2k1}n_k}\right)^2\frac{z_{+1k1}z_{+2k1}^2}{(n_k^c)^2}\right] \\
&= -E\left[\left(\frac{z_{+1k+}z_{+2k+}n_k^c}{z_{+1k1}z_{+2k1}n_k}\right)^2\frac{z_{+1k1}z_{+2k1}^2}{(n_k^c)^2}\right] \\
&= -\frac{1}{n_k^2}E\left[\left(\frac{z_{+1k+}^2z_{+2k+}^2}{z_{+1k1}}\right)^2\right] \\
&= -\frac{1}{n_k^2}O(n_k^3) \\
&= O(n_k) \text{ with negative leading coefficient.}
\end{aligned}
$$

Therefore, the expected value of the second term of (4.17) is

$$
\begin{aligned}
&E\left[\left(\frac{y_{EST}^{MAR}-\theta x_{EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)\left(\frac{x_{EST}^{MAR}-\mu_{x,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)\right] \\
&= \frac{1}{(\mu_{x,EST}^{MAR})^2}\sum_k A_{k,EST}^{MAR} \\
&= O(1/N^2)O(N) \text{ with negative leading coefficient} \\
&= O(1/N),\ \text{with negative leading coefficient,}
\end{aligned}
$$

as $\mu_{x,EST}^{MAR}$ is $O(N)$ when K and θ are fixed. When n_k and θ are fixed, the expected value of the second term of (4.17) is $O(1/K)$.

The expected value of the third term of (4.17) is

$$
\begin{aligned}
&E\left[\left(\frac{y_{EST}^{MAR}-\theta x_{MH,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)\left(\frac{x_{EST}^{MAR}-\mu_{x,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)^2\right] \\
&= \frac{1}{(\mu_{x,EST}^{MAR})^3}E\left[\left(\sum\frac{z_{11k1}z_{22k1}}{n_k^c}w_k-\theta\left(\sum\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right)\right. \\
&\qquad\left.\times\left(\sum\frac{z_{12k1}z_{21k1}}{n_k^c}w_k-\mu_{x,EST}^{MAR}\right)^2\right] \\
&= \frac{1}{(\mu_{x,EST}^{MAR})^3}E\left[\sum\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k-E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2\right] \\
&= \frac{1}{(\mu_{x,EST}^{MAR})^3}\sum E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right. \\
&\qquad\left.\times\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k-E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2\right] \qquad (4.18)
\end{aligned}
$$

Since the observations from different strata are independent, all of the expectations of cross product terms are zeros. The expected value of the kth term of the summation of (4.18) is

$$
\begin{aligned}
& E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2\right] \\
= \;& E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\frac{z_{12k1}^2 z_{21k1}^2}{(n_k^c)^2}w_k^2\right] \\
& - 2E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}\right)w_k^2\right] \\
& + \left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2 E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k}}{n_k^c}w_k\right] \\
= \;& E\left[\frac{w_k^3}{(n_k^c)^3}E[z_{11k1}z_{12k1}^2 z_{21k1}^2 z_{22k1} - \theta z_{12k1}^3 z_{21k1}^3 | Z]\right] \\
& - 2(n_k-1)\pi_{12k}\pi_{21k}E\left[\frac{w_k^2}{(n_k^c)^2}E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}|Z]\right] \\
& + 2\theta(n_k-1)\pi_{12k}\pi_{21k}E\left[\frac{w_k^2}{(n_k^c)^2}E[z_{12k1}^2 z_{21k1}^2|Z]\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
& E[z_{11k1}z_{12k1}^2 z_{21k1}^2 z_{22k1} - \theta z_{12k1}^3 z_{21k1}^3 | Z] \\
= \;& E[(z_{+1k1}-z_{21k1})z_{12k1}^2 z_{21k1}^2 (z_{+2k1}-z_{12k1}) - \theta z_{12k1}^3 z_{21k1}^3 | Z] \\
= \;& E[z_{+1k1}z_{+2k1}z_{12k1}^2 z_{21k1}^2 - z_{+1k1}z_{12k1}^3 z_{21k1}^2 - z_{+2k1}z_{12k1}^2 z_{21k1}^3 | Z] \\
& + (1-\theta)E[z_{12k1}^3 z_{21k1}^3|Z] \\
= \;& z_{+1k1}z_{+2k1}E[z_{12k1}^2 z_{21k1}^2|Z] - z_{+1k1}E[z_{12k1}^3 z_{21k1}^2|Z] \\
& - z_{+2k1}E[z_{12k1}^2 z_{21k1}^3|Z] + (1-\theta)E[z_{12k1}^3 z_{21k1}^3|Z]
\end{aligned}
$$

and

$$
\begin{aligned}
& E[z_{11k1}z_{12k1}z_{21k1}z_{22k1} - \theta z_{12k1}^2 z_{21k1}^2 | Z] \\
= \;& E[(z_{+1k1}-z_{21k1})z_{12k1}z_{21k1}(z_{+2k1}-z_{12k1}) - \theta z_{12k1}^2 z_{21k1}^2 | Z] \\
= \;& z_{+1k1}z_{+2k1}E[z_{12k1}z_{21k1}|Z] - z_{+1k1}E[z_{12k1}^2 z_{21k1}|Z] \\
& - z_{+2k1}E[z_{12k1}z_{21k1}^2|Z] + (1-\theta)E[z_{12k1}^2 z_{21k1}^2|Z]
\end{aligned}
$$

Furthermore,

$$E[z_{12k1}^3 z_{21k1}^3|Z]$$

$$
\begin{aligned}
&= E[z_{12k1}^3|Z]E[z_{21k1}^3|Z] \\
&= E[z_{12k1}^3 - 3z_{12k1}^2 E[z_{12k1}|Z] + 3z_{12k1}E[z_{12k1}|Z]^2 - E[z_{12k1}|Z]^3|Z]E[z_{21k1}^3|Z] \\
&\quad + E[3_{12k1}^2 E[z_{12k1}|Z] - 3z_{12k1}E[z_{12k1}|Z]^2 + E[z_{12k1}|Z]^3|Z]E[z_{21k1}^3|Z] \\
&= E[(z_{12k1} - E[z_{12k1}|Z])^3|Z]E[z_{21k1}^3|Z] + (3E[z_{12k1}^2|Z]E[z_{12k1}|Z] - 2E[z_{12k1}|Z]^3)E[z_{21k}^3|Z] \\
&= E[(z_{12k1} - E[z_{12k1}|Z])^3]E[(z_{21k1} - E[z_{21k1}|Z])^3|Z] \\
&\quad + E[(z_{12k1} - E[z_{1k21}|Z])|Z]^3(3E[z_{21k1}^2|z]E[z_{21k1}|Z] - 2E[z_{21k1}|Z]^3) \\
&\quad + (3E[z_{12k1}^2|Z]E[z_{12k1}|Z] - 2E[z_{12k1}|Z]^3)E[(z_{21k1}^3 - E[z_{21k1}|Z])^3|Z] \\
&\quad + (3E[z_{12k1}^2|Z]E[z_{12k1}|Z] - 2E[z_{12k1}|Z]^3)(3E[z_{21k1}^2|Z]E[z_{21k1}|Z] - 2E[z_{21k1}|Z]^3) \\
&= E[(z_{12k1} - E[z_{12k1}|Z])^3]E[(z_{21k1} - E[z_{21k1}|Z])^3|Z] \\
&\quad + E[(z_{12k1} - E[z_{12k1}|Z])^3|Z](3E[z_{21k1}|Z] \text{ Var }(z_{21k1}|Z) + E[z_{21k1}|Z]^3) \\
&\quad + (3E[z_{12k1}|Z] \text{ Var }(z_{12k1}|Z) + E[z_{12k1}|Z]^3)E[(z_{21k1} - E[z_{21k1}|Z])^3|Z] \\
&\quad + (3E[z_{12k1}|Z] \text{ Var }(z_{12k1}|Z) + E[z_{12k1}|Z]^3)(3E[z_{21k1}|Z] \text{ Var }(z_{21k1}|Z) + E[z_{21k1}|Z]^3)
\end{aligned}
$$

and

$$
\begin{aligned}
E[z_{12k1}^3 z_{21k1}^2|Z] &= E[(z_{12k1} - E[z_{12k1}|Z])^3|Z](\text{ Var }(z_{21k1}|Z) + E[z_{21k1}|Z]^2) \\
&\quad +(3E[z_{12k1}|Z] \text{ Var }(z_{12k1}|Z) + E[z_{12k1}|Z]^3)(\text{ Var }(z_{21k1}|Z) + E[z_{21k1}|Z]^2) \\
E[z_{12k1}^2 z_{21k1}^3|Z] &= (\text{ Var }(z_{12k1}|Z) + E[z_{12k1}|Z]^2)E[(z_{21k1} - E[z_{21k1}|Z])^3|Z] \\
&\quad +(\text{ Var }(z_{12k1}|Z) + E[z_{12k1}|Z]^2)(3 \text{ Var }(z_{21k1}|Z)E[z_{21k1}|Z] + E[z_{21k1}|Z]^3) \\
E[z_{12k1}^2 z_{21k1}^2|Z] &= (\text{ Var }(z_{12k1}|Z) + E[z_{12k1}|Z]^2)(\text{ Var }(z_{21k1}|Z) + E[z_{21k1}|Z]^2) \\
E[z_{12k1}^2 z_{21k1}|Z] &= \text{ Var }(z_{12k1}|Z)E[z_{21k1}|Z] + E[z_{12k1}|Z]^2 E[z_{21k1}|Z] \\
E[z_{12k1} z_{21k1}^2|Z] &= E[z_{12k1}|Z] \text{ Var }(z_{21k1}|Z) + E[z_{12k1}|Z]E[z_{21k1}|Z]^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E[z_{11k1}z_{12k1}^2 z_{21k1}^2 z_{22k1} - \theta z_{12k1}^3 z_{21k1}^3|Z] \\
&\quad = z_{+1k1}z_{+2k1}E[z_{12k1}^2 z_{21k1}^2|Z] - z_{+1k1}E[z_{12k1}^3 z_{21k1}^2|Z] \\
&\quad\quad -z_{+2k1}E[z_{12k1}^2 z_{21k1}^3|Z] + (1-\theta)E[z_{12k1}^3 z_{21k1}^3|Z]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & z_{+1k1}z_{+2k1}(\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^2)(\text{ Var }(z_{21k1}|Z)+E[z_{21k1}|Z]^2) \\
& -z_{+1k1}E[(z_{12k1}-E[z_{12k1}|Z])^3|Z](\text{ Var }(z_{21k1}|Z)+E[z_{21k1}|Z]^2) \\
& -z_{+1k1}(3E[z_{12k1}|Z]\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^3)(\text{ Var }(z_{21k1}|Z)+E[z_{21k1}|Z]^2) \\
& -z_{+2k1}(\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^2)E[(z_{21k1}-E[z_{21k1}|Z])^3|Z] \\
& -z_{+2k1}(\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^2)(3\text{ Var }(z_{21k1}|Z)E[z_{21k1}|Z]+E[z_{21k1}|Z]^3) \\
& +(1-\theta)E[(z_{12k1}-E[z_{12k1}|Z])^3]E[(z_{21k1}-E[z_{21k1}|Z])^3|Z] \\
& +(1-\theta)E[(z_{12k1}-E[z_{12k1}|Z])^3|Z](3E[z_{21k1}|Z]\text{ Var }(z_{21k1}|Z)+E[z_{21k1}|Z]^3) \\
& +(1-\theta)(3E[z_{12k1}|Z]\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^2)E[(z_{21k1}-E[z_{21k1}|Z])^3|Z] \\
& +(1-\theta)(3E[z_{12k1}|Z]\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^3)3E[z_{21k1}|Z]\text{ Var }(z_{21k1}|Z) \\
& +(1-\theta)(3E[z_{12k1}|Z]\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^3)E[z_{21k1}|Z]^3
\end{aligned}
$$

If one conditions on the marginal totals of z_{+1k1} and z_{+2k1}, the complete data have a product binomial distribution and z_{ijk1} have a binomial(z_{+jk1}, π_{ijk}/π_{+jk}). Then

$$
\begin{aligned}
& E[(z_{12k1}-E[z_{12k1}|Z])^3|Z] \\
= \quad & z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 - 3z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 + 2z_{+2k1}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 + 3z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \\
& + 3z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\left(1-\frac{\pi_{12k}}{\pi_{+2k}}\right) - 2z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}} - 3z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 \\
& - 3z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(1-\frac{\pi_{12k}}{\pi_{+2k}}\right) + 2\left(z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 \\
= \quad & 2z_{+2k1}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 - 3z_{+2k1}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 + z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}} \\
= \quad & z_{+2k1}\left(2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 - 3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 + \frac{\pi_{12k}}{\pi_{+2k}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E[(z_{21k1}-E[z_{21k1}|Z])^3|Z] \\
& \quad = \quad z_{+1k1}\left(2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 - 3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + \frac{\pi_{21k}}{\pi_{+1k}}\right).
\end{aligned}
$$

Therefore, we rewrite each individual term of $E[z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}-\theta z_{12k1}^3z_{21k1}^3|Z]$ as follows:

$$
z_{+1k1}z_{+2k1}(\text{ Var }(z_{12k1}|Z)+E[z_{12k1}|Z]^2)(\text{ Var }(z_{21k1}|Z)+E[z_{21k1}|Z]^2)
$$

$$
\begin{aligned}
&= z_{+1k1} z_{+2k1} \left(z_{+2k1} \frac{\pi_{12k}}{\pi_{+2k}} \left(1 - \frac{\pi_{12k}}{\pi_{+2k}}\right) + \left(z_{+2k1} \frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \right) \\
&\qquad \times \left(z_{+1k1} \frac{\pi_{21k}}{\pi_{+1k}} \left(1 - \frac{\pi_{21k}}{\pi_{+1k}}\right) + \left(z_{+1k1} \frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \right) \\
&= z_{+1k1}^2 z_{+2k1}^2 \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} + z_{+1k1}^2 z_{+2k1}^3 \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \frac{\pi_{21k}}{\pi_{+1k}} \\
&\quad + z_{+1k1}^3 z_{+2k1}^2 \frac{\pi_{12k}}{\pi_{+2k}} \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \frac{\pi_{22k}}{\pi_{+2k}} + z_{+1k1}^3 z_{+2k1}^3 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
&-z_{+1k1} E[(z_{12k1} - E[z_{12k1}|Z])^3|Z](\text{ Var } (z_{21k1}|Z) + E[z_{21k1}|Z]^2) \\
&= -z_{+1k1} \left(z_{+2k1} \left(2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 - 3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 + \frac{\pi_{12k}}{\pi_{+2k}} \right) \right) \\
&\quad \times \left(z_{+1k1} \frac{\pi_{21k}}{\pi_{+1k}} \left(\frac{\pi_{11k}}{\pi_{+1k}}\right) + z_{+1k1}^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \right) \\
&= -2 z_{+1k1}^2 z_{+2k1} \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 \frac{\pi_{21k}}{\pi_{+1k}} + 3 z_{+1k1}^2 z_{+2k1} \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \frac{\pi_{21k}}{\pi_{+1k}} \\
&\quad - z_{+1k1}^2 z_{+2k1} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} - 2 z_{+1k1}^3 z_{+2k1} \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \\
&\quad + 3 z_{+1k1}^3 z_{+2k1} \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 - z_{+1k1}^3 z_{+2k1} \frac{\pi_{21k}}{\pi_{+1k}} \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
&-z_{+1k1}(3E[z_{12k1}|Z] \text{ Var } (z_{12k1}|Z) + E[z_{12k1}|Z]^3)(\text{ Var } (z_{21k1}|Z) + E[z_{21k1}|Z]^2) \\
&= -z_{+1k1} \left(3 z_{+2k1}^2 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^3 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 \right) \\
&\qquad \times \left(z_{+1k1} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{21k}}{\pi_{+1k}} + z_{+1k1}^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \right) \\
&= -3 z_{+1k1}^2 z_{+2k1}^2 \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} - z_{+1k1}^2 z_{+2k1}^3 \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 \frac{\pi_{21k}}{\pi_{+1k}} \\
&\quad - 3 z_{+1k1}^3 z_{+2k1}^2 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \frac{\pi_{22k}}{\pi_{+2k}} - z_{+1k1}^3 z_{+2k1}^3 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
&-z_{+2k1}(\text{ Var } (z_{12k1}|Z) + E[z_{12k1}|Z]^2) E[(z_{21k1} - E[z_{21k1}|Z])^3|Z] \\
&= -z_{+2k1} \left(z_{+2k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^2 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \right) \\
&\qquad \times \left(z_{+1k1} \left(2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 - 3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + \frac{\pi_{21k}}{\pi_{+1k}} \right) \right)
\end{aligned}
$$

$$\begin{aligned}
&= -2z_{+1k1}z_{+2k1}^2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\frac{\pi_{22k}}{\pi_{+2k}} - 2z_{+1k1}z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 \\
&\quad +3z_{+1k1}z_{+2k1}^2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}} + 3z_{+1k1}z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \\
&\quad -z_{+1k1}z_{+2k1}^2\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}} - z_{+1k1}z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}},
\end{aligned}$$

$$\begin{aligned}
&-z_{+2k1}(\text{ Var }(z_{12k1}|Z) + E[z_{12k1}|Z]^2)(3\text{ Var }(z_{21k1}|Z)E[z_{21k1}|Z] + E[z_{21k1}|Z]^3) \\
&= -z_{+2k1}\left(z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\right) \\
&\qquad \times\left(3z_{+1k1}^2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + z_{+1k1}^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\right) \\
&= -3z_{+1k1}^2z_{+2k1}^2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}} - 3z_{+1k1}^2z_{+2k1}^3\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \\
&\quad -z_{+1k1}^3z_{+2k1}^2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\frac{\pi_{22k}}{\pi_{+2k}} - z_{+1k1}^3z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3,
\end{aligned}$$

$$\begin{aligned}
&(1-\theta)E[(z_{12k1} - E[z_{12k1}|Z])^3]E[(z_{21k1} - E[z_{21k1}|Z])^3|Z] \\
&= (1-\theta)z_{+2k1}z_{+1k1}\left(2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 - 3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 + \frac{\pi_{12k}}{\pi_{+2k}}\right) \\
&\qquad \times\left(2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 - 3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + \frac{\pi_{21k}}{\pi_{+1k}}\right) \\
&= (1-\theta)z_{+1k1}z_{+2k1}\left[4\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 - 6\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 + 2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\right] \\
&\quad -(1-\theta)z_{+1k1}z_{+2k1}\left[6\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 - 9\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + 3\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\right] \\
&\quad +(1-\theta)z_{+1k1}z_{+2k1}\left[2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\frac{\pi_{21k}}{\pi_{+1k}} - 3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}} + \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right],
\end{aligned}$$

$$\begin{aligned}
&(1-\theta)E[(z_{12k1} - E[z_{12k1}|Z])^3|Z](3E[z_{21k1}|Z]\text{ Var }(z_{21k1}|Z) + E[z_{21k1}|Z]^3) \\
&= (1-\theta)z_{+2k1}\left(2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3 - 3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 + \frac{\pi_{12k}}{\pi_{+2k}}\right) \\
&\qquad \times\left(3z_{+1k1}^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{11k}}{\pi_{+1k}} + z_{+1k1}^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\right) \\
&= (1-\theta)z_{+1k1}^2z_{+2k1}\left[6\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 - 9\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+3(1-\theta)z_{+1k1}^2 z_{+2k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \\
&+(1-\theta)z_{+1k1}^3 z_{+2k1}\left[2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 - 3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 + \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\right],
\end{aligned}
$$

$$
\begin{aligned}
&(1-\theta)(3E[z_{12k1}|Z]\ \text{Var}\ (z_{12k1}|Z) + E[z_{12k1}|Z]^3)E[(z_{21k1}-E[z_{21k1}|Z])^3|Z] \\
&= (1-\theta)\left(3z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\right) \\
&\qquad \times z_{+1k1}\left(2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 - 3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + \frac{\pi_{21k}}{\pi_{+1k}}\right) \\
&= (1-\theta)z_{+1k1}z_{+2k1}^2\left[6\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\frac{\pi_{22k}}{\pi_{+2k}} - 9\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\right] \\
&\quad +(1-\theta)z_{+1k1}z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}} \\
&\quad +(1-\theta)z_{+1k1}z_{+2k1}^3\left[2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 - 3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\frac{\pi_{21k}}{\pi_{+1k}}\right],
\end{aligned}
$$

$$
\begin{aligned}
&(1-\theta)(3E[z_{12k1}|Z]\ \text{Var}\ (z_{12k1}|Z) + E[z_{12k1}|Z]^3)3E[z_{21k1}|Z]\ \text{Var}\ (z_{21k1}|Z) \\
&= 3(1-\theta)\left(3z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\right) z_{+1k1}^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{11k}}{\pi_{+1k}} \\
&= 9(1-\theta)z_{+1k1}^2 z_{+2k1}^2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}} \\
&\quad +3(1-\theta)z_{+1k1}^2 z_{+2k1}^3\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
&(1-\theta)(3E[z_{12k1}|Z]\ \text{Var}\ (z_{12k1}|Z) + E[z_{12k1}|Z]^3)E[z_{21k1}|Z]^3 \\
&= (1-\theta)\left(3z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\right) z_{+1k1}^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 \\
&= 3(1-\theta)z_{+1k1}^3 z_{+2k1}^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\frac{\pi_{22k}}{\pi_{+2k}} \\
&\quad +(1-\theta)z_{+1k1}^3 z_{+2k1}^3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3.
\end{aligned}
$$

The coefficient of $z_{+1k1}^3 z_{+2k1}^3$ is

$$
\begin{aligned}
&\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 - \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \\
&\qquad - \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3 + (1-\theta)\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \left(1-\frac{\pi_{12k}}{\pi_{+2k}}-\frac{\pi_{21k}}{\pi_{+1k}}+3(1-\theta)\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \left(\left(1-\frac{\pi_{12k}}{\pi_{+2k}}\right)\left(1-\frac{\pi_{21k}}{\pi_{+1k}}\right)-\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \left(\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}-\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= 0.
\end{aligned}
$$

The coefficient of $z_{+1k1}^3 z_{+2k1}^2$ is

$$
\begin{aligned}
&\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}-3\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\\
&\qquad -\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\frac{\pi_{22k}}{\pi_{+2k}}+3(1-\theta)\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^3\frac{\pi_{22k}}{\pi_{+2k}}\\
&= \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\left(1-3\frac{\pi_{12k}}{\pi_{+2k}}-\frac{\pi_{21k}}{\pi_{+1k}}+3(1-\theta)\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\left(\left(1-\frac{\pi_{12k}}{\pi_{+2k}}\right)\left(1-\frac{\pi_{21k}}{\pi_{+1k}}\right)-\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&\qquad -2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}-(1-\theta)\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}-\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}-2\frac{\pi_{12k}}{\pi_{+2k}}+2(1-\theta)\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\left(-2\frac{\pi_{12k}}{\pi_{+2k}}+2\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}-2\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\left(-2\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{11k}}{\pi_{+1k}}-2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\right)\\
&= -2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}+\frac{\pi_{22k}}{\pi_{+2k}}\right)\right)\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}.
\end{aligned}
$$

Finally the coefficient of $z_{+1k1}^2 z_{+2k1}^3$ is

$$
\begin{aligned}
&\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}-\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\frac{\pi_{21k}}{\pi_{+1k}}\\
&\qquad -3\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2+3(1-\theta)\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\\
&= \frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\left(1-\frac{\pi_{12k}}{\pi_{+2k}}-3\frac{\pi_{21k}}{\pi_{+1k}}+3(1-\theta)\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= \frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\left(\left(1-\frac{\pi_{12k}}{\pi_{+2k}}\right)\left(1-\frac{\pi_{21k}}{\pi_{+1k}}\right)-\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&\qquad -2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}-(1-\theta)\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}(1-\frac{\pi_{12k}}{\pi_{+2k}})+\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}+\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\right)\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E[z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}-\theta z_{121k}^3z_{21k1}^3|Z]\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}z_{+1k1}^3z_{+2k1}^2-2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}z_{+1k1}^2z_{+2k1}^3\\
&\quad + \text{higher order terms}\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}z_{+1k1}^2z_{+2k1}^2\left(\frac{\pi_{21k}}{\pi_{+1k}}z_{+1k1}+\frac{\pi_{12k}}{\pi_{+2k}}z_{+2k1}\right)+\text{ higher order terms.}
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
&E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\frac{z_{12k1}^2z_{21k1}^2}{(n_k^c)^2}w_k^2\right]\\
&= E\left[\frac{w_k^3}{(n_k^c)^3}E[z_{11k1}z_{12k1}^2z_{21k1}^2z_{22k1}-\theta z_{121k}^3z_{21k1}^3|Z]\right]\\
&= E\left[\frac{z_{+1k+}^3z_{+2k+}^3}{z_{+1k1}^3z_{+2k1}^3n_k^3}\left(-2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}z_{+1k1}^2z_{+2k1}^2\left(\frac{\pi_{21k}}{\pi_{+1k}}z_{+1k1}+\frac{\pi_{12k}}{\pi_{+2k}}z_{+2k1}\right)\right)\right]\\
&\quad + \text{higher order terms}\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3z_{+2k+}^3}{z_{+1k1}^3z_{+2k1}^3}z_{+1k1}^3z_{+2k1}^2\right]\\
&\quad -2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3z_{+2k+}^3}{z_{+1k1}^3z_{+2k1}^3}z_{+1k1}^2z_{+2k1}^3\right]+\text{higher order terms}\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3z_{+2k+}^3}{z_{+2k1}}\right]\\
&\quad -2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3z_{+2k+}^3}{z_{+1k1}}\right]+\text{higher order terms}\\
&= -2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3z_{+2k+}^3}{z_{+2k1}}\right]\\
&\quad -2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3z_{+2k+}^3}{z_{+1k1}}\right]+\text{higher order terms}
\end{aligned}
$$

and

$$
E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}-\theta z_{12k1}^2z_{21k1}^2|Z]
$$

$$
\begin{aligned}
= \quad & z_{+1k1}^2 z_{+2k1}^2 \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} - z_{+1k1}^2 \left(z_{+2k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^2 \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \right) \frac{\pi_{21k}}{\pi_{+1k}} \\
& - z_{+2k1}^2 \frac{\pi_{12k}}{\pi_{+2k}} \left(z_{+1k1} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{21k}}{\pi_{+1k}} + z_{+1k1}^2 \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 \right) \\
& + (1-\theta) \left(z_{+2k1} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} + z_{+2k1}^2 \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \right) \left(z_{+1k1} \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{21k}}{\pi_{+1k}} + z_{+1k1}^2 \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 \right).
\end{aligned}
$$

The coefficient of $z_{+1k1}^2 z_{+2k1}^2$ is

$$
\frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} - \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \frac{\pi_{21k}}{\pi_{+1k}} - \frac{\pi_{12k}}{\pi_{+2k}} \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 + (1-\theta) \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 \quad = \quad 0.
$$

The coefficient of $z_{+1k1}^2 z_{+2k1}$ is

$$
\begin{aligned}
& -\frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} + (1-\theta) \frac{\pi_{12k}}{\pi_{+2k}} \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 \frac{\pi_{22k}}{\pi_{+2k}} \\
& = \quad \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} \left(-1 + \frac{\pi_{21k}}{\pi_{+1k}} - \theta \frac{\pi_{21k}}{\pi_{+1k}} \right) \\
& = \quad -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} - \theta \frac{\pi_{12k}}{\pi_{+2k}} \left(\frac{\pi_{21k}}{\pi_{+1k}} \right)^2 \frac{\pi_{22k}}{\pi_{+2k}} \\
& = \quad -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} - \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} \\
& = \quad -\frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}} + \frac{\pi_{22k}}{\pi_{+2k}} \right) \\
& = \quad -\frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} \frac{\pi_{11k}}{\pi_{+1k}},
\end{aligned}
$$

and the coefficient of $z_{+1k1} z_{+2k1}^2$ is

$$
\begin{aligned}
& -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} + (1-\theta) \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \frac{\pi_{21k}}{\pi_{+1k}} \\
& = \quad \frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \left(-1 + \frac{\pi_{12k}}{\pi_{+2k}} - \theta \frac{\pi_{12k}}{\pi_{+2k}} \right) \\
& = \quad -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \left(\frac{\pi_{22k}}{\pi_{+2k}} + \theta \frac{\pi_{21k}}{\pi_{+1k}} \right) \\
& = \quad -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} - \theta \frac{\pi_{11k}}{\pi_{+1k}} \left(\frac{\pi_{12k}}{\pi_{+2k}} \right)^2 \frac{\pi_{21k}}{\pi_{+1k}} \\
& = \quad -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{21k}}{\pi_{+1k}} \frac{\pi_{22k}}{\pi_{+2k}} - \left(\frac{\pi_{11k}}{\pi_{+1k}} \right)^2 \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \\
& = \quad -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}} \left(\frac{\pi_{21k}}{\pi_{+1k}} + \frac{\pi_{11k}}{\pi_{+1k}} \right) \\
& = \quad -\frac{\pi_{11k}}{\pi_{+1k}} \frac{\pi_{12k}}{\pi_{+2k}} \frac{\pi_{22k}}{\pi_{+2k}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&-2E\left[\frac{Z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}\right)w_k^2\right]\\
&= \quad 2(n_k-1)\pi_{12k}\pi_{21k}E\left[\left(\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}z_{+1k1}^2 z_{+2k1}+\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}z_{+1k1}z_{+2k1}^2\right)\frac{w_k^2}{(n_k^c)^2}\right]\\
&\qquad + \text{ higher order terms}\\
&= \quad 2(n_k-1)\pi_{12k}\pi_{21k}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}^2 z_{+2k1}^2 n_k^2}z_{+1k1}^2 z_{+2k1}\right]\\
&\qquad + 2(n_k-1)\pi_{12k}\pi_{21k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}^2 z_{+2k1}^2 n_k^2}z_{+1k1}z_{+2k1}^2\right] + \text{ higher order terms}\\
&= \quad 2(n_k-1)\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{+1k}^2\pi_{+2k}}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}^2 z_{+2k1}^2 n_k^2}z_{+1k1}^2 z_{+2k1}\right]\\
&\qquad + 2(n_k-1)\frac{\pi_{11k}\pi_{12k}^2\pi_{21k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}^2 z_{+2k1}^2 n_k^2}z_{+1k1}z_{+2k1}^2\right] + \text{ higher order terms}\\
&= \quad 2(n_k-1)\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{+1k}^2\pi_{+2k}}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}n_k^2}\right]\\
&\qquad + 2(n_k-1)\frac{\pi_{11k}\pi_{12k}^2\pi_{21k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}n_k^2}\right] + \text{ higher order terms.}
\end{aligned}
$$

Therefore the third term of (4.17) is

$$
\begin{aligned}
&E\left[\left(\frac{y_{EST}^{MAR}-\theta x_{MH,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)\left(\frac{x_{EST}^{MAR}-\mu_{x,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)^2\right]\\
&= \quad -\frac{2}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+2k1}}\right]\\
&\qquad -\frac{2}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+1k1}}\right]\\
&\qquad +\frac{2(n_k-1)}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{+1k}^2\pi_{+2k}}\frac{1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}n_k^2}\right]\\
&\qquad +\frac{2(n_k-1)}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k}\pi_{12k}^2\pi_{21k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}^2}\frac{1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}n_k^2}\right] + \text{higher order terms}\\
&= \quad \frac{-2}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{+1k1}^2\pi_{+2k1}}\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+2k1}}\right]\\
&\qquad +\frac{2}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{+1k1}^2\pi_{+2k1}}(n_k-1)\frac{1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}}\right]\\
&\qquad +\frac{-2}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k1}\pi_{12k}^2\pi_{21k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}^2}\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+1k1}}\right]\\
&\qquad +\frac{2}{(\mu_{x,EST}^{MAR})^3}\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{+1k1}\pi_{+2k1}^2}(n_k-1)\frac{1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}}\right]\\
&\qquad + \text{higher order terms}
\end{aligned}
$$

If the leading coefficient of

$$-\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+2k1}}\right]+(n_k-1)\frac{1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}}\right], \tag{4.19}$$

and the leading coefficient of

$$-\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+1k1}}\right]+(n_k-1)\frac{1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}}\right], \tag{4.20}$$

both equal 0, then the third term of (4.17) is

$$E\left[\left(\frac{y_{EST}^{MAR}-\theta x_{MH,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)\left(\frac{x_{EST}^{MAR}-\mu_{x,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)^2\right] \quad = \quad O(1/N^3)O(1/N^3)O(N^4)=O(1/N^2).$$

Conditioning on the total observations n_k, $(z_{+1k+},\ z_{+2k1})$ have a trinomial distribution with parameters n_k, π_{+1k} and $\pi_{+2k1} = \pi_{+2k}(1-p_{2k}^{missing})$, and $n_k - z_{+1k+} - z_{+2k1} = z_{+2k0}$ and $1-\pi_{+1k}-\pi_{+2k1}=\pi_{+2k0}=\pi_{+2k}p_{2k}^{missing}$. Therefore,

$$\begin{aligned}
&E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+2k1}}\right]\\
&= E\left[\frac{z_{+1k+}^3(z_{+2k1}+z_{+2k0})^3}{z_{+2k1}}\right]\\
&= E\left[z_{+1k+}^3\left(z_{+2k1}^2+3z_{+2k1}z_{+2k0}+3z_{+2k0}^2+\frac{z_{+2k0}^3}{z_{+2k1}}\right)\right]\\
&= E[z_{+1k+}^3 z_{+2k1}^2]+3E[z_{+1k+}^3 z_{+2k1}z_{+2k0}]+3E[z_{+1k+}^3 z_{+2k0}^2]\\
&\qquad +E\left[\frac{z_{+1k+}^3 z_{+2k0}^3}{z_{+2k1}}\right].
\end{aligned}$$

Expanding as before

$$\begin{aligned}
&z_{+1k+}^3 z_{+2k1}^2\\
&= z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}(z_{+2k1}-1)+z_{+1k+}^3 z_{+2k1}+3z_{+1k+}^2 z_{+2k1}^2\\
&\qquad -3z_{+1k+}^2 z_{+2k1}-2z_{+1k+}z_{+2k1}^2+2z_{+1k+}z_{+2k1}\\
&= z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}(z_{+2k1}-1)+z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}\\
&\qquad +3z_{+1k+}^2 z_{+2k1}-2z_{+1k+}z_{+2k1}+3z_{+1k+}(z_{+1k+}-1)z_{+2k1}(z_{+2k1}-1)+3z_{+1k+}^2 z_{+2k1}\\
&\qquad +3z_{+1k+}z_{+2k1}^2-3z_{+1k1}z_{+2k1}-3z_{+1k+}^2 z_{+2k1}-2z_{+1k+}z_{+2k1}^2+2z_{+1k+}z_{+2k1}\\
&= z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}(z_{+2k1}-1)+z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}\\
&\qquad +3z_{+1k+}(z_{+1k+}-1)z_{+2k1}(z_{+2k1}-1)+3z_{+1k+}(z_{+1k+}-1)z_{+2k1}+3z_{+1k+}z_{+2k1}
\end{aligned}$$

$$+ z_{+1k+}z_{+2k1}(z_{+2k1}-1) + z_{+1k+}z_{+2k1} - z_{+1k+}z_{+2k1}$$

$$= z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}(z_{+2k1}-1) + z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}$$

$$+ 3z_{+1k+}(z_{+1k+}-1)z_{+2k1}(z_{+2k1}-1) + 3z_{+1k+}(z_{+1k+}-1)z_{+2k1}$$

$$+ z_{+1k+}z_{+2k1}(z_{+2k1}-1) + z_{+1k+}z_{+2k1},$$

$$z^3_{+1k+}z_{+2k1}z_{+2k0}$$

$$= z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}z_{+2k0} + 3z^2_{+1k+}z_{+2k1}z_{+2k0} - 2z_{+1k+}z_{+2k1}z_{+2k0}$$

$$= z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}z_{+2k0} + 3z_{+1k+}(z_{+1k+}-1)z_{+2k1}z_{+2k0}$$

$$+z_{+1k+}z_{+2k1}z_{+2k0},$$

and

$$z^3_{+1k+}z^2_{+2k0}$$

$$= z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k0}(z_{+2k0}-1) + z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k0}$$

$$+ 3z_{+1k+}(z_{+1k+}-1)z_{+2k0}(z_{+2k0}-1) + 3z_{+1k+}(z_{+1k+}-1)z_{+2k0}$$

$$+ z_{+1k+}z_{+2k0}(z_{+2k0}-1) + z_{+1k+}z_{+2k0},$$

Therefore,

$$E\left[\frac{z^3_{+1k+}z^3_{+2k+}}{z_{+2k1}}\right]$$

$$= E[z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}(z_{+2k1}-1)]$$

$$+E[z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}] + 3E[z_{+1k+}(z_{+1k+}-1)z_{+2k1}(z_{+2k1}-1)]$$

$$+3E[z_{+1k+}(z_{+1k+}-1)z_{+2k1}] + E[z_{+1k+}z_{+2k1}(z_{+2k1}-1)] + E[z_{+1k+}z_{+2k1}]$$

$$+3E[z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k1}z_{+2k0}] + 9E[z_{+1k+}(z_{+1k+}-1)z_{+2k1}z_{+2k0}]$$

$$+3E[z_{+1k+}z_{+2k1}z_{+2k0}] + 3E[z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k0}(z_{+2k0}-1)]$$

$$+3E[z_{+1k+}(z_{+1k+}-1)(z_{+1k+}-2)z_{+2k0}]$$

$$+9E[z_{+1k+}(z_{+1k+}-1)z_{+2k0}(z_{+2k0}-1)] + 9E[z_{+1k+}(z_{+1k+}-1)z_{+2k0}]$$

$$+3E[z_{+1k+}z_{+2k0}(z_{+2k0}-1)] + 3E[z_{+1k+}z_{+2k0}]$$

$$
\begin{aligned}
& +E\left[\frac{z^3_{+1k+}z^3_{+2k0}}{z_{+2k1}}\right] \\
= \quad & n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi^3_{+1k}\pi^2_{+2k}(1-p^{missing}_{2k})^2 \\
& +n_k(n_k-1)(n_k-2)(n_k-3)\pi^3_{+1k}\pi_{+2k}(1-p^{missing}_{2k}) \\
& +3n_k(n_k-1)(n_k-2)(n_k-3)\pi^2_{+1k}\pi^2_{+2k}(1-p^{missing}_{2k})^2 \\
& +3n_k(n_k-1)(n_k-2)\pi^2_{+1k}\pi_{+2k}(1-p^{missing}_{2k}) \\
& +n_k(n_k-1)(n_k-2)\pi_{+1k}\pi^2_{+2k}(1-p^{missing}_{2k})^2+nk(n_k-1)\pi_{+1k}\pi_{+2k}(1-p^{missing}_{2k}) \\
& +3n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi^3_{+1k}\pi^2_{+2k}p^{missing}_{2k}(1-p^{missing}_{2k}) \\
& +9n_k(n_k-1)(n_k-2)(n_k-3)\pi^2_{+1k}\pi^2_{+2k}p^{missing}_{2k}(1-p^{missing}_{2k}) \\
& +3n_k(n_k-1)(n_k-2)\pi_{+1k}\pi^2_{+2k}p^{misising}_{2k}(1-p^{missing}_{2k}) \\
& +3n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi^3_{+1k}\pi^2_{+2k}(p^{missing}_{2k})^2 \\
& +3n_k(n_k-1)(n_k-2)(n_k-3)\pi^3_{+1k}\pi_{+2k}p^{missing}_{2k} \\
& +9n_k(n_k-1)(n_k-2)(n_k-3)\pi^2_{+1k}\pi^2_{+2k}(p^{missing}_{2k})^2+9n_k(n_k-1)(n_k-2)\pi^2_{+1k}\pi_{+2k}p^{missing}_{2k} \\
& +3n_k(n_k-1)(n_k-2)\pi_{+1k}\pi^2_{+2k}(p^{missing}_{2k})^2+3n_k(n_k-1)\pi_{+1k}\pi_{+2k}p^{missing}_{2k} \\
& +E\left[\frac{z^3_{+1k+}z^3_{+2k0}}{z_{+2k1}}\right] \\
= \quad & n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi^3_{+1k}\pi^2_{+2k}((1-p^{missing}_{2k})^2+3p^{missing}_{2k}(1-p^{missing}_{2k})) \\
& +3n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi^3_{+1k}\pi^2_{+2k}(p^{missing}_{2k})^2 \\
& +n_k(n_k-1)(n_k-2)(n_k-3)\pi^2_{+1k}\pi^2_{+2k}(3(1-p^{missing}_{2k})^2+9p^{missing}_{2k}(1-p^{missing}_{2k})) \\
& +9n_k(n_k-1)(n_k-2)(n_k-3)\pi^2_{+1k}\pi^2_{+2k}(p^{missing}_{2k})^2 \\
& +n_k(n_k-1)(n_k-2)(n_k-3)\pi^3_{+1k}\pi_{+2k}((1-p^{missing}_{2k})+3p^{missing}_{2k}) \\
& +n_k(n_k-1)(n_k-2)\pi_{+1k}\pi^2_{+2k}((1-p^{missing}_{2k})^2+3p^{misising}_{2k}(1-p^{missing}_{2k})+3(p^{missing}_{2k})^2) \\
& +n_k(n_k-1)(n_k-2)\pi^2_{+1k}\pi_{+2k}(3(1-p^{missing}_{2k})+9p^{missing}_{2k}) \\
& +nk(n_k-1)\pi_{+1k}\pi_{+2k}((1-p^{missing}_{2k})+3p^{missing}_{2k}) \\
& +E\left[\left.\frac{z^3_{+1k+}z^3_{+2k0}}{z_{+2k1}}\right|n_k\right] \\
= \quad & n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi^3_{+1k}\pi^2_{+2k}(1+p^{missing}_{2k}+(p^{missing}_{2k})^2)
\end{aligned}
$$

$$+3n_k(n_k-1)(n_k-2)(n_k-3)\pi_{+1k}^2\pi_{+2k}^2(1+p_{2k}^{missing}+(p_{2k}^{missing})^2)$$

$$+n_k(n_k-1)(n_k-2)(n_k-3)\pi_{+1k}^3\pi_{+2k}(1+2p_{2k}^{missing})$$

$$+n_k(n_k-1)(n_k-2)\pi_{+1k}\pi_{+2k}^2(1+p_{2k}^{missing}+(p_{2k}^{missing})^2)$$

$$+3n_k(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}(1+2p_{2k}^{missing})$$

$$+E\left[\frac{z_{+1k+}^3 z_{+2k0}^3}{z_{+2k1}}\right].$$

Also

$$\begin{aligned}
&E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}}\right] \\
&= E\left[\frac{z_{+1k+}^2(z_{+2k1}+z_{+2k0})^2}{z_{+2k1}}\right] \\
&= E\left[z_{+1k+}^2\left(z_{+2k1}+2z_{+2k0}+\frac{z_{+2k0}^2}{z_{+2k1}}\right)\right] \\
&= E[z_{+1k+}(z_{+1k+}-1)z_{+2k1}|n_k]+E[z_{+1k+}z_{+2k1}] \\
&\quad +2E[z_{+1k+}(z_{+1k+}-1)z_{+2k0}|n_k]+2E[z_{+1k+}z_{+2k0}] \\
&\quad +E\left[\frac{z_{+1k+}^2 z_{+2k0}^2}{z_{+2k1}}\right] \\
&= n_k(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}(1-p_{2k}^{missing})+n_k(n_k-1)\pi_{+1k}\pi_{+2k}(1-p_{2k}^{missing}) \\
&\quad +2n_k(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}p_{2k}^{missing}+2n_k(n_k-1)\pi_{+1k}\pi_{+2k}p_{2k}^{missing} \\
&\quad +E\left[\frac{z_{+1k+}^2 z_{+2k0}^2}{z_{+2k1}}\right] \\
&= n_k(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}(1-p_{2k}^{missing}+2p_{2k}^{missing}) \\
&\quad +n_k(n_k-1)\pi_{+1k}\pi_{+2k}(1-p_{2k}^{missing}+2p_{2k}^{missing})+E\left[\frac{z_{+1k+}^2 z_{+2k0}^2}{z_{+2k1}}\right] \\
&= n_k(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}(1+p_{2k}^{missing})+n_k(n_k-1)\pi_{+1k}\pi_{+2k}(1+p_{2k}^{missing}) \\
&\quad +E\left[\frac{z_{+1k+}^2 z_{+2k0}^2}{z_{+2k1}}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&-\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k+}^3}{z_{+2k1}}\right]+\frac{n_k-1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}}\right] \\
&= -\frac{n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)}{n_k^3\pi_{+1k}\pi_{+2k}}\pi_{+1k}^3\pi_{+2k}^2(1+p_{2k}^{missing}+(p_{2k}^{missing})^2) \\
&\quad -3\frac{n_k(n_k-1)(n_k-2)(n_k-3)}{n_k^3\pi_{+1k}\pi_{+2k}}\pi_{+1k}^2\pi_{+2k}^2(1+p_{2k}^{missing}+(p_{2k}^{missing})^2)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{n_k(n_k-1)(n_k-2)(n_k-3)}{n_k^3\pi_{+1k}\pi_{+2k}}\pi_{+1k}^3\pi_{+2k}(1+2p_{2k}^{missing}) \\
&- \frac{n_k(n_k-1)(n_k-2)}{n_k^3\pi_{+1k}\pi_{+2k}}\pi_{+1k}\pi_{+2k}^2(1+p_{2k}^{missing}+(p_{2k}^{missing})^2) \\
&- 3\frac{n_k(n_k-1)(n_k-2)}{n_k^3\pi_{+1k}\pi_{+2k}}\pi_{+1k}^2\pi_{+2k}(1+2p_{2k}^{missing}) \\
&+ \frac{n_k-1}{n_k^2}n_k(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}(1+p_{2k}^{missing}) \\
&+ \frac{n_k-1}{n_k^2}n_k(n_k-1)\pi_{+1k}\pi_{+2k}(1+p_{2k}^{missing}) \\
&- \frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k+}^3 z_{+2k0}^3}{z_{+2k1}}\right] \\
&+ \frac{n_k-1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k0}^2}{z_{+2k1}}\right].
\end{aligned}
$$

The leading coefficient is

$$
\begin{aligned}
&-\frac{1}{\pi_{+1k}\pi_{+2k}}\pi_{+1k}^3\pi_{+2k}^2(1+p_{2k}^{missing}+(p_{2k}^{missing})^2)+\pi_{+1k}^2\pi_{+2k}(1+p_{2k}^{missing}) \\
&\qquad - \text{leading coefficient of } \left(\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[z_{+1k+}^3\frac{z_{+2k0}^3}{z_{+2k1}}\right]\right) \\
&\qquad + \text{leading coefficient of } \left(\frac{n_k-1}{n_k^2}E\left[z_{+1k+}^2\frac{z_{+2k0}^2}{z_{+2k1}}\right]\right) \\
&= \quad -\pi_{+1k}^2\pi_{+2k}(p_{2k}^{missing})^2 \\
&\qquad - \text{leading coefficient of } \left(\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[z_{+1k+}^3\frac{z_{+2k0}^3}{z_{+2k1}}\right]\right) \\
&\qquad + \text{leading coefficient of } \left(\frac{n_k-1}{n_k^2}E\left[z_{+1k+}^2\frac{z_{+2k0}^2}{z_{+2k1}}\right]\right).
\end{aligned}
$$

Since

$$
E\left[\frac{z_{+1k+}^3 z_{+2k0}^3}{z_{+2k1}+1}\right] = n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\frac{\pi_{+1k}^3\pi_{+2k}^3(p_{2k}^{missing})^3}{\pi_{+2k}(1-p_{2k}^{missing})},
$$

and as we mentioned earlier that we study the cases conditional on $z_{+jk1} > 0$ for both $j = 1, 2$, then $E[z_{+1k+}^3 z_{+2k0}^3/z_{+2k1}]$ is finite. Also

$$
E\left[\frac{z_{+1k+}^3 z_{+2k0}^3}{z_{+2k1}} - \frac{z_{+1k+}^3 z_{+2k0}^3}{z_{+2k1}+1}\right] = E\left[\frac{z_{+1k+}^3 z_{+2k0}^3}{z_{+2k1}(z_{+2k1}+1)}\right] = O(N^4),
$$

but $E[z_{+1k+}^3 z_{+2k0}^3/z_{+2k1}]$ is $O(N^5)$, so the leading coefficient of $E[z_{+1k+}^3 z_{+2k0}^3/z_{+2k1}]$ is the same as the leading coefficient of $E[z_{+1k+}^3 z_{+2k0}^3/(z_{+2k1}+1)]$. The same argument implies that the leading coefficient of

$$
E\left[\frac{z_{+1k+}^2 z_{+2k0}^2}{z_{+2k1}}\right] = n_k^3\frac{\pi_{+1k}^2\pi_{+2k}^2(p_{2k}^{missing})^2}{\pi_{+2k}(1-p_{2k}^{missing})}
$$

So the leading coefficient of (4.19) is

$$\begin{aligned}
& -\pi_{+1k}^2\pi_{+2k}(p_{2k}^{missing})^2 - \frac{1}{\pi_{+1k}\pi_{+2k}}\frac{\pi_{+1k}^3\pi_{+2k}^3(p_{2k}^{missing})^3}{\pi_{+2k}(1-p_{2k}^{missing})} + \frac{\pi_{+1k}^2\pi_{+2k}^2(p_{2k}^{missing})^2}{\pi_{+2k}(1-p_{2k}^{missing})} \\
= \quad & \pi_{+1k}^2\pi_{+2k}(p_{2k}^{missing})^2\left(1 - \frac{p_{2k}^{missing}}{1-p_{2k}^{missing}} + \frac{1}{1-p_{2k}^{missing}}\right) \\
= \quad & \pi_{+1k}^2\pi_{+2k}(p_{2k}^{missing})^2\left(1 - \frac{1-p_{2k}^{missing}}{1-p_{2k}^{missing}}\right) \\
= \quad & 0
\end{aligned}$$

Similarly, the leading coefficient of (4.20) is

$$\begin{aligned}
& -\frac{1}{\pi_{+1k}\pi_{+2k}}\pi_{+1k}^2\pi_{+2k}^3(1+p_{1k}^{missing}+(p_{1k}^{missing})^2) + \pi_{+1k}\pi_{+2k}^2(1+p_{1k}^{missing}) \\
& \qquad - \text{leading coefficient of } \left(\frac{1}{\pi_{+1k}\pi_{+2k}}\frac{1}{n_k^3}E\left[\frac{z_{+1k0}^3}{z_{+1k1}}z_{+2k+}^3\right]\right) \\
& \qquad + \text{leading coefficient of } \left(\frac{n_k-1}{n_k^2}E\left[\frac{z_{+1k0}^2}{z_{+1k1}}z_{+2k+}^2\right]\right) \\
= \quad & -\pi_{+1k}\pi_{+2k}^2(p_{1k}^{missing})^2 - \frac{1}{\pi_{+1k}\pi_{+2k}}\frac{\pi_{+1k}^3(p_{1k}^{missing})^3\pi_{+2k}^3}{\pi_{+1k}(1-p_{1k}^{missing})} + \frac{\pi_{+1k}^2(p_{1k}^{missing})^2\pi_{+2k}^2}{\pi_{+1k}(1-p_{1k}^{missing})} \\
= \quad & \pi_{+1k}\pi_{+2k}^2(p_{1k}^{missing})^2\left(1 - \frac{p_{1k}^{missing}}{1-p_{1k}^{missing}} + \frac{1}{1-p_{1k}^{missing}}\right) \\
= \quad & 0.
\end{aligned}$$

So, the expected value of the third term of (4.17) is $O(1/N^2)$ for fixed K and θ and $O(1/K^2)$ for fixed n_k and θ with bounded cell probabilities. So the leading term of the bias is $O(1/N)$ for fixed K and θ and $O(1/K)$ for fixed n_k and θ with bounded cell probabilities. When the number of strata is fixed but the sizes of tables increase or when the sizes of tables are fixed but the number of strata increases, the bias of the common odds ratio estimator from the closed form data will tend to zero.

Now we investigate the variance. We have

$$\begin{aligned}
\text{Var }(\hat{\theta}_{MH,EST}^{MAR} - \theta) \quad & \doteq \quad \text{Var }\left(\sum_{k=1}^{K}\left(\frac{[(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})/n_k^c]w_k}{\mu_{x,EST}^{MAR}}\right)\right) \\
& = \quad \frac{1}{(\mu_{x,EST}^{MAR})^2}\sum_{k=1}^{K}\text{ Var }\left(\frac{(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})w_k}{n_k^c}\right) \\
& = \quad \frac{1}{(\mu_{x,EST}^{MAR})^2}\sum_{k=1}^{K}\text{ Var }\left(\frac{w_k}{n_k^c}E[z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|Z]\right)
\end{aligned}$$

$$
\begin{aligned}
& + \frac{1}{(\mu^{MAR}_{x,EST})^2} \sum_{k=1}^{K} E\left[\frac{w_k^2}{(n_k^c)^2} \text{ Var } (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|Z)\right] \\
= \quad & \frac{1}{(\mu^{MAR}_{x,EST})^2} \sum_{k=1}^{K} E\left[\frac{w_k^2}{(n_k^c)^2} \text{ Var } (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|Z)\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
& \text{Var } (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|Z) \\
= \quad & E[(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})^2|Z] - 0 \\
= \quad & E[z_{11k1}^2 z_{22k1}^2 - 2\theta z_{11k1}z_{12k1}z_{21k1}z_{22k1} + \theta^2 z_{12k1}^2 z_{21k1}^2|Z] \\
= \quad & \left(z_{+1k1}^2 \left(\frac{\pi_{11k}}{\pi_{+1k}}\right)^2 + z_{+1k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\left(z_{+2k1}^2 \left(\frac{\pi_{22k}}{\pi_{+2k}}\right)^2 + z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\right) \\
& -2\theta E[z_{+1k1}z_{21k1} - z_{21k1}^2|Z]E[z_{+2k1}z_{12k1} - z_{12k1}^2|Z] \\
& +\theta^2 \left(z_{+2k1}^2 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 + z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\right)\left(z_{+1k1}^2 \left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 + z_{+1k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\right) \\
= \quad & z_{+1k1}^2 z_{+2k1}^2 \left(\left(\frac{\pi_{11k}}{\pi_{+1k}}\right)^2\left(\frac{\pi_{22k}}{\pi_{+2k}}\right)^2 + \theta^2\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\right) \\
& +z_{+1k1}^2 z_{+2k1}\left(\left(\frac{\pi_{11k}}{\pi_{+1k}}\right)^2\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}} + \theta^2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\right) \\
& +z_{+1k1} z_{+2k1}^2\left(\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}\right)^2 + \theta^2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\right) \\
& +z_{+1k1}z_{+2k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}(1+\theta^2) \\
& +2\theta z_{+1k1}^2 z_{+2k1}^2\left(-\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}} + \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}} + \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 - \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\right) \\
& +2\theta z_{+1k1}^2 z_{+2k1}\left(\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}} - \frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}}\right) \\
& +2\theta z_{+1k1} z_{+2k1}^2\left(\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}} - \frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\right) \\
& -2\theta z_{+1k1}z_{+2k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}} \\
= \quad & z_{+1k1}^2 z_{+2k1}^2\left(2\left(\frac{\pi_{11k}}{\pi_{+1k}}\right)^2\left(\frac{\pi_{22k}}{\pi_{+2k}}\right)^2 - 2\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(1 - \frac{\pi_{12k}}{\pi_{+2k}} - \frac{\pi_{21k}}{\pi_{+1k}} + \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\right)\right) \\
& +z_{+1k1}^2 z_{+2k1}\left(\left(\frac{\pi_{11k}}{\pi_{+1k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}} + \theta^2\frac{\pi_{12k}}{\pi_{+2k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{22k}}{\pi_{+2k}} + 2\theta\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\frac{\pi_{11k}}{\pi_{+1k}}\right) \\
& +z_{+1k1} z_{+2k1}^2\left(\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}\right)^2 + \theta^2\frac{\pi_{11k}}{\pi_{+1k}}\left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{21k}}{\pi_{+1k}} + 2\theta\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}\right) \\
& +z_{+1k1}z_{+2k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}(1+\theta^2-2\theta)
\end{aligned}
$$

$$
\begin{aligned}
&= z_{+1k1}^2 z_{+2k1}^2(0) + z_{+1k1}^2 z_{+2k1}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2 \\
&\quad + z_{+1k1} z_{+2k1}^2\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 + z_{+1k1}z_{+2k1}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}(1-\theta)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
&E\left[\left(\frac{w_k}{n_k^c}\right)^2 \text{ Var }(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}|Z)\right] \\
&= E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 \text{ Var }(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}|Z)\right] \\
&= E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}n_k^2}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\right] \\
&\quad + E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}n_k^2}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\right] \\
&\quad + E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}z_{+2k1}n_k^2}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}(1-\theta)^2\right].
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}n_k^2}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\right] \\
&= \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{1}{n_k^2}E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+2k1}}\right] \\
&= \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{1}{n_k^2}E\left[z_{+1k+}^2 z_{+2k1}+2z_{+1k+}^2 z_{+2k0}+\frac{z_{+1k+}^2 z_{+2k0}^2}{z_{+2k1}}\right] \\
&> \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\left(\frac{(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}^2}{n_k\pi_{+2k}(1-p_{2k}^{missing})}+\frac{(n_k-1)\pi_{+1k}^2 p_{2k}^{missing}}{n_k(1-p_{2k}^{missing})}\right) \\
&\quad + \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\left(\frac{(n_k-1)\pi_{+1k}\pi_{+2k}^2}{n_k\pi_{+2k}(1-p_{2k}^{missing})}+\frac{\pi_{+1k}p_{2k}^{missing}}{n_k(1-p_{2k}^{missing})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}n_k^2}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\right] \\
&= \frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{1}{n_k^2}E\left[z_{+1k1}z_{+2k+}^2+2z_{+1k0}z_{+2k+}^2+\frac{z_{+1k0}^2 z_{+2k+}^2}{z_{+1k1}}\right] \\
&> \frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{(n_k-1)(n_k-2)\pi_{+1k}^2\pi_{+2k}^2}{n_k\pi_{+1k}(1-p_{1k}^{missing})}+\frac{(n_k-1)\pi_{+1k}^2\pi_{+2k}}{n_k\pi_{+1k}(1-p_{1k}^{missing})}\right) \\
&\quad + \frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{(n_k-1)\pi_{+2k}^2 p_{1k}^{missing}}{n_k(1-p_{1k}^{missing})}+\frac{\pi_{+2k}p_{1k}^{missing}}{n_k(1-p_{1k}^{missing})}\right),
\end{aligned}
$$

and

$$
E\left[\frac{z_{+1k+}^2 z_{+2k+}^2}{z_{+1k1}z_{+2k1}n_k^2}\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\frac{\pi_{22k}}{\pi_{+2k}}(1-\theta)^2\right]
$$

$$
\begin{aligned}
> \quad & \frac{\pi_{11k}^2\pi_{22k}^2(1-\theta)^2}{n_k\theta\pi_{+1k}^3\pi_{+2k}^3(1-p_{1k}^{missing})(1-p_{2k}^{missing})}((n_k-1)\pi_{+1k}^2\pi_{+2k}^2) \\
& +\frac{\pi_{11k}^2\pi_{22k}^2(1-\theta)^2}{n_k\theta\pi_{+1k}^3\pi_{+2k}^3(1-p_{1k}^{missing})(1-p_{2k}^{missing})}\pi_{+1k}\pi_{+2k}(\pi_{+2k}p_{1k}^{missing}\pi_{+1k}p_{2k}^{missing}) \\
& +\frac{\pi_{11k}^2\pi_{22k}^2(1-\theta)^2}{n_k\theta\pi_{+1k}^3\pi_{+2k}^3(1-p_{1k}^{missing})(1-p_{2k}^{missing})}\frac{\pi_{+1k}\pi_{+2k}p_{1k}^{missing}p_{2k}^{missing}}{n_k}.
\end{aligned}
$$

The coefficient of $(n_k-1)(n_k-2)/n_k$ is greater than

$$
\begin{aligned}
& \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{22k}}{\pi_{+2k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\frac{\pi_{+1k}^2\pi_{+2k}^2}{\pi_{+2k}(1-p_{2k}^{missing})} \\
& \quad +\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\frac{\pi_{+1k}^2\pi_{+2k}^2}{\pi_{+1k}(1-p_{1k}^{missing})} \\
> \quad & \frac{\pi_{+1k}^2\pi_{+2k}^2}{\pi_{+1k}^2\pi_{+2k}^3}\left(\pi_{12k}\pi_{22k}\left(\pi_{11k}+\frac{\pi_{11k}\pi_{22k}}{\pi_{12k}}\right)^2+\pi_{11k}\pi_{21k}\left(\pi_{22k}+\frac{\pi_{11k}\pi_{22k}}{\pi_{12k}}\right)^2\right) \\
= \quad & \frac{\pi_{+1k}^2\pi_{+2k}^2}{\pi_{+1k}^2\pi_{+2k}^3}\left(\pi_{11k}^2\pi_{12k}\pi_{22k}\left(\frac{\pi_{+2k}}{\pi_{12k}}\right)^2+\pi_{11k}\pi_{21k}\pi_{22k}^2\left(\frac{\pi_{+1k}}{\pi_{21k}}\right)^2\right) \\
= \quad & \pi_{11k}\pi_{22k}\left(\frac{\pi_{11k}\pi_{12k}\pi_{22k}+\pi_{12k}\pi_{21k}\pi_{22k}+\pi_{11k}\pi_{12k}\pi_{21k}+\pi_{11k}\pi_{21k}\pi_{22k}}{\pi_{12k}\pi_{21k}}\right) \\
= \quad & \pi_{11k}\pi_{22k}\left(\frac{\theta\pi_{12k}^2\pi_{21k}+\pi_{12k}\pi_{21k}\pi_{22k}+\pi_{11k}\pi_{12k}\pi_{21k}+\theta\pi_{12k}\pi_{21k}^2}{\pi_{12k}\pi_{21k}}\right) \\
= \quad & \pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}+\theta(\pi_{12k}+\pi_{21k})).
\end{aligned}
$$

and the coefficient of $(n_k-1)/n_k$ is greater than

$$
\begin{aligned}
& \frac{\pi_{12k}}{\pi_{+2k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{11k}}{\pi_{+1k}}+\theta\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\left(\frac{\pi_{+1k}^2p_{2k}^{missing}}{1-p_{2k}^{missing}}+\frac{\pi_{+1k}\pi_{+2k}}{1-p_{2k}^{missing}}\right) \\
& \quad +\frac{\pi_{11k}}{\pi_{+1k}}\frac{\pi_{21k}}{\pi_{+1k}}\left(\frac{\pi_{22k}}{\pi_{+2k}}+\theta\frac{\pi_{12k}}{\pi_{+2k}}\right)^2\left(\frac{\pi_{+1k}\pi_{+2k}}{1-p_{1k}^{missing}}+\frac{\pi_{+2k}^2p_{1k}^{missing}}{1-p_{1k}^{missing}}\right) \\
& \quad +\frac{\pi_{11k}^2\pi_{22k}^2(1-\theta)^2/\theta}{\pi_{+1k}\pi_{+2k}(1-p_{1k}^{missing})(1-p_{2k}^{missing})} \\
= \quad & \frac{\pi_{11k}^2\pi_{22k}}{\pi_{12k}\pi_{+1k}^2}\left(\frac{\pi_{+1k}^2p_{2k}^{missing}+\pi_{+1k}\pi_{+2k}}{1-p_{2k}^{missing}}\right)+\frac{\pi_{11k}\pi_{22k}^2}{\pi_{21k}\pi_{+2k}^2}\left(\frac{\pi_{+1k}^2+\pi_{+1k}\pi_{+2k}p_{1k}^{missing}}{1-p_{1k}^{missing}}\right) \\
& \quad +\frac{\pi_{11k}^2\pi_{22k}^2(1-\theta)^2/\theta}{\pi_{+1k}\pi_{+2k}(1-p_{1k}^{missing})(1-p_{2k}^{missing})} \\
= \quad & \frac{\pi_{11k}^2\pi_{22k}}{\pi_{12k}\pi_{+1k}(1-p_{2k}^{missing})}(\pi_{+1k}p_{2k}^{missing} \\
& \quad +\pi_{+2k})+\frac{\pi_{11k}\pi_{22k}^2}{\pi_{21k}\pi_{+2k}(1-p_{1k}^{missing})}(\pi_{+1k}+\pi_{+2k}p_{1k}^{missing}) \\
& \quad +\frac{\pi_{11k}^2\pi_{22k}^2(1-\theta)^2/\theta}{\pi_{+1k}\pi_{+2k}(1-p_{1k}^{missing})(1-p_{2k}^{missing})} \\
> \quad & \frac{\pi_{11k}^2\pi_{21k}\pi_{22k}(\pi_{+2k}^2+\pi_{+1k}\pi_{+2k}p_{2k}^{missing})}{\pi_{12k}\pi_{21k}\pi_{+1k}\pi_{+2k}}+\frac{\pi_{11k}\pi_{12k}\pi_{22k}^2(\pi_{+1k}^2+\pi_{+1k}\pi_{+2k}p_{1k}^{missing})}{\pi_{12k}\pi_{21k}\pi_{+1k}\pi_{+2k}}
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{\pi_{11k}^2\pi_{22k}^2\pi_{12k}\pi_{21k}(1-\theta)^2/\theta}{\pi_{12k}\pi_{21k}\pi_{+1k}\pi_{+2k}} \\
= & \frac{\pi_{11k}^2\pi_{21k}\pi_{22k}\pi_{+2k}^2 + \pi_{11k}\pi_{12k}\pi_{22k}^2\pi_{+1k}^2 + \pi_{11k}^2\pi_{12k}\pi_{21k}\pi_{22k}^2(1-\theta)/\theta}{\pi_{12k}\pi_{21k}\pi_{+1k}\pi_{+2k}} \\
& + \frac{\pi_{11k}^2\pi_{21k}\pi_{22k}\pi_{+1k}\pi_{+2k}p_{2k}^{missing} + \pi_{11k}\pi_{12k}\pi_{22k}^2\pi_{+1k}\pi_{+2k}p_{1k}^{missing}}{\pi_{12k}\pi_{21k}\pi_{+1k}\pi_{+2k}}
\end{aligned}
$$

Moreover

$$
\begin{aligned}
& \pi_{11k}^2\pi_{21k}\pi_{22k}\pi_{+2k}^2 + \pi_{11k}\pi_{12k}\pi_{22k}^2\pi_{+1k}^2 + \pi_{11k}^2\pi_{12k}\pi_{21k}\pi_{22k}^2(1-\theta)/\theta \\
= & \pi_{11k}^2\pi_{21k}\pi_{22k}(\pi_{12k}^2 + 2\pi_{12k}\pi_{22k} + \pi_{22k}^2) + \pi_{11k}\pi_{12k}\pi_{22k}^2(\pi_{11k}^2 + 2\pi_{11k}\pi_{21k} + \pi_{21k}^2) \\
& + \pi_{11k}^3\pi_{22k}^3\left(\frac{1-\theta}{\theta}\right)^2 \\
= & \pi_{11k}^2\pi_{12k}^2\pi_{21k}\pi_{22k} + 2\pi_{11k}^2\pi_{12k}\pi_{21k}\pi_{22k}^2 + \pi_{11k}^2\pi_{21k}\pi_{22k}^3 + \pi_{11k}^3\pi_{12k}\pi_{22k}^2 + 2\pi_{11k}^2\pi_{12k}\pi_{21k}\pi_{22k}^2 \\
& + \pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}^2 + \pi_{11k}^3\pi_{22k}^3\left(\frac{1-\theta}{\theta}\right)^2 \\
= & \frac{\pi_{11k}^3\pi_{12k}\pi_{22k}^2}{\theta} + 4\frac{\pi_{11k}^3\pi_{22k}^3}{\theta} + \pi_{11k}^2\pi_{21k}\pi_{22k}^3 + \pi_{11k}^3\pi_{12k}\pi_{22k}^2 + \frac{\pi_{11k}^2\pi_{21k}\pi_{22k}^3}{\theta} \\
& + \pi_{11k}^3\pi_{22k}^3\left(\frac{1-\theta}{\theta}\right)^2 \\
= & \pi_{11k}^2\pi_{21k}\pi_{22k}^3\left(1+\frac{1}{\theta}\right) + \pi_{11k}^2\pi_{12k}\pi_{22k}^2\left(1+\frac{1}{\theta}\right) + \pi_{11k}^3\pi_{22k}^3\left(1+\frac{1}{\theta}\right)^2 \\
= & \pi_{11k}^2\pi_{22k}^2\left(1+\frac{1}{\theta}\right)\left(\pi_{21k}\pi_{22k} + \pi_{11k}\pi_{12k} + \pi_{11k}\pi_{22k}\left(1+\frac{1}{\theta}\right)\right) \\
= & \pi_{11k}^2\pi_{22k}^2\left(1+\frac{1}{\theta}\right)(\pi_{21k}\pi_{22k} + \pi 11k\pi_{12k} + \pi_{11k}\pi_{22k} + \pi_{12k}\pi_{21k}) \\
= & \pi_{11k}^2\pi_{22k}^2\left(1+\frac{1}{\theta}\right)\pi_{+1k}\pi_{+2k}.
\end{aligned}
$$

The coefficient of $\frac{n_k - 1}{n_k}$ is greater than

$$
\begin{aligned}
& \frac{(1/\theta+1)\pi_{11k}^2\pi_{22k}^2\pi_{+1k}\pi_{+2k}}{\pi_{11k}\pi_{22k}\pi_{+1k}\pi_{+2k}/\theta} + \frac{\pi_{11k}^2\pi_{21k}\pi_{22k}\pi_{+1k}\pi_{+2k}p_{2k}^{missing}}{\pi_{12k}\pi_{21k}\pi_{+1k}\pi_{+2k}} \\
& + \frac{\pi_{11k}\pi_{12k}\pi_{22k}^2\pi_{+1k}\pi_{+2k}p_{1k}^{missing}}{\pi_{12k}\pi_{21k}\pi_{+1k}\pi_{+2k}} \\
> & (1+\theta)\pi_{11k}\pi_{22k}
\end{aligned}
$$

We know that $\mu_{x,EST}^{MAR} = \mu_x$. Therefore

$$
\text{Var}\ (\hat{\theta}_{MH,EST}^{MAR} - \theta) \doteq \frac{1}{(\mu_x)^2}\sum_{k=1}^{K} \text{Var}\ \left(\frac{(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})w_k}{n_k^c}\right)
$$

Table 4.1: closed form estimated cell means for Informative(A, C)

	General Closed-form	Boundary estimates	
		$\hat{a_{1.k}} = 0$	$\hat{a_{2.k}} = 0$
$\hat{m_{1jk}}$	z_{ijk1}	z_{1jk1}	$\frac{z_{1+k1}(z_{1jk1}+z_{+jk0})}{z_{1+k1}+z_{++k0}}$
$\hat{m_{2jk}}$	z_{2jk1}	$\frac{z_{2+k1}(z_{2jk1}+z_{+jk0})}{z_{2+k1}+z_{++k0}}$	z_{2jk1}
$\hat{a}_{1.k}$	$\frac{z_{22k1}z_{+1k0}-z_{12k1}z_{+2k0}}{z_{11k1}z_{22k1}-z_{12k1}z_{21k1}}$	0	$\frac{z_{++k0}}{z_{1+k1}}$
$\hat{a}_{2.k}$	$\frac{z_{11k1}z_{+2k0}-z_{21k1}z_{+1k0}}{z_{11k1}z_{22k1}-z_{12k1}z_{21k1}}$	$\frac{z_{++k0}}{z_{2+k1}}$	0

$$
\begin{aligned}
&> \frac{1}{\mu_x^2}\sum_1^K \Bigg(\frac{(n_k-1)(n_k-2)}{n_k}\pi_{11k}\pi_{22k}(\pi_{11k}+\pi_{22k}+\theta(\pi_{12k}+\pi_{21k})) \\
&\qquad \times \frac{n_k-1}{n_k}(1+\theta)\pi_{11k}\pi_{22k}\Bigg) \\
&= \text{Var } (\hat{\theta}_{MH}-\theta), \text{ the variance from the full data.}
\end{aligned}
$$

But Var $(\hat{\theta}_{MH,EST}^{MAR}-\theta)$ is still $O(1/N^2)O(N) = O(1/N)$ for fixed K and θ and $O(1/K^2)O(K) = O(1/K)$ for fixed n_k and θ and bounded away from 0 cell probabilities. That means the variances for the common odds ratio estimate using closed form estimators have larger variance than the full data, but when the table sizes become large or the number of tables increases, the variance will tend to zero. As mentioned before, we only study the cases where $z_{+1k1} > 0$ and $z_{+2k1} > 0$. That is, the number of fully observed data for $B = 1$ and 2 respectively in table k are positive. When N and K are fixed, the variance for the closed form is $O(\theta^3)$ as the full data or the only complete subtables estimates variances.

4.2.4 Closed Form Estimated Data for Informative Missingness Model

In the case that variable A is informatively missing depending on both A and C, both variable B and C are always observed as in (4.7). The correct closed forms for estimated cell mean are listed in Table 4.1. The odds ratio of the general closed form estimates in stratum k is

$$
\frac{z_{11k1}(1+\hat{a}_{1.k})z_{22k1}(1+\hat{a}_{2.k})}{z_{12k1}(1+\hat{a}_{1.k})z_{22k1}(1+\hat{a}_{2.k})} = \frac{z_{11k1}z_{22k1}}{z_{12k1}z_{21k1}}
$$

and the odds ratio of the strata with boundary condition of $\hat{a}_{1.k} = 0$ is

$$\frac{z_{11k1}z_{2+k1}(z_{22k1}+z_{+2k0})/(z_{2+k1}+z_{++k0})(1+z_{++k0}/z_{2+k1})}{z_{12k1}z_{2+k1}(z_{21k1}+z_{+1k0})/(z_{2+k1}+z_{++k0})(1+z_{++k0}/z_{2+k1})} = \frac{z_{11k1}(z_{22k1}+z_{+2k0})}{z_{12k1}(z_{21k1}+z_{+1k0})}$$

and the odds ratio of the strata with boundary condition of $\hat{a}_{2.k} = 0$ is

$$\frac{z_{1+k1}(z_{11k1}+z_{+1k0})/(z_{1+k1}+z_{++k0})(1+z_{++k0}/z_{1+k1})z_{22k1}}{z_{21k1}z_{1+k1}(z_{12k1}+z_{+2k0})/(z_{1+k1}+z_{++k0})(1+z_{++k0}/z_{1+k1})} = \frac{(z_{11k1}+z_{+1k0})z_{22k1}}{(z_{12k1}+z_{+2k0})z_{21k1}}.$$

When the variable A is informatively missing depending on A and the stratum variable C, the closed form estimator for the missing parameter $a_{i.k}$ might be out of the parameter space, and the boundary estimate must need to be used for the closed form. When the missing parameter $a_{i.k}$ is inside the parameter space, the general closed form estimates the odds ratio uses only fully observed subtable and still obeys the common odds ratio assumption. However, when the boundary condition is needed, there are extra terms and the common odds ratio assumption does not hold.

From the simulation experience, in about 75% of the strata one must use the boundary estimates. Moreover, the simulation results show the estimated common odds ratio from informative missingness model closed form either converges to an incorrect ratio or converge very slowly.

We have proved using complete only subdata estimates the common odds ratio consistently with expected leading bias term of order of $O(1/N)$ or $O(1/K)$. The simulations also show the MAR model closed form also estimates the common odds ratio with performance similar to the behavior of the complete only data when the data is missing informatively.

Informative Model Analyzed as MAR Model

If we analyze informatively missing data as an MAR Model, the closed forms are $\hat{m}_{ijk} = z_{ijk1}$ and $\hat{a}_{.jk} = z_{+jk0}/z_{+jk1}$ as in section 4.2.1, there is no boundary condition to be concerned with, and the simulation results show the estimators do converge to the correct common odds ratio. The estimated full counts table is the same as in Section 4.2.1. Conditioning on z_{i+k1}, z_{ijk1} is binomial(z_{i+k1}, π_{ijk}/π_{i+k}). Conditioning on the number of total observations n_k, (z_{1+k1}, z_{2+k1},

z_{+1k0}) have a multinomial distribution with parameters n_k, π_{1+k1}, π_{2+k1}, π_{+1k0} where $\pi_{1+k1} = \pi_{1+k}(1 - p_{1k}^{missing})$, $\pi_{2+k1} = \pi_{2+k}(1 - p_{2k}^{missing})$, and $\pi_{+1k0} = \pi_{11k}p_{1k}^{missing} + \pi_{21k}p_{2k}^{missing}$. The data are displayed below.

			Total
A is observed	z_{1+k1}	z_{2+k1}	z_{++k1}
A is unobserved	z_{+1k1}	z_{+2k1}	z_{++k0}
			n_k

The expected cell count is

$$\begin{aligned} E[\hat{\mu}_{ijk}] &= E\left[z_{ijk1}\frac{z_{+jk0}}{z_{+jk1}}\right] \\ &= E\left[\frac{z_{+jk0}}{z_{+jk1}}E[z_{ijk1}|z_{+jk1}, z_{+jk0}, z_{i+k1}]\right] \\ &= E\left[\frac{z_{+jk0}}{z_{+jk1}}z_{i+k1}\frac{\pi_{ijk}}{\pi_{i+k}}\right] \\ &= \frac{\pi_{ijk}}{\pi_{i+k}}E\left[\frac{z_{+jk0}z_{i+k1}}{z_{+jk1}}\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{\theta}_{MH,EST}^{INFM} &= \frac{\sum_1^K \hat{\mu}_{11k}\hat{\mu}_{22k}/n_k}{\sum_1^K \hat{\mu}_{12k}\hat{\mu}_{21k}/n_k} \\ &= \frac{\sum_1^K [(z_{11k1}z_{+1k+}/z_{+1k1})(z_{22k1}z_{+2k+}/z_{+2k1})]/n_k}{\sum_1^K [(z_{12k1}z_{+2k+}/z_{+2k1})(z_{21k1}z_{+1k+}/z_{+1k1})]/n_k} \\ &= \frac{\sum_1^K (z_{11k1}z_{22k1}/n_k^c)w_k}{\sum_1^K (z_{12k1}z_{21k1}/n_k^c)w_k}, \end{aligned}$$

as before,

$$w_k = \left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)\left(\frac{n_k^c}{n_k}\right).$$

Then

$$\hat{\theta}_{MH,EST}^{INFM} - \theta = \frac{\sum_1^K [(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})/n_k^c]w_k}{\sum_{1k}^K (z_{12k1}z_{21k1}/n_k^c)w_k},$$

and

$$E\left[\frac{z_{11k1}z_{22k1}}{n_k^c}w_k\right]$$

$$
\begin{aligned}
&= E\left[E\left[\frac{z_{11k1}z_{22k1}}{n_k^c}\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\frac{n_k^c}{n_k}\middle| z_{+1k1}, z_{+2k1}, z_{+1k0}, z_{+2k0}, z_{1+k1}, z_{2+k1}\right]\right] \\
&= E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}E[z_{11k1}z_{22k1}|z_{+1k1}, z_{+2k1}, z_{+1k0}, z_{+2k0}, z_{1+k1}, z_{2+k1}]\right] \\
&= E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}z_{1+k1}z_{2+k1}\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right] \\
&= \frac{\pi_{11k}\pi_{22k}}{\pi_{+1k}\pi_{+2k}}E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}z_{1+k1}z_{2+k1}\right] \\
&= O(n_k).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] &= E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}z_{1+k1}z_{2+k1}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right] \\
&= O(n_k)
\end{aligned}
$$

so that

$$
\begin{aligned}
E\left[\sum\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] &= \sum E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] \\
&= \mu_{x,EST}^{INFM}.
\end{aligned}
$$

Let $N = \sum_{k=1}^{K} n_k$, then when K and θ are fixed, $\mu_{x,EST}^{INFM,MAR}$ is $O(N)$ and when n_k and θ are fixed and cell probabilities are bounded away from 0, $\mu_{x,EST}^{INFM}$ is $O(K)$. When K and n_k are fixed, $\mu_{x,EST}^{INFM}$ is $O(1/\theta)$.

As in previous sections, writing $\hat{\theta}_{MH,EST}^{INFM} = y_{EST}^{INFM}/x_{EST}^{INFM}$ and using Taylor expansion,

$$
\begin{aligned}
\frac{y_{EST}^{INFM} - \theta x_{EST}^{INFM}}{x_{EST}^{INFM}} &= \frac{y_{EST}^{INFM} - \theta x_{EST}^{INFM}}{\mu_{x,EST}^{INFM}}\left[1 - \frac{x_{EST}^{INFM} - \mu_{x,EST}^{INFM}}{\mu_{x,EST}^{INFM}} + \cdots\right] \\
&= \frac{y_{EST}^{INFM} - \theta x_{EST}^{INFM}}{\mu_{x,EST}^{INFM}} \\
&\quad - \left(\frac{y_{EST}^{INFM} - \theta x_{EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right)\left(\frac{x_{EST}^{INFM} - \mu_{x,EST}^{Inf,MAR}}{\mu_{x,EST}^{INFM}}\right) \\
&\quad + \left(\frac{y_{EST}^{INFM} - \theta x_{MH,EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right)\left(\frac{x_{EST}^{INFM} - \mu_{x,EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right)^2 + \cdots. \quad (4.21)
\end{aligned}
$$

Let $\mathbf{Z_{inf}} = (z_{1+k1}, z_{2+k1}, z_{+1k1}, z_{+2k1}, z_{+1k0}, z_{+2k0})$. Then the expected value of the first term of (4.21) is

$$
E\left[\frac{y_{EST}^{INFM} - \theta x_{EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right]
$$

$$
\begin{aligned}
&= E\left[\frac{1}{\mu_{x,EST}^{INFM}}\sum_{k=1}^{K}\left(\frac{z_{11k1}z_{22k1}}{n_k^c}w_k - \theta\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right)\right] \\
&= \frac{1}{\mu_{x,EST}^{INFM}}\sum_{k=1}^{K}\left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - \theta\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right) \\
&= \frac{1}{\mu_{x,EST}^{INFM}}\sum_{k=1}^{K}E\left[\frac{w_k}{n_k^c}E[z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|\mathbf{Z_{inf}}]\right] \\
&= \frac{1}{\mu_{x,EST}^{INFM}}\sum_{k=1}^{K}E\left[\frac{w_k}{n_k^c}\left(z_{1+k1}\frac{\pi_{11k}}{\pi_{1+k}}z_{2+k1}\frac{\pi_{22k}}{\pi_{2+k}} - \theta z_{1+k1}\frac{\pi_{12k}}{\pi_{1+k}}z_{2+k1}\frac{\pi_{21k}}{\pi_{2+k}}\right)\right] \\
&= \frac{1}{\mu_{x,EST}^{INFM}}\sum_{k=1}^{K}E\left[\frac{w_k}{n_k^c}z_{1+k1}z_{2+k1}\left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}} - \theta\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right)\right] \\
&= 0.
\end{aligned}
$$

The expected value of the second term of (4.21) is

$$
\begin{aligned}
&E\left[\left(\frac{y_{EST}^{INFM} - \theta x_{EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right)\left(\frac{x_{EST}^{INFM} - \mu_{x,EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right)\right] \\
&= \frac{1}{(\mu_{x,EST}^{INFM})^2}E\left[\left(\sum\frac{z_{11k1}z_{22k1}}{n_k^c}w_k - \theta\left(\sum\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right)\right. \\
&\qquad \left.\times\left(\sum\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - \mu_{x,EST}^{INFM}\right)\right] \\
&= \frac{1}{(\mu_{x,EST}^{INFM})^2}E\left[\left(\sum\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right)\right. \\
&\qquad \left.\times\left(\sum\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)\right)\right] \\
&= \frac{1}{(\mu_{x,EST}^{INFM})^2}E\left[\sum\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)\right]\right] \\
&= \frac{1}{(\mu_{x,EST}^{INFM})^2}\sum E\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right] \\
&\quad + \frac{1}{(\mu_{x,EST}^{INFM})^2}\sum E\left[\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right] \\
&= \frac{1}{(\mu_{x,EST}^{INFM})^2}\sum E\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right] \\
&= \frac{1}{(\mu_{x,EST}^{INFM})^2}\left(\sum_k A_{k,EST}^{INFM}\right).
\end{aligned}
$$

Since tables k and k' are independent,

$$
\sum_{k\neq k'}E\left[\left(\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k'1}z_{21k'1}}{n_{k'}^c} - E\left[\frac{z_{12k'1}z_{21k'1}}{n_{k'}^c}\right]\right)w_k w_{k'}\right] = 0,
$$

and

$$
\sum\left(E\left[\frac{z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)
$$

$$= \sum \left(\text{(expected value of the first term of 4.21)} \times E\left[\frac{z_{12k1}z_{21k1}}{n_k^c} w_k\right]\right)$$
$$= 0.$$

Writing

$$\begin{aligned}
A_{k,EST}^{INFM} &= E\left[\frac{(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})(z_{12k1}z_{21k1})w_k^2}{(n_k^c)^2}\right] \\
&= E\left[\frac{w_k^2}{(n_k^c)^2} E[z_{11k1}z_{12k1}z_{21k1}z_{22k1} - \theta z_{12k1}^2 z_{21k1}^2 | \mathbf{Z_{inf}}]\right] \\
&= E\left[\frac{w_k^2}{(n_k^c)^2} E[(z_{1+k1} - z_{12k1})z_{12k1}z_{21k1}(z_{2+k1} - z_{21k1})|\mathbf{Z_{inf}}]\right]
\end{aligned}$$

We have

$$\begin{aligned}
&E[(z_{1+k1} - z_{12k1})z_{12k1}z_{21k1}(z_{2+k1} - z_{21k1})|\mathbf{Z_{inf}}] \\
&= E[z_{1+k1}z_{2+k1}z_{12k1}z_{21k1} - z_{1+k1}z_{12k1}z_{21k1}^2 - z_{2+k1}z_{12k1}^2 z_{21k1} + (1-\theta) z_{12k1}^2 z_{21k1}^2 |\mathbf{Z_{inf}}] \\
&= z_{1+k1}^2 z_{2+k1}^2 \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} - z_{1+k1}^2 \frac{\pi_{12k}}{\pi_{1+k}}\left[z_{2+k1}^2\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 + z_{2+k1}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right] \\
&\quad - z_{2+k1}^2 \frac{\pi_{21k}}{\pi_{2+k}}\left[z_{1+k1}^2\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 + z_{1+k1}\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\right] \\
&\quad + (1-\theta)\left(z_{1+k1}^2\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 + z_{1+k1}\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\right)\left(z_{2+k1}^2\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 + z_{2+k1}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right).
\end{aligned}$$

Therefore, the coefficient of $z_{1+k1}^2 z_{2+k1}^2$ is

$$\begin{aligned}
&\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} - \frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 - \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}} + (1-\theta)\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 \\
&= \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(1 - \frac{\pi_{21k}}{\pi_{2+k}} - \frac{\pi_{12k}}{\pi_{1+k}} + \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} - \theta\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(\left(1 - \frac{\pi_{21k}}{\pi_{2+k}}\right)\left(1 - \frac{\pi_{21k}}{\pi_{2+k}}\right) - \theta\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}} - \theta\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= 0.
\end{aligned}$$

So,

$$A_{k,EST}^{INFM} = E\left[\frac{w_k^2}{(n_k^c)^2}(G_k z_{1+k1} z_{2+k1}^2 + H_k z_{1+k1}^2 z_{2+k1}) + \text{ higher order terms}\right]$$

$$= O(1/n_k^2)O(n_k^3)$$

$$= O(n_k),$$

where

$$\begin{aligned}
G_k &= -\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} + (1-\theta)\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 \\
&= -\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(\frac{\pi_{12k}}{\pi_{1+k}} - \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} + \theta\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= -\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}} + \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right) \\
&= -\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}
\end{aligned}$$

and

$$\begin{aligned}
H_k &= -\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} + (1-\theta)\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} \\
&= -\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}} - \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} + \theta\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= -\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{11k}}{\pi_{1+k}} + \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right) \\
&= -\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}.
\end{aligned}$$

Therefore, the expected value of the second term of (4.21) is

$$\begin{aligned}
E\left[\left(\frac{y_{EST}^{INFM} - \theta x_{EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right)\left(\frac{x_{EST}^{INFM} - \mu_{x,EST}^{INFM}}{\mu_{x,EST}^{INFM}}\right)\right] &= \frac{1}{(\mu_{x,EST}^{INFM})^2}(\sum_k A_{k,EST}^{INFM}) \\
&= O(1/N^2)O(N) = O(1/N).
\end{aligned}$$

Where $N = \sum_1^K n_k$. Also, the expected value of the second term of (4.21) is $O(1/K^2)O(K) = O(1/K)$.

The expected value of the third term of (4.21) is

$$\begin{aligned}
&E\left[\left(\frac{y_{EST}^{MAR} - \theta x_{MH,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)\left(\frac{x_{EST}^{MAR} - \mu_{x,EST}^{MAR}}{\mu_{x,EST}^{MAR}}\right)^2\right] \\
&= \frac{1}{(\mu_{x,EST}^{INFM})^3}E\left[\left(\sum\frac{z_{11k1}z_{22k1}}{n_k^c}w_k - \theta\left(\sum\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& \times\left(\sum \frac{z_{12k1}z_{21k1}}{n_k^c}w_k - \mu_{x,EST}^{MAR}\right)^2\Bigg] \\
= \quad & \frac{1}{(\mu_{x,EST}^{INFM})^3} E\left[\sum\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2\right] \\
= \quad & \frac{1}{(\mu_{x,EST}^{INFM})^3}\sum E\Bigg[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right) \\
& \times\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2\Bigg]. \qquad (4.22)
\end{aligned}
$$

Since the observations from different strata are independent, all of the expectations of cross product terms are zeros. The expected value of the kth term of the summation in (4.22) is

$$
\begin{aligned}
& E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}w_k - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2\right] \\
= \quad & E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}w_k\right)\frac{z_{12k1}^2 z_{21k1}^2}{(n_k^c)^2}w_k^2\right] \\
& - 2E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\left(\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1}}{n_k^c}\right)\left(\frac{z_{12k1}z_{21k1}}{n_k^c}\right)w_k^2\right] \\
& + \left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2 E\left[\frac{z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k}}{n_k^c}w_k\right] \\
= \quad & E\left[\frac{w_k^3}{(n_k^c)^3}E[z_{11k1}z_{12k1}^2 z_{21k1}^2 z_{22k1} - \theta z_{12k1}^3 z_{21k1}^3|\mathbf{Z_{inf}}]\right] \\
& - E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{w_k^2}{(n_k^c)^2}E[z_{11k1}z_{12k1}z_{21k1}z_{22k1}|\mathbf{Z_{inf}}]\right] \\
& + 2\theta E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{w_k^2}{(n_k^c)^2}E[z_{12k1}^2 z_{21k1}^2|\mathbf{Z_{inf}}]\right] \\
= \quad & E\left[\frac{w_k^3}{(n_k^c)^3}E[z_{11k1}z_{12k1}^2 z_{21k1}^2 z_{22k1} - \theta z_{12k1}^3 z_{21k1}^3|\mathbf{Z_{inf}}]\right] \\
& - 2E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]A_{k,EST}^{INFM} \\
& + \left(E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]\right)^2 \times \text{ expected value of the first term of (4.21)}
\end{aligned}
$$

where $A_{k,EST}^{INFM}$ is $O(n_k)$ and expected value of the first term of (4.21) is zero. We also have

$$
\begin{aligned}
& E[z_{11k1}z_{12k1}^2 z_{21k1}^2 z_{22k1} - \theta z_{12k1}^3 z_{21k1}^3|\mathbf{Z_{inf}}] \\
= \quad & E[(z_{1+k1}-z_{12k1})z_{12k1}^2 z_{21k1}^2(z_{2+k1}-z_{21k1}) - \theta z_{12k1}^3 z_{21k1}^3|\mathbf{Z_{inf}}] \\
= \quad & E[z_{1+k1}z_{2+k1}z_{12k1}^2 z_{21k1}^2 - z_{1+k1}z_{12k1}^2 z_{21k1}^3 - z_{2+k1}z_{12k1}^3 z_{21k1}^2 + (1-\theta)z_{12k1}^3 z_{21k1}^3|\mathbf{Z_{inf}}] \\
= \quad & z_{1+k1}z_{2+k1}E[z_{12k1}^2 z_{21k1}^2|\mathbf{Z_{inf}}] - z_{1+k1}E[z_{12k1}^2 z_{21k1}^3|\mathbf{Z_{inf}}] \\
& - z_{2+k1}E[z_{12k1}^3 z_{21k1}^2|\mathbf{Z_{inf}}] + (1-\theta)E[z_{12k1}^3 z_{21k1}^3|\mathbf{Z_{inf}}]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & z_{1+k1}z_{2+k1}(\text{ Var }(z_{12k1}|\mathbf{Z_{inf}}) + E[z_{12k1}|\mathbf{Z_{inf}}]^2)(\text{ Var }(z_{21k1}|\mathbf{Z_{inf}}) + E[z_{21k1}|\mathbf{Z_{inf}}]^2) \\
& -z_{1+k1}(\text{ Var }(z_{12k1}|\mathbf{Z_{inf}}) + E[z_{12k1}|\mathbf{Z_{inf}}]^2)E[(z_{21k1} - E[z_{21k1}])^3|\mathbf{Z_{inf}}] \\
& -z_{1+k1}(\text{ Var }(z_{12k1}|\mathbf{Z_{inf}}) + E[z_{12k1}|\mathbf{Z_{inf}}]^2) \\
& \qquad \times (3 \text{ Var }(z_{21k1}|\mathbf{Z_{inf}})E[z_{21k1}|\mathbf{Z_{inf}}] + E[z_{21k1}|\mathbf{Z_{inf}}]^3) \\
& -z_{2+k1}E[(z_{12k1} - E[z_{12k1}])^3|\mathbf{Z_{inf}}](\text{ Var }(z_{21k1}|\mathbf{Z_{inf}}) + E[z_{21k1}|\mathbf{Z_{inf}}]^2) \\
& -z_{2+k1}(3 \text{ Var }(z_{12k1}|\mathbf{Z_{inf}})E[z_{12k1}|\mathbf{Z_{inf}}] + E[z_{12k1}|\mathbf{Z_{inf}}]^3) \\
& \qquad \times (\text{ Var }(z_{21k1}|\mathbf{Z_{inf}}) + E[z_{21k1}|\mathbf{Z_{inf}}]^2) \\
& +(1-\theta)E[(z_{12k1} - E[z_{12k1}|\mathbf{Z_{inf}}])^3]E[(z_{21k1} - E[z_{21k1}|\mathbf{Z_{inf}}])^3|\mathbf{Z_{inf}}] \\
& +(1-\theta)E[(z_{12k1} - E[z_{12k1}|\mathbf{Z_{inf}}])^3|Z] \\
& \qquad \times (3E[z_{21k1}|\mathbf{Z_{inf}}] \text{ Var }(z_{21k1}|\mathbf{Z_{inf}}) + E[z_{21k1}|\mathbf{Z_{inf}}]^3) \\
& +(1-\theta)(3E[z_{12k1}|\mathbf{Z_{inf}}] \text{ Var }(z_{12k1}|\mathbf{Z_{inf}}) + E[z_{12k1}|\mathbf{Z_{inf}}]^3) \\
& \qquad \times E[(z_{21k1} - E[z_{21k1}|\mathbf{Z_{inf}}])^3|\mathbf{Z_{inf}}] \\
& +3(1-\theta)(3E[z_{12k1}|\mathbf{Z_{inf}}] \text{ Var }(z_{12k1}|\mathbf{Z_{inf}}) + E[z_{12k1}|\mathbf{Z_{inf}}]^3) \\
& \qquad \times E[z_{21k1}|\mathbf{Z_{inf}}] \text{ Var }(z_{21k1}|\mathbf{Z_{inf}}) \\
& +(1-\theta)(3E[z_{12k1}|\mathbf{Z_{inf}}] \text{ Var }(z_{12k1}|\mathbf{Z_{inf}}) + E[z_{12k1}|\mathbf{Z_{inf}}]^3)E[z_{21k1}|\mathbf{Z_{inf}}]^3
\end{aligned}
$$

The highest order term is $z_{1+k1}^3 z_{2+k1}^3$ and its coefficient is

$$
\begin{aligned}
& \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 - \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^3 \\
& \qquad - \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^3 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 + (1-\theta) \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^3 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^3 \\
= \quad & \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 \left(1 - \frac{\pi_{21k}}{\pi_{2+k}} - \frac{\pi_{12k}}{\pi_{1+k}} + (1-\theta)\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
= \quad & \left(\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 \left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}} - \theta\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
= \quad & 0.
\end{aligned}
$$

The coefficient of $z_{1+k1}^2 z_{2+k1}^3$ is

$$
\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 - \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^3
$$

$$
\begin{aligned}
&\quad - 3\frac{\pi_{11k}}{\pi_{1+k}}\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 + 3(1-\theta)\frac{\pi_{11k}}{\pi_{1+k}}\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^3 \\
&= \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\left(1 - \frac{\pi_{21k}}{\pi_{2+k}} - 3\frac{\pi_{12k}}{\pi_{1+k}} + 3(1-\theta)\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\left(-2\frac{\pi_{12k}}{\pi_{1+k}} + 2(1-\theta)\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= -2\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\left(\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}} + \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right) \\
&= -2\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\frac{\pi_{22k}}{\pi_{2+k}}.
\end{aligned}
$$

Similarly, the coefficient of $z_{1+k1}^3 z_{2+k1}^2$ is

$$
\begin{aligned}
&\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} - 3\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\frac{\pi_{22k}}{\pi_{2+k}} \\
&\quad - \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^3\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} + 3(1-\theta)\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^3\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\frac{\pi_{22k}}{\pi_{2+k}} \\
&= \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\left(1 - 3\frac{\pi_{21k}}{\pi_{2+k}} - \frac{\pi_{12k}}{\pi_{1+k}} + 3(1-\theta)\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\left(-2\frac{\pi_{21k}}{\pi_{2+k}} + 2(1-\theta)\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right) \\
&= -2\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} + \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right) \\
&= -2\frac{\pi_{11k}}{\pi_{1+k}}\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E[z_{11k1}z_{12k1}^2 z_{21k1}^2 z_{22k1} - \theta z_{12k1}^3 z_{21k1}^3] \\
&\quad = E\left[\frac{w_k^3}{(n_k^c)^3}(C_k z_{1+k1}^2 z_{2+k1}^3 + D_k z_{1+k1}^3 z_{2+k1}^2)\right] + \text{ lower order term} \\
C_k &= -2\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\frac{\pi_{22k}}{\pi_{2+k}} \\
D_k &= -2\frac{\pi_{11k}}{\pi_{1+k}}\left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}.
\end{aligned}
$$

And the expected value of the third term of (4.21) is

$$
\begin{aligned}
&\frac{1}{(\mu_{x,EST}^{INFM})^3}E\left[\frac{w_k^3}{(n_k^c)^3}(C_k z_{1+k1}^2 z_{2+k1}^3 + D_k z_{1+k1}^3 z_{2+k1}^2)\right] \\
&\quad - 2\frac{1}{(\mu_{x,EST}^{INFM})^3}E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{w_k^2}{(n_k^c)^2}(G_k z_{1+k1}z_{2+k1}^2 + H_k z_{1+k1}^2 z_{2+k1})\right] \\
&\quad + \text{ lower order term.}
\end{aligned}
$$

If the coefficient of the second order term equals to zero, then the expected value of the third term of (4.21) is $O(1/N^3)O(N) = O(1/N^2)$. That is when the leading coefficient of

$$
\begin{aligned}
& E\left[\frac{w_k^3}{(n_k^c)^3} C_k z_{1+k1}^2 z_{2+k1}^3\right] - 2E\left[\frac{z_{12k1}z_{21k1}}{n_k^c} w_k\right] E\left[\frac{w_k^2}{(n_k^c)^2} G_k z_{1+k1} z_{2+k1}^2\right] \\
&= -2\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{1+k}^2\pi_{2+k}^3} E\left[\left(\frac{z_{+1k+}z_{+2k+}n_k^c}{z_{+1k1}z_{+2k1}n_k n_k^c}\right)^3 z_{1+k1}^2 z_{2+k1}^3\right] \\
&\quad + 2\frac{\pi_{11k}\pi_{12k}\pi_{21k}^2\pi_{22k}}{\pi_{1+k}^2\pi_{2+k}^3} E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k} z_{1+k1}z_{2+k1}\right] E\left[\left(\frac{z_{+1k+}z_{+2k+}n_k^c}{z_{+1k1}z_{+2k1}n_k n_k^c}\right)^2 z_{1+k1} z_{2+k1}^2\right].
\end{aligned}
$$

$$
\begin{aligned}
& E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^3 z_{1+k1}^2 z_{2+k1}^2\right] \\
&= E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^3 E[z_{1+k1}^2 z_{2+k1}^3 | n_k, z_{+1k1}, z_{+2k1}, z_{+1k0}]\right] \\
&= E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^3 n_k(n_k-1)(n_k-2)(n_k-3)(n_k-4)\pi_{1+k1}^2\pi_{2+k1}^3\right]. \qquad (4.23)
\end{aligned}
$$

The leading coefficient of (4.23) is

$$
\pi_{1+k}^2\pi_{2+k}^3(1-p_{1k}^{missing})^2(1-p_{2k}^{missing})^3 n_k^2 \times \left(\text{leading coefficient of } E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^3\right]\right)
$$

and

$$
\begin{aligned}
& E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k} z_{1+k1}z_{2+k1}\right] E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 z_{1+k1}z_{2+k1}^2\right] \\
&= E\left[\frac{w_k}{n_k^c} E[z_{1+k1}z_{2+k1} | n_k, z_{+1k1}, z_{+2k1}, z_{+1k0}]\right] \\
&\qquad \times E\left[\left(\frac{w_k}{n_k^c}\right)^2 E[z_{1+k1}z_{2+k1}^2 | n_k, z_{+1k1}, z_{+2k1}, z_{+1k0}]\right] \\
&= n_k(n_k-1)\pi_{1+k1}\pi_{2+k1} n_k(n_k-1)(n_k-2)\pi_{1+k1}\pi_{2+k1}^2 \\
&\qquad \times E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right] E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2\right]. \qquad (4.24)
\end{aligned}
$$

The leading coefficient of (4.24) is

$$
\begin{aligned}
& \pi_{1+k}^2\pi_{2+k}^3(1-p_{1k}^{missing})^2(1-p_{2k}^{missing})^3 n_k^2 \\
&\quad \times \left(\text{leading coefficient of } E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right] E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right]\right).
\end{aligned}
$$

Conditioning on n_k, z_{+1k1}, z_{+2k1}, z_{+1k0} have a multinomial distribution with parameters n_k, π_{+1k1}, π_{+2k1} and π_{+1k0}, where $\pi_{+jk1} = \pi_{1jk}(1-p_{1k}^{missing}) + \pi_{2jk})(1-p_{2k}^{missing})$, $\pi_{+1k0} = \pi_{11k}p_{1k}^{missing} +$

$\pi_{21k}p_{2k}^{missing}$. $\pi_{+2k0} = 1 - \pi_{+1k1} - \pi_{+2k1} - \pi_{+1k0}$ and $n_k - z_{+1k1} - z_{+2k1} - z_{+1k0} = z_{+2k0}$.

$$E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right] = E\left[1 + \frac{z_{+1k0}}{z_{+2k1}} + \frac{z_{+2k0}}{z_{+1k1}} + \frac{z_{+1k0}z_{+2k0}}{z_{+1k1}z_{+2k1}}\right]$$

By the same argument as in section 4.2.1, the leading coefficients of $E[z_{+1k0}/z_{+1k1}]$ is the same as $E[z_{+1k0}/(z_{+1k1}+1)]$, and $E[z_{+2k0}/z_{+2k1}]$ has the same leading coefficient as $E[z_{+2k0}/(z_{+2k1}+1)]$:

$$\begin{aligned}
&E\left[\frac{z_{+1k0}z_{+2k0}}{z_{+1k1}z_{+2k1}}\right] - E\left[\frac{z_{+1k0}z_{+2k0}}{(z_{+1k1}+1)(z_{+2k1}+1)}\right] \\
&= E\left[z_{+1k0}z_{+2k0}\frac{(z_{+1k1}+1)(z_{+2k1}+1) - z_{+1k1}z_{+2k1}}{z_{+1k1}z_{+2k1}(z_{+1k1}+1)(z_{+2k1}+1)}\right] \\
&= E\left[z_{+1k0}z_{+2k0}\frac{z_{+1k1}+z_{+2k1}+1}{z_{+1k1}z_{+2k1}(z_{+1k1}+1)(z_{+2k1}+1)}\right] \\
&= O(1/N).
\end{aligned}$$

But $E[z_{+1k0}z_{+2k0}/(z_{+1k1}z_{+2k1}))]$ is $O(1)$, so $E[z_{+1k0}z_{+2k0}/(z_{+1k1}z_{+2k1}))]$ and $E[z_{+1k0}z_{+2k0}/((z_{+1k1}+1)(z_{+2k1}+1))]$ has the same leading coefficient. Using the same technique, we can show that $z^a_{+1k0}z^b_{+2k0}/(z^c_{+1k1}z^d_{+2k1})$ has the same leading coefficient as $z^a_{+1k0}z^b_{+2k0}/((z_{+1k1}+1)(z_{+1k1}+2))\cdot(z_{+1k1}+c)(z_{+2k1}+1)(z_{+2k1}+2)\cdot(z_{+2k1}+d)$ for positive integers a, b, c, d. Then the leading coefficient of $E[z_{+1k+}z_{+2k+}/(z_{+1k1}z_{+2k1})]$ is

$$1 + \frac{\pi_{+1k0}}{\pi_{+1k1}} + \frac{\pi_{+2k0}}{\pi_{+2k1}} + \frac{\pi_{+1k0}\pi_{+2k0}}{\pi_{+1k1}\pi_{+2k1}}.$$

Since

$$\begin{aligned}
&E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right] \\
&= E\left[\left(1 + 2\frac{z_{+1k0}}{z_{+1k1}} + \frac{z^2_{+1k0}}{z^2_{+1k1}}\right)\left(1 + 2\frac{z_{+2k0}}{z_{+2k1}} + \frac{z^2_{+2k0}}{z^2_{+2k1}}\right)\right] \\
&= E\left[1 + 2\frac{z_{+2k0}}{z_{+2k1}} + \frac{z^2_{+2k0}}{z^2_{+2k1}} + 2\frac{z_{+1k0}}{z_{+1k1}} + 4\frac{z_{+1k0}z_{+2k0}}{z_{+1k1}z_{+2k1}} + 2\frac{z_{+1k0}z^2_{+2k0}}{z_{+1k1}z^2_{+2k1}}\right] \\
&\quad + E\left[\frac{z^2_{+1k0}}{z^2_{+1k1}} + 2\frac{z^2_{+1k0}z_{+2k0}}{z^2_{+1k1}z_{+2k1}} + \frac{z^2_{+1k0}z^2_{+2k0}}{z^2_{+1k1}z^2_{+2k1}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^3\right] \\
&= E\left[\left(1 + 3\frac{z_{+1k0}}{z_{+1k1}} + 3\frac{z^2_{+1k0}}{z^2_{+1k1}} + \frac{z^3_{+1k0}}{z^3_{+1k1}}\right)\left(1 + 3\frac{z_{+2k0}}{z_{+2k1}} + 3\frac{z^2_{+2k0}}{z^2_{+2k1}} + \frac{z^3_{+2k0}}{z^3_{+2k1}}\right)\right]
\end{aligned}$$

$$\begin{aligned} = \quad & E\left[1+3\frac{z_{+2k0}}{z_{+2k1}}+3\frac{z_{+2k0}^2}{z_{+2k1}^2}+\frac{z_{+2k0}^3}{z_{+2k1}^3}+3\frac{z_{+1k0}}{z_{+1k1}}+9\frac{z_{+1k0}z_{+2k0}}{z_{+1k1}z_{+2k1}}\right] \\ & +E\left[9\frac{z_{+1k0}z_{+2k0}^2}{z_{+1k1}z_{+2k1}^2}+3\frac{z_{+1k0}z_{+2k0}^3}{z_{+1k1}z_{+2k1}^3}+3\frac{z_{+1k0}^2}{z_{+1k1}^2}+9\frac{z_{+1k0}^2z_{+2k0}}{z_{+1k1}^2z_{+2k1}}+9\frac{z_{+1k0}^2z_{+2k0}^2}{z_{+1k1}^2z_{+2k1}^2}+3\frac{z_{+1k0}^2z_{+2k0}^3}{z_{+1k1}^2z_{+2k1}^3}\right] \\ & +E\left[\frac{z_{+1k0}^3}{z_{+1k1}^3}+3\frac{z_{+1k0}^3z_{+2k0}}{z_{+1k1}^3z_{+2k1}}+3\frac{z_{+1k0}^3z_{+2k0}^2}{z_{+1k1}^3z_{+2k1}^2}+\frac{z_{+1k0}^3z_{+2k0}^3}{z_{+1k1}^3z_{+2k1}^3}\right], \end{aligned}$$

the leading coefficient of

$$\begin{aligned} & E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right] \\ = \quad & 1+2\frac{\pi_{+2k0}}{\pi_{+2k1}}+\frac{\pi_{+2k0}^2}{\pi_{+2k1}^2}+2\frac{\pi_{+1k0}}{\pi_{+1k1}}+4\frac{\pi_{+1k0}\pi_{+2k0}}{\pi_{+1k1}\pi_{+2k1}}+2\frac{\pi_{+1k0}\pi_{+2k0}^2}{\pi_{+1k1}\pi_{+2k1}^2} \\ & +\frac{\pi_{+1k0}^2}{\pi_{+1k1}^2}+2\frac{\pi_{+1k0}^2\pi_{+2k0}}{\pi_{+1k1}^2\pi_{+2k1}}+\frac{\pi_{+1k0}^2\pi_{+2k0}^2}{\pi_{+1k1}^2\pi_{+2k1}^2} \end{aligned}$$

and the leading coefficient of

$$\begin{aligned} & E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^3\right] \\ = \quad & E\left[\left(1+3\frac{z_{+1k0}}{z_{+1k1}}+3\frac{z_{+1k0}^2}{z_{+1k1}^2}+\frac{z_{+1k0}^3}{z_{+1k1}^3}\right)\left(1+3\frac{z_{+2k0}}{z_{+2k1}}+3\frac{z_{+2k0}^2}{z_{+2k1}^2}+\frac{z_{+2k0}^3}{z_{+2k1}^3}\right)\right] \\ = \quad & 1+3\frac{\pi_{+2k0}}{\pi_{+2k1}}+3\frac{\pi_{+2k0}^2}{\pi_{+2k1}^2}+\frac{\pi_{+2k0}^3}{\pi_{+2k1}^3}+3\frac{\pi_{+1k0}}{\pi_{+1k1}}+9\frac{\pi_{+1k0}\pi_{+2k0}}{\pi_{+1k1}\pi_{+2k1}} \\ & +9\frac{\pi_{+1k0}\pi_{+2k0}^2}{\pi_{+1k1}\pi_{+2k1}^2}+3\frac{\pi_{+1k0}\pi_{+2k0}^3}{\pi_{+1k1}\pi_{+2k1}^3}+3\frac{\pi_{+1k0}^2}{\pi_{+1k1}^2}+9\frac{\pi_{+1k0}^2\pi_{+2k0}}{\pi_{+1k1}^2\pi_{+2k1}}+9\frac{\pi_{+1k0}^2\pi_{+2k0}^2}{\pi_{+1k1}^2\pi_{+2k1}^2} \\ & +3\frac{\pi_{+1k0}^2\pi_{+2k0}^3}{\pi_{+1k1}^2\pi_{+2k1}^3}+\frac{\pi_{+1k0}^3}{\pi_{+1k1}^3}+3\frac{\pi_{+1k0}^3\pi_{+2k0}}{\pi_{+1k1}^3\pi_{+2k1}}+3\frac{\pi_{+1k0}^3\pi_{+2k0}^2}{\pi_{+1k1}^3\pi_{+2k1}^2}+\frac{\pi_{+1k0}^3\pi_{+2k0}^3}{\pi_{+1k1}^3\pi_{+2k1}^3} \\ = \quad & \left(1+\frac{\pi_{+1k0}}{\pi_{+1k1}}+\frac{\pi_{+2k0}}{\pi_{+2k1}}+\frac{\pi_{+1k0}\pi_{+2k0}}{\pi_{+1k1}\pi_{+2k1}}\right) \\ & \times\left(1+2\frac{\pi_{+2k0}}{\pi_{+2k1}}+\frac{\pi_{+2k0}^2}{\pi_{+2k1}^2}+2\frac{\pi_{+1k0}}{\pi_{+1k1}}+4\frac{\pi_{+1k0}\pi_{+2k0}}{\pi_{+1k1}\pi_{+2k1}}+2\frac{\pi_{+1k0}\pi_{+2k0}^2}{\pi_{+1k1}\pi_{+2k1}^2}+\frac{\pi_{+1k0}^2}{\pi_{+1k1}^2}\right. \\ & \left.+2\frac{\pi_{+1k0}^2\pi_{+2k0}}{\pi_{+1k1}^2\pi_{+2k1}}+\frac{\pi_{+1k0}^2\pi_{+2k0}^2}{\pi_{+1k1}^2\pi_{+2k1}^2}\right) \\ = \quad & \text{leading coefficient of }\left(E\left[\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right]E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right]\right). \end{aligned}$$

Therefore,

$$\begin{aligned} E\left[\frac{w_k^3}{(n_k^c)^3}C_k z_{+1k1}^2 z_{+2k1}^3\right]-2E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{w_k^2}{(n_k^c)^2}G_k z_{1+k1}z_{2+k1}^2\right] &= O(n_k) \text{ and} \\ E\left[\frac{w_k^3}{(n_k^c)^3}D_k z_{+1k1}^3 z_{+2k1}^2\right]-2E\left[\frac{z_{12k1}z_{21k1}}{n_k^c}w_k\right]E\left[\frac{w_k^2}{(n_k^c)^2}G_k z_{1+k1}^2 z_{2+k1}\right] &= O(n_k) \end{aligned}$$

Therefore, the third term of (4.21) is $O(1/N^3O(N) = O(1/N^2)$ as $\mu_{x,EST}^{INFM} = O(N)$. So the leading term of the bias is $O(1/N)$. When total sample size increases for fixed K and θ, then the leading term of the bias converges to 0. Also, the expected values of the second term and third term of (4.21) are $O(1/K)$ and $O(1/K^2)$, so for fixed n_k and θ and bounded away from 0 cell probabilities, as $K \to \infty$ the estimator also converges to the true common odds ratio.

Next we investigate the variance. We have

$$\begin{aligned}
\text{Var } (\theta^{INFM}_{MH,EST} - \theta) &\doteq \text{Var } \left(\sum_{k=1}^{K} \left(\frac{[(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})/n^c_k]w_k}{\mu^{INFM}_{x,EST}} \right) \right) \\
&= \frac{1}{(\mu^{INFM}_{x,EST})^2} \sum_{k=1}^{K} \text{Var } \left(\frac{(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})w_k}{n^c_k} \right) \\
&= \frac{1}{(\mu^{INFM}_{x,EST})^2} \sum_{k=1}^{K} \text{Var } \left(\frac{w_k}{n^c_k} E[z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|\mathbf{Z_{inf}}] \right) \\
&\quad + \frac{1}{(\mu^{INFM}_{x,EST})^2} \sum_{k=1}^{K} E\left[\frac{w_k^2}{(n^c_k)^2} \text{ Var } (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|\mathbf{Z_{inf}}) \right] \\
&= \frac{1}{(\mu^{INFM}_{x,EST})^2} \sum_{k=1}^{K} E\left[\frac{w_k^2}{(n^c_k)^2} \text{ Var } (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|\mathbf{Z_{inf}}) \right]
\end{aligned}$$

We have

$$\begin{aligned}
&\text{Var } (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1}|\mathbf{Z_{inf}}) \\
&= E[(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})^2|\mathbf{Z_{inf}}] - 0 \\
&= E[z_{11k1}^2 z_{22k1}^2 - 2\theta z_{11k1}z_{12k1}z_{21k1}z_{22k1} + \theta^2 z_{12k1}^2 z_{21k1}^2|\mathbf{Z_{inf}}] \\
&= \left(z_{1+k1}^2 \left(\frac{\pi_{11k}}{\pi_{1+k}}\right)^2 + z_{1+k1}\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}} \right) \left(z_{2+k1}^2 \left(\frac{\pi_{22k}}{\pi_{2+k}}\right)^2 + z_{2+k1}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} \right) \\
&\quad -2\theta E[z_{1+k1}z_{12k1} - z_{12k1}^2|\mathbf{Z_{inf}}]E[z_{2+k1}z_{21k1} - z_{21k1}^2|\mathbf{Z_{inf}}] \\
&\quad +\theta^2 \left(z_{1+k1}^2 \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 + z_{1+k1}\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}} \right) \left(z_{2+k1}^2 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 + z_{2+k1}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} \right) \\
&= z_{1+k1}^2 z_{2+k1}^2 \left(\left(\frac{\pi_{11k}}{\pi_{1+k}}\right)^2 \left(\frac{\pi_{22k}}{\pi_{2+k}}\right)^2 + \theta^2 \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 \right) \\
&\quad +z_{1+k1}^2 z_{2+k1} \left(\left(\frac{\pi_{11k}}{\pi_{1+k}}\right)^2 \frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} + \theta^2 \left(\frac{\pi_{12k}}{\pi_{+2k}}\right)^2 \frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} \right) \\
&\quad +z_{1+k1} z_{2+k1}^2 \left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}} \left(\frac{\pi_{22k}}{\pi_{2+k}}\right)^2 + \theta^2 \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}} \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 \right) \\
&\quad +z_{1+k1}z_{2+k1}\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}(1+\theta^2)
\end{aligned}$$

$$
\begin{aligned}
& +2\theta z_{1+k1}^2 z_{2+k1}^2 \left(-\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} + \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \frac{\pi_{21k}}{\pi_{2+k}} + \frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 - \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\right) \\
& +2\theta z_{1+k1}^2 z_{2+k1} \left(\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} - \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right) \\
& +2\theta z_{1+k1} z_{2+k1}^2 \left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}} - \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{+1k}}\right)^2\right) \\
& -2\theta z_{1+k1} z_{2+k1} \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} \\
= \; & z_{1+k1}^2 z_{2+k1}^2 \left(2\left(\frac{\pi_{11k}}{\pi_{1+k}}\right)^2 \left(\frac{\pi_{22k}}{\pi_{2+k}}\right)^2 - 2\theta \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(1 - \frac{\pi_{12k}}{\pi_{1+k}} - \frac{\pi_{21k}}{\pi_{2+k}} + \frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\right)\right) \\
& +z_{1+k1}^2 z_{2+k1} \left(\left(\frac{\pi_{11k}}{\pi_{+1k}}\right)^2 \frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}} + \theta^2 \left(\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\right) \\
& +2\theta z_{1+k1}^2 z_{2+k1} \left(\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\left(1 - \frac{\pi_{12k}}{\pi_{1+k}}\right)\right) \\
& +z_{1+k1} z_{2+k1}^2 \left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(\frac{\pi_{22k}}{\pi_{2+k}}\right)^2 + \theta^2 \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{21k}}{\pi_{2+k}}\right)^2\right) \\
& +2\theta z_{1+k1} z_{2+k1}^2 \left(\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\left(1 - \frac{\pi_{21k}}{\pi_{2+k}}\right)\right) \\
& +z_{1+k1} z_{2+k1} \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}(1 + \theta^2 - 2\theta) \\
= \; & z_{1+k1}^2 z_{2+k1}^2 (0) + z_{+1k1}^2 z_{+2k1} \frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}\left(\frac{\pi_{11k}}{\pi_{1+k}} + \theta\frac{\pi_{12k}}{\pi_{1+k}}\right)^2 \\
& +z_{1+k1} z_{2+k1}^2 \frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\left(\frac{\pi_{22k}}{\pi_{2+k}} + \theta\frac{\pi_{21k}}{\pi_{2+k}}\right)^2 + z_{1+k1} z_{2+k1}\frac{\pi_{11k}}{\pi_{1+k}}\frac{\pi_{12k}}{\pi_{1+k}}\frac{\pi_{21k}}{\pi_{2+k}}\frac{\pi_{22k}}{\pi_{2+k}}(1-\theta)^2.
\end{aligned}
$$

We can see that

$$
\begin{aligned}
& E\left[\frac{w_k^2}{(n_k^c)^2} \operatorname{Var}\left(z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1} | \mathbf{Z_{inf}}\right)\right] \\
& = \frac{\pi_{21k}\pi_{22k}}{\pi_{1+k}^2\pi_{2+k}^2}(\pi_{11k} + \theta\pi_{12k})^2 E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 z_{1+k1}^2 z_{2+k1}\right] \\
& \quad + \frac{\pi_{11k}\pi_{12k}}{\pi_{1+k}^2\pi_{2+k}^2}(\pi_{22k} + \theta\pi_{21k})^2 E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 z_{1+k1} z_{2+k1}^2\right] \\
& \quad + (1-\theta)^2 \frac{\pi_{11k}\pi_{12k}\pi_{21k}\pi_{22k}}{\pi_{1+k}^2\pi_{2+k}^2} E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 z_{1+k1} z_{2+k1}\right]
\end{aligned}
$$

Since we only consider the cases where $z_{1+k1} > 0$ and $z_{2+k1} > 0$, we have

$$
\begin{aligned}
& E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 z_{1+k1}^2 z_{2+k1}\right] \\
& = E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 E[z_{1+k1}^2 z_{2+k1} | z_{+1k1}, z_{+2k1}, z_{+1k0}, z_{+2k0}, n_k]\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 (n_k(n_k-1)(n_k-2)\pi^2_{1+k1}\pi_{2+k1} + n_k(n_k-1)\pi_{1+k1}\pi_{2+k1})\right] \\
&= \left(\frac{(n_k-1)(n_k-2)}{n_k}\pi^2_{1+k1}\pi_{2+k1} + \frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}\right) E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2\right] \\
&> E\left[E\left[\frac{z^2_{+1k+}z^2_{+2k+}}{(z_{+1k1}+1)(z_{+1k1}+2)(z_{+2k1}+1)(z_{+2k1}+2)}\middle| n_k\right]\right] \\
&= \left(\frac{\pi^2_{+1k}\pi^2_{+2k}}{\pi^2_{+1k1}\pi^2_{+2k1}} + \frac{\pi^2_{+1k}\pi_{+2k}}{\pi^2_{+1k1}\pi^2_{+2k1}(n_k+1)}\right) \\
&\quad + \left(\frac{\pi_{+1k}\pi^2_{+2k}}{\pi^2_{+1k}\pi^2_{+2k}(n_k+1)} + \frac{\pi_{+1k}\pi_{+2k}}{\pi^2_{+1k}\pi^2_{+2k}(n_k+1)(n_k+2)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2 z_{1+k1}z^2_{2+k1}\right] \\
&= E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 E[z_{1+k1}z^2_{2+k1}|z_{+1k1}, z_{+2k1}, z_{+1k0}, z_{+2k0}, n_k]\right] \\
&= \left(\frac{(n_k-1)(n_k-2)}{n_k}\pi_{1+k1}\pi^2_{2+k1} + \frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}\right) E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}n_k}\right)^2 z_{1+k1}z_{2+k1}\right] \\
&= \frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right].
\end{aligned}
$$

So

$$
\begin{aligned}
&E\left[\frac{w_k^2}{(n_k^c)^2}\ \mathrm{Var}\ (z_{11k1}z_{22k1} - \theta z_{12k1}z_{21k1})\right] \\
&= \frac{\pi_{21k}\pi_{22k}}{\pi^2_{1+k}\pi^2_{2+k}}\left(\pi_{11k} + \frac{\pi_{11k}\pi_{22k}}{\pi_{21k}}\right)^2\left(\frac{(n_k-1)(n_k-2)}{n_k}\pi^2_{1+k1}\pi_{2+k1} + \frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}\right) \\
&\quad \times E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right] \\
&\quad + \frac{\pi_{11k}\pi_{12k}}{\pi^2_{1+k}\pi^2_{2+k}}\left(\pi_{22k} + \frac{\pi_{11k}\pi_{22k}}{\pi_{12k}}\right)^2\left(\frac{(n_k-1)(n_k-2)}{n_k}\pi_{1+k1}\pi^2_{2+k1} + \frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}\right) \\
&\quad \times E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right] \\
&\quad + \frac{(1-\theta)^2}{\theta}\frac{\pi^2_{11k}\pi^2_{22k}}{\pi^2_{1+k}\pi^2_{2+k}}\frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right].
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\pi_{21k}\pi_{22k}}{\pi_{1+k}^2}\left(\frac{\pi_{11k}}{\pi_{21k}}\right)^2\left(\frac{(n_k-1)(n_k-2)}{n_k}\pi_{1+k1}^2\pi_{2+k1}+\frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}\right) \\
& \times E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right] \\
& + \frac{\pi_{11k}\pi_{12k}}{\pi_{2+k}^2}\left(\frac{\pi_{22k}}{\pi_{12k}}\right)^2\left(\frac{(n_k-1)(n_k-2)}{n_k}\pi_{1+k1}\pi_{2+k1}^2+\frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}\right) \\
& \times E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right] \\
& + \frac{(1-\theta)^2}{\theta}\frac{\pi_{11k}^2\pi_{22k}^2}{\pi_{1+k}^2\pi_{2+k}^2}\frac{n_k-1}{n_k}\pi_{1+k1}\pi_{2+k1}E\left[\left(\frac{z_{+1k+}z_{+2k+}}{z_{+1k1}z_{+2k1}}\right)^2\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{Var}\ (\theta_{MH,EST}^{INFM}-\theta) \quad & \doteq \quad \text{Var}\left(\sum_{k=1}^{K}\left(\frac{[(z_{11k1}z_{22k1}-\theta z_{12k1}z_{21k1})/n_k^c]w_k}{\mu_{x,EST}^{INFM}}\right)\right) \\
& = \quad O(1/N^2)O(N) \\
& = \quad O(1/N).
\end{aligned}
$$

When the number of strata K and θ are fixed, and when n_k and θ are fixed, $\text{Var}\ (\theta_{MH,EST}^{INFM}-\theta)$ is $O(1/K)$. But when K and N are fixed, it's $O(\theta^2)O(\theta)=O(\theta^3)$. When K and N are fixed, the MAR closed from for informative missingness will increase the variance of the same order of the the variance estimated of the full data and complete only.

4.3 Independent binomial rows

In addition to the fixed the table size, if the total number of the observations in the first row is also fixed, the table consists of independent binomial rows and is a prospective study as in Agresti [1]. As the row sums are fixed in the independent rows binomial models, there are less biases and variabilities.

Chapter 5

Simulation Results

In our simulations, we have K 2×2 contingency tables, where rows are two independent binomials with parameters $(n_k/2, p_{1k})$ and $(n_k/2, p_{2k})$. The p_{1k} are uniform $(0,1)$ and $p_{2k} = p_{1k}/((1-p_{1k})\theta + p_{1k})$ so that the contingency tables satisfy the common odds ratio assumption. These K 2×2 contingency tables are the "Full Data" displayed below.

	$B=1$	$B=2$
$A=1$	z_{11k+}	z_{12k+}
$A=2$	z_{21k+}	z_{22k+}

Conditional on the full data, the incomplete data are the independent binomails z_{ijk0} with parameters $(z_{ijk+}, P(missing)_{jk})$ for $MAR(B,C)$ model and $(z_{ijk+}, P(missing)_{ik})$ for the informatively missing model (A,C). The remaining observable data are called the "Complete Data" and denoted as z_{ijk1}. Using the closed-form estimates of the full cell counts results in the "Estimated Data". Three different patterns of missingness probabilities are used in the simulations: the first case is $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for all k; the second case is $P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k = 1, \cdots K/2$ and $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k = K/2+1, \cdots K$; and the third case is $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, K/2$ and $p(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k = K/2+1, \cdots, K$. These probabilities were selected to assure that the effect of missingness would be substantial, but not overwhelming. We also wanted to see if varying patterns of missing value probabilities would affect the results.

5.1 Simulation

The simulations are based on 500 replications (Nrep=500). This value was selected because in many of our examples, the computational brade was heavy. The sample sizes and number of strata are varied to study the asymptotic behavior. We simulated 20, 40, and 80 strata with 20 observations per table, and 20 strata with 20, 40, and 80 observations per table. We generate data with $\theta = 1, 3, 5$ for both $MAR(B, C)$ and missing informatively(A, C). We also generate data with $\theta = 10$ for the $MAR(B, C)$ case. To avoid too many empty cell counts in the simulation, for $\theta = 10$ we used table size five times as large as the other simulations, that is, 100, 200 and 400 total observations for $\theta = 10$.

In the simulations, we compared estimates based on the "Full Data", "Complete Data" and "Estimated Data". In each data set, we study the Mantel-Haenszel estimator (MH), Mantel-Haenszel with one pair of pseudotables (PMH1) and Mantel-Haenszel with jackknifing (JK). From preliminary simulation results before the main study, adding one pair of pseudotables is the best choice, so we only studied one pair pseudotables case for different missingness models and common odds ratios.

The figures for the third missingness model, that is $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, K/2$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k = K/2+1, \cdots, K$, are displayed here and the detailed numerical results are available in the Appendices A - Appendices H. The simulation results show that the three missingness models have similar patterns.

5.2 Estimating the common odds ratio when A is $MAR(B, C)$

The first model we simulate specifies that the probability that A is missing depends on the observable column variable B and the stratum variable C. Four values of the common odds ratios are studied for this missingness model.

When the common odds ratio equals 1 ($\theta = 1$), estimation based on one pair of pseudotables performs almost the same as straight Mantel-Haenszel in the estimating θ, but the jackknifing reduces the bias for most of the cases. Using each method, when the table sizes increase, the

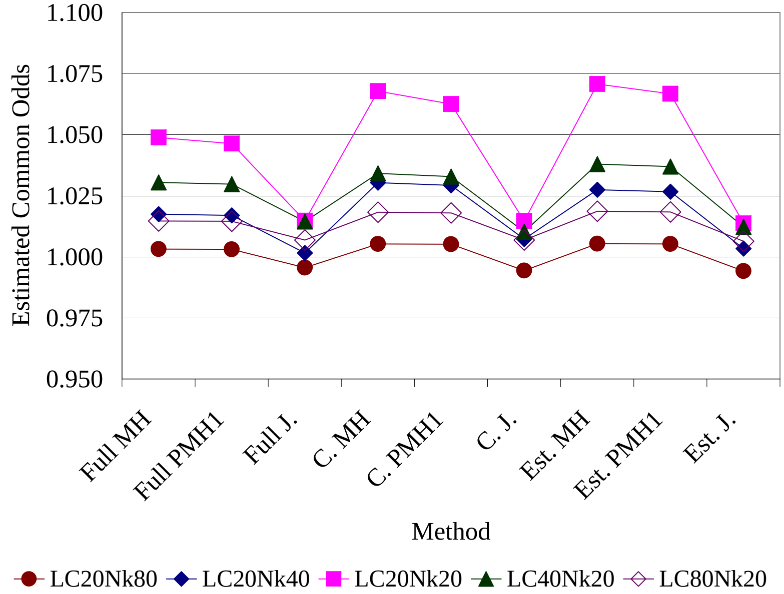

Figure 5.1: Mean: $\theta = 1$, MAR, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

biases decreases. When the number of strata is larger, the bias is smaller, except when missingness depends only on the stratum variable C with 80 strata. The bias of the "Complete Data" is slightly smaller than that of the "Estimated Data" and the difference is bigger in the case of 20 strata with 20 observations each. The Figure 5.1 is the figure of the common odds ratio estimated for $\theta = 1$. with $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, K/2$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k = K/2+1, \cdots, K$. The detail numbers for these three figures are shown in Table A.1.

When we double the total observations, either the table size or the number of the strata, the variance is halved according to Table A.2. These results show that the PMH1 and JK methods are not only reduce the bias, but also reduce the variabilities. "Complete Data" estimates have slightly smaller variabilities than "Estimated Data" estimates. The 20 strata with 20 observations case show the biggest differences between the "Complete Data" and "Estimated Data" variances, and the difference might be as much as 15%. The Figure 5.2 displays the variances of the es-

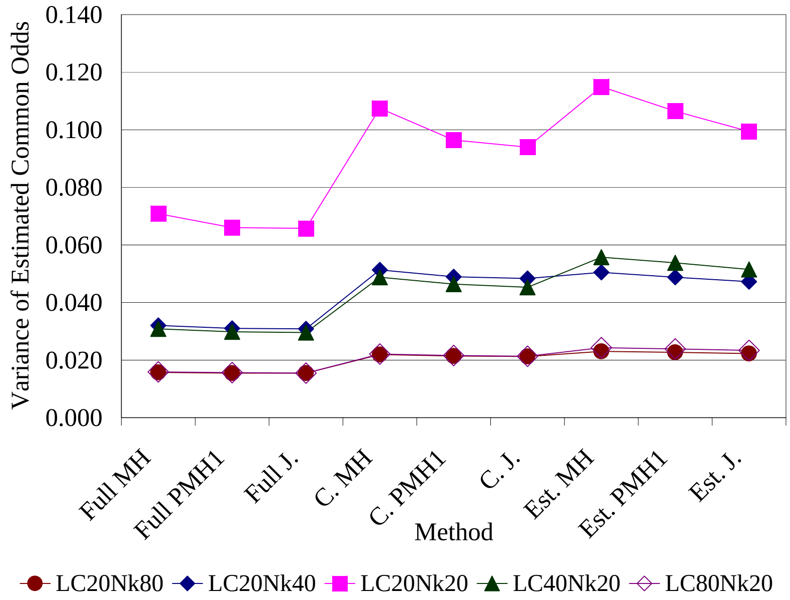

Figure 5.2: Variance: $\theta = 1$, MAR, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

timated common odds ratios for $\theta = 1$ with $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, K/2$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k = K/2+1, \cdots, K$.

Figure 5.3 and Table A.3 show the values of $T = (\hat{\theta} - \theta)/\sqrt{\text{ Var }(\hat{\theta})/Nrep}$ for $\theta = 1$. We see that the small bias in Figure 5.1 are real rather than Monte Carlo sample variation.

For $\theta = 3$, the simulations show that Mantel-Haenszel is an overestimate, but adding one pair of the pseudotables or jackknifing by dropping one table at a time reduces the bias, but might overcorrect and result in negative bias. The methods seem to converge to the true common odds ratio θ when either the number of strata or the size of each stratum goes to infinity.

Also the simulation results show that when using closed-form estimated data to estimate the common odds ratio, the biases after pseudotable or jackknifing adjustments are acceptable. Figure 5.4 displays the Monte Carlo averages of the estimators of $\theta = 3$ and Table A.4 has the

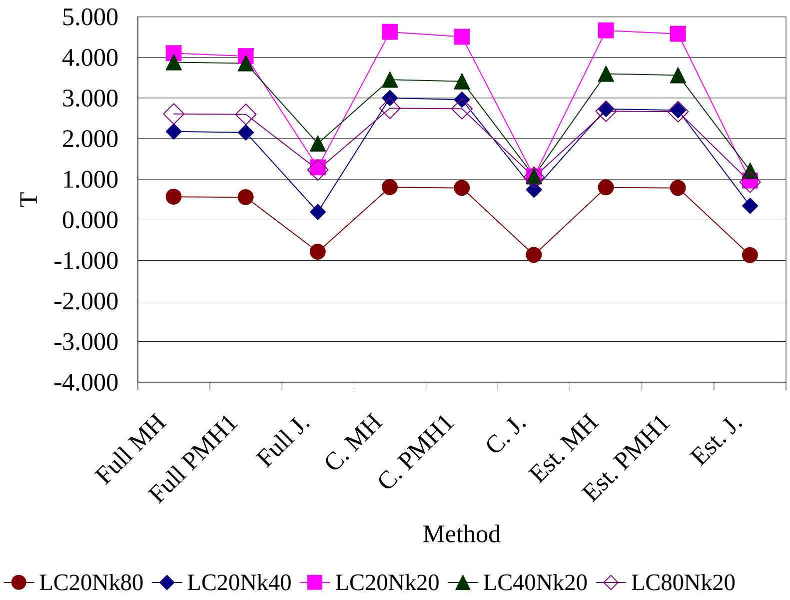

Figure 5.3: t: $\theta = 1$, MAR, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, and $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

detailed numerical values.

The simulation shows that adding one pair of the pseudotables or using jackknifing method not only reduces bias but also reduces the variance. The variance gets smaller when the number of strata or the size of each stratum becomes larger.

Using the "Complete Data" yields smaller variance than using "Estimated Data$''$. Figure 5.5 display the variances of the estimators and Table A.5 shows the numbers. Figure 5.6 shows the T values of the simulations and Table A.6 shows the numerical results.

The results for $\theta = 5$ and $\theta = 10$ are similar to the results for $\theta = 1$ or $\theta = 3$. except larger biases and variances in each individual cases. We do not have negative estimator in any of the cases for $\theta = 10$.

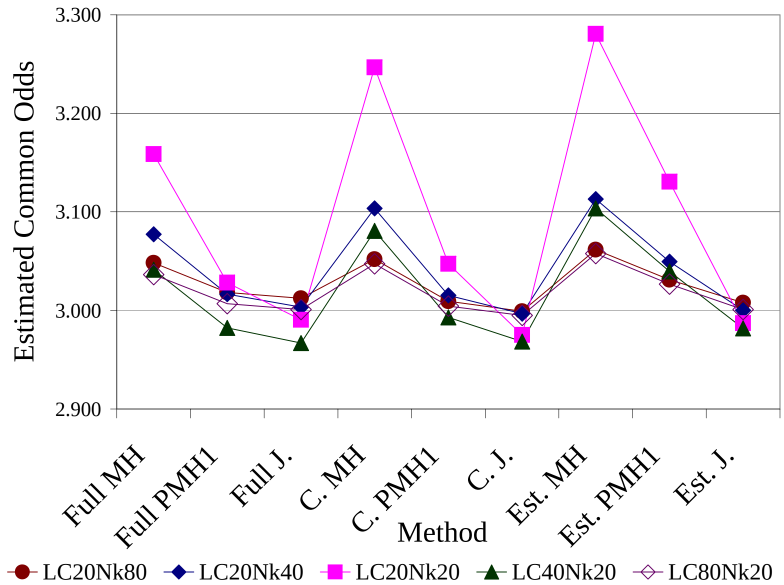

Figure 5.4: Mean: $\theta = 3$, MAR, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

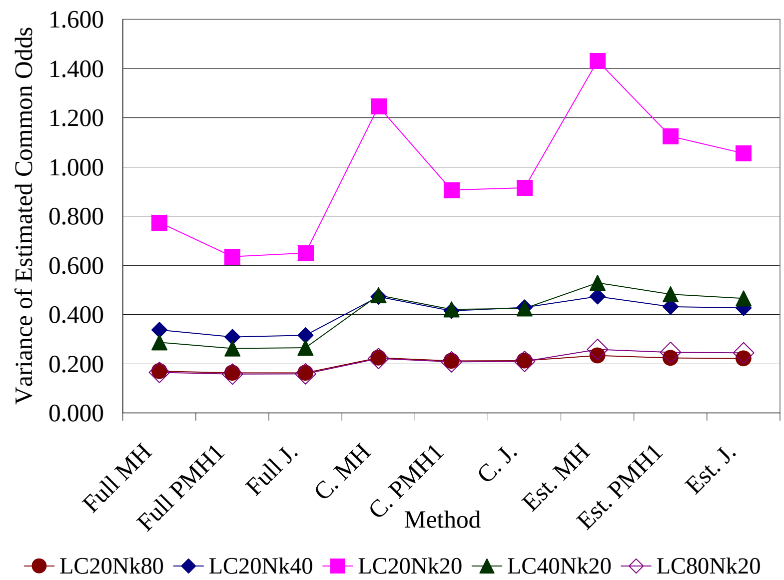

Figure 5.5: Variance: $\theta = 3$, MAR, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

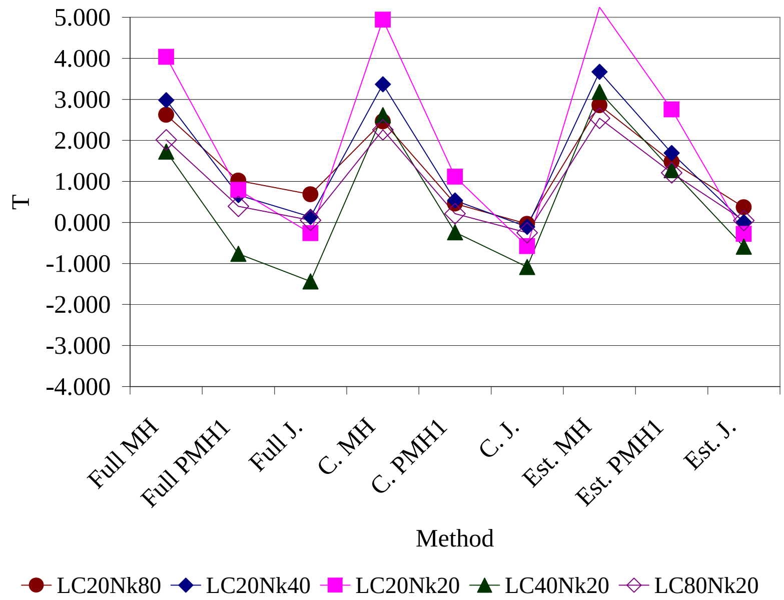

Figure 5.6: t: $\theta = 3$, MAR, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, and $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

5.3 Estimating the common odds ratio when A is missing informative(A, C)

When the row variable A is missing depending on the strata variable C and A itself, the boundary condition might have been used to solve for the closed-form estimator. In our simulations, about 3/4 of the cases one resulted in of the missingness parameters $a_{i.k}$ are falling onto the boundary. In some cases, both $a_{1.k}$ and $a_{2.k}$ are on the boundary, and the origin is the answer. Larger table sizes were more likely to produce real solutions. However, when the number of strata becomes larger the problem of boundary cases persists. In our simulation, under informative missingness the common odds is overestimated with big biases for our combination of numbers of strata and table sizes. When the table size increases, the estimators do converge to the correct θ but very slowly. When we fix the table size but increase the number of strata, the estimator doesn't seem to converge to the correct θ. The following table shows estimators of the common odds ratio and the average and the minimum and maximum number of the boundary tables in different combinations of the number of strata and the size of tables for $\theta = 3$ and $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$

for $k = 1, \cdots, K/2$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k = K/2+1, \cdots, K$. The large number of the boundary cases increase the bias.

Independent Binomial Rows, $\theta = 3$

$P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, K/2$

$P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k = K/2+1, \cdots, K$

Tables		Informative(A, C) Estimated Data			Boundary Cases			Analysized as MAR		
LC	N_k	MH[1]	PMH1[2]	JK[3]	Mean	Mim	Max	MH	PMH1	JK
Large Table (Large N_k)										
20	20	3.580	3.396	3.257	15.1	9	20	3.206	3.059	2.933
20	40	3.469	3.386	3.333	12.3	5	18	3.140	3.073	3.020
20	80	3.267	3.230	3.211	10.2	5	17	3.027	2.997	2.974
20	160	3.165	3.148	3.140	7.4	2	14	3.030	3.015	3.005
20	320	3.098	3.090	3.085	5.2	0	10	3.034	3.026	3.021
Large Number of Strata (Large LC)										
20	20	3.580	3.396	3.257	15.1	9	20	3.206	3.059	2.933
40	20	3.440	3.357	3.300	30.2	22	37	3.108	3.041	2.990
80	20	3.389	3.350	3.322	60.0	49	72	3.062	3.031	3.006
160	20	3.351	3.332	3.318	120.2	104	137	3.040	3.024	3.012
320	20	3.321	3.312	3.305	240.1	216	261	3.016	3.008	3.002
640	20	3.324	3.319	3.316	480.0	448	512	3.006	3.002	2.999

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

and for the cases where the "Full Data" are generated from multinomials with parameters $(n_k, p_{11k}, p_{12k}, p_{21k})$ are as following table.

Multinomial, $\theta = 3$

$P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$, for $k = 1, \cdots, K/2$

$P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$, for $k = K/2 + 1, \cdots, K$

Tables		Informative(A, C) Estimated Data			Boundary Cases			Analysized as MAR		
LC	N_k	MH[1]	PMH1[2]	JK[3]	Mean	Mim	Max	MH	PMH1	JK
Large Table (Large N_k)										
20	20	3.645	3.447	3.294	15.0	9	19	3.290	3.128	2.989
20	40	3.359	3.280	3.227	12.7	7	18	3.056	2.991	2.940
20	80	3.305	3.268	3.249	10.1	3	16	3.065	3.033	3.010
20	160	3.159	3.143	3.134	7.7	1	14	3.021	3.006	2.995
20	320	3.086	3.078	3.073	5.2	0	12	3.021	3.014	3.008
Large Number of Strata (Large LC)										
20	20	3.645	3.447	3.294	15.0	9	19	3.290	3.128	2.989
40	20	3.402	3.319	3.255	30.2	23	37	3.108	3.038	2.984
80	20	3.295	3.257	3.226	60.3	46	73	3.029	2.997	2.971
160	20	3.288	3.269	3.254	120.8	106	136	3.027	3.011	2.998
320	20	3.275	3.266	3.258	241.3	216	262	3.018	3.010	3.004
640	20	3.271	3.266	3.263	481.3	453	514	3.019	3.015	3.012

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

However, if we base our estimate on the "Complete Data" and using the same closedform as when missingness of A depends only on B and C as in the previous section, we do have reasonable estimators.

The common odds ratio is slightly overestimated by the Mantel-Haenszel statistic when the true common odds ratio is 1. Similar results occur if one adds one pair of the pseudotables (PMH1).

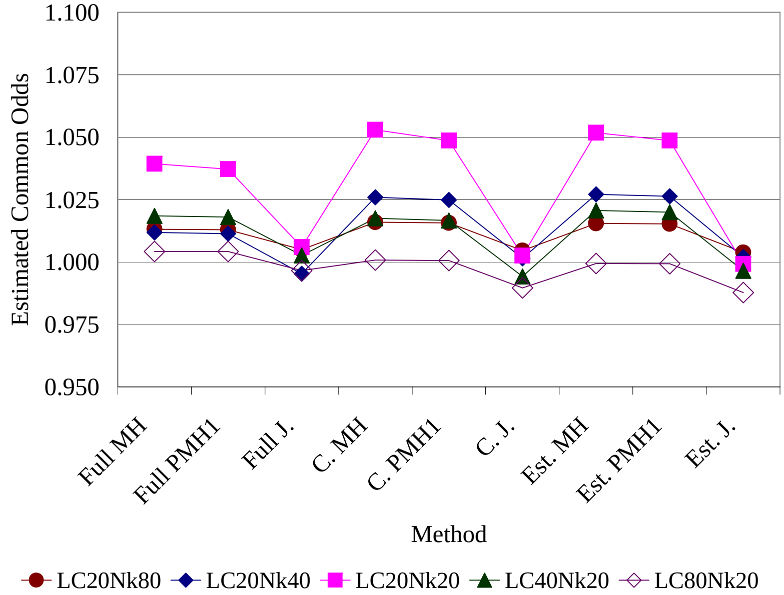

Figure 5.7: Mean: $\theta = 1$, Missing Informative, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

The jackknife estimator is smaller on the average than the other two estimators, but in most cases, it reduces the bias only when the number of strata is 80 and the table size is 20. In this case, the Mantel-Haenszel and PMH1 estimators are very close to the the true value and the jackknifing method underestimates θ. value. Using "Complete Data" is slightly better than using the "Estimated Data". The "Estimated Data" is based on a misspecified model closed-forms ($MAR(B, C)$) but false missingness model rather than the true ($Informative(A, C)$) model. When the number of strata or the table size gets larger, the estimators based on "Complete Data" and "Estimated Data" are both closed to the "Full Data". Figure 5.7 display the Monte Carlo means of the estimators. Table B.4 lists the simulation results.

The variances of the estimators have the same patterns as we saw in the $MAR(B, C)$ case. When we double the number of observations by doubling either the number of tables or the size of tables, we halve the variance. The variance of the "Complete Data" estimators are slightly smaller than the "Estimated Data" estimators. Figure 5.8 display the Monte Carlo variances of the estimators. Table B.5 lists the simulation results.

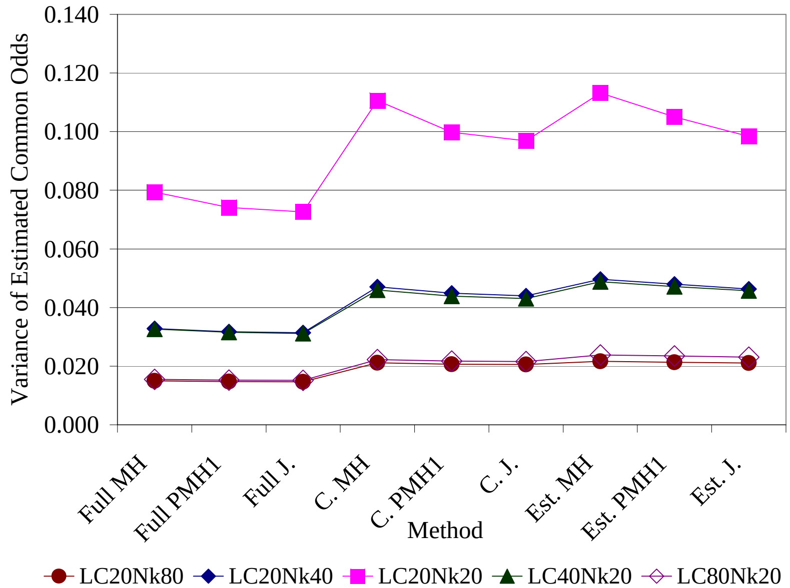

Figure 5.8: Variance: $\theta = 1$, Missing Informative, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

The most extreme T value for testing the significance of the bias of the jackknifing method is -1.805 for the "Estimated Data" when the number of strata is 80, the table size is 20 when the $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, K/2$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k = K/2+1, \cdots, K$. The t value for the "Complete Data" of the same case is -1.579. All of the t values for Jackknifing are between -1.85 and 1.204 which means the average of the estimators are not significantly different from the true value. However, for 20 strata with 20 observations in each table, the T values of the Mantel-Haenszel and PMH1 estimators are much larger than 2. The average of estimator is significantly different from the true value. Figure 5.9 display the t statistics for the estimators. Table B.6 lists the simulation results.

Figure 5.10 and Table B.4 display the results of the common odds ratio estimators for $\theta = 3$ of the missing Missing Informatively (A, C). As we mentioned before, the true informative(A, C) model closed-form estimators do not estimate the true frequencies correctly, but the biases of the $MAR(B, C)$ model closed-form estimator are in the acceptable range and are indicated as

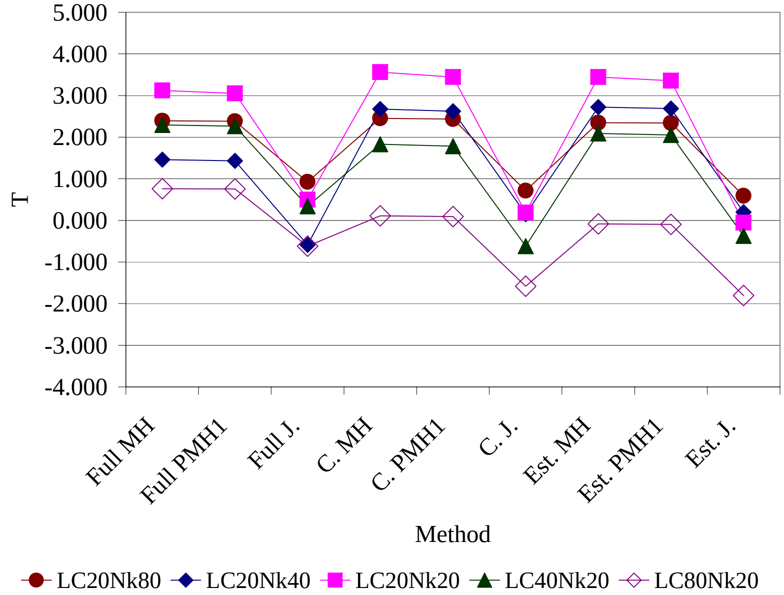

Figure 5.9: t: $\theta = 1$, Missing Informative, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

"Estimated Data". The "Complete Data" estimators are less bias than the "Estimated Data". Jackknifing corrects the bias better than adding one pair of the pseudo-tables in most of the cases, but the estimators of the both methods have less bias than Mantel-Haenszel estimators.

The "Complete Data" estimators have smaller variances than the "Estimated Data" estimators. Adding pseudotables yields smaller variances than Jackknifing for "Complete Data" and "Full Data", but the results are opposite for "Estimated Data". The different between two methods with the same data set are very small when the total observations are large, either large tables or lots of the strata. In the large tables or lots of the strata cases, the variances different between the "Complete Data" and "Estimated Data" are small as well, but the different gets larger when the total observations are reduced. Figure 5.11 display the Monte Carlo means of the estimators and Table B.5 shows the result numbers.

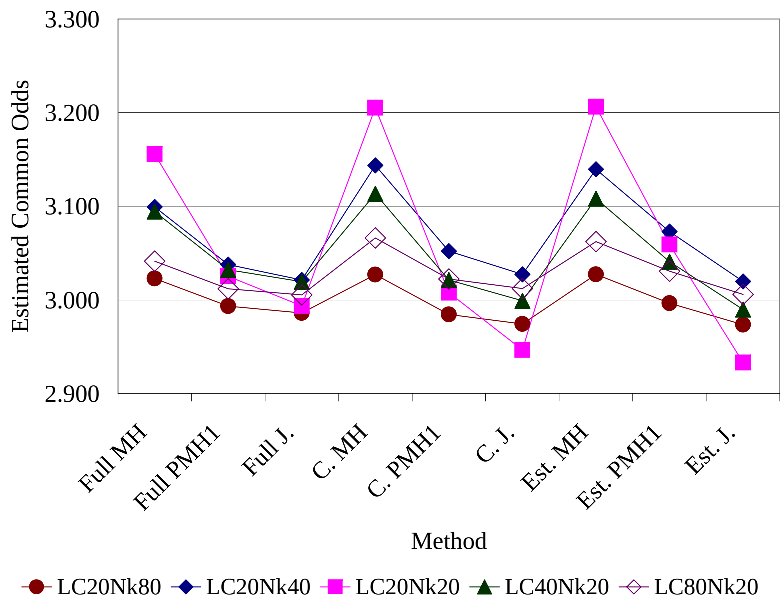

Figure 5.10: Mean: $\theta = 3$, Missing Informative, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

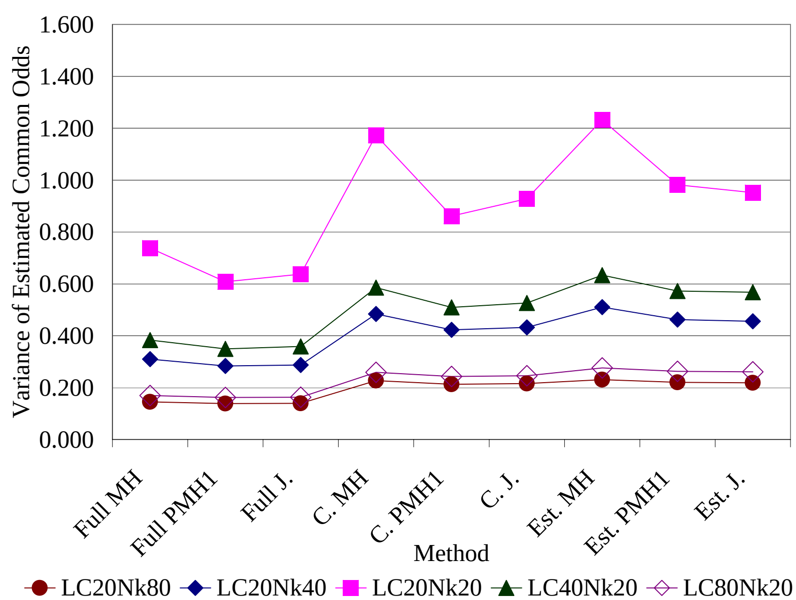

Figure 5.11: Variance: $\theta = 3$, Missing Informative, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

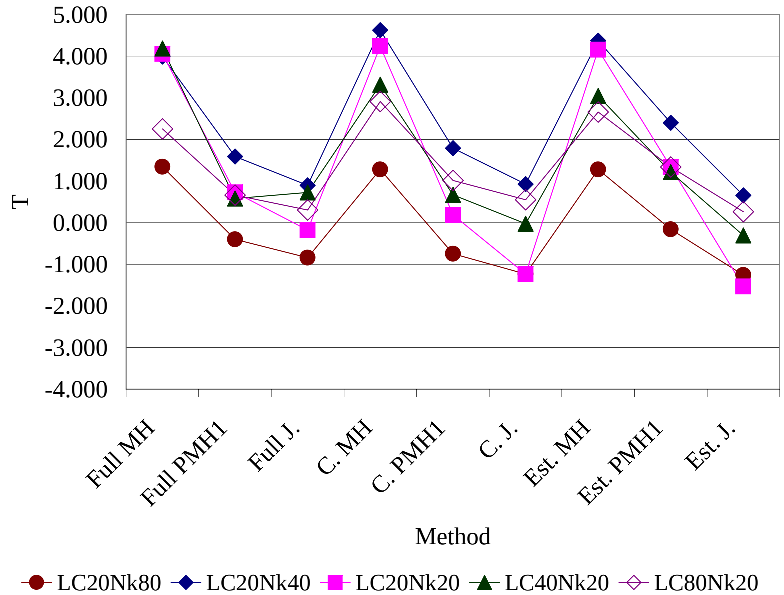

Figure 5.12: t: $\theta = 3$, Missing Informative, $P(missing)_{1k} = 0.15, P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

The T value range of using the adding one pair of pseudo-tables method for "Full Data" is between -0.839 and 1.58 and for "Complete Data" is between -1.236 and 1.753 both show that adding one pair of the pseudo-table method estimators are within the 95% confidence interval. The t value of using the adding one pair of pseduo-table method for "Estimated Data" are bigger than 1.96 which are significantly different from the true value for few cases, but Jackknifing method's t value range is between -1.709 and 1.097 which. The cases that $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k \geq K/2 + 1$ of the "Complete Data" and "Full Data" of Jackknifing method estimators are significantly different to the true values except when the number of strata is 20 and the table size is 80. The t values of these cases are very big with range of 2.657 to 4.373 for "Complete Data" and 2.908 to 4.624 for "Full Data". Figure 5.12 and Table B.6 display the Monte Carlo variances results.

5.4 Variance Estimation When A is MAR(B, C)

To estimate the variabilities of the estimators of the common odds ratio, we employ jackknifing, bootstrapping and the asymptotic variance formula for ratio estimates from Cochran [12]. Tables C.1 and C.3 are the results for the "Full Data" and the "Complete Data" of Mantel-Haenszel estimators and Mantel-Haenszel with one pair of the pseudo-tables estimators respectively. Table C.2 and C.4 are the results for the "Full Data" and the "Estimated Data" of Mantel-Haenszel estimators and Mantel-Haenszel with one pair of the pseudo-tables respectively. Figures 5.13, and 5.14 display the values in these tables of the cases that $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, K/2$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k = K/2 + 1, \cdots, K$. graphically. We compare the variance from the simulations for the common odds ratio when use Mantel-Haenszel method and note as "True". There is one simulation gets NA when using Bootstrapping method to estimate the variance for "Complete Data" or "Estimated Data" for $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.15$ for $k = 1, \cdots, 10$ and $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k = 11, \cdots, 20$ with 20 observations in each stratum. We excluded these cases to calculate Bootstrapping method average of the estimators and the variances.

The asymptotic formula estimators are very close to the "True" with bias range between 0.2% to 10% and average of 4% for "Full Data". The Jackknifing estimators are also not too far away from the "True" with 1% to 13% range of the bias for "Full Data". The bias for the Bootstrapping estimators was as large as 25% for the "Complete Data" and "Estimated Data" for 20 strata with 20 observations each cases. The variances were usually overestimated for all of the three estimating methods. The exceptions were when the asymptotic formula was used with when 20 strata with 40 observations each, no matter what data set and missingness model was used. In these cases, the Jackknifing and Bootstrapping methods sometimes overestimated and sometimes underestimated the variances, but the three estimators are all closed to the "True". When using "Full Data", the bias for Bootstrapping method was 10% and the other two methods had even smaller biases.

Adding one pair of pseudotables reduced the highest bias percentage from 25% to 20% when use "Complete Data" and from 27% to 23% when use "Estimated Data" when using Bootstrapping method and eliminated the zero divisor problem.

We Also examined the variances of the estimated variances in the simulation. The largest variances of the variance estimators are 0.019 without pseudotables and 0.0149 with pseudotables overall different missingness models, estimating methods and datasets. In general, the asymptotic formula estimators have smallest variation and the Bootstrapping method has the largest variances but all of the variances for the variance estimators are very small. Adding pseudotables might have larger variance for $\theta = 1$ but adding pseudotables get rid of the cases with zero divisors.

In general, the variances are increased when we use the estimated data. However, the increase in variance is quite small when the number of tables or the size of the tables is large. Figures 5.15 and 5.16 are the figures for the variance of the variance estimates and Tables C.5 and C.6 list the results.

Table C.7 and C.9 are the results for the "Full Data" and the "Complete Data" of Mantel-Haenszel estimators and Mantel-Haenszel with one pair of the pseudo-tables estimators respectively. Table C.8 and C.8 are the results for the "Full Data" and the "Estimated Data" of Mantel-Haenszel estimators and Mantel-Haenszel with one pair of the pseudo-tables respectively. Figures 5.17, and 5.18 are the figures for these tables.

The results for $\theta = 3$ are similar to $\theta = 1$. All of the three methods tend to overestimate the variance. The biases of the estimators increase when the total observations are reduced and the estimators are closed when the total observations are closed, regardless of the missingness models or the number of strata. However, there are several simulations get zero divisor cases when using

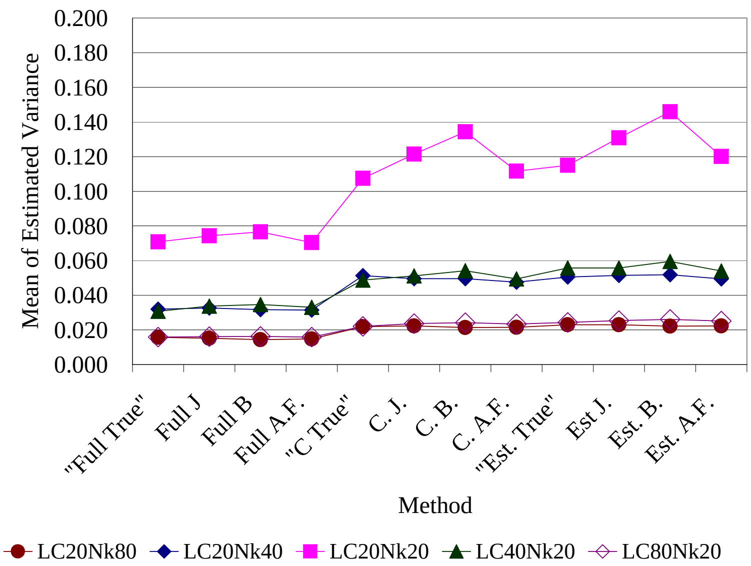

Figure 5.13: Estimating Variance: $\theta = 1$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

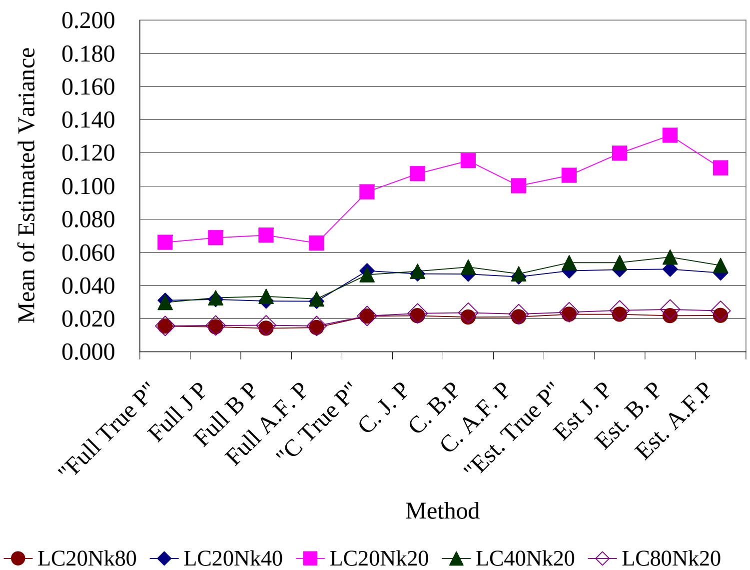

Figure 5.14: Estimating Variance: $\theta = 1$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$, With Pseudo-Tables

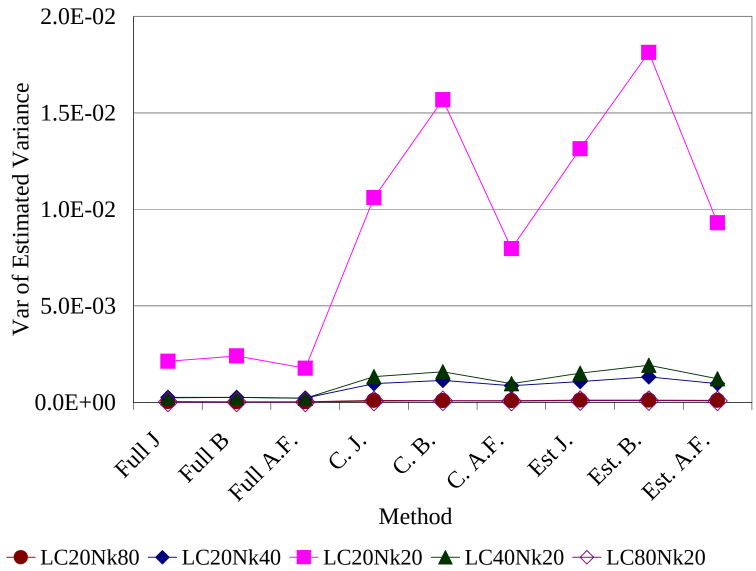

Figure 5.15: Variance of Estimating Variance: $\theta = 1$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

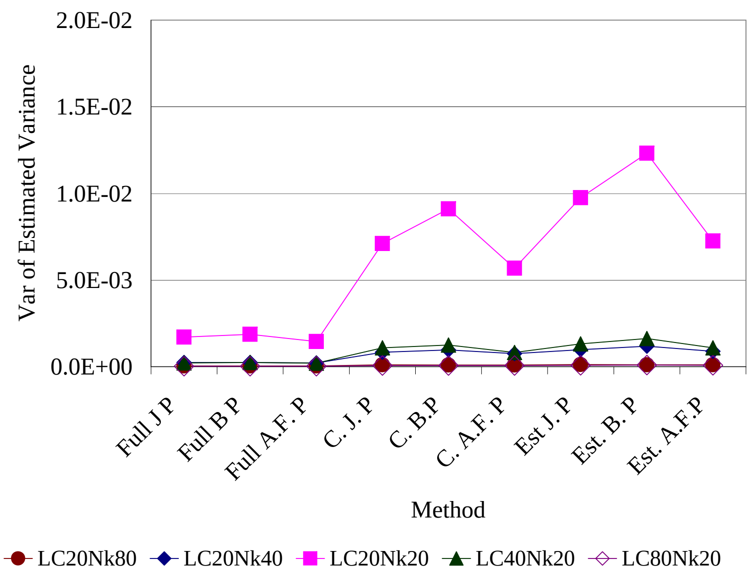

Figure 5.16: Variance of Estimating Variance: $\theta = 1$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$ With Pseudo-Table

Bootstrapping method to estimate the variance for "Complete Data" or "Estimated Data", when there are 20 tables with 20 observations each regardless of the missingness models.

The biases of the variance estimators are larger for $\theta = 3$ than $\theta = 1$. The zero divisors problems are cost more when $\theta = 3$ and increase the bias of the estimators when using Bootstrapping method for 20 strata with 20 observations each for all of the three data sets. The biases for Jackknifing estimators are also large for 20 strata with 20 observations each cases. The biases percentage ranges are wider for these two resampling methods, but not very different for the Asymptotic formula. Adding one pair of the pseudotables reduce the highest bias percentage from 68.22% to 45.7% for "Estimated Data", from 68.02% to 40.68% for "Complete Data" and 38.62% to 29.77% for "Full Data". The highest biases are different with or without the pseudo-tables. Also, there are no zero divisor cases when adding one pair of the pseudo-tables.

The asymptotic formula estimators have smallest variation and the Bootstrapping method has the largest variances. The variance of the estimated variances are instable for the 20 strata with 20 observations each cases for Bootstrapping when "Complete Data" or "Estimated Data" and were about 5 times as large as the asymptotic formula. The variance of the variance estimation for Jackknifing were as large as twice of the asymptotic formula. The variances for "Estimated Data" are larger than the "Complete Data" and not surprised, they are larger than the "Full Data" results. When the total observations are large, the differences between the "Estimated Data" and "Complete Data" are closed but the differences increase when the sample sizes are reduced. Adding one pair of the pseudotables halved of the variance estimation for "Complete Data" and "Estimated Data" for the asymptotic formula. The variance of variance estimations of Bootstrapping were reduced to twice to three times as large as the asymptotic formula results when one pair of pseudotables were added. Jackknifing results were about 25% to 50% higher than the asymptotic formula for "Complete Only" and 50% higher to double the variance of the variance estimations of asymptotic formula for "Estimated Data" when one pair of the pseudotables were

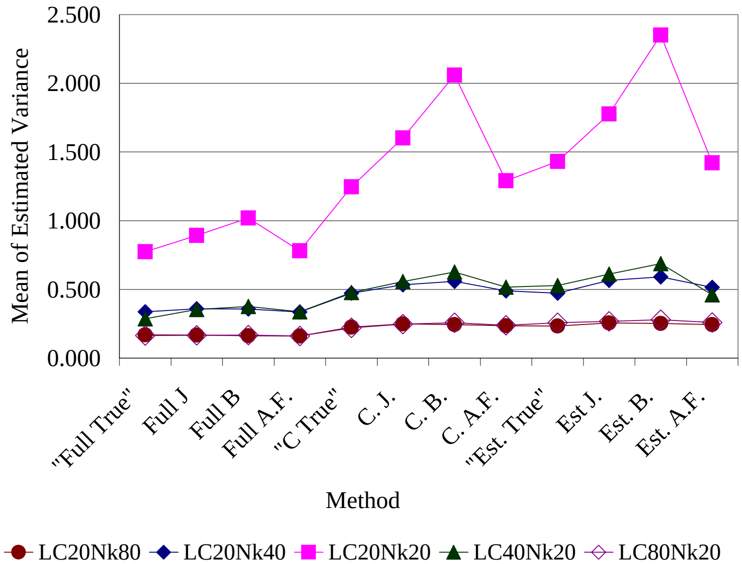

Figure 5.17: Estimating Variance: $\theta = 3$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

added.

Figures 5.19 and 5.20 are the figures for the variance of the Estimating Variances and Tables C.11 and C.12 list the results.

The results for $\theta = 5$ similar to $\theta = 3$ but the bias of the variance and the variabilities of the variance were both larger. Depending on the missingness model, there were 17 to 23 zero divisors cases when we simulated the 20 strata with 20 observations each when we used Bootstrapping method to estimated the variances for “Complete Only” and “Estimated Data”. There were few simulations had zero divisors problems when we use Bootstrapping for “Full Data” in all of the three missingness models. The zero divisor problem also happened once when there are 20 strata with 40 observations with missingness model of $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for $k = 1, \cdots, 10$ and $P(missing)_{1k} = 0.4$ and $P(missing)_{2k} = 0.15$ for $k = 11, \cdots, 20$ for “Complete Only” and “Estimated Data”. The variance of the variance estimations were more unstable than $\theta = 3$ and was larger than 1000 for the “Estimated Data” when the $P(missing)_{1k} = 0.15$ and $P(missing)_{2k} = 0.4$ for all of the strata when Jackknifing result was 111.5 and the asymptotic

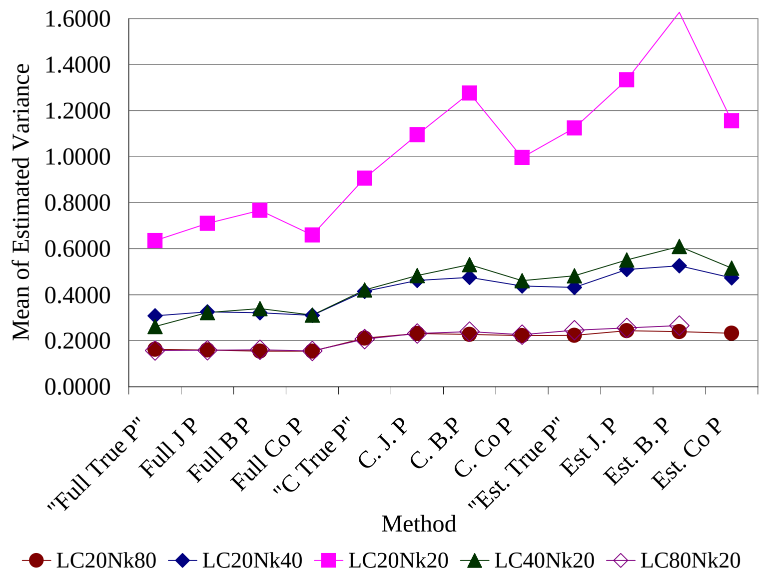

Figure 5.18: Estimating Variance: $\theta = 3$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$, With Pseudo-Tables

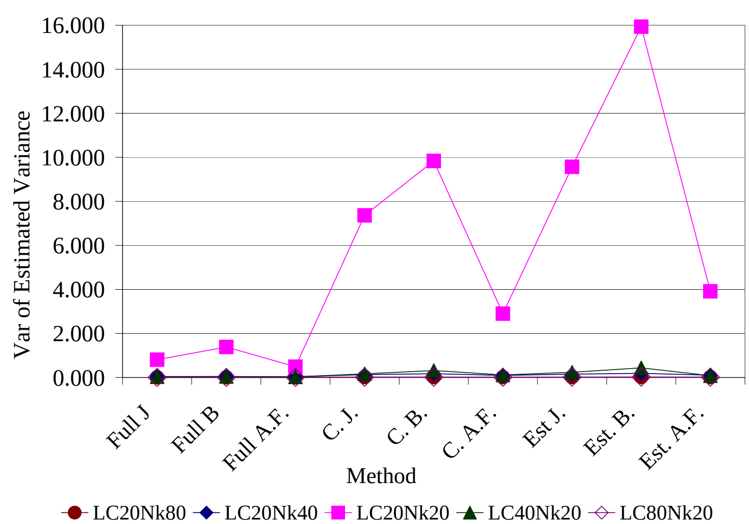

Figure 5.19: Variance of Estimating Variance: $\theta = 3$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$

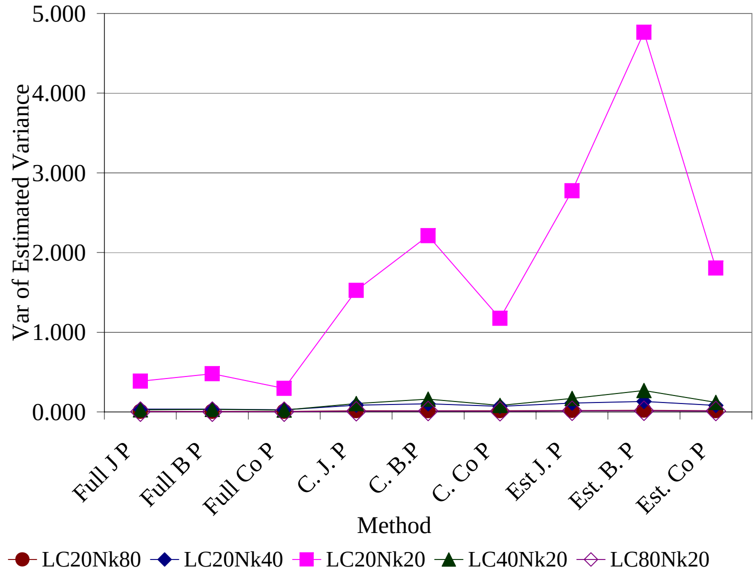

Figure 5.20: Variance of Estimating Variance: $\theta = 3$, MAR, $P(missing)_{1k} = 0.15$, $P(missing)_{2k} = 0.4$ for $k \leq K/2$, $P(missing)_{1k} = 0.4$, $P(missing)_{2k} = 0.15$ for $k > K/2$ With Pseudo-Table

formula result was 47.5. As before, adding one pair of the pseudotables reduced the biases and the variance of the variance estimations to reasonable range, especial for the Bootstrapping.

The table sizes we used for $\theta = 10$ were five times as large as the table sizes in the other three. That were 100, 200 and 400 instead of 20, 40 and 80 to prevent too many zero cell counts. The results for $\theta = 10$ were similar to the results we saw before except there were no zero divisors cases. The biases and the variance of the variance estimations for the 20 strata with 20 observations each cases were not as extreme as in $\theta = 5$ cases, but still much larger than the other 4 combinations.

5.5 Estimating Variance When A is Missing Informative(A, C)

The results for the estimated variance and the variance of the estimated variance were not much different between MAR(B, C) and missing informative (A, C).

All of the three methods tend to overestimate the variance and in general, the asymptotic formula estimators had smallest bias and variance and bootstrapping had the largest. Division by zero cases occurred in the simulations of bootstrapping for 20 strata with 20 observations each for all of the three common odds ratio, that is $\theta = 1, 3$ and 5. These 20 strata with 20 observations had large biases and unreasonable large variance when estimated the variance when we used bootstrapping method. Instability of the variance of the variance estimation situation also happened in the jackknifing estimators, but not as serious as bootstrapping. As we saw in MAR(B, C), adding one pair of the pseudotables reduce the biases and variance of the variance estimations.

The results tables for the variance estimations and the variances of the variance estimations for missing informative(A, C) are list in the Appendix D.

5.6 Multinomial Data

We also simulated cases where the "Full Data" were generated by the multinomial with parameters n_k, p_{11k}, p_{12k}, p_{21k}. To better compare with the independent binomial rows simulations, we simulated the cases that $p_{11k}+p_{12k} = 0.5$. The conditional probability $p(\text{column 1}|\text{row 1}) = p(1|1)$ is uniform $(0, 1)$, so $p_{11k} = 0.5p(1|1)$. To satisfied the common odds ration assumption, $p_{21k} = p_{11k}(1-p_{11k}-p_{12k})/(p_{11k}+p_{12k}\theta)$ and $p_{22k} = 1-p_{11k}-p_{12k}-p_{21k}$. As independent binomial rows cases, we simulated $\theta = 1, 3, 5$, and 10 for MAR(B, C) and $\theta = 1, 3$ and 5 for missing informative (A, C).

The results for the two generation methods were similar. The results tables for the estimators, variances and t-values for the common odds ratios when A is MAR(B, C) are in Appendix E and when A is missing informative (A, C) are in Appendix F. The variance estimators and variances of the variances estimations are in Appendix G for A is MAR(B, C) and Appendix H for A is missing informative(A, C).

Chapter 6

Summary and Future Research

6.1 Summary

The inputation method of Baker, Rosenberger and Dersimonian[5] was generalized to $I \times J \times K$ tables where the stratum variable C is always observed. In these cases, the closed-form estimates for $I \times J \times K$ tables are the same as the closed-form estimates for each individual $I \times J$ table.

Using Taylor expansions and exact multinomial moment calculations, expressions were derived for large-sample bias and variance of Mantel-Haenszel estimator and modified versions for full data, MAR, and informative missingness. All estimates considered in thesis have the following behaviors: the bias and variance are $O(1/N)$ for fixed K and θ when total number of observations $N \to \infty$ or $O(1/K)$ for fixed table size n_k and θ with $K \to \infty$; however, the bias and variance both will increase as the θ increases.

6.2 Simulation Findings

When variable A is MAR, $\hat{\theta}_{PMH1}$ and $\hat{\theta}_{JK}$ had reduced bias relative to $\hat{\theta}_{MH}$, and in most cases the bias was near 0. The estimators $\hat{\theta}_{PMH1}$ had smaller variance than $\hat{\theta}_{MH}$ and generally, $\hat{\theta}_{JK}$ and $\hat{\theta}_{PMH1}$ had similar variances.

For informative missingness, estimates based on complete data performed as they did in the MAR case. Estimates based on imputed data and correct missingness model (Baker, Rosenberger and Dersimonian[5]) showed nonnegligible bias. The MAR imputation when missingness was actually informative performed comparably to estimates based on complete data only: their bias and variance decreased with the sample size.

Asymptotic formulas gave more accurate estimates of $\text{Var}(\hat{\theta})$ than jackknifing or bootstrapping in almost all of the cases and estimators. Variance estimates based on bootstrapping were extremely unstable.

6.3 Conclusions

When the stratum variable C and the column variable B are always observed and only the row variable A might be missing, either MAR(B, C) or Informative(A, C), in a $2 \times 2 \times K$ contingency table, Mantel-Haenszel estimators with one pair of pseudotables or Mantel-Haenszel estimators with Jackknifing for "Complete Data" both perform well and have the advantage that we don't need to know a model of missingness.

On the other hand, where the row variable A is missing informatively (A, C), the common odds ratio estimators using the closed-form estimated data converge very slowly for large strata cases and don't converge to correct common odds ratio for sparse data. MAR(B, C) closed-form estimated data performs well in this case and no boundary estimates occur in the MAR(B, C) formula.

The asymptotic formula for variance of ratio with one pair of pseudotables generally have the smallest variance estimators and variances of variance estimators. However, the bootstrapping method generally have the largest estimators and might have zero divisors problems and unstable variances of variance estimators.

Our advice to practitioners is: use complete data estimators, using either pseudotable or jackknife to reduce both bias and variance, and use the asymptotic variance formula, not resampling methods. The easiest estimation for the common odds ratio estimate is Mantel-Haenszel with one pair of pseudotables, based on complete data and use asymptotic variance formula to estimate the variance.

6.4 Future Research

The behaviors of the common odds ratio estimators for "Complete Data" and the "Estimated Data" with the more complicated missingness models might be very different than when only one variable may be missing. Common odds ratio estimators for $I \times J \times K$ tables or higher dimension contingency tables are also of interest. The multiple logistic regressions with missing covariates problem is another interesting study topic.

Appendix A

Simulation Result Tables – MAR Model

Table A.1: Mean of Estimated Common Odds Ratio – MAR Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	1.013	1.013	1.005	1.016	1.016	1.005	1.015	1.015	1.003
40	20	1.008	1.008	0.993	1.012	1.011	0.989	1.012	1.011	0.987
20	20	1.013	1.012	0.980	1.034	1.031	0.986	1.038	1.036	0.986
20	40	1.012	1.012	0.997	1.017	1.016	0.994	1.019	1.018	0.994
20	80	1.006	1.006	0.999	1.008	1.007	0.997	1.007	1.007	0.996
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	1.008	1.008	1.001	1.018	1.018	1.007	1.019	1.019	1.007
40	20	1.007	1.007	0.992	1.012	1.011	0.990	1.017	1.016	0.992
20	20	1.037	1.035	1.004	1.056	1.052	1.006	1.062	1.058	1.005
20	40	1.015	1.014	0.998	1.018	1.017	0.996	1.019	1.018	0.995
20	80	1.001	1.000	0.993	0.996	0.996	0.986	0.998	0.998	0.987
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	1.015	1.015	1.007	1.018	1.018	1.007	1.019	1.018	1.006
40	20	1.030	1.030	1.014	1.034	1.033	1.010	1.038	1.037	1.012
20	20	1.049	1.046	1.015	1.068	1.063	1.015	1.071	1.067	1.014
20	40	1.017	1.017	1.002	1.030	1.029	1.007	1.027	1.027	1.003
20	80	1.003	1.003	0.996	1.005	1.005	0.994	1.005	1.005	0.994

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Appendix B

Simulation Result Tables – Informative Model

Table A.2: Variance of Estimated Common Odds Ratio – MAR Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.015	0.015	0.015	0.020	0.019	0.019	0.022	0.021	0.021
40	20	0.035	0.034	0.034	0.048	0.046	0.045	0.052	0.050	0.048
20	20	0.065	0.061	0.060	0.097	0.088	0.087	0.106	0.099	0.093
20	40	0.035	0.034	0.034	0.049	0.047	0.046	0.051	0.049	0.048
20	80	0.016	0.015	0.015	0.022	0.021	0.021	0.022	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.015	0.015	0.015	0.024	0.024	0.023	0.026	0.026	0.026
40	20	0.031	0.030	0.030	0.043	0.041	0.040	0.047	0.045	0.044
20	20	0.067	0.063	0.063	0.109	0.099	0.096	0.124	0.115	0.106
20	40	0.035	0.033	0.033	0.046	0.044	0.043	0.049	0.047	0.046
20	80	0.014	0.014	0.013	0.020	0.020	0.019	0.021	0.021	0.021
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$ and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.016	0.016	0.015	0.022	0.022	0.021	0.024	0.024	0.023
40	20	0.031	0.030	0.030	0.049	0.046	0.045	0.056	0.054	0.051
20	20	0.071	0.066	0.066	0.107	0.096	0.094	0.115	0.106	0.099
20	40	0.032	0.031	0.031	0.051	0.049	0.048	0.050	0.049	0.047
20	80	0.016	0.016	0.015	0.022	0.021	0.021	0.023	0.023	0.022

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table A.3: T value for Estimated Common Odds Ratio – MAR Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.373	2.364	0.975	2.524	2.508	0.743	2.346	2.335	0.505
40	20	0.991	0.959	-0.888	1.184	1.131	-1.212	1.154	1.115	-1.307
20	20	1.120	1.044	-1.842	2.436	2.308	-1.074	2.641	2.551	-1.013
20	40	1.485	1.455	-0.407	1.716	1.665	-0.617	1.853	1.817	-0.572
20	80	1.151	1.141	-0.262	1.158	1.141	-0.537	1.067	1.055	-0.651
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	1.508	1.497	0.112	2.597	2.580	0.997	2.637	2.624	0.962
40	20	0.902	0.871	-1.093	1.311	1.263	-1.137	1.745	1.710	-0.802
20	20	3.223	3.155	0.359	3.810	3.692	0.403	3.917	3.828	0.372
20	40	1.758	1.727	-0.232	1.903	1.853	-0.475	1.902	1.865	-0.516
20	80	0.103	0.093	-1.341	-0.556	-0.572	-2.284	-0.325	-0.337	-2.094
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$ and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.609	2.599	1.230	2.749	2.732	1.047	2.672	2.660	0.928
40	20	3.879	3.854	1.881	3.454	3.408	1.075	3.595	3.560	1.209
20	20	4.102	4.030	1.289	4.628	4.505	1.069	4.663	4.576	0.967
20	40	2.177	2.149	0.193	3.001	2.956	0.743	2.731	2.699	0.342
20	80	0.565	0.555	-0.791	0.802	0.784	-0.865	0.798	0.785	-0.875

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table A.4: Mean of Estimated Common Odds Ratio – MAR Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	3.043	3.013	3.007	3.079	3.036	3.026	3.087	3.056	3.030
40	20	3.047	2.988	2.973	3.072	2.986	2.962	3.087	3.024	2.970
20	20	3.102	2.977	2.943	3.188	2.997	2.941	3.191	3.050	2.924
20	40	3.029	2.971	2.957	3.062	2.977	2.956	3.071	3.010	2.962
20	80	3.051	3.021	3.015	3.054	3.013	3.004	3.054	3.024	3.002
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	3.031	3.002	2.995	3.062	3.020	3.009	3.061	3.030	3.003
40	20	3.083	3.022	3.007	3.080	2.994	2.970	3.091	3.028	2.970
20	20	3.155	3.027	2.995	3.298	3.097	3.037	3.311	3.158	3.018
20	40	3.070	3.009	2.997	3.079	2.994	2.977	3.079	3.016	2.967
20	80	3.045	3.016	3.010	3.056	3.015	3.007	3.055	3.025	3.004
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$ and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	3.036	3.007	3.001	3.047	3.004	2.995	3.058	3.027	3.001
40	20	3.041	2.982	2.967	3.081	2.993	2.968	3.104	3.040	2.982
20	20	3.159	3.028	2.990	3.247	3.047	2.975	3.281	3.131	2.987
20	40	3.077	3.017	3.003	3.104	3.015	2.997	3.113	3.050	3.000
20	80	3.048	3.018	3.012	3.052	3.009	2.999	3.062	3.031	3.008

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table A.5: Variance of Estimated Common Odds Ratio – MAR Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.163	0.156	0.158	0.252	0.237	0.239	0.270	0.258	0.256
40	20	0.358	0.327	0.333	0.520	0.458	0.469	0.580	0.528	0.522
20	20	0.668	0.557	0.573	1.144	0.853	0.877	1.227	0.989	0.930
20	40	0.289	0.265	0.272	0.439	0.387	0.403	0.464	0.424	0.425
20	80	0.147	0.141	0.143	0.205	0.193	0.196	0.211	0.202	0.201
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.147	0.141	0.143	0.248	0.233	0.236	0.268	0.257	0.254
40	20	0.308	0.282	0.288	0.476	0.419	0.431	0.514	0.469	0.461
20	20	0.715	0.592	0.623	1.285	0.955	0.983	1.501	1.196	1.136
20	40	0.303	0.278	0.284	0.441	0.390	0.400	0.502	0.458	0.456
20	80	0.162	0.156	0.158	0.230	0.217	0.222	0.235	0.226	0.226
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$ and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.164	0.157	0.159	0.221	0.207	0.210	0.258	0.246	0.244
40	20	0.286	0.262	0.265	0.478	0.421	0.425	0.529	0.482	0.466
20	20	0.774	0.635	0.650	1.246	0.907	0.915	1.432	1.125	1.055
20	40	0.337	0.309	0.315	0.472	0.415	0.429	0.473	0.432	0.426
20	80	0.169	0.162	0.163	0.224	0.211	0.212	0.234	0.224	0.221

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table A.6: T value for Estimated Common Odds Ratio – MAR Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.383	0.762	0.384	3.529	1.657	1.203	3.733	2.445	1.337
40	20	1.773	-0.473	-1.035	2.227	-0.479	-1.240	2.546	0.723	-0.931
20	20	2.789	-0.678	-1.687	3.933	-0.079	-1.401	3.847	1.115	-1.760
20	40	1.220	-1.272	-1.843	2.083	-0.825	-1.552	2.343	0.336	-1.315
20	80	2.971	1.270	0.883	2.690	0.655	0.204	2.605	1.172	0.104
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	1.817	0.120	-0.289	2.785	0.918	0.433	2.642	1.345	0.150
40	20	3.324	0.913	0.293	2.586	-0.213	-1.008	2.844	0.900	-0.996
20	20	4.104	0.780	-0.156	5.876	2.222	0.841	5.671	3.229	0.385
20	40	2.842	0.400	-0.114	2.666	-0.203	-0.814	2.481	0.538	-1.082
20	80	2.510	0.880	0.575	2.620	0.702	0.321	2.546	1.181	0.177
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$ and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.013	0.393	0.053	2.256	0.216	-0.248	2.540	1.210	0.045
40	20	1.730	-0.767	-1.437	2.616	-0.239	-1.086	3.181	1.277	-0.591
20	20	4.032	0.790	-0.264	4.939	1.113	-0.581	5.246	2.755	-0.283
20	40	2.981	0.669	0.138	3.369	0.536	-0.108	3.672	1.690	0.003
20	80	2.625	1.017	0.685	2.458	0.460	-0.036	2.854	1.475	0.371

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table B.1: Mean of Estimated Common Odds Ratio – Informative (A, C) for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	1.001	1.001	0.994	1.006	1.006	0.995	1.009	1.008	0.997
40	20	1.026	1.025	1.010	1.038	1.036	1.014	1.040	1.039	1.015
20	20	1.032	1.030	0.998	1.049	1.045	0.997	1.058	1.054	1.001
20	40	1.015	1.015	0.999	1.029	1.028	1.006	1.028	1.027	1.004
20	80	1.010	1.010	1.002	1.009	1.008	0.997	1.008	1.008	0.996
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.999	0.999	0.992	1.000	1.000	0.989	1.001	1.001	0.989
40	20	1.013	1.013	0.997	1.018	1.017	0.996	1.024	1.023	0.999
20	20	1.024	1.023	0.991	1.044	1.040	0.995	1.053	1.049	0.999
20	40	1.021	1.021	1.005	1.035	1.034	1.012	1.035	1.034	1.010
20	80	1.003	1.003	0.995	1.006	1.006	0.995	1.006	1.006	0.995
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	1.004	1.004	0.997	1.001	1.001	0.990	0.999	0.999	0.988
40	20	1.019	1.018	1.003	1.018	1.017	0.994	1.021	1.020	0.996
20	20	1.039	1.037	1.006	1.053	1.049	1.003	1.052	1.049	0.999
20	40	1.012	1.011	0.995	1.026	1.025	1.001	1.027	1.026	1.002
20	80	1.013	1.013	1.005	1.016	1.016	1.005	1.015	1.015	1.004

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table B.2: Variance of Estimated Common Odds Ratio – Informative (A, C) for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.015	0.014	0.014	0.021	0.021	0.021	0.024	0.024	0.023
40	20	0.033	0.032	0.032	0.047	0.045	0.044	0.050	0.049	0.047
20	20	0.065	0.060	0.058	0.100	0.090	0.086	0.106	0.099	0.089
20	40	0.036	0.035	0.034	0.052	0.050	0.049	0.052	0.051	0.049
20	80	0.019	0.018	0.018	0.025	0.024	0.024	0.025	0.024	0.024
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \le K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.014	0.014	0.014	0.020	0.019	0.019	0.022	0.021	0.021
40	20	0.034	0.033	0.033	0.052	0.050	0.049	0.057	0.055	0.053
20	20	0.060	0.056	0.055	0.095	0.086	0.083	0.109	0.101	0.095
20	40	0.034	0.033	0.033	0.050	0.048	0.047	0.052	0.051	0.049
20	80	0.016	0.015	0.015	0.022	0.021	0.021	0.023	0.023	0.022
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \le K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.015	0.015	0.015	0.022	0.022	0.022	0.024	0.023	0.023
40	20	0.033	0.032	0.031	0.046	0.044	0.043	0.049	0.047	0.046
20	20	0.079	0.074	0.073	0.111	0.100	0.097	0.113	0.105	0.098
20	40	0.033	0.032	0.031	0.047	0.045	0.044	0.050	0.048	0.046
20	80	0.015	0.015	0.015	0.021	0.021	0.021	0.022	0.021	0.021

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table B.3: T value for Estimated Common Odds Ratio – Informative (A, C) for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.252	0.241	-1.160	0.956	0.938	-0.781	1.232	1.218	-0.489
40	20	0.142	0.141	0.056	0.173	0.171	0.065	0.180	0.178	0.070
20	20	0.125	0.121	-0.010	0.156	0.150	-0.012	0.177	0.173	0.005
20	40	0.080	0.078	-0.004	0.127	0.124	0.025	0.123	0.121	0.018
20	80	0.076	0.075	0.017	0.055	0.054	-0.018	0.050	0.050	-0.023
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	-0.114	-0.124	-1.572	0.013	-0.003	-1.757	0.188	0.175	-1.637
40	20	1.586	1.555	-0.309	1.788	1.737	-0.433	2.235	2.196	-0.096
20	20	2.235	2.157	-0.816	3.191	3.063	-0.387	3.573	3.479	-0.095
20	40	2.592	2.563	0.675	3.487	3.442	1.204	3.456	3.421	0.995
20	80	0.524	0.513	-0.947	0.915	0.897	-0.793	0.928	0.916	-0.788
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.764	0.754	-0.618	0.111	0.093	-1.579	-0.082	-0.096	-1.805
40	20	2.295	2.266	0.339	1.830	1.781	-0.620	2.090	2.053	-0.373
20	20	3.123	3.053	0.496	3.560	3.441	0.186	3.443	3.356	-0.056
20	40	1.459	1.430	-0.589	2.674	2.625	0.153	2.724	2.688	0.200
20	80	2.392	2.384	0.923	2.451	2.436	0.714	2.349	2.338	0.589

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Appendix C

Simulation Result Tables – Variance Estimation for MAR(B, C) Model

Table B.4: Simulation Results – Informative (A, C): Estimated Common Odds Ratio for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	3.022	2.993	2.986	3.045	3.002	2.991	3.045	3.014	2.989
40	20	3.104	3.042	3.028	3.153	3.058	3.033	3.163	3.093	3.037
20	20	3.131	3.003	2.966	3.201	3.003	2.924	3.213	3.063	2.922
20	40	3.054	2.994	2.978	3.102	3.013	2.986	3.111	3.045	2.992
20	80	3.033	3.004	2.998	3.038	2.996	2.984	3.043	3.012	2.990
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	3.040	3.010	3.003	3.051	3.009	2.999	3.041	3.010	2.984
40	20	3.097	3.036	3.022	3.119	3.031	3.010	3.148	3.082	3.026
20	20	3.182	3.051	3.018	3.219	3.027	2.966	3.268	3.121	2.984
20	40	3.076	3.015	3.000	3.099	3.012	2.991	3.115	3.051	2.998
20	80	3.021	2.992	2.986	3.055	3.014	3.005	3.057	3.026	3.003
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	3.041	3.012	3.005	3.066	3.022	3.012	3.062	3.031	3.006
40	20	3.094	3.032	3.019	3.113	3.021	2.999	3.108	3.041	2.990
20	20	3.156	3.025	2.994	3.205	3.008	2.947	3.206	3.059	2.933
20	40	3.099	3.038	3.021	3.144	3.052	3.027	3.140	3.073	3.020
20	80	3.023	2.993	2.986	3.027	2.985	2.974	3.027	2.997	2.974

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table B.5: Variance of Estimated Common Odds Ratio – Informative (A, C) for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.151	0.145	0.146	0.220	0.207	0.210	0.226	0.216	0.215
40	20	0.385	0.352	0.358	0.636	0.555	0.572	0.641	0.581	0.575
20	20	0.750	0.620	0.640	1.324	0.988	0.978	1.401	1.124	1.029
20	40	0.337	0.309	0.315	0.524	0.459	0.473	0.547	0.496	0.491
20	80	0.164	0.158	0.159	0.250	0.236	0.237	0.254	0.243	0.241
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.179	0.172	0.173	0.269	0.254	0.257	0.285	0.273	0.271
40	20	0.296	0.271	0.278	0.469	0.413	0.427	0.531	0.483	0.479
20	20	0.792	0.655	0.668	1.228	0.922	0.968	1.376	1.114	1.073
20	40	0.364	0.332	0.337	0.490	0.430	0.442	0.525	0.478	0.472
20	80	0.176	0.169	0.172	0.252	0.237	0.241	0.266	0.254	0.254
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.169	0.162	0.163	0.259	0.243	0.246	0.275	0.263	0.261
40	20	0.383	0.349	0.358	0.586	0.509	0.526	0.633	0.572	0.568
20	20	0.738	0.608	0.637	1.173	0.861	0.928	1.232	0.982	0.952
20	40	0.310	0.283	0.287	0.483	0.423	0.432	0.510	0.462	0.456
20	80	0.145	0.139	0.140	0.226	0.212	0.216	0.230	0.220	0.219

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table B.6: T value for Estimated Common Odds Ratio – Informative (A, C) for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	1.248	-0.432	-0.803	2.135	0.093	-0.449	2.125	0.664	-0.545
40	20	3.758	1.580	1.061	4.296	1.753	0.987	4.550	2.728	1.097
20	20	3.393	0.086	-0.945	3.898	0.065	-1.716	4.019	1.329	-1.709
20	40	2.092	-0.226	-0.879	3.162	0.415	-0.456	3.354	1.417	-0.257
20	80	1.846	0.231	-0.132	1.706	-0.184	-0.712	1.930	0.567	-0.473
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.092	0.541	0.179	2.184	0.383	-0.052	1.712	0.447	-0.708
40	20	3.999	1.547	0.938	4.216	1.087	0.339	4.553	2.654	0.836
20	20	4.575	1.405	0.481	4.414	0.637	-0.772	5.114	2.558	-0.336
20	40	2.802	0.566	0.002	3.150	0.410	-0.319	3.556	1.663	-0.055
20	80	1.111	-0.456	-0.750	2.463	0.628	0.220	2.458	1.166	0.150
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.253	0.659	0.303	2.908	1.014	0.548	2.657	1.336	0.261
40	20	4.182	0.582	0.728	3.314	0.669	-0.027	3.043	1.211	-0.305
20	20	4.053	0.724	-0.177	4.237	0.184	-1.236	4.157	1.337	-1.532
20	40	3.989	1.588	0.893	4.624	1.790	0.927	4.373	2.399	0.654
20	80	1.340	-0.402	-0.839	1.276	-0.746	-1.236	1.276	-0.160	-1.259

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table C.1: Variance Estimation – MAR Model for $\theta = 1$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"*True*"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.015	0.016	0.016	0.016	0.020	0.023	0.024	0.023
40	20	0.035	0.032	0.033	0.032	0.048	0.048	0.051	0.047
20	20	0.065	0.070	0.073	0.067	0.097	0.108	0.119	0.101
20	40	0.035	0.033	0.032	0.032	0.049	0.049	0.049	0.047
20	80	0.016	0.016	0.015	0.016	0.022	0.023	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.015	0.016	0.016	0.016	0.024	0.023	0.024	0.023
40	20	0.031	0.033	0.034	0.032	0.043	0.047	0.050	0.046
20	20	0.067	0.073	0.076	0.070	0.109	0.116	0.126	0.107
20	40	0.035	0.034	0.033	0.033	0.046	0.048	0.048	0.046
20	80	0.014	0.015	0.015	0.015	0.020	0.022	0.021	0.021
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.016	0.016	0.016	0.016	0.022	0.024	0.024	0.023
40	20	0.031	0.034	0.035	0.033	0.049	0.051	0.054	0.049
20	20	0.071	0.074	0.077	0.070	0.107	0.121	0.134	0.112
20	40	0.032	0.033	0.032	0.031	0.051	0.050	0.050	0.047
20	80	0.016	0.015	0.014	0.015	0.022	0.022	0.021	0.022

[1] "*True*"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.2: Variance Estimation – MAR Model for $\theta = 1$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"$True$"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.015	0.016	0.016	0.016	0.022	0.025	0.026	0.025
40	20	0.035	0.032	0.033	0.032	0.052	0.052	0.055	0.050
20	20	0.065	0.070	0.073	0.067	0.106	0.119	0.134	0.111
20	40	0.035	0.033	0.032	0.032	0.051	0.051	0.052	0.049
20	80	0.016	0.016	0.015	0.016	0.022	0.024	0.023	0.023
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.015	0.016	0.016	0.016	0.026	0.026	0.026	0.025
40	20	0.031	0.033	0.034	0.032	0.047	0.052	0.055	0.051
20	20	0.067	0.073	0.076	0.070	0.124	0.131	0.146	0.121
20	40	0.035	0.034	0.033	0.033	0.049	0.050	0.051	0.048
20	80	0.014	0.015	0.015	0.015	0.021	0.023	0.022	0.023
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.016	0.016	0.016	0.016	0.024	0.025	0.026	0.025
40	20	0.031	0.034	0.035	0.033	0.056	0.056	0.060	0.054
20	20	0.071	0.074	0.077	0.070	0.115	0.131	0.146	0.120
20	40	0.032	0.033	0.032	0.031	0.050	0.051	0.052	0.049
20	80	0.016	0.015	0.014	0.015	0.023	0.023	0.022	0.022

[1] "$True$"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.3: Variance Estimation – MAR Model for $\theta = 1$ Full Data vs. Complete Only With Pseudo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"*True*"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.015	0.016	0.016	0.016	0.019	0.023	0.023	0.022
40	20	0.034	0.031	0.032	0.031	0.046	0.046	0.048	0.045
20	20	0.061	0.065	0.067	0.062	0.088	0.097	0.104	0.091
20	40	0.034	0.032	0.031	0.031	0.047	0.046	0.046	0.044
20	80	0.015	0.016	0.015	0.015	0.021	0.022	0.021	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.015	0.016	0.016	0.016	0.024	0.023	0.023	0.022
40	20	0.030	0.032	0.032	0.031	0.041	0.045	0.047	0.044
20	20	0.063	0.068	0.070	0.065	0.099	0.103	0.110	0.096
20	40	0.033	0.033	0.032	0.032	0.044	0.045	0.045	0.043
20	80	0.014	0.015	0.014	0.015	0.020	0.021	0.020	0.021
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.016	0.016	0.016	0.016	0.022	0.023	0.024	0.023
40	20	0.030	0.033	0.033	0.032	0.046	0.049	0.051	0.047
20	20	0.066	0.069	0.070	0.066	0.096	0.107	0.115	0.100
20	40	0.031	0.032	0.031	0.030	0.049	0.047	0.047	0.045
20	80	0.016	0.015	0.014	0.015	0.021	0.022	0.021	0.021

[1] "*True*"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.4: Variance Estimation – MAR Model for $\theta = 1$ Full Data vs. Estimated Data With Pseudo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"$True$"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.015	0.016	0.016	0.016	0.021	0.025	0.025	0.024
40	20	0.034	0.031	0.032	0.031	0.050	0.050	0.053	0.049
20	20	0.061	0.065	0.067	0.062	0.099	0.109	0.121	0.103
20	40	0.034	0.032	0.031	0.031	0.049	0.050	0.050	0.048
20	80	0.015	0.016	0.015	0.015	0.022	0.023	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.015	0.016	0.016	0.016	0.026	0.025	0.026	0.025
40	20	0.030	0.032	0.032	0.031	0.045	0.050	0.053	0.049
20	20	0.063	0.068	0.070	0.065	0.115	0.120	0.133	0.112
20	40	0.033	0.033	0.032	0.032	0.047	0.048	0.049	0.047
20	80	0.014	0.015	0.014	0.015	0.021	0.023	0.022	0.022
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.016	0.016	0.016	0.016	0.024	0.025	0.026	0.025
40	20	0.030	0.033	0.033	0.032	0.054	0.054	0.057	0.052
20	20	0.066	0.069	0.070	0.066	0.106	0.120	0.131	0.111
20	40	0.031	0.032	0.031	0.030	0.049	0.050	0.050	0.048
20	80	0.016	0.015	0.014	0.015	0.023	0.023	0.022	0.022

[1] "$True$"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.5: Variance of Variance Estimation – MAR Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.3E-5	2.4E-5	2.2E-5	5.7E-5	6.3E-5	5.5E-5	7.0E-5	7.7E-5	6.7E-5
40	20	2.4E-4	2.6E-4	2.2E-4	6.7E-4	7.9E-4	6.0E-4	7.8E-4	9.6E-4	7.1E-4
20	20	2.2E-3	2.6E-3	1.9E-3	7.0E-3	9.6E-3	5.3E-3	1.1E-2	1.7E-2	7.9E-3
20	40	3.1E-4	3.1E-4	2.8E-4	7.9E-4	8.7E-4	7.0E-4	1.0E-3	1.2E-3	9.2E-4
20	80	5.4E-5	4.7E-5	4.8E-5	1.2E-4	1.1E-4	1.0E-4	1.3E-4	1.2E-4	1.1E-4
		$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$								
80	20	2.2E-5	2.5E-5	2.2E-5	7.4E-5	8.2E-5	7.1E-5	9.3E-5	1.1E-4	9.0E-5
40	20	2.3E-4	2.7E-4	2.2E-4	6.1E-4	7.2E-4	5.4E-4	7.9E-4	9.7E-4	7.1E-4
20	20	2.2E-3	2.6E-3	1.9E-3	9.6E-3	1.3E-2	7.2E-3	1.4E-2	1.9E-2	1.0E-2
20	40	4.2E-4	4.2E-4	3.6E-4	9.8E-4	1.1E-3	8.2E-4	1.1E-3	1.2E-3	9.2E-4
20	80	4.9E-5	4.4E-5	4.4E-5	1.2E-4	1.1E-4	1.0E-4	1.3E-4	1.3E-4	1.2E-4
		$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$								
80	20	2.7E-5	2.8E-5	2.6E-5	7.3E-5	7.9E-5	7.0E-5	9.1E-5	1.0E-4	8.7E-5
40	20	2.4E-4	2.7E-4	2.2E-4	1.3E-3	1.6E-3	9.8E-4	1.5E-3	1.9E-3	1.2E-3
20	20	2.1E-3	2.4E-3	1.8E-3	1.1E-2	1.6E-2	8.0E-3	1.3E-2	1.8E-2	9.3E-3
20	40	2.6E-4	2.6E-4	2.4E-4	9.7E-4	1.1E-3	8.6E-4	1.1E-3	1.3E-3	9.7E-4
20	80	4.4E-5	3.9E-5	4.0E-5	1.1E-4	1.0E-4	9.7E-5	1.2E-4	1.2E-4	1.1E-4

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.6: Variance of Variance Estimation – MAR Model for $\theta = 1$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.2E-5	2.3E-5	2.1E-5	5.4E-5	5.8E-5	5.1E-5	6.7E-5	7.4E-5	6.4E-5
40	20	2.2E-4	2.4E-4	2.0E-4	5.8E-4	6.7E-4	5.2E-4	7.1E-4	8.6E-4	6.4E-4
20	20	1.8E-3	2.0E-3	1.5E-3	4.9E-3	6.1E-3	3.9E-3	8.4E-3	1.2E-2	6.2E-3
20	40	2.8E-4	2.8E-4	2.5E-4	6.8E-4	7.4E-4	6.2E-4	9.4E-4	1.1E-3	8.3E-4
20	80	5.2E-5	4.5E-5	4.6E-5	1.1E-4	1.0E-4	9.8E-5	1.2E-4	1.2E-4	1.1E-4
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.2E-5	2.4E-5	2.1E-5	6.9E-5	7.6E-5	6.6E-5	8.9E-5	1.0E-4	8.5E-5
40	20	2.1E-4	2.4E-4	2.0E-4	5.3E-4	6.2E-4	4.8E-4	7.1E-4	8.6E-4	6.5E-4
20	20	1.8E-3	2.0E-3	1.6E-3	6.7E-3	8.4E-3	5.2E-3	1.1E-2	1.5E-2	8.2E-3
20	40	3.7E-4	3.7E-4	3.3E-4	8.4E-4	9.0E-4	7.2E-4	9.4E-4	1.1E-3	8.3E-4
20	80	4.7E-5	4.2E-5	4.2E-5	1.1E-4	1.0E-4	9.8E-5	1.2E-4	1.2E-4	1.1E-4
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.6E-5	2.7E-5	2.5E-5	6.8E-5	7.3E-5	6.5E-5	8.6E-5	9.5E-5	8.3E-5
40	20	2.2E-4	2.4E-4	2.0E-4	1.1E-3	1.3E-3	8.2E-4	1.3E-3	1.6E-3	1.1E-3
20	20	1.7E-3	1.9E-3	1.5E-3	7.1E-3	9.1E-3	5.7E-3	9.8E-3	1.2E-2	7.3E-3
20	40	2.4E-4	2.4E-4	2.2E-4	8.4E-4	9.7E-4	7.5E-4	9.8E-4	1.2E-3	8.8E-4
20	80	4.2E-5	3.8E-5	3.8E-5	1.0E-4	9.6E-5	9.1E-5	1.2E-4	1.1E-4	1.0E-4

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.7: Variance Estimation – MAR Model for $\theta = 3$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"*True*"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.163	0.168	0.171	0.164	0.252	0.251	0.260	0.243
40	20	0.358	0.355	0.375	0.337	0.520	0.537	0.590	0.500
20	20	0.668	0.817	0.926	0.727	1.144	1.406	1.805	1.174
20	40	0.289	0.346	0.346	0.325	0.439	0.517	0.535	0.475
20	80	0.147	0.168	0.162	0.161	0.205	0.240	0.234	0.228
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.147	0.166	0.169	0.162	0.248	0.247	0.256	0.238
40	20	0.308	0.362	0.380	0.344	0.476	0.537	0.586	0.497
20	20	0.715	0.835	0.942	0.749	1.285	1.557	2.159	1.286
20	40	0.303	0.353	0.354	0.332	0.441	0.512	0.527	0.468
20	80	0.162	0.165	0.159	0.159	0.230	0.237	0.231	0.225
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.164	0.165	0.168	0.162	0.221	0.247	0.257	0.239
40	20	0.286	0.354	0.375	0.337	0.478	0.557	0.626	0.517
20	20	0.774	0.892	1.020	0.780	1.246	1.602	2.060	1.291
20	40	0.337	0.359	0.357	0.335	0.472	0.534	0.559	0.490
20	80	0.169	0.168	0.163	0.161	0.224	0.249	0.244	0.236

[1] "*True*"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.8: Variance Estimation – MAR Model for $\theta = 3$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"*True*"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.163	0.168	0.171	0.164	0.270	0.269	0.279	0.259
40	20	0.358	0.355	0.375	0.337	0.580	0.579	0.638	0.537
20	20	0.668	0.817	0.926	0.727	1.227	1.543	2.006	1.262
20	40	0.289	0.346	0.346	0.325	0.464	0.540	0.559	0.495
20	80	0.147	0.168	0.162	0.161	0.211	0.244	0.239	0.232
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \le K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.147	0.166	0.169	0.162	0.268	0.271	0.281	0.261
40	20	0.308	0.362	0.380	0.344	0.514	0.601	0.659	0.550
20	20	0.715	0.835	0.942	0.749	1.501	1.783	2.525	1.439
20	40	0.303	0.353	0.354	0.332	0.502	0.556	0.575	0.508
20	80	0.162	0.165	0.159	0.159	0.235	0.248	0.242	0.236
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \le K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.164	0.165	0.168	0.162	0.258	0.268	0.280	0.259
40	20	0.286	0.354	0.375	0.337	0.529	0.611	0.688	0.461
20	20	0.774	0.892	1.020	0.780	1.432	1.777	2.352	1.421
20	40	0.337	0.359	0.357	0.335	0.473	0.565	0.591	0.515
20	80	0.169	0.168	0.163	0.161	0.234	0.256	0.252	0.243

[1] "*True*"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.9: Variance Estimation – MAR Model for $\theta = 3$ Full Data vs. Complete Data Only with Pseduo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"*True*"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.156	0.161	0.164	0.158	0.237	0.235	0.243	0.230
40	20	0.327	0.323	0.339	0.313	0.458	0.468	0.505	0.448
20	20	0.557	0.660	0.711	0.619	0.853	0.988	1.154	0.918
20	40	0.265	0.315	0.313	0.301	0.387	0.450	0.459	0.426
20	80	0.141	0.161	0.154	0.155	0.193	0.224	0.219	0.216
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \le K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.141	0.159	0.162	0.156	0.233	0.231	0.239	0.226
40	20	0.282	0.330	0.344	0.319	0.419	0.467	0.502	0.445
20	20	0.592	0.675	0.729	0.639	0.955	1.088	1.248	1.003
20	40	0.278	0.320	0.319	0.307	0.390	0.445	0.452	0.421
20	80	0.156	0.158	0.151	0.153	0.217	0.222	0.216	0.213
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \le K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.157	0.158	0.161	0.156	0.207	0.231	0.240	0.226
40	20	0.262	0.323	0.340	0.312	0.421	0.483	0.531	0.461
20	20	0.635	0.710	0.766	0.660	0.907	1.096	1.276	0.996
20	40	0.309	0.326	0.322	0.310	0.415	0.462	0.476	0.438
20	80	0.162	0.160	0.155	0.155	0.211	0.232	0.227	0.223

[1] "*True*"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.10: Variance Estimation – MAR Model for $\theta = 3$ Full Data vs. Estimated Data with Pseduo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"$True$"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.156	0.161	0.164	0.158	0.258	0.256	0.266	0.249
40	20	0.327	0.323	0.339	0.313	0.528	0.524	0.570	0.493
20	20	0.557	0.660	0.711	0.619	0.989	1.182	1.441	1.038
20	40	0.265	0.315	0.313	0.301	0.424	0.488	0.500	0.455
20	80	0.141	0.161	0.154	0.155	0.202	0.233	0.228	0.223
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.141	0.159	0.162	0.156	0.257	0.258	0.268	0.250
40	20	0.282	0.330	0.344	0.319	0.469	0.542	0.586	0.505
20	20	0.592	0.675	0.729	0.639	1.196	1.351	1.602	1.175
20	40	0.278	0.320	0.319	0.307	0.458	0.501	0.513	0.467
20	80	0.156	0.158	0.151	0.153	0.226	0.236	0.231	0.226
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.157	0.158	0.161	0.156	0.246	0.256	0.266	0.249
40	20	0.262	0.323	0.340	0.312	0.482	0.551	0.610	0.516
20	20	0.635	0.710	0.766	0.660	1.125	1.334	1.627	1.156
20	40	0.309	0.326	0.322	0.310	0.432	0.509	0.526	0.473
20	80	0.162	0.160	0.155	0.155	0.224	0.244	0.240	0.233

[1] "$True$"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.11: Variance of Variance Estimation – MAR Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.004	0.004	0.004	0.013	0.015	0.012	0.016	0.018	0.014
40	20	0.039	0.047	0.033	0.114	0.157	0.092	0.144	0.197	0.114
20	20	0.423	0.748	0.282	2.808	7.123	1.384	3.836	10.313	1.761
20	40	0.043	0.048	0.034	0.116	0.150	0.087	0.129	0.166	0.096
20	80	0.007	0.006	0.006	0.016	0.016	0.013	0.017	0.017	0.014
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.003	0.003	0.003	0.011	0.012	0.010	0.013	0.015	0.012
40	20	0.031	0.036	0.026	0.118	0.157	0.087	0.189	0.259	0.118
20	20	0.398	0.610	0.284	4.484	14.479	2.160	6.211	23.215	2.780
20	40	0.040	0.043	0.032	0.138	0.137	0.084	0.156	0.173	0.104
20	80	0.006	0.006	0.005	0.015	0.016	0.013	0.017	0.017	0.015
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.003	0.004	0.003	0.011	0.013	0.010	0.016	0.019	0.014
40	20	0.035	0.045	0.028	0.163	0.316	0.114	0.236	0.436	0.081
20	20	0.806	1.387	0.489	7.364	9.836	2.898	9.569	15.936	3.915
20	40	0.045	0.046	0.034	0.130	0.172	0.095	0.151	0.190	0.104
20	80	0.007	0.007	0.006	0.019	0.020	0.016	0.021	0.022	0.017

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table C.12: Variance of Variance Estimation – MAR Model for $\theta = 3$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.003	0.004	0.003	0.011	0.012	0.010	0.014	0.015	0.012
40	20	0.030	0.035	0.027	0.076	0.098	0.067	0.107	0.139	0.089
20	20	0.225	0.305	0.179	0.869	1.423	0.653	1.514	3.105	0.939
20	40	0.032	0.035	0.027	0.079	0.093	0.064	0.097	0.118	0.076
20	80	0.006	0.005	0.005	0.013	0.013	0.011	0.015	0.015	0.013
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.003	0.003	0.002	0.009	0.010	0.008	0.012	0.013	0.011
40	20	0.024	0.027	0.021	0.078	0.095	0.064	0.136	0.165	0.092
20	20	0.218	0.283	0.185	1.290	1.822	0.975	2.329	3.739	1.432
20	40	0.031	0.032	0.026	0.089	0.085	0.062	0.113	0.120	0.082
20	80	0.005	0.005	0.005	0.013	0.013	0.012	0.015	0.015	0.013
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.003	0.003	0.003	0.009	0.011	0.009	0.014	0.016	0.012
40	20	0.026	0.033	0.023	0.104	0.162	0.081	0.169	0.268	0.121
20	20	0.387	0.480	0.294	1.525	2.212	1.173	2.775	4.762	1.804
20	40	0.034	0.034	0.028	0.086	0.102	0.070	0.112	0.130	0.082
20	80	0.006	0.006	0.005	0.016	0.016	0.013	0.018	0.019	0.015

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Appendix D

Simulation Result Tables – Variance Estimation for Informative Model

Table D.1: Variance Estimation– Informative (A, C) for $\theta = 1$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.015	0.015	0.016	0.015	0.021	0.023	0.024	0.023
40	20	0.033	0.033	0.034	0.032	0.047	0.051	0.054	0.049
20	20	0.065	0.075	0.077	0.071	0.100	0.120	0.134	0.109
20	40	0.036	0.033	0.032	0.032	0.052	0.050	0.050	0.048
20	80	0.019	0.016	0.015	0.016	0.025	0.023	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.014	0.016	0.016	0.015	0.020	0.022	0.023	0.022
40	20	0.034	0.033	0.033	0.032	0.052	0.048	0.050	0.046
20	20	0.060	0.070	0.072	0.067	0.095	0.110	0.119	0.101
20	40	0.034	0.034	0.033	0.033	0.050	0.050	0.050	0.048
20	80	0.016	0.016	0.015	0.016	0.022	0.022	0.021	0.022
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.015	0.016	0.016	0.015	0.022	0.023	0.023	0.022
40	20	0.033	0.033	0.034	0.032	0.046	0.049	0.051	0.047
20	20	0.079	0.074	0.077	0.070	0.111	0.116	0.129	0.107
20	40	0.033	0.034	0.033	0.033	0.047	0.052	0.052	0.049
20	80	0.015	0.016	0.015	0.016	0.021	0.023	0.023	0.023

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.2: Variance Estimation– Informative (A, C) for $\theta = 1$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.015	0.015	0.016	0.015	0.024	0.025	0.025	0.024
40	20	0.033	0.033	0.034	0.032	0.050	0.054	0.058	0.053
20	20	0.065	0.075	0.077	0.071	0.106	0.131	0.148	0.119
20	40	0.036	0.033	0.032	0.032	0.052	0.052	0.052	0.050
20	80	0.019	0.016	0.015	0.016	0.025	0.023	0.023	0.023
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.014	0.016	0.016	0.015	0.022	0.024	0.025	0.024
40	20	0.034	0.033	0.033	0.032	0.057	0.053	0.056	0.052
20	20	0.060	0.070	0.072	0.067	0.109	0.123	0.136	0.113
20	40	0.034	0.034	0.033	0.033	0.052	0.055	0.055	0.053
20	80	0.016	0.016	0.015	0.016	0.023	0.023	0.022	0.023
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.015	0.016	0.016	0.015	0.024	0.024	0.025	0.024
40	20	0.033	0.033	0.034	0.032	0.049	0.051	0.054	0.050
20	20	0.079	0.074	0.077	0.070	0.113	0.121	0.137	0.112
20	40	0.033	0.034	0.033	0.033	0.050	0.053	0.054	0.051
20	80	0.015	0.016	0.015	0.016	0.022	0.024	0.023	0.023

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.3: Variance Estimation– Informative (A, C) for $\theta = 1$ Full Data vs. Complete Only With Pseudo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"$True$"[1]	JK[2]	$B.$[3]	$A.F.$[4]	"$True$"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.014	0.015	0.015	0.015	0.021	0.023	0.023	0.022
40	20	0.032	0.032	0.033	0.031	0.045	0.048	0.051	0.047
20	20	0.060	0.069	0.075	0.066	0.090	0.106	0.117	0.098
20	40	0.035	0.032	0.031	0.031	0.050	0.048	0.048	0.046
20	80	0.018	0.016	0.015	0.015	0.024	0.022	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.014	0.015	0.015	0.015	0.019	0.022	0.022	0.021
40	20	0.033	0.032	0.032	0.031	0.050	0.045	0.048	0.044
20	20	0.056	0.065	0.066	0.062	0.086	0.098	0.104	0.091
20	40	0.033	0.033	0.032	0.032	0.048	0.048	0.048	0.046
20	80	0.015	0.016	0.015	0.015	0.021	0.022	0.021	0.021
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.015	0.015	0.016	0.015	0.022	0.022	0.023	0.022
40	20	0.032	0.032	0.033	0.031	0.044	0.046	0.049	0.045
20	20	0.074	0.068	0.070	0.065	0.100	0.103	0.111	0.096
20	40	0.032	0.033	0.032	0.031	0.045	0.049	0.049	0.047
20	80	0.015	0.016	0.015	0.015	0.021	0.023	0.022	0.022

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.4: Variance Estimation– Informative (A, C) for $\theta = 1$ Full Data vs. Estimated Data With Pseudo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.014	0.015	0.015	0.015	0.024	0.024	0.025	0.024
40	20	0.032	0.032	0.033	0.031	0.049	0.052	0.056	0.051
20	20	0.060	0.069	0.075	0.066	0.099	0.119	0.136	0.109
20	40	0.035	0.032	0.031	0.031	0.051	0.050	0.050	0.048
20	80	0.018	0.016	0.015	0.015	0.024	0.023	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.014	0.015	0.015	0.015	0.021	0.024	0.024	0.023
40	20	0.033	0.032	0.032	0.031	0.055	0.051	0.054	0.050
20	20	0.056	0.065	0.066	0.062	0.101	0.113	0.122	0.105
20	40	0.033	0.033	0.032	0.032	0.051	0.053	0.053	0.051
20	80	0.015	0.016	0.015	0.015	0.023	0.023	0.022	0.022
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.015	0.015	0.016	0.015	0.023	0.023	0.024	0.023
40	20	0.032	0.032	0.033	0.031	0.047	0.049	0.052	0.048
20	20	0.074	0.068	0.070	0.065	0.105	0.111	0.122	0.104
20	40	0.032	0.033	0.032	0.031	0.048	0.051	0.052	0.049
20	80	0.015	0.016	0.015	0.015	0.021	0.023	0.023	0.023

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.5: Variance of Variance Estimation – Informative (A, C) for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.0E-5	2.1E-5	2.0E-5	6.6E-5	7.0E-5	6.3E-5	8.3E-5	9.0E-5	8.0E-5
40	20	2.3E-4	2.6E-4	2.2E-4	7.4E-4	9.1E-4	6.7E-4	9.1E-4	1.1E-3	8.1E-4
20	20	3.7E-3	4.3E-3	2.7E-3	1.6E-2	2.6E-2	9.3E-3	2.1E-2	3.7E-2	1.1E-2
20	40	3.2E-4	3.2E-4	2.9E-4	9.9E-4	1.1E-3	8.4E-4	1.1E-3	1.2E-3	9.4E-4
20	80	6.7E-5	5.9E-5	5.9E-5	1.5E-4	1.3E-4	1.3E-4	1.5E-4	1.4E-4	1.3E-4
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.3E-5	2.4E-5	2.2E-5	5.9E-5	6.4E-5	5.5E-5	7.0E-5	7.7E-5	6.7E-5
40	20	2.2E-4	2.4E-4	2.0E-4	6.5E-4	7.7E-4	5.9E-4	8.5E-4	1.0E-3	7.7E-4
20	20	2.0E-3	2.3E-3	1.7E-3	8.3E-3	1.0E-2	5.7E-3	1.1E-2	1.5E-2	8.0E-3
20	40	3.7E-4	3.6E-4	3.3E-4	1.0E-3	1.1E-3	9.0E-4	1.3E-3	1.3E-3	1.1E-3
20	80	6.4E-5	5.8E-5	5.5E-5	1.5E-4	1.4E-4	1.3E-4	1.6E-4	1.6E-4	1.4E-4
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.3E-5	2.5E-5	2.3E-5	6.2E-5	6.8E-5	6.0E-5	7.2E-5	7.9E-5	6.9E-5
40	20	2.5E-4	2.8E-4	2.3E-4	6.6E-4	7.7E-4	5.9E-4	6.9E-4	8.2E-4	6.2E-4
20	20	2.9E-3	3.4E-3	2.4E-3	1.0E-2	1.8E-2	7.6E-3	1.0E-2	1.8E-2	7.8E-3
20	40	3.6E-4	3.7E-4	3.2E-4	1.0E-3	1.2E-3	8.7E-4	1.1E-3	1.3E-3	9.3E-4
20	80	5.5E-5	5.0E-5	5.0E-5	1.2E-4	1.2E-4	1.1E-4	1.2E-4	1.2E-4	1.1E-4

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.6: Variance of Variance Estimation – Informative (A, C) for $\theta = 1$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.0E-5	2.0E-5	1.9E-5	6.1E-5	6.6E-5	5.9E-5	7.9E-5	8.6E-5	7.7E-5
40	20	2.1E-4	2.4E-4	2.0E-4	6.3E-4	7.7E-4	5.7E-4	8.0E-4	9.9E-4	7.3E-4
20	20	2.8E-3	1.2E-2	2.2E-3	1.0E-2	2.0E-2	6.6E-3	1.5E-2	3.5E-2	8.9E-3
20	40	2.9E-4	2.9E-4	2.6E-4	8.5E-4	9.4E-4	7.3E-4	9.7E-4	1.1E-3	8.4E-4
20	80	6.4E-5	5.6E-5	5.6E-5	1.4E-4	1.3E-4	1.2E-4	1.4E-4	1.3E-4	1.2E-4
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.2E-5	2.3E-5	2.1E-5	5.5E-5	6.0E-5	5.2E-5	6.7E-5	7.4E-5	6.4E-5
40	20	2.0E-4	2.2E-4	1.9E-4	5.6E-4	6.6E-4	5.1E-4	7.7E-4	9.2E-4	7.0E-4
20	20	1.6E-3	1.8E-3	1.4E-3	5.7E-3	6.3E-3	4.2E-3	8.3E-3	1.1E-2	6.3E-3
20	40	3.3E-4	3.3E-4	3.0E-4	8.9E-4	9.2E-4	7.8E-4	1.1E-3	1.2E-3	9.8E-4
20	80	6.1E-5	5.5E-5	5.3E-5	1.4E-4	1.3E-4	1.2E-4	1.6E-4	1.5E-4	1.4E-4
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.2E-5	2.4E-5	2.2E-5	5.8E-5	6.3E-5	5.6E-5	6.9E-5	7.5E-5	6.6E-5
40	20	2.2E-4	2.5E-4	2.1E-4	5.7E-4	6.6E-4	5.1E-4	6.2E-4	7.3E-4	5.6E-4
20	20	2.3E-3	2.6E-3	2.0E-3	6.9E-3	1.0E-2	5.5E-3	7.8E-3	1.2E-2	6.2E-3
20	40	3.3E-4	3.3E-4	2.9E-4	8.6E-4	1.0E-3	7.5E-4	9.6E-4	1.1E-3	8.4E-4
20	80	5.3E-5	4.8E-5	4.7E-5	1.2E-4	1.1E-4	1.0E-4	1.2E-4	1.1E-4	1.1E-4

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.7: Variance Estimation– Informative (A, C) for $\theta = 3$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.151	0.163	0.166	0.159	0.220	0.248	0.258	0.240
40	20	0.385	0.369	0.388	0.350	0.636	0.597	0.665	0.553
20	20	0.750	0.866	1.000	0.764	1.324	1.640	2.489	1.314
20	40	0.337	0.368	0.368	0.344	0.524	0.572	0.608	0.523
20	80	0.164	0.168	0.162	0.162	0.250	0.251	0.249	0.237
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.179	0.167	0.171	0.163	0.269	0.244	0.254	0.236
40	20	0.296	0.360	0.384	0.343	0.469	0.534	0.593	0.498
20	20	0.792	0.891	0.997	0.782	1.228	1.438	2.093	1.193
20	40	0.364	0.368	0.367	0.344	0.490	0.540	0.558	0.491
20	80	0.176	0.163	0.156	0.156	0.252	0.242	0.234	0.228
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.169	0.167	0.171	0.163	0.259	0.255	0.265	0.247
40	20	0.383	0.364	0.385	0.347	0.586	0.575	0.635	0.534
20	20	0.738	0.850	0.972	0.756	1.173	1.458	2.099	1.212
20	40	0.310	0.375	0.375	0.349	0.483	0.588	0.620	0.534
20	80	0.145	0.171	0.164	0.163	0.226	0.249	0.243	0.235

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.8: Variance Estimation– Informative (A, C) for $\theta = 3$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.151	0.163	0.166	0.159	0.226	0.260	0.272	0.252
40	20	0.385	0.369	0.388	0.350	0.641	0.631	0.705	0.583
20	20	0.750	0.866	1.000	0.764	1.401	1.742	2.639	1.386
20	40	0.337	0.368	0.368	0.344	0.547	0.592	0.635	0.542
20	80	0.164	0.168	0.162	0.162	0.254	0.253	0.251	0.240
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.179	0.167	0.171	0.163	0.285	0.268	0.280	0.258
40	20	0.296	0.360	0.384	0.343	0.531	0.492	0.526	0.460
20	20	0.792	0.891	0.997	0.782	1.376	1.695	2.417	1.358
20	40	0.364	0.368	0.367	0.344	0.525	0.588	0.613	0.536
20	80	0.176	0.163	0.156	0.156	0.266	0.254	0.248	0.241
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.169	0.167	0.171	0.163	0.275	0.267	0.278	0.258
40	20	0.383	0.364	0.385	0.347	0.633	0.600	0.665	0.557
20	20	0.738	0.850	0.972	0.756	1.232	1.577	2.342	1.292
20	40	0.310	0.375	0.375	0.349	0.510	0.605	0.642	0.549
20	80	0.145	0.171	0.164	0.163	0.230	0.253	0.248	0.240

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.9: Variance Estimation– Informative (A, C) for $\theta = 3$ Full Data vs. Complete Only With Pseudo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.145	0.156	0.159	0.154	0.207	0.232	0.241	0.227
40	20	0.352	0.335	0.350	0.324	0.555	0.514	0.562	0.492
20	20	0.620	0.693	0.753	0.648	0.988	1.121	1.333	1.010
20	40	0.309	0.334	0.333	0.318	0.459	0.493	0.514	0.467
20	80	0.158	0.161	0.155	0.155	0.236	0.234	0.231	0.224
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$; $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.172	0.160	0.163	0.158	0.254	0.229	0.237	0.223
40	20	0.271	0.328	0.348	0.318	0.413	0.466	0.509	0.447
20	20	0.655	0.711	0.764	0.663	0.922	1.020	1.188	0.939
20	40	0.332	0.333	0.330	0.318	0.430	0.468	0.475	0.440
20	80	0.169	0.155	0.149	0.150	0.237	0.226	0.219	0.216
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$; $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.162	0.160	0.163	0.157	0.243	0.238	0.247	0.233
40	20	0.349	0.331	0.347	0.321	0.509	0.496	0.538	0.475
20	20	0.608	0.682	0.734	0.642	0.861	1.024	1.175	0.948
20	40	0.283	0.339	0.337	0.322	0.423	0.505	0.522	0.475
20	80	0.139	0.163	0.156	0.157	0.212	0.232	0.226	0.222

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.10: Variance Estimation– Informative (A, C) for $\theta = 3$ Full Data vs. Estimated Data With Pseudo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.145	0.156	0.159	0.154	0.216	0.248	0.259	0.241
40	20	0.352	0.335	0.350	0.324	0.581	0.566	0.623	0.532
20	20	0.620	0.693	0.753	0.648	1.124	1.307	1.620	1.125
20	40	0.309	0.334	0.333	0.318	0.496	0.531	0.560	0.496
20	80	0.158	0.161	0.155	0.155	0.243	0.246	0.240	0.236
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.172	0.160	0.163	0.158	0.273	0.256	0.266	0.248
40	20	0.271	0.328	0.348	0.318	0.483	0.550	0.604	0.517
20	20	0.655	0.711	0.764	0.663	1.114	1.298	1.553	1.118
20	40	0.332	0.333	0.330	0.318	0.478	0.529	0.545	0.491
20	80	0.169	0.155	0.149	0.150	0.254	0.242	0.236	0.231
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.162	0.160	0.163	0.157	0.263	0.255	0.264	0.248
40	20	0.349	0.331	0.347	0.321	0.572	0.539	0.589	0.509
20	20	0.608	0.682	0.734	0.642	0.982	1.208	1.430	1.061
20	40	0.283	0.339	0.337	0.322	0.462	0.541	0.566	0.502
20	80	0.139	0.163	0.156	0.157	0.220	0.241	0.236	0.229

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.11: Variance of Variance Estimation – Informative (A, C) for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.003	0.003	0.003	0.009	0.010	0.008	0.010	0.012	0.009
40	20	0.045	0.052	0.037	0.159	0.225	0.127	0.174	0.248	0.136
20	20	0.589	1.223	0.380	4.954	39.693	2.173	5.765	44.197	2.384
20	40	0.050	0.049	0.038	0.157	0.203	0.116	0.179	0.248	0.133
20	80	0.007	0.007	0.006	0.027	0.042	0.020	0.024	0.036	0.019
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.003	0.004	0.003	0.011	0.012	0.010	0.014	0.016	0.012
40	20	0.027	0.036	0.024	0.091	0.124	0.072	0.135	0.193	0.107
20	20	1.232	1.492	0.583	2.591	22.729	1.382	5.112	32.504	1.812
20	40	0.070	0.073	0.051	0.157	0.179	0.110	0.193	0.234	0.135
20	80	0.006	0.006	0.005	0.020	0.018	0.016	0.021	0.020	0.017
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.004	0.004	0.004	0.012	0.014	0.011	0.014	0.016	0.012
40	20	0.045	0.054	0.037	0.161	0.224	0.127	0.180	0.250	0.143
20	20	0.483	0.893	0.309	2.113	13.942	1.148	2.815	22.496	1.464
20	40	0.061	0.064	0.044	0.236	0.307	0.156	0.230	0.314	0.151
20	80	0.008	0.007	0.006	0.019	0.019	0.015	0.020	0.020	0.016

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table D.12: Variance of Variance Estimation – Informative (A, C) for $\theta = 3$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.003	0.003	0.002	0.007	0.009	0.007	0.009	0.010	0.008
40	20	0.034	0.038	0.030	0.104	0.136	0.092	0.128	0.172	0.106
20	20	0.303	0.407	0.236	1.544	2.724	1.006	2.367	4.846	1.294
20	40	0.038	0.037	0.031	0.103	0.123	0.085	0.131	0.167	0.104
20	80	0.006	0.006	0.005	0.021	0.030	0.017	0.021	0.025	0.017
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.003	0.003	0.003	0.009	0.010	0.008	0.012	0.014	0.011
40	20	0.021	0.027	0.019	0.061	0.079	0.054	0.100	0.136	0.084
20	20	0.532	0.549	0.341	0.932	1.416	0.702	2.179	3.053	1.032
20	40	0.051	0.051	0.040	0.103	0.109	0.080	0.141	0.157	0.105
20	80	0.006	0.005	0.005	0.016	0.015	0.014	0.019	0.018	0.016
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.003	0.004	0.003	0.010	0.011	0.009	0.012	0.013	0.011
40	20	0.033	0.039	0.030	0.102	0.130	0.091	0.129	0.169	0.110
20	20	0.248	0.314	0.194	0.708	1.026	0.572	1.224	1.973	0.815
20	40	0.045	0.046	0.035	0.147	0.171	0.110	0.163	0.201	0.116
20	80	0.007	0.007	0.006	0.016	0.015	0.013	0.017	0.017	0.014

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Appendix E

Simulation Result Tables – MAR Model – Multinomial

Table E.1: Mean of Mean of Estimated Common Odds Ratio – Multinomial MAR Model : for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	1.001	1.001	0.993	1.005	1.005	0.993	1.009	1.009	0.996
40	20	1.016	1.016	1.000	1.011	1.011	0.987	1.016	1.015	0.990
20	20	1.036	1.034	1.000	1.051	1.046	0.998	1.058	1.055	1.001
20	40	1.015	1.014	0.999	1.018	1.018	0.995	1.020	1.019	0.995
20	80	1.000	1.000	0.993	0.998	0.998	0.987	0.999	0.999	0.988
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	1.017	1.017	1.009	1.019	1.019	1.008	1.021	1.021	1.008
40	20	1.014	1.013	0.997	1.025	1.023	1.001	1.031	1.030	1.005
20	20	1.031	1.029	0.996	1.030	1.026	0.980	1.026	1.024	0.970
20	40	1.023	1.022	1.006	1.039	1.037	1.014	1.042	1.040	1.016
20	80	1.006	1.006	0.997	1.007	1.007	0.996	1.009	1.008	0.997
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	1.015	1.014	1.006	1.018	1.018	1.006	1.020	1.020	1.007
40	20	1.002	1.002	0.985	1.016	1.016	0.992	1.017	1.016	0.991
20	20	1.022	1.021	0.987	1.033	1.030	0.980	1.031	1.028	0.973
20	40	1.032	1.031	1.014	1.046	1.044	1.021	1.048	1.047	1.022
20	80	1.012	1.012	1.005	1.019	1.019	1.008	1.020	1.020	1.009

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Appendix F

Simulation Result Tables – Informative Model – Multinomial

Table E.2: Variance of Estimated Common Odds Ratio – Multinomial MAR Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.017	0.017	0.017	0.028	0.027	0.027	0.030	0.029	0.029
40	20	0.034	0.033	0.033	0.047	0.044	0.044	0.051	0.050	0.048
20	20	0.072	0.067	0.066	0.116	0.103	0.101	0.124	0.114	0.106
20	40	0.037	0.035	0.035	0.052	0.050	0.049	0.054	0.052	0.051
20	80	0.014	0.014	0.014	0.022	0.021	0.021	0.022	0.021	0.021
$P_{1k}(missing) = P^{missing}2k = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.018	0.018	0.018	0.025	0.025	0.024	0.027	0.027	0.027
40	20	0.038	0.037	0.036	0.054	0.051	0.051	0.060	0.058	0.057
20	20	0.071	0.066	0.065	0.107	0.095	0.093	0.111	0.102	0.094
20	40	0.035	0.034	0.034	0.053	0.050	0.050	0.057	0.055	0.054
20	80	0.017	0.017	0.017	0.024	0.024	0.024	0.025	0.025	0.024
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.018	0.017	0.017	0.024	0.023	0.023	0.025	0.024	0.024
40	20	0.032	0.031	0.031	0.047	0.045	0.044	0.051	0.049	0.048
20	20	0.063	0.059	0.058	0.101	0.091	0.087	0.108	0.100	0.091
20	40	0.029	0.028	0.028	0.049	0.047	0.047	0.052	0.051	0.049
20	80	0.016	0.016	0.016	0.027	0.026	0.026	0.028	0.027	0.027

[1] MH – Mantel-Haenszel Estimator
[2] PMH1 – Mantel-Haenszel with Pseudo-Tables
[3] JK – Jackknifing Method

Table E.3: T value for Estimated Common Odds Ratio – Multinomial MAR Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.130	0.118	-1.259	0.655	0.633	-0.947	1.171	1.156	-0.486
40	20	1.937	1.907	-0.039	1.178	1.125	-1.361	1.566	1.527	-1.042
20	20	2.985	2.905	0.022	3.368	3.228	-0.165	3.715	3.614	0.036
20	40	1.752	1.723	-0.128	1.807	1.759	-0.480	1.929	1.894	-0.450
20	80	0.090	0.080	-1.360	-0.250	-0.267	-1.960	-0.095	-0.107	-1.857
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.851	2.841	1.500	2.727	2.709	1.102	2.819	2.806	1.109
40	20	1.573	1.541	-0.305	2.366	2.317	0.074	2.810	2.773	0.473
20	20	2.589	2.500	-0.366	2.047	1.907	-1.446	1.756	1.646	-2.156
20	40	2.699	2.669	0.679	3.756	3.709	1.438	3.877	3.841	1.510
20	80	0.957	0.947	-0.433	1.001	0.985	-0.606	1.211	1.199	-0.436
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.462	2.451	1.098	2.638	2.619	0.934	2.803	2.790	1.018
40	20	0.224	0.193	-1.852	1.699	1.646	-0.860	1.702	1.662	-0.941
20	20	1.969	1.891	-1.226	2.334	2.197	-1.525	2.102	2.000	-1.970
20	40	4.128	4.101	1.920	4.625	4.582	2.171	4.707	4.676	2.237
20	80	2.172	2.161	0.830	2.578	2.559	1.072	2.747	2.734	1.249

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table E.4: Mean of Estimated Common Odds Ratio – Multinomial MAR Model : for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	3.057	3.026	3.019	3.084	3.039	3.028	3.081	3.049	3.022
40	20	3.052	2.990	2.973	3.107	3.014	2.991	3.103	3.036	2.980
20	20	3.159	3.023	2.987	3.226	3.019	2.952	3.251	3.097	2.962
20	40	3.110	3.046	3.031	3.100	3.011	2.988	3.097	3.033	2.980
20	80	2.997	2.968	2.962	3.000	2.960	2.951	3.005	2.975	2.954
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	3.043	3.012	3.006	3.073	3.028	3.019	3.080	3.047	3.020
40	20	3.061	2.997	2.982	3.103	3.012	2.985	3.135	3.066	3.004
20	20	3.232	3.089	3.058	3.296	3.082	3.022	3.337	3.172	3.031
20	40	3.096	3.033	3.020	3.146	3.054	3.037	3.141	3.074	3.023
20	80	3.045	3.015	3.009	3.075	3.032	3.023	3.080	3.048	3.025
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	3.044	3.013	3.006	3.064	3.019	3.008	3.064	3.032	3.004
40	20	3.051	2.988	2.974	3.090	2.996	2.974	3.101	3.033	2.976
20	20	3.195	3.054	3.019	3.329	3.103	3.031	3.364	3.195	3.041
20	40	3.042	2.980	2.967	3.086	2.995	2.975	3.089	3.023	2.970
20	80	3.016	2.986	2.979	3.013	2.971	2.959	3.010	2.980	2.954

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table E.5: Variance of Estimated Common Odds Ratio – Multinomial MAR Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.180	0.172	0.174	0.255	0.239	0.242	0.267	0.254	0.252
40	20	0.310	0.282	0.287	0.538	0.469	0.486	0.564	0.511	0.505
20	20	0.811	0.666	0.682	1.491	1.091	1.122	1.623	1.291	1.216
20	40	0.405	0.369	0.378	0.545	0.479	0.494	0.561	0.511	0.507
20	80	0.161	0.154	0.156	0.232	0.218	0.222	0.245	0.235	0.234
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.180	0.172	0.174	0.260	0.243	0.246	0.294	0.280	0.277
40	20	0.338	0.309	0.316	0.499	0.437	0.446	0.554	0.503	0.493
20	20	0.867	0.710	0.740	1.454	1.065	1.132	1.673	1.324	1.277
20	40	0.350	0.320	0.326	0.581	0.507	0.527	0.616	0.559	0.554
20	80	0.162	0.155	0.157	0.233	0.220	0.223	0.252	0.241	0.240
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$ and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.168	0.160	0.162	0.238	0.223	0.225	0.257	0.245	0.243
40	20	0.353	0.320	0.328	0.570	0.492	0.506	0.636	0.571	0.558
20	20	0.790	0.642	0.659	1.511	1.067	1.064	1.569	1.225	1.072
20	0	0.365	0.333	0.338	0.542	0.472	0.486	0.575	0.521	0.511
20	80	0.178	0.170	0.171	0.242	0.227	0.229	0.246	0.235	0.232

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table E.6: T value for Estimated Common Odds Ratio – Multinomial MAR Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	3.021	1.394	1.009	3.737	1.772	1.287	3.524	2.153	0.970
40	20	2.097	-0.440	-1.115	3.271	0.457	-0.283	3.068	1.116	-0.622
20	20	3.943	0.624	-0.347	4.132	0.407	-1.012	4.410	1.900	-0.768
20	40	3.862	1.707	1.131	3.017	0.356	-0.396	2.888	1.020	-0.621
20	80	-0.174	-1.841	-2.165	0.021	-1.935	-2.332	0.206	-1.158	-2.117
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.275	0.650	0.296	3.210	1.283	0.851	3.313	2.004	0.848
40	20	2.328	-0.103	-0.705	3.271	0.391	-0.512	4.054	2.083	0.142
20	20	5.576	2.361	1.502	5.482	1.774	0.470	5.829	3.350	0.617
20	40	3.637	1.290	0.782	4.288	1.683	1.130	4.020	2.206	0.704
20	80	2.502	0.838	0.496	3.481	1.544	1.100	3.550	2.204	1.152
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$ and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	2.408	0.723	0.351	2.936	0.880	0.374	2.844	1.447	0.199
40	20	1.928	-0.466	-1.006	2.660	-0.138	-0.824	2.839	0.978	-0.721
20	20	4.894	1.501	0.510	5.977	2.232	0.674	6.494	3.943	0.878
20	40	1.570	-0.757	-1.262	2.622	-0.175	-0.814	2.614	0.713	-0.935
20	80	0.853	-0.746	-1.119	0.605	-1.365	-1.901	0.460	-0.913	-2.144

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table F.1: Mean of Estimated Common Odds Ratio – Multinomial Informative (A, C) for $\theta = 1$.

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	1.012	1.012	1.004	1.014	1.014	1.002	1.014	1.014	1.002
40	20	1.021	1.021	1.004	1.040	1.038	1.014	1.044	1.042	1.017
20	20	1.019	1.017	0.984	1.036	1.032	0.983	1.038	1.035	0.983
20	40	1.031	1.030	1.014	1.042	1.041	1.018	1.044	1.042	1.018
20	80	0.999	0.999	0.992	0.999	0.999	0.988	1.001	1.001	0.990
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$										
80	20	1.003	1.003	0.995	1.002	1.002	0.989	1.000	1.000	0.986
40	20	1.016	1.016	0.999	1.028	1.027	1.004	1.029	1.028	1.003
20	20	1.042	1.039	1.006	1.067	1.062	1.015	1.065	1.061	1.007
20	40	1.021	1.020	1.004	1.029	1.028	1.006	1.030	1.029	1.004
20	80	1.002	1.002	0.995	1.012	1.012	1.002	1.013	1.013	1.002
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$										
80	20	1.012	1.012	1.004	1.020	1.019	1.008	1.019	1.019	1.007
40	20	1.011	1.011	0.995	1.012	1.011	0.988	1.013	1.012	0.987
20	20	1.064	1.060	1.027	1.076	1.070	1.023	1.079	1.074	1.022
20	40	1.020	1.019	1.004	1.026	1.025	1.002	1.024	1.023	1.000
20	80	1.007	1.007	0.999	1.012	1.012	1.000	1.011	1.011	1.000

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table F.2: Variance of Estimated Common Odds Ratio – Multinomial Informative (A, C) Model for $\theta = 1$.

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.019	0.018	0.018	0.029	0.028	0.028	0.029	0.029	0.028
40	20	0.035	0.034	0.034	0.057	0.054	0.054	0.063	0.060	0.059
20	20	0.067	0.062	0.061	0.097	0.087	0.084	0.101	0.094	0.087
20	40	0.034	0.033	0.033	0.055	0.053	0.052	0.058	0.056	0.054
20	80	0.016	0.015	0.015	0.023	0.023	0.023	0.024	0.023	0.023
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$										
80	20	0.014	0.013	0.013	0.022	0.021	0.021	0.024	0.024	0.023
40	20	0.034	0.032	0.032	0.050	0.048	0.047	0.056	0.054	0.052
20	20	0.072	0.066	0.066	0.121	0.108	0.106	0.137	0.125	0.117
20	40	0.036	0.035	0.034	0.054	0.051	0.050	0.059	0.056	0.054
20	80	0.017	0.017	0.017	0.022	0.022	0.022	0.024	0.024	0.024
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$										
80	20	0.018	0.017	0.017	0.025	0.024	0.024	0.027	0.026	0.026
40	20	0.031	0.030	0.029	0.046	0.044	0.043	0.048	0.046	0.045
20	20	0.081	0.075	0.074	0.129	0.115	0.114	0.137	0.126	0.120
20	40	0.034	0.033	0.033	0.053	0.050	0.050	0.053	0.052	0.050
20	80	0.016	0.016	0.016	0.024	0.024	0.024	0.025	0.024	0.024

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table F.3: T value for Estimated Common Odds Ratio – Multinomial Informative (A, C) Model for $\theta = 1$.

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	1.957	1.945	0.634	1.899	1.879	0.333	1.860	1.844	0.217
40	20	0.114	0.112	0.023	0.167	0.164	0.062	0.174	0.173	0.068
20	20	0.072	0.068	-0.066	0.115	0.108	-0.057	0.119	0.115	-0.059
20	40	0.165	0.164	0.077	0.179	0.177	0.077	0.181	0.180	0.079
20	80	-0.004	-0.005	-0.068	-0.004	-0.004	-0.079	0.006	0.006	-0.068
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$										
80	20	0.629	0.620	-0.914	0.294	0.274	-1.645	0.006	-0.007	-1.993
40	20	1.989	1.959	-0.065	2.787	2.739	0.424	2.770	2.733	0.285
20	20	3.501	3.419	0.508	4.329	4.219	1.000	3.939	3.847	0.464
20	40	2.418	2.389	0.455	2.829	2.778	0.567	2.756	2.719	0.413
20	80	0.417	0.406	-0.901	1.840	1.823	0.239	1.855	1.842	0.220
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$										
80	20	2.034	2.022	0.656	2.811	2.792	1.109	2.654	2.640	0.918
40	20	1.419	1.391	-0.692	1.267	1.215	-1.275	1.281	1.244	-1.344
20	20	4.997	4.922	2.253	4.746	4.623	1.501	4.758	4.668	1.421
20	40	2.392	2.364	0.435	2.549	2.499	0.239	2.347	2.312	0.009
20	80	1.235	1.223	-0.156	1.709	1.689	0.056	1.609	1.595	-0.055

[1] MH – Mantel-Haenszel Estimator

[2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table F.4: Mean of Estimated Common Odds Ratio – Multinomial Informative (A, C) for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	3.039	3.008	3.002	3.071	3.025	3.014	3.066	3.033	3.007
40	20	3.016	2.955	2.939	3.048	2.957	2.928	3.060	2.992	2.933
20	20	3.164	3.025	2.984	3.237	3.024	2.951	3.246	3.086	2.938
20	40	3.092	3.029	3.016	3.134	3.039	3.017	3.128	3.059	3.010
20	80	3.026	2.997	2.989	3.072	3.029	3.018	3.073	3.042	3.018
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	3.011	2.980	2.973	3.039	2.990	2.976	3.048	3.015	2.979
40	20	3.089	3.024	3.010	3.162	3.066	3.043	3.177	3.106	3.045
20	20	3.147	3.009	2.972	3.202	2.997	2.940	3.212	3.059	2.923
20	40	3.045	2.984	2.967	3.079	2.991	2.969	3.070	3.006	2.952
20	80	3.030	3.000	2.995	3.050	3.008	3.000	3.055	3.025	3.002
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	3.017	2.987	2.980	3.033	2.989	2.978	3.029	2.997	2.971
40	20	3.101	3.035	3.022	3.109	3.013	2.990	3.108	3.038	2.984
20	20	3.175	3.038	2.995	3.286	3.069	3.000	3.290	3.128	2.989
20	40	3.046	2.984	2.969	3.052	2.963	2.938	3.056	2.991	2.940
20	80	3.067	3.037	3.030	3.062	3.019	3.008	3.065	3.033	3.010

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table F.5: Variance of Estimated Common Odds Ratio – Multinomial Informative (A, C) Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.177	0.169	0.171	0.272	0.254	0.257	0.290	0.276	0.274
40	20	0.298	0.273	0.277	0.482	0.421	0.424	0.512	0.463	0.450
20	20	0.816	0.654	0.667	1.403	0.981	1.031	1.370	1.073	0.965
20	40	0.408	0.369	0.379	0.608	0.525	0.542	0.620	0.559	0.554
20	80	0.167	0.160	0.161	0.240	0.226	0.228	0.238	0.228	0.226
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.157	0.150	0.151	0.263	0.244	0.246	0.292	0.278	0.273
40	20	0.344	0.314	0.320	0.506	0.444	0.453	0.598	0.543	0.534
20	20	0.888	0.715	0.759	1.435	1.011	1.037	1.461	1.145	1.077
20	40	0.316	0.288	0.292	0.450	0.394	0.403	0.491	0.446	0.441
20	80	0.157	0.150	0.151	0.220	0.207	0.209	0.227	0.217	0.214
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.175	0.168	0.169	0.253	0.237	0.240	0.261	0.249	0.247
40	20	0.391	0.355	0.364	0.544	0.473	0.487	0.590	0.532	0.528
20	20	0.642	0.529	0.536	1.330	0.969	0.987	1.412	1.118	1.025
20	40	0.329	0.300	0.306	0.491	0.430	0.443	0.506	0.460	0.454
20	80	0.163	0.156	0.158	0.224	0.210	0.212	0.233	0.222	0.220

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Table F.6: T value for Estimated Common Odds Ratio – Multinomial Informative (A, C) Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	MH[1]	PMH1[2]	JK[3]	MH	PMH1	JK	MH	PMH1	JK
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.083	0.446	0.084	3.033	1.088	0.597	2.759	1.398	0.288
40	20	0.654	-1.938	-2.603	1.558	-1.484	-2.482	1.878	-0.254	-2.240
20	20	4.057	0.702	-0.451	4.481	0.535	-1.086	4.692	1.867	-1.412
20	40	3.237	1.056	0.588	3.833	1.205	0.526	3.645	1.773	0.300
20	80	1.435	-0.182	-0.594	3.300	1.357	0.849	3.362	1.945	0.849
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.605	-1.136	-1.549	1.696	-0.466	-1.104	1.972	0.624	-0.900
40	20	3.384	0.972	0.391	5.078	2.215	1.432	5.112	3.207	1.390
20	20	3.478	0.237	-0.709	3.764	-0.061	-1.316	3.920	1.230	-1.650
20	40	1.801	-0.660	-1.367	2.618	-0.325	-1.082	2.235	0.213	-1.617
20	80	1.706	0.020	-0.277	2.397	0.407	-0.004	2.597	1.182	0.108
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.904	-0.733	-1.109	1.487	-0.510	-1.018	1.279	-0.133	-1.315
40	20	3.599	1.324	0.812	3.289	0.425	-0.312	3.133	1.159	-0.486
20	20	4.884	1.177	-0.142	5.543	1.571	-0.009	5.454	2.714	-0.249
20	40	1.778	-0.642	-1.263	1.655	-1.259	-2.085	1.748	-0.302	-1.998
20	80	3.725	2.086	1.674	2.941	0.918	0.405	2.998	1.574	0.486

[1] MH – Mantel-Haenszel Estimator [2] PMH1 – Mantel-Haenszel with Pseudo-Tables

[3] JK – Jackknifing Method

Appendix G

Simulation Result Tables – Variance Estimation for MAR Model –Multinomial

Table G.1: Variance Estimation – Multinomial MAR Model for $\theta = 1$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.017	0.016	0.017	0.016	0.028	0.024	0.025	0.024
40	20	0.034	0.034	0.035	0.033	0.047	0.051	0.054	0.049
20	20	0.072	0.078	0.081	0.074	0.116	0.124	0.141	0.114
20	40	0.037	0.034	0.033	0.033	0.052	0.050	0.050	0.048
20	80	0.014	0.016	0.015	0.015	0.022	0.023	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.017	0.025	0.024	0.025	0.024
40	20	0.038	0.034	0.035	0.033	0.054	0.051	0.054	0.049
20	20	0.071	0.077	0.080	0.073	0.107	0.115	0.127	0.106
20	40	0.035	0.035	0.034	0.034	0.053	0.052	0.051	0.049
20	80	0.017	0.016	0.015	0.016	0.024	0.022	0.021	0.022
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.017	0.024	0.024	0.025	0.024
40	20	0.032	0.034	0.034	0.033	0.047	0.052	0.055	0.050
20	20	0.063	0.075	0.079	0.071	0.101	0.120	0.135	0.110
20	40	0.029	0.036	0.035	0.034	0.049	0.053	0.053	0.051
20	80	0.016	0.016	0.015	0.016	0.027	0.023	0.022	0.022

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Appendix H

Simulation Result Tables – Variance Estimation for Informative Model –Multinomial

Table G.2: Variance Estimation – Multinomial MAR Model for $\theta = 1$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.017	0.016	0.017	0.016	0.030	0.026	0.027	0.026
40	20	0.034	0.034	0.035	0.033	0.051	0.055	0.059	0.054
20	20	0.072	0.078	0.081	0.074	0.124	0.137	0.158	0.126
20	40	0.037	0.034	0.033	0.033	0.054	0.053	0.054	0.051
20	80	0.014	0.016	0.015	0.015	0.022	0.024	0.023	0.023
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.017	0.027	0.027	0.027	0.026
40	20	0.038	0.034	0.035	0.033	0.060	0.056	0.060	0.054
20	20	0.071	0.077	0.080	0.073	0.111	0.128	0.145	0.118
20	40	0.035	0.035	0.034	0.034	0.057	0.055	0.055	0.052
20	80	0.017	0.016	0.015	0.016	0.025	0.023	0.022	0.023
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.017	0.025	0.026	0.027	0.026
40	20	0.032	0.034	0.034	0.033	0.051	0.056	0.059	0.054
20	20	0.063	0.075	0.079	0.071	0.108	0.130	0.149	0.119
20	40	0.029	0.036	0.035	0.034	0.052	0.056	0.056	0.053
20	80	0.016	0.016	0.015	0.016	0.028	0.024	0.023	0.023

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.3: Variance Estimation – Multinomial MAR Model for $\theta = 1$ Full Data vs. Complete Only With Pseudo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.017	0.016	0.016	0.016	0.027	0.024	0.024	0.023
40	20	0.033	0.033	0.034	0.032	0.044	0.048	0.051	0.047
20	20	0.067	0.072	0.074	0.069	0.103	0.109	0.119	0.102
20	40	0.035	0.033	0.032	0.032	0.050	0.048	0.047	0.046
20	80	0.014	0.016	0.015	0.015	0.021	0.022	0.021	0.021
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.016	0.025	0.024	0.024	0.023
40	20	0.037	0.033	0.034	0.032	0.051	0.048	0.051	0.046
20	20	0.066	0.071	0.073	0.067	0.095	0.101	0.109	0.095
20	40	0.034	0.034	0.033	0.033	0.050	0.049	0.048	0.046
20	80	0.017	0.015	0.014	0.015	0.024	0.021	0.020	0.021
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.017	0.016	0.017	0.016	0.023	0.024	0.024	0.023
40	20	0.031	0.032	0.033	0.032	0.045	0.049	0.052	0.047
20	20	0.059	0.070	0.072	0.066	0.091	0.106	0.115	0.099
20	40	0.028	0.034	0.033	0.033	0.047	0.050	0.050	0.048
20	80	0.016	0.016	0.015	0.016	0.026	0.023	0.022	0.022

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.4: Variance Estimation – Multinomial MAR Model for $\theta = 1$ Full Data vs. Estimated Data With Pseudo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.017	0.016	0.016	0.016	0.029	0.026	0.027	0.026
40	20	0.033	0.033	0.034	0.032	0.050	0.053	0.057	0.052
20	20	0.067	0.072	0.074	0.069	0.114	0.125	0.140	0.116
20	40	0.035	0.033	0.032	0.032	0.052	0.051	0.051	0.049
20	80	0.014	0.016	0.015	0.015	0.021	0.023	0.022	0.022
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.016	0.027	0.026	0.027	0.026
40	20	0.037	0.033	0.034	0.032	0.058	0.054	0.058	0.052
20	20	0.066	0.071	0.073	0.067	0.102	0.117	0.129	0.108
20	40	0.034	0.034	0.033	0.033	0.055	0.053	0.053	0.050
20	80	0.017	0.015	0.014	0.015	0.025	0.023	0.022	0.022
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.017	0.016	0.017	0.016	0.024	0.026	0.027	0.025
40	20	0.031	0.032	0.033	0.032	0.049	0.054	0.057	0.052
20	20	0.059	0.070	0.072	0.066	0.100	0.119	0.133	0.110
20	40	0.028	0.034	0.033	0.033	0.051	0.053	0.054	0.051
20	80	0.016	0.016	0.015	0.016	0.027	0.023	0.023	0.023

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.5: Variance of Variance Estimation – Multinomial MAR Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.9E-5	3.1E-5	2.8E-5	9.4E-5	1.0E-4	9.0E-5	1.2E-4	1.2E-4	1.1E-4
40	20	2.5E-4	2.8E-4	2.4E-4	7.0E-4	8.3E-4	6.2E-4	9.7E-4	1.2E-3	8.7E-4
20	20	2.7E-3	3.3E-3	2.2E-3	1.2E-2	2.1E-2	9.2E-3	1.8E-2	3.2E-2	1.3E-2
20	40	3.9E-4	3.8E-4	3.4E-4	1.1E-3	1.2E-3	9.3E-4	1.3E-3	1.4E-3	1.1E-3
20	80	5.4E-5	5.0E-5	4.8E-5	1.4E-4	1.3E-4	1.2E-4	1.5E-4	1.5E-4	1.3E-4
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.9E-5	3.2E-5	2.8E-5	8.1E-5	8.9E-5	7.7E-5	1.0E-4	1.1E-4	9.7E-5
40	20	2.7E-4	3.1E-4	2.5E-4	7.3E-4	9.4E-4	6.7E-4	1.0E-3	1.4E-3	9.5E-4
20	20	3.5E-3	4.2E-3	2.8E-3	1.5E-2	2.0E-2	1.2E-2	1.7E-2	2.7E-2	1.3E-2
20	40	4.5E-4	5.0E-4	3.8E-4	9.8E-4	1.0E-3	8.2E-4	1.2E-3	1.2E-3	9.9E-4
20	80	5.7E-5	5.1E-5	5.0E-5	1.4E-4	1.3E-4	1.2E-4	1.5E-4	1.4E-4	1.3E-4
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing)0.15$ for $k > K/2$										
80	20	3.1E-5	3.3E-5	3.0E-5	7.7E-5	8.8E-5	7.3E-5	9.4E-5	1.1E-4	9.0E-5
40	20	2.1E-4	2.3E-4	2.0E-4	7.3E-4	9.0E-4	6.5E-4	8.9E-4	1.1E-3	8.0E-4
20	20	2.4E-3	2.8E-3	2.0E-3	1.1E-2	1.6E-2	8.0E-3	1.5E-2	2.3E-2	1.0E-2
20	40	3.4E-4	3.3E-4	3.0E-4	1.0E-3	1.1E-3	8.7E-4	1.2E-3	1.5E-3	1.1E-3
20	80	5.1E-5	4.6E-5	4.6E-5	1.3E-4	1.3E-4	1.2E-4	1.4E-4	1.4E-4	1.3E-4

[1] JK– Jackknifing Method [2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.6: Variance of Variance Estimation – Multinomial MAR Model Estimate for $\theta = 1$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	2.8E-5	3.0E-5	2.7E-5	8.7E-5	9.5E-5	8.4E-5	1.1E-4	1.2E-4	1.1E-4
40	20	2.3E-4	2.5E-4	2.1E-4	6.0E-4	7.0E-4	5.4E-4	8.6E-4	1.0E-3	7.8E-4
20	20	2.1E-3	2.5E-3	1.8E-3	8.3E-3	1.2E-2	6.5E-3	1.3E-2	2.0E-2	9.8E-3
20	40	3.5E-4	3.4E-4	3.1E-4	9.2E-4	9.8E-4	8.0E-4	1.1E-3	1.2E-3	9.6E-4
20	80	5.1E-5	4.8E-5	4.6E-5	1.3E-4	1.2E-4	1.1E-4	1.4E-4	1.4E-4	1.3E-4
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	2.7E-5	3.0E-5	2.7E-5	7.6E-5	8.3E-5	7.2E-5	9.7E-5	1.1E-4	9.3E-5
40	20	2.4E-4	2.8E-4	2.3E-4	6.3E-4	8.0E-4	5.8E-4	9.2E-4	1.2E-3	8.5E-4
20	20	2.7E-3	3.1E-3	2.3E-3	9.3E-3	1.1E-2	7.9E-3	1.2E-2	1.7E-2	9.5E-3
20	40	4.0E-4	4.3E-4	3.4E-4	8.4E-4	8.6E-4	7.1E-4	1.0E-3	1.1E-3	8.9E-4
20	80	4.7E-5	4.2E-5	4.2E-5	1.1E-4	1.0E-4	9.8E-5	1.2E-4	1.2E-4	1.1E-4
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	3.0E-5	3.2E-5	2.9E-5	7.1E-5	8.1E-5	6.8E-5	9.0E-5	1.0E-4	8.5E-5
40	20	1.9E-4	2.1E-4	1.8E-4	6.2E-4	7.5E-4	5.6E-4	7.9E-4	9.7E-4	7.2E-4
20	20	1.9E-3	2.2E-3	1.6E-3	7.4E-3	9.5E-3	5.7E-3	1.1E-2	1.5E-2	8.2E-3
20	40	3.1E-4	3.0E-4	2.7E-4	8.7E-4	9.4E-4	7.5E-4	1.1E-3	1.3E-3	9.7E-4
20	80	4.9E-5	4.4E-5	4.4E-5	1.2E-4	1.2E-4	1.1E-4	1.4E-4	1.3E-4	1.2E-4

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.7: Variance Estimation – Multinomial MAR Model for $\theta = 3$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.180	0.180	0.184	0.176	0.255	0.266	0.277	0.257
40	20	0.310	0.379	0.404	0.360	0.538	0.580	0.646	0.540
20	20	0.811	0.929	1.071	0.814	1.491	1.673	2.136	1.339
20	40	0.405	0.388	0.391	0.363	0.545	0.563	0.596	0.516
20	80	0.161	0.161	0.155	0.155	0.232	0.230	0.226	0.220
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.180	0.176	0.179	0.171	0.260	0.256	0.266	0.248
40	20	0.338	0.376	0.397	0.357	0.499	0.588	0.650	0.542
20	20	0.867	0.947	1.119	0.844	1.454	1.631	2.370	1.352
20	40	0.350	0.375	0.380	0.353	0.581	0.563	0.598	0.519
20	80	0.162	0.170	0.163	0.163	0.233	0.249	0.240	0.234
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.168	0.175	0.179	0.171	0.238	0.264	0.275	0.256
40	20	0.353	0.370	0.393	0.352	0.570	0.582	0.667	0.541
20	20	0.790	0.959	1.085	0.827	1.511	1.929	2.386	1.488
20	40	0.365	0.363	0.367	0.340	0.542	0.560	0.597	0.513
20	80	0.178	0.171	0.163	0.163	0.242	0.254	0.247	0.240

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.8: Variance Estimation – Multinomial MAR Model for $\theta = 3$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.180	0.180	0.184	0.176	0.267	0.283	0.294	0.273
40	20	0.310	0.379	0.404	0.360	0.564	0.616	0.689	0.572
20	20	0.811	0.929	1.071	0.814	1.623	1.799	2.317	1.439
20	40	0.405	0.388	0.391	0.363	0.561	0.587	0.622	0.535
20	80	0.161	0.161	0.155	0.155	0.245	0.236	0.232	0.225
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.180	0.176	0.179	0.171	0.294	0.286	0.298	0.275
40	20	0.338	0.376	0.397	0.357	0.554	0.660	0.733	0.605
20	20	0.867	0.947	1.119	0.844	1.673	1.924	3.008	1.541
20	40	0.350	0.375	0.380	0.353	0.616	0.607	0.649	0.556
20	80	0.162	0.170	0.163	0.163	0.252	0.260	0.252	0.246
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.168	0.175	0.179	0.171	0.257	0.283	0.295	0.273
40	20	0.353	0.370	0.393	0.352	0.636	0.637	0.740	0.587
20	20	0.790	0.959	1.085	0.827	1.569	2.155	2.681	1.624
20	40	0.365	0.363	0.367	0.340	0.575	0.600	0.644	0.545
20	80	0.178	0.171	0.163	0.163	0.246	0.264	0.257	0.249

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.9: Variance Estimation – Multinomial MAR Model for $\theta = 3$ Full Data vs. Complete Only With Pseudo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.172	0.172	0.175	0.169	0.239	0.249	0.258	0.243
40	20	0.282	0.343	0.364	0.332	0.469	0.501	0.547	0.481
20	20	0.666	0.735	0.798	0.686	1.091	1.128	1.327	1.027
20	40	0.369	0.351	0.351	0.335	0.479	0.485	0.505	0.461
20	80	0.154	0.154	0.148	0.149	0.218	0.216	0.211	0.208
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.172	0.168	0.171	0.165	0.243	0.240	0.248	0.234
40	20	0.309	0.341	0.358	0.330	0.437	0.508	0.551	0.482
20	20	0.710	0.748	0.821	0.709	1.065	1.120	1.315	1.042
20	40	0.320	0.339	0.341	0.325	0.507	0.485	0.505	0.463
20	80	0.155	0.162	0.155	0.156	0.220	0.232	0.224	0.221
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.160	0.167	0.170	0.164	0.223	0.247	0.256	0.241
40	20	0.320	0.335	0.353	0.325	0.492	0.498	0.554	0.479
20	20	0.642	0.748	0.796	0.693	1.067	1.217	1.445	1.108
20	40	0.333	0.327	0.329	0.313	0.472	0.479	0.500	0.456
20	80	0.170	0.163	0.156	0.157	0.227	0.237	0.230	0.226

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.10: Variance Estimation – Multinomial MAR Model for $\theta = 3$ Full Data vs. Estimated Data With Pseudo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.172	0.172	0.175	0.169	0.254	0.269	0.279	0.261
40	20	0.282	0.343	0.364	0.332	0.511	0.554	0.611	0.523
20	20	0.666	0.735	0.798	0.686	1.291	1.344	1.708	1.167
20	40	0.369	0.351	0.351	0.335	0.511	0.527	0.550	0.490
20	80	0.154	0.154	0.148	0.149	0.235	0.225	0.221	0.216
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.172	0.168	0.171	0.165	0.280	0.273	0.283	0.264
40	20	0.309	0.341	0.358	0.330	0.503	0.592	0.648	0.552
20	20	0.710	0.748	0.821	0.709	1.324	1.431	1.734	1.243
20	40	0.320	0.339	0.341	0.325	0.559	0.543	0.572	0.508
20	80	0.155	0.162	0.155	0.156	0.241	0.248	0.240	0.236
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.160	0.167	0.170	0.164	0.245	0.269	0.280	0.262
40	20	0.320	0.335	0.353	0.325	0.571	0.568	0.642	0.535
20	20	0.642	0.748	0.796	0.693	1.225	1.523	1.932	1.287
20	40	0.333	0.327	0.329	0.313	0.521	0.536	0.565	0.498
20	80	0.170	0.163	0.156	0.157	0.235	0.251	0.244	0.238

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.11: Variance of Variance Estimation – Multinomial MAR Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.004	0.005	0.004	0.013	0.014	0.011	0.014	0.016	0.013
40	20	0.038	0.050	0.032	0.129	0.181	0.104	0.152	0.215	0.123
20	20	1.256	3.375	0.550	6.369	12.811	2.603	8.123	15.168	3.266
20	40	0.065	0.070	0.051	0.180	0.240	0.124	0.233	0.304	0.150
20	80	0.006	0.005	0.005	0.015	0.015	0.013	0.016	0.016	0.014
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.004	0.005	0.004	0.013	0.014	0.011	0.019	0.021	0.016
40	20	0.036	0.045	0.031	0.164	0.233	0.116	0.192	0.290	0.142
20	20	0.640	1.391	0.432	3.595	17.851	1.987	7.759	103.630	3.133
20	40	0.053	0.063	0.043	0.174	0.234	0.134	0.228	0.327	0.173
20	80	0.007	0.007	0.006	0.019	0.018	0.015	0.022	0.020	0.017
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.004	0.004	0.004	0.013	0.015	0.011	0.015	0.017	0.013
40	20	0.044	0.054	0.036	0.253	0.691	0.179	0.453	1.442	0.304
20	20	1.219	2.102	0.554	16.990	25.045	5.428	21.863	29.898	5.712
20	40	0.051	0.057	0.040	0.182	0.247	0.131	0.284	0.367	0.177
20	80	0.008	0.008	0.007	0.023	0.022	0.018	0.025	0.024	0.020

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table G.12: Variance of Variance Estimation – Multinomial MAR Model Estimate for $\theta = 3$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.004	0.004	0.004	0.010	0.011	0.010	0.013	0.014	0.011
40	20	0.029	0.036	0.026	0.084	0.109	0.075	0.111	0.148	0.095
20	20	0.560	0.791	0.324	1.775	3.240	1.163	3.154	10.654	1.715
20	40	0.049	0.051	0.041	0.113	0.134	0.089	0.163	0.193	0.115
20	80	0.005	0.005	0.004	0.013	0.013	0.011	0.014	0.014	0.012
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.004	0.004	0.003	0.011	0.012	0.010	0.016	0.018	0.014
40	20	0.028	0.033	0.025	0.103	0.132	0.082	0.140	0.194	0.110
20	20	0.316	0.444	0.264	1.106	1.765	0.929	3.095	4.923	1.636
20	40	0.040	0.046	0.035	0.112	0.137	0.097	0.164	0.214	0.133
20	80	0.006	0.006	0.005	0.016	0.015	0.013	0.019	0.018	0.015
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.003	0.004	0.003	0.010	0.012	0.010	0.013	0.015	0.012
40	20	0.032	0.039	0.029	0.136	0.248	0.115	0.273	0.605	0.207
20	20	0.490	0.535	0.310	2.657	4.608	1.875	5.446	12.115	2.510
20	40	0.038	0.041	0.032	0.114	0.138	0.093	0.200	0.234	0.135
20	80	0.007	0.007	0.006	0.019	0.018	0.016	0.022	0.021	0.018

[1] JK– Jackknifing Method [2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.1: Variance Estimation – Multinomial Informative (A, C) Model: for $\theta = 1$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.019	0.017	0.017	0.016	0.029	0.024	0.025	0.024
40	20	0.035	0.036	0.037	0.035	0.057	0.055	0.058	0.053
20	20	0.067	0.075	0.078	0.071	0.097	0.117	0.131	0.109
20	40	0.034	0.034	0.033	0.032	0.055	0.049	0.050	0.047
20	80	0.016	0.016	0.015	0.015	0.023	0.023	0.022	0.022
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$; $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$									
80	20	0.014	0.016	0.016	0.016	0.022	0.026	0.027	0.025
40	20	0.034	0.035	0.036	0.034	0.050	0.051	0.054	0.049
20	20	0.072	0.079	0.083	0.075	0.121	0.124	0.139	0.114
20	40	0.036	0.035	0.035	0.034	0.054	0.051	0.051	0.048
20	80	0.017	0.016	0.015	0.015	0.022	0.022	0.021	0.021
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$; $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.017	0.025	0.025	0.025	0.025
40	20	0.031	0.034	0.035	0.033	0.046	0.050	0.053	0.049
20	20	0.081	0.082	0.086	0.077	0.129	0.125	0.141	0.116
20	40	0.034	0.034	0.033	0.033	0.053	0.051	0.050	0.048
20	80	0.016	0.016	0.015	0.016	0.024	0.024	0.023	0.023

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.2: Variance Estimation – Multinomial Informative (A, C) Model:for $\theta = 1$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.019	0.017	0.017	0.016	0.029	0.026	0.027	0.026
40	20	0.035	0.036	0.037	0.035	0.063	0.059	0.063	0.057
20	20	0.067	0.075	0.078	0.071	0.101	0.125	0.143	0.116
20	40	0.034	0.034	0.033	0.032	0.058	0.052	0.052	0.050
20	80	0.016	0.016	0.015	0.015	0.024	0.023	0.022	0.022
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$									
80	20	0.014	0.016	0.016	0.016	0.024	0.028	0.029	0.028
40	20	0.034	0.035	0.036	0.034	0.056	0.056	0.060	0.054
20	20	0.072	0.079	0.083	0.075	0.137	0.138	0.154	0.126
20	40	0.036	0.035	0.035	0.034	0.059	0.055	0.055	0.052
20	80	0.017	0.016	0.015	0.015	0.024	0.023	0.022	0.023
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$									
80	20	0.018	0.017	0.017	0.017	0.027	0.026	0.027	0.026
40	20	0.031	0.034	0.035	0.033	0.048	0.053	0.056	0.051
20	20	0.081	0.082	0.086	0.077	0.137	0.133	0.152	0.124
20	40	0.034	0.034	0.033	0.033	0.053	0.051	0.052	0.049
20	80	0.016	0.016	0.015	0.016	0.025	0.024	0.023	0.023

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.3: Variance Estimation – Multinomial Informative (A, C) Model:for $\theta = 1$ Full Data vs. Complete Only With Pseudo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.01	0.01	0.016	0.01	0.02	0.024	0.02	0.02
40	20	0.03	0.03	0.035	0.03	0.05	0.052	0.05	0.05
20	20	0.06	0.06	0.072	0.06	0.08	0.104	0.11	0.09
20	40	0.03	0.03	0.031	0.03	0.05	0.048	0.04	0.04
20	80	0.01	0.01	0.015	0.01	0.02	0.022	0.02	0.02
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$									
80	20	0.01	0.01	0.016	0.01	0.02	0.025	0.02	0.02
40	20	0.03	0.03	0.035	0.03	0.04	0.048	0.05	0.04
20	20	0.06	0.07	0.076	0.06	0.10	0.109	0.11	0.10
20	40	0.03	0.03	0.033	0.03	0.05	0.048	0.04	0.04
20	80	0.01	0.01	0.014	0.01	0.02	0.021	0.02	0.02
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$									
80	20	0.01	0.01	0.017	0.01	0.02	0.024	0.02	0.02
40	20	0.03	0.03	0.034	0.03	0.04	0.048	0.05	0.04
20	20	0.07	0.07	0.078	0.07	0.11	0.110	0.12	0.10
20	40	0.03	0.03	0.032	0.03	0.05	0.048	0.04	0.04
20	80	0.01	0.01	0.015	0.01	0.02	0.023	0.02	0.02

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.4: Variance Estimation – Multinomial Informative (A, C) Model:for $\theta = 1$ Full Data vs. Estimated Data With Pseudo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.01	0.01	0.016	0.01	0.02	0.025	0.02	0.02
40	20	0.03	0.03	0.035	0.03	0.06	0.056	0.06	0.05
20	20	0.06	0.06	0.072	0.06	0.09	0.114	0.12	0.10
20	40	0.03	0.03	0.031	0.03	0.05	0.050	0.05	0.04
20	80	0.01	0.01	0.015	0.01	0.02	0.023	0.02	0.02
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$									
80	20	0.01	0.01	0.016	0.01	0.02	0.027	0.02	0.02
40	20	0.03	0.03	0.035	0.03	0.05	0.054	0.05	0.05
20	20	0.06	0.07	0.076	0.06	0.12	0.125	0.13	0.11
20	40	0.03	0.03	0.033	0.03	0.05	0.053	0.05	0.05
20	80	0.01	0.01	0.014	0.01	0.02	0.023	0.02	0.02
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$									
80	20	0.01	0.01	0.017	0.01	0.02	0.026	0.02	0.02
40	20	0.03	0.03	0.034	0.03	0.04	0.051	0.05	0.04
20	20	0.07	0.07	0.078	0.07	0.12	0.121	0.13	0.11
20	40	0.03	0.03	0.032	0.03	0.05	0.049	0.04	0.04
20	80	0.01	0.01	0.015	0.01	0.02	0.024	0.02	0.02

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.5: Variance of Variance Estimation – Multinomial Informative (A, C) Model for $\theta = 1$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	3.1E-5	3.3E-5	3.0E-5	8.7E-5	9.5E-5	8.3E-5	9.8E-5	1.1E-4	9.4E-5
40	20	2.6E-4	2.9E-4	2.4E-4	8.4E-4	1.1E-3	7.6E-4	1.1E-3	1.4E-3	9.9E-4
20	20	2.4E-3	2.8E-3	2.0E-3	8.1E-3	1.2E-2	6.2E-3	9.8E-3	1.6E-2	7.5E-3
20	40	3.1E-4	3.1E-4	2.7E-4	9.8E-4	1.1E-3	8.4E-4	1.1E-3	1.3E-3	1.0E-3
20	80	5.0E-5	4.7E-5	4.6E-5	1.5E-4	1.4E-4	1.3E-4	1.5E-4	1.4E-4	1.3E-4
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$										
80	20	2.2E-5	2.3E-5	2.1E-5	7.9E-5	8.5E-5	7.5E-5	1.0E-4	1.1E-4	9.7E-5
40	20	2.5E-4	2.9E-4	2.4E-4	7.5E-4	9.2E-4	6.7E-4	9.6E-4	1.2E-3	8.7E-4
20	20	2.9E-3	3.7E-3	2.4E-3	1.3E-2	2.0E-2	9.8E-3	2.0E-2	2.6E-2	1.4E-2
20	40	5.0E-4	5.0E-4	4.2E-4	1.2E-3	1.3E-3	1.0E-3	1.5E-3	1.7E-3	1.3E-3
20	80	5.2E-5	4.7E-5	4.7E-5	1.1E-4	1.1E-4	9.8E-5	1.4E-4	1.3E-4	1.3E-4
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$										
80	20	2.8E-5	2.9E-5	2.7E-5	8.3E-5	9.0E-5	7.9E-5	9.4E-5	1.0E-4	9.0E-5
40	20	2.2E-4	2.4E-4	2.0E-4	6.5E-4	7.9E-4	5.9E-4	7.1E-4	8.8E-4	6.4E-4
20	20	3.3E-3	4.2E-3	2.7E-3	9.7E-3	1.7E-2	7.8E-3	1.2E-2	1.8E-2	9.5E-3
20	40	3.3E-4	3.2E-4	2.9E-4	9.9E-4	1.0E-3	8.5E-4	1.1E-3	1.1E-3	9.2E-4
20	80	6.3E-5	5.8E-5	5.7E-5	1.6E-4	1.4E-4	1.4E-4	1.6E-4	1.5E-4	1.4E-4

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.6: Variance of Variance Estimation – Multinomial Informative (A, C) Model for $\theta = 1$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	3.0E-5	3.2E-5	2.9E-5	8.1E-5	8.9E-5	7.8E-5	9.3E-5	1.0E-4	8.9E-5
40	20	2.3E-4	2.6E-4	2.2E-4	7.1E-4	8.7E-4	6.5E-4	9.7E-4	1.2E-3	8.8E-4
20	20	1.9E-3	2.2E-3	1.6E-3	5.5E-3	7.2E-3	4.4E-3	7.3E-3	1.1E-2	5.8E-3
20	40	2.9E-4	2.9E-4	2.6E-4	8.5E-4	9.4E-4	7.3E-4	9.7E-4	1.1E-3	8.4E-4
20	80	4.8E-5	4.5E-5	4.4E-5	1.4E-4	1.3E-4	1.2E-4	1.4E-4	1.3E-4	1.3E-4
$P(missing)_{1k} = P(missing)_{2k} = 0.15$ for $k \leq K/2$ $P(missing)_{1k} = P(missing)_{2k} = 0.4$ for $k > K/2$										
80	20	2.1E-5	2.2E-5	2.0E-5	7.3E-5	7.9E-5	7.0E-5	9.7E-5	1.1E-4	9.3E-5
40	20	2.3E-4	2.6E-4	2.1E-4	6.4E-4	7.8E-4	5.8E-4	8.5E-4	1.1E-3	7.8E-4
20	20	2.3E-3	2.8E-3	1.9E-3	8.4E-3	1.1E-2	6.9E-3	1.4E-2	1.7E-2	1.0E-2
20	40	4.5E-4	4.5E-4	3.8E-4	1.0E-3	1.1E-3	8.8E-4	1.4E-3	1.5E-3	1.2E-3
20	80	5.0E-5	4.5E-5	4.5E-5	1.0E-4	9.9E-5	9.2E-5	1.3E-4	1.3E-4	1.2E-4
$P(missing)_{1k} = 0.15$, and $P(missing)_{2k} = 0.4$ for $k \leq K/2$ $P(missing)_{1k} = 0.4$, and $P(missing)_{2k} = 0.15$ for $k > K/2$										
80	20	2.7E-5	2.8E-5	2.6E-5	7.8E-5	8.4E-5	7.4E-5	8.9E-5	9.8E-5	8.5E-5
40	20	2.0E-4	2.2E-4	1.8E-4	5.6E-4	6.7E-4	5.1E-4	6.4E-4	7.9E-4	5.8E-4
20	20	2.6E-3	3.2E-3	2.2E-3	6.5E-3	8.9E-3	5.6E-3	8.8E-3	1.2E-2	7.4E-3
20	40	3.0E-4	2.9E-4	2.6E-4	8.5E-4	8.9E-4	7.4E-4	9.6E-4	9.9E-4	8.3E-4
20	80	6.1E-5	5.5E-5	5.4E-5	1.5E-4	1.3E-4	1.3E-4	1.5E-4	1.4E-4	1.3E-4

[1] JK– Jackknifing Method   [2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.7: Variance Estimation – Multinomial Informative (A, C) Model:for $\theta = 3$ Full Data vs. Complete Only

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.177	0.175	0.179	0.171	0.272	0.269	0.280	0.260
40	20	0.298	0.366	0.386	0.347	0.482	0.588	0.660	0.541
20	20	0.816	0.974	1.155	0.839	1.403	1.707	2.686	1.376
20	40	0.408	0.375	0.382	0.352	0.608	0.590	0.639	0.538
20	80	0.167	0.175	0.168	0.167	0.240	0.260	0.256	0.246
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.157	0.172	0.176	0.168	0.263	0.294	0.308	0.282
40	20	0.344	0.380	0.405	0.360	0.506	0.594	0.662	0.550
20	20	0.888	0.925	1.070	0.815	1.435	1.638	1.951	1.289
20	40	0.316	0.372	0.373	0.345	0.450	0.537	0.569	0.489
20	80	0.157	0.166	0.160	0.160	0.220	0.240	0.235	0.228
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.175	0.172	0.176	0.168	0.253	0.260	0.271	0.251
40	20	0.391	0.384	0.408	0.364	0.544	0.593	0.657	0.550
20	20	0.642	0.940	1.126	0.821	1.330	1.703	2.326	1.382
20	40	0.329	0.368	0.370	0.343	0.491	0.562	0.595	0.510
20	80	0.163	0.176	0.170	0.168	0.224	0.256	0.252	0.242

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.8: Variance Estimation – Multinomial Informative (A, C) Model:for $\theta = 3$ Full Data vs. Estimated Data

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.177	0.175	0.179	0.171	0.290	0.282	0.294	0.272
40	20	0.298	0.366	0.386	0.347	0.512	0.627	0.707	0.575
20	20	0.816	0.974	1.155	0.839	1.370	1.931	2.966	1.462
20	40	0.408	0.375	0.382	0.352	0.620	0.601	0.651	0.549
20	80	0.167	0.175	0.168	0.167	0.238	0.265	0.261	0.251
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.157	0.172	0.176	0.168	0.292	0.322	0.338	0.308
40	20	0.344	0.380	0.405	0.360	0.598	0.666	0.741	0.611
20	20	0.888	0.925	1.070	0.815	1.461	1.782	2.249	1.410
20	40	0.316	0.372	0.373	0.345	0.491	0.578	0.609	0.524
20	80	0.157	0.166	0.160	0.160	0.227	0.254	0.249	0.242
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.175	0.172	0.176	0.168	0.261	0.273	0.286	0.264
40	20	0.391	0.384	0.408	0.364	0.590	0.623	0.692	0.576
20	20	0.642	0.940	1.126	0.821	1.412	1.830	2.555	1.469
20	40	0.329	0.368	0.370	0.343	0.506	0.577	0.614	0.526
20	80	0.163	0.176	0.170	0.168	0.233	0.260	0.256	0.247

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.9: Variance Estimation – Multinomial Informative (A, C) Model:for $\theta = 3$ Full Data vs. Complete Only With Pseudo-Tables

Tables		Full Data				Complete Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.169	0.167	0.170	0.164	0.254	0.251	0.260	0.245
40	20	0.273	0.332	0.348	0.320	0.421	0.505	0.555	0.480
20	20	0.654	0.757	0.829	0.700	0.981	1.130	1.359	1.041
20	40	0.369	0.338	0.341	0.324	0.525	0.503	0.530	0.477
20	80	0.160	0.167	0.160	0.161	0.226	0.242	0.238	0.233
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.150	0.165	0.168	0.162	0.244	0.272	0.284	0.265
40	20	0.314	0.344	0.364	0.333	0.444	0.512	0.560	0.489
20	20	0.715	0.731	0.796	0.686	1.011	1.053	1.224	0.978
20	40	0.288	0.336	0.334	0.318	0.394	0.464	0.476	0.436
20	80	0.150	0.158	0.152	0.153	0.207	0.225	0.219	0.216
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.168	0.165	0.168	0.162	0.237	0.243	0.252	0.237
40	20	0.355	0.347	0.366	0.336	0.473	0.510	0.554	0.488
20	20	0.529	0.746	0.826	0.691	0.969	1.146	1.393	1.053
20	40	0.300	0.333	0.333	0.317	0.430	0.483	0.500	0.455
20	80	0.156	0.168	0.162	0.162	0.210	0.239	0.234	0.229

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.10: Variance Estimation – Multinomial Informative (A, C) Model:for $\theta = 3$ Full Data vs. Estimated Data With Pseudo-Tables

Tables		Full Data				Estimated Data			
LC	N_k	"True"[1]	JK[2]	B.[3]	A.F.[4]	"True"	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k									
80	20	0.169	0.167	0.170	0.164	0.276	0.268	0.279	0.260
40	20	0.273	0.332	0.348	0.320	0.463	0.561	0.621	0.524
20	20	0.654	0.757	0.829	0.700	1.073	1.388	1.709	1.169
20	40	0.369	0.338	0.341	0.324	0.559	0.535	0.569	0.500
20	80	0.160	0.167	0.160	0.161	0.228	0.252	0.247	0.240
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$									
80	20	0.150	0.165	0.168	0.162	0.278	0.306	0.321	0.294
40	20	0.314	0.344	0.364	0.333	0.543	0.597	0.655	0.558
20	20	0.715	0.731	0.796	0.686	1.145	1.308	1.608	1.135
20	40	0.288	0.336	0.334	0.318	0.446	0.519	0.537	0.480
20	80	0.150	0.158	0.152	0.153	0.217	0.242	0.237	0.232
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$									
80	20	0.168	0.165	0.168	0.162	0.249	0.260	0.271	0.253
40	20	0.355	0.347	0.366	0.336	0.532	0.558	0.611	0.525
20	20	0.529	0.746	0.826	0.691	1.118	1.355	1.745	1.182
20	40	0.300	0.333	0.333	0.317	0.460	0.517	0.541	0.481
20	80	0.156	0.168	0.162	0.162	0.222	0.247	0.243	0.236

[1] "True"– The real variance of the Common Odds Ratio Estimate from simulations

[2] JK– Jackknifing Method [3] B. – Bootstrap Method [4] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.11: Variance of Variance Estimation – Multinomial Informative (A, C) Model for $\theta = 3$

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.004	0.004	0.004	0.015	0.017	0.013	0.017	0.020	0.015
40	20	0.033	0.039	0.028	0.177	0.273	0.126	0.196	0.305	0.142
20	20	1.363	2.899	0.702	7.156	45.004	3.162	15.476	78.718	3.774
20	40	0.079	0.096	0.058	0.250	0.394	0.173	0.243	0.379	0.177
20	80	0.009	0.009	0.008	0.025	0.027	0.020	0.026	0.029	0.021
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.004	0.004	0.003	0.020	0.023	0.017	0.025	0.029	0.021
40	20	0.036	0.044	0.030	0.127	0.174	0.099	0.177	0.241	0.135
20	20	0.842	1.637	0.559	27.527	6.812	6.163	19.569	11.375	5.407
20	40	0.057	0.059	0.042	0.143	0.235	0.103	0.162	0.217	0.118
20	80	0.007	0.007	0.006	0.017	0.017	0.014	0.019	0.019	0.016
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.004	0.004	0.004	0.012	0.013	0.010	0.013	0.015	0.012
40	20	0.043	0.052	0.037	0.153	0.208	0.119	0.184	0.248	0.142
20	20	0.672	1.693	0.385	5.296	14.402	2.290	7.797	21.650	3.093
20	40	0.050	0.055	0.038	0.177	0.241	0.119	0.197	0.274	0.134
20	80	0.009	0.009	0.008	0.024	0.024	0.019	0.024	0.024	0.019

[1] JK– Jackknifing Method

[2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

Table H.12: Variance Estimation – Informative (A, C) Model Multinomial: Variance of Variance Estimate for $\theta = 3$ With Pseudo-Tables

Tables		Full Data			Complete Data			Estimated Data		
LC	N_k	JK[1]	B.[2]	A.F.[3]	JK	B.	A.F.	JK	B.	A.F.
$P_{1k}(missing) = 0.15$ and $P_{2k}(missing) = 0.4$ for all k										
80	20	0.004	0.004	0.003	0.012	0.014	0.011	0.014	0.017	0.013
40	20	0.025	0.029	0.022	0.111	0.152	0.089	0.139	0.198	0.108
20	20	0.521	0.706	0.378	1.525	2.802	1.254	4.229	6.947	1.743
20	40	0.053	0.060	0.044	0.144	0.183	0.118	0.167	0.226	0.133
20	80	0.008	0.008	0.007	0.021	0.022	0.017	0.023	0.024	0.019
$P_{1k}(missing) = P_{2k}^{missing} = 0.15$ for $k \leq K/2$ $P_{1k}(missing) = P_{2k}^{missing} = 0.4$ for $k > K/2$										
80	20	0.003	0.003	0.003	0.016	0.018	0.014	0.021	0.025	0.019
40	20	0.027	0.033	0.024	0.084	0.108	0.072	0.131	0.169	0.105
20	20	0.372	0.503	0.327	2.492	3.462	1.759	4.432	11.221	2.162
20	40	0.042	0.043	0.033	0.093	0.109	0.074	0.120	0.143	0.092
20	80	0.006	0.006	0.006	0.014	0.014	0.012	0.017	0.017	0.014
$P_{1k}(missing) = 0.15$, and $P_{2k}(missing) = 0.4$ for $k \leq K/2$ $P_{1k}(missing) = 0.4$, and $P_{2k}(missing) = 0.15$ for $k > K/2$										
80	20	0.003	0.004	0.003	0.010	0.011	0.009	0.011	0.013	0.010
40	20	0.032	0.038	0.029	0.098	0.121	0.085	0.131	0.165	0.108
20	20	0.334	0.509	0.235	1.347	2.616	0.984	2.532	6.046	1.504
20	40	0.038	0.040	0.030	0.112	0.130	0.085	0.140	0.171	0.102
20	80	0.008	0.008	0.007	0.020	0.020	0.016	0.020	0.020	0.017

[1] JK– Jackknifing Method [2] B. – Bootstrap Method

[3] A.F. – $V(\hat{R}) \doteq \frac{1-f}{n\bar{X}^2} \frac{\sum_1^N (y_i - Rx_i)^2}{N-1}$

BIBLIOGRAPHY

[1] Agresti, A. *Categorical Data Analysis* (John Wiley & sons, Inc New York, 1990).

[2] Baker, S. G. "Marginal Regression for Repeated Binary Data with Outcome Subject to Non-ignorable Nonresponse." Biometrics, **51** 1042 (1995)

[3] Baker, S. G. "Analyzing a Randomized Cancer Prevention Trial With a Missing Binary Outcome, an Auxiliary Variable, and All-or-None Compliance" Journal of the American Statistical Association, **95**(449) 43 (2000)

[4] Baker, S. G. and N. M. Laird, "Regression Analysis With Categorical Data Subject to Nonignorable Nonresponse" Journal of the American Statistical Association, **83** 62 (1988) HA1.A6

[5] Baker, S. G., W. F. Rosenberger and R. Dersimonian, " Closed-Form Estimates for Missing counts in Two-Way contingency Tables" Statistics in Medicine, **11** 643 (1992)

[6] Bishop, Y. M. N., S. E. Fienberg, and P. W. Holland, (with the collaboration of Richard J. Light and Frederick Mosteller) *Discrete Multivariate Analysis* (The MIT Press, 1975)

[7] Breslow, N. E. "Odds Ratio Estimators When the Data are Sparse" Biometrika, **68**(1) 73 (1981)

[8] Breslow, N. E., J. M. Robins and J. A. Wellner. "On the semi-parametric efficiency of logistic regression under case-control sampling Bernoulli" **6**(3) 447 (2000)

[9] Breslow, N. E. and K. Y. Liang "The Variance of the Mantel-Haenszel Estimator" Biometrics, **38** 943 (1982)

[10] Brown, C. H., A. Indurkhya and S. G. Kellam. "Power Calculations for Data Missing by Design: Applications to a Follow-Up Study of Lead Exposure and Attention" Journal of the American Statistical Association, **95** 383 (2000)

[11] Chambers, R.L. and A.H. Welsch "Log-Linear Models for Survey Data With Non-Ignorable Non-Response" Journal of the Royal Statistical Society, Ser. B,**55** 157 (1993)

[12] Cochran, W. G., *Sampling Techniques, 3rd Edition* (John Wiley & Sons Inc. 1977)

[13] D'Agostino, R. B. Jr. and D. B. Rubin, "Estimating and Using Propensity Scores With Partially Missing Data" Journal of the American Statistical Association, **95** 749 (2000)

[14] David, F. N. and D. E. Barton. *Combinatorial Chance* (Hafner Publishing Company New York 1962)

[15] Follmann, D. A. and M. Wu "An approximate generalized linear model with random effects for informative missing data." Biometrics, **51**(1) 151 (1995)

[16] Follmann, D. A. and D. Lambert "Identifiability of finite mixtures of logistic regression models" Journal of Statistical Planning Inference, **27**(3) 375 (1991)

[17] Follmann, D. A. and D. Lambert " Generalizing Logistics Regression by Nonparametric Mixing" Journal of American Statistical Association, **84**(405) 295 (1989)6

[18] Greenland, S. and J. M. Robins "Estimation of a Common Effect Parameter from Sparse Follow-Up Data" Biometrics, **41**(1) 55 (1985)

[19] Hauck, W. W., "The Large Sample Variance of the Mantel-Haenszel Estimator of a Common Odds Ratio" Biometrics, **35** 817 (1979)

[20] Hauck, W. W. "A comparative Study of Conditional Maximum Likelihood Estimation of a Common Odds Ratio" Biometrics **40** 1117 (1984)

[21] Hauck, W. W., S. Anderson, and F. J. Keahy, III "Finite-Sample Properties of Some Old and Some New Estimators of a Common Odds Ratio from Multiple 2×2 Tables" Journal of the American Statistical Association **77** 145 (1982)

[22] Heitjan, D.F. "Causal Inference in Clinical Trials, A Comparative Example" Controlled Clinical Trials, **20** 309 (1999)

[23] Horowitz, J. L. and C. F. Manski, " Nonparametric Analysis of Randomized Experiments With Missing Covariate and Outcome Data" Journal of American Statistical Association, **95** 77 (2000)

[24] Huber, P. J. *Robust Statistics* (John Wiley & Sons, Inc. 1981)

[25] Huber, P. J. *Robust Statistical Procedures, Second Edition* (Society for Industrial and Applied Mathematics, 1996)

[26] Ibrahim, J. G., S. R. Lipsitz, and M-H Chen, " Missing Covariates in Generalized Linear Models When the Missing Data Mechanism is Non-ignorable" Journal of the Royal Statistical Society, Ser. B., **61** 173 (1999)

[27] Mantel, N. and Haenszel, W. "Statistical aspects of the analysis of data from retrospective studies of disease." J. Natl. Cancer Inst., **22** 719 (1959)

[28] Kendall, M. and A. Stuart *The Advanced Theory of Statistics, Volume 1, Fourth Edition* (Macmillan Publishing Co., Inc. 1977)

[29] Gupta, V. K., K. Lal and L. Bhar "Robustness of designed experiments against missing data" Journal of Applied Statistics, **28**(1) 63 (2001)

[30] Little, R. J. A. and D. B. Rubin, *Statistical Analysis With Missing Data* (John Wiley & Sons, Inc. New York, 1987).

[31] Meng, X.-L. "Missing Data: Dial M for ???" Journal of American Statistical Association, **95** 1325 (2000)

[32] Molenberghs, G., E. J. T. Goetghebeur, S. R. Lipsitz and M. G. Kenward " Nonrandom Missingness in Categorical Data: Strengths and Limitations" The American Statistician, **53**(2) 110 (1999)

[33] Molenberghs, G., M. G. Kenward and E. Lesaffre "The Analysis of Longitudinal Ordinal Data With Nonrandom Drop-Out" Biometrika, **84** 33 (1997)

[34] Murdoch, D. J. and P. J. Green "Exact Sampling From a Continuous State Space Scandinavian" Journal of Statistics, **25** 483 (1998)

[35] Pigeot, I. and H. Strugholtz, "Jackknifing the Mantel-Haenszel Estimator in Sparse contingency Tables" Austral. & New Zealand J. Statist, **40**(4), 433 (1998)

[36] Phillips, A. and P.W. Holland "Estimators of the Variance of the Mantel-Haenszel Log-Odds-Ratio Estimate" Biometrics, **43**(2) 425 (1987)

[37] Preisser, J. S., A. T. Galecki, K. K. Lohman, and L. E. Wagenknecht "Analysis of Smoking Trends With Incomplete Longitudinal Binary Responses" Journal of American Statistical Association, **95** 1021 (2000)

[38] Propp, J. G. and D. B. Wilson "Exact Sampling With Coupled Markov Chains and Applications to Statistical Mechanics" Random Structures and Algorithms, **9** 223 (1996)

[39] Quenouile, M. H. "Notes on Bias in Estimation" Biometrika, **43** 353 (1956)

[40] Robins, J., N. Breslow and S. Greenland "Estimators of the Mantel-Haenszel Variance Consisten in Both Sparse Data and Large-Strata Limiting Models" Biomatrics, **42** 311 (1986)

[41] Rohatgi, V. K. *An Introduction to Probability Theory and Mathematical Statistics* (John Wiley & sons, Inc New York, 1975).

[42] Santner, T. J. and D. E. Duffy *The Statistical analysis of Discrete Data* (Springer-Verlag, New York, 1989)

[43] Scharfstein, D. O. and A. Rotnitzky and J. M. Robins "Adjusting for Nonignorable Drop-out Using Semiparametric Nonresponse Models" (with Discussion) Journal of American Statistical Association, **94** 1096 (1999)

[44] Schluchter, M. D. "Methods for the Analysis of Informatively Censored Longitudinal Data" Statistics in Medicine, **11** 1861 (1992)

[45] Slud, E. V. and L. V. Rubinstein "Dependent Censoring Completing Risks and Summary Survival Curves" Biometrika, **70** 643 (1983)

[46] Wu, M. C. and K. R. Bailey "Estimation and Comparison of Changes in the Presence of Informative Right Censoring: Conditional Linear Model" Biometrics, **45** 939 (1989)

[47] Wu, M. C. and R. J. Carroll "Estimation and Comparison of Changes in the Presence of Informative Right Censoring by Modeling the Censoring Process" Biometrics, **44** 175 (1988)

[48] Wypij, D. and T. J. Santner, "Pseudotable Methods for the Analysis of 2×2 Tables" Computational Statistics & Data Analysis, **13** 173 (1992)