ABSTRACT

Title of dissertation:	A TANNAKIAN FRAMEWORK FOR G-DISPLAYS AND RAPOPORT-ZINK SPACES
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We develop a Tannakian framework for group-theoretic analogs of displays, originally introduced by Bültel and Pappas, and further studied by Lau. We use this framework to define Rapoport-Zink functors associated to triples $(G, \{\mu\}, [b])$, where G is a flat affine group scheme over $\mathbb{Z}_p$ and μ is a cocharacter of G defined over a finite unramified extension of $\mathbb{Z}_p$. We prove these functors give a quotient stack presented by Witt vector loop groups, thereby showing our definition generalizes the group-theoretic definition of Rapoport-Zink spaces given by Bültel and Pappas. As an application, we prove a special case of a conjecture of Bültel and Pappas by showing their definition coincides with that of Rapoport and Zink in the case of unramified EL-type local Shimura data.

A TANNAKIAN FRAMEWORK FOR G-DISPLAYS AND RAPOPORT-ZINK SPACES

by

Patrick James Daniels

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Advisory Committee:
Professor Thomas J. Haines, Chair/Advisor
Professor Lawrence C. Washington
Professor Niranjan Ramachandran
Professor Patrick Brosnan
Professor Kaustubh Agashe, Dean's Representative

Dedication

For Ali and Malcolm.

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Chapter 1: Introduction

Let p be a fixed prime. The theory of displays, developed by Zink in [1], is a generalization of Dieudonné theory for formal p-divisible groups. By results of Zink and Lau [2], formal p-divisible groups over p-adically complete and separated $\mathbb{Z}_p$-algebras are classified by their associated displays. When G is a reductive group over $\mathbb{Z}_p$ and μ is a minuscule cocharacter of G, Bültel and Pappas [3] defined group-theoretic analogs of displays, called (G, μ)-displays, with the intention of using these objects to stand in for p-divisible groups with G-structure in a general definition of Rapoport-Zink spaces. When $G = \mathrm{GL}_n$ and μ is the cocharacter $t \mapsto (1^{(d)}, t^{(n-d)})$, the category of (G, μ)-displays over a p-nilpotent $\mathbb{Z}_p$-algebra R is equivalent to the category of Zink displays over R. In this way the theory of (G, μ)-displays naturally generalizes Zink's theory of displays.

In a different direction, Langer and Zink [4] defined a category of higher displays, which contains Zink's displays as a full subcategory. Recently, Lau [5] reformulated the framework for higher displays in such a way as to allow for a uniform treatment of a number of display-like objects, including Dieudonné displays as in [6], F-zips as in [7], and Frobenius gauges as in [8]. Further, Lau used his framework to give a general definition of G-displays of type μ for an arbitrary smooth group scheme G and cocharacter μ. When G is reductive and μ is minuscule, the category of G-displays of type μ coincides with the category of (G, μ)-displays defined by Bültel and Pappas. Hence Lau's work can be seen as a way to link Langer and Zink's generalization of the theory of displays with that of Bültel and Pappas.

In this thesis we offer another way to relate these two theories by developing a Tannakian framework for (G, μ)-displays. More precisely, we define a Tannakian G-display to be an exact tensor functor from the category of representations of G on finite free $\mathbb{Z}_p$-modules to the category of higher displays. That such a definition is reasonable is suggested by the following general mantra. Let G be a group, and let **Cat** be an exact tensor category. Then an object in **Cat** endowed with G-structure should manifest itself in two ways: as a torsor for G (or for some closely related group) perhaps with some additional structures, and as an exact tensor functor from the category of representations of G to **Cat**. The relation between these two interpretations is well-known when **Cat** is the category of vector bundles over a scheme S, cf. [9]. This principle has been notably applied in the case where **Cat** is the category of isocrystals over a field k in [10], and where **Cat** is the category of F-zips over a field k in [11]. In our situation, Lau's theory offers the torsor-theoretic definition of G-displays, and we contribute a Tannakian version of the theory.

When G is a classical group, there is often a third interpretation: An object in **Cat** is said to be endowed with G-structure if it is equipped with some additional structures corresponding to the group G, such as a bilinear form or actions on the object by a semisimple algebra. The Tannakian framework for objects with G-structure is closely related to this third interpretation. Indeed, given an exact tensor functor $\mathscr{F} : \mathbf{Rep}(G) \to \mathbf{Cat}$, we can obtain an object in $\mathbf{Cat}$ with additional structures by evaluating $\mathscr{F}$ on the faithful representation which gives the embedding of G into some GL_n. In the case where G is an orthogonal group, Lau applies this principle to prove that G-displays correspond to displays equipped with a perfect symmetric bilinear form, cf. [5, Proposition 5.5.2]. This can be seen as a special case of the Tannakian framework we develop in this paper.

Since our definition of Tannakian (G, μ)-displays extends the definition of (G, μ)-displays in [3], we can use it to define a natural generalization of the Rapoport-Zink functor defined there. Our Tannakian framework proves advantageous in this regard, because it brings the theory closer to Zink's original theory of displays, and therefore to the theory of p-divisible groups. In particular, we prove that Bültel and Pappas's Rapoport-Zink functor is representable by the classical Rapoport-Zink space in the case where the data of definition is of EL-type. This proves a conjecture of Bültel and Pappas in this special case.

Let us describe our results in more detail. Let R be a p-adic ring, and denote by $W(R)$ the ring of p-typical Witt vectors for R, which is equipped with Frobenius f and Verschiebung v. Then, following Lau [5], we define a graded variant of the Witt ring $W(R)$, which we denote by $W(R)^{\oplus}$ (cf. Definition 2.1.2). This ring is equipped with homomorphisms $\sigma, \tau : W(R)^{\oplus} \to W(R)$ such that the triple $\underline{W}(R) = (W(R)^{\oplus}, \sigma, \tau)$ becomes a higher frame in the sense of *loc. cit.* Pairs (M, F) consisting of a finite projective graded $W(R)^{\oplus}$-module M and a σ-linear bijection of $W(R)^{\oplus}$-modules $F : M \to M \otimes_{W(R)^{\oplus}, \tau} W(R)$ are called displays over $\underline{W}(R)$, cf. Definition 2.3.2. The categories of finite projective graded $W(R)^{\oplus}$-modules and of displays over $\underline{W}(R)$ are exact rigid tensor categories.

Let G be a flat affine group scheme of finite type over $\mathbb{Z}_p$, let k_0 be a finite extension of $\mathbb{F}_p$, and let $\mu : \mathbb{G}_{m,W(k_0)} \to G_{W(k_0)}$ be a cocharacter of G defined over $W(k_0)$. Lau associates to G and μ a group, called the display group, as follows. Let $G = \text{Spec } A$. Then $\mathbb{G}_{m,W(k_0)}$ acts on A via conjugation and therefore determines a $\mathbb{Z}$-grading on A. The display group $L^+_\mu G(R)$ (denoted $G(W(R)^\oplus)_\mu$ in [5]) is the subset of $G(W(R)^\oplus)$ consisting of homomorphisms $A \to W(R)^\oplus$ which preserve the respective gradings. Our first result is to give an interpretation of this group as the collection of tensor automorphisms of a certain fiber functor. This result can be seen as an analog of Tannakian duality in this setting.

Denote by $\mathbf{PGrMod}(W(R)^\oplus)$ the category of finite projective graded $W(R)^\oplus$-modules. Associated to G and μ we define a canonical graded fiber functor for every p-adic ring R

$$\mathscr{C}_{\mu,R} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}(W(R)^\oplus)$$

by assigning to (V,ρ) the $W(R)^\oplus$-module $V_{W(k_0)} \otimes_{W(k_0)} W(R)^\oplus$, where $V_{W(k_0)}$ is endowed with a natural $\mathbb{Z}$-grading via the action of $\mathbb{G}_{m,W(k_0)}$. Define $\text{Aut}^\otimes(\mathscr{C}_{\mu,R})$ to be the collection of tensor automorphisms of this functor. As R varies, this defines an fpqc sheaf in groups, denoted $\underline{\text{Aut}}^\otimes(\mathscr{C}_\mu)$. For any $h \in L^+_\mu G(R)$, the collection $\{\rho(h)\}_{(V,\rho)}$, where (V,ρ) varies over all representations of G on finite free $\mathbb{Z}_p$-modules, defines an element of $\text{Aut}^\otimes(\mathscr{C}_{\mu,R})$.

Theorem 1.0.1. *The association $h \mapsto \{\rho(h)\}$ defines an isomorphism of fpqc* *sheaves of groups*

$$L^+_\mu G \xrightarrow{\sim} \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}_\mu).$$

Let $\mathbf{Disp}(\underline{W}(R))$ be the category of displays over the frame $\underline{W}(R)$. We define a Tannakian G-display to be an exact tensor functor

$$\mathscr{D} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{Disp}(\underline{W}(R)).$$

We say $\mathscr{D}$ is a Tannakian (G, μ)-display if the underlying functor to the category of finite projective $W(R)^{\oplus}$-modules is fpqc-locally isomorphic to $\mathscr{C}_{\mu,R}$. Our second main result is the connection between this definition and that of Lau [5]. Let us recall his definition of G-displays of type μ.

Denote by L^+G the Witt loop group, i.e. the functor on p-nilpotent $W(k_0)$-algebras $R \mapsto G(W(R))$. Both the display group $L^+_\mu G$ and the Witt loop group L^+G are representable by [5, Lemma 5.4.1]. The ring homomorphisms σ and τ induce group homomorphisms $\sigma, \tau : L^+_\mu G \to L^+G$, and the display group acts naturally upon the Witt loop group via

$$L^+G(R) \times L^+_\mu G(R) \mapsto L^+G(R),\ (g,h) \mapsto \tau(h)^{-1} \cdot g \cdot \sigma(h). \tag{1.0.1}$$

Lau defines the stack of G-displays of type μ to be the fpqc-quotient stack $[L^+G/L^+_\mu G]$ with respect to this action. Explicitly, a G-display of type μ over a p-nilpotent $W(k_0)$-algebra R can be interpreted as a pair (Q, α), where Q is an $L^+_\mu G$-torsor over

R, and $\alpha : Q \to L^+G$ is a map which is $L^+_\mu G$-equivariant with respect to the action (1.0.1).

Now we can naturally associate a G-display of type μ to any Tannakian (G,μ)-display $\mathscr{D}$ over a p-nilpotent $W(k)$-algebra R. Denote by υ_R the forgetful functor from the category of displays over $\underline{W}(R)$ to the category of finite projective graded $W(R)^\oplus$-modules. Then the fpqc sheaf in groups

$$Q_{\mathscr{D}} := \underline{\mathrm{Isom}}^\otimes(\mathscr{C}_{\mu,R}, \upsilon_R \circ \mathscr{D})$$

consisting of isomorphisms of tensor functors between $\mathscr{C}_{\mu,R}$ and $\upsilon_R \circ \mathscr{D}$ is naturally an $L^+_\mu G$-torsor. For any representation (V,ρ), write $\mathscr{D}(V,\rho) = (M(\rho), F(\rho))$ for the corresponding display over $\underline{W}(R)$. Given an isomorphism of tensor functors $\lambda : \mathscr{C}_{\mu,R'} \xrightarrow{\sim} \upsilon_{R'} \circ \mathscr{D}_{R'}$ defined over an R-algebra R', the collection $\{F(\rho)\}_{(V,\rho)}$ determines an element $\alpha_{\mathscr{D}}(\lambda)$ of $L^+G(R')$. One checks that the assignment $\lambda \mapsto \alpha_{\mathscr{D}}(\lambda)$ is $L^+_\mu G$-equivariant, so the pair $(Q_{\mathscr{D}}, \alpha_{\mathscr{D}})$ determines a G-display of type μ.

Theorem 1.0.2. *The association $\mathscr{D} \mapsto (Q_{\mathscr{D}}, \alpha_{\mathscr{D}})$ defines an equivalence of categories between Tannakian (G,μ)-displays over R and G-displays of type μ over $\underline{W}(R)$.*

This equivalence follows from Theorem 3.3.5 below. In Appendix A we prove the analogous theorem for frames which naturally form an étale sheaf on Spec R. This includes most frames of interest in Dieudonné theory, including a truncated variant of the Witt frame, and the so-called Zink frame, which is used in the study of Dieudonné displays [6].

When G is reductive and μ is minuscule, the category of G-displays of type μ over $\underline{W}(R)$ is equivalent to the category of (G,μ)-displays over R by [5, Remark 6.3.4]. Hence the same holds for the category of Tannakian (G,μ)-displays over R.

Finally, let us discuss the connection with Rapoport-Zink spaces. If R is a p-adic ring, an isodisplay over R is a pair (N,φ) consisting of a finitely generated projective $W(R)[1/p]$-module N and an f-linear automorphism φ of N. By generalizing a construction in [1], we can associate to any display over $\underline{W}(R)$ an isodisplay over R. This association defines an exact tensor functor $\mathbf{Disp}(\underline{W}(R)) \to \mathbf{Isodisp}(R)$, and by composing a Tannakian (G,μ)-display with this functor we obtain a functor $\mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{Isodisp}(R)$, which denote $\mathscr{D}[1/p]$. Such an object is called a G-isodisplay. A G-quasi-isogeny of Tannakian (G,μ)-displays is an isomorphism of their corresponding G-isodisplays.

Now let k be an algebraic closure of $\mathbb{F}_p$, and let $(G,\{\mu\},[b])$ be a local Shimura datum, so G is a smooth affine group scheme over $\mathbb{Z}_p$ whose generic fiber is reductive, $\{\mu\}$ is a geometric conjugacy class of cocharacters, and $[b]$ is a σ-conjugacy class of elements in $G(W(k)[1/p])$. The triple $(G,\{\mu\},[b])$ is required to satisfy certain axioms, see Definition 4.1.2. To a choice of (μ,b) satisfying some conditions (cf. Definition 4.1.3) we associate a Tannakian (G,μ)-display $\mathscr{D}_0$ over k. Following [3], we then define $\mathrm{RZ}_{G,\mu,b}$ as the functor on p-nilpotent $W(k)$-algebras which assigns to any R the set of isomorphism classes of pairs $(\mathscr{D},\iota)$, where

- $\mathscr{D}$ is a Tannakian (G,μ)-display over R,
- $\iota : \mathscr{D}_{R/pR} \dashrightarrow (\mathscr{D}_0)_{R/pR}$ is a G-quasi-isogeny.

The functor $\mathrm{RZ}_{G,\mu,b}$ can be interpreted as the functor of isomorphism classes associated to an fpqc stack $\mathbf{RZ}_{G,\mu,b}$ on the site of p-nilpotent $W(k)$-algebras. This stack can, in turn, be expressed explicitly as a quotient stack in terms of Witt vector loop groups. The functor $R \mapsto G(W(R)[1/p])$ is representable by an group ind-scheme LG over $\mathbb{Z}_p$ (cf. [12]), so we can form the fiber product $L^+G \times_{m_\mu, c_b} LG$ whose points in a $W(k)$-algebra R are pairs (U, g) with $U \in L^+G(R)$ and $g \in LG(R)$ such that

$$g^{-1} \cdot b \cdot f(g) = U \cdot \mu^\sigma(p).$$

From this we form the quotient stack $[(L^+G \times_{m_\mu, c_b} LG)/L^+_\mu G]$ with respect to the following action of $L^+_\mu G$, which is well-defined by Lemma 4.2.1:

$$(U, g) \cdot h = (\tau(h)^{-1} \cdot U \cdot \sigma(h), g \cdot \tau(h)).$$

The following, which is Theorem 4.2.2 below, is a generalization of [3, §4.2.3].

Theorem 1.0.3. *There is an isomorphism of stacks*

$$\mathbf{RZ}_{G,\mu,b} \xrightarrow{\sim} [(L^+G \times_{m_\mu, c_b} LG)/L^+_\mu G].$$

It is a consequence of this theorem that when G is reductive and μ is minuscule, the Rapoport-Zink functor defined above coincides with the one defined in [3], cf. Proposition 5.1.1. In *loc. cit.* it is conjectured that, under mild assumptions, the functor $\mathrm{RZ}_{G,\mu,b}$ is representable by a formal scheme which is formally smooth and locally formally of finite type over Spf $W(k)$. When $G = \mathrm{GL}_n$ and μ is the cocharacter $t \mapsto (1^{(d)}, t^{(n-d)})$, the category of (G,μ)-displays over a ring R coincides the category of Zink displays over R, so in this case representability can be proved by explicit connection with the original functor defined by Rapoport and Zink [13]. This is stated in [3], and we provide details in §5.2. Further, Bültel and Pappas prove that, when $(G, \{\mu\}, [b])$ is of Hodge type (again with some additional assumptions), the restriction of $\mathrm{RZ}_{G,\mu,b}$ to Noetherian p-nilpotent $W(k)$-algebras is representable by a formal scheme with the desired properties.

The Tannakian framework we develop in this paper allows us to compare the functor $\mathrm{RZ}_{G,\mu,b}$ with that of Rapoport and Zink outside of the case $G = \mathrm{GL}_n$. In particular, we consider the case of unramified EL-type local Shimura data. In particular, let B be a semisimple $\mathbb{Q}_p$-algebra whose simple factors are all matrix algebras over unramified extensions of $\mathbb{Q}_p$, let $\mathcal{O}_B$ be a maximal order in B, let Λ be a finite free $\mathbb{Z}_p$-module equipped with an action of $\mathcal{O}_B$, and define $G = \mathrm{GL}_{\mathcal{O}_B}(\Lambda)$. Such a tuple $\mathbf{D} = (B, \mathcal{O}_B, \Lambda)$ is called an unramified integral EL-type datum. There is a natural embedding $\eta : G \hookrightarrow \mathrm{GL}(\Lambda)$, and if $(\mathscr{D}, \iota) \in \mathrm{RZ}_{G,\mu,b}(R)$, then evaluating $\mathscr{D}$ on the representation (Λ, η) determines a Zink display equipped with an $\mathcal{O}_B$-action. In turn, by applying the functor BT_R from nilpotent Zink displays to formal p-divisible groups (cf. [1] and Theorem 2.4.5 below), we obtain a formal p-divisible group X over R with $\mathcal{O}_B$-action. A key result which allows the comparison of $\mathrm{RZ}_{G,\mu,b}$ with the classical EL-type RZ-space is the following lemma, which reinterprets the Kottwitz determinant condition on $\mathrm{Lie}(X)$ (cf. [13, 3.23(a)]) as a condition on the

Zink display associated to X:

Lemma 1.0.4. *Let $\underline{M} = (M, F)$ be the Zink display associated to a formal p-divisible group X, so $BT_R(\underline{M}) = X$. Then $\mathrm{Lie}(X)$ satisfies the determinant condition with respect to $\mathbf{D}$ if and only if M is fpqc-locally isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} W(R)^{\oplus}$ as a graded $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R)^{\oplus}$-module.*

We remark that the condition on M in the lemma is automatic if M comes from evaluating a Tannakian (G, μ)-display $\mathscr{D}$ on (Λ, η). As a result of this lemma and the above discussion, we obtain a map from $\mathrm{RZ}_{G,\mu,b}$ to the EL-type Rapoport-Zink functor defined in [13], denoted $\mathrm{RZ}_{\mathbf{D}}(X_0)$, where X_0 is the p-divisible group corresponding to the Tannakian (G, μ)-display $\mathscr{D}_0$.

Theorem 1.0.5. *If $(G, \{\mu\}, [b])$ is of unramified EL-type, and $\eta(b)$ has no slopes equal to 0, the map $\mathrm{RZ}_{G,\mu,b} \to \mathrm{RZ}_{\boldsymbol{D}}(X_0)$ is an isomorphism. In particular, the functor $\mathrm{RZ}_{G,\mu,b}$ is representable by a formal scheme which is formally smooth and locally formally of finite type over $\mathrm{Spf}\, W(k)$.*

This proves [3, Conjecture 4.2.1] in the case of EL-type local Shimura data. We remark that a similar analysis should prove the conjecture in the case where the data is of PEL-type.

The paper [14] makes up the majority of this thesis, and the remainder of the thesis will appear as part of an upcoming work.

1.1 Notation

- Let p be a prime. A ring or abelian group will be called p-adic if it is complete and separated with respect to the p-adic topology.

- If $f : A \to B$ is a ring homomorphism, and M is an A-module, we write M^f for $M \otimes_{A,f} B$. If the map f is understood, we sometimes also write $M_B = M \otimes_A B$. If N is a B-module, we say a map $\varphi : M \to N$ is f-linear if $\varphi(am) = f(a)\varphi(m)$ for all $a \in A, m \in M$. In other words, if we denote by $N_{[f]}$ the A-module obtained from N via restriction of scalars along f, then φ is an A-module homomorphism $M \to N_{[f]}$. In this case we write $\varphi^\sharp$ for the linearization $M^f \to N$ given by $m \otimes b \mapsto \varphi(m)b$. We say φ is an f-linear bijection if $\varphi^\sharp$ is a B-module isomorphism.

- For a $\mathbb{Z}_p$-algebra $\mathcal{O}$, denote by $\mathbf{Nilp}_{\mathcal{O}}$ the site consisting of the category of $\mathcal{O}$-algebras in which p is nilpotent, endowed with the fpqc topology. We will refer to such an $\mathcal{O}$-algebra as a p-nilpotent $\mathcal{O}$-algebra.

- If $\varphi : G \to H$ is a morphism of groups in a topos and P is an G-torsor, then P^φ is the pushforward of P to H, which is the G-torsor defined as the quotient of $P \times H$ by the action $(p, h) \mapsto (pg^{-1}, gh)$.

- Let $\bigoplus S_n$ be a $\mathbb{Z}$-graded ring. For a ring homomorphism $\varphi : \bigoplus S_n \to R$, we write φ_n for the restriction of φ to S_n.

- For a group scheme G defined over a ring A, we write $\mathbf{Rep}_A(G)$ for the category of representations of G on finite projective A-modules.

- Suppose **Cat** is a tensor category, and $\mathscr{F}_1$ and $\mathscr{F}_2$ are tensor functors $\mathbf{Rep}_A(G) \to \mathbf{Cat}$. Then if $\lambda : \mathscr{F}_1 \to \mathscr{F}_2$ is a tensor morphism, and (V, ρ) is an object in $\mathbf{Rep}_A(G)$, we write λ_ρ for the induced morphism $\mathscr{F}_1(V, \rho) \to \mathscr{F}_2(V, \rho)$ in **Cat**.

Chapter 2: Preliminaries

2.1 Higher frames

Let us recall the formalism of higher frames, following [5].

Definition 2.1.1. A *pre-frame* $\underline{S} = (S, \sigma, \tau)$ consists of a $\mathbb{Z}$-graded ring

$$S = \bigoplus_{n \in \mathbb{Z}} S_n$$

along with ring homomorphisms $\sigma : S \to S_0$ and $\tau : S \to S_0$. A pre-frame is a *frame* if the following conditions are satisfied:

- The endomorphism τ_0 of S_0 is the identity, and $\tau_{-n} : S_{-n} \to S_0$ is bijective for all $n \geq 1$.
- The endomorphism σ_0 of S_0 induces the p-power Frobenius $s \mapsto s^p$ on S_0/pS_0, and if t is the unique element in S_{-1} such that $\tau_{-1}(t) = 1$, then $\sigma_{-1}(t) = p$.
- We have $p \in \mathrm{Rad}(S_0)$, the Jacobson radical of S_0.

We say $\underline{S}$ is a frame for $R = S_0/tS_1$. A morphism of pre-frames $(S, \sigma, \tau) \to (S', \sigma', \tau')$ is a morphism of graded rings $\psi : S \to S'$ such that $\sigma' \circ \psi = \psi \circ \sigma$ and $\tau' \circ \psi = \psi \circ \tau$.

As in [5], we remark that a frame is equivalent to a triple $(\bigoplus_{n\geq 0} S_n, \sigma, \{t_n\}_{n\geq 0})$ consisting of an $\mathbb{Z}_{\geq 0}$-graded ring $S_{\geq 0} = \bigoplus_{n\geq 0} S_n$, a ring homomorphism $\sigma : S_{\geq 0} \to S_0$ and a collection of maps $t_n : S_{n+1} \to S_n$ for $n \geq 0$ such that

- For every $n \geq 0$, the homomorphism $t_n : S_{n+1} \to S_n$ is $S_{\geq 0}$-linear.
- The endomorphism $\sigma_0 : S_0 \to S_0$ induces the p-power Frobenius $s \mapsto s^p$ on S_0/pS_0, and $\sigma_n(t_n(a)) = p\sigma_{n+1}(a)$ for all $a \in S_{n+1}$.
- We have $p \in \mathrm{Rad}(S_0)$.

The equivalence is given as follows. Define $S_{\leq 0} = S_0[t]$ where t is an indeterminate with degree -1, and let $S = S_{\leq 0} \oplus \bigoplus_{n>0} S_n$. To give S a ring structure it suffices to define multiplication by t on S_n for $n > 0$. For this we use the maps t_n, i.e. if $s \in S_n$, $n > 0$, define $t \cdot s := t_{n-1}(s)$. The homomorphism σ extends uniquely to all of S by defining $\sigma(t^n \cdot s_0) = p^n \sigma_0(s_0)$ for $s_0 \in S_0$. It remains only to define $\tau : S \to S_0$. The restriction of τ to S_0 is necessarily given by the identity. Since multiplication by t is bijective on $S_0[t]$, we can define $\tau_n : S_{-n} \to S_0$ by multiplication by t^{-n} for $n > 0$. Lastly, if $s \in S_n$ for $n > 0$, then

$$\tau(s) = (t_0 \circ \cdots \circ t_{n-1})(s) = t^n \cdot s.$$

Let us take a further moment to recall some notations and definitions concerning Witt vectors. Attached to a ring R is the ring $W(R)$ of p-typical Witt vectors $W(R)$. Elements of $W(R)$ are tuples $(\xi_0, \xi_1, \dots) \in R^{\mathbb{Z}_{\geq 0}}$, and the ring structure is characterized as the unique one which is functorial in R and for which the maps

$$w_i : W(R) \to R,\ (\xi_i)_{i \in \mathbb{Z}_{\geq 0}} \mapsto \xi_0^{p^i} + p\xi_1^{p^{i-1}} + \cdots + p^i \xi_i$$

are ring homomorphisms. Additionally, the ring $W(R)$ is equipped with Frobenius and Verschiebung maps $f, v : W(R) \to W(R)$. The Verschiebung is the additive map given by shifting: $v(\xi_0, \xi_1, \dots) = (0, \xi_0, \xi_1, \dots)$, and the Frobenius is a ring homomorphism characterized by its compatibility with the maps w_i:

$$w_i(f(x)) = w_{i+1}(x).$$

We will denote by I_R the kernel of $w_0 : W(R) \to R$. Equivalently, $I_R = v(W(R))$.

The following is the frame of primary interest in this paper.

Definition 2.1.2. Let R be a p-adic ring. The *Witt frame* for R is defined as follows (cf. [5, Example 2.1.3]). By the above remarks, it suffices to define $S_{\geq 0}$, $\sigma : S_{\geq 0} \to S_0$, and $t_n : S_{n+1} \to S_n$ for every $n > 0$. Let $S_0 = W(R)$, and define $S_n = I_R$ viewed as a $W(R)$-module for $n \geq 1$. If $n, m \geq 1$, then multiplication for $S_n \times S_m \to S_{n+m}$ is given by

$$I_R \times I_R \to I_R,\ (v(a), v(b)) \mapsto v(ab).$$

The homomorphism $t_0 : S_1 \to S_0$ is the inclusion of the submodule $I_R \hookrightarrow W(R)$, and for $n \geq 1$, $t_n : S_{n+1} \to S_n$ is multiplication by p. The endomorphism σ_0 of

$S_0 = W(R)$ is the Witt vector Frobenius, f. For every $n \geq 1$, define $\sigma_n(v(a)) = a$ for all $v(a) \in S_n = I_R$.

We will denote the graded ring S for this frame by $S = W(R)^{\oplus}$, and write $\underline{S} = \underline{W}(R)$ to denote the frame $(W(R)^{\oplus}, \sigma, \tau)$. If $R \to R'$ is a ring homomorphism, then the induced map $\underline{W}(R) \to \underline{W}(R')$ is a morphism of frames. By [5, 4.1] the functor $\underline{W}$ which sends R to $\underline{W}(R)$ is an fpqc sheaf of frames on $\mathbf{Nilp}_{\mathbb{Z}_p}$.

2.2 Graded modules

Let S be a $\mathbb{Z}$-graded ring, and denote by $\mathbf{GrMod}(S)$ the category of graded S-modules. If M and N are objects in this category, denote the morphisms between M and N in $\mathbf{GrMod}(S)$ by $\mathrm{Hom}_S^0(M, N)$. Then $\mathrm{Hom}_S^0(M, N)$ is the set of S-module homomorphisms which preserve the gradings of M and N, i.e. the set of $\varphi \in \mathrm{Hom}_S(M, N)$ such that $\varphi(M_i) \subseteq N_i$.

Let $\mathbf{PGrMod}(S)$ be the full subcategory of $\mathbf{GrMod}(S)$ consisting of finite projective graded S-modules. By [5, Lemma 3.0.1], this is equivalent to the full subcategory of projective objects in $\mathbf{GrMod}(S)$ which are finitely generated.

For reference, let us review the exact tensor structure of $\mathbf{PGrMod}(S)$. Note $\mathbf{GrMod}(S)$ is an abelian category, so $\mathbf{PGrMod}(S)$ inherits additivity and a notion of exactness: a sequence of finite projective graded S-modules is exact if and only if it is exact in $\mathbf{GrMod}(S)$. The category $\mathbf{GrMod}(S)$ is also endowed with a tensor product: if $M = \bigoplus_i M_i$ and $N = \bigoplus_i N_i$ are graded S-modules, then $M \otimes_S N$ is a graded S-module with graded pieces

$$(M \otimes_S N)_\ell = \{\sum_i m_i \otimes n_i \in M \otimes N \mid \deg(m_i) + \deg(n_i) = \ell\}.$$

The ring S viewed as a graded module over itself is the unit object in both **Gr-Mod**(S) and **PGrMod**(S). Since the tensor product of two finite projective S-module is again a finite projective S-module, **PGrMod**(S) is a tensor subcategory of **GrMod**(S). The dual of M in **GrMod**(S) is the dual S-module $M^\vee = \mathrm{Hom}_S(M, S)$ with grading $(M^\vee)_i = (M_{-i})^\vee$.

Lemma 2.2.1. *The category* **PGrMod**(S) *is an exact rigid tensor category.*

Proof. After the above remarks, it remains only to show that **PGrMod**(S) is rigid. Since every object in **PGrMod**(S) admits a dual, it is enough to show that $M^{\vee\vee} \cong M$ (cf. the footnote under [15, Definition 1.7]). But this is clear because projectivity is preserved under taking duals and because finite projective S-modules are reflexive. $\square$

Suppose now $\underline{S} = (S, \sigma, \tau)$ is a frame for a ring R. Let

$$\nu : S \to S_0/tS_1 = R \tag{2.2.1}$$

be the natural projection $S_0 \to R$ extended by zero on S_n for $n \neq 0$ (this map is called ρ in [5]). By considering R to be a graded ring concentrated in degree zero, we can view ν as a homomorphism of graded rings. Then for any finite projective graded S-module M, the base change $\overline{L} = M \otimes_{S,\nu} R$ along ν is a finite projective graded R-module. Write

$$\overline{L} = \bigoplus_{i \in \mathbb{Z}} \overline{L}_i$$

for the decomposition of $\overline{L}$ into its graded pieces.

Recall the rank function associated to a finite projective R-module is the locally constant function on Spec R defined by $\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p}))$, where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is the residue field of $\mathfrak{p}$. In particular, if Spec R is connected, any finite projective R-module has constant rank.

Definition 2.2.2. Let $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$ be a collection of integers such that $i_1 \leq i_2 \leq \cdots \leq i_n$. We say a finite projective graded S-module M is *of type* I if $\mathrm{rk}(\overline{L}_k)$ is equal to the multiplicity of k in I for all k.

For example, for any collection $I = (i_r)$, the finite free graded S-module $\bigoplus_r S(-i_r)$ has type I. If Spec R is connected, every finite projective graded S-module has a unique type. Note that our convention on the ordering of the components of I differs from that of [5].

Definition 2.2.3. Let M be a finite projective graded S-module and write $M \otimes_{S,\nu} R = \bigoplus_i \overline{L}_i$.

(i) The *depth* of M, denoted $d(M)$, is the minimal integer i such that $\overline{L}_i \neq 0$.

(ii) The *altitude* of M, denoted $a(M)$, is the maximal integer i such that $\overline{L}_i \neq 0$.

If a finite projective graded S-module M is of type $I = (i_1, \dots, i_n)$, then $d(M) = i_1$ and $a(M) = i_n$.

Definition 2.2.4. A *normal decomposition* for a finite projective graded S-module M is a finite projective graded S_0-module $L = \bigoplus_i L_i$ such that $L \subseteq M$ and $L \otimes_{S_0} S = M$ as graded S-modules.

It follows from [5, Lemma 3.1.4] that every finite projective graded S-module has a normal decomposition if every finite projective R-module lifts to S_0. This holds in particular when S_0 is p-adic. It follows that every finite projective graded $W(R)^{\oplus}$-module has a normal decomposition.

Lemma 2.2.5. *Let M be a finite projective graded S-module, and write $M \otimes_{S,\nu} R = \bigoplus_i \overline{L}_i$. If $L = \bigoplus_i L_i$ is a normal decomposition for M, then $L_i = 0$ if and only if $\overline{L}_i = 0$. In particular, if M is of type $I = (i_1, \ldots, i_n)$, then $L_i \neq 0$ if and only if the multiplicity of i in I is nonzero.*

Proof. We have an isomorphism of graded R-modules

$$M \otimes_{S,\nu} R \cong L \otimes_{S_0} S \otimes_{S,\nu} R \cong L \otimes_{S_0} R,$$

using that $S_0 \to S \xrightarrow{\nu} R$ is the natural quotient $S_0 \to R$. Hence for every i we have an isomorphism of R-modules $L_i \otimes_{S_0} R \cong \overline{L}_i$. Clearly $\overline{L}_i = 0$ if $L_i = 0$. On the other hand, by the proof of [5, Lemma 3.1.1], $tS_1 \subseteq \mathrm{Rad}(S_0)$. Then $L_i \otimes_{S_0} R = 0$ implies $L_i = 0$ by Nakayama's lemma. $\square$

It follows from Lemma 2.2.5 that, if M has normal decomposition $L = \bigoplus_i L_i$, then the depth (resp. altitude) of M is equal to the minimal (resp. maximal) i such that $L_i \neq 0$. For any finite projective graded S-module, we have a natural homomorphism of S_0-modules $\theta_n : M_n \to M^\tau$ given by the composition of $M_n \hookrightarrow M$ and $M \to M^\tau$. We remark that τ defines an isomorphism $S/(t-1)S \xrightarrow{\sim} S_0$; to see that $\ker\tau \subseteq (t-1)S$ it is enough to check $(\ker\tau) \cap S_{\leq 0} \subseteq (t-1)S$, which is easy. Then

$$M^\tau \cong \varinjlim M_i,$$

where the colimit is taken along $t : M_n \to M_{n-1}$.

Lemma 2.2.6. *Let M be a finite projective graded S-module with normal decomposition L. Then $\theta_n : M_n \to M^\tau$ is an isomorphism of S_0-modules for all $n \leq d(M)$.*

Proof. Let $n \leq d(M)$, and let $L = \bigoplus_i L_i$ be a normal decomposition for M. Observe

$$M_n = \bigoplus_i L_i \otimes_{S_0} S_{n-i},$$

and under $M^\tau \cong L$, the map θ_n is given by

$$\bigoplus_i L_i \otimes_{S_0} S_{n-i} \to \bigoplus_i L_i,\ \ (\ell_i \otimes s_{n-i})_i \mapsto (\tau(s_{n-i})\ell_i)_i.$$

By Lemma 2.2.5, $L_i = 0$ for all $i < n$, so S_{n-i} occurs in the above decomposition only when $n-i \leq 0$. In this case we have an isomorphism of S_0-modules $S_{n-i} \cong S_0 \cdot t^{i-n}$, where $t \in S_{-1}$ is the unique element such that $\tau(t) = 1$. Then any $\ell_i \otimes s_{n-i} \in L_i \otimes_{S_0} S_{n-i}$ can be written as $\ell_i' \otimes t^{i-n}$, and θ_n is given by $(\ell_i' \otimes t^{i-n}) \mapsto \ell_i'$. From this description the conclusion is clear. $\square$

Let us now focus on the case where $S = W(R)^\oplus$ for a p-adic ring R. Denote by $\mathbf{PGrMod}^W$ the fibered category over $\mathbf{Nilp}_{\mathbb{Z}_p}$ whose fiber over R is the category $\mathbf{PGrMod}(W(R)^\oplus)$.

Lemma 2.2.7. *The fibered category* $\mathbf{PGrMod}^W$ *is an fpqc stack over* $\mathbf{Nilp}_{\mathbb{Z}_p}$.

Proof. This is [5, Lemma 4.3.2]. □

In the following we collect some properties of finite projective graded $W(R)^\oplus$-modules which can be checked fpqc-locally on Spec R.

Lemma 2.2.8. *Let M be a finite projective $W(R)^\oplus$-module. Let $R \to R'$ be a faithfully flat homomorphism of $\mathbb{Z}_p$-algebras, and let M' be the base change of M to $W(R')^\oplus$. Then*

(i) $a(M) = a(M')$,

(ii) $d(M) = d(M')$, *and*

(iii) M *is of type* $I = (i_1, \dots, i_n)$ *if and only if* M' *is of type* I.

Proof. Denote by ν' the map $W(R')^\oplus \to R'$ defined as in (2.2.1). Write $\overline{L} = \bigoplus_i \overline{L}_i$ for the base change of M along ν and $\overline{L}' = \bigoplus_i \overline{L}'_i$ for the base change of M' along ν'. By functoriality of the maps ν and ν' we have an isomorphism of graded R'-modules

$$\overline{L} \otimes_R R' \cong \overline{L}'$$

and therefore an isomorphism of their graded pieces $\overline{L}_i \otimes_R R' \cong \overline{L}'_i$. Faithful flatness of $R \to R'$ implies that $\overline{L}_i = 0$ if and only if $\overline{L}'_i = 0$. This proves (i) and (ii).

Part (iii) follows because the rank of a projective module is invariant under base change. $\square$

The following is an analog of [1, Lemma 30].

Lemma 2.2.9. *Let M be a finite projective graded $W(R)^{\oplus}$-module, and let $R \to R'$ be a faithfully flat extension of $W(k_0)$-algebras. Then there is an exact sequence*

$$0 \longrightarrow M \longrightarrow M \otimes_{W(R)^{\oplus}} W(R')^{\oplus} \rightrightarrows M \otimes_{W(R)^{\oplus}} W(R' \otimes_R R')^{\oplus} \Rrightarrow \cdots$$

where the arrows are induced by applying the functor W to the usual exact sequence

$$0 \longrightarrow R \longrightarrow R' \rightrightarrows R' \otimes_R R' \Rrightarrow \cdots$$

Proof. Since M is a direct summand of a free $W(R)^{\oplus}$-module we can reduce to the case $M = W(R)^{\oplus}$. In that case the result follows because $R \mapsto W(R)$ and $R \mapsto I_R$ are fpqc sheaves. $\square$

To close this section, we prove that exactness is a property of finite projective $W(R)^{\oplus}$-modules which is fpqc local in R.

Lemma 2.2.10. *Let M, N, and P be finite projective $W(R)^{\oplus}$-modules equipped with $W(R)^{\oplus}$-module homomorphisms $N \to M \to P$. The following are equivalent:*

(*i*) *The sequence*

$$0 \to N \to M \to P \to 0$$

is exact.

(*ii*) *For some faithfully flat $\mathbb{Z}_p$-algebra homomorphism $R \to R'$ the sequence*

$$0 \to N_{W(R')^{\oplus}} \to M_{W(R')^{\oplus}} \to P_{W(R')^{\oplus}} \to 0$$

is exact.

(*iii*) *For every faithfully flat $\mathbb{Z}_p$-algebra homomorphism $R \to R'$ the sequence in (ii) is exact.*

Proof. If the sequence in (i) is exact then it is split exact, so it will remain exact after tensoring by any extension. Then (i) implies (iii). Obviously (iii) implies (ii), so it remains only to show (ii) implies (i).

Suppose the sequence in (ii) is exact for some faithfully flat extension $R \to R'$. Let us write R'' for $R' \otimes_R R'$, Consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_{W(R')^{\oplus}} \arrow[r] \arrow[d] & M_{W(R')^{\oplus}} \arrow[r] \arrow[d] & P_{W(R')^{\oplus}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_{W(R'')^{\oplus}} \arrow[r] & M_{W(R'')^{\oplus}} \arrow[r] & P_{W(R'')^{\oplus}} \arrow[r] & 0
\end{tikzcd}$$

Here the bottom map in each column is induced by the difference of the two maps $W(R')^{\oplus} \to W(R' \otimes_R R')^{\oplus}$ induced by the two canonical ring homomorphisms $R' \to R' \otimes_R R'$. Then the columns are exact by Lemma 2.2.9. The middle row is exact by assumption and the bottom row is exact because it is obtained by tensoring the middle row over $W(R')^{\oplus} \to W(R' \otimes_R R')^{\oplus}$. Injectivity of $N \to M$ and exactness at M follow immediately from the diagram.

It remains to check $M \to P$ is surjective. By Nakayama's Lemma, cf. [5, Lemma 3.1.1], it is enough to show $M \otimes_{W(R)^\oplus} R \to P \otimes_{W(R)^\oplus} R$ is surjective. But $M \otimes_{W(R)^\oplus} R' \to P \otimes_{W(R)^\oplus} R'$ is surjective, so the result follows because $R \to R'$ is faithfully flat. □

Remark 2.2.11. In an earlier draft of this paper, before [5] was announced, we defined a category of objects called canvases, which we used to develop the subsequent theory of Tannakian (G, μ)-displays. The category of canvases over R is equivalent to the category of finite projective graded $W(R)^\oplus$-modules. Let us briefly explain the equivalence. A *precanvas* is a collection of $W(R)$-modules $\{P_i\}_{i\in\mathbb{Z}}$ along with $W(R)$-module homomorphisms $\iota_i : P_{i+1} \to P_i$ and $\alpha_i : I_R \otimes_{W(R)} P_i \to P_{i+1}$ such that $\iota_i \circ \alpha_i = \alpha_{i-1} \circ (\mathrm{id}_{I_R} \otimes \iota_{i-1}) = \mathrm{mult} : I_R \otimes_{W(R)} P_i \to P_i$. Given a collection $\{L_i\}_{i\in\mathbb{Z}}$ of finite projective $W(R)$-modules there is a standard construction of a precanvas, essentially following the construction for displays given in [4]. A *canvas* is a precanvas isomorphic to one resulting from this construction. Given a finite projective graded $W(R)^\oplus$-module M, we define a precanvas over R as follows:

- M_i is the ith graded piece of M, regarded as a $W(R)$-module;
- $\iota_i : M_{i+1} \to M_i$ is multiplication by $t \in W(R)^\oplus$;
- $\alpha_i : I_R \otimes_{W(R)} M_i \to M_{i+1}$ is the action of $W(R)^\oplus_1 = I_R$ on M_i.

One checks that this construction satisfies the desired compatibilities and defines a functor from $\mathbf{PGrMod}(W(R)^\oplus)$ to the category of canvases over R, which is an equivalence of categories since every finite projective graded $W(R)^\oplus$-module has a normal decomposition. Defining basic constructions such as duals and tensor products using canvases requires some work in this category, as it is tedious to check all the necessary compatibilities. In the framework developed in [5] this work is no longer necessary because in this case the constructions follow from the well-known constructions for graded modules. Therefore we have adopted the more streamlined approach using the graded ring $W(R)^{\oplus}$ in this paper.

2.3 Displays

In this section we review the definitions and elementary properties of displays over a frame. As in the previous sections, our main reference for this section is [5]. Let $\underline{S} = (S, \sigma, \tau)$ be a frame.

Definition 2.3.1. A *predisplay* $\underline{M} = (M, F)$ over $\underline{S}$ consists of a graded S-module M and a σ-linear map $F : M \to M^{\tau}$.

A morphism $(M, F) \to (M', F')$ of predisplays is a homomorphism of graded S-modules $M \to M'$ which is compatible with the maps F and F'. Denote the resulting category of predisplays over $\underline{S}$ by $\mathbf{Predisp}(\underline{S})$. This is an abelian category, because the same is true of $\mathbf{GrMod}(S)$.

Definition 2.3.2. A *display* over $\underline{S}$ is a predisplay $\underline{M} = (M, F)$ over $\underline{S}$ such that M is a finite projective graded S-module and $F : M \to M^{\tau}$ is a σ-linear bijection.

Displays over $\underline{S}$ form a full subcategory of $\mathbf{Predisp}(\underline{S})$ which we will denote by $\mathbf{Disp}(\underline{S})$. Note a σ-linear bijection $M \to M^{\tau}$ is by definition a σ-linear homomorphism whose linearization $F^{\sharp} : M^{\sigma} \to M^{\tau}$ is an S_0-module isomorphism. In this way we see that endowing a finite projective S-module M with the structure of a display is equivalent to giving an S_0-module isomorphism $M^\sigma \xrightarrow{\sim} M^\tau$.

Definition 2.3.3. Suppose $\underline{S}$ is a frame for R and let $\underline{M} = (M, F)$ be a predisplay (resp. display) over $\underline{S}$.

(i) $\underline{M}$ is *effective* if $d(M) \geq 0$.

(ii) $\underline{M}$ is an *n-predisplay* (resp. n-display) if it is effective and $a(M) = n$.

One can check that our definition of an effective predisplay agrees with that of [5]. As in the previous section, we collect here some notions regarding the exact tensor category structure of $\mathbf{Disp}(\underline{S})$. Morphisms of displays over $\underline{S}$ are, in particular, morphisms of the underlying finite projective graded S-modules, so exactness is inherited from that category (or from $\mathbf{GrMod}(S)$). The tensor product of displays is the tensor product in the category of predisplays:

$$(M, F) \otimes (M', F') := (M \otimes_S M', F \otimes F').$$

Since $M \otimes_S M'$ is a finite projective graded S-module, this tensor product preserves the category of displays. The unit object is (S, σ).

The dual of a display $\underline{M} = (M, F)$ is the display $\underline{M}^\vee = (M^\vee, F^\vee)$, where $M^\vee$ is the dual of M in $\mathbf{PGrMod}(S)$, and $F^\vee$ corresponds to the dual of the inverse of $F^\sharp : M^\sigma \xrightarrow{\sim} M^\tau$. It is clear that $\underline{M}$ is reflexive with respect to this notion, so the following analog of Lemma 2.2.1 is immediate.

Lemma 2.3.4. *The category* $\mathbf{Disp}(\underline{S})$ *is an exact rigid tensor category.*

Displays carry a good notion of bilinear form, which characterizes the tensor product of displays.

Definition 2.3.5. Let $\underline{M} = (M, F)$, $\underline{M}' = (M', F')$, and $\underline{M}'' = (M'', F'')$ be displays over $\underline{S}$. A *bilinear form* $\beta : \underline{M} \times \underline{M}' \to \underline{M}''$ is a bilinear form of the underlying graded S-modules $M \times M' \to M''$ such that

$$F''(\beta(x, y)) = \beta^\tau(F(x), F'(y)),$$

where $\beta^\tau : M^\tau \times (M')^\tau \to (M'')^\tau$ is the induced bilinear map of S_0-modules.

The tensor product of $\underline{M}$ and $\underline{M}'$ is characterized by the following universal property: it admits a bilinear form $\beta_0 : \underline{M} \times \underline{M}' \to \underline{M} \otimes \underline{M}'$, and any other bilinear form $\underline{M} \times \underline{M}' \to \underline{M}''$ factors uniquely through β_0.

Let us review some other useful constructions for displays.

Definition 2.3.6. Let $\underline{M} = (M, F)$ be a predisplay over a frame $\underline{S} = (S, \sigma, \tau)$, and suppose $\underline{S} \to \underline{S}'$ is a morphism to another frame $\underline{S}' = (S', \sigma', \tau')$. Then the *base change* of $\underline{M}$ to $\underline{S}'$ is the predisplay $\underline{M}_{\underline{S}'} = (M \otimes_S S', F \otimes \sigma')$.

Base change defines a functor $\mathbf{Predisp}(\underline{S}) \to \mathbf{Predisp}(\underline{S}')$ which preserves the full subcategories of displays.

The definition of type for a finite projective S-module extends in a natural way to displays.

Definition 2.3.7. Let I be a collection of integers as in Definition 2.2.2. We say a display $\underline{M} = (M, F)$ is *of type I* if the finite projective graded S-module M is of type I.

Definition 2.3.8. A *standard datum* for a display is a pair (L, Φ) consisting of a finite projective graded S_0-module $L = \bigoplus_i L_i$ and a σ_0-linear automorphism $\Phi : L \to L$.

From a standard datum (L, Φ), we define a display (M, F) by taking $M := L \otimes_{S_0} S$ and $F(x \otimes s) = \sigma(s)\Phi(x)$. On the other hand, if $\underline{M} = (M, F)$ is a display, then any normal decomposition L of M determines a standard datum by viewing L as a submodule of $M = L \otimes_{S_0} S$ via $x \mapsto x \otimes 1$ and taking $\Phi = F\big|_L$. It is clear that the display resulting from this (L, Φ) is indeed (M, F). Hence if every finite projective graded S-module has a normal decomposition then every display over $\underline{S}$ is standard, i.e. is defined from a standard datum. Note also that if (L, Φ) is a standard datum for a display $\underline{M} = (M, F)$, then $M^\tau \cong L$ and $M^\sigma \cong L^{\sigma_0}$, cf. [5, Remark 3.2.5].

Let us remark that both the tensor product and base change can be defined in a natural way using standard data. Indeed, if $\underline{M} = (M, F)$ and $\underline{M}' = (M', F')$ are displays over $\underline{S}$ with standard data (L, Φ) and (L', Φ') respectively, then $(L \otimes_{S_0} L', \Phi \otimes \Phi')$ is a standard datum for $\underline{M} \otimes \underline{M}'$. Similarly $(L \otimes_{S_0} S_0', \Phi \otimes \sigma_0')$ is a standard datum for $\underline{M}_{\underline{S}'}$. The characterizations of the tensor product and the base change above prove that the resulting object is independent of the choice of standard data.

Now we return our focus to the Witt frame, which is the case of interest in the remainder of the paper. Note that if R and R' are two p-adic rings, then a ring homomorphism $R \to R'$ induces a morphism of frames $\underline{W}(R) \to \underline{W}(R')$. Let

$\mathbf{Disp}^W$ be the category fibered over $\mathbf{Nilp}_{\mathbb{Z}_p}$ whose fiber over R is $\mathbf{Disp}(\underline{W}(R))$. As in the case of finite projective graded $W(R)^{\oplus}$-modules, this fibered category satisfies effective descent for morphisms and for objects.

Lemma 2.3.9. *The fibered category $\mathbf{Disp}^W$ is an fpqc stack over $\mathbf{Nilp}_{\mathbb{Z}_p}$.*

Proof. This is [5, Lemma 4.4.2]. □

Suppose $\underline{M} = (M, F)$ and $\underline{M'} = (M', F')$ are displays over $W(R)^{\oplus}$. Let $\psi : M \to M'$ be a homomorphism of graded $W(R)^{\oplus}$-modules, and write $\psi_{R'}$ for the base change of ψ to $W(R')^{\oplus}$. We have the following lemma, which says that the property of "being a morphism of displays" can be checked fpqc locally.

Lemma 2.3.10. *Let R be a p-nilpotent $\mathbb{Z}_p$-algebra, and let $R \to R'$ be a faithfully flat extension of $\mathbb{Z}_p$-algebras. With the set-up as above, $\psi_{R'}$ is a morphism of displays over $\underline{W}(R')$ if and only if ψ is a morphism of displays over $\underline{W}(R)$.*

Proof. Obviously $\psi_{R'}$ is a morphism of displays if ψ is. For the converse, we want the following diagram to commute:

$$\begin{array}{ccc} M^\sigma & \xrightarrow{\psi^\sigma} & (M')^\sigma \\ \big\downarrow{\scriptstyle F^\sharp} & & \big\downarrow{\scriptstyle (F')^\sharp} \\ M^\tau & \xrightarrow{\psi^\tau} & (M')^\tau \end{array}$$

The diagram commutes after base change to $W(R')$, so the result follows from Witt vector descent for finitely generated projective modules, [1, Corollary 34]. □

2.4 1-displays

Fix a p-adic ring R. Following the notation in Definition 2.1.2, let us denote the Witt vector Frobenius on $W(R)$ by f.

Definition 2.4.1. A *Zink display* over R is a quadruple $\underline{P} = (P_0, P_1, F_0, F_1)$ consisting of a finitely generated projective $W(R)$-module P_0, $P_1 \subseteq P_0$ is a submodule and $F_i : P_i \to P_0$ is an f-linear map, for $i = 0$ and 1, such that the following conditions are satisfied:

(i) $I_R P_0 \subseteq P_1 \subseteq P_0$, and the filtration

$$0 \subseteq P_1/I_R P_0 \subseteq P_0/I_R P_0$$

has finitely generated projective R-modules as graded pieces.

(ii) $F_1 : P_1 \to P_0$ is an f-linear epimorphism.

(iii) For $x \in P_0$ and $\xi \in W(R)$ we have

$$F_1(v(\xi) \cdot x) = \xi \cdot F_0(x).$$

We remark that Zink displays are frequently referred to only as "displays" in the literature, and in [1] they are called "not-necessarily-nilpotent" (or 3n-) displays. The filtration in (i) of Definition 2.4.1 is called the Hodge filtration of $\underline{P}$.

Lemma 2.4.2. *The category of Zink displays over R is equivalent to the category of 1-displays over $\underline{W}(R)$.*

Proof. Let $\underline{M} = (M, F)$ be a 1-display over $\underline{W}(R)$. Since $d(M) \geq 0$, we know $\theta_0 : M_0 \to M^\tau$ is bijective, cf. Lemma 2.2.6. We claim additionally that $\theta_1 : M_1 \to M^\tau$ is injective. Indeed, by [5, Lemma 3.1.4] and Lemma 2.2.5, we can choose a normal decomposition L for M such that $L_i = 0$ if $i \notin 0, 1$. Then

$$M_1 = (L_0 \otimes_{W(R)} I_R) \oplus (L_1 \otimes_{W(R)} W(R)), \tag{2.4.1}$$

and $\theta_1 : M_1 \to M^\tau = L$ is given by the natural inclusion.

Now let $P_0 = M^\tau$ and let P_1 be the image of M_1 under θ_1. Then P_0 is a finitely generated projective $W(R)$-module, and P_1 is a submodule. The restrictions of F to M_0 and M_1 are f-linear homomorphisms of $W(R)$-modules $M_i \to M^\tau$. Then $F_i := F\big|_{M_i} \circ \theta_i^{-1} : P_i \to P_0$ is also f-linear for $i = 0, 1$. We claim (P_0, P_1, F_0, F_1) is a Zink display.

The first part of condition (i) in Definition 2.4.1 is immediate. To verify the remaining conditions, choose a normal decomposition $L = L_0 \oplus L_1$ for M as above, and let $\Phi = F\big|_L$, so (L, Φ) is a standard datum for $\underline{M}$. The $W(R)$-module M_1 can be written as in (2.4.1), and we see

$$M_0 = (L_0 \otimes_{W(R)} W(R)) \oplus (L_1 \otimes_{W(R)} W(R) \cdot t).$$

The second part of condition (i) follows because $P_0/P_1 \cong L_0 \otimes_{W(R)} R$, and $P_1/I_R P_0 \cong$

$L_1 \otimes_{W(R)} R$. Condition (ii) is equivalent to the condition that $(F\big|_{M_1})^\sharp : (M_1)^f \to M^\tau$ is surjective. Since $\Phi^\sharp : L^f \to L$ is a $W(R)$-module isomorphism and L is naturally identified with M^τ, it is enough to show for any $x \otimes \xi \in L^f$, there is some $y \in (M_1)^f$ with $F^\sharp(y) = \Phi^\sharp(x \otimes \xi)$. First suppose $x \otimes \xi \in (L_0)^f$. Then $x \otimes v(\xi) \otimes 1 \in (M_1)^f$, and

$$F^\sharp(x \otimes v(\xi) \otimes 1) = F(x \otimes v(\xi)) = \sigma(v(\xi))F(x) = \xi F(x) = \Phi^\sharp(x \otimes \xi).$$

Now if $x \otimes \xi \in (L_1)^f$, then $x \otimes 1 \otimes \xi \in (M_1)^f$, and

$$F^\sharp(x \otimes 1 \otimes \xi) = \xi F(x) = \Phi^\sharp(x \otimes \xi).$$

This completes the proof of condition (ii). For condition (iii), let $x = x_0 + x_1 \in P_0 = M^\tau = L$ with $x_0 \in L_0$ and $x_1 \in L_1$. If $\xi \in W(R)$, then

$$\theta_1^{-1}(v(\xi) \cdot x) = x_0 \otimes v(\xi) + x_1 \otimes v(\xi)$$

with the first $v(\xi)$ viewed as an element of $(W(R)^\oplus)_1 = I_R$ and the second as an element in $(W(R)^\oplus)_0 = W(R)$. Then

$$F_1(v(\xi) \cdot x) = \xi(\Phi(x_0) + p\Phi(x_1)).$$

But $\theta_0^{-1}(x) = x_0 \otimes 1 + x_1 \otimes t$, so this is the same as $\xi \cdot F_0(x)$.

If $\psi : (M, F) \to (M', F')$ is a morphism of 1-displays, then the $W(R)$-module homomorphism ψ^τ defines a morphism of the corresponding Zink displays. Hence the association $(M, F) \mapsto (P_0, P_1, F_0, F_1)$ determines a functor from the category of 1-displays over $\underline{W}(R)$ to the category of Zink displays over R, which we claim is an equivalence of categories. Choosing a normal decomposition L for M as above, we see

$$M = (L_0 \otimes_{W(R)} W(R)^{\oplus}) \oplus (L_1 \otimes_{W(R)} W(R)^{\oplus}(-1)).$$

It follows that any morphism $\underline{M} \to \underline{M'}$ of 1-displays is uniquely determined by its restriction to L_0 and L_1, and therefore by its restriction to M_0 and M_1, so the functor is faithful.

Now let $\underline{P}$ and $\underline{P'}$ be the Zink displays associated to 1-displays (M, F) and (M', F'), and suppose $\varphi : \underline{P} \to \underline{P'}$ is a morphism of Zink displays. In particular, φ is a $W(R)$-module homomorphism $P_0 = M^\tau \to (M')^\tau = P_0'$ which sends $P_1 = \theta_1(M_1)$ to $P_1' = \theta_1'(M_1')$, and which satisfies

$$F_0' \circ \varphi = \varphi \circ F_0, \quad \text{and} \quad F_1' \circ \varphi = \varphi \circ F_1. \tag{2.4.2}$$

For $i = 0, 1$, let ψ_i be the composition

$$L_i \hookrightarrow P_i \xrightarrow{\psi} P_i' \xrightarrow{(\theta_i')^{-1}} M_i'.$$

Then $\psi_0 + \psi_1$ defines a $W(R)$-module homomorphism $L \to M'$ which sends L_i to M_i' for every i. Therefore it induces a graded $W(R)^{\oplus}$-module homomorphism

$\psi : M \to M'$. By construction $\psi^\tau = \varphi$, and it follows from the identities (2.4.2) that $F' \circ \psi = \psi^\tau \circ F$. Hence the functor is full.

Now suppose (P_0, P_1, F_0, F_1) is a Zink display. By [1, Lemma 2], condition (i) in the definition of Zink displays implies the existence of finitely generated projective $W(R)$-modules L_0 and L_1 such that $P_0 = L_0 \oplus L_1$ and $P_1 = I_R L_0 \oplus L_1$. If we define $\Phi = F_0\big|_{L_0} \oplus F_1\big|_{L_1}$, then (P_0, Φ) constitutes a standard datum for a 1-display whose resulting Zink display is isomorphic to (P_0, P_1, F_0, F_1). $\square$

If (P_0, P_1, F_0, F_1) is a Zink display, then by [1, Lemma 10] there exists a unique linear map $V^\sharp : F_0 \to F_0^f$ characterized by

$$V^\sharp(\xi \cdot F_0(x)) = p\xi \otimes x, \quad \text{and} \quad V^\sharp(\xi \cdot F_1(y)) = \xi \otimes y$$

for all $\xi \in W(R)$, $x \in P_0, y \in P_1$. Denote by $V_i^\sharp$ the induced map $P_0^{f^i} \to P_0^{f^{i+1}}$.

Definition 2.4.3. A Zink display (P_0, P_1, F_0, F_1) over R is *nilpotent* if there exists an N such that the composition

$$V_N^\sharp \circ V_{N-1}^\sharp \circ \cdots \circ V^\sharp : P_0 \to P_0^{f^{N+1}}$$

is zero modulo $I_R + pW(R)$.

Remark 2.4.4. If $R = k$ is a perfect field of characteristic p, then displays over k are equivalent to Dieudonné modules over k, and the nilpotence condition on a display corresponds to topological nilpotence of the Verschiebung operator on the corresponding Dieudonné module, cf. [1, Proposition 15].

For a general p-adic ring R, Zink defines a functor BT_R from the category of Zink displays over R to the category of formal groups over R, and he shows that the restriction of BT_R to the full subcategory of nilpotent displays has essential image contained in the category of p-divisible formal groups. The following is the main theorem connecting displays and formal p-divisible groups. For many R it was proved by Zink in [1], and for all R which are p-adic this was proved by Lau in [2].

Theorem 2.4.5 (Zink, Lau). *The functor BT_R induces an equivalence of categories between nilpotent Zink displays over R and formal p-divisible groups over R. Further, BT_R has the following properties: if $\underline{P} = (P_0, P_1, F_0, F_1)$ is a nilpotent Zink display, then*

(*i*) $\mathrm{Lie}(BT_R(\underline{P})) = P_0/P_1$

(*ii*) *The height of $BT_R(\underline{P})$ is equal to the rank of P_0 over $W(R)$.*

Finally let us mention some aspects the of the connection between the theory of displays and the theory of crystals associated to p-divisible groups. Suppose X is a formal p-divisible group over a p-nilpotent $\mathbb{Z}_p$-algebra R. Associated to X is its covariant Dieudonné crystal $\mathbb{D}(X)$. Evaluating $\mathbb{D}(X)$ on the trivial PD-thickening $\mathrm{id}_R : R \to R$, we obtain a finite projective R-module $\mathbb{D}(X)_R$ equipped with a functorial exact sequence

$$0 \to \mathrm{Lie}(X^\vee)^* \to \mathbb{D}(X)_R \to \mathrm{Lie}(X) \to 0$$

which is compatible with base change. Here $\mathrm{Lie}(X)$ is the Lie algebra of X, which is a finite projective R-module, $X^\vee$ is the Serre dual of X, and $(-)^*$ denotes linear dual. The filtration

$$\mathbb{D}(X)_R \supset \mathrm{Lie}(X^\vee)^* \supset 0$$

is the Hodge filtration of X. If $X = BT_R(\underline{P})$ for a nilpotent Zink display $\underline{P} = (P_0, P_1, F_0, F_1)$, then it follows from [1, Theorem 94] that there is a canonical isomorphism of finite projective R-modules

$$P_0 \otimes_{W(R)} R \cong \mathbb{D}(X)_R$$

which sends the Hodge filtration of $\underline{P}$ to the Hodge filtration of X.

Chapter 3: G-displays

3.1 G-displays of type μ and (G, μ)-displays

Let G be a flat affine group scheme of finite type over $\mathbb{Z}_p$, let k_0 be a finite extension of $\mathbb{F}_p$, and let $\mu : \mathbb{G}_{m,W(k_0)} \to G_{W(k_0)}$ be a cocharacter defined over $W(k_0)$. If R is a $W(k_0)$-algebra, then $W(R)$ is endowed with the structure of a $W(k_0)$-algebra via composition

$$W(k_0) \xrightarrow{\Delta} W(W(k_0)) \to W(R),$$

where the first map is the Cartier homomorphism (cf. [16, Ch VII, Prop 4.2]) and the second is induced by functoriality from the $W(k_0)$-algebra structure homomorphism. Then $W(R)^{\oplus}$ is a graded $W(k_0)$-algebra, and $\sigma : W(R)^{\oplus} \to W(R)$ extends the Frobenius on $W(k_0)$. A frame whose graded ring and Frobenius satisfy these properties is called a $W(k_0)$-frame, cf. [5, Definition 5.0.1].

Using the cocharacter μ we define a (right) action of $\mathbb{G}_{m,W(k_0)}$ on $G_{W(k_0)}$ as follows: if $\lambda \in \mathbb{G}_{m,W(k_0)}(R)$ and $g \in G_{W(k_0)}(R)$ for some $W(k_0)$-algebra R, define

$$g \cdot \lambda := \mu(\lambda)^{-1} g \mu(\lambda).$$

Write $G_{W(k_0)} = \text{Spec } A$ for a $W(k_0)$-algebra A. Then the action defined above also defines an action on A. Since $G_{W(k_0)}(R) = \text{Hom}_{W(k_0)}(A, R)$ for any $W(k_0)$-algebra R, there is a canonical bijection between elements of A and natural transformations $G_{W(k_0)} \to \mathbb{A}^1_{W(k_0)}$ given by sending $f \in A$ to evaluation on f. Hence we can make the action on A explicit: if $f \in A$ and $\lambda \in \mathbb{G}_{m,W(k_0)}(R)$, define $\lambda \cdot f$ to be the function $G(R) \to R$ given by

$$(\lambda \cdot f)(g) := f(\mu(\lambda)^{-1} g \mu(\lambda))$$

for $g \in G_{W(k_0)}(R)$. Giving the collection of these actions as R varies corresponds to a $\mathbb{Z}$-grading on A. In particular, A_n is the set of $f \in A$ with $(\lambda \cdot f)(g) = \lambda^n f(g)$ for all $\lambda \in \mathbb{G}_{m,W(k_0)}(R)$, $g \in G(R)$.

Following [5, §5], for any $\mathbb{Z}$-graded $W(k_0)$-algebra S, let $G(S)_\mu$ be the set of $\mathbb{G}_m$-equivariant morphisms $\text{Spec } S \to G$ over $W(k_0)$. Equivalently, we have

$$G(S)_\mu = \text{Hom}^0_{W(k_0)}(A, S),$$

i.e. $G(S)_\mu$ is the subset of $G_{W(k_0)}(S)$ consisting of $W(k_0)$-algebra homomorphisms which preserve the respective gradings. The Hopf algebra structure for A preserves the grading induced by μ, so $G(S)_\mu$ forms a subgroup of $G_{W(k_0)}(S)$.

Suppose now $\underline{S} = (S, \sigma, \tau)$ is a $W(k_0)$-frame. The $\mathbb{Z}_p$-algebra homomorphisms

$\sigma, \tau : S \to S_0$ induce group homomorphisms

$$\sigma, \tau : G(S)_\mu \to G(S_0).$$

Indeed, if $g \in G(S)_\mu$, then $\sigma(g)$ (resp. $\tau(g)$) is defined by post-composing $g \in \mathrm{Hom}_{W(k_0)}(A, S)$ with $\sigma : S \to S_0$ (resp. $\tau : S \to S_0$). Using these homomorphisms we define a group action of $G(S)_\mu$ on $G(S_0)$ as follows:

$$G(S_0) \times G(S)_\mu \to G(S_0),\ (x, g) \mapsto \tau(g)^{-1} x \sigma(g). \tag{3.1.1}$$

Let us restrict our focus to the Witt frame, $\underline{W}(R)$ (cf. 2.1.2). We define two group-valued functors on $W(k_0)$-algebras as follows: if R is a $W(k_0)$-algebra, let

$$L^+G(R) := G(W(R)),\ \text{and}\ L^+_\mu G(R) := G(W(R)^\oplus)_\mu.$$

By [5, Lemma 5.4.1] these are representable functors. We will refer to the $W(k_0)$-group scheme L^+G as the *positive Witt loop group scheme*, and to $L^+_\mu G$ as the *display group* for the pair (G, μ).

Definition 3.1.1 (Lau)**.** The stack of G-*displays of type* μ is the fpqc quotient stack

$$G\text{-}\mathbf{Disp}^W_\mu = [L^+G/L^+_\mu G].$$

over $\mathbf{Nilp}_{W(k_0)}$, where $L^+_\mu G$ acts on L^+G via the action (A.2.1).

Explicitly, for a p-nilpotent $W(k_0)$-algebra R, $G\text{-}\mathbf{Disp}^W_\mu(R)$ is the groupoid of pairs (Q, α), where Q is an fpqc-locally trivial $L^+_\mu G$-torsor over Spec R and $\alpha : Q \to L^+G$ is an $L^+_\mu G$-equivariant map. If (G, μ) and (G', μ') are two pairs as above, and $\varphi : G \to G'$ is a $\mathbb{Z}_p$-group scheme homomorphism such that $\varphi \circ \mu = \mu'$, then φ induces a morphism of stacks

$$G\text{-}\mathbf{Disp}^W_\mu \to G'\text{-}\mathbf{Disp}^W_{\mu'}.$$

Indeed, $\mathbb{G}_m$-equivariance of φ furnishes us with a group homomorphism $G(S)_\mu \to G'(S)_{\varphi\circ\mu}$, and since φ is defined over $\mathbb{Z}_p$ it commutes with σ and τ. On the level of objects, the pair (Q, α) is sent to (Q^φ, α'), where Q^φ is the pushforward of Q along $\varphi : L^+_\mu G \to L^+_{\mu'} G'$ and α' is the induced $L^+_{\mu'} G'$-equivariant morphism $Q^\varphi \to L^+G'$.

Remark 3.1.2. Let $I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$ with $i_1 \le i_2 \le \cdots \le i_n$, and define a cocharacter

$$\mu_I : \mathbb{G}_m \to \mathrm{GL}_n,\ \lambda \mapsto \mathrm{diag}(\lambda^{i_1}, \lambda^{i_2}, \ldots, \lambda^{i_n}).$$

Then GL_n-$\mathbf{Disp}^W_{\mu_I}$ is the stack of displays of type $I = (i_1, \ldots, i_n)$, cf. [5, Example 5.3.5].

As a particular example, consider the case where $I = (0^{(r)}, 1^{(n-r)})$ for some r. Then $\mu = \mu_{r,n}$ is the minuscule cocharacter $\lambda \mapsto \mathrm{diag}(1^{(r)}, \lambda^{(n-r)})$, and $L^+_\mu \mathrm{GL}_n(R)$

consists of block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

- A is an $r \times r$-matrix whose entries are in $W(R)_0^{\oplus} = W(R)$,
- B is an $r \times (n-r)$-matrix whose entries are in $W(R)_1^{\oplus} = I_R$,
- C is an $(n-r) \times r$-matrix whose entries are in $W(R)_{-1}^{\oplus} = W(R) \cdot t$,
- D is an $(n-r) \times (n-r)$-matrix whose entries are in $W(R)_0^{\oplus} = W(R)$.

In this case, GL_n-$\mathbf{Disp}_{\mu_{r,n}}^W$ is isomorphic to the stack of Zink displays (P_0, P_1, F_0, F_1) with $\mathrm{rk}_{W(R)} P_0 = n$ and $\mathrm{rk}_R(P_0/P_1) = d$, cf. Lemma 2.4.2.

Bültel and Pappas define [3] an alternative category of (G, μ)-displays over R in the case where G is reductive over $\mathbb{Z}_p$ and μ is a minuscule cocharacter defined over $W(k_0)$. Let us briefly explain. Let P_μ be the parabolic sub-group scheme of G defined by μ (see [3, Appendix 1]). Then Bültel and Pappas define a closed sub-group scheme H^μ of L^+G whose points in a $W(k_0)$-algebra R are those elements of $L^+G(R)$ which map to $P_\mu(R)$ under the canonical map $L^+G(R) \to G(R)$. By [3, Proposition 3.1.2], there is a group scheme homomorphism

$$\Phi_{G,\mu} : H^\mu \to L^+G$$

such that $\Phi_{G,\mu}(h) = F(\mu(p)h\mu(p)^{-1}) \in G(W(R)[1/p])$, where F is induced from the Witt vector Frobenius. Then a (G,μ)-display over a $W(k_0)$-scheme S is a triple (Q, P, u), where Q is a torsor for H^μ over S, P is the pushforward of Q to L^+G, and $u : Q \to P$ is a morphism such that $u(q \cdot h) = u(q)\Phi_{G,\mu}(h)$ for all $h \in H^\mu$, $q \in Q$. In [3, 3.2.7] it is shown that the stack of (G,μ)-displays over $\mathbf{Nilp}_{W(k_0)}$ is isomorphic to the fpqc quotient stack $[L^+G/_{\Phi_{G,\mu}} H^\mu]$, where H^μ acts on L^+G via

$$g \cdot h = h^{-1} g \Phi_{G,\mu}(h)$$

for R a p-nilpotent $\mathbb{Z}_p$-algebra, $h \in H^\mu(R)$, $g \in L^+G(R)$. In [5, Remark 6.3.4] Lau proves the following lemma by showing that τ induces an isomorphism $L^+_\mu G \xrightarrow{\sim} H^\mu$, which is compatible with the actions of $L^+_\mu G$ resp. H^μ on L^+G.

Lemma 3.1.3. *The stack of (G,μ)-displays as in* [3] *is isomorphic to G-$\mathbf{Disp}^W_\mu$.*

$\square$

3.2 Graded fiber functors

Let G be a flat affine group scheme of finite type over $\mathbb{Z}_p$. Denote by $\mathbf{Rep}_{\mathbb{Z}_p}(G)$ the category of representations of G on finite free $\mathbb{Z}_p$-modules. Let R be a $W(k_0)$-algebra.

Definition 3.2.1. A *graded fiber functor* over $W(R)^\oplus$ is an exact tensor functor

$$\mathscr{F} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}^W(R).$$

Denote by $\mathbf{GFF}^W(R)$ the category of graded fiber functors over $W(R)^{\oplus}$. Morphisms in this category are morphisms of tensor functors. If $R \to R'$ is a ring homomorphism and $\mathscr{F}$ is a graded fiber functor over $W(R)^{\oplus}$, then we define the base change of $\mathscr{F}$ to $W(R')^{\oplus}$, written $\mathscr{F}_{R'}$, as the composition $\mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}^W(R) \to \mathbf{PGrMod}^W(R')$. As R varies in $\mathbf{Nilp}_{W(k_0)}$ we obtain a fibered category $\mathbf{GFF}^W$ whose fiber over R is $\mathbf{GFF}^W(R)$.

Lemma 3.2.2. *The fibered category $\mathbf{GFF}^W$ is an fpqc stack in groupoids over $\mathbf{Nilp}_{W(k_0)}$.*

Proof. The proof is essentially the same as that of [11, Proposition 7.2]. We repeat the argument here for completeness.

Both $\mathbf{Rep}_{\mathbb{Z}_p}(G)$ and $\mathbf{PGrMod}^W(R)$ are rigid tensor categories (cf. Lemma 2.2.1), so by [15, Proposition 1.13], if $\mathscr{F}_1$ and $\mathscr{F}_2$ are graded fiber functors over $W(R)^{\oplus}$, then every morphism of tensor functors $\mathscr{F}_1 \to \mathscr{F}_2$ is an isomorphism. Hence $\mathbf{GFF}^W$ is fibered in groupoids.

It remains to show $\mathbf{GFF}^W$ satisfies effective descent for morphisms and for objects. Let R be a p-nilpotent $W(k_0)$-algebra, let R' be a faithfully flat extension, and let $R'' = R' \otimes_R R'$. Suppose $\lambda' : (\mathscr{F}_1)_{R'} \to (\mathscr{F}_2)_{R'}$ is a morphism over R' such that the two pullbacks to R'' agree. Then for each $(V, \rho) \in \mathrm{Ob}(\mathbf{Rep}_{\mathbb{Z}_p}(G))$, the same holds for ${\lambda'}_{\rho}$. By Lemma 2.2.7, morphisms of finite projective graded $W(R)^{\oplus}$-modules descend, so we obtain unique morphisms λ_ρ for every (V, ρ). We need these morphisms to piece together to form a natural transformation $\lambda : \mathscr{F}_1 \to \mathscr{F}_2$ which is compatible with the tensor product. Let $\alpha : (V, \rho) \to (U, \pi)$ be a morphism in

$\mathbf{Rep}_{\mathbb{Z}_p}(G)$. Then we must show the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}_1(V,\rho) & \xrightarrow{\mathscr{F}_1(\alpha)} & \mathscr{F}_1(U,\pi) \\ \downarrow{\scriptstyle \lambda_\rho} & & \downarrow{\scriptstyle \lambda_\pi} \\ \mathscr{F}_2(V,\rho) & \xrightarrow{\mathscr{F}_2(\alpha)} & \mathscr{F}_2(U,\pi) \end{array}$$

By assumption the diagram commutes after base change to R'. Then the morphism $(\mathscr{F}_1)_{R'}(V,\rho) \to (\mathscr{F}_2)_{R'}(U,\pi)$ descends uniquely to a morphism $\mathscr{F}_1(V,\rho) \to \mathscr{F}_2(U,\pi)$. Since both $\lambda_\pi \circ \mathscr{F}_1(\pi)$ and $\mathscr{F}_2(\pi) \circ \lambda_\rho$ satisfy this property, they agree by uniqueness, and the diagram commutes. Hence λ is a natural transformation. A similar argument shows that λ is compatible with tensor products, so we conclude morphisms descend.

Finally we prove $\mathbf{GFF}^W$ satisfies effective descent for objects. Let $\mathscr{F}'$ be a graded fiber functor over $W(R')^{\oplus}$ equipped with a descent datum, i.e. equipped with an isomorphism

$$p_1^*\mathscr{F}' \xrightarrow{\sim} p_2^*\mathscr{F}'$$

of tensor functors over $W(R'')^{\oplus}$ satisfying the cocycle condition, where p_1^* and p_2^* denote base change along the maps induced by $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ from $R' \to R' \otimes R'$, respectively. The given descent datum induces a descent datum on $\mathscr{F}'(V,\rho)$ for each (V,ρ) in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, so, by Lemma 2.2.7, for every (V,ρ) we obtain a finite projective $W(R)^{\oplus}$-module $\mathscr{F}(V,\rho)$ over R whose base change to R' is $\mathscr{F}'(V,\rho)$. We claim the assignment $(V,\rho) \to \mathscr{F}(V,\rho)$ is functorial. Suppose $\alpha : (V,\rho) \to (U,\pi)$ is a morphism in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$. Then we obtain a morphism $\mathscr{F}'(\alpha) : \mathscr{F}'(V,\rho) \to \mathscr{F}'(U,\pi)$. Because the descent datum on $\mathscr{F}'$ is a natural transformation, we obtain a commutative diagram:

$$\begin{tikzcd}
p_1^*\mathscr{F}'(V,\rho) \arrow[r, "\sim"] \arrow[d, "p_1^*\mathscr{F}'(\alpha)"] & p_2^*\mathscr{F}'(V,\rho) \arrow[d, "p_2^*\mathscr{F}'(\alpha)"] \\
p_1^*\mathscr{F}'(U,\pi) \arrow[r, "\sim"] & p_2^*\mathscr{F}'(U,\pi)
\end{tikzcd}$$

From this we see the two pullbacks of $\mathscr{F}'(\alpha)$ coincide, so by descent for finite projective modules over a graded ring we obtain a unique homomorphism $\mathscr{F}(V,\rho) \to \mathscr{F}(U,\pi)$. The uniqueness part of this assertion is enough to prove that $\mathscr{F}$ preserves compositions and the identity, so $\mathscr{F}$ is, in fact, a functor. Now, being a tensor functor, $\mathscr{F}'$ is equipped with isomorphisms $\mathscr{F}'(\mathbb{1}) \cong \mathbb{1}$ and $\mathscr{F}'(V,\rho) \otimes \mathscr{F}'(U,\pi) \cong \mathscr{F}'(V \otimes U, \rho \otimes \pi)$. Since the descent datum on $\mathscr{F}'$ is a tensor morphism, it is compatible with these isomorphisms. Hence these isomorphisms descend as above to $\mathscr{F}$. The isomorphisms are compatible with the associativity and commutativity restraints on $\mathscr{F}'$, so by uniqueness the same holds for $\mathscr{F}$, and $\mathscr{F}$ is a tensor functor. By Lemma 2.2.10 exactness is an fpqc local property for finite projective $W(R)^{\oplus}$-modules, so $\mathscr{F}$ is exact. $\square$

If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two graded fiber functors over $W(R)^{\oplus}$, denote by $\underline{\mathrm{Isom}}^{\otimes}(\mathscr{F}_1, \mathscr{F}_2)$ the functor which assigns to an R-algebra R' the set $\mathrm{Isom}^{\otimes}((\mathscr{F}_1)_{R'}, (\mathscr{F}_2)_{R'})$ of isomorphisms of tensor functors $(\mathscr{F}_1)_{R'} \to (\mathscr{F}_2)_{R'}$. By the Lemma 3.2.2 this is an fpqc sheaf over $\mathbf{Nilp}_R$. We write $\underline{\mathrm{Aut}}^{\otimes}(\mathscr{F}) := \underline{\mathrm{Isom}}^{\otimes}(\mathscr{F}, \mathscr{F})$ and $\mathrm{Aut}^{\otimes}(\mathscr{F}_R) := \mathrm{Isom}^{\otimes}(\mathscr{F}_R, \mathscr{F}_R)$. There is a natural action of $\underline{\mathrm{Aut}}^{\otimes}(\mathscr{F}_1)$ on $\underline{\mathrm{Isom}}^{\otimes}(\mathscr{F}_1, \mathscr{F}_2)$ by precomposition.

If R is a $W(k_0)$-algebra, let us define a tensor functor

$$\mathscr{C}_{\mu,R} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}^W(R)$$

attached to any cocharacter μ of G defined over $W(k_0)$. Given a representation (V, ρ) in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, μ induces a canonical decomposition

$$V_{W(k_0)} = \bigoplus_{i \in \mathbb{Z}} V^i_{W(k_0)}, \tag{3.2.1}$$

where $V_{W(k_0)} := V \otimes_{\mathbb{Z}_p} W(k_0)$, and

$$V^i_{W(k_0)} = \{v \in V_{W(k_0)} \mid (\rho \circ \mu)(z) \cdot v = z^i v \text{ for all } z \in \mathbb{G}_m(W(k_0))\}.$$

By base change along $W(k_0) \to W(R)^\oplus$ we obtain a finite projective graded $W(R)^\oplus$ module

$$V \otimes_{\mathbb{Z}_p} W(R)^\oplus = V_{W(k_0)} \otimes_{W(k_0)} W(R)^\oplus.$$

Any morphism $\varphi : (V, \rho) \to (U, \pi)$ in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$ preserves the grading induced by μ, so we have defined a functor

$$\mathscr{C}_{\mu,R} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}^W(R),\ V \mapsto V_{W(k_0)} \otimes_{W(k_0)} W(R)^\oplus.$$

The resulting functor obviously preserves the tensor product, and it is exact because the underlying modules of objects in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$ are free, so all short exact sequences remain exact after tensoring over $W(R)^{\oplus}$. Then $\mathscr{C}_{\mu,R}$ is a graded fiber functor over $W(R)^{\oplus}$.

Remark 3.2.3. For any $W(k_0)$-algebra R, the $W(k_0)$-algebra structure homomorphism for $W(R)$ factors through $\Delta : W(k_0) \to W(W(k_0))$ by definition. Then we see $\mathscr{C}_{\mu,R}$ is the base change of $\mathscr{C}_{\mu,W(k_0)}$ along $\mathbf{PGrMod}^W(W(k_0)) \to \mathbf{PGrMod}^W(R)$. We will denote $\mathscr{C}_{\mu,W(k_0)}$ simply by $\mathscr{C}_\mu$.

Definition 3.2.4. A graded fiber functor $\mathscr{F}$ over $W(R)^{\oplus}$ is *of type* μ if for some faithfully flat extension $R \to R'$ there is an isomorphism $\mathscr{F}_{R'} \cong \mathscr{C}_{\mu,R'}$.

Remark 3.2.5. Let (V, ρ) be a representation of G on a finite free $\mathbb{Z}_p$-algebra of rank n, and suppose the weights of $\rho \circ \mu$ on V are $\{w_1, \dots, w_r\}$ with $w_1 \leq w_2 \leq \cdots \leq w_r$. Let r_i be the rank of $V^{w_i}_{W(k_0)}$. Then define $I := I_\mu(V, \rho) = (i_1, \cdots, i_n)$ as follows: First let $i_1 = \cdots = i_{r_1} = w_1$. Then for $j \geq 1$ and any r satisfying

$$\sum_{k=1}^{j} r_k < r \leq \sum_{k=1}^{j+1} r_k,$$

let $i_r = w_r$. We see that $\mathscr{C}_{\mu,R}(V, \rho)$ is of type I. If $\mathscr{F}$ is any graded fiber functor over $W(R)^{\oplus}$ of type μ, then it follows from Lemma 2.2.8 that $\mathscr{F}(V, \rho)$ is of type I as well.

Denote by $\mathbf{GFF}_\mu(W(R)^{\oplus})$ the category of graded fiber functors of type μ over $W(R)^{\oplus}$. Base change preserves graded fiber functors of type μ, so we obtain a fibered category $\mathbf{GFF}^W_\mu$ whose fiber over R is $\mathbf{GFF}_\mu(W(R)^{\oplus})$. Since the property of "being type μ" is an fpqc-local property, $\mathbf{GFF}_{\mu}^{W}$ forms a substack of GFF^{W}.

For any $\mathbb{Z}_p$-algebra R let $\mathbf{PMod}(R)$ be the category of finite projective R-modules. Associated to this category we have the canonical fiber functor

$$\omega_R : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PMod}(R),\ (V, \rho) \mapsto V \otimes_{\mathbb{Z}_p} R.$$

Define $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ to be the fpqc sheaf in groups over $\mathbf{Nilp}_{W(k_0)}$ which associates to an $W(k_0)$-algebra R the group of automorphisms of ω_R. By Tannakian duality, the assignment $g \mapsto \{\rho(g)\}_{(V,\rho)}$ defines an isomorphism of fpqc sheaves in groups

$$G \xrightarrow{\sim} \underline{\mathrm{Aut}}^{\otimes}(\omega),$$

cf. [17, Theorem 44] for the statement in this generality.

Lemma 3.2.6. *Let R be a $W(k_0)$-algebra. For all $g \in L_{\mu}^{+}G(R)$, the collection $\{\rho(g)\}_{(V,\rho)}$ comprises an element of $\underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}_{\mu,R})$.*

Proof. For every (V, ρ), we have

$$\rho(g) \in \mathrm{GL}(V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus})_{\rho\circ\mu},$$

so it is enough to show that any $h \in \mathrm{GL}(V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus})_{\rho\circ\mu}$ preserves the grading on $V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus}$ for all (V, ρ).

Suppose $\mathrm{rk}_{\mathbb{Z}_p}(V) = r$, and choose an ordered basis $\{v_1, \ldots, v_r\}$ for $V_{W(k_0)}$ over $W(k_0)$ such that each $v_i \in V_{W(k_0)}^{n_i}$ and $n_1 \leq n_2 \leq \cdots \leq n_r$. Relative to this basis we have

$$\mathrm{GL}(V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus}) \cong \mathrm{GL}_r(W(R)^{\oplus}),$$

and $(\rho \circ \mu)(z) = \mathrm{diag}(z^{n_1}, \ldots, z^{n_r})$. Let A be the coordinate ring of $\mathrm{GL}_{r,W(k_0)}$, so

$$A = W(k_0)[X_{ij}, Y]_{i,j=1}^{r}/(\det(X_{ij})Y - 1).$$

If $\lambda \in \mathbb{G}_m(W(k_0))$, the action of λ on A is given by

$$X_{ij} \mapsto \lambda^{n_j - n_i} X_{ij}.$$

Then any $h \in \mathrm{GL}_r(W(R)^{\oplus})_{\rho\circ\mu}$ is represented by a matrix $(h_{ij})_{ij}$ with $h_{ij} \in W(R)^{\oplus}_{n_j - n_i}$. Let $v \in (V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus})_{\ell}$. We need to show $h(v) \in (V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus})_{\ell}$. Write

$$v = \sum_{j=1}^{r} v_j \otimes \xi_j,$$

where $\xi_j \in W(R)^{\oplus}_{\ell - n_j}$ for all j. Then

$$h(v) = \sum_{j=1}^{r} \sum_{i=1}^{r} v_i \otimes h_{ij}\xi_j,$$

and $v_i \otimes h_{ij}\xi_j$ is of degree $n_i + (n_j - n_i) + (\ell - n_j) = \ell$ as desired. $\square$

It follows from the lemma that the assignment $g \mapsto \{\rho(g)\}_{(V,\rho)}$ induces a homomorphism of fpqc sheaves of groups on $\mathbf{Nilp}_{W(k_0)}$

$$\Psi : L^+_\mu G \to \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}_\mu). \tag{3.2.2}$$

Theorem 3.2.7. *The homomorphism (3.2.2) is an isomorphism.*

Proof. For every $W(k_0)$-algebra R, Ψ_R is the restriction of the map $G(W(R)^\oplus) \to \mathrm{Aut}^{\otimes}(\omega_{W(R)^\oplus})$ given by $g \mapsto \{\rho(g)\}_{(V,\rho)}$. An inverse to this map is constructed in [17, Theorem 44] (cf. also [18, Theorem 9.2]). We need only verify the restriction of this inverse to $\mathrm{Aut}^{\otimes}(\mathscr{C}_{\mu,R})$ respects the grading. Let us review the construction of this map.

Denote by $\mathbf{Rep}'_{\mathbb{Z}_p}(G)$ the category whose objects are the representations (V,ρ) of G on $\mathbb{Z}_p$-modules such that

$$(V,\rho) = \varinjlim (W,\pi),$$

where (W,π) runs through the partially ordered set of all G-sub-representations of (V,ρ) belonging to $\mathbf{Rep}_{\mathbb{Z}_p}(G)$. The functor $\mathscr{C}_{\mu,R}$ extends to a functor

$$\mathscr{C}'_{\mu,R} : \mathbf{Rep}'_{\mathbb{Z}_p}(G) \to \mathbf{GrMod}(W(R)^\oplus),\ V \mapsto V_{W(k_0)} \otimes_{W(k_0)} W(R)^\oplus.$$

If we denote by $\underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}'_\mu)$ the group-valued functor on $W(k_0)$-algebras given by

$R \mapsto \mathrm{Aut}^{\otimes}(\mathscr{C}'_{\mu,R})$, then there is a canonical isomorphism

$$\underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}_{\mu}) \xrightarrow{\sim} \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}'_{\mu}).$$

We will abuse notation and also denote the composition $L^+_{\mu}G \to \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}'_{\mu})$ by Ψ. It is enough to define an inverse to this composition.

Write $G = \mathrm{Spec}\ A$. Recall the regular representation ρ_{reg} is the representation of G on A, viewed as a $\mathbb{Z}_p$-module, whose comodule morphism is the comultiplication for A. Explicitly, if R is a $\mathbb{Z}_p$-algebra, $g \in G(R)$, and $a \in A \otimes_{\mathbb{Z}_p} R$, then $\rho_{\mathrm{reg}}(g) \cdot a$ is defined by

$$(\rho_{\mathrm{reg}}(g) \cdot a)(h) = a(hg).$$

Since G is a flat affine group scheme over a Noetherian ring, [19, Cor. to Proposition 2] implies that ρ_{reg} is an object of $\mathbf{Rep}'_{\mathbb{Z}_p}(G)$. One checks that the morphisms

$$(\mathbb{Z}_p, \mathbb{1}) \to (A, \rho_{\mathrm{reg}}) \quad \text{and} \quad (A, \rho_{\mathrm{reg}}) \otimes (A, \rho_{\mathrm{reg}}) \to (A, \rho_{\mathrm{reg}}), \tag{3.2.3}$$

given by the unit and multiplication respectively, are G-equivariant.

Now let R be a $W(k_0)$-algebra and let $\lambda \in \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}'_{\mu,R})$, so for every (V, ρ) in $\mathbf{Rep}'_{\mathbb{Z}_p}(G)$,

$$\lambda_{\rho} : V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus} \to V_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus}$$

is a graded $W(R)^{\oplus}$-module automorphism. In particular, λ determines a graded $W(R)^{\oplus}$-module automorphism $\lambda_{\rho_{\mathrm{reg}}}$ of $A \otimes_{\mathbb{Z}_p} W(R)^{\oplus}$. Moreover, since λ is a morphism of tensor functors, functoriality applied to (3.2.3) implies $\lambda_{\rho_{\mathrm{reg}}}$ is a graded $W(R)^{\oplus}$-algebra homomorphism.

Define $\Phi_R : \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}_{\mu,R}) \to G(W(R))^{\oplus}$ by assigning to $\lambda \in \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}_{\mu,R})$ the composition

$$A_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus} \xrightarrow{\lambda_{\rho_{\mathrm{reg}}}} A_{W(k_0)} \otimes_{W(k_0)} W(R)^{\oplus} \xrightarrow{\varepsilon \otimes \mathrm{id}_{W(R)^{\oplus}}} W(R)^{\oplus},$$

where the ε is the counit for $A_{W(k_0)}$. We claim this composition is a graded $W(R)^{\oplus}$-module homomorphism, so $\Phi_R(\lambda) \in G(W(R)^{\oplus})_{\mu} = L_{\mu}^{+}G(R)$. By assumption $\lambda_{\rho_{\mathrm{reg}}}$ respects the grading, so it remains only to show $\varepsilon \otimes \mathrm{id}_{W(R)^{\oplus}}$ is a graded $W(R)^{\oplus}$-homomorphism. Because the zero element of $W(R)^{\oplus}$ is homogeneous of degree n for all n, it is enough to show $\varepsilon(a) = 0$ if $a \in (A_{W(k_0)})_n$ for $n \neq 0$. Let $a \in (A_{W(k_0)})_n$ and $\lambda \in \mathbb{G}_m(W(k_0))$. Then

$$(\lambda \cdot a)(e) = a(\mu(\lambda)^{-1} e \mu(\lambda)) = a(e) = \varepsilon(a),$$

where e is the identity element of $G(W(k_0))$. But since $a \in (A_{W(k_0)})_n$,

$$(\lambda \cdot a)(e) = \lambda^n a(e) = \lambda^n \varepsilon(a).$$

Hence $(\lambda^n - 1)\varepsilon(a) = 0$ in $W(k_0)$ for all $\lambda \in W(k_0)^{\times}$, so $\varepsilon(a) = 0$ if $n \neq 0$.

The construction of Φ_R is functorial in R, so as R varies we obtain a natural transformation

$$\Phi : \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}'_\mu) \to L^+_\mu G,$$

and the verifications in [17, Theorem 44] show Φ and Ψ compose to the identity in both directions. $\square$

It follows from the theorem that if $\mathscr{F}$ is a graded fiber functor of type μ over $W(R)^{\oplus}$, then $\underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \mathscr{F})$ is an $L^+_\mu G$-torsor, with $L^+_\mu G$ acting by pre-composition. This defines a functor

$$\mathbf{GFF}^W_\mu(R) \to \mathbf{Tors}_{L^+_\mu G}(R),\ \mathscr{F} \mapsto \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \mathscr{F}), \tag{3.2.4}$$

where $\mathbf{Tors}_{L^+_\mu G}$ is the fpqc stack of $L^+_\mu G$-torsors over $\mathbf{Nilp}_{W(k_0)}$.

Corollary 3.2.8. *The functor (3.2.4) induces an isomorphism of stacks*

$$\mathbf{GFF}^W_\mu \xrightarrow{\sim} \mathbf{Tors}_{L^+_\mu G}.$$

Proof. Fix a p-nilpotent $W(k_0)$-algebra R. Suppose $\mathscr{F}$ and $\mathscr{F}'$ are graded fiber functors of type μ over $W(R)^{\oplus}$, and let ψ_1 and ψ_2 be morphisms $\mathscr{F} \to \mathscr{F}'$ which induce the same morphism

$$\varphi : \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \mathscr{F}) \to \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \mathscr{F}').$$

Let $R \to R'$ be a faithfully flat extension and choose $\lambda \in \mathrm{Isom}^{\otimes}(\mathscr{C}_{\mu,R'}, \mathscr{F}_{R'})$. Then

$$(\psi_1)_{R'} \circ \lambda = (\varphi_{R'})(\lambda) = (\psi_2)_{R'} \circ \lambda.$$

Since λ is an isomorphism, $(\psi_1)_{R'} = (\psi_2)_{R'}$. Then by descent, $\psi_1 = \psi_2$, and the functor $\mathbf{GFF}^W_\mu(R) \to \mathbf{Tors}_{L^+_\mu G}(R)$ is faithful.

Now consider an $L^+_\mu G$-equivariant morphism

$$\varphi : \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \mathscr{F}) \to \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \mathscr{F}')$$

As above, choose $R \to R'$ faithfully flat. Then for any $\lambda \in \mathrm{Isom}^{\otimes}(\mathscr{C}_{\mu,R'}, \mathscr{F}_{R'})$, the element

$$\psi' = \varphi_{R'}(\lambda) \circ \lambda^{-1} \in \mathrm{Isom}^{\otimes}(\mathscr{F}_{R'}, \mathscr{F}'_{R'})$$

is determined uniquely by φ by $L^+_\mu G$-equivariance. Indeed, if $\lambda' : \mathscr{C}_{\mu,R'} \xrightarrow{\sim} \mathscr{F}_{R'}$ is another choice, then $\lambda^{-1} \circ \lambda'$ is a tensor automorphism of $\mathscr{C}_{\mu,R'}$, hence is an element of $L^+_\mu G(R')$. Then

$$\varphi_{R'}(\lambda') = \varphi_{R'}(\lambda \circ \lambda^{-1} \circ \lambda') = \varphi_{R'}(\lambda) \circ (\lambda^{-1} \circ \lambda')$$

by $L^+_\mu G(R')$-equivariance, so $\varphi_{R'}(\lambda) \circ \lambda^{-1} = \varphi_{R'}(\lambda') \circ (\lambda')^{-1}$. Similarly, by $L^+_\mu G(R'')$-

equivariance,

$$p_1^*\psi' = \varphi_{R''}(p_1^*\lambda) \circ (p_1^*\lambda)^{-1} = \varphi_{R''}(p_2^*\lambda) \circ (p_2^*\lambda)^{-1} = p_2^*\psi',$$

so ψ' descends to some $\psi : \mathscr{F} \to \mathscr{F}'$. It is clear that ψ' maps to $\varphi_{R'}$, so by descent we have ψ mapping to φ, and we see the functor is full.

Then we are done by [20, Lemma 046N] since any $L_\mu^+ G$-torsor is fpqc-locally trivial, and the trivial $L_\mu^+ G$-torsor is in the essential image of this functor. $\square$

3.3 Tannakian (G, μ)-displays

Let G and μ be as in the previous section. In this section we give a Tannakian definition of G-displays of type μ and show that the resulting stack coincides with those defined by Lau and Bültel-Pappas. Let R be a p-adic $W(k_0)$-algebra.

Definition 3.3.1. A *Tannakian G-display* over R is an exact tensor functor

$$\mathscr{D} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{Disp}^W(R).$$

As in the previous section such functors form a fibered category over $\mathbf{Nilp}_{W(k_0)}$, which we will denote by $G\text{-}\mathbf{Disp}^{W,\otimes}$.

Lemma 3.3.2. *The fibered category $G\text{-}\mathbf{Disp}^{W,\otimes}$ is an fpqc stack in groupoids.*

Proof. The proof is essentially the same as that of Lemma 3.2.2, after replacing $\mathbf{PGrMod}^W$ by $\mathbf{Disp}^W$ everywhere. $\square$

Denote by υ_R the natural forgetful functor which sends a display $\underline{M} = (M, F)$ to its underlying finite projective graded $W(R)^{\oplus}$-module M. If $\mathscr{D}$ is a Tannakian G-display over R, then by composing with the forgetful functor υ_R we obtain a graded fiber functor $\upsilon_R \circ \mathscr{D} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}^W(R)$.

Definition 3.3.3. A *Tannakian* (G, μ)*-display* over R is a Tannakian G-display $\mathscr{D}$ over R such that $\upsilon_R \circ \mathscr{D}$ is a graded fiber functor of type μ.

Denote by $G\text{-}\mathbf{Disp}_\mu^{W,\otimes}$ the fibered category over $\mathbf{Nilp}_{W(k_0)}$ whose fiber over R is the category of Tannakian (G, μ)-displays over R. Evidently $G\text{-}\mathbf{Disp}_\mu^{W,\otimes}$ is a substack of $G\text{-}\mathbf{Disp}^{W,\otimes}$.

Construction 3.3.4. Suppose $\mathscr{D}$ is a Tannakian (G, μ)-display over R. We will associate to $\mathscr{D}$ a G-display of type μ. By Corollary 3.2.8,

$$Q_{\mathscr{D}} := \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \upsilon_R \circ \mathscr{D})$$

is an $L_\mu^+ G$-torsor over R. Let R' be an R-algebra and suppose $\lambda : \mathscr{C}_{\mu,R'} \xrightarrow{\sim} \upsilon_{R'} \circ \mathscr{D}_{R'}$ is an isomorphism of tensor functors. If (V, ρ) is in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, write $\mathscr{D}_{R'}(V, \rho) = (M(\rho)', F(\rho)')$. Define $\alpha_{\mathscr{D}}(\lambda)_\rho$ as the composition

$$V \otimes_{\mathbb{Z}_p} W(R') \xrightarrow{(\lambda_\rho)^\sigma} (M(\rho)')^\sigma \xrightarrow{(F(\rho)')^\sharp} (M(\rho)')^\tau \xleftarrow{(\lambda_\rho)^\tau} V \otimes_{\mathbb{Z}_p} W(R').$$

On the left we are implicitly identifying

$$(V \otimes_{\mathbb{Z}_p} W(R')^{\oplus})^\sigma \cong V \otimes_{\mathbb{Z}_p} W(R')$$

using the isomorphism induced by the natural isomorphism of rings $W(R')^{\oplus}\otimes_{W(R')^{\oplus},\sigma} W(R') \xrightarrow{\sim} W(R')$. We have a similar identification on the right when we replace σ by τ.

Because $\lambda : \mathscr{C}_{\mu,R'} \to \upsilon_{R'}\circ\mathscr{D}_{R'}$ is a tensor morphism, it follows that $\{\alpha_{\mathscr{D}}(\lambda)_\rho\}_{(V,\rho)}$ is an element of $\mathrm{Aut}^{\otimes}(\omega_{W(R')})$, which is isomorphic to $G(W(R')) = L^+G(R')$ by Tannakian duality. Hence there is some $\alpha_{\mathscr{D}}(\lambda) \in L^+G(R)$ such that $\rho(\alpha_{\mathscr{D}}(\lambda)) = \alpha_{\mathscr{D}}(\lambda)_\rho$ for every (V,ρ). Altogether we have a morphism of fpqc sheaves

$$\alpha_{\mathscr{D}} : Q_{\mathscr{D}} \to L^+G.$$

It remains to show $\alpha_{\mathscr{D}}$ is $L^+_\mu G$-equivariant. For this let $h \in L^+_\mu G(R')$. Then $(\lambda\cdot h)_\rho$ is the composition

$$V\otimes_{\mathbb{Z}_p} W(R')^{\oplus} \xrightarrow{\rho(h)} V\otimes_{\mathbb{Z}_p} W(R')^{\oplus} \xrightarrow{\lambda_\rho} M(\rho)'.$$

Hence we see $\alpha_{\mathscr{D}}(\lambda\cdot h)_\rho$ is given by

$$(\rho(h)^\tau)^{-1}\circ((\lambda_\rho)^\tau)^{-1}\circ F(\rho)^\sharp\circ(\lambda_\rho)^\sigma\circ(\rho(h))^\sigma = \rho(\tau(h^{-1})\cdot\alpha_{\mathscr{D}}(\lambda)\cdot\sigma(h)).$$

By Tannakian duality again we obtain $\alpha_{\mathscr{D}}(\lambda\cdot h) = \tau(h^{-1})\cdot\alpha_{\mathscr{D}}(\lambda)\cdot\sigma(h)$, so $\alpha_{\mathscr{D}}$ is $L^+_\mu G$-equivariant.

The pair $(Q_{\mathscr{D}},\alpha_{\mathscr{D}})$ comprises a G-display of type μ in the sense of Definition A.2.2. Suppose now $\mathscr{D}_1$ and $\mathscr{D}_2$ are Tannakian (G,μ)-displays over R, and write

$\mathscr{D}_1(V,\rho) = (M_1(\rho), F_1(\rho))$ and $\mathscr{D}_2(V,\rho) = (M_2(\rho), F_2(\rho))$. Given a morphism of Tannakian (G,μ)-displays $\psi : \mathscr{D}_1 \to \mathscr{D}_2$, we get a morphism

$$Q_{\mathscr{D}_1} = \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \upsilon_R \circ \mathscr{D}_1) \to \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \upsilon_R \circ \mathscr{D}_2) = Q_{\mathscr{D}_2}$$

by post-composition with ψ. This is obviously a morphism of torsors, and if (V,ρ) is a representation of G, $\lambda \in Q_{\mathscr{D}_1}(R')$, then $\alpha_{\mathscr{D}_2}(\upsilon_R(\psi) \circ \lambda)_\rho$ is given by

$$(\lambda_\rho^\tau)^{-1} \circ ((\psi_{R'})_\rho^\tau)^{-1} \circ (F_2(\rho)')^\sharp \circ (\psi_{R'})_\rho^\sigma \circ \lambda_\rho^\sigma,$$

where $(\mathscr{D}_i)_{R'}(V,\rho) = (M_i(\rho)', F_i(\rho)')$. But because $(\psi_{R'})\rho$ is a morphism of displays $M_1(\rho)_{\underline{W}(R')} \to M_2(\rho)_{\underline{W}(R')}$, this becomes

$$(\lambda_\rho^\tau)^{-1} \circ (F_1(\rho)')^\sharp \circ \lambda_\rho^\sigma = \alpha_{\mathscr{D}_1}(\lambda)_\rho.$$

We conclude that the morphism $Q_{\mathscr{D}_1} \to Q_{\mathscr{D}_2}$ is a morphism of G-displays of type μ, so the construction $\mathscr{D} \mapsto (Q_{\mathscr{D}}, \alpha_{\mathscr{D}})$ is functorial. Denote the resulting functor by T_R.

This construction is evidently compatible with base change, so we obtain a morphism of stacks

$$T : G\text{-}\mathbf{Disp}_\mu^{W,\otimes} \to G\text{-}\mathbf{Disp}_\mu^{W},\ \mathscr{D} \mapsto (Q_{\mathscr{D}}, \alpha_{\mathscr{D}}). \tag{3.3.1}$$

Theorem 3.3.5. *The morphism (3.3.1) is an isomorphism of fpqc stacks over*

$\mathbf{Nilp}_{W(k_0)}$.

Proof. Fix a p-nilpotent $W(k_0)$-algebra R. It is immediate from Corollary 3.2.8 that T_R is faithful. Let us prove it is full. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be Tannakian (G,μ)-displays over R, and write

$$\mathscr{D}_1(V,\rho) = (M_1(\rho), F_1(\rho)) \text{ and } \mathscr{D}_2(V,\rho) = (M_2(\rho), F_2(\rho))$$

for every representation (V,ρ) of G. Suppose

$$\eta : (Q_{\mathscr{D}_1}, \alpha_{\mathscr{D}_1}) \to (Q_{\mathscr{D}_2}, \alpha_{\mathscr{D}_2})$$

is a morphism of G-displays of type μ. By Corollary 3.2.8 there exists some $\psi : \upsilon_R \circ \mathscr{D}_1 \to \upsilon_R \circ \mathscr{D}_2$ which induces η. For every representation (V,ρ) in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, we obtain a morphism of graded $W(R)^{\oplus}$-modules

$$\psi_\rho : M_1(\rho) \to M_2(\rho).$$

The collection of these morphisms is functorial and compatible with tensor product, so it remains only to show ψ_ρ is compatible with $F_1(\rho)$ and $F_2(\rho)$. By Lemma 2.3.10 it is enough to check this condition after a faithfully flat extension of rings $R \to R'$. Choose such an extension with the property that $Q_{\mathscr{D}_1}(R')$ is nonempty, and suppose $\lambda : \mathscr{C}_{\mu,R'} \to \upsilon_{R'} \circ (\mathscr{D}_1)_{R'}$ is an isomorphism of graded fiber functors.

Let (V,ρ) be a representation of G. For brevity, let us write $M_i(\rho)' :=$

$M_i(\rho)_{W(R')^{\oplus}}$ and $F_i(\rho)'$ for the base change of $F_i(\rho)$ to $W(R')^{\oplus}$. Consider the following diagram:

$$\begin{tikzcd}
(M_1(\rho)')^{\sigma} \arrow[r, "(\lambda_\rho^\sigma)^{-1}"] \arrow[d, "(F_1(\rho)')^\sharp"'] & V \otimes_{\mathbb{Z}_p} W(R') \arrow[r, "="] \arrow[d, "\rho(\alpha_{\mathscr{D}_1}(\lambda))"'] & V \otimes_{\mathbb{Z}_p} W(R') \arrow[r, "(\psi_{R'}\circ\lambda)_\rho^\sigma"] \arrow[d, "\rho(\alpha_{\mathscr{D}_2}(\psi_{R'}\circ\lambda))"'] & (M_2(\rho)')^{\sigma} \arrow[d, "(F_2(\rho)')^\sharp"'] \\
(M_1(\rho)')^{\tau} \arrow[r, "(\lambda_\rho^\tau)^{-1}"] & V \otimes_{\mathbb{Z}_p} W(R') \arrow[r, "="] & V \otimes_{\mathbb{Z}_p} W(R') \arrow[r, "(\psi_{R'}\circ\lambda)_\rho^\tau"] & (M_2(\rho)')^{\tau}
\end{tikzcd}$$

The left- and right-most squares commute by definition of $\alpha_{\mathscr{D}_i}$. Because η is a morphism of G-displays of type μ, we have

$$\alpha_{\mathscr{D}_1}(\lambda) = \alpha_{\mathscr{D}_2}(\eta(\lambda)) = \alpha_{\mathscr{D}_2}(\psi_{R'} \circ \lambda).$$

Therefore the middle square and hence the whole diagram commutes. But composition across the top is $(\psi_{R'})_\rho^\sigma$ and across the bottom is $(\psi_{R'})_\rho^\tau$, so commutativity of this diagram means that $(\psi_{R'})_\rho$ is a morphism of displays for every (V, ρ), i.e. that ψ is a morphism of Tannakian (G, μ)-displays which induces η. We conclude T_R is full.

It remains to show T_R is essentially surjective. Let $\underline{Q} = (Q, \alpha)$ be a G-display of type μ over R. By Corollary 3.2.8, there is some graded fiber functor $\mathscr{F}$ of type μ such that $Q \cong \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \mathscr{F})$. Write $\mathscr{F}(V, \rho) = M(\rho)$. By [20, Lemma 046N] it is enough to show the base change $\underline{Q}_{R'}$ is in the essential image of $T_{R'}$ for some faithfully flat extension $R \to R'$.

Suppose $R \to R'$ is a faithfully flat extension such that $\mathrm{Isom}^{\otimes}(\mathscr{C}_{\mu,R'}, \mathscr{F}_{R'})$ is nonempty. Let $\lambda : \mathscr{C}_{\mu,R'} \to \mathscr{F}_{R'}$ be an isomorphism of graded fiber functors of type μ. Then $\alpha(\lambda) \in L^+G(R')$, so $\rho(\alpha(\lambda))$ is an automorphism of $V \otimes_{\mathbb{Z}_p} W(R')$ for every

(V, ρ). Define $F(\rho)'$ to be the σ-linear homomorphism $M(\rho)' \to (M(\rho)')^\tau$ such that

$$(F(\rho)')^\sharp := \lambda_\rho^\tau \circ \rho(\alpha(\lambda)) \circ (\lambda_\rho^\sigma)^{-1}.$$

Then $\underline{M(\rho)'} = (M(\rho)', F(\rho)')$ is a display over $W(R')^\oplus$. We claim the association $\mathscr{D}'_{\underline{Q}} : (V, \rho) \mapsto \underline{M(\rho)'}$ is a Tannakian (G, μ)-display over R'.

First let us show $\mathscr{D}'_{\underline{Q}}$ is functorial. Suppose $\varphi : (V, \rho) \to (U, \pi)$ is a morphism in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$. Then $\mathscr{F}(\varphi)$ is a homomorphism of graded $W(R')^\oplus$-modules $M(\rho) \to M(\pi)$, and we need to show that $\mathscr{F}(\varphi)$ is compatible with $F(\rho)'$ and $F(\pi)'$. Consider the following diagram:

$$\begin{array}{ccccccc}
(M(\rho)')^\sigma & \xrightarrow{(\lambda_\rho^\sigma)^{-1}} & V \otimes_{\mathbb{Z}_p} W(R') & \xrightarrow{\varphi_{W(R')}} & U \otimes_{\mathbb{Z}_p} W(R') & \xrightarrow{\lambda_\pi^\sigma} & (M(\pi)')^\sigma \\
\Big\downarrow{\scriptstyle (F(\rho)')^\sharp} & & \Big\downarrow{\scriptstyle \rho(\alpha(\lambda))} & & \Big\downarrow{\scriptstyle \pi(\alpha(\lambda))} & & \Big\downarrow{\scriptstyle (F(\pi)')^\sharp} \\
(M(\rho)')^\tau & \xrightarrow{(\lambda_\rho^\tau)^{-1}} & V \otimes_{\mathbb{Z}_p} W(R') & \xrightarrow{\varphi_{W(R')}} & U \otimes_{\mathbb{Z}_p} W(R) & \xrightarrow{\lambda_\pi^\tau} & (M(\pi)')^\tau
\end{array}$$

Again, the outside squares commute by definition of $F(\rho)'$ and $F(\pi)'$. The middle square commutes because φ is a morphism in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$. Since λ is a natural transformation, composition across the top is $\mathscr{F}(\varphi)^\sigma$, and composition across the bottom is $\mathscr{F}(\varphi)^\tau$. Hence $\mathscr{D}'_{\underline{Q}}$ is a functor. A completely analogous argument proves that it is compatible with the tensor product, so $\mathscr{D}'_{\underline{Q}}$ is a Tannakian (G, μ)-display over R'.

Now consider $\underline{Q}_{\mathscr{D}'_{\underline{Q}}}$, the G-display of type μ associated to $\mathscr{D}_{\underline{Q}}$. By definition of $Q_{\mathscr{D}'_{\underline{Q}}}$ and construction of $\mathscr{D}'_{\underline{Q}}$, we have

$$Q_{\mathscr{D}'_{\underline{Q}}} = \underline{\mathrm{Isom}}^\otimes(\mathscr{C}_{\mu,R'}, \upsilon_{R'} \circ \mathscr{D}'_{\underline{Q}}) = \underline{\mathrm{Isom}}^\otimes(\mathscr{C}_{\mu,R'}, \mathscr{F}_{R'}) \cong Q_{R'}.$$

By Tannakian duality, under this identification we have $\alpha_{R'} = \alpha_{\mathscr{D}'_{\underline{Q}}}$. Hence $\underline{Q}_{R'} \cong T_{R'}(\mathscr{D}'_{\underline{Q}})$, and T is essentially surjective. $\square$

Combining the theorem with Lemma 3.1.3 we obtain the following corollary:

Corollary 3.3.6. *If G is a reductive group scheme over $\mathbb{Z}_p$ and μ is a minuscule cocharacter defined over $W(k_0)$ then the stack of (G, μ)-displays (as in [3]) is isomorphic to the stack of Tannakian (G, μ)-displays.*

In [3], a (G, μ)-display is called banal if the underlying torsor is trivial. We close this section by defining the analogous notion for Tannakian (G, μ)-displays, and by giving a local description of the stack of Tannakian (G, μ)-displays which is formally very similar to that of [3, 3.2.7]. Fix a p-nilpotent $W(k_0)$-algebra R.

Definition 3.3.7. A Tannakian (G, μ)-display $\mathscr{D}$ over R is *banal* if there is an isomorphism $\upsilon_R \circ \mathscr{D} \cong \mathscr{C}_{\mu,R}$.

Banal Tannakian (G, μ)-displays over R form a full subcategory of Tannakian (G, μ)-displays over R, and by definition any Tannakian (G, μ)-display is fpqc-locally banal.

Construction 3.3.8. To any $U \in L^+G(R)$ we can associate a banal (G, μ)-display $\mathscr{D}_U$ as follows. Let (V, ρ) be a representation of G on a finite free $\mathbb{Z}_p$-module. To (V, ρ) we associate the following standard datum:

- $L = V \otimes_{\mathbb{Z}_p} W(R)$ viewed as a graded $W(R)$-module with $L_i = V^i_{W(k_0)} \otimes_{W(k_0)} W(R)$, where $V^i_{W(k_0)}$ is the decomposition (3.2.1) of $V \otimes_{\mathbb{Z}_p} W(k_0)$ induced by μ;

- $\Phi_U : L \to L$ is the composition $\rho(U) \circ (\mathrm{id}_V \otimes f)$.

Write $F_U : L \otimes_{W(R)} W(R)^{\oplus} \to L \otimes_{W(R)} W(R)^{\oplus}$ for the resulting σ-linear map, defined explicitly as

$$F_U(x \otimes \xi \otimes s) = \sigma(s)\rho(U)(x \otimes f(\xi))$$

for $x \in V$, $\xi \in W(R)$ and $s \in W(R)^{\oplus}$. Then $\mathscr{D}_U$ is the Tannakian (G, μ)-display

$$\mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{Disp}^W(R),\ (V, \rho) \mapsto (L \otimes_{W(R)} W(R)^{\oplus}, F_U).$$

By construction it is clear that $\upsilon_R \circ \mathscr{D}_U = \mathscr{C}_{\mu,R}$, so $\mathscr{D}_U$ is indeed banal.

In fact, the following lemma shows any banal Tannakian (G, μ)-display is isomorphic to $\mathscr{D}_U$ for some $U \in L^+G(R)$.

Lemma 3.3.9. *Let $\mathscr{D}$ be a banal Tannakian (G, μ)-display over R, and let and let $\lambda : \mathscr{C}_{\mu,R} \xrightarrow{\sim} \upsilon_R \circ \mathscr{D}$ be an isomorphism of graded fiber functors. Define $U = \alpha_{\mathscr{D}}(\lambda)$ as in Construction 3.3.4. Then $\mathscr{D} \cong \mathscr{D}_U$.*

Proof. It is clear that λ provides an isomorphism between the underlying graded fiber functors. Let (V, ρ) be a representation, and write $\mathscr{D}(V, \rho) = (M(\rho), F(\rho))$. In order to see that λ is compatible with the display structures, we need to check $\lambda_\rho^\tau \circ \rho(U) = F(\rho)^\sharp \circ \lambda_\rho^\sigma$. But $U = \alpha_{\mathscr{D}}(\lambda)$, so this follows from Construction 3.3.4. $\square$

We can now give an explicit description of the category of banal Tannakian (G, μ)-displays. This corresponds to the description of banal (G, μ)-displays given in

[3, 3.2.7] when G is reductive and μ is minuscule. Define a category $[L^+G/L^+_\mu G]^{\mathrm{pre}}(R)$ as follows:

- The objects in $[L^+G/L^+_\mu G]^{\mathrm{pre}}(R)$ are elements $U \in L^+G(R)$;
- given U, U' in $L^+G(R)$, the set of morphisms U to U' is given by

$$\mathrm{Hom}(U, U') = \{h \in L^+G_\mu(R) \mid \tau(h)^{-1}U'\sigma(h) = U\}.$$

Proposition 3.3.10. *The category of banal Tannakian (G, μ)-displays over R is equivalent to the category $[L^+G/L^+_\mu G]^{pre}(R)$.*

Proof. We claim the assignment $U \mapsto \mathscr{D}_U$ determines a functor from $[L^+G/L^+_\mu G]^{\mathrm{pre}}(R)$ to the category of banal Tannakian (G, μ)-displays over R. Let U, U' in $L^+G(R)$ and $h \in \mathrm{Hom}(U, U')$. Applying the homomorphism Ψ (cf. (3.2.2)) to h we obtain a morphism $\Psi(h)$ of the underlying graded fiber functors of $\mathscr{D}_U$ and $\mathscr{D}_{U'}$. These are both equal to $\mathscr{C}_{\mu,R}$, so $\Psi(h) \in \mathrm{Aut}^{\otimes}(\mathscr{C}_{\mu,R}) = L^+_\mu G(R)$. The condition $\tau(h)^{-1}U'\sigma(h) = U$ exactly corresponds to the condition that $\Psi(h)$ determines a morphism of Tannakian (G, μ)-displays $\mathscr{D}_U \to \mathscr{D}_{U'}$, so the above functor is well-defined. That the functor is fully faithful is an immediate consequence of Theorem 3.2.7, and that it is essentially surjective follows from Lemma 3.3.9. □

3.4 G-quasi-isogenies

Suppose R is a p-adic $\mathbb{Z}_p$-algebra. Then $W(R)$ is endowed with a natural structure of a $\mathbb{Z}_p$-algebra via

$$\mathbb{Z}_p \xrightarrow{\Delta} W(\mathbb{Z}_p) \to W(R).$$

The Frobenius and Verschiebung for $W(R)$ extend in a natural way to $W(R)[1/p]$.

Definition 3.4.1. An *isodisplay* over R is a pair $\underline{N} = (N, \varphi)$ where N is a finitely generated projective $W(R)[1/p]$-module and $\varphi : N \to N$ is an f-linear isomorphism.

The category of isodisplays over R has a natural structure of an exact tensor category, with tensor product defined by $(N_1, \varphi_1) \otimes (N_2, \varphi_2) := (N_1 \otimes N_2, \varphi_1 \otimes \varphi_2)$, and with exactness inherited from the analogous category defined by omitting the finitely generated projective condition for the $W(R)[1/p]$-modules N.

Let $\underline{M} = (M, F)$ be a display over $\underline{W}(R)$, and suppose the depth of M is d (cf. Definition 2.2.3). Then we can associate to $\underline{M}$ an isodisplay as follows. Because $d(M) = d$, we have an isomorphism of $W(R)$-modules $M_d \xrightarrow{\theta_d} M^\tau$, cf. Lemma 2.2.6. Define φ as follows: first consider the composition

$$\varphi' : M^\tau \xrightarrow{\theta_d^{-1}} M_d \xrightarrow{F_d} M^\tau,$$

where F_d is the restriction of F to M_d. This is an f-linear endomorphism of M^τ. We claim it induces an f-linear automorphism of $M^\tau[1/p]$. Indeed, we can choose a standard datum (L,Φ), with $L=\bigoplus_{i=d}^{a} L_i$, so $M = L\otimes_{W(R)} W(R)^{\oplus}$ and $F(x\otimes s) = \sigma(s)\Phi(x)$ for $x\in L$, $s\in W(R)^{\oplus}$. Then

$$M_d = \bigoplus_{n=0}^{a-d} \left(L_{d+n} \otimes_{W(R)} W(R)^{\oplus}_{-n}\right),$$

and F_d becomes the composition

$$\bigoplus_{n=0}^{a-d} L_{d+n}\otimes_{W(R)} W(R)^{\oplus}_{-n} \xrightarrow{\bigoplus \mathrm{id}\otimes p^n} \bigoplus_{n=0}^{a-d} L_{d+n}\otimes_{W(R)} W(R)^{\oplus}_{-n} \xrightarrow{\bigoplus \mathrm{id}\otimes\tau_{-n}} \bigoplus_{n=0}^{a-d} L_{d+n} \xrightarrow{\Phi} \bigoplus_{n=0}^{a-d} L_{d+n}$$

The last map is an f-linear bijection by assumption, and the second map is a bijection by definition of τ. The first becomes a bijection after we invert p, so this proves the claim.

Now define $\varphi := p^d\varphi'[1/p]$. By this procedure we obtain an isodisplay $\underline{M}[1/p] = (M^\tau[1/p],\varphi)$. This construction is evidently functorial, so if we denote the category of isodisplays over R by $\mathbf{Isodisp}(R)$, we obtain an exact tensor functor

$$\mathbf{Disp}^W(R)\to \mathbf{Isodisp}(R). \tag{3.4.1}$$

This procedure generalizes the one given for assigning an isodisplay to a 1-display in [1, Example 63].

Remark 3.4.2. If $\underline{S}$ is a frame over R, we can define an analogous category of isodisplays over $\underline{S}$. If every finite projective graded S-module M admits a normal decomposition, then Lemma 2.2.6 gives an isomorphism of S_0-modules $M_d \xrightarrow{\sim} M^\tau$, where $d = d(M)$, and we can mimic the construction above to define a functor analogous to (3.4.1). This holds in particular when S_0 is p-adic by [5, Lemma 3.1.4]. However, without the guaranteed existence of normal decompositions, it is unclear to the author whether θ_d is an isomorphism in general.

Definition 3.4.3. Let $\underline{M} = (M, F)$ and $\underline{M}' = (M', F')$ be displays over $\underline{W}(R)$.

(i) A *quasi-isogeny* $\gamma : \underline{M} \dashrightarrow \underline{M'}$ is an isomorphism of isodisplays $\underline{M}[1/p] \xrightarrow{\sim} \underline{M'}[1/p]$.

(ii) A quasi-isogeny is an *isogeny* if it is induced by a morphism of displays.

We say $\underline{M}$ is *isogenous* to $\underline{M'}$ if there exists an isogeny $\underline{M} \to \underline{M'}$.

Now suppose G is a flat affine group scheme over $\mathbb{Z}_p$, and that μ is a cocharacter for G defined over $W(k_0)$. Let R be a $W(k_0)$-algebra.

Definition 3.4.4. A G*-isodisplay* over R is an exact tensor functor $\mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{Isodisp}(R)$.

If $\mathscr{D}$ is a Tannakian (G, μ)-display, then we obtain a G-isodisplay by composition with the natural functor (3.4.1). Denote the resulting G-isodisplay by $\mathscr{D}[1/p]$.

Definition 3.4.5. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be Tannakian (G, μ)-displays. A G*-quasi-isogeny* $\mathscr{D}_1 \dashrightarrow \mathscr{D}_2$ is an isomorphism of G-isodisplays $\mathscr{D}_1[1/p] \xrightarrow{\sim} \mathscr{D}_2[1/p]$.

Suppose $\mathscr{D} \cong \mathscr{D}_U$ is a banal Tannakian (G, μ)-display over R, given by $U \in L^+G(R)$ as in the previous section. Then we can explicitly compute the resulting

G-isodisplay. In this case, if (V,ρ) is a representation of G, we have $\mathscr{D}[1/p](V,\rho) = (N(\rho),\varphi(\rho))$, where

$$N(\rho) = V \otimes_{\mathbb{Z}_p} W(R)[1/p].$$

Lemma 3.4.6. *For every (V,ρ) in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, the Frobenius on $\mathscr{D}[1/p](V,\rho)$ is given by*

$$\varphi = \rho\left(U\mu^\sigma(p)\right) \circ (\mathrm{id}_V \otimes f).$$

Proof. Fix (V,ρ) in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, and let $\mathscr{D}(V,\rho) = (M(\rho),F(\rho))$, where $M(\rho) = V \otimes_{\mathbb{Z}_p} W(R)^\oplus$. Because $\mathscr{D}$ is banal and defined from $U \in L^+G(R)$, F is defined from the f-linear automorphism

$$\Phi_U = \rho(U) \circ (\mathrm{id}_V \otimes f)$$

of $V \otimes_{\mathbb{Z}_p} W(R)$. Suppose the weights of $\rho \circ \mu$ are $\{w_1,\dots,w_r\}$ with $w_1 \le w_2 \le \cdots \le w_r$. Then $d(M(\rho)) = w_1$.

Let $L_i = V^i_{W(k_0)} \otimes_{W(k_0)} W(R)$. Then we have

$$M(\rho)^\tau \cong V \otimes_{\mathbb{Z}_p} W(R) = \bigoplus_{i=1}^r L_{w_i}.$$

The f-linear automorphism φ of $N(\rho) = \left(M(\rho)^\tau\right)[1/p]$ is constructed in two steps.

First we consider the composition

$$M(\rho)^\tau \xrightarrow{\theta_{w_1}^{-1}} M(\rho)_{w_1} \xrightarrow{F_{w_1}} M(\rho)^\tau.$$

In our case, if $x \in L_{w_i}$, then

$$\theta_{w_1}^{-1}(x) = x \otimes t^{w_i - w_1} \in L_{w_i} \otimes_{W(R)} W(R)^{\oplus}_{w_1 - w_i},$$

where $t \in W(R)^{\oplus}_1$ is the indeterminate from Definition 2.1.2. Applying F_{w_1}, we obtain

$$F_{w_1}(x \otimes t^{w_i - w_1}) = p^{w_i - w_1}\rho(U)(\mathrm{id}_V \otimes f)(x).$$

Multiplying by $p^{d(M)}$, we have

$$\varphi(x) = p^{w_i}\rho(U)(\mathrm{id}_V \otimes f)(x).$$

for $x \in V^{w_i}_{W(k_0)} \otimes_{W(k_0)} W(R)[1/p]$. This is evidently the same as $\rho(U)(\mathrm{id}_V \otimes f)\rho(\mu(p))(x)$, so the result follows from the identity

$$(\mathrm{id}_V \otimes f) \circ \rho(\mu(p)) = \rho(\mu^\sigma(p)) \circ (\mathrm{id}_V \otimes f).$$

□

Chapter 4: RZ spaces

4.1 Local Shimura data and the RZ functor

We recall the formalism for local Shimura data developed in [21], and we give a purely group theoretic definition of an RZ functor, following [3]. Our definition relies on the framework developed in the previous section, and as such, allows for a more general development than that in [3]. In particular, we do not need to assume G is reductive or μ minuscule in order to formulate the definitions in this section.

Let k be an algebraic closure of $\mathbb{F}_p$, and let $W(k)$ be the ring of Witt vectors over k. Write $K = W(k)[1/p]$, and let $\bar{K}$ be an algebraic closure of K. In this section we write σ for the automorphism of K coming from a lift of the absolute Frobenius $x \mapsto x^p$ of k. Let G be a smooth affine group scheme over $\mathbb{Z}_p$ whose generic fiber $G_{\mathbb{Q}_p}$ is reductive. Consider pairs $(\{\mu\}, [b])$ such that

- $\{\mu\}$ is a $G(\bar{K})$-conjugacy class of cocharacters $\mathbb{G}_{m\bar{K}} \to G_{\bar{K}}$;
- $[b]$ is a σ-conjugacy class elements $b \in G(K)$.

Let $E \subseteq \bar{K}$ be the field of definition of the conjugacy class $\{\mu\}$. Denote by $\mathcal{O}_E$ its valuation ring and k_E its residue field. We make the following assumption:

Assumption 4.1.1. The field $E \subseteq \bar{K}$ is contained in K, and there exists a cocharacter $\mu : \mathbb{G}_{mE} \to G_E$ in the conjugacy class $\{\mu\}$ which is defined over E and which extends to an integral cocharacter

$$\mu : \mathbb{G}_{m\mathcal{O}_E} \to G_{\mathcal{O}_E}.$$

When the assumption is satisfied we may identify $\mathcal{O}_E \cong W(k_E)$ and $E \cong W(k_E)[1/p]$.

Definition 4.1.2. A *local integral Shimura datum* is a triple $(G, \{\mu\}, [b])$ as above such that

(i) $\{\mu\}$ is minuscule and satisfies Assumption 4.1.1, and

(ii) for any integral representative μ of $\{\mu\}$ as in Assumption 4.1.1, the σ-conjugacy class $[b]$ has a representative

$$b \in G(W(k))\mu^\sigma(p)G(W(k)).$$

Definition 4.1.3. Let $(G, \{\mu\}, [b])$ be a local integral Shimura datum. A *framing pair* for $(G, \{\mu\}, [b])$ is a pair (μ, b) where

- $\mu : \mathbb{G}_{m,W(k_E)} \to G_{W(k_E)}$ is a representative of the conjugacy class $\{\mu\}$ as in Assumption 4.1.1,
- b is a representative of the σ-conjugacy class $[b]$ such that, for some $u \in$

$L^+G(k)$,

$$b = u\mu^\sigma(p). \tag{4.1.1}$$

It follows from Definition 4.1.2 that a framing pair always exists for a local integral Shimura datum $(G, \{\mu\}, [b])$. If (μ, b) is a framing pair, then the element $u \in L^+G(k)$ such that $b = u\mu^\sigma(p)$ is uniquely determined.

Definition 4.1.4. Let (μ, b) be a framing pair for $(G, \{\mu\}, [b])$, and let $u \in L^+G(k)$ be the unique element such that $b = u\mu^\sigma(p)$. The *framing object* associated to (μ, b) is the banal Tannakian (G, μ)-display $\mathscr{D}_u$ associated to u by Construction 3.3.8.

Definition 4.1.5. Fix a framing pair (μ, b) for $(G, \{\mu\}, [b])$, and let $\mathscr{D}_0$ be the associated framing object. The *RZ-functor* associated to the triple (G, μ, b) is the functor on $\mathbf{Nilp}_{W(k)}$ which assigns to a p-nilpotent $W(k)$-algebra R the set of isomorphism classes of pairs $(\mathscr{D}, \iota)$, where

- $\mathscr{D}$ is a Tannakian (G, μ)-display over R,
- $\iota : \mathscr{D}_{R/pR} \dashrightarrow (\mathscr{D}_0)_{R/pR}$ is a G-quasi-isogeny.

Two pairs $(\mathscr{D}_1, \iota_1)$ and $(\mathscr{D}_2, \iota_2)$ are isomorphic if there is an isomorphism $\mathscr{D}_1 \xrightarrow{\sim} \mathscr{D}_2$ lifting $\iota_2^{-1} \circ \iota_1$. Denote the RZ functor associated to (G, μ, b) by $\mathrm{RZ}_{G,\mu,b}$.

Associated to $\mathrm{RZ}_{G,\mu,b}$ we have a category $\mathbf{RZ}_{G,\mu,b}$ fibered over $\mathbf{Nilp}_{W(k)}$, such that if R is a p-nilpotent $W(k)$-algebra, then $\mathbf{RZ}_{G,\mu,b}(R)$ is the groupoid of pairs $(\mathscr{D}, \iota)$ over R as in Definition 4.1.5. It follows from Lemma 3.3.2 that $\mathbf{RZ}_{G,\mu,b}$ is an fpqc stack in groupoids.

4.2 Realization as a quotient stack

In this section we will reinterpret $\mathbf{RZ}_{G,\mu,b}$ as a quotient stack. From this we obtain an equivalence between our RZ functor and the one defined in [3], in the case where both are defined.

First we recall the definition of the Witt loop scheme as in [3, Section 2.2]. Let R be a ring and let X be an affine scheme of finite type over $W(R)$. Then the functor on R-algebras

$$R' \mapsto X(W(R')[1/p])$$

is representable by an ind-scheme over R by [12, Proposition 32]. If X is a $W(k_0)$-scheme, we can apply this to the base change of X along the Cartier homomorphism $W(k_0) \xrightarrow{\Delta} W(W(k_0))$ to obtain an ind-scheme over $W(k_0)$. We will denote this ind-scheme by LX.

For such a scheme X, denote by ${}^{\sigma}X$ the base change of X via the automorphism σ of $W(k_0)$:

$${}^{\sigma}X = X \times_{\mathrm{Spec}\ W(k_0),\sigma} \mathrm{Spec}\ W(k_0).$$

There is a natural isomorphism ${}^{\sigma}(LX) \xrightarrow{\sim} L({}^{\sigma}X)$. If R is a $W(k_0)$-algebra, then the Witt vector frobenius f on $W(R)$ induces a map on R points $f : LX(R) \to$

${}^{\sigma}(LX)(R)$ as follows: if $x \in LX(R)$, then $f(x)$ is the composition

$$\mathrm{Spec}\ W(R)[1/p] \xrightarrow{f} \mathrm{Spec}\ W(R)[1/p] \xrightarrow{x} X.$$

This map is functorial in R and hence defines a morphism of ind-schemes $f : LX \to {}^{\sigma}LX$ over $W(k_0)$. If X is defined over $\mathbb{Z}_p$, there is a natural isomorphism ${}^{\sigma}LX \xrightarrow{\sim} LX$, so in this case f defines an endomorphism $f : LX \to LX$. If G is a group scheme over $W(k_0)$, then LG is a group ind-scheme over $W(k_0)$, and in this case f is a group ind-scheme homomorphism.

Let $(G, \{\mu\}, [b])$ be a local integral Shimura datum, and choose a framing pair (μ, b) for $(G, \{\mu\}, [b])$, so $b = u\mu^{\sigma}(p)$ for some $u \in L^+G(k)$. To b and μ we associate two morphisms:

$$c_b : LG \to LG,\ g \mapsto g^{-1} \cdot b \cdot f(g),$$

$$m_{\mu} : L^+G \to LG,\ U \mapsto U \cdot \mu^{\sigma}(p).$$

Using these morphisms we form the fiber product $L^+G \times_{m_\mu, c_b} LG$, defined by the following Cartesian diagram:

$$\begin{tikzcd} L^+G \times_{m_\mu, c_b} LG \arrow[r] \arrow[d] & LG \arrow[d, "c_b"] \\ L^+G \arrow[r, "m_\mu"] & LG \end{tikzcd}$$

Lemma 4.2.1. *Suppose $h \in L^+_{\mu}G(R)$, and $(U, g) \in (L^+G \times_{m_\mu, c_b} LG)(R)$. Then*

$$(U, g) \cdot h := (\tau(h)^{-1} \cdot U \cdot \sigma(h), g \cdot \tau(h)) \tag{4.2.1}$$

is an element of $(L^+G \times_{m_\mu, c_b} LG)(R)$.

Proof. We need to show

$$c_b(g\tau(h)) = m_\mu(\tau(h)^{-1}U\sigma(h)).$$

This reduces to showing

$$\sigma(h) = \mu^\sigma(p)f(\tau(h))\mu^\sigma(p)^{-1}$$

when viewed as an element of $LG(R)$. The proof of this fact is contained in the proof of [5, Lemma 5.2.1]. Let us give the argument for completeness. Consider $W(R)[1/p][t, t^{-1}]$ as a graded ring with $\deg t = -1$. Define τ' and σ' : $W(R)[1/p][t, t^{-1}] \to W(R)[1/p]$ by

$$\tau'(t^n\xi) = \xi, \text{ and } \sigma'(t^n\xi) = p^n f(\xi) \text{ for } \xi \in W(R)[1/p].$$

Then the triple $(W(R)[1/p][t, t^{-1}], \sigma', \tau')$ constitutes a pre-frame in the sense of Definition 2.1.1. The map of graded rings

$$\psi : W(R)^\oplus \to W(R)[1/p][t, t^{-1}]$$

sending $\xi \in W(R)^\oplus_n$ to $t^{-n}\tau(\xi)$ determines a homomorphism of pre-frames, which induces commutative diagrams

$$
\begin{tikzcd}
L^+_\mu G(R) \arrow[r, "\sigma"] \arrow[d, "\psi"] & L^+G(R) \arrow[d, "\psi_0"] \\
G(W(R)[1/p][t,t^{-1}])_\mu \arrow[r, "\sigma'"] & LG(R)
\end{tikzcd}
\qquad
\begin{tikzcd}
L^+_\mu G(R) \arrow[r, "\tau"] \arrow[d, "\psi"] & L^+G(R) \arrow[d, "\psi_0"] \\
G(W(R)[1/p][t,t^{-1}])_\mu \arrow[r, "\tau'"] & LG(R)
\end{tikzcd}
$$

Here $G(W(R)[1/p][t,t^{-1}])_\mu$ is defined as in Section 3.1, and ψ_0 is induced by the natural map $W(R) \to W(R)[1/p]$. From this we see it is enough to prove

$$\sigma'(h) = \mu^\sigma(p) f(\tau'(h)) \mu^\sigma(p)^{-1},$$

for $h \in G(W(R)[1/p][t,t^{-1}])_\mu$.

Now, τ' induces a bijection $G(W(R)[1/p][t,t^{-1}])_\mu \xrightarrow{\sim} LG(R)$. Indeed, if $G = \operatorname{Spec} A$ and $g \in G(W(R)[1/p]) = LG(R)$, then g factors uniquely as $\tau' \circ h$, where h is the graded homomorphism

$$h : A \to W(R)[1/p][t,t^{-1}],\ a \mapsto t^{-n} g(a)\ a \in A_n.$$

On the other hand, by definition, σ' factors as $f \circ \tau'_p$, where $\tau'_p : W(R)[1/p][t,t^{-1}] \to W(R)[1/p]$ is determined by $t \mapsto p$. In the same manner as τ' and σ', τ'_p induces a group homomorphism

$$\tau'_p : G(W(R)[1/p][t,t^{-1}]) \to LG(R)$$

for every R. Putting this together, it is enough to show

$$(\tau'_p \circ (\tau')^{-1})(g) = \mu(p) g \mu(p)^{-1}$$

for $g \in LG(R)$.

Let $a \in A_n$. Then a can be viewed as a function $G(W(R)[1/p]) \to W(R)[1/p]$ given by evaluating $g \in G(W(R)[1/p]) = \mathrm{Hom}(A, W(R)[1/p])$ at a, cf. Section 3.1. By the definitions of τ'_p and $(\tau')^{-1}$, we have

$$a\left((\tau'_p \circ (\tau')^{-1})(g)\right) = p^{-n}g(a).$$

Since $a \in A_n$, we see

$$p^{-n}g(a) = (p^{-1} \cdot a)(g),$$

where the dot represents the action of $\mathbb{G}_m(W(R)[1/p])$ on A. But by definition of this action, we have

$$(p^{-1} \cdot a)(g) = a\left(\mu(p)g\mu(p)^{-1}\right).$$

Hence $(\tau'_p \circ (\tau')^{-1})(g)$ and $\mu(p)g\mu(p)^{-1}$ yield the same result when they are evaluated on any $a \in A$, so they are equal. □

It follows from the lemma (4.2.1) determines an action of $L^+_\mu G$ on $L^+G \times_{m_\mu, c_b} LG$. Using this action we form the quotient stack

$$[(L^+G \times_{m_\mu, c_b} LG)/L^+_\mu G]$$

over $\mathbf{Nilp}_{W(k)}$. Explicitly, if R is a $W(k)$-algebra, an object in $[(L^+G \times_{m_\mu, c_b} LG)/L^+_\mu G]$

is a pair (Q, β) consisting of an $L^+_\mu G$-torsor Q and a morphism $\beta : Q \to L^+G \times_{m_\mu, c_b} LG$ which is equivariant with respect to the action (4.2.1).

Theorem 4.2.2. *Let (μ, b) be a framing pair for $(G, \{\mu\}, [b])$. Then there is an isomorphism of stacks*

$$\mathbf{RZ}_{G,\mu,b} \xrightarrow{\sim} [(L^+G \times_{m_\mu, c_b} LG)/L^+_\mu G].$$

Proof. Let us begin by defining a morphism between the two stacks. Fix a p-nilpotent $W(k)$-algebra R, and let $(\mathscr{D}, \iota) \in \mathbf{RZ}_{G,\mu,b}(R)$. Then we can associate to $\mathscr{D}$ an $L^+_\mu G$-torsor as in Construction 3.3.4:

$$Q_{\mathscr{D}} = \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \upsilon_R \circ \mathscr{D}).$$

We want to define an $L^+_\mu G$-equivariant morphism

$$Q_{\mathscr{D}} \to L^+G \times_{m_\mu, c_b} LG.$$

Let $\lambda \in Q_{\mathscr{D}}(R')$ for some $R \to R'$. We need to assign to λ a pair $(U, g) \in (L^+G \times_{m_\mu, c_b} LG)(R')$. For U we can take the element $\alpha_{\mathscr{D}}(\lambda)$ defined in Construction 3.3.4. It remains only to define g.

The quasi-isogeny ι corresponds to an isomorphism of G-isodisplays

$$\mathscr{D}_{R/pR}[1/p] \xrightarrow{\sim} (\mathscr{D}_0)_{R/pR}[1/p],$$

where $\mathscr{D}_0$ is the framing object associated to (μ, b). If (V, ρ) is a representation of G, write $M(\rho)$ and $M_0(\rho)$ for the finite graded $W(R)^{\oplus}$-modules underlying $\mathscr{D}(V, \rho)$ and $\mathscr{D}_0(V, \rho)$, respectively. We also write $\overline{M}(\rho)$ and $\overline{M}_0(\rho)$ for their base changes to $W(R/pR)^{\oplus}$. Then for all (V, ρ), the quasi-isogeny ι defines an isomorphism of $W(R/pR)[1/p]$-modules

$$\iota_\rho : (\overline{M}(\rho))^\tau[1/p] \xrightarrow{\sim} (\overline{M}_0(\rho))^\tau[1/p].$$

Now $M_0(\rho) = V \otimes_{\mathbb{Z}_p} W(R)^{\oplus}$, so

$$(\overline{M}_0(\rho))^\tau[1/p] = V \otimes_{\mathbb{Z}_p} W(R/pR)[1/p].$$

Since p is nilpotent in R, we have an identification $W(R)[1/p] \cong W(R/pR)[1/p]$ (cf. [3, pg. 29]). Then we can also identify

$$(\overline{M}_0(\rho))^\tau[1/p] \cong V \otimes_{\mathbb{Z}_p} W(R)[1/p].$$

On the other hand, λ induces an isomorphism

$$V \otimes_{\mathbb{Z}_p} W(R')[1/p] \xrightarrow{\lambda_\rho^\tau} (\overline{M}(\rho)_{W(R'/pR')^{\oplus}})^\tau[1/p].$$

Write $\{\iota'_\rho\}$ for the base change of $\{\iota_\rho\}$ to $W(R'/pR')$. Then $\{\iota'_\rho \circ \lambda_\rho^\tau\}_{(V,\rho)}$ becomes an element of $\mathrm{Aut}^{\otimes}(\omega_{W(R')[1/p]})$, which, by Tannakian duality, is isomorphic to $G(W(R')[1/p]) = LG(R')$. Hence we obtain $g_{(\mathscr{D},\iota)}(\lambda) \in LG(R')$ with

$\rho(g_{(\mathscr{D},\iota)}(\lambda)) = \iota'_\rho \circ \lambda^\tau_\rho$ for every representation (V, ρ).

Now we define

$$\beta_{(\mathscr{D},\iota)} : Q_{\mathscr{D}} \to L^+G \times_{m_\mu, c_b} LG,\ \lambda \mapsto (\alpha_{\mathscr{D}}(\lambda), g_{(\mathscr{D},\iota)}(\lambda)).$$

In order to see this is well-defined, we need to check

$$c_b(g_{(\mathscr{D},\iota)}(\lambda)) = m_\mu(\alpha_{\mathscr{D}}(\lambda)). \tag{4.2.2}$$

For every representation (V, ρ), write φ_ρ for the Frobenius of $\mathscr{D}_{R'/pR'}[1/p](V, \rho)$ and $(\varphi_0)_\rho$ for the Frobenius of $(\mathscr{D}_0)_{R'/pR'}[1/p](V, \rho)$. Then we have

$$(\varphi_0)_\rho \circ \iota'_\rho = \iota'_\rho \circ \varphi_\rho. \tag{4.2.3}$$

But if $\lambda \in Q_{\mathscr{D}}(R') = \mathrm{Isom}^{\otimes}(\mathscr{C}_{\mu,R'}, \upsilon_{R'} \circ \mathscr{D}_{R'})$, then $\mathscr{D}_{R'}$ is banal, and it is isomorphic to $\mathscr{D}_{\alpha_{\mathscr{D}_{R'}}(\lambda)}$ via λ by Lemma 3.3.9. By Lemma 3.4.6, we have

$$\varphi_\rho = \lambda^\tau_\rho \circ \rho(\alpha_{\mathscr{D}_{R'}}(\lambda)\mu^\sigma(p)) \circ (\mathrm{id}_V \otimes f) \circ (\lambda^\tau_\rho)^{-1},$$

and

$$(\varphi_0)_\rho = \rho(b) \circ (\mathrm{id}_V \otimes f).$$

Now $\iota_{2\rho} = \rho(g_{(\mathscr{D},\iota)}(\lambda)) \circ (\lambda_\rho^\tau)^{-1}$, so (4.2.3) becomes

$$\rho(bf(g_{(\mathscr{D},\iota)}(\lambda))) = \rho(g_{(\mathscr{D},\iota)}(\lambda)\alpha_{\mathscr{D}_{R'}}(\lambda)\mu^\sigma(p)).$$

Then (4.2.2) follows from Tannakian duality.

One checks using the definition of $\alpha_{\mathscr{D}}$ (cf. Construction 3.3.4) that $\beta_{(\mathscr{D},\iota)}$ is equivariant for the action (4.2.1), so $(Q_{\mathscr{D}}, \beta_{(\mathscr{D},\iota)})$ determines an element of $[(L^+G \times_{m_\mu, c_b} LG)/L_\mu^+G](R)$. Therefore we have defined the desired morphism of stacks

$$\Psi : \mathbf{RZ}_{G,\mu,b} \to [(L^+G \times_{m_\mu,c_b} LG)/L_\mu^+G].$$

That Ψ is faithful follows from Corollary 3.2.8. Let us prove it is full. Suppose $(\mathscr{D}_1, \iota_1)$ and $(\mathscr{D}_2, \iota_2)$ are Tannakian (G,μ)-displays over R, equipped with G-quasi-isogenies over R/pR, and that

$$\theta : (Q_{\mathscr{D}_1}, \beta_{(\mathscr{D}_1,\iota_1)}) \to (Q_{\mathscr{D}_2}, \beta_{(\mathscr{D}_2,\iota_2)})$$

is a morphism of the associated objects in $[(L^+G \times_{m_\mu,c_b} LG)/L_\mu^+G](R)$. This means, for any $\lambda \in Q_{\mathscr{D}_1}(R')$,

$$\beta_{(\mathscr{D}_2,\iota_2)}(\theta(\lambda)) = \beta_{(\mathscr{D}_1,\iota_1)}(\lambda). \tag{4.2.4}$$

By forgetting the LG factor, we obtain a morphism of G-displays of type μ, which we also denote by θ,

$$\theta : (Q_{\mathscr{D}_1}, \alpha_{\mathscr{D}_1}) \to (Q_{\mathscr{D}_2}, \alpha_{\mathscr{D}_2}).$$

By Theorem 3.3.5, there is an isomorphism of Tannakian (G, μ)-displays $\psi : \mathscr{D}_1 \to \mathscr{D}_2$ inducing θ. It remains to show that ψ lifts the quasi-isogeny $(\iota_2)^{-1} \circ \iota$, i.e. that $\psi_{R/pR} = (\iota_2)^{-1} \circ \iota$. By descent it is enough to show $\psi_{R'/pR'} = (\iota_2)^{-1}_{R'} \circ \iota_{R'}$ for some $R \to R'$ faithfully flat.

Let R' be a faithfully flat extension of R such that there is some $\lambda \in Q_{\mathscr{D}_1}(R')$. Denote by $\theta(\lambda)$ the corresponding element of $Q_{\mathscr{D}_2}(R')$. Because θ is induced by ψ, we have

$$\theta(\lambda)_\rho = (\psi_{R'})_\rho \circ \lambda_\rho \tag{4.2.5}$$

for every representation (V, ρ) of G. Write $M_1(\rho)$ and $M_2(\rho)$ for the finite graded $W(R')^{\oplus}$-modules underlying $(\mathscr{D}_1)_{R'}(V, \rho)$ and $(\mathscr{D}_2)_{R'}(V, \rho)$, respectively, and write $\overline{M}_1(\rho)$ and $\overline{M}_2(\rho)$ for their base changes to $W(R'/pR')^{\oplus}$. For any (V, ρ), consider the following diagram:

$$\begin{tikzcd}
 & (\overline{M}_1(\rho))^\tau[1/p] \arrow[dd, "(\psi_{R'/pR'})^\tau_\rho"] \arrow[dr, "(\iota_1)_\rho"] & \\
V \otimes_{\mathbb{Z}_p} W(R')[1/p] \arrow[ur, "\lambda^\tau_\rho"] \arrow[dr, "\theta(\lambda)^\tau_\rho"'] & & V \otimes_{\mathbb{Z}_p} W(R')[1/p] \\
 & (\overline{M}_2(\rho))^\tau[1/p] \arrow[ur, "(\iota_2)_\rho"'] &
\end{tikzcd}$$

Composition across the top of this diagram is $\rho(g_{(\mathscr{D}_1,\iota_1)}(\lambda))$, and composition across the bottom is $\rho(g_{(\mathscr{D}_2,\iota_2)}(\theta(\lambda)))$, so the outer square of the diagram commutes by

(4.2.4). The left-hand triangle commutes by (4.2.5), so $\upsilon_{R'/pR'}(\psi)_\rho = (\iota_2)_\rho^{-1} \circ \iota_\rho$ for every (V, ρ), as desired. Therefore ψ is a morphism in $\mathbf{RZ}_{G,\mu,b}(R)$ and Ψ is full.

Finally let us verify the essential surjectivity of Ψ. As usual, we can apply [20, Lemma 046N], so it is enough to show that every object (Q, β) of $[(L^+G \times_{m_\mu, c_b} LG)/L^+_\mu G](R)$ is fpqc-locally in the essential image of Ψ. Let (Q, β) be an object over R, and choose a faithfully flat extension $R \to R'$ such that $Q(R')$ is nonempty. Denote by α the composition

$$Q \xrightarrow{\beta} L^+G \times_{m_\mu, c_b} LG \to L^+G.$$

Then (Q, α) is a G-display of type μ over R, so by Theorem 3.3.5, there is a Tannakian (G, μ)-display $\mathscr{D}$ with $T_R(\mathscr{D}) \cong (Q, \alpha)$. Moreover, we can make $\mathscr{D}$ explicit after fpqc localization. Choose $\lambda \in Q(R')$, and let $U = \alpha(\lambda)$. Then the proof of Theorem 3.3.5 implies that $\mathscr{D}_{R'} \cong \mathscr{D}_U$.

Now denote by g the composition

$$Q \xrightarrow{\beta} L^+G \times_{m_\mu, c_b} LG \to LG.$$

Let λ be as above, let $\mathscr{D}_U(V, \rho) = (M(\rho), F(\rho))$, and as before let $\mathscr{D}_0(V, \rho) = (M_0(\rho), F_0(\rho))$. Then for every (V, ρ), $g(\lambda)$ defines a $W(R)[1/p]$-module isomorphism

$$(\overline{M}(\rho))^\tau[1/p] \xrightarrow{\lambda_\rho^\tau} V \otimes_{\mathbb{Z}_p} W(R)[1/p] \xrightarrow{\rho(g(\lambda))} V \otimes_{\mathbb{Z}_p} W(R)[1/p] \xrightarrow{=} (\overline{M}_0(\rho))^\tau[1/p],$$

which we will denote by ι_ρ. We claim the collection $\{\iota_\rho\}$ defines an isomorphism of G-isodisplays $\mathscr{D}_U[1/p] \xrightarrow{\sim} (\mathscr{D}_0)[1/p]$. We need only see that ι_ρ is compatible with the Frobenius on each of $\mathscr{D}_U[1/p](V,\rho)$ and $(\mathscr{D}_0)[1/p](V,\rho)$ for every (V,ρ). By Lemma 3.4.6, this is equivalent to the condition

$$\rho(bf(g(\lambda)))(\mathrm{id}_V \otimes f) = \rho(g(\lambda)U\mu^\sigma(p))(\mathrm{id}_V \otimes f)$$

for all (V,ρ). By Tannakian duality, this is in turn equivalent to $c_b(g(\lambda)) = m_\mu(U)$, which holds by the assumption that $\beta(\lambda) \in (L^+G \times_{m_\mu, c_b} LG)(R')$. Therefore $(\mathscr{D}_U, \{\iota_\rho\})$ is an element of $\mathbf{RZ}_{G,\mu,b}(R')$ which satisfies $\Psi_{R'}(\mathscr{D}_U, \{\iota_\rho\}) \cong (Q_{R'}, \beta_{R'})$. We conclude Ψ is essentially surjective. $\square$

Chapter 5: Representability in some cases

5.1 The representability conjecture

Suppose $(G, \{\mu\}, [b])$ is a local integral Shimura datum, and let (μ, b) be a framing pair. In this section, suppose additionally that G is reductive over $\mathbb{Z}_p$. Then the framing object $\mathscr{D}_0$ corresponds to a (G, μ)-display $\mathcal{D}_0$ over k in the sense of [3]. In *loc. cit.*, a functor is associated to the data (G, μ, b) which classifies isomorphism classes of deformations of $\mathcal{D}_0$ up to G-quasi-isogeny. Let us call this functor $\mathrm{RZ}^{\mathrm{BP}}_{G,\mu,b}$, and denote by $\mathbf{RZ}^{\mathrm{BP}}_{G,\mu,b}$ the corresponding fpqc stack in groupoids.

Proposition 5.1.1. *If G is a reductive group scheme over $\mathbb{Z}_p$, then the stacks $\mathbf{RZ}^{\mathrm{BP}}_{G,\mu,b}$ and $\mathbf{RZ}_{G,\mu,b}$ are isomorphic.*

Proof. Combine Theorem 4.2.2 with [3, §4.2.3] and [5, Remark 6.3.4]. □

Bültel and Pappas make the following representability conjecture, see [3, Conjecture 4.2.1].

Conjecture 5.1.2 (Bültel-Pappas)**.** Assume that G is reductive and that -1 is not a slope of $\mathrm{Ad}^G(b)$. Then the functor $\mathrm{RZ}^{\mathrm{BP}}_{G,\mu,b}$ is representable by a formal scheme which is formally smooth and formally locally of finite type over $W(k)$.

See [3, 3.2.16] for details on the slope condition. For $G = \mathrm{GL}_n$, and for b with no slopes equal to zero, Conjecture 5.1.2 follows essentially from the equivalence between Zink displays and formal p-divisible groups due to Zink and Lau and the representability of the classical Rapoport-Zink functor from [13]. This is explained in the next section. In [3], in the case where G is reductive and the data (G, μ, b) is of Hodge type (see Definition 5.5.1), Conjecture 5.1.2 is proven for the restriction of the functor $\mathrm{RZ}_{G,\mu,b}$ to Noetherian rings in $\mathbf{Nilp}_{W(k)}$. Further, Bültel and Pappas show [3, Remark 5.2.7] that the resulting formal scheme is isomorphic to the formal schemes constructed in [22] and [23]. In particular, this shows that, when R is Noetherian, $\mathrm{RZ}_{G,\mu,b}(R)$ agrees with the points in the classical Rapoport-Zink space defined in [13] in the unramified EL- and PEL-type cases. In Section 5.4 we will take this one step further and prove that $\mathrm{RZ}_{G,\mu,b}$ is naturally isomorphic to the classical Rapoport-Zink functor in the unramified EL-type case. This combined with the known representability of the classical functor will prove Conjecture 5.1.2 in that case.

5.2 Representability for GL_n

In this section we summarize the proof of Conjecture 5.1.2 in the case $G = \mathrm{GL}_n$ by comparing the moduli spaces of deformations of p-divisible groups up to quasi-isogeny from [13] to an analogous moduli space defined using Zink displays (equivalently, 1-displays). The results of this section are well known and alluded to in [3], but we found it prudent to work out some details for use in Section 5.4. Let us begin by recalling a few definitions from [13].

Definition 5.2.1. Let X and X' be p-divisible groups over a base scheme S.

- An *isogeny* of p-divisible groups $\varphi : X \to X'$ is an epimorphism of fppf sheaves such that the kernel of f is representable by a finite locally free group scheme over S.

- A *quasi-isogeny* $X \dashrightarrow X'$ is a global section ρ of the Zariski sheaf $\underline{\mathrm{Hom}}_S(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}$ such that, Zariski locally, there exists an integer n such that $[p^n]\rho$ is an isogeny.

We will be primarily interested in the case where S is affine. In this case we can simplify both definitions.

Lemma 5.2.2. *Suppose $S = \mathrm{Spec}\ R$ is an affine scheme, and let X and X' be p-divisible groups over S. Then*

(i) $\varphi : X \to X'$ is an isogeny if and only if there exists a morphism $\psi : X' \to X$ and a natural number m such that $\varphi \circ \psi = [p^m]_{X'}$ and $\psi \circ \varphi = [p^m]_X$, and

(ii) $\rho : X \dashrightarrow X'$ is a quasi-isogeny if and only if it is an invertible element of $\mathrm{Hom}_S(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. For (i), the "if" direction is [24, Lemma 3.6], and the "only if" direction is [25, Corollary 3.3.4].

For (ii), observe that any affine scheme is quasi-compact, so if ρ is a quasi-isogeny we can choose some n large enough that $[p^n]\rho$ is an isogeny globally on S, and hence ρ is an invertible element of $\mathrm{Hom}_S(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}$. Conversely, if ρ is an invertible element of $\mathrm{Hom}_S(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}$, then $[p^n]\rho$ is a morphism $X \to X'$ for some n which is an isogeny by the criterion in (i). $\square$

The following lemma says that our notion of isogeny of 1-displays agrees with this notion via the functor BT_R.

Lemma 5.2.3. *Let $\alpha : \underline{M} \to \underline{M'}$ be a morphism of nilpotent 1-displays over $\underline{W}(R)$, and let $\varphi : BT_R(\underline{M}) \to BT_R(\underline{M'})$ be the corresponding morphism of formal p-divisible groups over R. Then α is an isogeny of displays if and only if φ is an isogeny of p-divisible groups.*

Proof. First suppose φ is an isogeny of p-divisible groups. By Lemma 5.2.2 there is a natural number m and a morphism $\psi : BT_R(\underline{M'}) \to BT_R(\underline{M})$ such that $\varphi \circ \psi = [p^m]_{BT_R(\underline{M'})}$ and $\psi \circ \varphi = [p^m]_{BT_R(\underline{M})}$. Since BT_R is fully faithful, there exists some $\beta : \underline{M'} \to \underline{M}$ such that $BT_R(\beta) = \psi$, $\alpha \circ \beta = p^m$ on $\underline{M'}$ and $\beta \circ \alpha = p^m$ on $\underline{M}$. Then α induces an isomorphism after inverting p, so it is an isogeny of displays.

Conversely, suppose α is an isogeny. By [26, Corollary 5.12] and [26, Corollary 5.14] we reduce to the case where R is reduced and M^τ and $(M')^\tau$ are free modules over $W(R)$. In that case $W(R)$ is p-torsion free, so $M^\tau \to M^\tau[1/p]$ is injective. After inverting p, there is a $\beta : (M'^\tau)[1/p] \to M^\tau[1/p]$ which is an inverse to α. Then $p^m\beta$ defines a homomorphism of displays $\underline{M'} \to \underline{M}$ for some m. If we let $\psi = BT_R(p^m\beta)$, then ψ and and m satisfy $\varphi \circ \psi = [p^m]_{BT_R(\underline{M'})}$ and $\psi \circ \varphi = [p^m]_{BT_R(\underline{M})}$, so φ is an isogeny. $\square$

If $\underline{M}$ and $\underline{M}'$ are 1-displays, by Lemma 2.4.2 and [1, Proposition 66] we have

$$\mathrm{Hom}_{\mathrm{Disp}^{W}(R)}(\underline{M}, \underline{M}') \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}_{\mathrm{Isodisp}(R)}(\underline{M}[1/p], \underline{M}'[1/p]),$$

so a quasi-isogeny of 1-displays is an invertible element of $\mathrm{Hom}_{\mathrm{Disp}^{W}(R)}(\underline{M}, \underline{M}') \otimes_{\mathbb{Z}} \mathbb{Q}$. If $\underline{M}, \underline{M}'$ are nilpotent, then by 2.4.5,

$$\mathrm{Hom}_{\mathrm{Disp}^{W}(R)}(\underline{M}, \underline{M}') \cong \mathrm{Hom}(BT_R(\underline{M}), BT_R(\underline{M}')),$$

and in particular quasi-isogenies $\underline{M} \dashrightarrow \underline{M}'$ correspond bijectively to quasi-isogenies $BT_R(\underline{M}) \dashrightarrow BT_R(\underline{M}')$.

Now we observe that in the above discussion it is enough to take either $\underline{M}$ or $\underline{M}'$ nilpotent, that is, the nilpotence condition for 1-displays behaves well with respect to isogenies.

Lemma 5.2.4. *Suppose p is nilpotent in R. Let $\underline{M} = (M, F)$ and $\underline{M}' = (M', F')$ be 1-displays over $\underline{W}(R)$, and suppose $\underline{M}$ and $\underline{M}'$ are quasi-isogenous. Then $\underline{M}$ is nilpotent if and only if $\underline{M}'$ is nilpotent.*

Proof. By [1, Lemma 21] it is enough to check the nilpotence condition after base change along an extension $\alpha : R \to R'$ such that any element in $\ker \alpha$ is nilpotent. Then we can reduce to the case where R is a reduced ring and $pR = 0$. In that case R embeds into a product of fields, so applying [1, Lemma 21] again we can reduce to the case where R is a product of algebraically closed fields of characteristic p. This in turn reduces to the case where R is a single algebraically closed field k of characteristic p. But 1-displays over k are equivalent to Dieudonné modules over k, and the nilpotence condition for $\underline{M}'$ corresponds to topological nilpotence of the Verschiebung operator on the Dieudonné module associated to $\underline{M}'$ (cf. Remark 2.4.4), which in turn is controlled by the slopes of the associated isocrystal. But since $\underline{M}$ and $\underline{M}'$ are quasi-isogenous, their associated isocrystals are isomorphic. Hence $\underline{M}'$ is nilpotent if and only if $\underline{M}$ is. $\square$

Suppose k is an algebraically closed field of characteristic p, and fix a p-divisible group X_0 over k.

Definition 5.2.5. Define $\mathrm{RZ}(X_0)$ to be the set-valued functor on the category of $W(k)$-schemes in which p is locally nilpotent which associates to any such S the set of isomorphism classes of pairs (X, ρ), where

- X is a p-divisible group over S, and
- $\rho : X \times_S \overline{S} \dashrightarrow X_0 \times_{\mathrm{Spec}\, k} \overline{S}$ is a quasi-isogeny.

Here $\overline{S}$ is the closed subscheme of S defined by $p\mathcal{O}_S$, and we say two pairs (X_1, ρ_1), (X_2, ρ_2) are isomorphic if $\rho_2^{-1} \circ \rho_1$ lifts to an isomorphism $X_1 \to X_2$.

By [13, Theorem 2.16], the functor $\mathrm{RZ}(X_0)$ is representable by a formal scheme which is formally smooth and locally formally of finite type over $\mathrm{Spf}\, W(k)$. We can define an exactly analogous functor using displays.

Definition 5.2.6. Define $\mathrm{RZ}(\underline{M_0})$ to be the set-valued functor on $\mathbf{Nilp}_{W(k)}$ which associates to a ring R in $\mathbf{Nilp}_{W(k)}$ the set of isomorphism classes of pairs $(\underline{M}, \gamma)$, where

- $\underline{M}$ is a 1-display over $\underline{W}(R)$, and
- $\gamma : \underline{M}_{\underline{W}(R/pR)} \dashrightarrow \underline{M_0}_{\underline{W}(R/pR)}$ is a quasi-isogeny.

We say two pairs $(\underline{M_1}, \gamma_1)$ and $(\underline{M_2}, \gamma_2)$ are isomorphic if $\gamma_2^{-1} \circ \gamma_1$ lifts to an isomorphism $\underline{M_1} \to \underline{M_2}$.

Proposition 5.2.7. *Let $\underline{M}_0$ be a nilpotent 1-display, and let $X_0 = BT_k(\underline{M_0})$. Then the functor* $\mathrm{RZ}(\underline{M_0})$ *and the restriction of* $\mathrm{RZ}(X_0)$ *to* $\mathbf{Nilp}_{W(k)}$ *are naturally isomorphic. In particular,* $\mathrm{RZ}(\underline{M_0})$ *is representable by a formal scheme which is formally smooth and locally formally of finite over* $\mathrm{Spf}\, W(k)$*.*

Proof. After one checks that the nilpotence condition for displays is preserved by isogenies, and that the functor BT_R of Theorem 2.4.5 sends isogenies of nilpotent displays to isogenies of p-divisible groups, the theorem is an immediate corollary of Theorem 2.4.5. $\square$

Let $G = \mathrm{GL}_n$, let $1 \leq d \leq n$, and let $\mu_{d,n}$ be the cocharacter

$$\mu_{d,n} : \mathbb{G}_m \to \mathrm{GL}_n \ a \mapsto \mathrm{diag}(1^{(d)}, a^{(n-d)}).$$

Let $[b]$ be a σ-conjugacy class of elements in $\mathrm{GL}_n(K)$, and choose a representative b as in Section 4.1. This determines a $(\mathrm{GL}_n, \mu_{d,n})$-display $\mathscr{D}_0$. By Remark 3.1.2, the stack of $(\mathrm{GL}_n, \mu_{d,n})$-displays is isomorphic to the stack of displays of type $(0^{(d)}, 1^{(n-d)})$. Such a display has depth 0 and altitude 1, and is therefore a 1-display. Then the functor $\mathrm{RZ}_{\mathrm{GL}_n, \mu_{d,n}, b}$ is naturally isomorphic to the functor $\mathrm{RZ}(\underline{M_0})$, where $\underline{M_0}$ is the 1-display associated to $\mathscr{D}_0$. If the slopes of b are all different from 0, then the resulting 1-display is nilpotent, so the functor is representable and isomorphic to $\mathrm{RZ}(BT_k(\underline{M_0}))$ by Proposition 5.2.7. This proves the following corollary.

Corollary 5.2.8. *If the slopes of* b *are all different from 0, the functor* $\mathrm{RZ}_{GL_n,\mu_{d,n},b}$ *is naturally isomorphic to* $\mathrm{RZ}(BT_k(\underline{M_0}))$*. In particular, it is representable by a formal scheme which is formally smooth and formally locally of finite type over* $\mathrm{Spf}\ W(k)$*.*

5.3 The determinant condition

In the next two sections our goal is to give a generalization of Corollary 5.2.8 when the local Shimura datum $(G, \{\mu\}, [b])$ is of EL-type. We begin by recalling the definition of RZ data in the EL case, cf. [13, 1.38], [21, 4.1]. As in 4.1, let k be an algebraic closure of $\mathbb{F}_p$, and let $W(k)$ be the ring of Witt vectors over k. Write $K = W(k)[1/p]$, and let $\bar{K}$ be an algebraic closure of K.

Definition 5.3.1. An *unramified integral EL-datum* is a tuple

$$\mathbf{D} = (B, \mathcal{O}_B, \Lambda)$$

such that

- B is a semisimple $\mathbb{Q}_p$-algebra whose simple factors are matrix algebras over unramified extensions of $\mathbb{Q}_p$;
- $\mathcal{O}_B$ is a maximal order in B;
- Λ is a finite free $\mathbb{Z}_p$-module with an $\mathcal{O}_B$-action.

To an unramified integral EL-datum $\mathbf{D} = (B, \mathcal{O}_B, \Lambda)$ we associate a reductive algebraic group $G = G_{\mathbf{D}}$ over $\mathbb{Z}_p$ whose points in a $\mathbb{Z}_p$-algebra R are given by

$$G(R) = \mathrm{GL}_{\mathcal{O}_B}(\Lambda \otimes_{\mathbb{Z}_p} R).$$

Definition 5.3.2. An *unramified integral RZ-datum of EL-type* is a tuple

$$(\mathbf{D}, \{\mu\}, [b])$$

such that $\mathbf{D}$ is an EL-datum with associated group G and $(\{\mu\}, [b])$ is a pair as in §4.1 such that $\{\mu\}$ satisfies Assumption 4.1.1 and $(G, \{\mu\}, [b])$ is a local integral Shimura datum. We require the following condition:

- For any μ in $\{\mu\}$, the only weights occurring in the weight decomposition for $\Lambda \otimes_{\mathbb{Z}_p} W(k)$ are 0 and 1, i.e.

$$\Lambda \otimes_{\mathbb{Z}_p} W(k) = \Lambda^0 \oplus \Lambda^1.$$

For the remainder of this section let us fix an unramified integral RZ-datum of EL-type $(\mathbf{D}, \{\mu\}, [b])$. Before we can formulate the corresponding moduli problem of p-divisible groups, we must first recall the determinant condition following [13, 3.23]. Let $\mathbb{V}$ be the scheme over $\mathbb{Z}_p$ whose points in a $\mathbb{Z}_p$-algebra R are given by

$$\mathbb{V}(R) = \mathcal{O}_B \otimes_{\mathbb{Z}_p} R.$$

For any $W(k)$-algebra R define

$$\delta_{\mathbf{D}} : \mathbb{V}(R) \to \mathbb{A}^1_{W(k)}(R),\ a \mapsto \det\left(a \mid \Lambda^0 \otimes_{W(k)} R\right).$$

This determines a morphism of $W(k)$-schemes $\mathbb{V}_{W(k)} \to \mathbb{A}^1_{W(k)}$. Now, let R be a $W(k)$-algebra and L be a finite projective R-module endowed with an $\mathcal{O}_B$-action. Then we can define similarly, for any R-algebra R',

$$\delta_L : \mathbb{V}_R(R') \to \mathbb{A}^1_R(R'),\ a \mapsto \det\left(a \mid L \otimes_R R'\right),$$

which determines a morphism of R-schemes $\mathbb{V}_R \to \mathbb{A}^1_R$.

Definition 5.3.3. We say that L *satisfies the determinant condition with respect to* $\boldsymbol{D}$ if the morphisms of R-schemes $\delta_{\mathbf{D}} \otimes \mathrm{id}_{\mathrm{Spec}(R)}$ and δ_L are equal.

Let X be a p-divisible group over a p-nilpotent $W(k)$-algebra R which is equipped with an action of $\mathcal{O}_B$, i.e. a homomorphism $\mathcal{O}_B \to \mathrm{End}(X)$. Then the Lie algebra $\mathrm{Lie}(X)$ is endowed with the structure of an $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-module, so one can ask whether $\mathrm{Lie}(X)$ satisfies the determinant condition with respect to $\mathbf{D}$.

Let R be a p-nilpotent $W(k)$-algebra, and suppose X is a formal p-divisible group with an action by $\mathcal{O}_B$. Let $\underline{M} = (M, F)$ be the nilpotent 1-display with $\mathcal{O}_B$-action corresponding to X, so $X = BT_R(\underline{M})$. We would like to reinterpret the determinant condition as a condition on the projective graded $W(R)^{\oplus}$-module M.

Suppose the height of X is equal to $\mathrm{rk}_{\mathbb{Z}_p}\Lambda$, so by Theorem 2.4.5,

$$\mathrm{rk}_{W(R)}M^\tau = \mathrm{rk}_{\mathbb{Z}_p}\Lambda.$$

By the recollections from Section 2.4, we have an identification

$$M^\tau \otimes_{W(R)} R \cong \mathbb{D}(X)_R$$

which identifies the Hodge filtrations, i.e.

$$(M^\tau \otimes_{W(R)} R \supset E_1 \supset 0) \xrightarrow{\sim} (\mathbb{D}(X)_R \supset \mathrm{Lie}(X^\vee)^* \supset 0)$$

is an isomorphism of filtered $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$-modules. Here E_1 is the image of M_1 under the composition

$$M_1 \xrightarrow{\theta_1} M^\tau \to M^\tau \otimes_{W(R)} R.$$

In particular, we have an identification

$$\mathrm{Lie}(X) \cong M^\tau/\theta_1(M_1).$$

Viewing $\mathcal{O}_B$ as a graded $\mathbb{Z}_p$-algebra concentrated in degree zero, we can view $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R)^\oplus$ as a graded ring. Then $\Lambda \otimes_{\mathbb{Z}_p} W(R)^\oplus$ inherits the structure of a graded $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R)^\oplus$-module.

Lemma 5.3.4. *The following are equivalent:*

(i) *For some faithfully flat extension $R \to R'$ there is an isomorphism of graded $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$-modules*

$$M_{W(R')^{\oplus}} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$$

(ii) *For some faithfully flat extension $R \to R'$ there is an isomorphism of filtered $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R'$-modules*

$$(M^{\tau} \otimes_{W(R)} R' \supset E_1 \otimes_R R' \supset 0) \xrightarrow{\sim} (\Lambda \otimes_{\mathbb{Z}_p} R' \supset \Lambda^1 \otimes_{W(k)} R' \supset 0) \quad (5.3.1)$$

(iii) Lie(X) *satisfies the determinant condition with respect to* $\boldsymbol{D}$.

Proof. We start by proving (i) and (ii) are equivalent. Suppose (i) holds, so for some faithfully flat $R \to R'$ there is a $\varphi : M_{W(R')^{\oplus}} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$ which is compatible with the $\mathcal{O}_B$-action and the grading. Then in particular, M_1', the first graded piece of $M_{W(R')^{\oplus}}$, is carried by φ into the first graded piece of $\Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$. That is,

$$\varphi(M_1') \cong (\Lambda^0 \otimes_{W(k)} I_{R'}) \oplus (\Lambda^1 \otimes_{W(k)} W(R')).$$

By reducing modulo $I_{R'}(M_{W(R')^{\oplus}})^{\tau}$ we see that (ii) holds.

Now let us prove (ii) implies (i). Let

$$\bar{\varphi} : M^{\tau} \otimes_{W(R)} R' \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} R'$$

be an isomorphism preserving the filtration. First we lift $\bar{\varphi}$ to an $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R)$-module isomorphism

$$\varphi : (M_{W(R')^{\oplus}})^{\tau} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} W(R').$$

By restricting to simple factors and applying Morita equivalence we may assume $\mathcal{O}_B = \mathcal{O}_L$ is the ring of integers in an unramified extension L of degree n over $\mathbb{Q}_p$. In that case we have an isomorphism

$$\mathcal{O}_L \otimes_{\mathbb{Z}_p} W(k) \cong \prod_{j \in \mathbb{Z}/n\mathbb{Z}} W(k),$$

which gives decompositions

$$(M_{W(R')^{\oplus}})^{\tau} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M'(j), \quad \text{and} \quad \Lambda \otimes_{\mathbb{Z}_p} W(R') = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \Lambda'(j),$$

where $M'(j) = \{m \in (M_{W(R')^{\oplus}})^{\tau} \mid a \cdot m = \sigma^j(a)m, \ a \in \mathcal{O}_L\}$, and similarly for $\Lambda'(j)$. Since $\bar{\varphi}$ is an isomorphism of $\mathcal{O}_L \otimes_{\mathbb{Z}_p} R'$-modules, it induces isomorphisms

$$M'(j) \otimes_{W(R')^{\oplus}} R' \xrightarrow{\sim} \Lambda'(j) \otimes_{W(k)} R'$$

for every j. Each $M'(j)$ is projective as a $W(R')$-module, so these isomorphisms can be lifted to $W(R')$-module homomorphisms

$$M'(j) \to \Lambda'(j),$$

which are surjective by Nakayama's lemma. Summing these together we obtain a map

$$(M_{W(R')^{\oplus}})^{\tau} \to \Lambda \otimes_{\mathbb{Z}_p} W(R'),$$

which is an $\mathcal{O}_L \otimes_{\mathbb{Z}_p} W(R')$-module homomorphism since it preserves the decompositions above. At the same time, it is a surjective homomorphism between projective $W(R')$-modules of the same rank, so it must be an isomorphism.

If M' is a finite projective graded $W(R')^{\oplus}$-module of non-negative depth and altitude one, then the assignment $M' \mapsto ((M')^{\tau}, \theta_1'(M_1'))$ determines a functor F to the category $\mathbf{Pairs}(R)$ of pairs (P, Q), where P is a finite projective $W(R)$-module and $Q \subset P$ is a submodule. By the proof of Lemma 2.4.2 this functor is fully faithful. In the case at hand, because $\bar{\varphi}$ preserves the filtration (5.3.1), it follows that φ sends $\theta_1'(M_1')$ into $(\Lambda^0 \otimes_{W(k)} I_{R'}) \oplus (\Lambda^1 \otimes_{W(k)} W(R'))$, so φ determines an isomorphism

$$((M_{W(R')^{\oplus}})^{\tau}, \theta_1'(M_1')) \xrightarrow{\sim} (\Lambda \otimes_{\mathbb{Z}_p} W(R'), (\Lambda^0 \otimes_{W(k)} I_{R'}) \oplus (\Lambda^1 \otimes_{W(k)} W(R')))$$

in $\mathbf{Pairs}(R)$. Since the left-hand side is $F(M_{W(R')^{\oplus}})$ and the right-hand side is $F(\Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus})$, full faithfulness of F implies the existence of an isomorphism of graded $W(R')^{\oplus}$-modules

$$\tilde{\varphi} : M_{W(R')^{\oplus}} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$$

lifting φ. Since the $\mathcal{O}_B$-action on $(M_{W(R')^{\oplus}})^{\tau}$ is induced by the given action on $M_{W(R')^{\oplus}}$, we can once again apply full faithfulness of F to see $\tilde{\varphi}$ is compatible with the respective $\mathcal{O}_B$-actions.

Now let us prove the equivalence of (ii) and (iii). Suppose (ii) holds. The determinant condition can be checked fpqc-locally because morphisms of schemes can be glued locally for the fpqc topology. But by (ii) and the above identifications there is an isomorphism of $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R'$-modules

$$\mathrm{Lie}(X_{R'}) \xrightarrow{\sim} \Lambda^0 \otimes_{W(k)} R',$$

where $R \to R'$ is a faithfully flat extension. This implies the determinant condition holds over R', and therefore over R as well.

Conversely, suppose (iii) holds, i.e. $\mathrm{Lie}(X)$ satisfies the determinant condition with respect to $\mathbf{D}$. Again by restricting to simple factors and applying Morita equivalence we can assume $\mathcal{O}_B = \mathcal{O}_L$ is the ring of integers in an unramified extension L of degree n over $\mathbb{Q}_p$. As in the proof of (ii) implies (i), we have decompositions

$$\Lambda^0 = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \Lambda^0(j), \quad \text{and} \quad \mathrm{Lie}(X) = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} L(j).$$

In this case the determinant condition is equivalent to

$$\mathrm{rk}_{W(k)} \Lambda^0(j) = \mathrm{rk}_R L(j)$$

for every j, cf. [13, 3.23(b)]. Hence $L(j)$ and $\Lambda^0(j) \otimes_{W(k)} R$ are projective R-

modules of the same rank, so after some localization we can find an isomorphism $\alpha_j : L(j) \xrightarrow{\sim} \Lambda^0(j) \otimes_{W(k)} R$ for every j. Similarly we can decompose

$$\Lambda^1 = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \Lambda^1(j) \quad \text{and} \quad \mathrm{Lie}(X^\vee)^* = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} L'(j),$$

and, perhaps after another localization, we can find isomorphisms $\beta_j : L'(j) \xrightarrow{\sim} \Lambda^1(j) \otimes_{W(k)} R$ for every j. Now, write

$$\mathbb{D}(X)_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} D(j) \quad \text{and} \quad \Lambda = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \Lambda(j).$$

Since $L(j)$ and $\Lambda^0(j)$ are projective, the short exact sequences

$$0 \to L'(j) \to D(j) \to L(j) \to 0$$

and

$$0 \to \Lambda^1(j) \otimes_{W(k)} R \to \Lambda(j) \otimes_{W(k)} R \to \Lambda^0(j) \otimes_{W(k)} R \to 0$$

split over R. Hence we can piece together the isomorphisms α_j and β_j to obtain $\phi_j : D(j) \xrightarrow{\sim} \Lambda(j) \otimes_{W(k)} R$, preserving the filtration by direct summands. Combining these for all j gives an isomorphism

$$\varphi : \mathbb{D}(X)_R \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} R.$$

By construction, φ is an isomorphism of $\mathcal{O}_L \otimes_{\mathbb{Z}_p} R$-modules which identifies the Hodge filtrations of the two sides. By the remarks before the statement of the lemma we have a natural identification

$$(M^\tau \otimes_{W(R)} R \supset E_1 \supset 0) \xrightarrow{\sim} (\mathbb{D}(X)_R \supset \mathrm{Lie}(X^\vee)^* \supset 0).$$

Combining this with φ gives the desired isomorphism. $\square$

5.4 Representability in the EL-type case

In this section we prove that $\mathrm{RZ}_{G,\mu,b}$ is representable by a formal scheme when (μ, b) is a framing pair for a local integral Shimura datum $(G, \{\mu\}, [b])$ associated to an unramified integral RZ-datum of EL-type. Throughout this section we continue to assume k is an algebraic closure of $\mathbb{F}_p$, that $\mathbf{D} = (B, \mathcal{O}_B, \Lambda)$ is an unramified integral EL-datum, and that $(\mathbf{D}, \{\mu\}, [b])$ is an unramified integral RZ-datum of EL-type. Let X_0 be a p-divisible group over k equipped with an action by $\mathcal{O}_B$. Let us begin by recalling the definition of the unramified EL-type Rapoport-Zink space associated to X_0 and $\mathbf{D}$.

Definition 5.4.1. Define $\mathrm{RZ}_{\mathbf{D}}(X_0)$ to be the set-valued functor on the category of $W(k)$-schemes in which p is locally nilpotent which associates to any such S the set of isomorphism classes of pairs (X, ρ), where

- X is a p-divisible group over S equipped with an action of $\mathcal{O}_B$, such that $\mathrm{Lie}(X)$ satisfies the determinant condition with respect to $\mathbf{D}$, and

- $\rho : X \times_S \overline{S} \dashrightarrow X_0 \times_{\mathrm{Spec}(k)} \overline{S}$ is a quasi-isogeny which commutes with the action of $\mathcal{O}_B$.

Here again $\overline{S}$ denotes the closed subscheme of S defined by $p\mathcal{O}_S$, and two pairs (X_1, ρ_1) and (X_2, ρ_2) are isomorphic if $\rho_2^{-1} \circ \rho_1$ lifts to an isomorphism $X_1 \to X_2$ respecting the $\mathcal{O}_B$-actions.

By [13, Theorem 3.25], the inclusion $\mathrm{RZ}_{\mathbf{D}}(X_0) \hookrightarrow \mathrm{RZ}(X_0)$ is a closed immersion, so the functor $\mathrm{RZ}_{\mathbf{D}}(X_0)$ is representable by a formal scheme which is formally smooth and formally locally of finite type over Spf $W(k)$. If $\underline{M_0}$ is a 1-display over k equipped with an $\mathcal{O}_B$-action, we obtain an analogous subfunctor of $\mathrm{RZ}(\underline{M_0})$ by replacing all p-divisible groups that arise in Definition 5.4.1 with displays.

Definition 5.4.2. Let $\mathrm{RZ}_{\mathbf{D}}(\underline{M_0})$ be the set-valued functor on $\mathbf{Nilp}_{W(k)}$ which associates to a ring R in $\mathbf{Nilp}_{W(k)}$ the set of isomorphism classes of pairs $(\underline{M}, \gamma)$, where

- $\underline{M} = (M, F)$ is a 1-display over $\underline{W}(R)$ equipped with an action by $\mathcal{O}_B$ such that the underlying graded $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R)^{\oplus}$-module M is fpqc-locally isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} W(R)^{\oplus}$, and

- $\gamma : \underline{M}_{\underline{W}(R/pR)} \dashrightarrow \underline{M_0}_{\underline{W}(R/pR)}$ is a quasi-isogeny of displays which commutes with the $\mathcal{O}_B$-actions.

As in Definition 5.2.6, two pairs $(\underline{M_1}, \gamma_1)$ and $(\underline{M_2}, \gamma_2)$ are isomorphic if $\gamma_2^{-1} \circ \gamma_1$ lifts to an isomorphism $\underline{M_1} \to \underline{M_2}$.

Lemma 5.4.3. *Let $\underline{M}_0$ be a nilpotent 1-display endowed with an $\mathcal{O}_B$-action, and let $X_0 = BT_k(\underline{M_0})$. Then there is a natural isomorphism*

$$\mathrm{RZ}_{\boldsymbol{D}}(\underline{M_0}) \xrightarrow{\sim} \mathrm{RZ}_{\boldsymbol{D}}(X_0)$$

of functors on $\mathbf{Nilp}_{W(k)}$.

Proof. The natural isomorphism $\mathrm{RZ}(\underline{M_0}) \xrightarrow{\sim} \mathrm{RZ}(X_0)$ from Proposition 5.2.7 restricts to a natural isomorphism $\mathrm{RZ}_{\mathbf{D}}(\underline{M_0}) \xrightarrow{\sim} \mathrm{RZ}_{\mathbf{D}}(X_0)$ by Lemma 5.3.4. □

Now, suppose G is the group associated to $(\mathbf{D}, \{\mu\}, [b])$, and denote by η the natural closed embedding

$$\eta : G \hookrightarrow GL(\Lambda).$$

Let (μ, b) be a framing pair for $(G, \{\mu\}, [b])$, and denote by $\mathscr{D}_0$ the associated framing object, cf. Definition 4.1.4. Evaluating $\mathscr{D}_0$ on (Λ, η), we obtain a display over $\underline{W}(k)$, which we will denote by $\underline{M_0}$. The condition in Definition 5.3.2 implies $\mathscr{C}_{\mu,k}(\Lambda, \eta)$ is of type $(0^{(d)}, 1^{(n-d)})$, where $d = \mathrm{rk}_{W(k)}\Lambda^0$ and $n = \mathrm{rk}_{\mathbb{Z}_p}\Lambda$. Hence $\underline{M_0}$ is also of type $(0^{(d)}, 1^{(n-d)})$, and therefore $\underline{M_0}$ is a 1-display.

Now, notice that $\mathrm{RZ}_{\mathbf{D}}(\underline{M_0})$ is the functor of isomorphism classes associated to a category fibered in groupoids $\mathbf{RZ}_{\mathbf{D}}(\underline{M_0})$. By Lemma 2.3.9 (or [1, Theorem 37]), $\mathbf{RZ}_{\mathbf{D}}(\underline{M_0})$ is an fpqc stack. Evaluation on the embedding $\eta : G \hookrightarrow GL(\Lambda)$ induces a natural morphism of stacks

$$\mathbf{RZ}_{G,\mu,b} \to \mathbf{RZ}_{\mathrm{D}}(\underline{M_0}).$$

Let us explain. If R is a p-nilpotent $W(k)$-algebra and $(\mathscr{D}, \iota)$ is an object of $\mathbf{RZ}_{G,\mu,b}(R)$, then by evaluating $\mathscr{D}$ on (Λ, η) we obtain a display $\underline{M} = (M, F)$ over $\underline{W}(R)$. Since there is an isomorphism $\lambda : \upsilon_{R'}\mathscr{D}_{R'} \xrightarrow{\sim} \mathscr{C}_{\mu,R'}$ after some faithfully flat $R \to R'$, we have an isomorphism of graded $W(R')^{\oplus}$-modules $\lambda_\eta : M_{W(R')^{\oplus}} \cong \Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$. By Lemma 2.2.8, then, $\underline{M}$ is a 1-display. Because the endomorphism on Λ induced by any $a \in \mathcal{O}_B$ is G-equivariant, by functoriality we obtain an action of $\mathcal{O}_B$ on $\underline{M}$. Furthermore, by functoriality of λ, for each $a \in \mathcal{O}_B$, we have $\mathscr{C}_{\mu,R'}(a) \circ \lambda_\eta = \lambda_\eta \circ (\upsilon_{R'} \circ \mathscr{D}_{R'})(a)$, i.e. the isomorphism $M_{W(R')^{\oplus}} \cong \Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$ is compatible with the $\mathcal{O}_B$-action. Now the G-quasi-isogeny ι induces a quasi-isogeny

$$\gamma := \iota_\eta : \underline{M}_{\underline{W}(R/pR)} \dashrightarrow \underline{M_0}_{\underline{W}(R/pR)}$$

by evaluation on (Λ, η). Again, because the action by $\mathcal{O}_B$ is G-equivariant, we see that γ is compatible with the $\mathcal{O}_B$-actions. Hence $(\underline{M}, \gamma)$ is an element of $\mathbf{RZ}_{\mathbf{D}}(\underline{M_0})(R)$. This construction is compatible with base change, so we have defined a morphism of stacks

$$\mathbf{RZ}_{G,\mu,b} \to \mathbf{RZ}_{\mathbf{D}}(\underline{M_0}). \tag{5.4.1}$$

We will show that (5.4.1) is an isomorphism, but first we prove a useful lemma.

If $(\underline{M}, \gamma)$ is any object in $\mathbf{RZ_D}(\underline{M_0})(R)$, define

$$Q_M := \underline{\mathrm{Isom}}^0(\Lambda \otimes_{\mathbb{Z}_p} W(R)^{\oplus}, M) \tag{5.4.2}$$

as the functor on R-algebras taking R' to the group of graded $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$-isomorphisms between $M_{W(R')^{\oplus}}$ and $\Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$. Then Q_M is an $L_\mu^+ G$-torsor over R because M is locally isomorphic to $\Lambda \otimes_{\mathbb{Z}_p} W(R)^{\oplus}$, and the group of graded $\mathcal{O}_B \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$ automorphisms of $\Lambda \otimes_{\mathbb{Z}_p} W(R')^{\oplus}$ is isomorphic to $L_\mu^+ G(R')$. By mimicking Construction 3.3.4, we obtain a morphism

$$\alpha_F : Q_M \to L^+ G \tag{5.4.3}$$

such that the pair (Q_M, α_F) determines a G-display of type μ over R.

Lemma 5.4.4. *Let R be a p-nilpotent $W(k)$-algebra, let $(\mathscr{D}, \iota) \in RZ_{G,\mu,b}(R)$, and let $(\underline{M}, \gamma)$ be the image of $(\mathscr{D}, \iota)$ under (5.4.1). If $(Q_{\mathscr{D}}, \alpha_{\mathscr{D}})$ is the G-display of type μ associated to $\mathscr{D}$ by Construction 3.3.4, then evaluation on (Λ, η) induces a natural isomorphism of G-displays of type μ*

$$(Q_{\mathscr{D}}, \alpha_{\mathscr{D}}) \xrightarrow{\sim} (Q_M, \alpha_F).$$

Proof. By definition, $Q_{\mathscr{D}} = \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}_{\mu,R}, \upsilon_R \circ \mathscr{D})$, so evaluation on (Λ, η) defines an isomorphism of $L_\mu^+ G$-torsors $\delta : Q_{\mathscr{D}} \to Q_M$. We need to show that δ is an isomorphism of G-displays of type μ, i.e. that $\alpha_F \circ \delta = \alpha_{\mathscr{D}}$. Let $\lambda \in Q_{\mathscr{D}}(R')$

for some R-algebra R'. Then $\eta((\alpha_F \circ \delta)(\lambda)) = \eta(\alpha_F(\lambda_\eta)) \in L^+\mathrm{GL}(\Lambda)(R')$ is the composition $(\lambda_\eta^\tau)^{-1} \circ F^\sharp \circ \lambda_\eta^\sigma$. On the other hand, by definition of $\alpha_{\mathscr{D}}$, this is exactly $\alpha_{\mathscr{D}}(\lambda)_\eta$. But

$$\alpha_{\mathscr{D}}(\lambda)_\eta = \eta(\alpha_{\mathscr{D}}(\lambda)),$$

so we have $\eta(\alpha_{\mathscr{D}}(\lambda)) = \eta(\alpha_F(\lambda_\eta))$. Hence $\alpha_{\mathscr{D}}(\lambda) = \alpha_F(\delta(\lambda))$. $\square$

The following is the main theorem of this section.

Theorem 5.4.5. *The morphism (5.4.1) is an isomorphism of fpqc stacks.*

Proof. To see that it is faithful, let R be a p-nilpotent $W(k)$-algebra, and suppose ψ_1 and ψ_2 are two morphisms $(\mathscr{D}, \iota) \to (\mathscr{D}_2, \iota_2)$ in $\mathbf{RZ}_{G,\mu,b}$ which agree after applying (5.4.1). By descent, it is enough to check that ψ_1 and ψ_2 are equal fpqc-locally. But after some faithfully flat $R \to R'$, ψ_1 and ψ_2 correspond to elements $h_1, h_2 \in \mathrm{Aut}^{\otimes}(\mathscr{C}_{\mu,R'}) = L^+_\mu G(R')$, and to say that ψ_1 and ψ_2 agree after applying (5.4.1) means $\eta(h_1) = \eta(h_2)$. But η is a faithful representation, so this implies $h_1 = h_2$, hence $\psi_1 = \psi_2$.

To show the morphism is full, suppose $(\mathscr{D}_1, \iota_1)$, $(\mathscr{D}_2, \iota_2)$ are objects in $\mathbf{RZ}_{G,\mu,b}(R)$ corresponding to objects $(\underline{M_1}, \gamma_1)$, $(\underline{M_2}, \gamma_2)$ in $\mathbf{RZ}_{\mathbf{D}}(\underline{M_0})(R)$, and suppose $\varphi : (\underline{M_1}, \gamma_1) \xrightarrow{\sim} (\underline{M_2}, \gamma_2)$ is a morphism in $\mathbf{RZ}_{\mathbf{D}}(\underline{M_0})(R)$. By Lemma 5.4.4, we have isomorphisms of G-displays of type μ

$$\delta_1 : (Q_{\mathscr{D}_1}, \alpha_{\mathscr{D}_1}) \xrightarrow{\sim} (Q_{M_1}, \alpha_{F_1}) \text{ and } \delta_2 : (Q_{\mathscr{D}_2}, \alpha_{\mathscr{D}}) \xrightarrow{\sim} (Q_{M_2}, \alpha_{F_2}),$$

where the Q_{M_i} and α_{F_i} are defined as in (5.4.2) and (5.4.3), respectively. The map of underlying $W(R)^\oplus$-modules $\varphi : M_1 \to M_2$ induces a morphism $\psi : Q_{M_1} \to Q_{M_2}$. As in the proof of Lemma 5.4.4, one checks that $\alpha_{F_2} \circ \psi = \alpha_{F_1}$, so the composition

$$(Q_{\mathscr{D}_1}, \alpha_{\mathscr{D}_1}) \xrightarrow{\delta_1} (Q_{M_1}, \alpha_{F_1}) \xrightarrow{\psi} (Q_{M_2}, \alpha_{F_2}) \xrightarrow{(\delta_2)^{-1}} (Q_{\mathscr{D}_2}, \alpha_{\mathscr{D}_2})$$

is a morphism of G-displays of type μ, which we call ζ. By Theorem 3.3.5, ζ is induced by a unique morphism of Tannakian (G,μ)-displays $\xi : \mathscr{D}_1 \to \mathscr{D}_2$. We claim ξ induces φ via (5.4.1), i.e. that $\xi_\eta = \varphi$. Once we know this, it will follow that ξ lifts $(\iota_2)^{-1} \circ \iota_1$. Indeed, if ξ induces φ, then ξ_η lifts $(\iota_2)^{-1}_\eta \circ (\iota_1)_\eta$, and then we can argue as in the proof of faithfulness.

By descent it is enough to check ξ_η and φ agree fpqc-locally. Suppose $R \to R'$ is a faithfully flat extension, and let $\lambda \in \underline{\mathrm{Isom}}^\otimes(\mathscr{C}_{\mu,R'}, \upsilon_{R'} \circ (\mathscr{D}_1)_{R'})$. Then because ξ induces ζ, we have $\zeta(\lambda) = \xi_{R'} \circ \lambda$. On the other hand, by definition of ζ, $\zeta(\lambda)$ is the unique morphism $\mathscr{C}_{\mu,R'} \xrightarrow{\sim} \upsilon_{R'} \circ (\mathscr{D}_2)_{R'}$ such that $\zeta(\lambda)_\eta = \psi(\delta(\lambda)) = \varphi_{R'} \circ \lambda_\eta$. Hence

$$\varphi_{R'} \circ \lambda_\eta = \zeta(\lambda)_\eta = (\xi_{R'})_\eta \circ \lambda_\eta.$$

But λ_η is an isomorphism, so $\varphi_{R'} = (\xi_{R'})_\eta$, i.e. $\varphi = \xi_\eta$. This shows (5.4.1) is full.

Last we prove (5.4.1) is essentially surjective. As usual, by [20, Lemma 046N], it is enough to show any $(\underline{M}, \gamma)$ over R is in the essential image of (5.4.1) fpqc-locally. But after some faithfully flat $R \to R'$ we have an isomorphism $\lambda : \Lambda \otimes_{\mathbb{Z}_p} W(R')^\oplus \xrightarrow{\sim} M_{W(R')^\oplus}$. Then the composition $(\lambda^\tau)^{-1} \circ F^\sharp_{R'} \circ \lambda^\sigma$ is obtained by applying η to some

$U \in L^+G(R')$. Similarly, we obtain $g \in LG(R')$ such that $\eta(g)$ is given by the composition

$$\Lambda \otimes_{\mathbb{Z}_p} W(R')[1/p] \xrightarrow{\lambda^\tau} (M_{W(R'/pR')^\oplus})^\tau[1/p] \xrightarrow{\gamma} ((M_0)_{W(R'/pR')^\oplus})^\tau[1/p] \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} W(R')[1/p],$$

after identifying

$$\Lambda \otimes_{\mathbb{Z}_p} W(R')[1/p] \cong \Lambda \otimes_{\mathbb{Z}_p} W(R'/pR')[1/p]$$

as in the proof of Theorem 4.2.2. Let $\mathscr{D}_U$ be the banal Tannakian (G, μ)-display obtained from U. Then g induces a G-quasi-isogeny $\iota_g : (\mathscr{D}_U)_{R/pR} \dashrightarrow (\mathscr{D}_0)_{R/pR}$, and $(\mathscr{D}_U, \iota_g)$ is an object in $\mathbf{RZ}_{G,\mu,b}(R')$ whose image under (5.4.1) is isomorphic to $(\underline{M}, \gamma)$.

□

Suppose now that all slopes of $\eta(b)$ are nonzero. Then the 1-display $\underline{M_0}$ obtained by evaluating $\mathscr{D}_0$ on (Λ, η) is nilpotent. Hence $X_0 = BT_k(\underline{M_0})$ is a formal p-divisible group, and both $\underline{M_0}$ and X_0 inherit $\mathcal{O}_B$-actions from the action of $\mathcal{O}_B$ on Λ. Combining Lemma 5.4.3 and Theorem 5.4.5, we obtain the following corollary.

Corollary 5.4.6. *Suppose all slopes of $\eta(b)$ are different from 0. Then there is a natural isomorphism of functors*

$$\mathrm{RZ}_{G,\mu,b} \xrightarrow{\sim} \mathrm{RZ}_{\boldsymbol{D}}(X_0).$$

In particular, $\mathrm{RZ}_{G,\mu,b}$ *is representable by a formal scheme which is formally smooth and formally locally of finite type over* $\operatorname{Spf} W(k)$*.*

This proves Conjecture 5.1.2 in the unramified EL-type case.

5.5 Remarks on the Hodge-type case

In this section we specialize the study of our RZ functor to the Hodge-type case. In this case, our Tannakian approach further allows us to prove that the $\mathbf{RZ}_{G,\mu,b}$ is a stack in setoids, i.e. that objects in $\mathbf{RZ}_{G,\mu,b}(R)$ have no nontrivial automorphisms for any R. This is known when G is reductive and R is Noetherian by [3, Proposition 3.6.1].

Definition 5.5.1. A local Shimura datum $(G, \{\mu\}, [b])$ is *of Hodge-type* if there exists a faithful representation Λ of G on a finite free $\mathbb{Z}_p$-module such that the corresponding closed embedding of group schemes $\eta : G \hookrightarrow \mathrm{GL}(\Lambda)$ satisfies the following property: after a choice of basis $\Lambda_{\mathcal{O}_E} \xrightarrow{\sim} \mathcal{O}_E^n$, the composite

$$\eta \circ \mu : \mathbb{G}_{m,\mathcal{O}_E} \to \mathrm{GL}_{n,\mathcal{O}_E}$$

is the cocharacter $a \mapsto \operatorname{diag}(1^{(r)}, a^{(n-r)})$ for some $1 \leq r < n$.

Definition 5.5.2. A *local Hodge embedding datum* for a local Shimura datum $(G, \{\mu\}, [b])$ of Hodge-type consists of

- a group scheme embedding $\eta : G \hookrightarrow \mathrm{GL}(\Lambda)$ as above,
- a framing pair (μ, b) for $(G, \{\mu\}, [b])$.

Let (Λ, η, b) be a local Hodge embedding datum for a local Shimura datum $(G, \{\mu\}, [b])$ of Hodge-type. Let R be a p-nilpotent $W(k)$-algebra and $(\mathscr{D}, \iota)$ be an object in $\mathbf{RZ}_{G,\mu,b}(R)$. By evaluating $\mathscr{D}$ on (Λ, η), we obtain a display $\underline{M}(\eta)$ over $\underline{W}(R)$. Because $\upsilon_R \circ \mathscr{D}$ is fpqc-locally of type μ, it follows from Lemma 2.2.8 that $\underline{M}(\eta)$ is of type $(0^{(r)}, 1^{(n-r)})$. In particular, it is a 1-display over $\underline{W}(R)$.

Let $\mathscr{D}_0$ be the framing object associated to (μ, b), and denote by $\underline{M_0}$ the evaluation of $\mathscr{D}_0$ on (Λ, η). Then ι induces a quasi-isogeny of 1-displays

$$\iota_\eta : \underline{M}(\eta)_{\underline{W}(R/pR)} \dashrightarrow \underline{M_0}_{\underline{W}(R/pR)}.$$

Hence the pair $(\underline{M}(\eta), \iota_\eta)$ determines an object in $\mathbf{RZ}(\underline{M_0})(R)$, where $\mathbf{RZ}(\underline{M_0})$ is the fpqc-stack whose functor of isomorphism classes is $\mathrm{RZ}(\underline{M_0})$. This construction defines a morphism of stacks

$$\mathbf{RZ}_{G,\mu,b} \to \mathbf{RZ}(\underline{M_0}).$$

If $\eta(b)$ has no slopes equal to zero, then $\underline{M_0}$ is a nilpotent 1-display over $\underline{W}(k)$, so by Proposition 5.2.7, $\mathrm{RZ}(\underline{M_0})$ is represented by the classical RZ-space $\mathrm{RZ}(X_0)$, where $X_0 = BT_k(\underline{M_0})$.

Proposition 5.5.3. *Assume that $(G, \{\mu\}, [b])$ is a Hodge-type local Shimura datum with a local Hodge embedding datum such that $\eta(b)$ has no slopes equal to 0. Then $\mathbf{RZ}_{G,\mu,b}$ is a stack in setoids.*

Proof. Let R be in $\mathbf{Nilp}_{W(k)}$ and let $(\mathscr{D}, \iota)$ be an object in $\mathbf{RZ}_{G,\mu,b}(R)$. Suppose ψ

is an automorphism of the pair $(\mathscr{D}, \iota)$. Then ψ is a natural transformation of tensor functors $\mathscr{D} \to \mathscr{D}$ which lifts the identity $\mathscr{D}_{R/pR} \to \mathscr{D}_{R/pR}$. Evaluating $\mathscr{D}$ on (Λ, η), we obtain an automorphism ψ_η of $(\underline{M}(\eta), \iota_\eta)$ in $\mathbf{RZ}(\underline{M_0})$. But by Proposition 5.2.7, $\mathrm{RZ}(\underline{M_0})$ is representable, hence its objects have no nontrivial automorphisms. Then $\psi_\eta = \mathrm{id}_{\underline{M}(\eta)}$. This implies that $\psi = \mathrm{id}_{\mathscr{D}}$, since any endomorphism of an exact tensor functor from $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ to an exact rigid tensor category which is the identity on a faithful representation is itself the identity. $\square$

Appendix A: Tannakian G-displays over other frames

A.1 Other examples of frames

In addition to the Witt frame, there are a variety of frames of interest in the study of Dieudonné theory, see [5, §2.1] for a number of examples. If (S, σ, τ) is a frame which has the property that S_n is a p-adically complete abelian group for every n and $R = S_0/\tau S_1$ is a p-nilpotent $\mathbb{Z}_p$-algebra, then Lau formulates a framework for G-displays over (S, σ, τ), see [5, §5.3]. Such frames, called p-adic frames, can naturally be arranged into étale sheaves of frames on Spec R, and it is trivial to generalize Lau's theory to arbitrary étale sheaves of frames. In this appendix, we review the framework of Lau, and develop a Tannakian framework for this situation analogous to the framework developed in Section 3.3. Most of the proofs from Section 3.3 carry over into this situation, but we remark where modifications need to be made.

Let us begin by reviewing a two additional examples of frames.

Example A.1.1 (The truncated Witt frame)**.** Fix an $\mathbb{F}_p$-algebra R and a non-negative integer m. The truncated Witt frame $\underline{W}_m(R)$ is defined as follows. Let $S_0 = W_n(R)$ be the truncated Witt ring of length m. Denote by $I_{n+1}(R)$ the kernel of

$$w_0 : W_{m+1}(R) \to R.$$

Then define an $\mathbb{Z}$-graded ring S by taking $S_0 = W_m(R)$, $S_n = W_m(R) \cdot t^{-n}$ for $n \leq -1$, and $S_n = I_{m+1}(R)$ for every $n \geq 1$. We endow S with the structure of a frame by defining σ and τ as the unique morphisms such that $\underline{W}(R) \to \underline{W}_m(R)$ defines a morphism of frames.

Definition A.1.2. A ring R is *admissible* if the following hold

(i) R is a local ring and there exists some n such that $x^n = 0$ for all $x \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R.

(ii) The residue field κ of R is perfect of characteristic p.

(iii) Either $p \geq 3$ or $pR = 0$.

Example A.1.3 (The Zink frame)**.** Let R be an admissible ring, and let $W(\mathfrak{m}) = \ker(W(R) \to W(\kappa))$. By [24, Lemma 1.4] (see also [27, Chapitre IV, Proposition 4.3]), the homomorphism $W(\kappa) \to \kappa$ lifts to a homomorphism $W(\kappa) \to R$. By applying the functor W and composing with the Cartier homomorphism $\Delta_\kappa : W(\kappa) \to W(W(\kappa))$, we obtain a section $s : W(\kappa) \to W(R)$ of the short exact sequence

$$0 \to W(\mathfrak{m}) \to W(R) \to W(\kappa) \to 0.$$

Denote by $\hat{W}(\mathfrak{m})$ the subset of $W(\mathfrak{m})$ which consists of sequences $(x_0, x_1, \dots)$ having only finitely many nonzero elements. Then the direct sum

$$\mathbb{W}(R) = s(W(\kappa)) \oplus \hat{W}(\mathfrak{m})$$

is an f and v-stable subring of $W(R)$ (see [6, §2]), called the Zink ring of R. The ring $\mathbb{W}(R)$ determines a subframe $\underline{\mathbb{W}}(R)$ of $\underline{W}(R)$ as follows. Define a graded ring S by $S_0 = \mathbb{W}(R)$, $S_n = \mathbb{I}(R) = \mathbb{W}(R) \cap I(R)$ for $n \geq 1$ and $S_n = \mathbb{W}(R) \cdot t^{-n}$ for all $n \leq -1$. The maps σ and τ are determined by restriction. By [24, §1F], $\mathbb{W}(R)$ is p-adically complete, so $p \in \mathrm{Rad}(\mathbb{W}(R))$, and $\underline{\mathbb{W}}(R)$ determines a frame over R.

Denote by $\mathbf{A\acute{E}t}_R$ the category of étale R-algebras equipped with the étale topology. Each of the above frames determines a functor from $\mathbf{A\acute{E}t}_R$ to the category of frames.

Lemma A.1.4. *(i) If R is an $\mathbb{F}_p$-algebra, the assignment $R' \mapsto \underline{W}_m(R')$ defines a sheaf of frames on $\mathbf{A\acute{E}t}_R$.*

(ii) If R is an admissible ring, then the assignment $R' \mapsto \underline{\mathbb{W}}(R')$ defines a sheaf of frames on $\mathbf{A\acute{E}t}_R$.

We will denote by $\underline{W}_m$ and $\underline{\mathbb{W}}$ the étale sheaves of frames defined in the lemma.

Proof. Each of these frames are p-adic frames in the sense of [5]. Let $\underline{S}$ be $\underline{W}_m(R)$ or $\mathbb{W}(R)$. Then by [5, Lemma 4.2.3], the assignment $R' \mapsto \underline{S}(R')$ determines a sheaf on $\mathbf{A\acute{E}t}_R$, where $\underline{S}(R')$ denotes the base change of $\underline{S}$ to R' in the sense of [5, Lemma 4.2.3]. Then it remains only to show $\underline{S}(R') = W_m(R')$ when $\underline{S} = W_m(R)$

and $\underline{S}(R') = \mathbb{W}(R')$ when $\underline{S} = \mathbb{W}(R)$. The first follows from [5, Example 4.2.6], and the second from [5, Lemma 4.2.5]. $\square$

A.2 Tannakian G-displays over étale sheaves of frames

Recall [5, §5] a frame $\underline{S}$ is a frame over $W(k_0)$ if S is a graded $W(k_0)$-algebra and $\sigma : S \to S_0$ extends the Frobenius of $W(k_0)$. If R is a $W(k_0)$-algebra, then the frames $\underline{W}(R)$, $\underline{W}_m(R)$, and $\underline{\mathbb{W}}(R)$ are $W(k_0)$-frames. See [5, Example 5.0.2] for details.

If $X = \operatorname{Spec} R'$ is an affine $W(k_0)$-scheme, then an action of $\mathbb{G}_m$ on X is equivalent to a $\mathbb{Z}$-grading on R' (see §3.1). If $\mathbb{G}_m$ acts on X, and S is a $\mathbb{Z}$-graded $W(k_0)$-algebra, then let $X(S)^0 \subseteq X(S)$ be the set of $\mathbb{G}_m$-equivariant sections $\operatorname{Spec} S \to X$ over $W(k_0)$. In other words, $X(S)^0$ is the set of homomorphisms of graded $W(k_0)$-algebras $R' \to S$.

Suppose $\underline{S}$ is an étale sheaf of frames on Spec R. If $R \to R'$ is étale, write

$$\underline{S}(R') = (S(R'), \sigma(R'), \tau(R')).$$

To X and $\underline{S}$ we associate two functors on étale R-algebras:

$$X(\underline{S})^0 : R' \mapsto X(S(R'))^0, \text{ and } X(\underline{S_0}) : R' \mapsto X(S(R')_0).$$

Lemma A.2.1. *Let $\underline{S}$ be an étale sheaf of frames on* Spec R*, and let $X =$* Spec A *be an affine scheme of finite type over $W(k_0)$ with a $\mathbb{G}_m$-action. Then the functors*

$X(\underline{S})^0$ *and* $X(\underline{S}_0)$ *are étale sheaves on* Spec R*.*

Proof. The proof is formally the same as that of [5, Lemma 5.3.1]. Let us repeat it here and fill in some details. First we note that if X and Y are affine schemes of finite type over $W(k_0)$, then

$$(X \times Y)(\underline{S})^0 = X(\underline{S})^0 \times Y(\underline{S})^0,$$

and

$$\mathrm{Eq}(X \rightrightarrows Y)(\underline{S})^0 = \mathrm{Eq}\left(X(\underline{S})^0 \rightrightarrows Y(\underline{S})^0\right).$$

It follows that $X(\underline{S})^0$ commutes with finite products and equalizers. The analogous statements for $X(\underline{S}_0)$ hold as well.

Now we claim that if $X = \mathrm{Spec}\ A$ is any affine scheme of finite type over $W(k_0)$ which is equipped with a $\mathbb{G}_m$-action, then X can be realized as the equalizer of two $\mathbb{G}_m$-equivariant morphisms $\mathbb{A}^n_{W(k_0)} \rightrightarrows \mathbb{A}^m_{W(k_0)}$. Indeed, by [28, Lemma 2.21], there exists a $\mathbb{G}_m$-equivariant closed immersion $X \hookrightarrow \mathbb{A}^n_{W(k_0)}$, where $\mathbb{A}^n_{W(k_0)} = \mathrm{Spec}\ W(k_0)[X_1, \ldots, X_n]$ with X_i homogeneous of degree s_i. This corresponds to a homomorphism of graded $W(k_0)$-algebras

$$W(k_0)[X_1, \ldots, X_n] \to A.$$

The kernel of this homomorphism is graded, and therefore it is generated by finitely many homogeneous elements $(f_1, \ldots, f_m)$, say with $\deg(f_i) = r_i$. Let $Y_1, \ldots, Y_m$ be indeterminates with $\deg(Y_i) = r_i$, and define graded $W(k_0)$-algebra homomorphisms

$$\varphi, \psi : W(k_0)[Y_1, \ldots, Y_m] \to W(k_0)[X_1, \ldots, X_n]$$

by $\varphi(Y_i) = f_i$ and $\psi(Y_i) = 0$. Evidently A is the coequalizer of φ and ψ, which proves the claim.

Now, using the claim and the commutativity of these functors with finite products and equalizers, we reduce to the case where $X = \mathbb{A}^1_{W(k_0)} = \text{Spec } W(k_0)[u]$ with u homogeneous of degree r. But if $R \to R'$ is étale,

$$\mathbb{A}^1_{W(k_0)}(\underline{S})^0(R') = S(R')_r, \text{ and } \mathbb{A}^1_{W(k_0)}(\underline{S}_0)(R') = S(R')_0.$$

The result follows because $\underline{S}$ is an étale sheaf of frames. $\square$

Let $G = \text{Spec } \mathcal{O}_G$ be a flat affine group scheme of finite type over $\mathbb{Z}_p$, and let $\mu : \mathbb{G}_{m,W(k_0)} \to G_{W(k_0)}$ be a cocharacter of G defined over $W(k_0)$. As in §3.1, we can define the display group associated to G and μ with values in a $\mathbb{Z}$-graded ring S by

$$G(S)_\mu := G(S)^0,$$

i.e. $G(S)_\mu$ is the subset of $G_{W(k_0)}(S) = \text{Hom}_{W(k_0)}(\mathcal{O}_G, S)$ consisting of $W(k_0)$-algebra homomorphisms which preserve the respective gradings. Similarly, if $\underline{S}$ is an étale sheaf of frames on Spec R, then define

$$G(\underline{S})_\mu := G(\underline{S})^0,$$

so $G(\underline{S})_\mu$ is an étale sheaf of groups on Spec R. If (S, σ, τ) is a $W(k_0)$-frame, then the $\mathbb{Z}_p$-algebra homomorphisms again induce $\sigma, \tau : G(S)_\mu \to G(S_0)$, which we can use to define an action of $G(S)_\mu$ on $G(S_0)$:

$$G(S_0) \times G(S)_\mu \to G(S_0),\ (x, g) \mapsto \tau(g)^{-1} x \sigma(g). \tag{A.2.1}$$

If $\underline{S}$ is an étale sheaf of $W(k_0)$-frames on Spec R, this action sheafifies to provide an action of $G(\underline{S})_\mu$ on $G(\underline{S}_0)$.

Definition A.2.2. Let R be a p-nilpotent $W(k_0)$-algebra, and suppose $\underline{S}$ is an étale sheaf of $W(k_0)$-frames on Spec R. The *stack of Tannakian G-displays of type μ over $\underline{S}$* is the étale quotient stack

$$G\text{-}\mathbf{Disp}_{\underline{S},\mu} := [G(\underline{S}_0)/G(\underline{S})_\mu]$$

over $\mathbf{A\acute{E}t}_R$, where $G(\underline{S})_\mu$ acts on $G(\underline{S}_0)$ via the action (A.2.1).

Explicitly, for an étale R-algebra R', $G\text{-}\mathbf{Disp}_{\underline{S},\mu}(R')$ is the groupoid of pairs (Q, α), where Q is an étale locally trivial $G(\underline{S})_\mu$-torsor over Spec R and $\alpha : Q \to G(\underline{S}_0)$ is a $G(\underline{S})_\mu$-equivariant morphism for the action (A.2.1).

A.3 Descent

Let us begin by checking some étale-local properties of finite projective graded modules over étale sheaves of frames. Suppose $\underline{S}$ is an étale sheaf frames on Spec R, and write $\underline{S}(R) = (S(R), \tau(R), \sigma(R))$, so S determines an étale sheaf of $\mathbb{Z}$-graded rings. The following lemma containes analogs of Lemma 2.2.8, Lemma 2.2.9, and Lemma 2.2.10.

Lemma A.3.1. *Let $R \to R'$ be a faithfully flat étale ring homomorphism.*

(i) If M is a finite projective graded $S(R)$-module, then there is an exact sequence

$$0 \longrightarrow M \longrightarrow M \otimes_{S(R)} S(R') \rightrightarrows M \otimes_{S(R)} S(R' \otimes_R R') \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} \cdots$$

where the arrows are induced by applying S to the usual exact sequence

$$0 \longrightarrow R \longrightarrow R' \rightrightarrows R' \otimes_R R' \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} \cdots$$

(ii) A finite projective graded $S(R)$-module M is of type $I = (i_1, \dots, i_n)$ if and only if $M_{S(R')}$ is of type I.

(iii) A sequence $0 \to M \to N \to P \to 0$ of finite projective graded $S(R)$-modules is exact if and only if it is exact after base change to $S(R')$.

(iv) If $\underline{M} = (M, F)$ and $\underline{M}' = (M', F')$ are displays over $\underline{S}$, then a homomorphism $\psi : M \to M'$ of graded $S(R)$-modules is a morphism of displays if and only if $\psi_{R'}$ is a morphism of displays.

Proof. For (i), since M is finite projective we can reduce to the case where M is finite free as a graded $S(R)$-module. This in turn reduces to the case $M = S(R)$, for which the result holds because S is an étale sheaf of $\mathbb{Z}$-graded rings.

The proof for (ii) follows from the fact that the rank of a finite projective module is invariant under base change, and the proof of (iii) is formally the same as that of Lemma 2.2.10.

Let us prove (iv). If ψ is a morphism of displays then $\psi_{R'}$ is as well. For the converse, we need to prove $(F')^\sharp \circ \sigma^*\psi$ and $\tau^*\psi \circ F^\sharp$ agree as homomorphism of finite projective $S(R)_0$-modules. We know this holds after base change to $S(R')_0$, so it is enough to prove the base change functor from the category of finite projective $S(R)_0$-modules to the category of finite projective $S(R')_0$-modules is faithful. But this is easy to see because by (i) the homomorphism $M \to M \otimes_{S(R)_0} S(R')_0$ is injective. $\square$

Let R be a ring, and let S be an étale sheaf of $\mathbb{Z}$-graded rings over Spec R. Then we will denote by $\mathbf{PGrMod}_S$ the fibered category over $\mathbf{A\acute{E}t}_R$ whose fiber over an étale R-algebra R' is $\mathbf{PGrMod}(S(R'))$. Further, if $\underline{S}$ is a sheaf of frames, let $\mathbf{Disp}_{\underline{S}}$ denote the fibered category of displays over $\underline{S}$.

Definition A.3.2. We say:

- An étale sheaf of $\mathbb{Z}$-graded rings S on Spec R *satisfies descent for modules* if $\mathbf{PGrMod}_S$ is an étale stack over $\mathbf{A\acute{E}t}_R$.
- An étale sheaf of frames $\underline{S}$ on Spec R *satisfies descent for displays* if $\mathbf{Disp}_{\underline{S}}$ is an étale stack over $\mathbf{A\acute{E}t}_R$.

Lemma A.3.3. *Let $\underline{S}$ be an étale sheaf of frames on* Spec R *such that S satisfies descent for modules. Then $\underline{S}$ satisfies descent for displays.*

Proof. Since morphisms of displays are in particular morphisms of the underlying modules, it follows immediately that morphisms descend. To prove that objects descend it remains only to show that isomorphisms $\sigma^* M \to \tau^* M$ form an étale sheaf. But since $\underline{S}$ is an étale sheaf of frames, the functor $S_0 : R' \mapsto S(R')_0$ is an étale sheaf of rings on Spec R, and from this one deduces that for any finite projective R-module N the following sequence is exact:

$$0 \to N \to N \otimes_{S(R)_0} S(R')_0 \rightrightarrows N \otimes_{S(R)_0} S(R' \otimes_R R')_0$$

Then it is easy to see that isomorphisms of S_0-modules $\sigma^* M \to \tau^* M$ form an étale sheaf. □

Lemma A.3.4. *The étale sheaves of frames $\underline{W}_m$ and $\underline{\mathbb{W}}$ satisfy descent for displays.*

Proof. Follows from Lemma A.3.3 and [5, Lemma 4.3.1]. □

A.4 Tannakian G-displays over étale sheaves of frames

In this section, we develop a Tannakian framework for the étale stack of Tannakian G-displays of type μ, analogous to the one developed in §3.3. Let G be a flat affine group scheme of finite type over $\mathbb{Z}_p$, and denote by $\mathbf{Rep}_{\mathbb{Z}_p}(G)$ the category of representations of G on finite free $\mathbb{Z}_p$-modules.

Definition A.4.1. Let S be a $\mathbb{Z}$-graded $\mathbb{Z}_p$-algebra. A *graded fiber functor* over S

is an exact tensor functor

$$\mathscr{F} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}(S).$$

Denote by $\mathbf{GFF}(S)$ the category of graded fiber functors over S. Suppose S is an étale sheaf of $\mathbb{Z}$-graded rings on Spec R. If $R \to R'$ is a homomorphism of étale R-algebras, the natural base change $M \mapsto M \otimes_{S(R)} S(R')$ induces a base change functor $\mathbf{GFF}_S(R) \to \mathbf{GFF}_S(R')$. In this way we obtain a fibered category $\mathbf{GFF}_S$ over $\mathbf{A\acute{E}t}_R$. Explicitly, if R' is in $\mathbf{A\acute{E}t}_R$, then $\mathbf{GFF}_S(R') = \mathbf{GFF}(S(R'))$.

Lemma A.4.2. *Let $\underline{S}$ be an étale sheaf of quasi-frames such that the underlying sheaf of graded rings S satisfies descent for modules. Then the fibered category $\mathbf{GFF}_S$ is an étale stack over $\mathbf{A\acute{E}t}_R$.*

Proof. The proof is the same as that of Lemma 3.2.2, with Lemma A.3.1 (iii) replacing Lemma 2.2.10. $\square$

Now suppose R is a $W(k_0)$-algebra, and that $\underline{S}$ is an étale sheaf of $W(k_0)$-frames over Spec R which satisfies descent for modules. For any cocharacter μ of G defined over $W(k_0)$ and any étale R-algebra R', we define a distinguished graded fiber functor over $S(R')$. Given a representation (V, ρ) in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$, μ induces a canonical weight decomposition

$$V_{W(k_0)} = \bigoplus_{i \in \mathbb{Z}} V^i_{W(k_0)}, \tag{A.4.1}$$

where $V_{W(k_0)} = V \otimes_{\mathbb{Z}_p} W(k_0)$, and

$$V^i_{W(k_0)} = \{v \in V_{W(k_0)} \mid (\rho \circ \mu)(z) \cdot v = z^i v \text{ for all } z \in \mathbb{G}_m(W(k_0))\}.$$

By tensoring $V_{W(k_0)}$ over $W(k_0) \to S(R')$, we obtain a finite projective graded $S(R')$-module. Since any morphism of representations preserves the grading induced by μ, we obtain an exact tensor functor

$$\mathscr{C}(\underline{S})_{\mu,R'} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{PGrMod}_S(R'),\ V \mapsto V_{W(k_0)} \otimes_{W(k_0)} S(R'). \tag{A.4.2}$$

Notice if R' is an étale R-algebra, then $\mathscr{C}(\underline{S})_{\mu,R'}$ is given by the composition of functors

$$\mathbf{Rep}_{\mathbb{Z}_p}(G) \xrightarrow{\mathscr{C}(\underline{S})_{\mu,R}} \mathbf{PGrMod}_S(R) \to \mathbf{PGrMod}_S(R'),$$

where the second functor is the canonical base change. If R is understood, we will suppress it in the notation and write $\mathscr{C}(\underline{S})_\mu$ for $\mathscr{C}(\underline{S})_{\mu,R}$.

Definition A.4.3. A graded fiber functor $\mathscr{F}$ over $S(R)$ is *of type* μ if for some faithfully flat étale extension $R \to R'$ there is an isomorphism $\mathscr{F}_{R'} \cong \mathscr{C}(\underline{S})_{\mu,R'}$.

Let $\mathbf{GFF}_{S,\mu}$ denote the fibered category of graded fiber functors of type μ. Since the property of being type μ is étale-local, $\mathbf{GFF}_{S,\mu}$ forms a substack of $\mathbf{GFF}_S$. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two graded fiber functors over $\underline{S}$, denote by $\underline{\mathrm{Isom}}^{\otimes}(\mathscr{F}_1, \mathscr{F}_2)$ the étale sheaf of isomorphisms of graded fiber functors functors $\mathscr{F}_1 \xrightarrow{\sim} \mathscr{F}_2$. Let

$\underline{\mathrm{Aut}}^{\otimes}(\mathscr{F}) = \underline{\mathrm{Isom}}^{\otimes}(\mathscr{F}, \mathscr{F})$. Now we have the analogs of the main theorems of §3.2.

Theorem A.4.4. *Let $\underline{S}$ be an étale sheaf of $W(k_0)$-frames which satisfies descent for modules. The assignment $g \mapsto \{\rho(g)\}_{(V,\rho)}$ defines an isomorphism of étale sheaves on* Spec R

$$G(\underline{S})_\mu \xrightarrow{\sim} \underline{\mathrm{Aut}}^{\otimes}(\mathscr{C}(\underline{S})_\mu),$$

which, in turn, induces an isomorphism of stacks

$$\mathbf{GFF}_{\underline{S},\mu} \xrightarrow{\sim} \mathbf{Tors}_{G(\underline{S})_\mu}.$$

Proof. The arguments of §3.2 go through nearly verbatim, after replacing the Witt frame with $\underline{S}$, and the fpqc topology with the étale topology. $\square$

Let us now make the definitions of Tannakian (G, μ)-displays over arbitrary étale sheaves of $W(k_0)$-frames. The main result of this section will be the analog of Theorem 3.3.5. For any étale R-algebra R' we have a forgetful functor

$$\upsilon_{S(R')} : \mathbf{Disp}_{\underline{S}}(R') \to \mathbf{PGrMod}_S(R'),\ (M, F) \mapsto M. \tag{A.4.3}$$

Definition A.4.5. Let R be a p-nilpotent $\mathbb{Z}_p$-algebra.

- A *Tannakian G-display over* $\underline{S}(R)$ is an exact tensor functor

$$\mathscr{P} : \mathbf{Rep}_{\mathbb{Z}_p}(G) \to \mathbf{Disp}_{\underline{S}}(R).$$

- A *Tannakian* (G,μ)*-display* over $\underline{S}(R)$ is a Tannakian G-display $\mathscr{P}$ over $\underline{S}(R')$ such that $\upsilon_{S(R)} \circ \mathscr{P}$ is a graded fiber functor of type μ.

If $R \to R'$ is étale, denote by $G\text{-}\mathbf{Disp}^{\otimes}(\underline{S}(R'))$, resp. $G\text{-}\mathbf{Disp}^{\otimes}_{\mu}(\underline{S}(R'))$ the category of Tannakian G-displays, resp. the full subcategory of Tannakian (G,μ)-displays over $\underline{S}(R')$. We see that Tannakian G-displays form an étale stack $G\text{-}\mathbf{Disp}^{\otimes}_{\underline{S}}$ over $\mathbf{A\acute{E}t}_R$, and Tannakian (G,μ)-displays define a substack $G\text{-}\mathbf{Disp}^{\otimes}_{\underline{S},\mu}$.

To any Tannakian (G,μ)-display we can associate a G-display of type μ. Let us summarize the construction, which is analogous to Construction 3.3.4. Let $\mathscr{P}$ be a Tannakian (G,μ)-display over $\underline{S}(R)$. By Theorem A.4.4,

$$Q_{\mathscr{P}} := \underline{\mathrm{Isom}}^{\otimes}(\mathscr{C}(\underline{S})_{\mu,R}, \upsilon_{S(R)} \circ \mathscr{P})$$

is a $G(\underline{S})_{\mu}$-torsor over R. If R' is an étale R-algebra and $\lambda : \mathscr{C}(\underline{S})_{\mu,R'} \xrightarrow{\sim} \upsilon_{S(R')} \circ \mathscr{P}_{R'}$ is an isomorphism of tensor functors, then for every (V,ρ) in $\mathbf{Rep}_{\mathbb{Z}_p}(G)$ we obtain an automorphism

$$\alpha_{\mathscr{P}}(\lambda)_{\rho} := (\lambda_{\rho})^{\tau} \circ (F(\rho)')^{\sharp} \circ (\lambda_{\rho})^{\sigma}$$

of $V \otimes_{\mathbb{Z}_p} S(R')_0$, where $\mathscr{P}_{R'}(V,\rho) = (M(\rho)', F(\rho)')$. As (V,ρ) varies, the collection $\{\alpha(\lambda)_{\rho}\}_{(V,\rho)}$ constitutes an element of $\mathrm{Aut}^{\otimes}(\omega_{S(R')_0})$, where $\omega_{R'}$ is the canonical fiber functor $(V,\rho) \mapsto V \otimes_{\mathbb{Z}_p} R'$ associated to any $\mathbb{Z}_p$-algebra R'. By duality,

$$\mathrm{Aut}^{\otimes}(\omega_{S(R')_0}) \cong G(S(R')_0) = G(\underline{S}_0)(R'),$$

so there is some $\alpha_{\mathscr{P}}(\lambda) \in G(\underline{S}_0)(R')$ such that $\rho(\alpha_{\mathscr{P}}(\lambda)) = \alpha_{\mathscr{P}}(\lambda)_\rho$ for every (V, ρ). All together this defines a morphism of étale sheaves

$$\alpha_{\mathscr{P}} : Q_{\mathscr{P}} \to G(\underline{S}_0). \tag{A.4.4}$$

As in Construction 3.3.4 one checks that the association $\mathscr{P} \mapsto (Q_{\mathscr{P}}, \alpha_{\mathscr{P}})$ is functorial and compatible with base change, so we obtain a morphism of stacks

$$G\text{-}\mathbf{Disp}^{\otimes}_{\underline{S},\mu} \to G\text{-}\mathbf{Disp}_{\underline{S},\mu}, \ \mathscr{P} \mapsto (Q_{\mathscr{P}}, \alpha_{\mathscr{P}}). \tag{A.4.5}$$

The following is the analog of Theorem 3.3.5.

Theorem A.4.6. *If $\underline{S}$ satisfies descent for modules, then the morphism (A.4.5) is an isomorphism of étale stacks over $\mathbf{A\acute{E}t}_R$.*

Proof. The proof of Theorem 3.3.5 goes through here as well, after replacing the Witt frame by the frame $\underline{S}$, and the fpqc topology by the étale topology. Let us sketch the argument.

By the first part of Theorem A.4.4, the functor is faithful. If $\mathscr{P}_1$ and $\mathscr{P}_2$ are Tannakian (G, μ)-displays over R, and $\eta : (Q_{\mathscr{P}_1}, \alpha_{\mathscr{P}_1}) \to (Q_{\mathscr{P}_2}, \alpha_{\mathscr{P}_2})$ is a morphism, then the second part of Theorem A.4.4 provides us with a morphism $\psi : \upsilon_R \circ \mathscr{P}_1 \to \upsilon_R \circ \mathscr{P}_2$ which induces $Q_{\mathscr{P}_1} \to Q_{\mathscr{P}_2}$. It remains only to check this morphism is compatible with the respective Frobeneii, but by Lemma A.3.1 (iv) it is enough to check this after some faithfully flat étale extension $R \to R'$. By choosing an extension such that $Q_{\mathscr{P}_1}(R')$ is nonempty, the result follows from the definitions of the $\alpha_{\mathscr{P}_i}$.

Finally, to complete the proof it is enough to show that every G-display of type μ over $\underline{S}(R)$ is étale locally in the essential image of (A.4.5), which is easily done using Theorem A.4.4. $\square$

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