APPROVAL SHEET

Title of Thesis: A Combinatorial Representation for Oriented Polyhedral Surfaces

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Master of Arts, 1960

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Date approved: May 13, 1960

A COMBINATORIAL REPRESENTATION FOR ORIENTED POLYHEDRAL SURFACES

by

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Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Arts
1960

SECTION I

Any connected linear graph can be imbedded without self-intersections in a closed, two-sided surface so that the components of the complement of the graph in the surface are topological discs (1). Our purpose is to strengthen this well known theorem (in Theorem 2) and then apply the stronger result to formulating (in Theorems 3 and 4) a simple combinatorial manner for representing an oriented polyhedral surface, i.e., a surface with an imbedded graph as described above.

Each disc and its closure may be considered a (topological) polygon whose edges and vertices are those shared by the graph. Every edge of the graph belongs to exactly two polygons or perhaps to the same polygon twice. On the other hand, any collection of such polygons with no intersections except that each edge belongs exactly twice to member polygons forms one or more closed (not necessarily orientable) polyhedral surfaces. When in no proper subcollection of the polygons do the edges have this property of coinciding in pairs, the collection forms a single polyhedron, in which case its edges and vertices are a connected graph.

It is intuitively evident that: Theorem 1. If a polyhedron is two-sided -- that is orientable -- then with respect to one of these sides, the edge-ends to each vertex have a uniquely determined cyclic ordering clockwise around the vertex.

We will not formally prove this theorem. For a more precise and thorough treatment of polyhedral surfaces, including essentially a proof of this theorem, the reader is referred to (2); though the polydra there have only triangular faces, the point-set considerations apply to the polyhedra of this paper. For our purposes, however, the following remarks about the ordering are more useful.

Every vertex of a polygon is an endpoint of exactly two of its edges or is an endpoint of the same edge twice. The vertices of a polyhedron arise by the coincidences of the polygonal vertices induced by the coincidences of the polygonal edges in pairs. In particular, a certain polygonal vertex will coincide twice with other polygonal vertices by virtue of the coincidences of the two polygonal edges in which it occurs (or else will coincide only with itself by virtue of a coincidence with each other of the polygonal edges containing it). These two other vertices will each coincide in the same manner with another polygonal vertex, and so on. We thus have, except for orientation, a cyclic ordering of polygonal vertices, each coinciding with the two vertices adjacent to it in the ordering. Because of the transitivity of coincidence, these polygonal vertices will thereby all coincide with each other to form a single vertex of the polyhedron.

No other polygonal vertex will coincide in the polyhedron with these because the coincidences of the polygonal edges in which they occur are all accounted for and therefore there are no further edge coincidences to give rise to further vertex coincidences with these vertices. This discussion indicates why a polyhedron, formed by the coincidence of edges of polygons in pairs, is locally euclidean, even at its vertices, and is hence a surface.

Now, where v_1 and v_2 are polygonal vertices which are adjacent in the ordered set of vertices which coincide to form vertex v of a polyhedron, pair v_1 v_2 corresponds to the edge-end to v formed by the coincidence of the polygonal edges which gives rise to the coincidence of v_1 and v_2. The ordered set of polygonal vertices determines a similar ordering of its adjacent pairs of vertices and hence an ordering of the edge-ends to v. On the basis of our remarks so far (which apply as well to non-orientable polyhedra), this ordering of the edge-ends to a polyhedral vertex is cyclic but without orientation -- that is, no distinction is made between the ordering and its reverse.

The edges of a polygon are cyclically ordered without orientation by saying that adjacent edges in the ordering are those having a common (polygonal vertex) end point. To orient a polygon is to orient this ordering, i.e., to specify one of the two possible cyclic orderings whose adjacent edges are the same as above. Pairs of adjacent edges are thus directed so that for each edge there exists one right-adjacent edge and one left-adjacent edge. (b is right-adjacent to a if and only if a is left-adjacent to b.)

By specifying an "upper side" of the polygon we specify a clockwise orientation of the polygon, and conversely; that is, we can orient a surface element (or a surface if it has two sides) by specifying an "upper side". To formalize this concept of "the sides" of a surface element one must, of course, employ neighborhoods of points of the surface element imbedded in some 3 dimensional space. Nevertheless, the device of "sides" is intuitively convenient without definition.

We now have for any polyhedron, the sets of the edges of its polygons and the sets of the edges to its vertices all cyclically ordered without orientation. By the manner of construction of these ordered sets, it is clear that two edges are adjacent in some vertex set if and only if they are adjacent in some polygon set.

An oriented polyhedron is one whose polygons are oriented so that the directed adjacency of pairs of edges in the polygon sets induces a directed adjacency of the corresponding pairs in the vertex sets which gives rise to a consistent orientation of each of the vertex set orderings. That is, an oriented polyhedron is one for which the orderings of all the polygon sets and vertex sets are oriented so that edge a is right-adjacent to edge b in a polygon set if and only if for the corresponding adjacent pair in a vertex set b is right-adjacent to a. This description may be taken as a definition for "oriented polyhedron" or else can be proven from some other definition. It should be geometrically evident that the description is equivalent to forming an oriented polyhedron by joining edges of oriented polygons so that the upper sides of the polygons are locally on the same side of their union.

SECTION II

It has apparently not been observed in the literature that, indeed, conversely to Theorem I:

Theorem 2. Given a connected linear graph with an arbitrarily specified cyclic ordering of the edges to each vertex, there exists a topologically unique, two-sided polyhedron whose edges and vertices are the given graph and whose clockwise edge orderings at each vertex (with respect to one of the sides) are as specified.

Proof. We may see that Theorem 2 is true by using the concept of the dual of a polyhedron: by selecting in each face of a polyhedron a point and by replacing each edge, which belongs to two faces or to the same face twice, by a new edge intersecting it and joining the corresponding selected points of the faces, we obtain a new polyhedral division of the surface where faces have been replaced by vertices and vertices have been replaced by faces. Each of these two polyhedra is the topologically unique "dual" of the other.

Suppose we are given a graph and a specified cyclic ordering of the edges to each vertex. Corresponding to each vertex, let there be a polygon such that the edges of the polygon correspond one-to-one to the edges of the graph going to that vertex and such that the polygon edges are arranged with respect to a certain "upper side" of the polygon clockwise around the boundary in the order specified for the corresponding edges to the vertex. Since an edge of the graph has two ends, exactly two polygonal edges correspond to each edge of the graph. Now we join the polygons together along pairs of edges corresponding to the same graph edge so that upper sides of the polygons all become part of the same side of the surface formed. (At any step of the process of joining, any remaining pair of corresponding edges may be joined in wrong way and one right way.) In the process, the polygonal vertices are made to coincide with each other as described in the remarks following Theorem 1. We thus obtain a polyhedron the 1-skeleton of whose dual is the given graph imbedded in the desired manner. This dual is therefore the topologically unique polyhedron whose existence is asserted in the theorem -- unique because any other such polyhedron is constructable in the same unambiguous manner. Though this definitely informal proof is instructive, it is actually not necessary to employ the dual in order to prove Theorem 2. The theorem follows immediately from the remarks following Theorem 1 by noticing that the steps for obtaining, for a polyhedron, the orderings of the vertex sets (of edges) are reversible. By reversing the procedure, one finds for a graph with cyclic ordered vertex sets a unique set of circuits of edges such that a surface is formed when each circuit is made to bound a 2 cell. We shall essentially describe this reverse procedure in Section 4 where we give a combinatorial construction for the dual of a polyhedron.

SECTION III

There is a simple combinatorial representation for polyhedral surfaces, based on the remarks following Theorem 1 or, alternatively, based on Theorem 2. Topologically, an oriented polygon is determined by a cyclically ordered set of elements which correspond one-to-one to the edges of the polygon so that element b is right-adjacent to element a in the set when corresponding edges a and b meet at a common vertex so that edge b follows edge a clockwise around the boundary with respect to the orientation. The polygon may be one-edged in which case, of course, $a = b$.

An oriented polyhedral surface $\overline{P}$ is then representable by a class of such cyclically ordered sets where each element occurs exactly twice in sets of the class, perhaps in the same set twice. (T): Two occurrences e_1 and e_2 of the same element e will correspond to polygon edges e_1 and e_2 which coincide in the polyhedron so that the vertex (end-point) of edge e_1 shared with its left-adjacent edge coincides with the vertex shared by edge e_2 with its right adjacent edge. Adjacency here is, of course, with respect to the set representations.

Conversely, every such class of ordered sets, in which, (A), every element of these sets occurs exactly twice, determines in the manner (T) a set of oriented polyhedra. If, (B), for no proper subclass, does every element occurring once in a set of the subclass occur twice in sets of the subclass, then no proper subclass determines a polyhedron and hence the class represents a single polyhedron.

On the other hand, a class of <u>unordered</u> sets such that, (A), every element of a set in the class occurs exactly twice in sets of the class, perhaps to the same set twice, is a convenient representation of a graph. The sets correspond to the vertices of the graph. Each element corresponds to and edge whose endpoints are the vertices corresponding to the sets in which the element occurs. If, (B), no proper sub- class of the class has property (A), then the corresponding graph is connected.

A class with properties (A) and (B) such that, (C), the sets of the class are cyclically ordered, represents on the one hand a polyhedron $\overline{P}$ as described earlier. On the other hand, it specifies a cyclic ordering of the edge ends to each vertex of the represented graph. The polyhedron P, whose 1-skeleton is this graph so arranged clockwise around the vertice, is the dual of $\overline{P}$.

We prefer to regard the class of ordered sets as rep- resenting P, rather than $\overline{P}$:

<u>Theorem 3</u>. A class of sets with properties (A), (B), and (C) is a representation for a topologically unique ori- ented polyhedral surface (polyhedron) where the sets of the class correspond to its vertices and the distinct elements correspond to its edges. Every oriented polyhedral surface has such a representation.

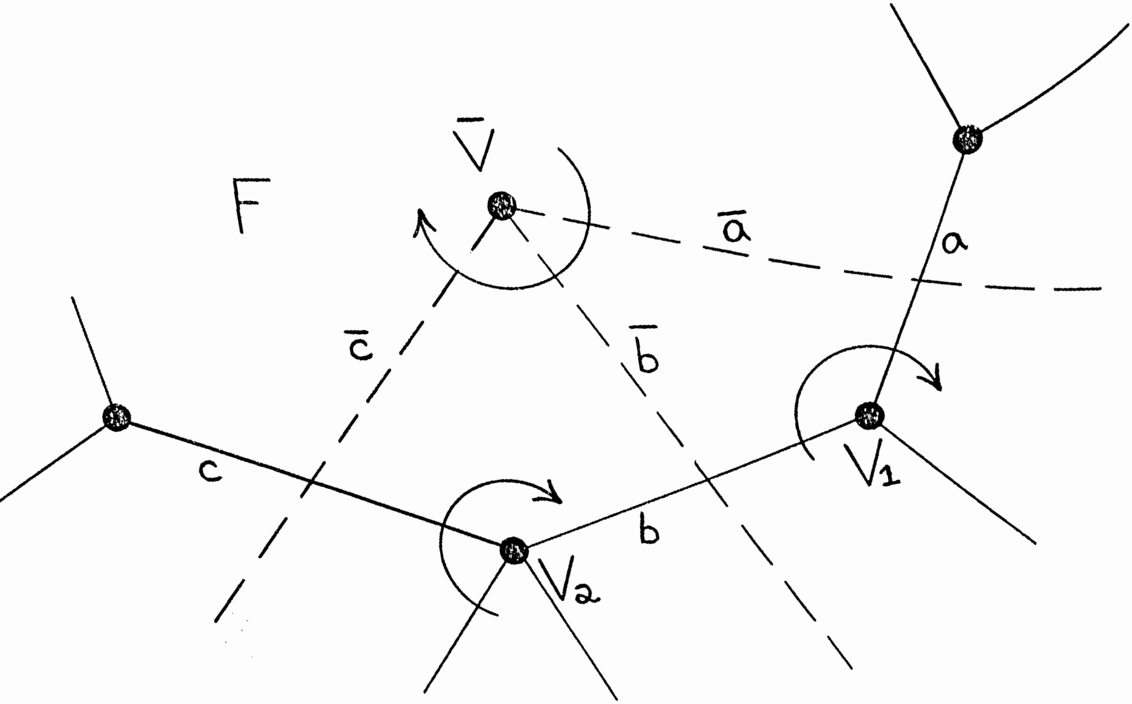

Figure 1.

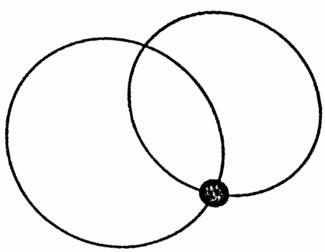

Figure 2.

SECTION IV

In the dual, $\overline{P}$, of polyhedron P there is an edge, $\overline{a}$, corresponding to each edge, a, of P. Edge $\overline{a}$ of $\overline{P}$ is the one which intersects a and joins the (arbitrarily selected) vertices of $\overline{P}$ lying in the cell-faces of P on whose boundary lies a. If a lies twice in the boundary of the same face F of P, i.e., if F lies on both sides of a, then $\overline{a}$ is a closed curve joining to itself the vertex of $\overline{P}$ corresponding to F.

Since P is regarded as having a particular orientation (given by the representation) and since $\overline{P}$ is geometrically constructed on the same surface on which P has some geometric existence, we want $\overline{P}$ as a surface to have the same orientation as P. Therefore, the edges to a vertex $\overline{V}$ of $\overline{P}$ will be arranged clockwise around $\overline{V}$ with the same cyclic order as that with which the corresponding edges are arranged clockwise around the face F of P to which $\overline{V}$ corresponds.

According to Section I, edges a and b of face F of P have endpoints at a common vertex V_1 of F so that a and b are adjacent components of the boundary of F and b follows a clockwise around the boundary if and only if a follows b clockwise around V, i.e., if and only if a is right-adjacent to b in the ordered set V of the representation of P. See figure 1. In this case, $\overline{b}$ is then right-adjacent to $\overline{a}$ in the set $\overline{V}$ of the representation of $\overline{P}$.

Now, similarly, the edge $\bar{c}$ which is right adjacent to $\bar{b}$ in set $\bar{V}$ corresponds to the c following b clockwise around the boundary of F. Edge b follows this c in the clockwise arrangement of edges around the other endpoint V_2 of b, the vertex which separates b and c as components of the boundary of F. That is, c is the left-adjacent edge to the other set occurrence (in set V_2) of b in the representation of P. Therefore, making no verbal distinction between a polyhedron P and its representation, the dual of P may be combinatorially computed by using:

<u>Theorem 4.</u> For a vertex-set occurrence of any edge $\bar{b}$ in the dual $\bar{P}$ of P the edges left-adjacent and right-adjacent to this occurrence are respectively $\bar{a}$ and $\bar{c}$, where edge a is right adjacent to one occurrence of b in P and c is left adjacent to the other occurrence of b in P. The other occurrence of $\bar{b}$ follows the same rule in the only other possible instance of the rule.

We can compute $\bar{P}$ by letting occurrences of $\bar{x}$ in $\bar{P}$ correspond respectively to particular occurrences of x in P. Then to the right of an occurrence of $\bar{a}$ in $\bar{P}$, we place an occurrence of $\bar{b}$ corresponding to the b to the left of the a occurrence other than the one to which this $\bar{a}$ occurrence corresponds.

Example:

P [(adefg)(b)(cdf)(bac)(ge)]

$\bar{P}$ [(abbcfegfd)(dcage)]

The vertices of $\bar{P}$ are combinatorially the faces of P. Since in the example P there are five vertices, two faces and seven edges, Euler's formula, V-E+F, tells us that the characteristic is zero and hence that the polyhedron is a torus. A simpler torus is [(abab)] with one vertex and one face.

[(aa)] is a sphere on which there is one edge whose endpoints are at the same vertex. This polyhedron has two faces, both one-edged. Its dual is [(a)(a)], a sphere on which there is a single edge with separate endpoints.

SECTION V

The class representation of polyhedra provides convenient machinery for their classification according to their mutual homeomorphisms as surfaces. There is a set of simple operations on the representations for getting from one polyhedral representation to another where the polyhedra represented have their geometric existence on homeomorphic surfaces. Using these operations, the polyhedra can be reduced to canonical forms which correspond to the orientable surfaces of various genuses.

Another useful application of the theorems in this paper is to the study of the Cayley diagram of a finite group. For this topic see (3).

Since a self-intersecting imbedding of a graph in a plane also arranges the edges to each vertex in a definite clockwise order around the vertex, any imbedding of any connected graph in a plane determines a topologically unique oriented polyhedral surface. For example figure 2 corresponds to the torus, [(abab)].

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