

## ABSTRACT

Title of dissertation: Essays on Market Microstructure and  
Asset Pricing  
Bo Hu, Doctor of Philosophy, 2019

Dissertation directed by: Professor Albert S. Kyle  
Department of Finance

This dissertation contains three essays that explore various topics in market microstructure and asset pricing. These topics include statistical arbitrage, algorithmic trading, market manipulation, and term-structure modeling.

Chapter 1 studies a model of statistical arbitrage trading in an environment with “fat-tailed” information. I show that if risk-neutral arbitrageurs are uncertain about the variance of fat-tail shocks and if they implement max-min robust optimization, they will choose to ignore a wide range of pricing errors. Although model risk hinders their willingness to trade, arbitrageurs can capture the most profitable opportunities because they follow a linear momentum strategy beyond the inaction zone. This is exactly equivalent to a famous machine-learning algorithm called LASSO. Arbitrageurs can also amass market power due to their conservative trading under this strategy. Their uncoordinated exercise of robust control facilitates tacit collusion, protecting their profits from being competed away even if their number goes to infinity. This work sheds light on how algorithmic trading by arbitrageurs may adversely affect the competitiveness and efficiency of financial markets.

Chapter 2 extends the basic model in Chapter 1 by considering an insider who strategically interacts with a group of algorithmic arbitrageurs who follow machine-learning-type

trading strategies. When market liquidity is good enough, arbitrageurs may be induced to trade too aggressively, giving the insider a reversal trading opportunity. In this case, the insider may play a pump-and-dump strategy to trick those arbitrageurs. This strategy is very similar to those controversial trading practices (such as momentum ignition and stop-loss hunting) in reality. We show that such strategies can largely distort price informativeness and threaten market stability at the expense of common investors. This study reveals a list of economic conditions under which this type of trade-based manipulations are likely to occur. Policy implications are discussed as well.

Chapter 3 provides a simple proof for the long-run pricing kernel decomposition developed by Hansen and Scheinkman (Econometrica, 2009). In a stationary Markovian economy, the long forward rate should be flat so that the pricing kernel can be easily factorized in a multiplicative form of the transitory and permanent components. The permanent (martingale) component plays a key role as it induces the change of probabilities to the long forward measure where the long-maturity discount bond serves as the numeraire. I derive an explicit expression for this martingale component. It reveals a strong restriction on the market prices of risk in a popular approach of interest rates modeling. This approach neglects the permanent martingale component and restricts risk premia in a way undesirable for model calibration. Further analysis demonstrates the advantages of equilibrium modeling of a production economy since it is featured with a path-dependent pricing kernel that has a non-degenerate permanent martingale.

# ESSAYS ON MARKET MICROSTRUCTURE AND ASSET PRICING

by

Bo Hu

Dissertation submitted to the Faculty of the Graduate School of the  
University of Maryland, College Park in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy  
2019

Advisory Committee:

Professor Albert S. Kyle, Chair/Advisor

Professor Mark Loewenstein, Co-Advisor

Professor Steven L Heston

Professor Yajun Wang

Professor Shrihari Santosh

Professor John Chao, Dean's Representative

© Copyright by  
Bo Hu  
2019

## Acknowledgments

This doctoral dissertation would not have been possible without the guidance of my mentors or the support from my family. I would like to express my gratitude to my committee members: Albert Pete Kyle, Mark Loewenstein, Steve Heston, Yajun Wang, Shri Santosh, and John Chao. I also owe a lot to my family, including my wife Wen Chen and my daughter Chloe.

I am tremendously grateful to my advisor Pete Kyle, who is the best scholar and mentor I have ever met. It is Pete who brought me to this fascinating field. His brilliant mind and great personality makes him a magnet. He sets a role model for me what makes an ideal scholar. I feel so fortunate to learn from Pete in the past five years and will keep working with him in the future. I am also deeply grateful to my co-advisor Mark. He is very kind to his students and always willing to help us in research. Mark is also a true scholar and great mentor whom I enjoy interacting with intellectually. Just like Pete, he emphasizes research independence and encourages research novelty. I appreciate all their guidance and support throughout my PhD life.

I am also deeply grateful to Steve. I have been working with him for many years in both teaching and research. Steve has such a charming personality and he has so many life experiences to share with. I always enjoy chatting with him on anything interesting. Despite his sharp mind, he is very nice to make his students feel at ease. Steve gives a lot of stimulating comments on my work and helps me quite a lot to deepen my thinking.

I am really grateful to Yajun. She has given me very strong support ever since I entered this PhD program. Her authentic enthusiasm invites me to work with her. She has such a

rigor mind that also encourages me to take new levels of intellectual challenges. My family is so lucky to have Yajun as a very dear friend and colleague.

I am extremely grateful to Shri who has helped me go through many difficulties in my dissertation. Shri is such a great scholar. I benefit from his suggestions in many aspects, including how to shape research questions, how to write good papers, how to make better presentations. He encouraged me to take challenges in research and inspired me at every stage of my dissertation.

Special thanks goes to Prof. John Chao who taught me many econometric courses. John is my favorite teacher. He gave me many valuable suggestions on my dissertation. I really appreciate his kindness to be on my dissertation committee.

I would like to thank many other professors from our department, including Gurdip Bakshi, Cecilia Bustamante, Julien Cujean, Vojislav Maksimovic, Richmond Mathews, Claudia Moise, Geoffrey Tate, Nagpurnanand Prabhala, Alberto Rossi, Lemma Senbet, Pablo Slutzky, Haluk Unal, Russ Wermers, and Liu Yang. I also thank many friends for their kind help, including Sylvain Delalay, Danmo Lin, Jinming Xue, Wei Zhou, and so on.

Finally, I must thank my parents for their love and support throughout my life. This dissertation is dedicated to them!



# Contents

- 1 Statistical Arbitrage with Uncertain Fat Tails** **1**
- 1.1 Introduction 1
- 1.2 Model 7
- 1.2.1 Model Setup 7
- 1.2.2 Equilibrium Definition 12
- 1.3 Results 16
- 1.3.1 Optimal Strategy without Model Risk 16
- 1.3.2 Robust Strategy under Model Risk 18
- 1.3.3 Equivalent Learning Rule and Alternative Interpretations 24
- 1.3.4 Cartel Effect and Market Inefficiency 28
- 1.4 Conclusion 33
- 1.5 Appendix 34
- 1.5.1 Proof of Proposition 1 34
- 1.5.2 Proof of Proposition 2 35
- 1.5.3 Proof of Proposition 3 40
- 1.5.4 Proof of Proposition 4 and Corollary 4 42

1.5.5	Proof of Corollary 5	44
1.5.6	Proof of Corollary 6	45
<b>2</b>	<b>Strategic Trading with Algorithmic Arbitrageurs</b>	<b>49</b>
2.1	Introduction	49
2.2	Model	52
2.3	Results	57
2.3.1	Equilibrium with Linear-Triggering Strategies	57
2.3.2	Disruptive Strategies and Price Manipulations	63
2.4	Conclusion	67
2.5	Appendix	68
2.5.1	Nonlinear Rational-Expectations Equilibrium	68
2.5.2	Asymptotic Linearity	71
2.5.3	Learning Bias and Strategic Informed Trading	74
2.5.4	Proof of Proposition 6	76
2.5.5	Proof of Corollary 9	82
2.5.6	Economic Conditions for Trade-Based Manipulations	83
<b>3</b>	<b>What if the Long Forward Rate is Flat?</b>	<b>87</b>
3.1	Introduction	87
3.2	Long-Run Pricing Kernel Factorization	92
3.2.1	General Setup and Assumptions	92
3.2.2	Long-run SDF Factorization	94
3.2.3	When will the Long Forward Rate be Flat?	98

3.2.4	How to Find the Long Forward Rate? . . . . .	100
3.3	Long Forward Measure and SDF Decomposition . . . . .	104
3.3.1	The $T$ -Forward Measure . . . . .	104
3.3.2	The Long Forward Measure . . . . .	105
3.3.3	Long-Forward Rate under Bond Price Bubbles . . . . .	109
3.3.4	The Markov Potential Approach for Term Structure Models . . . . .	113
3.4	Path-Dependence of Pricing Kernels . . . . .	119
3.4.1	Ross' Recovery . . . . .	119
3.4.2	Lucas' Tree Model of an Endowment Economy . . . . .	121
3.4.3	Cox-Ingersoll-Ross Equilibrium of a Production Economy . . . . .	123
3.4.4	Extension and Future Research . . . . .	127
3.5	Conclusion . . . . .	129
3.6	Appendix . . . . .	130
3.6.1	Proof of Proposition 7 . . . . .	130
3.6.2	Proof of Lemma 2 . . . . .	131
3.6.3	Proof of Proposition 9 . . . . .	132
3.6.4	Proof of Proposition 10 . . . . .	134
3.6.5	Proof of Corollary 14 . . . . .	135

# Chapter 1

## Statistical Arbitrage with Uncertain Fat Tails

### 1.1 Introduction

In finance, extreme movements of asset prices occur much more frequently than predicted by the tail probabilities of a Gaussian distribution. Such *fat-tail* events have caused many problems, as exemplified by the failure of *Long Term Capital Management*. It is error-prone to predict fat-tail events or to deal with their higher-order statistics. These difficulties give rise to model risk<sup>1</sup> and drive traders to implement robust control. Model risk is a prominent concern for arbitrageurs whose activities are essential for market efficiency. Little is known about how model risk affects arbitrage trading in a fat-tail environment. This topic is both practically relevant and theoretically challenging. Answers to this question can provide new insights into many topics in asset pricing, risk management, and market regulation.

---

<sup>1</sup>Model risk is the risk of loss when traders use the wrong model or deal with uncertain model parameters.

The existence of various anomalies such as momentum suggests that financial markets are not completely efficient<sup>2</sup>. Statistical arbitrage opportunities are also indicative of price inefficiency, because arbitrageurs can make profits given only public information<sup>3</sup>. To study statistical arbitrage trading, I introduce random fat-tail shocks to disrupt the efficient market of a two-period Kyle economy (101). In the standard Kyle model setup, an informed trader privately observes the stock liquidation value and trades sequentially to maximize her profits, under the camouflage of noise traders and against competitive market makers. A Gaussian information structure permits a unique linear equilibrium with an efficient linear pricing rule.

In this chapter, I model the stock value as a random realization drawn from either a Gaussian or a Laplacian distribution, which have the same mean and variance. It is only observed by an informed trader. The choice of a Laplacian distribution is empirically well-grounded<sup>4</sup>. It has fat tails on both sides since its probability density decays exponentially. This mixture setup allows the stock value to be *fat-tailed* with some probability. Market makers believe that they live in the Gaussian world and also regard it as the common belief among all agents. Market makers have the correct prior about the mean, variance, and skewness, but incorrect beliefs about higher moments of the stock value distribution. With Gaussian beliefs, they keep using a linear pricing rule<sup>5</sup>, which can result in estimation bias if fat-tail shocks occur. This invites arbitrageurs to correct pricing errors. By assumption,

---

<sup>2</sup>As documented by Jegadeesh and Titman (89), the momentum strategy could earn abnormal returns.

<sup>3</sup>See Ref. (106), (22), (16), (78), and (61) for discussions.

<sup>4</sup>The Laplace distribution can well characterize the distributions of stock returns sampled at different time horizons. This is documented, for example, by Silva et al. (123).

<sup>5</sup>The empirical price impact function, which measures the average price change in response to the size of an incoming order, is roughly linear with slight concavity. See Ref. (108), (65) [p. 453], (58), and (102).

arbitrageurs are sophisticated enough to distinguish the distribution types (i.e., mispricing cases), but they face uncertainty about the dispersion of Laplace priors. For robust control, arbitrageurs make trading decisions under the criterion of *max-min expected utility*<sup>6</sup>.

My main finding is that model risk can motivate risk-neutral arbitrageurs to implement a machine-learning algorithm which mitigates their competition and ignores many mispricings. This result contains three points that are discussed in greater details below.

First, arbitrageurs' maximin robust strategy has a wide inaction zone: they start trading only when the observed order flow exceeds three standard deviations of noise trading. Yet this strategy is effective in catching the most profitable trades: arbitrageurs trade less than 2% of the time but can capture over 60% of the maximum profits they could earn in the absence of model risk. Under this strategy, arbitrageurs choose to ignore small mispricings. They focus on large events that involve little uncertainty about the trading direction. *Ex post*, an econometrician may find a lot of mispricings that persist in this economy and question arbitrageurs' rationality or capacity. In fact, arbitrageurs are rational and risk-neutral in my setting. They leave money on the table because of their aversion to *uncertainty*.

Second, my work rationalizes a famous machine-learning method widely used in finance. The above-mentioned robust strategy is operationally equivalent to a simple algorithm called the *Least Absolute Shrinkage and Selection Operator* (LASSO)<sup>7</sup>. This is a powerful tool that

---

<sup>6</sup>The theory of max-min expected utility is a standard treatment for ambiguity-averse preferences. It is axiomatized by Gilboa and Schmeidler (63), as a framework for robust decision making under uncertainty. Related discussions can be found in papers by Dow and Werlang (43), Hansen and Sargent (72), and so on.

<sup>7</sup>LASSO is a machine-learning technique developed by Tibshirani (124) to improve prediction accuracy and model interpretability. It is popular among algorithmic traders. This technique has recently been

can select a few key factors from a large set of regression coefficients. The standard statistical interpretation of LASSO involves a different mechanism, namely, the *Maximum a Posteriori* estimate. This learning rule lacks Bayesian rationality because it uses the posterior mode as point estimate, without summarizing all relevant information. In my setup, arbitrageurs are Bayesian-rational when they decide to use LASSO: they evaluate all possible states using Bayes' rule and dynamically maximize a well-defined utility with sequential rationality.

Last, the maximin robust strategy supports tacit collusion and impairs market efficiency. Arbitrageurs trade conservatively beyond the inaction zone. This enables them to accumulate market power, which is most prominent near the kinks of their robust strategy. Thus, uncoordinated exercise of individual robust control facilitates tacit collusion among traders, without any communication device or explicit agreement. Remarkably, even as the number of arbitrageurs goes to infinity, their total profit does not vanish but converges to a finite level. This non-competitive payoff is due to the “cartel” effect which hinders price efficiency.

**Contributions to the literature.** This chapter investigates statistical arbitrage trading in an uncertain fat-tail environment. This topic requires new methods and inspires fresh thinking. Results discussed here can contribute in multiple ways to the vast literature of asset pricing, market microstructure, and behavioral finance.

First, this work develops a new modeling framework for statistical arbitrage. The semi-strong-form market efficiency holds in the standard Kyle (1985) model where traders have common Gaussian beliefs about the economy. This simple assumption has been followed

---

employed in many financial studies, such as Ref. (82), (99), (29), and (56).

by most subsequent studies<sup>8</sup>. The present work deviates from the literature by introducing fat-tail shocks to disrupt the Kyle equilibrium when market makers stick to Gaussian beliefs. Unexpected changes in the underlying distribution cause mispricings in the market. This gives room for arbitrageurs if they can foresee fat-tail shocks. Due to model risk, arbitrageurs are uncertain about the extent of mispricings. If they simply follow the maximin criterion, they may overemphasize the least favorable prior and become overly pessimistic in decision making. I implement a rational procedure that prevents such biases. Similar to the spirit of *rational expectations*, an internally consistent assumption is that arbitrageurs inside this model have the correct belief *on average* about the model structure, despite their uncertainty about some prior parameter. Recognizing this consistency, a rational arbitrageur only considers those strategies that converge to the optimal strategy (as averaged across all possible priors) and that preserve the convexity of their optimal strategy. Such constraints make their admissible strategies comparable to the ideal rational-expectations strategy<sup>9</sup>.

Second, this work is the first to study how market efficiency gets hindered by model risk when arbitrageurs have fat-tail beliefs. This angle distinguishes the present work from the existing literature on *limits to arbitrage*<sup>10</sup>. Previous studies have suggested various important frictions, including short-selling costs, leverage constraints, and wealth effects, which limit arbitrageurs' *ability* to eliminate mispricings. Excluding those frictions, the present work identifies another mechanism that can strongly affect the *willingness* of arbitrageurs to trade. Specifically, model uncertainty of fat-tail priors make arbitrageurs hesitate to eradicate small

---

<sup>8</sup>The literature includes Ref. 6, 79, 53, 54, 128, 7, 129, 85, and 32, among many others.

<sup>9</sup>The rational-expectations strategy is the one that traders would use if they knew the true Laplace prior.

<sup>10</sup>Gromb and Vayanos (66) is an excellent survey on this subject. See also 122, 131, 59, 97, among others.

mispricings, because of ambiguity about the trading direction.

Finally, this work sheds light on interesting topics at the interface of behavioral finance and machine learning. I use the max-min decision rule to rationalize the LASSO (“*soft-thresholding*”) strategy, which was taken by Gabaix (57) as a behavioral assumption of the *anchoring-and-adjustment* mechanism. The LASSO algorithm has an inaction zone where agents choose to ignore whatever happened, similar to the *status quo bias*<sup>11</sup>. The strategy of arbitrageurs also resembles the behavior of feedback traders discussed in behavioral finance<sup>12</sup>. In the eyes of an observer who has a Gaussian prior, arbitrageurs are “irrational” because they show up randomly and all perform feedback trading based on historical prices. The observer’s view is incorrect, given his misspecified prior in this economy.

---

<sup>11</sup>See Ref. 93 and 120.

<sup>12</sup>For behavioral interpretations of feedback traders, see Ref. 40, 12, and 13.

## 1.2 Model

In this section, I present an equilibrium model to study how arbitrageurs' prior uncertainty about mispricing shocks affects their arbitrage trading strategy and the efficiency of financial markets. This model adds random fat-tail shocks to disturb the efficient market of a two-period Kyle model. Pricing errors arise if market makers use the wrong prior, giving room for arbitrageurs to make profits on average.

### 1.2.1 Model Setup

Table 1.1: The timeline and market participants in an economy of two auctions.

	$t = 0$	$t = 1$	$t = 2$	$t = 3$
Informed Trader	observe $v$	submit $x_1$	submit $x_2$	receive $\pi_x$
Noise Traders	...	submit $u_1$	submit $u_2$	...
Arbitrageurs	observe $s$	submit $z_{1,n}$	submit $z_{2,n}$	receive $\pi_{z,n}$
Market Makers	prior $\mathcal{N}(0, \sigma_v^2)$	set $p_1$	set $p_2$	observe $v$

*Structure and Notation.* Consider the market in Table 1.1 with two rounds of trading, indexed by  $t = 1, 2$ . The liquidation value of a risky asset, denoted  $\tilde{v}$ , is either Gaussian or Laplacian:

$$\tilde{v} = (1 - \tilde{s}) \cdot \tilde{v}_G + \tilde{s} \cdot \tilde{v}_L, \quad \text{where} \quad \tilde{v}_G \sim \mathcal{N}(0, \sigma_v^2), \quad \tilde{v}_L \sim \mathcal{L}(0, \xi_v), \quad \xi_v \equiv \frac{\sigma_v}{\sqrt{2}}. \quad (1.1)$$

Here,  $\tilde{s}$  takes the integer value 1 with probability  $\alpha$  and takes the value 0 with probability  $1 - \alpha$ . The true Laplace scale parameter is set to be  $\xi_v = \frac{\sigma_v}{\sqrt{2}}$  so that the variance of  $\tilde{v}$  is always  $\sigma_v^2$ . The initial asset price is set as  $p_0 = 0$  without loss of generality. The quantities traded by noise traders are Gaussian, denoted  $\tilde{u}_1 \sim \mathcal{N}(0, \sigma_u^2)$  and  $\tilde{u}_2 \sim \mathcal{N}(0, \gamma\sigma_u^2)$ . The noise variances can be different, as tuned by the parameter  $\gamma > 0$ . All the random variables  $\tilde{v}$ ,  $\tilde{s}$ ,  $\tilde{u}_1$ , and  $\tilde{u}_2$  are mutually independent. The parameters  $\{\sigma_v, \sigma_u, \gamma\}$  are common knowledge.

A risk-neutral informed trader privately observes  $\tilde{v}$  at  $t = 0$ , submits market orders,  $\tilde{x}_1$  and  $\tilde{x}_2$ , to buy or sell this asset before her private signal becomes public at  $t = 3$ . The strategy is denoted by a vector of real-valued functions,  $\mathbf{X} = \langle X_1, X_2 \rangle$ . Prices and volumes become public information right after the auctions take place. The information sets of informed trader before trading at  $t = 1, 2$  are  $\mathcal{I}_{1,x} = \{\tilde{v}\}$  and  $\mathcal{I}_{2,x} = \{\tilde{v}, \tilde{p}_1\}$  where  $\tilde{p}_1$  is the asset price at  $t = 1$ . It is justified to write  $\tilde{x}_1 = X_1(\tilde{v})$  and  $\tilde{x}_2 = X_2(\tilde{v}, \tilde{p}_1)$ . The informed trader's total profit from both periods can be written as  $\tilde{\pi}_x = \sum_{t=1}^2 (\tilde{v} - \tilde{p}_t) \tilde{x}_t$ .

A number of risk-neutral arbitrageurs (indexed by  $n = 1, \dots, N$ ) observe  $\tilde{s}$ , which encodes the distribution type of  $\tilde{v}$ . Each arbitrageur can place market orders,  $\tilde{z}_{1,n}$  and  $\tilde{z}_{2,n}$ , to exploit potential market inefficiency. Their strategy profile is represented by a matrix of real-valued functions,  $\mathbf{Z} = [\mathbf{Z}_1, \dots, \mathbf{Z}_N]$  where  $\mathbf{Z}_n = \langle Z_{1,n}, Z_{2,n} \rangle$  is the  $n$ -th arbitrageur's strategy for  $n = 1, \dots, N$ . The information sets of arbitrageurs are  $\mathcal{I}_{1,z} = \{\tilde{s}\}$  and  $\mathcal{I}_{2,z} = \{\tilde{s}, \tilde{p}_1\}$  before

their trading at  $t = 1, 2$ . The quantities traded by the  $n$ -th arbitrageur are  $\tilde{z}_{1,n} = Z_{1,n}(\tilde{s})$  and  $\tilde{z}_{2,n} = Z_{2,n}(\tilde{s}, \tilde{p}_1)$ . The total profit for the  $n$ -th trader is denoted  $\tilde{\pi}_{z,n} = \sum_{t=1}^2 (\tilde{v} - \tilde{p}_t) \tilde{z}_{t,n}$ .

Uninformed competitive market makers clear the market by setting prices at which they strive to break even. Their pricing strategy is denoted by the vector of real-valued functions,  $\mathbf{P} = \langle P_1, P_2 \rangle$ . The total order flow  $\tilde{y}_t \equiv \tilde{x}_t + \sum_{n=1}^N \tilde{z}_{t,n} + \tilde{u}_t$  is observed by market makers before they set the price  $\tilde{p}_t$  at period  $t \in \{1, 2\}$ . We can write  $\tilde{p}_1 = P_1(\tilde{y}_1)$  and  $\tilde{p}_2 = P_2(\tilde{y}_1, \tilde{y}_2)$ .

*Belief System.* Several assumptions are needed to clarify traders' beliefs in this model:

**Assumption 1.** *The informed trader and market makers think that it was common belief among all traders that the asset liquidation value was normally distributed,  $\tilde{v} \sim \mathcal{N}(0, \sigma_v^2)$ .*

**Assumption 2.** *Arbitrageurs have the correct Gaussian prior when  $\tilde{s} = 0$ , but they face uncertainty about the variance of fat-tail shocks when  $\tilde{s} = 1$ . Their Laplace prior is modeled as  $\mathcal{L}(0, \tilde{\xi})$  where  $\tilde{\xi} \in \Omega$  is a positive random variable. Arbitrageurs are ambiguity-averse and maximize the minimum expected payoff over all possible priors. On average, arbitrageurs are correct about the information structure, despite their prior uncertainty.*

**Assumption 3.** *Arbitrageurs know that market makers and the informed trader obey Assumption 1. Moreover, Assumption 2 is held as common knowledge among arbitrageurs.*

Since fat-tail shocks occur with probability  $\alpha$  in this market, the higher-order moments of  $\tilde{v}$  can differ from those of the Gaussian counterpart  $\tilde{v}_G$ . When  $\alpha = 0$ , the asset value  $\tilde{v}$  is exactly Gaussian and the model reduces to the standard two-period Kyle model. The Laplace probability density,  $f_L(v) = \frac{1}{2\xi_v} \exp\left(-\frac{|v|}{\xi_v}\right)$ , has fat tails as it decays to zero at

an exponential rate. Thus, the likelihood of observing extreme events under the Laplace distribution is much higher than under the Gaussian distribution with identical variance.

Knowledge of  $\tilde{s}$  is valuable since it tells traders the distribution type of stock value. If market makers have fat-tail beliefs and observe  $\tilde{s} = 1$ , they should use a convex pricing rule (which is rarely seen in real data). The Gaussian prior in Assumption 1 permits linear pricing schedules compatible with empirical observations. Despite its simplicity, the linear pricing function can underestimate the fat-tail information in large order flows. This opens the door to arbitrageurs because market makers have mistakes with probability  $\alpha$ .

Arbitrageurs are sophisticated traders who may use advanced technology to detect mispricings. Their privilege of observing  $\tilde{s}$  represents their superior ability to identify statistical arbitrage opportunities. Nonetheless, arbitrageurs often face uncertainty about their trading models. The failure of *Long-Term Capital Management* (LTCM) demonstrates the critical role of model risk and the disastrous impact when the worst-case scenario hit. This motivates Assumption 2 that arbitrageurs care about the worst-case expected profits for robustness. As proved by Gilboa and Schmeidler (63), the *max-min expected utility* theory rationalizes ambiguity-averse preferences. However, decisions derived from maximin optimization tend to follow the least favorable prior regardless of its likelihood. This appears too pessimistic. A more realistic assumption is that arbitrageurs' admissible strategies converge, in a rational manner<sup>13</sup>, to the average of optimal strategies evaluated across all possible priors. Similar to the concept of *rational expectations*, I assume that arbitrageurs inside this model are correct on average about the model structure. Without systematic bias, the average of optimal

---

<sup>13</sup>To avoid overfitting, their admissible strategy should preserve the convexity of their optimal strategies.

strategies across all possible priors should converge to the *rational-expectations equilibrium* (REE) strategy which corresponds to the ideal case that they know the true prior  $\xi_v$ .

Assumptions 1, 2, and 3 capture salient features of real-life arbitrage. In a nearly efficient market, arbitrage opportunities should be rare and thus overlooked by most market participants. Such opportunities may be identified and exploited by a small number of traders (i.e., arbitrageurs who observe  $\tilde{s}$ ). What may limit their trading is the model risk and their imperfect competition. Arbitrageurs are likely to have similar priors and preferences, given that they have similar forecasting technology and face similar pressures of robust control.

The belief system described in Assumption 1 can be denoted as  $\mathcal{B} = \{\tilde{s} = 0\}$ , which is shared by the informed trader and market makers. They think that it is common knowledge among all traders that  $\tilde{v} \sim \mathcal{N}(0, \sigma_v^2)$ . Arbitrageurs are aware of their Gaussian belief  $\mathcal{B}$ . By Assumptions 2 and 3, the belief system shared by arbitrageurs can be expressed as  $\mathcal{A} = \{\tilde{s}, \tilde{\xi}\}$ , where  $\tilde{\xi}$  denotes the uncertain Laplace prior. Arbitrageurs' belief depends on the observed  $\tilde{s}$  which tells them the type of prior to use:

$$\tilde{v} \sim \mathcal{N}(0, \sigma_v^2) \text{ if } \mathcal{A} = \{\tilde{s} = 0, \tilde{\xi}\} \quad \text{and} \quad \tilde{v} \sim \mathcal{L}(0, \tilde{\xi}) \text{ if } \mathcal{A} = \{\tilde{s} = 1, \tilde{\xi}\}. \quad (1.2)$$

Obviously,  $\mathcal{A}$  and  $\mathcal{B}$  are consistent when  $\tilde{s} = 0$  but they are at odds when  $\tilde{s} = 1$ . Market makers believe that any uninformed trader holds the same Gaussian prior as they do. In fact, arbitrageurs can infer that market makers use the wrong prior when  $\tilde{s} = 1$ .<sup>14</sup>

---

<sup>14</sup>This is not “agreement to disagree” because traders have inconsistent belief structures here. Han and Kyle (67) discussed the situation where traders have inconsistent beliefs about the mean. In my model, traders agree on the mean but hold inconsistent beliefs about higher moments of  $\tilde{v}$ .

## 1.2.2 Equilibrium Definition

The trading of arbitrageurs affects the realized profit of informed trader  $\tilde{\pi}_x$ . To emphasize its dependence on all traders' strategies, we write  $\tilde{\pi}_x = \tilde{\pi}_x(\mathbf{X}, \mathbf{P}, \mathbf{Z})$ . Similarly, each arbitrageur takes into account the strategies played by other traders. To stress such dependence, we write  $\tilde{z}_{t,n} = \tilde{z}_{t,n}(\mathbf{X}, \mathbf{P}, \mathbf{Z})$  and  $\tilde{\pi}_{z,n} = \tilde{\pi}_{z,n}(\mathbf{X}, \mathbf{P}, \mathbf{Z})$  for  $n = 1, \dots, N$ . By Assumption 2, each arbitrageur seeks to maximize the minimum expected profit over all possible priors:

$$\max_{\mathbf{Z}_n \in \mathcal{Z}^2} \min_{\xi \in \Omega} \mathbb{E}^{\mathcal{A}} \left[ \tilde{\pi}_{z,n} \middle| \tilde{s}, \tilde{\xi} = \xi \right] = \max_{\mathbf{Z}_n \in \mathcal{Z}^2} \min_{\xi \in \Omega} \mathbb{E}^{\mathcal{A}} \left[ \sum_{t=1}^2 (\tilde{v} - \tilde{p}_t) z_{t,n} \middle| \tilde{s}, \tilde{\xi} = \xi \right], \quad (1.3)$$

where  $\mathbf{Z}_n = \langle z_{1,n}, z_{2,n} \rangle$ . Both  $z_{1,n}$  and  $z_{2,n}$  are in the admissible set  $\mathcal{Z}$  which requires asymptotic convergence to the REE without losing the convexity/concavity of the REE strategy.

*Definition of Equilibrium.* A sequential trading equilibrium in this model is defined as a tuple of strategies  $(\mathbf{X}, \mathbf{P}, \mathbf{Z})$  such that the following conditions hold:

1. For any alternative strategy  $\mathbf{X}' = \langle X'_1, X'_2 \rangle$  differing from  $\mathbf{X} = \langle X_1, X_2 \rangle$ , the strategy  $\mathbf{X}$  yields an expected total profit no less than  $\mathbf{X}'$ , and also  $X_2$  yields an expected profit in the second period no less than the single deviation  $X'_2$ :

$$\mathbb{E}^{\mathcal{B}}[\tilde{\pi}_x(\mathbf{X}, \mathbf{P}, \mathbf{Z}) | \tilde{v}] \geq \mathbb{E}^{\mathcal{B}}[\tilde{\pi}_x(\mathbf{X}', \mathbf{P}, \mathbf{Z}) | \tilde{v}], \quad (1.4)$$

$$\mathbb{E}^{\mathcal{B}}[(\tilde{v} - \tilde{p}_2(\langle X_1, X_2 \rangle, \mathbf{P}, \mathbf{Z})) X_2 | \tilde{v}, \tilde{p}_1] \geq \mathbb{E}^{\mathcal{B}}[(\tilde{v} - \tilde{p}_2(\langle X_1, X'_2 \rangle, \mathbf{P}, \mathbf{Z})) X'_2 | \tilde{v}, \tilde{p}_1]. \quad (1.5)$$

2. For all  $n = 1, \dots, N$  and any alternative strategy profile  $\mathbf{Z}'$  differing from  $\mathbf{Z}$  only in the  $n$ -th component  $\mathbf{Z}'_n = \langle Z'_{1,n}, Z'_{2,n} \rangle$ , the strategy profile  $\mathbf{Z}$  yields a utility level (i.e., the minimum expected profit over all possible priors) no less than  $\mathbf{Z}'$ , and also  $Z_{2,n}$  yields

a utility level in the second period no less than the single deviation  $Z'_{2,n}$ :

$$\begin{aligned} \min_{\xi \in \Omega} \mathbb{E}^{\mathcal{A}}[\tilde{\pi}_{z,n}(\mathbf{X}, \mathbf{P}, \mathbf{Z}) | \tilde{s}, \tilde{\xi} = \xi] &\geq \min_{\xi \in \Omega} \mathbb{E}^{\mathcal{A}}[\tilde{\pi}_{z,n}(\mathbf{X}, \mathbf{P}, \mathbf{Z}') | \tilde{s}, \tilde{\xi} = \xi]; & (1.6) \\ \min_{\xi \in \Omega} \mathbb{E}^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\cdot, Z_{2,n}))Z_{2,n} | \tilde{s}, \tilde{p}_1, \tilde{\xi} = \xi] &\geq \min_{\xi \in \Omega} \mathbb{E}^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\cdot, Z'_{2,n}))Z'_{2,n} | \tilde{s}, \tilde{p}_1, \tilde{\xi} = \xi]; & (1.7) \end{aligned}$$

where the strategy profile on the right hand side of Eq. (1.7) only differs from  $(\mathbf{X}, \mathbf{P}, \mathbf{Z})$  at  $Z_{2,n}$ . Any strategy considered by arbitrageurs has to be in the admissible set  $\mathcal{Z}$ .

3. The prices,  $\mathbf{P} = \langle P_1, P_2 \rangle$ , are set by the market makers' posterior expectation of  $\tilde{v}$ :

$$\tilde{p}_1 = P_1(\tilde{y}_1) = \mathbb{E}^{\mathcal{B}}[\tilde{v} | \tilde{y}_1], \quad \text{and} \quad \tilde{p}_2 = P_2(\tilde{y}_1, \tilde{y}_2) = \mathbb{E}^{\mathcal{B}}[\tilde{v} | \tilde{y}_1, \tilde{y}_2]. \quad (1.8)$$

*Equilibrium Conjecture.* The full equilibrium  $(\mathbf{X}, \mathbf{P}, \mathbf{Z})$  can be characterized separately. The informed trader and market makers believe that they were living in a two-period Kyle model (Assumption 1). They think that arbitrageurs held the same Gaussian belief and would not trade in a conjectured equilibrium with (semi-strong-form) market efficiency. This inspires them to conjecture a subgame perfect linear equilibrium  $(\mathbf{X}, \mathbf{P})$ .

**Proposition 1.** *Under Assumptions 1, there exists a unique subgame perfect linear equilibrium  $(\mathbf{X}, \mathbf{P})$  identical to the linear equilibrium of a two-period Kyle model with normally distributed random variables. Market makers set the linear pricing rule:*

$$\tilde{p}_1 = P_1(\tilde{y}_1) = \lambda_1 \tilde{y}_1, \quad \tilde{p}_2 = P_2(\tilde{y}_1, \tilde{y}_2) = \tilde{p}_1 + \lambda_2 \tilde{y}_2, \quad \lambda_1 = \frac{\sqrt{2\delta(2\delta - 1)} \sigma_v}{4\delta - 1} \frac{\sigma_v}{\sigma_u}, \quad \lambda_2 = \delta \lambda_1 \quad (1.9)$$

The equilibrium ratio  $\delta = \frac{\lambda_2}{\lambda_1}$  is determined by the largest root to the cubic equation:

$$8\gamma\delta^3 - 4\gamma\delta^2 - 4\delta + 1 = 0. \quad (1.10)$$

The informed trader follows the linear trading strategy:

$$\tilde{x}_1 = X_1(\tilde{v}) = \frac{\tilde{v}}{\rho\lambda_1} = \frac{2\delta - 1}{4\delta - 1} \cdot \frac{\tilde{v}}{\lambda_1}, \quad \tilde{x}_2 = X_2(\tilde{v}, \tilde{y}_1) = \frac{\tilde{v} - \lambda_1 \tilde{y}_1}{2\delta\lambda_1}, \quad (1.11)$$

where  $\rho \equiv \frac{4\delta-1}{2\delta-1}$  is a liquidity-dependent parameter that reflects the trading intensity at  $t = 1$ .

Informed trader and market makers believe that no arbitrageurs would trade under  $(\mathbf{X}, \mathbf{P})$ .

*Proof.* This is an extension of Proposition 1 in Huddart et al. (85). See Appendix 1.5.1.  $\square$

To break even under different liquidity conditions, market makers can adjust the slopes of linear pricing schedules. For example, when noise trading volatility is constant (i.e.,  $\gamma = 1$ ), they can solve from Eq. (1.10) that  $\delta \approx 0.901$ ; when  $\gamma = \frac{3}{4}$ , they can find that  $\delta = 1$  and  $\lambda_1 = \lambda_2 = \frac{\sqrt{2}}{3} \frac{\sigma_v}{\sigma_u}$ ; when liquidity evaporates ( $\gamma \rightarrow 0$ ), the solution explodes:  $\delta \rightarrow \infty$  so that  $\lambda_1 = \frac{\sigma_v}{2\sigma_u}$  and  $\lambda_2 \rightarrow \infty$ . It is convenient to introduce a dimensionless parameter to denote the liquidity condition. Market depth is usually measured by the inverse of price impact parameter. To quantify the change of market depth in the second period, I define

$$\mu \equiv \frac{\lambda_1^{-1} - \lambda_2^{-1}}{\lambda_1^{-1}} = 1 - \frac{1}{\delta}. \quad (1.12)$$

In general,  $\mu \in [-1, 1]$ . For example,  $\mu = 0.5$  indicates a 50% drop of market depth, while  $\mu = 0$  reflects constant depth. Market depth becomes higher (i.e.,  $\mu < 0$ ) if  $\gamma > \frac{3}{4}$ .

If market makers know that  $\tilde{v}$  is drawn from the mixture distribution, the linear pricing rule in Eq. (1.9) can still help them to break even, regardless of the mixture parameter  $\alpha$ . Linear pricing preserves the symmetry of probability distributions so that market makers' unconditional expected profits are zero:  $E[(\tilde{p}_2 - \tilde{v})\tilde{y}_2] = 0$  and  $E[(\tilde{p}_1 - \tilde{v})\tilde{y}_1 + (\tilde{p}_2 - \tilde{v})\tilde{y}_2] = 0$ . This shows the robustness of linear pricing strategy and may explain its popularity.

By Proposition 1, the informed trader and market makers believe that no arbitrageurs would trade in this market. Thus, any strategy profile  $\mathbf{Z}$  does not affect the linear equilibrium strategies  $\mathbf{X}$  and  $\mathbf{P}$ . Arbitrageurs can take Proposition 1 as given when solving their own dynamic optimization problems Eq. (1.6) and Eq. (1.7). Arbitrageurs know that the informed trader and market makers do not anticipate their trading. Arbitrageurs take into account the price impacts of all traders in the market. When  $s = 0$ , the belief structure of all traders is consistent and correct. In this case, arbitrageurs have no advantage over market makers.

**Corollary 1.** *When  $s = 0$ , arbitrageurs do not trade because the market is indeed efficient.*

Arbitrageurs are better “informed” than market makers in the presence of fat-tail shocks. Will they trade immediately? Let us conjecture now and verify later that arbitrageurs would not trade in the first period. This is intuitive given the symmetry of their priors and the linearity of pricing rule. It simplifies the procedure to solve this equilibrium. First, Eq. (1.7) can be used to derive the optimal strategy profile  $\langle Z_{2,1}, \dots, Z_{2,N} \rangle$  in the next period under the conjecture that  $Z_{1,n} = 0$  for all  $n = 1, \dots, N$ . Second, Eq. (1.6) can be used to verify that it is not a profitable deviation for any arbitrageur to trade in the first period. If no one would deviate,  $\mathbf{Z} = [\langle 0, Z_{2,1} \rangle, \dots, \langle 0, Z_{2,N} \rangle]$  will indeed be the equilibrium strategy for arbitrageurs.

## 1.3 Results

### 1.3.1 Optimal Strategy without Model Risk

The linearity of informed trader's strategy  $X_1(v) = \frac{v}{\rho\lambda_1}$  simplifies arbitrageurs' inference. Intuitively, the quantities traded by them in the presence of fat-tail shocks are proportional to their conditional expectation of the stock value mispriced by the market. Of course, the posterior estimate of  $\tilde{v}$  depends on their fat-tail priors. It is helpful to study the ideal case that model risk vanishes. If there is no ambiguity in their prior, arbitrageurs become *subjective expected utility* optimizers, under their Laplace prior  $\mathcal{L}(0, \xi)$  when  $s = 1$ .

**Proposition 2.** *In the absence of model risk, arbitrageurs maximize their expected profits. Over the liquidity regime  $\mu > \mu_\epsilon \approx -0.2319$  where  $\mu_\epsilon$  is the largest root to the cubic equation  $\mu^3 + 21\mu^2 + 35\mu + 7 = 0$ , arbitrageurs do not trade at  $t = 1$  and their optimal strategy at  $t = 2$  is proportional to their posterior expectation of  $\tilde{\theta} = \tilde{v} - p_1$  under the prior  $\mathcal{L}(0, \xi)$ :*

$$Z_{2,n}^o(s, y_1; \xi) = s \frac{1 - \mu}{N + 1} \cdot \frac{\hat{v}(y_1; \xi) - \lambda_1 y_1}{2\lambda_1} = s \frac{1 - \mu}{N + 1} \cdot \frac{\hat{\theta}(y_1; \xi)}{2\lambda_1}, \quad n = 1, \dots, N. \quad (1.13)$$

The estimator  $\hat{v}(y_1; \xi)$  is the posterior mean of  $\tilde{v}$  under the prior that  $\tilde{v}$  is drawn from  $\mathcal{L}(0, \xi)$ :

$$\hat{v} = E^A[\tilde{v} | y_1 = y' \sigma_u, \xi] = \frac{\kappa \xi (y' - \kappa) \operatorname{erfc}\left(\frac{\kappa - y'}{\sqrt{2}}\right)}{\operatorname{erfc}\left(\frac{\kappa - y'}{\sqrt{2}}\right) + e^{2\kappa y'} \operatorname{erfc}\left(\frac{\kappa + y'}{\sqrt{2}}\right)} + \frac{\kappa \xi (y' + \kappa) \operatorname{erfc}\left(\frac{\kappa + y'}{\sqrt{2}}\right)}{\operatorname{erfc}\left(\frac{\kappa + y'}{\sqrt{2}}\right) + e^{-2\kappa y'} \operatorname{erfc}\left(\frac{\kappa - y'}{\sqrt{2}}\right)} \quad (1.14)$$

The rescaled estimator  $\hat{v}/\xi$  is an increasing function of the rescaled quantity  $y' = y_1/\sigma_u$ , with one dimensionless shape parameter,  $\kappa \equiv \frac{\rho\lambda_1\sigma_u}{\xi}$ . The rational-expectations equilibrium (REE) corresponds to the case that their prior is correct, i.e.,  $\xi = \xi_v$ . Under REE,  $\kappa = \frac{2}{\sqrt{1+\mu}}$ .

*Proof.* See Appendix 1.5.2. □

Arbitrageurs only trade when fat-tail shocks occur. In the eyes of some econometrician who holds the Gaussian belief and trusts in market efficiency, those arbitrageurs seem to be “irrational” because they show up randomly and behave like feedback traders. This may raise various behavioral arguments, without recognizing the misspecification of priors.

Arbitrageurs’ prior is symmetric (non-directional) at the beginning. They postpone arbitrage trading until they could tell the trading direction from past price movements, or equivalently, until their posterior beliefs become skewed. Proposition 2 confirms this no-trade conjecture in the first period. It also explains why this study starts from a two-period setup. Even though arbitrageurs are better informed (with the knowledge of  $\tilde{s}$ ) than market makers, their prior expectation of the stock value is identical to market makers’. Arbitrageurs have to watch the market first to see in which direction market makers incur pricing errors. This “wait-and-see” strategy suggests that arbitrage trading can be delayed for learning purposes so that mispricings may sustain for a longer period of time. The mechanism here is different from the delayed arbitrage discussed in the paper by Abreu and Brunnermeier (1) where arbitrageurs face uncertainty about when their peers will exploit a common arbitrage.

The optimal strategy is symmetric with the past order flow:  $Z_{2,n}^o(s, -y_1) = -Z_{2,n}^o(s, y_1)$ . The rescaled strategy,  $Z_{2,n}^o/\sigma_u$ , is a function of the rescaled order flow  $y' = y_1/\sigma_u$  in the fat-tail case. The optimal strategy becomes almost linear at large order flows. Its asymptotic slope is equal to the slope of linear strategy for traders who have a uniform prior ( $\xi \rightarrow \infty$ ). Examination of the first and second derivatives leads to the following statement.

**Corollary 2.** *When  $s = 1$ , the optimal strategy  $Z_{2,n}^o(s, y_1)$  is convex in the positive domain of  $y_1$  and concave otherwise. It is asymptotically linear with a limit slope of  $\frac{1-\mu}{(1+\mu)(N+1)}$ .*

### 1.3.2 Robust Strategy under Model Risk

As indicated by Eq. (1.14), the estimator  $\hat{v}$  depends on the dispersion of Laplace prior,  $\xi$ . How would arbitrageurs trade when they have uncertain priors? Model risk is a critical issue in statistical arbitrage, because using a wrong prior could yield a business disaster like the failure of LTCM. In the real world, traders often face the pressure to test the performance of their strategies in the worst-case scenario. This pressure can drive them to adopt alternative strategies that sacrifice some optimality for robustness.

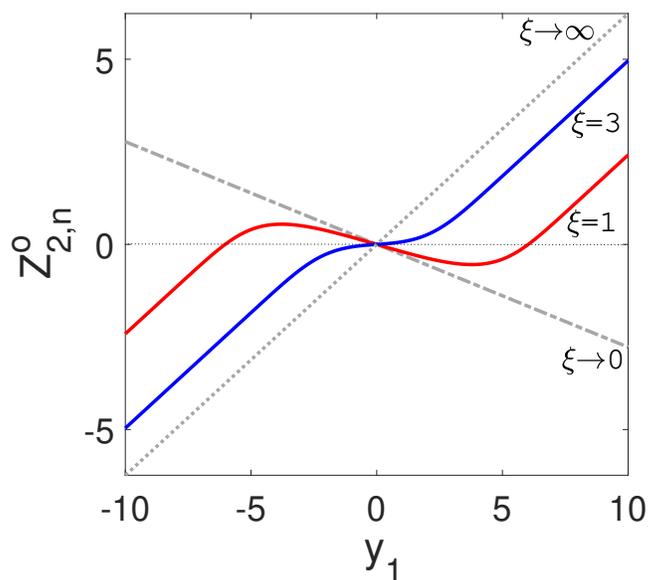


Figure 1.1: The optimal strategy  $Z_{2,n}^o(s = 1, y_1; \xi)$  in Eq. (1.13) under different values of  $\xi$ .

Fig. 1.1 shows the optimal strategy under different values of the Laplace parameter  $\xi$ . An arbitrageur with the prior  $\xi \rightarrow 0$  believes that the stock value is unchanged (i.e.,  $\tilde{v} = 0$ ). This trader will attribute all the order flow  $y_1$  to noise trading and trade against any price change. In contrast, an arbitrageur with the extreme prior  $\xi \rightarrow \infty$  believes that

the past order flow is dominated by informed trading and thus will chase the price trend straightly. For small  $\xi$ , arbitrageurs will engage in contrarian trading on small order flows which are dominated by noise trading under their belief. For large  $\xi$ , arbitrageurs always use a momentum strategy.

Suppose that arbitrageurs' uncertain prior  $\tilde{\xi}$  is in the interval  $[\xi_L, \xi_H]$ , where both the highest and lowest priors,  $\xi_H$  and  $\xi_L$ , have non-zero chances. If the divergence between  $\xi_H$  and  $\xi_L$  is large enough, arbitrageurs can face ambiguity about the trading direction conditional on small order flows<sup>15</sup>: they may want to buy the asset under a high prior (for example,  $\xi = 3$  in Fig. 1.1) but sell it under a low prior (for example,  $\xi = 1$  in Fig. 1.1). If they use the wrong prior, they may trade in the wrong direction and undergo adverse fat-tail shocks.

By Assumption 2, arbitrageurs rank strategies based on the maximin decision criterion, i.e., each arbitrageur maximizes the minimum expected profit over a set of multiple priors. Pure maximin optimization can give very pessimistic decisions which stick to the least favorable prior even if it has a tiny chance to occur. To avoid over-pessimistic responses, I assume that arbitrageurs' admissible strategies converge to the averaged optimal strategy (across all priors) in a rational manner that preserves its convexity and/or concavity. Let's also enforce internal consistency: arbitrageurs inside this model "know" its structure in a statistical sense. On average, they are correct about the economy without systematic bias.

First, it is reasonable and important to invoke the convergence condition. If arbitrageurs observe an extremely large order flow  $y_1$ , they will be pretty sure that  $y_1$  was dominated by informed trading in the fat-tail scenario. This resolves their ambiguity about

---

<sup>15</sup>If the extreme priors satisfy  $y_1 Z_{2,n}^o(s, y_1; \xi_H) > 0$  for any  $y_1 \neq 0$  and  $y_1 Z_{2,n}^o(s, y_1; \xi_L) \leq 0$  for a nonzero measure of  $y_1$ , then different fat-tail priors can give opposite trading directions at small order flows.

trading directions and boosts their confidence to follow the averaged optimal strategy,  $E^A[Z_{2,n}^o(\tilde{s}, y_1; \tilde{\xi}) | \tilde{s} = 1]$ . Let  $Z^\infty$  denote the asymptotes of this averaged strategy. Simple derivation yields

$$Z^\infty(y_1, K_\xi) = \frac{1 - \mu}{1 + \mu} \cdot \frac{y_1 - \text{sign}(y_1)K_\xi}{N + 1}, \quad \text{where } K_\xi = \frac{\lambda_1 \rho^2 \sigma_u^2}{\rho - 1} E^A[\tilde{\xi}^{-1}]. \quad (1.15)$$

To ensure internal consistency, Eq. (1.15) should coincide with the asymptotes of the *rational-expectations equilibrium* (REE) strategy given the true prior  $\xi_v$ . This requires  $E^A[\tilde{\xi}^{-1}] = \xi_v^{-1}$  under which the asymptotes becomes  $Z^\infty(y_1, K^*)$  where

$$K^* = \frac{\lambda_1 \rho^2 \sigma_u^2}{(\rho - 1)} \xi_v^{-1} = \frac{3 + \mu}{\sqrt{1 + \mu}} \sigma_u = \frac{\sqrt{2} \sigma_v}{\lambda_1}. \quad (1.16)$$

The condition  $E^A[\tilde{\xi}^{-1}] = \xi_v^{-1}$  means that arbitrageurs' average belief is correct regarding the *precision* of Laplace prior. Similar to the concept of *rational expectations*, arbitrageurs inside this model make unbiased predictions on average, despite their uncertainty about the model structure. Any candidate strategy should converge to  $Z^\infty(y_1, K^*)$ . This condition ensures that the strategy space of arbitrageurs is anchored to their REE strategy (benchmark).

Second, the admissible strategies should rationally preserve the convexity and/or concavity of the optimal strategy. By Corollary 2, any optimal strategy (without model risk) is convex in the positive domain and concave otherwise (Fig. 1.1). Thus, any candidate strategy must be convex in the regime of  $y_1 > 0$  and concave in the regime of  $y_1 < 0$ . Without this convexity-preserving condition, traders would consider strategies with arbitrarily complex curvatures. This may cause over-fitting problems and make model interpretation difficult.

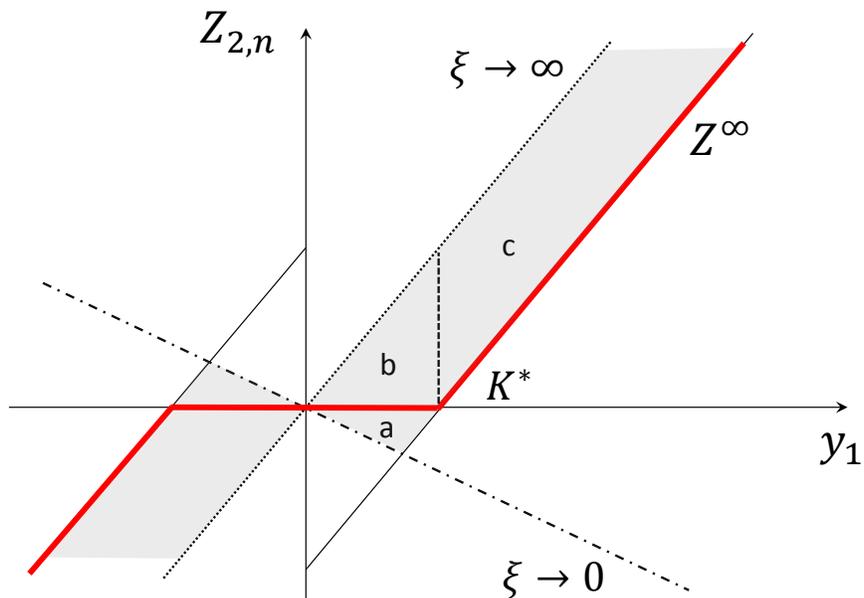


Figure 1.2: The robust strategy  $Z_{2,n}$  in the presence of model risk.

Any strategy that converges to the REE strategy without losing its convex property must lie in the shaded areas of Fig. 1.2. Any strategy running outside this area violates either the convergence condition or the convexity-preserving rule. We can focus on the positive domain and divide the shaded area into three regions. For any  $y_1 \in [0, K^*]$ , arbitrageurs will not sell against  $y_1$ , because they may lose money if the highest prior  $\xi_H$  is true. This rules out any decision point inside the triangle “a”. Similarly, arbitrageurs will not buy the stock since they may also lose money if the lowest prior  $\xi_L$  is true. This rules out any decision point inside the triangle “b”. So the max-min choice criteria indicate a no-trade zone over  $y_1 \in [0, K^*]$ . Next, for any  $y_1 > K^*$ , ambiguity-averse traders should not trade a quantity more than the one prescribed by the REE asymptotes  $Z^\infty(y_1, K^*)$ ; otherwise they may lose in the worst-case scenario. This argument rules out any decision point inside the region “c”. By symmetry, the robust strategy turns out to be a piecewise linear function of  $y_1$ , with the

trading threshold  $K^*$ . This simple strategy is labeled by the red line in Fig. 1.2.

**Proposition 3.** *If arbitrageurs face sufficient model uncertainty about the fat-tail priors and if they follow the max-min choice criteria to rank the admissible strategies defined before, then their robust strategy at  $t = 2$  is a piece-wise linear function of the order flow at  $t = 1$ :*

$$Z_{2,n}(s, y_1; K^*) = sZ^\infty(y_1, K^*)\mathbf{1}_{|y_1| > K^*} = s \frac{1 - \mu}{1 + \mu} \cdot \frac{y_1 - \text{sign}(y_1)K^*}{N + 1} \cdot \mathbf{1}_{|y_1| > K^*}, \quad (1.17)$$

which is along the REE asymptotes with the trading threshold  $K^*$  given by Eq. (1.16).

*Proof.* See Appendix 1.5.3. □

The endogenous decision boundary  $K^*$  is independent of the number of arbitrageurs ( $N$ ) or the variance of asset value ( $\sigma_u^2$ ). For constant noise trading volatility ( $\gamma = 1$ ), one can find  $K^* \approx 3.063\sigma_u$  which is roughly three standard deviations of noise order flows. This indicates a very large inaction zone for the robust strategy. To see how inactive it is, let us examine the unconditional variance of the first-period total order flow,  $\sigma_y^2 = \frac{\sigma_v^2}{(\rho\lambda_1)^2} + \sigma_u^2 = \frac{3+\mu}{2}\sigma_u^2$ , which implies  $K^* \approx 2.5483\sigma_y$ . When the asset value  $\tilde{v}$  is Laplacian, the probability that arbitrageurs get triggered to trade is very small,  $P(|y_1| > K^*) \approx 1.33\%$ . One might think that such a strategy is too inert to be profitable. This is not true. Numerically, the robust strategy can capture about 60% of the maximum profit recouped by the ideal REE strategy. This performance is surprisingly good given the idleness of the robust strategy. Fat-tail shocks create a disproportionate distribution of mispricings. The robust strategy is effective in picking up most profitable trades which correspond to those large fat-tail events.

So far, I have discussed various belief-related reasons for arbitrageurs' inaction. Their no-trade conditions are summarized as follows:

**Corollary 3.** *Arbitrageurs do not trade if any of the following conditions holds:*

- (1) the market is efficient in the semi-strong form under their belief;*
- (2) their prior expectation of  $\tilde{v}$  is identical to market makers' expectation;*
- (3) the past price change cannot drive them out of their inaction (ambiguity) zone.*

*Proof.* Condition (1) holds at  $\tilde{s} = 0$ , Condition (2) holds for their decision making at  $t = 1$ , and Condition (3) is implied by Proposition 3. □

Given their idleness, it may well be the case that arbitrageurs are overlooked by the rest of the market. This is self-consistent with the implication of Assumptions 1, 2, and 3.

More importantly, given their no-trade strategy in the first period and inaction region in the second period, a lot of pricing errors can persist in this market. *Ex post*, an econometrician can run regressions on historical data to discover many mispricings in this economy. The econometrician may question the rationality or capability of arbitrageurs as they apparently leave money on the table. *Ex ante*, arbitrageurs assess all possible states using Bayes' rule. They are risk-neutral but ambiguity averse. For maximin robustness, they rationally ignore small profit opportunities which involve ambiguity about the trading direction. Neither financial constraints nor trading frictions exist here. There is no limit to arbitrageurs' trading ability. It is model risk that reduces their willingness to eliminate mispricings. This intrinsic friction is especially important in the fat-tail world where it leads to a large no-trade zone.

### 1.3.3 Equivalent Learning Rule and Alternative Interpretations

The optimal strategy without model risk uses the posterior *mean* estimate in Bayesian learning (Proposition 2). What is the learning mechanism behind the robust strategy? Arbitrageurs are Bayesian rational when they solve their maximin objectives, Eq. (1.6) and Eq. (1.7). It is noteworthy that the derived (robust) strategy is observationally equivalent to the *Least Absolute Shrinkage and Selection Operator* (LASSO), a famous machine-learning technique developed by Tibshirani (124). The LASSO estimate can be interpreted as the posterior *mode* under independent Laplace prior. In statistics, the posterior *mode* is formally known as the *Maximum a Posteriori* (MAP) estimate. This learning rule itself lacks Bayesian rationality because it does not use all relevant information in forming expectations of unknown variables<sup>16</sup>. Nonetheless, the MAP estimate can “produce” the robust strategy.

**Proposition 4.** *If arbitrageurs know the true Laplace prior  $\xi_v$  but directly use the MAP learning rule to estimate the mispricing signal  $\tilde{\theta} = \tilde{v} - p_1$ , then their strategy in the second period will be operationally equivalent to the robust strategy in Proposition 3:*

$$Z_{2,n}(s = 1, y_1; K^*) = \frac{\hat{\theta}_{map}}{2(N + 1)\lambda_2} = \frac{(\hat{v}_{map} - \lambda_1 y_1)\mathbf{1}_{|y_1| > K^*}}{2(N + 1)\lambda_2}. \quad (1.18)$$

Here,  $\hat{\theta}_{map}$  is the MAP estimate of  $\tilde{\theta}$ . It contains  $\hat{v}_{map}$  which is the MAP estimate of  $\tilde{v}$  under the prior  $\mathcal{L}(0, \xi_v)$ . This is a soft-thresholding function with a threshold  $\kappa\sigma_u = \frac{\rho\lambda_1\sigma_u^2}{\xi_v} = \frac{2\sigma_u}{\sqrt{1+\mu}}$ :

$$\hat{v}_{map}(y_1; \xi_v) = \rho\lambda_1[y_1 - \text{sign}(y_1)\kappa\sigma_u]\mathbf{1}_{|y_1| > \kappa\sigma_u}. \quad (1.19)$$

*Proof.* See Appendix 1.5.4. □

---

<sup>16</sup>The MAP estimate of a variable equals the mode of the posterior distribution. As a point estimate, it does not summarize all relevant information in the posterior distribution.

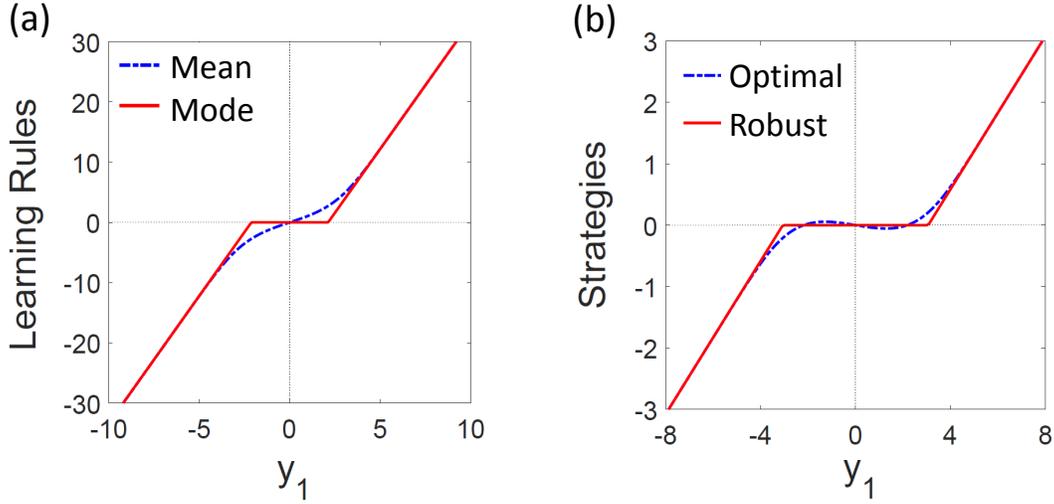


Figure 1.3: (a) The posterior mean versus the posterior mode of  $\tilde{v}$  under the Laplace prior  $\mathcal{L}(0, \xi_v)$ . (b) the optimal (REE) strategy versus the robust strategy at  $t = 2$  when  $s = 1$ .

Fig. 1.3 compares the learning rules and their associated strategies. Both the posterior mean estimate  $\hat{v}$  and the REE strategy  $Z_{2,n}^o(s = 1, y_1; \xi_v)$  are smooth and nonlinear. In contrast, the posterior mode estimate  $\hat{v}_{map}$  is zero for  $y_1 \in [-\kappa\sigma_u, \kappa\sigma_u]$  and linear beyond that zone. The robust strategy  $Z_{2,n}$  has a similar pattern as it performs linear momentum trading beyond the inaction zone  $[-K^*, K^*]$ . Traders who follow this strategy only respond to large events and deliberately ignore small ones. This rational response is similar to various behavioral patterns, including limited attention, status quo bias, anchoring and adjustment, among others<sup>17</sup>. Again, it is worth stressing that arbitrageurs are Bayesian-rational here: they evaluate all possible states using Bayes rule and maximize their well-defined utility with sequential rationality. One can apply Propositions 3 and 4 to rationalize the behavioral assumption of Gabaix (57). In his model, the *soft-thresholding* function like Eq. (1.19) is

<sup>17</sup>Barberis and Thaler (14) provide an excellent survey on those topics in behavioral finance.

used to describe the anchoring bias. Such behavior also permits a rational interpretation.

In a multi-asset economy subject to uncertain fat-tail shocks, Proposition 4 implies that arbitrageurs can directly incorporate the LASSO algorithm into their trading system:

**Corollary 4.** *Suppose that arbitrageurs identify  $M \geq 1$  assets with independent and identically distributed liquidation values,  $\tilde{v}_i \sim \mathcal{L}(0, \xi_v)$  for  $i = 1, \dots, M$ , and each of these assets is traded by a single informed trader in the two-period Kyle model with constant noise trading. For robust learning, arbitrageurs solve the LASSO objective in the Lagrangian form:*

$$\min_{\{v_1, \dots, v_M\}} \sum_{i=1}^M \left| p_{1,i} - \frac{v_i}{\rho} \right|^2 + \frac{2(\lambda_1 \sigma_u)^2}{\xi_v} |v_i|, \quad (1.20)$$

where  $p_{1,i} = \lambda_1 y_{1,i}$  is the price change of the  $i$ -th asset and  $\rho^{-1}$  is the percentage of private signal that has been incorporated into the asset price at  $t = 1$ . This leads to a simple strategy

$$Z_{2,n}(p_{1,i}, \xi_v) = \frac{\rho - 1}{N + 1} \cdot \frac{p_{1,i} \pm 2\xi_v}{2\lambda_2} \cdot \mathbf{1}_{|p_{1,i}| \geq 2\xi_v}, \quad \text{for } i = 1, \dots, M, \quad (1.21)$$

which is automatically triggered to trade the  $i$ -th asset if its price change  $p_{1,i}$  exceeds  $\pm 2\xi_v$ .

*Proof.* See Appendix 1.5.4. □

The objective of maximizing the posterior (under MAP) is equivalent to the minimization problem Eq. (1.20). It involves an  $l^1$  penalty term that comes from the Laplace prior  $\mathcal{L}(0, \xi_v)$ . LASSO shrinks certain estimation coefficients to zero and effectively selects a simpler model that exclude those coefficients. This is a popular tool among quantitative traders because it picks up a small number of key features (factors) from a large set of candidate features. For traders who use LASSO, their trading models shall involve fat-tail (typically Laplace) priors. If traders use the Gaussian prior instead, they will incur an  $l^2$  penalty in

their objective. The resulted algorithm is *ridge regression* which uniformly shrinks the size of all coefficients but does not send any coefficients to zero. Even with parameter uncertainty about the Gaussian prior, traders will not get an inaction zone. This is because signal inference is linear when the posterior is Gaussian. For symmetric unimodal distributions, the mean coincides with the mode; the two learning rules will give identical predictions. Since different Gaussian priors only change the slopes of linear responses, the maximin robust strategy in a pure Gaussian-mixture model will be linear; see Appendix 1.5.4 for more details.

Corollary 4 can help explain the momentum strategy and anomaly in asset pricing<sup>18</sup>. Short-term momentum traders can be viewed as statistical arbitrageurs who have uncertain fat-tail priors about mispriced stocks. Their robust trading is exactly the momentum strategy of buying winners and selling losers. Those traders usually focus on top market gainers and losers, instead of the entire universe of equities. Corollary 4 can also be used to interpret rule-based algorithmic trading which gets triggered at some predefined price levels. At first glance, such trading behavior seems to be mechanical and at odds with Bayesian rationality. It is possible that algorithmic traders are Bayesian-rational. They may use machine-learning techniques (such as LASSO) to manage unknown risks or improve prediction accuracy.

The robust LASSO strategy can also be used by market makers for error self-correction. Market makers can split their pricing logic into two programs. The first one is the linear pricing strategy which allows them to almost break even, despite their occasional mistakes. The second program uses the fat-tail prior to correct the errors of linear pricing strategy,

---

<sup>18</sup>See Ref. 89, 28, 23, 80, 38, 105, among others.

just like the actions of arbitrageurs. This leads to the LASSO algorithm. Integrating both programs, market makers can keep using the linear pricing rule until their inventory exceeds the endogenous thresholds. At that point, they will switch to momentum trading and reduce excessive inventories. The no-trade zone in the second program is the ambiguity zone where they hesitate to correct uncertain pricing errors; this no-trade zone is also their comfortable zone to do market making. This new interpretation differs from conventional arguments that market makers' inventory limits are due to their high risk aversion or large inventory costs.

### 1.3.4 Cartel Effect and Market Inefficiency

Arbitrageurs trade conservatively beyond the endogenous inaction zone. Their conservative trading facilitates their tacit collusion which mitigates their competition and impedes market efficiency. This has interesting implications for limits to arbitrage.

**Proposition 5.** *As  $N \rightarrow \infty$ , the total profit of arbitrageurs vanishes if they use the REE strategy. However, their total profit has a positive limit if they follow the robust strategy.*

*Proof.* If arbitrageurs all follow the optimal REE strategy  $Z_{2,n}^o(s, y_1; \xi_v)$ , they will compete away their total arbitrage profit when  $N$  goes to infinity:

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E}^{\mathcal{A}} \left[ \sum_{n=1}^N (\tilde{v} - \tilde{p}_2) Z_{2,n}^o \right] &= \lim_{N \rightarrow \infty} \mathbb{E}^{\mathcal{A}} \left[ N (\tilde{v} - \lambda_1 \tilde{y}_1 - \lambda_2 X_2(\tilde{v}, \tilde{y}_1) - N \lambda_2 Z_{2,n}^o) Z_{2,n}^o \right] \\
&= \lim_{N \rightarrow \infty} \mathbb{E}^{\mathcal{A}} \left[ \frac{(N+1)(\tilde{v} - \lambda_1 \tilde{y}_1) - N(\hat{v} - \lambda_1 \tilde{y}_1)}{2(N+1)} \cdot \frac{N(\hat{v} - \lambda_1 \tilde{y}_1)}{2(N+1)\lambda_2} \right] \\
&= \lim_{N \rightarrow \infty} \frac{N}{4(N+1)^2 \lambda_2} \mathbb{E}^{\mathcal{A}} [(\hat{v}(\tilde{y}_1) - \lambda_1 \tilde{y}_1)^2] = 0, \tag{1.22}
\end{aligned}$$

where in the above derivation we have used Eq. (1.11) and  $\mathbb{E}^{\mathcal{A}}[\tilde{v}] = \mathbb{E}^{\mathcal{A}}[\mathbb{E}^{\mathcal{A}}[\tilde{v}|\tilde{y}_1]] = \mathbb{E}^{\mathcal{A}}[\hat{v}(\tilde{y}_1)]$ .

In contrast, if arbitrageurs follow the robust strategy  $Z_{2,n}(s, y_1; K^*)$ , their total arbitrage profit will converge to a positive value, indicating a cartel effect:

$$\begin{aligned}
\lim_{N \rightarrow \infty} \mathbb{E}^{\mathcal{A}} \left[ \sum_{n=1}^N (\tilde{v} - \tilde{p}_2) Z_{2,n} \right] &= \lim_{N \rightarrow \infty} \mathbb{E}^{\mathcal{A}} \left[ \frac{(N+1)(\tilde{v} - \lambda_1 \tilde{y}_1) - N \hat{\theta}_{map}}{2(N+1)} \cdot \frac{N \hat{\theta}_{map}}{2(N+1) \lambda_2} \right] \\
&= \lim_{N \rightarrow \infty} \frac{\mathbb{E}^{\mathcal{A}} [N(N+1)(\hat{v} - \hat{v}_{map} + \hat{v}_{map} - \lambda_1 \tilde{y}_1) \hat{\theta}_{map} - N^2 \hat{\theta}_{map}^2]}{4(N+1)^2 \lambda_2} \\
&= \frac{\mathbb{E}^{\mathcal{A}} [(\hat{v} - \hat{v}_{map}) \hat{\theta}_{map}]}{4 \lambda_2} > 0,
\end{aligned} \tag{1.23}$$

where in the above derivation we have used Eq. (1.18) and  $\mathbb{E}^{\mathcal{A}}[\tilde{v}] = \mathbb{E}^{\mathcal{A}}[\hat{v}(\tilde{y}_1)]$ . The expression of the MAP estimate  $\hat{\theta}_{map} \equiv (\hat{v}_{map} - \lambda_1 \tilde{y}_1) \mathbf{1}_{|\tilde{y}_1| > K^*}$  implies  $(\hat{v}_{map} - \lambda_1 \tilde{y}_1) \hat{\theta}_{map} = \hat{\theta}_{map}^2$ . The last expression is strictly positive because  $(\hat{v} - \hat{v}_{map})$  and  $\hat{\theta}_{map}$  has the same sign for  $|\tilde{y}_1| > K^*$ .  $\square$

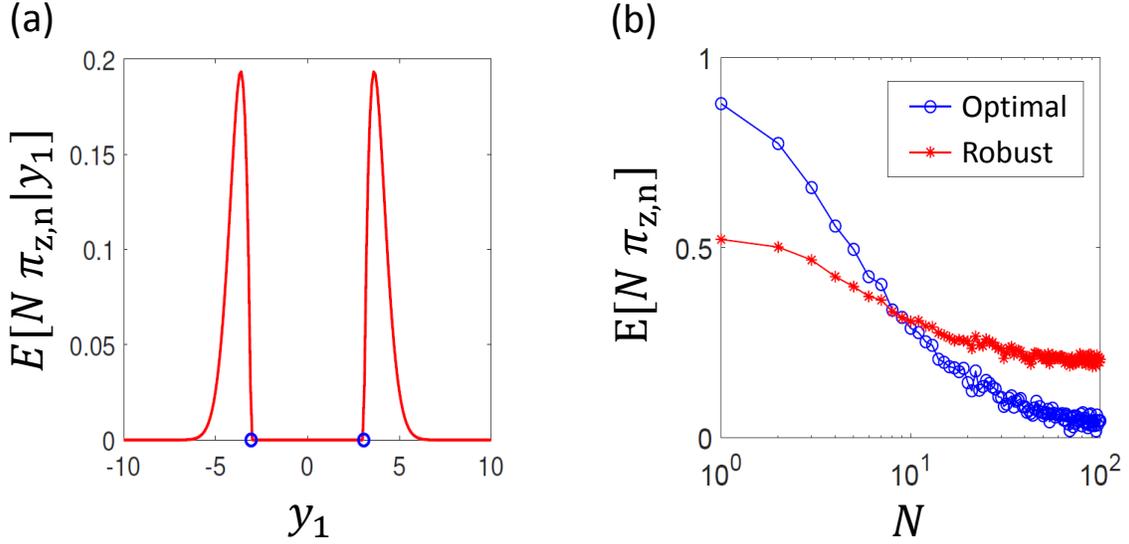


Figure 1.4: (a) The arbitrageurs' total profit under the robust strategy conditional on  $y_1$ . (b) The total arbitrage profit under the REE strategy vs. that under the robust strategy.

Fig. 1.4(a) shows the total profit of a hundred arbitrageurs who follow the robust

strategy, conditional on the observed order flow  $y_1$ . This profit profile (red curve) is proportional to the term  $(\hat{v} - \hat{v}_{map}) \cdot \hat{\theta}_{map}$  in Eq. (1.23). It exhibits two spikes of profits beyond the trading thresholds (labeled by blue circles). These spikes indicate the major source of their extra profits. Intuitively, arbitrageurs' under-trading is most prominent near the "kinks" of their robust strategy. Their non-competitive profits must be strongest there.

Fig. 1.4(b) compares the total payoffs to arbitrageurs when they follow different types of strategies. In the oligopolistic case (i.e., small  $N$ ), the REE strategy allows them to earn higher profits, because the robust strategy ignores a wide range of profit opportunities. As  $N$  increases, the profitability of the REE strategy decays faster. In the competitive limit, arbitrageurs compete away their profits under REE and restore market efficiency at  $t = 2$ .

In contrast, arbitrageurs' total payoff converges to a positive value when they follow the robust strategy [Fig. 1.4(b)]. This confirms Proposition 5 and indicates a non-competitive effect. Their positive limiting payoff is attributable to the market power they amass beyond the inaction zone, where they trade less aggressively than they would do under REE [Fig. 1.3 and Fig. 1.4(a)]. This collusive behavior does not involve any communication device or explicit agreement. Their tacit collusion is not a result of financial constraints or trading frictions. It is due to traders' robust control for (non-Gaussian) model risk. Outside their inaction region, the cartel effect will prevent the market from being fully efficient .

**Corollary 5.** *In the limit  $N \rightarrow \infty$ , arbitrageurs will restore market efficiency when they follow the REE strategy, i.e.,  $\lim_{N \rightarrow \infty} E^A[P_2(\tilde{y}_1, \tilde{y}_2)|\tilde{y}_1] = E^A[\tilde{v}|\tilde{y}_1]$  under  $Z_{2,n}^o(s, y_1; \xi_v)$  for  $n = 1, \dots, N$ ; however, market efficiency is hindered when a finite fraction of arbitrageurs follow the robust strategy, i.e.,  $\lim_{N \rightarrow \infty} E^A[P_2(\tilde{y}_1, \tilde{y}_2)|\tilde{y}_1] \neq E^A[\tilde{v}|\tilde{y}_1]$  under  $Z_{2,n}(s, y_1; K^*)$ .*

*Proof.* See Appendix 1.5.5. □

By Corollary 5, it is difficult to restore market efficiency even if the economy hosts an infinite number of risk-neutral arbitrageurs. To restore price efficiency in the second period, it requires that (almost) every arbitrageur follows the REE strategy, that is, (almost) every arbitrageur knows *on average* the correct fat-tail prior and has no aversion to uncertainty. This is practically impossible because real-life arbitrageurs face different levels of model risks. Moreover, there exist both internal and external pressures that force them to manage such risks. Their robust control easily translates to their ambiguity aversion, which significantly limits their *willingness* to eliminate mispricings. As reviewed by Gromb and Vayanos (66), existing studies mostly focus on different costs that limits arbitrageurs' *ability* in trading. Those frictions could be eased by injecting sufficient capital or removing certain constraints. The mechanism here is different. First, model risk is an intrinsic problem which may not be resolved easily. Second, arbitrageurs here are *able* to eliminate pricing errors; they hesitate to do so because of their aversion to uncertainty<sup>19</sup>. Third, arbitrageurs' hesitation in arbitrage has two characteristics: (1) the large inaction region tells them to leave money on the table; (2) their undertrading beyond the inaction region supports them as a "cartel". Consequently, even with an infinite number of risk-neutral arbitrageurs, a wide range of pricing errors can persist in this economy. This is an endogenous outcome of model risk.

Nowadays, financial markets have been largely occupied by algorithmic traders. The surge of quantitative modeling and machine-learning techniques can bring about hidden

---

<sup>19</sup>Arbitrageurs are risk-neutral but ambiguity-averse in this setup. Their hesitation to perform arbitrage trading is not due to their risk aversion.

issues. The present work demonstrates that statistical arbitrageurs can use machine-learning tools to combat model uncertainty and similar algorithmic “kinks” in their strategy can mitigate their competition at the expense of market efficiency. This is a general implication, given that many machine-learning algorithms have inaction regions and decision “kinks”.

*Equilibrium Condition.* In the liquidity regime  $\mu < 0$ , an arbitrageur may find it profitable to trade in the first period and take advantage of the aggressive feedback trading of other arbitrageurs. One can verify Eq. (1.6) to see whether this unilateral deviation is profitable.

**Corollary 6.** *The conjectured equilibrium strategy profile may fail in the liquidity regime  $\mu < \mu^*(N)$ , where  $\mu^*(N)$  is the largest root that solves  $1 + \frac{N-1}{N+1} \cdot \frac{2}{1+\mu} = \frac{4}{\sqrt{1-\mu}}$ . Given a large number of arbitrageurs using the same robust strategy, it can be profitable for an individual trader to deviate from the conjectured no-trade strategy in the first period. This deviation involves trading a large quantity  $z_1 \gg K^*$  to trigger the other arbitrageurs and then unwinding the position at more favorable prices supported by the over-aggressive trading of others.*

*Proof.* See Appendix 1.5.6

□

## 1.4 Conclusion

This chapter studies an equilibrium model of strategic arbitrage in the fat-tail environment. The presence of arbitrageurs is rationalized by applying random fat-tail shocks to the standard Kyle model where market makers adhere to Gaussian beliefs. If arbitrageurs are uncertain about the various of fat-tail shocks, their robust strategy under the max-min choice criteria is operationally equivalent to the LASSO algorithm in machine learning. For robustness, arbitrageurs choose to ignore a wide range of small (uncertain) mispricings and take actions only on large (certain) ones. This strategy is highly effective given its infrequent trading activity. As a result, many anomalies may be detected *ex post* by an external econometrician based on historical data in this economy. The econometrician may conclude that market inefficiency is due to arbitrageurs' behavioral bias as they overlook those anomalies. In fact, arbitrageurs are rational under their robust-control objective. They use Bayes rule to carefully evaluate all possible states over their multiple priors. Arbitrageurs can amass significant market power due to their under-trading beyond the kinks of robust strategy. This cartel effect allows them to earn noncompetitive profits which do not vanish even if their number goes to infinity. Therefore, price efficiency is further impaired.

## 1.5 Appendix

### 1.5.1 Proof of Proposition 1

Under their common belief, the informed trader and market makers first conjecture that arbitrageurs do not trade if the market is efficient. As in the two-period Kyle model, they can seek a linear equilibrium  $(\mathbf{X}, \mathbf{P})$ , where  $\mathbf{P} = \langle P_1, P_2 \rangle$  is the linear pricing strategy of market makers. Let  $P_1(y_1) = \lambda_1 y_1$  and  $P_2(y_1, y_2) = \lambda_1 y_1 + \lambda_2 y_2$ . The information set of informed trader before trading at  $t = 2$  is  $\mathcal{I}_{2,x} = \{v, y_1\}$ . After  $t = 1$ , she conjectures the price at  $t = 2$  as

$$\tilde{p}_2 = P_2(\tilde{y}_1, \tilde{y}_2) = \lambda_1 y_1 + \lambda_2 [X_2(v, y_1) + \tilde{u}_2], \quad \text{under } \{\mathcal{I}_{2,x}, \mathcal{B}\}. \quad (1.24)$$

Her optimal strategy at  $t = 2$  under the information set  $\mathcal{I}_{2,x}$  and belief system  $\mathcal{B}$  is

$$X_2(v, y_1) = \arg \max_{x_2} \mathbb{E}^{\mathcal{B}} [(v - \tilde{p}_2)x_2 | \mathcal{I}_{2,x}] = \frac{v - \lambda_1 y_1}{2\lambda_2}. \quad (1.25)$$

The informed trader conjectures the price at  $t = 1$  to be  $\tilde{p}_1 = \lambda_1 [X_1(v) + \tilde{u}_1]$  under  $\{\mathcal{I}_{1,x}, \mathcal{B}\}$ .

With this notion and  $X_2(v, y_1)$ , her subjective expected profit is a quadratic function of  $x_1$ :

$$\Pi_x(v, x_1) = x_1(v - \lambda_1 x_1) + \mathbb{E}^{\mathcal{B}} \left[ \frac{(v - \lambda_1(x_1 + \tilde{u}_1))^2}{4\lambda_2} \middle| \mathcal{I}_{1,x} = \{v\} \right]. \quad (1.26)$$

The first order condition is  $0 = v - 2\lambda_1 x_1 - \frac{v - \lambda_1 x_1}{2\delta}$ , where  $\delta \equiv \frac{\lambda_2}{\lambda_1}$ . The optimal strategy is

$$X_1(v) = \frac{2\delta - 1}{4\delta - 1} \cdot \frac{v}{\lambda_1} = \frac{v}{\rho\lambda_1}, \quad (1.27)$$

where  $\rho = \frac{4\delta - 1}{2\delta - 1}$ . The above results constitute Eq. (1.11) in Proposition 1. Market makers hold the same Gaussian belief. As an extension of Proposition 1 in Huddart et al. (85), it takes some similar calculations to derive that  $\lambda_1 = \frac{\sqrt{2\delta(2\delta - 1)}}{4\delta - 1} \frac{\sigma_v}{\sigma_u}$ , where the ratio  $\delta$  is given

by the largest root to the cubic equation:

$$8\gamma\delta^3 - 4\gamma\delta^2 - 4\delta + 1 = 0. \quad (1.28)$$

Here,  $\gamma > 0$  is the ratio of noise trading volatilities over time. Under this pair of linear strategies  $\mathbf{X}$  and  $\mathbf{P}$ , prices are conditional expectations of public information under market makers' belief  $\mathcal{B}$ . So the informed trader and market makers believe that if they play  $\mathbf{X}$  and  $\mathbf{P}$  no arbitrageurs would trade. This confirms the initial conjecture and completes the proof.

### 1.5.2 Proof of Proposition 2

Arbitrageurs know that they are not anticipated to trade by the informed trader and market makers. In the Gaussian case ( $s = 0$ ), they have no informational advantage over market makers. The market is efficient under the subgame perfect equilibrium  $(\mathbf{X}, \mathbf{P})$  in Proposition 1. Indeed, arbitrageurs will not trade when  $s = 0$ . In the Laplacian case ( $s = 1$ ), they can exploit the pricing bias because market makers use the wrong prior. To solve the equilibrium, I conjecture first and verify later that arbitrageurs do not trade in the first period, i.e.,  $Z_{1,n} = 0$  for  $n = 1, \dots, N$ . Under this conjecture, I solve their optimal strategy at  $t = 2$ . Arbitrageurs anticipate the informed trader's linear strategy and the market-clearing price,

$$P_2(\tilde{y}_1, \tilde{y}_2) = \lambda_1 \tilde{y}_1 + \lambda_2 \left( X_2(\tilde{v}, \tilde{y}_1) + \sum_{n=1}^N Z_{2,n}(\tilde{s}, \tilde{y}_1) + \tilde{u}_2 \right). \quad (1.29)$$

They estimate  $\tilde{v}$  based on the observed  $y_1$  and their Laplace prior  $\mathcal{L}(0, \tilde{\xi})$ . In the absence of model risk (i.e.,  $\tilde{\xi} = \xi$ ), the  $n$ -th arbitrageur solves her optimal strategy,

$$Z_{2,n}^o(s = 1, y_1; \xi) = \arg \max_{z_{2,n}} \mathbb{E}^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2)z_{2,n} | \mathcal{I}_{2,z}], \quad (1.30)$$

under  $\mathcal{I}_{2,z} \equiv \{s, y_1\}$  and the belief  $\mathcal{A} = \{s, \xi\}$ . Let  $Z_{2,-n}^o = \sum_{m \neq n} Z_{2,m}^o$  be the their aggregate trading except the  $n$ -th arbitrageur's. The first order condition for  $z_{2,n}$  is

$$\mathbb{E}^{\mathcal{A}}[\tilde{v}|\mathcal{I}_{2,z}] - \lambda_1 y_1 = \lambda_2 \left( \mathbb{E}^{\mathcal{A}}[X_2|\mathcal{I}_{2,z}] + 2z_{2,n} + \mathbb{E}^{\mathcal{A}}[Z_{2,-n}^o|\mathcal{I}_{2,z}] \right). \quad (1.31)$$

Since  $\mathbb{E}^{\mathcal{A}}[X_2|\mathcal{I}_{2,z}] = \frac{\hat{v} - \lambda_1 y_1}{2\delta\lambda_1}$  where  $\hat{v} = \hat{v}(y_1; \xi) = \mathbb{E}^{\mathcal{A}}[\tilde{v}|\mathcal{I}_{2,z}]$ , the solution is

$$Z_{2,n}^o(s = 1, y_1; \xi) = \frac{\hat{v} - \lambda_1 y_1}{2\delta\lambda_1} - \frac{\mathbb{E}^{\mathcal{A}}[X_2|\mathcal{I}_{2,z}] + \mathbb{E}^{\mathcal{A}}[Z_{2,-n}^o|\mathcal{I}_{2,z}]}{2} = \frac{\hat{v} - \lambda_1 y_1}{4\delta\lambda_1} - \frac{\mathbb{E}^{\mathcal{A}}[Z_{2,-n}^o|\mathcal{I}_{2,z}]}{2}. \quad (1.32)$$

The  $n$ -th arbitrageur conjectures that every other arbitrageur solves the same problem and trades  $Z_{2,m}^o = \eta \cdot (\hat{v} - p_1)$  for any  $m \neq n$ , with a coefficient  $\eta$  to be solved. Eq. (1.32) becomes

$$\begin{aligned} Z_{2,n}^o(s = 1, y_1; \xi) &= \frac{\hat{v} - \lambda_1 y_1}{4\delta\lambda_1} - (N-1) \frac{\eta(\hat{v} - \lambda_1 y_1)}{2} \\ &= [\delta^{-1} - 2\lambda_1 \eta(N-1)] \frac{(\hat{v} - \lambda_1 y_1)}{4\lambda_1}. \end{aligned} \quad (1.33)$$

Since each arbitrageur makes the same conjecture in a symmetric equilibrium, they find that

$\eta = \frac{\delta^{-1} - 2\lambda_1 \eta(N-1)}{4\lambda_1}$ , which has a unique solution

$$\eta = \frac{1}{2\delta\lambda_1(N+1)} > 0. \quad (1.34)$$

Without model risk, the optimal strategy of arbitrageurs under the Laplace prior  $\mathcal{L}(0, \xi)$  is

$$Z_{2,n}^o(s, y_1; \xi) = \frac{\hat{v}(y_1; \xi) - \lambda_1 y_1}{2(N+1)\delta\lambda_1} = \frac{1-\mu}{N+1} \cdot \frac{\hat{\theta}(y_1; \xi)}{2\lambda_1}, \quad n = 1, \dots, N. \quad (1.35)$$

Since  $X_1(v) = \frac{v}{\rho\lambda_1}$ , arbitrageurs have a Laplace prior for  $\tilde{x}_1$ , denoted  $f_L(x_1) = \frac{\rho\lambda_1}{2\xi} \exp\left(-\frac{\rho\lambda_1|x_1|}{\xi}\right)$ .

By Bayes' rule, the posterior probability of the informed trading  $x_1$  conditional on  $y_1$  is

$$f(x_1|y_1) = \frac{f(y_1|x_1)f_L(x_1)}{f(y_1)} = \frac{\rho\lambda_1}{2\xi f(y_1) \sqrt{2\pi\sigma_u^2}} \exp\left[-\frac{(y_1 - x_1)^2}{2\sigma_u^2} - \frac{\rho\lambda_1|x_1|}{\xi}\right]. \quad (1.36)$$

The probability density function of  $\tilde{y}_1 = \frac{\tilde{v}}{\rho\lambda_1} + \tilde{u}_1$  is found to be:

$$f(y_1) = \frac{\rho\lambda_1}{4\xi} \exp\left(\frac{\rho^2\lambda_1^2\sigma_u^2}{2\xi^2}\right) \left[ e^{-\frac{\rho\lambda_1 y_1}{\xi}} \operatorname{erfc}\left(\frac{\rho\lambda_1\sigma_u^2/\xi - y_1}{\sqrt{2}\sigma_u}\right) + e^{\frac{\rho\lambda_1 y_1}{\xi}} \operatorname{erfc}\left(\frac{\rho\lambda_1\sigma_u^2/\xi + y_1}{\sqrt{2}\sigma_u}\right) \right]. \quad (1.37)$$

I define a dimensionless parameter  $\kappa \equiv \frac{\rho\lambda_1\sigma_u}{\xi}$  and rewrite  $f(y_1)$  in a dimensionless form

$$f(y_1 = y'\sigma_u) = \frac{\kappa e^{\frac{\kappa^2}{2}}}{4\sigma_u} \left[ e^{-\kappa y'} \operatorname{erfc}\left(\frac{\kappa - y'}{\sqrt{2}}\right) + e^{\kappa y'} \operatorname{erfc}\left(\frac{\kappa + y'}{\sqrt{2}}\right) \right], \quad (1.38)$$

which is symmetric and decays exponentially at large  $|y'|$ . Bayes' rule implies that

$$\mathbb{E}^A[\tilde{x}_1 = x'\sigma_u | y_1 = y'\sigma_u, \xi] = \sigma_u \int_{-\infty}^{\infty} x f(x|y) dx = \sigma_u \int_{-\infty}^{\infty} \frac{x f(y|x) f(x)}{f(y)} dx, \quad (1.39)$$

Given that  $X_1(v) = \frac{v}{\rho\lambda_1}$ , it is easy to derive the posterior expectation of  $\tilde{v}$  explicitly:

$$\begin{aligned} \hat{v} &= \mathbb{E}^A[\tilde{v} | y_1 = y'\sigma_u, \xi] \\ &= \frac{\kappa\xi(y' - \kappa) \operatorname{erfc}\left(\frac{\kappa - y'}{\sqrt{2}}\right)}{\operatorname{erfc}\left(\frac{\kappa - y'}{\sqrt{2}}\right) + e^{2\kappa y'} \operatorname{erfc}\left(\frac{\kappa + y'}{\sqrt{2}}\right)} + \frac{\kappa\xi(y' + \kappa) \operatorname{erfc}\left(\frac{\kappa + y'}{\sqrt{2}}\right)}{\operatorname{erfc}\left(\frac{\kappa + y'}{\sqrt{2}}\right) + e^{-2\kappa y'} \operatorname{erfc}\left(\frac{\kappa - y'}{\sqrt{2}}\right)}. \end{aligned} \quad (1.40)$$

The rescaled  $\hat{v}/\xi$  is an increasing function of  $y'$  with a single shape parameter  $\kappa$ . Asymptotic linearity holds at  $|y'| \gg \kappa$  that  $\hat{v} \rightarrow \rho\lambda_1[y_1 - \operatorname{sign}(y_1)\kappa\sigma_u]$ . All the second-order conditions are easy to check. The REE corresponds to the equilibrium where all arbitrageurs have the correct prior. Under REE, we have  $\xi = \xi_v = \frac{\sigma_v}{\sqrt{2}}$  such that the shape parameter becomes

$$\kappa(\xi = \xi_v) = \frac{\rho\lambda_1\sigma_u}{\xi_v} = \frac{4\delta - 1}{2\delta - 1} \frac{\sqrt{4\delta(2\delta - 1)}}{4\delta - 1} = \frac{2}{\sqrt{1 + \mu}}, \quad (1.41)$$

where  $\mu \equiv 1 - \frac{1}{\delta}$  quantifies the percentage change of market depth in the second period.

To verify that no arbitrageurs would trade in the first period, I examine the condition Eq. (1.6). Suppose the  $n$ -th arbitrageur deviates from the conjectured strategy by trading a nonzero quantity  $Z_{1,n}^{o,d} = z_1 \neq 0$  in the first period. Then the actual total order flow at  $t = 1$  is  $\tilde{y}'_1 = \tilde{x}_1 + \tilde{z}_1 + \tilde{u}_1$ , instead of  $\tilde{y}_1 = \tilde{x}_1 + \tilde{u}_1$  in the conjectured equilibrium. Taking  $\mathbf{X}$ ,  $\mathbf{P}$ , and  $Z_{2,m}^o(s, y'_1; \xi) = s \frac{\hat{v}(y'_1) - \lambda_1 y'_1}{2(N+1)\delta\lambda_1}$  for any  $m \neq n$  as given, the  $n$ -th arbitrageur's optimal strategy at  $t = 2$  conditional on the information set  $\mathcal{I}'_{2,z} = \{s, y_1, z_1\}$  is

$$\begin{aligned}
Z_{2,n}^{o,d}(s, y'_1; \xi) &= s \frac{\hat{v}(y_1; \xi) - \lambda_1 y'_1}{2\delta\lambda_1} - s \frac{\text{E}^A[X_2(\tilde{v}, y'_1)|\mathcal{I}'_{2,z}] + \text{E}^A[Z_{2,-n}^o(s, y'_1; \xi)|\mathcal{I}'_{2,z}]}{2} \\
&= s \frac{\hat{v}(y_1; \xi) - \lambda_1 y'_1}{4\delta\lambda_1} - s \frac{Z_{2,-n}^o(s, y'_1; \xi)}{2} \\
&= s \frac{\hat{v}(y_1; \xi) - \lambda_1 y'_1}{4\delta\lambda_1} - s \frac{(N-1)[\hat{v}(y'_1; \xi) - \lambda_1 y'_1]}{4(N+1)\delta\lambda_1} \\
&= s \frac{\hat{v}(y_1; \xi) - \lambda_1 y'_1}{2(N+1)\lambda_2} + s \frac{N-1}{4(N+1)\lambda_2} [\hat{v}(y_1; \xi) - \hat{v}(y'_1; \xi)]. \tag{1.42}
\end{aligned}$$

If the  $n$ -th trader does not deviate from the no-trade strategy in the first period, her optimal strategy should be  $Z_{2,n}^o(s, y_1; \xi) = s \frac{\hat{v}(y_1; \xi) - \lambda_1 y_1}{2(N+1)\delta\lambda_1}$ , where  $y_1 = x_1 + u_1$ . For convenience, we just need to consider the case  $s = 1$ . Let's add the notation that  $\Delta P_1 \equiv \lambda_1(\tilde{y}'_1 - \tilde{y}_1) = \lambda_1 z_1$ ,  $\Delta \hat{v} \equiv \hat{v}(\tilde{y}'_1; \xi) - \hat{v}(\tilde{y}_1; \xi)$ ,  $\Delta Z \equiv Z_{2,n}^{o,d}(s, y'_1; \xi) - Z_{2,n}^o(s, y_1; \xi) = -\frac{\lambda_1 z_1}{2(N+1)\lambda_2} - \frac{N-1}{4(N+1)\lambda_2} \Delta \hat{v}$  and

$$\begin{aligned}
\Delta P_2 &\equiv P_2(\mathbf{X}, \mathbf{Z}') - P_2(\mathbf{X}, \mathbf{Z}) \\
&= \lambda_1 z_1 + \lambda_2 [\Delta Z + X_2(\tilde{v}, \tilde{y}'_1) - X_2(\tilde{v}, \tilde{y}_1) + Z_{2,-n}^o(\tilde{y}'_1) - Z_{2,-n}^o(\tilde{y}_1)] \\
&= \lambda_1 z - \frac{\lambda_1 z}{2(N+1)} - \frac{N-1}{4(N+1)} \Delta \hat{v} - \frac{\lambda_1 z}{2} + \frac{(N-1)(\Delta \hat{v} - \lambda_1 z)}{2(N+1)} \\
&= \frac{\lambda_1 z_1}{2(N+1)} + \frac{N-1}{4(N+1)} \Delta \hat{v} = -\lambda_2 \Delta Z, \tag{1.43}
\end{aligned}$$

where  $\mathbf{Z}'$  differs from  $\mathbf{Z} \equiv [\langle 0, Z_{2,1}^o \rangle, \dots, \langle 0, Z_{2,N}^o \rangle]$  only in the  $n$ -th element  $(\mathbf{Z}')_n = \langle z_1, Z_{2,n}^{o,d} \rangle$ .

Since  $\tilde{y}_1 = X_1(\tilde{v}) + \tilde{u}_1$ , we have  $E^{\mathcal{A}}[\tilde{y}_1 \cdot z_1] = 0$  and  $E^{\mathcal{A}}[\hat{v}(\tilde{y}_1) \cdot z_1] = 0$ . The payoff difference is

$$\begin{aligned}
\Delta\Pi_{z,n}^d &= E^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\mathbf{X}, \mathbf{Z}'))Z_{2,n}^{o,d} + (\tilde{v} - \tilde{p}_1(\mathbf{X}, \mathbf{Z}'))z_1 - (\tilde{v} - \tilde{p}_2(\mathbf{X}, \mathbf{Z}))Z_{2,n}^o | \tilde{s} = 1, \tilde{\xi} = \xi] \\
&= E^{\mathcal{A}}[\tilde{v}z_1 - z_1\tilde{p}_1(\mathbf{X}, \mathbf{Z}') + \tilde{v}\Delta Z - \Delta P_2 \cdot Z_{2,n}^{o,d} - \tilde{p}_2(\mathbf{X}, \mathbf{Z}) \cdot \Delta Z | \tilde{s} = 1, \tilde{\xi} = \xi] \\
&= -\lambda_1 z_1^2 + E^{\mathcal{A}}[E^{\mathcal{A}}[(\tilde{v} - \tilde{p}_2(\mathbf{X}, \mathbf{Z}) + \lambda_2 Z_{2,n}^{o,d})\Delta Z | \tilde{y}_1]] \\
&= -\lambda_1 z_1^2 + E^{\mathcal{A}}\left[\left(\frac{\hat{v}(\tilde{y}_1; \xi) - \lambda_1 \tilde{y}_1}{N+1} + \lambda_2 \Delta Z\right) \cdot \Delta Z\right] \\
&= -\lambda_1 z_1^2 + \frac{E^{\mathcal{A}}[(\lambda_1 z_1 + \frac{1}{2}(N-1)\Delta\hat{v})^2]}{4(N+1)^2\lambda_2} - \frac{N-1}{4(N+1)^2\lambda_2} E^{\mathcal{A}}[(\hat{v}(\tilde{y}_1; \xi) - \lambda_1 \tilde{y}_1) \cdot \Delta\hat{v}]. \quad (1.44)
\end{aligned}$$

One can rewrite Eq. (1.44) in a symmetric form with respect to  $z_1$ :

$$\begin{aligned}
\Delta\Pi_{z,n}^d &= -\lambda_1 z_1^2 + \frac{E^{\mathcal{A}}[(\lambda_1 z_1)^2 + \frac{1}{4}(N-1)^2(\Delta\hat{v})^2 - (N-1)[\hat{v}(\tilde{y}_1; \xi) - \lambda_1(\tilde{y}_1 + z_1)]\Delta\hat{v}]}{4(N+1)^2\lambda_2} \\
&= -\lambda_1 z_1^2 + \frac{(\lambda_1 z_1)^2 + E^{\mathcal{A}}[\frac{1}{4}(N-1)^2(\Delta\hat{v})^2 - (N-1)(\hat{\theta}(\tilde{y}_1 + z_1; \xi) - \Delta\hat{v})\Delta\hat{v}]}{4(N+1)^2\lambda_2}. \quad (1.45)
\end{aligned}$$

This is an even function of  $z_1$  because one can use the symmetry of  $\tilde{y}_1$  and  $\hat{v}(\cdot)$  to prove

$$\begin{aligned}
E^{\mathcal{A}}[\hat{\theta}(\tilde{y}_1 - z_1; \xi) \cdot \Delta\hat{v}(\tilde{y}_1, -z_1; \xi)] &= E^{\mathcal{A}}[\hat{\theta}(\tilde{y}_1 - z_1; \xi)(\hat{v}(\tilde{y}_1 - z_1; \xi) - \hat{v}(\tilde{y}_1; \xi))] \\
&= E^{\mathcal{A}}[\hat{\theta}(-\tilde{y}_1 - z_1; \xi)(\hat{v}(-\tilde{y}_1 - z_1; \xi) - \hat{v}(-\tilde{y}_1; \xi))] \\
&= E^{\mathcal{A}}[-\hat{\theta}(\tilde{y}_1 + z_1; \xi)(-\hat{v}(\tilde{y}_1 + z_1; \xi) + \hat{v}(\tilde{y}_1; \xi))] = E^{\mathcal{A}}[\hat{\theta}(\tilde{y}_1 + z_1; \xi) \cdot \Delta\hat{v}(\tilde{y}_1, z_1; \xi)].
\end{aligned}$$

The first term of Eq. (1.45) is the average cost to play  $z_1$  at  $t = 1$ , whereas the second term represents the average profit from exploiting the biased response of other traders at  $t = 2$ . The profit of this strategic exploitation has an upper limit which is achieved when all the arbitrageurs have the extreme fat-tail prior  $\xi \rightarrow \infty$ . In this limit, their response to the past order flow is the strongest and exactly linear with  $y_1$ :  $\lim_{\xi \rightarrow \infty} Z_{2,n}^o = \frac{y_1}{(N+1)(2\delta-1)}$ . Since  $\Delta\Pi_{z,n}^d(-z_1) = \Delta\Pi_{z,n}^d(z_1)$ , we only need to consider the positive deviation. For any  $z_1 > 0$ ,

$$\Delta\hat{v}(\tilde{y}_1, z_1; \xi) \equiv \hat{v}(\tilde{y}_1 + z_1; \xi) - \hat{v}(\tilde{y}_1; \xi) \leq \rho\lambda_1(\tilde{y}_1' - \tilde{y}_1) = \rho\lambda_1 z_1, \quad (1.46)$$

where the equality holds at  $\xi \rightarrow \infty$ . Given that  $\lim_{\xi \rightarrow \infty} \hat{\theta}(\tilde{y}'_1; \xi) = \lambda_1(\rho - 1)(\tilde{y}_1 + z_1)$ , I find

$$\begin{aligned} \Delta \Pi_{z,n}^d &< \lim_{\xi \rightarrow \infty} \Delta \Pi_{z,n}^d = -\lambda_1 z_1^2 + \lambda_1^2 \frac{z_1^2 + \frac{1}{4}(N-1)^2 \rho^2 z_1^2 - (N-1)\rho E^A[(\rho-1)(\tilde{y}_1 + z_1) - \rho z_1] z_1}{4(N+1)^2 \lambda_2} \\ &= -\lambda_1 z_1^2 + (1-\mu)\lambda_1 z_1^2 \frac{[(N-1)\rho + 2]^2}{16(N+1)^2} \end{aligned} \quad (1.47)$$

The last expression of Eq. (1.47) is negative for any  $\mu > \mu^*(N)$  where  $\mu^*(N)$  is the largest root to the equation:  $1 + \left(\frac{N-1}{N+1}\right) \frac{2}{1+\mu} = \frac{4}{\sqrt{1-\mu}}$ . The maximum of  $\mu^*(N)$  is found to be  $\mu_\epsilon \equiv \lim_{N \rightarrow \infty} \mu^*(N) \approx -0.23191$ , which is the largest root to the cubic equation:

$$\mu^3 + 21\mu^2 + 35\mu + 7 = 0. \quad (1.48)$$

In the liquidity regime of  $\mu > \mu_\epsilon \approx -0.23191$ , it is indeed unprofitable for any individual arbitrageur to trade in the first period, i.e.,  $\Delta \Pi_{z,n}^d(z_1) < 0$  for any  $z_1 \neq 0$ . This confirms the no-trade conjecture at  $t = 1$  and completes the proof of Proposition 2.

### 1.5.3 Proof of Proposition 3

All admissible strategies must lie in the area enclosed by  $Z_{2,n}^o(y_1, \xi \rightarrow 0)$ ,  $Z_{2,n}^o(y_1, \xi \rightarrow \infty)$ , and the REE asymptotes  $Z^\infty(y_1, K^*)$ . Any strategy that runs outside this region will violate either the asymptotic requirement or the condition of convexity/concavity preservation. By symmetry, we just discuss the positive domain where the REE strategy is always convex. To satisfy the convexity-preservation rule, the first derivative of an admissible strategy,  $\frac{\partial Z_{2,n}^o}{\partial y_1}$ , can never decrease in the domain of  $y_1 > 0$ . With a non-decreasing first derivative, the admissible strategy can never go beyond the asymptote  $Z^\infty(y_1, K^*)$  and curve back to it.

For  $y_1 \in [0, K^*]$ , any selling decision located in the bottom triangle “a” would lose money in the worst-case scenario (i.e., if the highest prior  $\xi_H$  is true, under which one should

buy). Similarly, any buying decision located in the up triangle “b” would lose money in the worst-case scenario (i.e., if the lowest prior  $\xi_L$  is true, under which one should buy). This argument indicates a no-trade strategy over  $y_1 \in [0, K^*]$ . For any  $y_1 > K^*$ , I will prove that any buying decision  $Z'_{2,n}(y_1)$  located inside the area “c” may either lose more money or earn less money than the buying decision  $Z^\infty(y_1, K^*)$  determined by the REE asymptotes. Let  $Z_\Delta \equiv Z'_{2,n}(y_1) - Z^\infty(y_1, K^*)$ . The difference of their payoffs under the lowest prior  $\xi_L$  is

$$\begin{aligned} \mathbb{E}^{\mathcal{A}}[\Delta\tilde{\pi}_{z,n}|y_1, \tilde{\xi} = \xi_L] &= \mathbb{E}^{\mathcal{A}} \left[ (\tilde{v} - \lambda_1 y_1 - \lambda_2 (X_2 + Z'_{2,n} + Z_{2,-n} + \tilde{u}_2)) Z'_{2,n} \middle| y_1, \tilde{\xi} = \xi_L \right] \\ &\quad - \mathbb{E}^{\mathcal{A}} \left[ (\tilde{v} - \lambda_1 y_1 - \lambda_2 (X_2 + Z^\infty + Z_{2,-n} + \tilde{u}_2)) Z^\infty \middle| y_1, \tilde{\xi} = \xi_L \right] \\ &= \mathbb{E}^{\mathcal{A}} \left[ Z_\Delta \left[ \frac{\tilde{\theta}}{2} - \lambda_2 (Z^\infty + Z_{2,-n} + \tilde{u}_2) \right] \middle| y_1, \tilde{\xi} = \xi_L \right] - \lambda_2 Z'_{2,n} Z_\Delta. \end{aligned} \quad (1.49)$$

The worst-case scenario is that  $\xi_L$  is true and every other arbitrageur trades  $Z^\infty(y_1, K^*)$ .

Let  $\hat{\theta}_L(y_1; \xi_L) \equiv \mathbb{E}^{\mathcal{A}}[\tilde{\theta}|y_1, \xi_L]$  and  $Z^L \equiv \frac{\hat{\theta}_L}{2(N+1)\lambda_2}$ . Obviously,  $Z^L < Z^\infty < Z'_{2,n}$  and  $Z_\Delta > 0$ .

It is not a profitable deviation for anyone to trade more than  $Z^\infty(y_1, K^*)$ , since

$$\begin{aligned} \mathbb{E}^{\mathcal{A}}[\Delta\tilde{\pi}_{z,n}|y_1, \tilde{\xi} = \xi_L] &= \lambda_2 Z_\Delta [(N+1)Z^L - Z^\infty - (N-1)Z^\infty] - \lambda_2 Z'_{2,n} Z_\Delta \\ &= \lambda_2 Z_\Delta ((N+1)Z^L - NZ^\infty - Z'_{2,n}) < 0. \end{aligned} \quad (1.50)$$

So the robust strategy is to follow the REE asymptote,  $Z^\infty(y_1, K^*)$ , for any  $y_1 > K^*$ .

By symmetry, the robust strategy is exactly Eq. (1.17). It remains to verify that no arbitrageur would find it profitable to trade in the first period, given that the other arbitrageurs only trade at  $t = 2$  using the same robust strategy. The proof of no-trade condition Eq. (1.6) will be similar to the proof in Proposition 2; see Appendix 1.5.6 for more details.

### 1.5.4 Proof of Proposition 4 and Corollary 4

Under the prior  $\mathcal{L}(0, \xi_v)$ , the Maximum a Posteriori (MAP) estimate of  $\tilde{v}$  given  $y_1$  is

$$\hat{v}_{map} = \arg \max_v f(v|y_1) = \arg \max_v f(y_1|v)f_L(v) = \arg \max_v \exp \left[ -\frac{(y_1 - \frac{v}{\rho\lambda_1})^2}{2\sigma_u^2} - \frac{|v|}{\xi_v} \right], \quad (1.51)$$

We need to find the point of  $v$  that minimizes  $(y_1 - \frac{v}{\rho\lambda_1})^2 + \frac{2\sigma_u^2|v|}{\xi_v}$  whose first order condition is  $y_1 = \frac{v}{\rho\lambda_1} + \kappa\sigma_u \text{sign}(v)$ . Graphically inverting this function  $y_1(v)$  leads to the MAP estimator:

$$\hat{v}_{map}(y_1; \xi_v) = \text{sign}(y_1)\rho\lambda_1 \max[|y_1| - \kappa\sigma_u, 0] = \rho\lambda_1 [y_1 - \text{sign}(y_1)\kappa\sigma_u] \mathbf{1}_{|y_1| > \kappa\sigma_u}, \quad (1.52)$$

which has a learning threshold  $\kappa\sigma_u = \frac{\rho\lambda\sigma_u^2}{\xi_v}$ . Eq. (1.52) is also known as “soft-thresholding” in statistics. This gives a Bayesian interpretation for the LASSO algorithm. LASSO has a similar objective function that involves an  $l^1$  penalty arising from the Laplace prior. The MAP estimate  $\hat{v}_{map}$  is a continuous and piecewise-linear function of  $y_1$ . One can also apply the MAP learning procedure to directly estimate the residual signal  $\tilde{\theta} = \tilde{v} - p_1$ :

$$\hat{\theta}_{map} = \arg \max_{\theta} \exp \left[ -\frac{(y_1 - \frac{\theta + \lambda_1 y_1}{\rho\lambda_1})^2}{2\sigma_u^2} - \frac{|\theta + \lambda_1 y_1|}{\xi_v} \right] = \arg \min_{\theta} \frac{(y_1 - \frac{\theta + \lambda_1 y_1}{\rho\lambda_1})^2}{2\sigma_u^2} + \frac{|\theta + \lambda_1 y_1|}{\xi_v}. \quad (1.53)$$

The first order condition of this objective leads to

$$y_1(\theta) = \frac{\theta}{\rho\lambda_1} + \text{sign}(\theta) \frac{\rho\kappa\sigma_u}{\rho - 1}. \quad (1.54)$$

Graphically inverting the function  $y_1(\theta)$  yields the MAP estimator of  $\tilde{\theta}$ :

$$\hat{\theta}_{map} = (\rho - 1)\lambda_1 [y_1 - \text{sign}(y_1)K^*] \mathbf{1}_{|y_1| > K^*}, \quad \text{where } K^* = \frac{\rho\kappa\sigma_u}{\rho - 1} = \frac{\lambda_1\rho^2\sigma_u^2}{(\rho - 1)\xi_v} = \frac{\sqrt{2}\sigma_v}{\lambda_1}. \quad (1.55)$$

Since  $K^* = \frac{\rho}{\rho-1}\kappa\sigma_u > \kappa\sigma_u$ , one can also write  $\hat{\theta}_{map} = (\hat{v}_{map} - \lambda_1 y_1) \mathbf{1}_{|y_1| > K^*}$ . This establishes an *observational equivalence* to the robust strategy, since we find the following identity

$$Z_{2,n}(s, y_1; K^*) = sZ^\infty(y_1, K^*)\mathbf{1}_{|y_1| > K^*} = s \frac{(\hat{v}_{map} - \lambda_1 y_1) \mathbf{1}_{|y_1| > K^*}}{2(N+1)\lambda_2} = \frac{s \cdot \hat{\theta}_{map}}{2(N+1)\lambda_2}. \quad (1.56)$$

Therefore, if arbitrageurs directly use the MAP rule to estimate the mispricing signal  $\tilde{\theta}$ , they will get the same strategy  $Z_{2,n}(s, y_1; K^*)$ . This MAP rule (posterior mode estimate) differs from the posterior mean  $\hat{v}(y_1; \xi_v)$  which drives the REE strategy  $Z_{2,n}^o(s, y_1; \xi_v)$ .

Proof of Corollary 4: The MAP estimate for each asset value under the prior  $\mathcal{L}(0, \xi_v)$  is:

$$\hat{v}_{i,map} = \arg \max_{v_i} f(v_i | y_{1,i}) = \exp \left[ -\frac{(y_{1,i} - \frac{v_i}{\rho\lambda_1})^2}{2\sigma_u^2} - \frac{|v_i|}{\xi_v} \right] = \arg \min_{v_i} \left| p_{1,i} - \frac{v_i}{\rho} \right|^2 + \frac{2(\lambda_1\sigma_u)^2}{\xi_v} |v_i|, \quad (1.57)$$

which amounts to the LASSO objective in the Lagrangian form for  $i \in \{1, \dots, M\}$ . This leads to the trading algorithm below, which takes the price change  $p_{1,i}$  for each stock as input:

$$Z_{2,n}(p_{1,i}, \xi_v) = \frac{(\rho-1) [\lambda_1 y_{1,i} - \text{sign}(y_{1,i}) \lambda_1 K^*] \mathbf{1}_{|\lambda_1 y_{1,i}| > \lambda_1 K^*}}{2(N+1)\lambda_2} = \frac{\rho-1}{N+1} \cdot \frac{p_{1,i} \pm 2\xi_v}{2\lambda_2} \mathbf{1}_{|p_{1,i}| > 2\xi_v} \quad (1.58)$$

where we have used Eq. (1.16) to derive  $\lambda_1 K^* = \sqrt{2}\sigma_v = 2\xi_v$  given  $\xi_v = \sigma_v/\sqrt{2}$ . Q.E.D.

What if arbitrageurs all adhere to the Gaussian prior? First, they will not trade if their Gaussian prior is identical to market makers' Gaussian prior because they will find out the market is efficient in the semi-strong sense. Arbitrageurs only trade when they have different prior beliefs. Let's model their Gaussian prior as  $\tilde{v} \sim \mathcal{N}(0, \tilde{\zeta}^2)$ , where  $\tilde{\zeta}$  is a random variable reflecting the model uncertainty about the Gaussian prior dispersion. The assumption of prior distribution only changes how arbitrageurs learn from prices without

affecting the informed trader's strategy by Assumption 1. For any specific value of  $\tilde{\zeta} = \zeta$ , the arbitrageurs' posterior belief about  $\tilde{v}$  conditional on  $\tilde{y}_1 = \frac{\tilde{v}}{\rho\lambda_1} + \tilde{u}_1$  is still Gaussian:

$$f(v|y_1) = \frac{f(y_1|v)f_G(v)}{f(y_1)} = \frac{1}{2\pi\zeta\sigma_u f(y_1)} \exp\left[-\frac{(y_1 - v/(\rho\lambda_1))^2}{2\sigma_u^2} - \frac{v^2}{2\zeta^2}\right]. \quad (1.59)$$

Under the Gaussian prior of  $\tilde{v}$ , arbitrageurs believe that  $y_1 = \frac{\tilde{v}}{\rho\lambda_1} + \tilde{u}_1 \sim \mathcal{N}(0, \zeta^2/(\rho\lambda_1)^2 + \sigma_u^2)$  for a given value of  $\zeta$ . By projection theorem, they obtain a linear estimator,

$$\hat{v}(y_1; \zeta) = \mathbb{E}^{\mathcal{N}}[\tilde{v}|y_1, \zeta] = \frac{\zeta^2/(\rho\lambda_1)}{\zeta^2/(\rho\lambda_1)^2 + \sigma_u^2} y_1 = \frac{\rho\lambda_1\zeta^2}{\zeta^2 + (\rho\lambda_1\sigma_u)^2} y_1. \quad (1.60)$$

The mean of a Gaussian distribution is the same as its mode. So the MAP estimate of  $\tilde{v}$  coincides with the posterior mean, i.e.,  $\hat{v}_{map} = \hat{v}$  in this case. The rational strategy for arbitrageurs with Gaussian priors is always a linear function of the order flow  $y_1$ :

$$Z_{2,n}^o(y_1; \zeta) = \frac{1}{N+1} \frac{\hat{v} - \lambda_1 y_1}{2\delta\lambda_1} = \frac{(\rho-1)\zeta^2 - (\rho\lambda_1\sigma_u)^2}{\zeta^2 + (\rho\lambda_1\sigma_u)^2} \cdot \frac{y_1}{2(N+1)}, \quad \text{for } n = 1, \dots, N. \quad (1.61)$$

Any uncertainty about the prior  $\zeta$  only changes the slope of this linear strategy. Therefore, the robust strategy must be linear under the max-min choice criteria.

### 1.5.5 Proof of Corollary 5

If arbitrageurs follow the REE strategy when  $s = 1$ , the price at  $t = 2$  is

$$\tilde{p}_2 = \lambda_1 \tilde{y}_1 + \lambda_2 \left[ X_2 + \sum_{n=1}^N Z_{2,n}^o(s, \tilde{y}_1; \xi_v) + \tilde{u}_2 \right] = \frac{\tilde{v} + \lambda_1 \tilde{y}_1}{2} + \frac{N}{N+1} \frac{\hat{v} - \lambda_1 \tilde{y}_1}{2} + \lambda_2 \tilde{u}_2. \quad (1.62)$$

As  $N \rightarrow \infty$ , the expectation of  $\tilde{p}_2 = P_2(\tilde{y}_1, \tilde{y}_2)$  under arbitrageurs' information and belief is

$$\lim_{N \rightarrow \infty} \mathbb{E}^{\mathcal{A}}[\tilde{p}_2 | \mathcal{I}_{2,z}] = \frac{\hat{v} + \lambda_1 y_1}{2} + \frac{\hat{v} - \lambda_1 y_1}{2} = \hat{v} = \mathbb{E}^{\mathcal{A}}[\tilde{v} | \mathcal{I}_{2,z}]. \quad (1.63)$$

When arbitrageurs use the robust strategy, the price at  $t = 2$  is

$$\tilde{p}_2 = \lambda_1 \tilde{y}_1 + \lambda_2 \left[ X_2 + \sum_{n=1}^N Z_{2,n}(s, \tilde{y}_1; K^*) + \tilde{u}_2 \right] = \frac{\tilde{v} + \lambda_1 \tilde{y}_1}{2} + \frac{N}{N+1} \frac{\hat{v}_{map} - \lambda_1 \tilde{y}_1}{2} \mathbf{1}_{|\tilde{y}_1| > K^*} + \lambda_2 \tilde{u}_2. \quad (1.64)$$

The (ex ante) expected price under arbitrageurs' information and belief has a positive limit:

$$\lim_{N \rightarrow \infty} \mathbb{E}^A[\tilde{p}_2 | \mathcal{I}_{2,z}] = \frac{\hat{v} + \lambda_1 y_1}{2} + \frac{\hat{v}_{map} - \lambda_1 y_1}{2} \mathbf{1}_{|y_1| > K^*} = \frac{\hat{v} + \hat{v}_{map}}{2} - \frac{\hat{v}_{map} - \lambda_1 y_1}{2} \mathbf{1}_{|y_1| > K^*} \neq \hat{v}, \quad (1.65)$$

indicating price inefficiency in the limit of  $N \rightarrow \infty$ .

### 1.5.6 Proof of Corollary 6

If arbitrageurs only trade at  $t = 2$  and follow the robust strategy we derived, each of them may find that the total trading of other arbitrageurs has a response slope greater than one, i.e.,  $\frac{N-1}{N+1} \cdot \frac{1-\mu}{1+\mu} > 1$  if  $-1 < \mu < 0$  and  $N > -\frac{1}{\mu}$ . It may become profitable for any arbitrageur to disrupt the equilibrium by trading a large quantity,  $z_1 \gg K^*$ , in the first period so that the other arbitrageurs will be triggered almost surely. If  $z_1 > \frac{(N-1)(\mu-1)}{2(N\mu+1)} K^*$ , the momentum trading of arbitrageurs at  $t = 2$  can overwhelm the trade  $z_1$ . This may create opportunities for the initial instigator to unwind her position at favorable prices.

Suppose the  $n$ -th arbitrageur (instigator) secretly trades  $z_1 \neq 0$  when  $s = 1$  to trick other traders. Her objective at  $t = 2$  is to maximize the minimum expected profit over all possible priors:  $\max_{z'_{2,n} \in \mathcal{Z}} \min_{\xi \in \Omega} \mathbb{E}^A[(\tilde{v} - \lambda_1 \tilde{y}'_1 - \lambda_2 \tilde{y}'_2) z'_{2,n} | \mathcal{I}_{2,z}]$ , where  $\tilde{y}'_1 = X_1(\tilde{v}) + z_1 + \tilde{u}_1$  and  $\tilde{y}'_2 = X_2(\tilde{v}, \tilde{y}'_1) + z'_{2,n} + Z_{2,-n}(\tilde{y}'_1, K^*) + \tilde{u}_2$ . Here,  $Z_{2,-n} = \sum_{m \neq n} Z_{2,m}(y'_1, K^*) = \frac{(N-1)\hat{\theta}_{map}(y'_1)}{2(N+1)\lambda_2}$  is the total quantity traded by the other arbitrageurs (excluding the  $n$ -th one) who form the estimate of  $\tilde{\theta} = \tilde{v} - \lambda_1 y'_1$  based on  $y'_1$  without knowing that  $y'_1$  contains the secret trade  $z_1$ . The

instigator's estimate,  $\hat{\theta}_{map}(y_1) = [\hat{v}_{map}(y_1) - \lambda_1 y_1] \mathbf{1}_{|y_1| > K^*} = (\rho - 1) \lambda_1 [y_1 - \text{sign}(y_1) K^*] \mathbf{1}_{|y_1| > K^*}$ , is however based on  $y_1 = x_1 + u_1$  instead of  $y'_1$ , because she is aware of the order flow  $z_1$  secretly placed by herself. The strategy of this instigator in the second period reflects how she strategically exploits the other traders' overreaction due to her trade  $z_1$ :

$$\begin{aligned} Z'_{2,n}(y_1, z_1) &= \frac{\hat{v}_{map}(y_1) - \lambda_1 y'_1}{4\lambda_2} \mathbf{1}_{|y_1| > K^*} - \frac{N-1}{4(N+1)\lambda_2} [\hat{v}_{map}(y'_1) - \lambda_1 y'_1] \mathbf{1}_{|y'_1| > K^*} \\ &= \frac{\hat{\theta}_{map}(y_1)}{4\lambda_2} - \frac{z_1}{4\delta} - \frac{(N-1)(\rho-1)(y_1 + z_1 - K^*)}{4(N+1)\delta} \\ &= \frac{\hat{\theta}_{map}(y_1)}{2(N+1)\lambda_2} - \frac{(N+1)z_1 + (N-1)(\rho-1)[z_1 + (y_1 - K^*) \mathbf{1}_{|y_1| < K^*}]}{4(N+1)\delta} \end{aligned} \quad (1.66)$$

where we used the condition  $z_1 \gg K^*$  so that  $\mathbf{1}_{|y'_1 = y_1 + z_1| > K^*} = 1$  with probability arbitrarily close to 1. Her expected total profit is  $\Pi_{z,n}^d = E^A[(\tilde{v} - \lambda_1 \tilde{y}'_1) z_1 + (\tilde{v} - \lambda_1 \tilde{y}'_1 - \lambda_2 \tilde{y}'_2) \cdot Z'_{2,n} | \mathcal{I}_{1,z}]$  and the extra profit attributable to her unilateral deviation  $(z_1, Z'_{2,n})$  is

$$\Delta \Pi_{z,n}^d = \Pi_{z,n}^d - E^A[(\tilde{v} - \lambda_1 \tilde{y}_1 - \lambda_2 \tilde{y}_2) \cdot Z_{2,n} | \tilde{s} = 1], \quad (1.67)$$

where  $\tilde{y}_1 = X_1(\tilde{v}) + \tilde{u}_1$ ,  $\tilde{y}_2 = X_2(\tilde{v}, \tilde{y}_1) + \sum_{n=1}^N Z_{2,n}(\tilde{y}_1, K^*) + \tilde{u}_2$ , and  $Z_{2,n}(\tilde{y}_1, K^*) = \frac{\hat{\theta}_{map}(\tilde{y}_1)}{2(N+1)\lambda_2}$ .

Using the results  $E^A[\tilde{y}_1 \cdot z_1] = 0$ ,  $E^A[\hat{\theta}_{map}(\tilde{y}_1) \cdot z_1] = 0$  and  $\hat{\theta}_{map} \mathbf{1}_{|y_1| < K^*} = 0$ , we derive that

$$\begin{aligned} \Delta \Pi_{z,n}^d &= -\lambda_1 z_1^2 + \lambda_2 E^A[(Z'_{2,n}(\tilde{y}_1, z_1))^2] - \lambda_2 E^A[(Z_{2,n}(\tilde{y}_1, K^*))^2] \\ &= -\lambda_1 z_1^2 + \lambda_2 E^A[(Z'_{2,n}(\tilde{y}_1, z_1) + Z_{2,n}(\tilde{y}_1, K^*))(Z'_{2,n}(\tilde{y}_1, z_1) - Z_{2,n}(\tilde{y}_1, K^*))] \\ &= -\lambda_1 z_1^2 + \lambda_2 \left( \frac{\lambda_1}{\lambda_2} \right)^2 E^A \left[ \left( \frac{(N-1)\rho + 2}{4(N+1)} z_1 + \frac{(N-1)(\rho-1)}{4(N+1)} (\tilde{y}_1 - K^*) \mathbf{1}_{|\tilde{y}_1| < K^*} \right)^2 \right] \\ &= -\lambda_1 z_1^2 + (1-\mu) \lambda_1 z_1^2 \left[ \frac{(N-1)\rho + 2}{4(N+1)} \right]^2 + (1-\mu) \lambda_1 \frac{(N-1)^2 (\rho-1)^2}{16(N+1)^2} E^A[(\tilde{y}_1 - K^*)^2 \mathbf{1}_{|\tilde{y}_1| < K^*}]. \end{aligned}$$

Since  $\delta = \frac{1}{1-\mu}$  and  $\rho = \frac{3+\mu}{1+\mu}$  by definition, the above expression is positive if the coefficient in front of  $z_1^2$  is positive. This is equivalent to the condition:

$$1 + \frac{N-1}{N+1} \cdot \frac{2}{1+\mu} > \frac{4}{\sqrt{1-\mu}}. \quad (1.68)$$

Given any  $N > 1$ , there exists a critical liquidity point  $\mu^*(N)$  below which  $\Delta\Pi_{z,n}^d > 0$ . For example,  $\mu^*(N = 2) \approx -0.68037$ ,  $\mu^*(N = 3) \approx -0.54843$ ,  $\mu^*(N = 10) \approx -0.33525$ ,  $\lim_{N \rightarrow \infty} \mu^* = \mu_\epsilon \approx -0.23191$ . Thus, in the liquidity regime  $\mu < \mu_\epsilon \approx -0.23191$ , if the number of arbitrageurs is large enough, the conjectured equilibrium  $\mathbf{Z} = [\langle 0, Z_{2,1} \rangle, \dots, \langle 0, Z_{2,N} \rangle]$  may fail, because it may permit profitable deviations (or disruptive strategies) at  $t = 1$ .



# Chapter 2

## Strategic Trading with Algorithmic Arbitrageurs

### 2.1 Introduction

In this chapter, I extend the previous model to allow for strategic interaction between the informed trader and the group of arbitrageurs. Considering the nonlinear responses of arbitrageurs, the informed trader will twist his strategy to disrupt their learning. We solve a simplified model where arbitrageurs adhere to linear-triggering strategies and individually choose the optimal trading thresholds. Such “kinks” encourage the informed trader to hide his private signal by trading a quantity almost equal to their equilibrium threshold (under the camouflage of noise trading). This limits the amount of information revealed to his opponents and mitigates the competitive pressure.

The informed trader entices arbitrageurs to mimic past order flows; arbitrageurs’ trend-

following responses also tempt the informed trader to trick them: she may first trade a large quantity to trigger those arbitrageurs and then unwind her position against them. This strategy resembles several controversial schemes in reality. One example is *momentum ignition*, a trading algorithm that attempts to trigger many other algorithmic traders to run in the same direction so that the instigator can profit from trading against the momentum she ignited. Another scheme is *stop-loss hunting* which attempts to force some traders out of their positions by pushing the asset price to certain levels where they have set stop-loss orders. In my setup, this sort of strategies can impair pricing accuracy, exaggerate price volatility, and raise the average trading costs for common investors. Numerical results also generate empirically testable patterns regarding price overreactions and volatility spikes.

This work can have multiple contributions to the literature:

First, it can help us to interpret empirical results about high-frequency traders (HFTs)<sup>1</sup>. My primary model of statistical arbitrage can describe the situation where an informed institutional investor executed large orders over time without anticipating that HFTs detected her footprints to catch the momentum train; see Ref. (107) for a historical account. As an extension, I consider strategic interaction between an informed trader and a group of arbitrageurs. This extended model can describe the situation where institutional investors anticipate those HFTs and optimize their execution algorithms with strategic considerations. My model is consistent with the empirical implications reported in Ref. (126) on HFTs around institutional trading: (1)“*HFTs appear to lean against the wind when an order starts executing but if it executes more than seven hours, they seem to reverse course and trade with wind.*” (2)“*Institutional orders appear mostly information-motivated, in particular*

---

<sup>1</sup>For recent research on high-frequency trading, see Ref. 75, 126, 96, and 98.

*the ones with long-lasting executions that HFTs eventually trade along with.”* (3)“*Investors are privately informed and optimally trade on their signal in full awareness of HFTs preying on the footprint they leave in the market.*”

Second, the extended model contributes to the body of literature on market manipulations<sup>2</sup>. In Allen and Gale (4), a trade-based price manipulation is played by an *uninformed* trader who attempts to trick other traders into believing the existence of informed trading. In my model, the manipulative strategy is performed by an *informed* trader who trades in an unexpected way to distort the learning of other traders. The informed trader may hide her signal when it is strong and bluff when it is weak. In the linear equilibrium of Foster and Viswanathan (53), the better informed trader may also hide her information in early periods and even trade against the direction of her superior signal.<sup>3</sup> My analysis focuses on a nonlinear equilibrium where the informed trader hides her information to reduce competitive pressure from arbitrageurs. Several articles by Chakraborty and Yilmaz<sup>4</sup> show that if market makers face uncertainty about the existence of informed trades, then the informed trader will bluff in every equilibrium by directly adding noise to other traders’ inference problem. The disruptive strategy in my model is different because (1) it occurs under a set of specific conditions, not state-by-state in every equilibrium; (2) it is a pure strategy that distorts the learning of other traders, not a mixed strategy that adds some noise<sup>5</sup>; (3) it produces

---

<sup>2</sup>See Ref. 4, 100, 87, 125, 84, 85, 95, 90, 20, 21, 103, 64, 88, 55, among many others.

<sup>3</sup>The unique linear equilibrium in Kyle (1985) model “*rules out a situation in which the insider can make unbounded profits by first destabilizing prices with unprofitable trades made at the  $n$ th auction, then recouping the losses and much more with profitable trades at future auctions.*”

<sup>4</sup>See Ref. 25, 26, 27.

<sup>5</sup>Mixed strategies are studied in modified Kyle models; see Ref. 85 and 132 for example.

bimodal distributions of prices, thereby magnifying both price volatility and trading costs.

Finally, the disruptive strategy in this chapter shows that asset price “bubbles and crashes” can take place in a strategic environment where speculators have fat-tail beliefs. Under good enough liquidity conditions, a better-informed savvy trader may trade very aggressively to trigger those speculators whose subsequent momentum responses can give this savvy trader a reversal trading opportunity. This finding is related to the literature on market instability<sup>6</sup>. The mechanism here shares some similarity with the model of Scheinkman and Xiong (121) where asset price bubbles reflect resale options due to traders’ overconfidence. In my setup, speculators’ over-aggressive trading implicitly grants the informed trader a “resale option” which could be exercised if condition permits. It is however worth remarking that traders in my (extended) model share a common fat-tail prior, without any overconfidence bias.

## 2.2 Model

In this chapter, I extend the previous model to investigate how strategic interaction between the informed trader and the arbitrageurs affect equilibrium outcomes. This model extension can be interpreted as an institutional informed trader optimizes the dynamic order-execution algorithm by taking in account the responses of algorithmic arbitrageurs who use simple machine-learning strategies to exploit her trades. The extended model can be used, for example, to analyze controversial issues in algorithmic trading. It can shed light on hidden risks when algorithmic traders pervade financial markets. Such risks may account for market

---

<sup>6</sup>See Ref. 104, 2, 81, and 121, among others.

vulnerability and deserve more attention from regulators.

Let us consider a *savvy informed trader* who observes simultaneously the asset value  $\tilde{v}$  and the distribution-type signal  $\tilde{s}$  at the beginning. She anticipates the momentum trading of arbitrageurs and behaves strategically. In the Laplacian case, she will consider how her initial trading affects arbitrageurs' next responses. By backward induction, her expected total profit contains a nonlinear term reflecting her consideration of arbitrageurs' nonlinear inference. As a result, her first-period trading strategy is no longer linear and the *rational-expectations equilibrium* (REE) becomes intractable; more discussions are available in Appendix 2.5.1.

To gain insights, the analysis in this chapter is devoted to a tractable model where strategic arbitrageurs only consider linear-triggering strategies that converge to the REE. This model keeps the basic structure (Table 1.1) elaborated in the previous chapter. I present a set of new assumptions to clarify traders' belief systems and information sets.

**Assumption 4.** *As common knowledge, this market has fixed linear pricing schedules,  $\tilde{p}_1 = P_1(\tilde{y}_1) = \lambda_1 \tilde{y}_1$  and  $\tilde{p}_2 = P_2(\tilde{y}_1, \tilde{y}_2) = \lambda_1 \tilde{y}_1 + \lambda_2 \tilde{y}_2$ , that are exogenously given by Eq. (1.9).*

**Assumption 5.** *Arbitrageurs observe  $\tilde{s}$  and have the correct priors:  $\mathcal{N}(0, \sigma_v^2)$  at  $\tilde{s} = 0$  and  $\mathcal{L}(0, \xi_v)$  at  $\tilde{s} = 1$ . For simplicity, arbitrageurs only consider linear-triggering strategies of the form<sup>7</sup>:  $Z_{2,n}(s = 1, y_1; K_n) = Z^\infty(y_1, \xi_v) \mathbf{1}_{|y_1| > K_n}$ , where  $Z^\infty$  denotes the asymptotes of their REE strategy to be determined in the limit REE. Each arbitrageur chooses the optimal threshold, taking as given the best responses of other arbitrageurs and the informed trader.*

**Assumption 6.** *The risk-neutral informed trader observes both  $\tilde{v}$  and  $\tilde{s}$  at  $t = 0$ . This*

---

<sup>7</sup>Using linear-triggering strategies, arbitrageurs implicitly conjecture that the informed trader's strategy increases with her private signal. However, the Bayesian-rational strategy is not necessarily monotone.

*fact and Assumption 6 are held as common knowledge among the informed trader and arbitrageurs. In other words, the informed trader knows everything known by the arbitrageurs, including their prior belief and their adherence to linear-triggering strategies. Arbitrageurs also know everything known by the informed trader except the private information  $\tilde{v}$ .*

The above assumptions put our focus on the strategic interplay between informed trader and arbitrageurs. The linear pricing rule in Assumption 4 can hold when market makers believe that they are living in the two-period Kyle model with the Gaussian prior  $\tilde{v} \sim \mathcal{N}(0, \sigma_v^2)$ . Arbitrageurs' adherence to linear-triggering strategies in Assumption 5 is motivated by the robust strategy discovered in Section 1.2. If traders worry about the complexity or overtrading of the REE strategy, they may favor such simple algorithms. The suggested linear-triggering strategies are determined by three parameters: slope, intercept, and threshold. These provide well-defined trading rules amenable for computerized executions. Assumption 6 explains the "savviness" of this informed trader who is Bayesian-rational, has correct knowledge about the information structure, and anticipates the strategy space of arbitrageurs.

The timeline of this model is identical to Table 1.1, except that the informed trader observes both  $\tilde{v}$  and  $\tilde{s}$  at  $t = 0$ . The strategies of informed trader and arbitrageurs are denoted by  $\mathbf{X} = \langle X_1, X_2 \rangle$  and  $\mathbf{Z} = [\mathbf{Z}_1, \dots, \mathbf{Z}_N]$ , where  $\mathbf{Z}_n = \langle Z_{1,n}, Z_{2,n} \rangle$  is the  $n$ -th arbitrageur's strategy for  $n = 1, \dots, N$ . The informed trader knows  $\mathcal{I}_{1,x} = \{\tilde{v}, \tilde{s}\}$  before trading at  $t = 1$  and  $\mathcal{I}_{2,x} = \{\tilde{v}, \tilde{s}, \tilde{y}_1\}$  before trading at  $t = 2$ . We can write  $\tilde{x}_1 = X_1(\tilde{v}, \tilde{s})$  and  $\tilde{x}_2 = X_2(\tilde{v}, \tilde{s}, \tilde{y}_1)$ . Given the information sets of arbitrageurs,  $\mathcal{I}_{1,z} = \{\tilde{s}\}$  and  $\mathcal{I}_{2,z} = \{\tilde{s}, \tilde{y}_1\}$ , it is justified to write  $\tilde{z}_{1,n} = Z_{1,n}(\tilde{s})$  and  $\tilde{z}_{2,n} = Z_{2,n}(\tilde{s}, \tilde{y}_1)$  for  $n = 1, \dots, N$ . Let  $\tilde{\pi}_x = \sum_{t=1}^2 (\tilde{v} - \tilde{p}_t) \tilde{x}_t$  be the

informed trader's profit, and  $\tilde{\pi}_{z,n} = \sum_{t=1}^2 (\tilde{v} - \tilde{p}_t) \tilde{z}_{t,n}$  be the  $n$ -th arbitrageur's profit. It is common knowledge that the market-clearing prices are

$$\tilde{p}_1 = P_1(\tilde{y}_1) = \lambda_1 \tilde{y}_1 = \lambda_1 \left( X_1(\tilde{s}, \tilde{v}) + \sum_{n=1}^N Z_{1,n}(\tilde{s}) + \tilde{u}_1 \right), \quad (2.1)$$

$$\tilde{p}_2 = P_2(\tilde{y}_1, \tilde{y}_2) = \tilde{p}_1 + \lambda_2 \tilde{y}_2 = \lambda_1 \tilde{y}_1 + \lambda_2 \left( X_2(\tilde{s}, \tilde{v}, \tilde{y}_1) + \sum_{n=1}^N Z_{2,n}(\tilde{s}, \tilde{y}_1) + \tilde{u}_2 \right). \quad (2.2)$$

To stress the dependence of prices on the strategies of traders, we write  $\tilde{p}_t = \tilde{p}_t(\mathbf{X}, \mathbf{Z})$  for  $t = 1, 2$ . We also write  $\tilde{\pi}_x = \tilde{\pi}_x(\mathbf{X}, \mathbf{Z})$  and  $\tilde{\pi}_{z,n} = \tilde{\pi}_{z,n}(\mathbf{X}, \mathbf{Z})$  because the strategy of informed trader will affect the trading profits of arbitrageurs through direct competition and learning interference, and arbitrageurs' strategies also affect the informed trader's profits through competition and strategic interaction.

In this model, the informed trader and arbitrageurs have the same (consistent) belief system. In particular, they have correct common knowledge about the mixture distribution of  $\tilde{v}$ . Since  $\tilde{s}$  is observed by all of them at  $t = 0$ , the informed trader is aware of the time at which arbitrageurs may trade. However, the informed trader cannot fool arbitrageurs into believing a different type of  $\tilde{v}$ . It is also common knowledge among them that every arbitrageur adheres to the linear-triggering strategy with only one choice variable: the trading threshold.

*Definition of Equilibrium.* The equilibrium here is defined as a pair of strategies  $(\mathbf{X}, \mathbf{Z})$  such that, under the market-clearing prices Eq. (2.1) and Eq. (2.2), the following conditions hold:

1. For any alternative strategy  $\mathbf{X}' = \langle X'_1, X'_2 \rangle$  differing from  $\mathbf{X} = \langle X_1, X_2 \rangle$ , the strategy  $\mathbf{X}$  yields an expected total profit no less than  $\mathbf{X}'$ , and also  $X_2$  yields an expected profit

in the second period no less than any single deviation  $X'_2$ :

$$\mathbb{E}[\tilde{\pi}_x(\mathbf{X}, \mathbf{Z})|\tilde{v}, \tilde{s}] \geq \mathbb{E}[\tilde{\pi}_x(\mathbf{X}', \mathbf{Z})|\tilde{v}, \tilde{s}], \quad (2.3)$$

$$\mathbb{E}[(\tilde{v} - \tilde{p}_2(\langle X_1, X_2 \rangle, \mathbf{Z}))X_2|\tilde{v}, \tilde{s}, \tilde{y}_1] \geq \mathbb{E}[(\tilde{v} - \tilde{p}_2(\langle X_1, X'_2 \rangle, \mathbf{Z}))X'_2|\tilde{v}, \tilde{s}, \tilde{y}_1] \quad (2.4)$$

2. For all  $n = 1, \dots, N$  and for any alternative strategy profile  $\mathbf{Z}'$  differing from  $\mathbf{Z}$  only in the  $n$ -th component  $\mathbf{Z}'_n = \langle Z'_{1,n}, Z'_{2,n} \rangle$ , the strategy  $\mathbf{Z}$  yields an expected profit no less than  $\mathbf{Z}'$ , and also  $Z_{2,n}$  yields an expected profit in the second period no less than  $Z'_{2,n}$ :

$$\mathbb{E}[\tilde{\pi}_{z,n}(\mathbf{X}, \mathbf{Z})|\tilde{s}] \geq \mathbb{E}[\tilde{\pi}_{z,n}(\mathbf{X}, \mathbf{Z}')|\tilde{s}], \quad (2.5)$$

$$\mathbb{E}[(\tilde{v} - \tilde{p}_2(\cdot, Z_{2,n}))Z_{2,n}|\tilde{s}, \tilde{y}_1] \geq \mathbb{E}[(\tilde{v} - \tilde{p}_2(\cdot, Z'_{2,n}))Z'_{2,n}|\tilde{s}, \tilde{y}_1]. \quad (2.6)$$

The strategy profile on the right hand side of Eq. (2.6) only differs from  $(\mathbf{X}, \mathbf{Z})$  at  $Z_{2,n}$ .

In the Gaussian case, the informed trader's strategy remains the same as those in Proposition 1; arbitrageurs find no trading opportunity in this efficient market. To solve the equilibrium in the fat-tail case, it is useful to conjecture first and verify later that arbitrageurs will not trade in the first period. We first solve their second-period optimal strategy under this no-trade conjecture and then check if it is indeed unprofitable for any arbitrageur to trade in the first period. There is another implicit conjecture in the model development. To follow the linear-triggering strategies, arbitrageurs think that the informed trader plays a monotone strategy which increases with her private signal. This needs to be verified too.

## 2.3 Results

### 2.3.1 Equilibrium with Linear-Triggering Strategies

In the fat-tail case, large order flows at  $t = 1$  are mostly attributable to the informed trading. This simplifies the inference problem for arbitrageurs as they can conjecture that the informed trader's strategy is asymptotically linear:

$$X_1(s = 1, v) \rightarrow \frac{v}{\rho\lambda_1} + \text{sign}(v)c\kappa\sigma_u, \quad (2.7)$$

where  $\rho$  and  $c$  are parameters to be determined in the limit equilibrium. The intercept term,  $c\kappa\sigma_u$ , reflects how the informed trader exploits her opponents' learning bias,  $\kappa\sigma_u = \frac{\rho\lambda_1\sigma_u^2}{\xi_v}$ . If Eq. (2.7) holds, the arbitrageurs' estimate of  $\tilde{v}$  will be asymptotically linear with the past order flow. In Appendix 2.5.2, I solve the asymptotic  $X_1(s = 1, v)$  and match the coefficients with Eq. (2.7). This yields two algebraic equations for  $\rho$  and  $c$  whose solutions are given by

$$\rho(\mu, N) = \frac{2 + 5N + N^2 + 2\mu - N\mu - (N + 2)\sqrt{N^2 + (1 + \mu)^2 + 2N(3\mu - 1)}}{2N(1 - \mu)}, \quad (2.8)$$

$$c(\mu, N) = -\frac{3 + N - \mu - \sqrt{N^2 + (1 + \mu)^2 + 2N(3\mu - 1)}}{1 + N + \mu + \sqrt{N^2 + (1 + \mu)^2 + 2N(3\mu - 1)}} \cdot \frac{N}{2}. \quad (2.9)$$

Here, the parameter  $\rho$  decreases with  $\mu$  and  $N$ , because poorer liquidity or higher competitive pressure tomorrow can stimulate more aggressive informed trading today. The parameter  $c$  increases (with  $\mu$ ) from  $-1$  to  $0$ , because poor future liquidity tends to discourage strategic actions; as shown in Appendix 2.5.3, this parameter reflects the extent of how the informed trader strategically exploits the estimation bias of arbitrageurs. These two parameters can determine the REE asymptotes,  $Z^\infty$ , which helps us to pin down the following equilibrium.

**Proposition 6.** *In the liquidity regime of  $\mu > \mu_\epsilon$  where  $\mu_\epsilon \approx 0.005$  according to numerical results, the following equilibrium  $(\mathbf{X}, \mathbf{Z})$  holds. First, arbitrageurs do not trade in the first period, i.e.,  $Z_{1,n} = 0$  for  $n = 1, \dots, N$ . Their optimal linear-triggering strategy at  $t = 2$  is*

$$Z_{2,n}(s, y_1; K^*) = sZ^\infty(y_1, \xi_v)\mathbf{1}_{|y_1| > K^*} = s \frac{(1-\mu)(\rho-1)}{N+2} \left[ y_1 - \text{sign}(y_1) \frac{\rho(1+c)\kappa\sigma_u}{\rho-1} \right] \mathbf{1}_{|y_1| > K^*}, \quad (2.10)$$

$$K^*(\mu, N) = \max \left[ \kappa\sigma_u, \frac{\rho(1+c)\kappa\sigma_u}{\rho-1} \right] = \sigma_u \frac{2\sqrt{1+\mu}}{3+\mu} \max \left[ \rho, \frac{\rho^2(1+c)}{\rho-1} \right]. \quad (2.11)$$

For the informed trader, the equilibrium strategy at  $t = 2$  is to trade

$$X_2(v, s, y_1; K^*) = (1-\mu) \frac{v - \lambda_1 y_1}{2\lambda_1} - s \frac{NZ^\infty(y_1)\mathbf{1}_{|y_1| > K^*}}{2}. \quad (2.12)$$

The strategy at  $t = 1$  is monotone with her signal, given by Eq. (2.61) in Appendix 2.5.4.

*Proof.* See Appendix 2.5.4. □

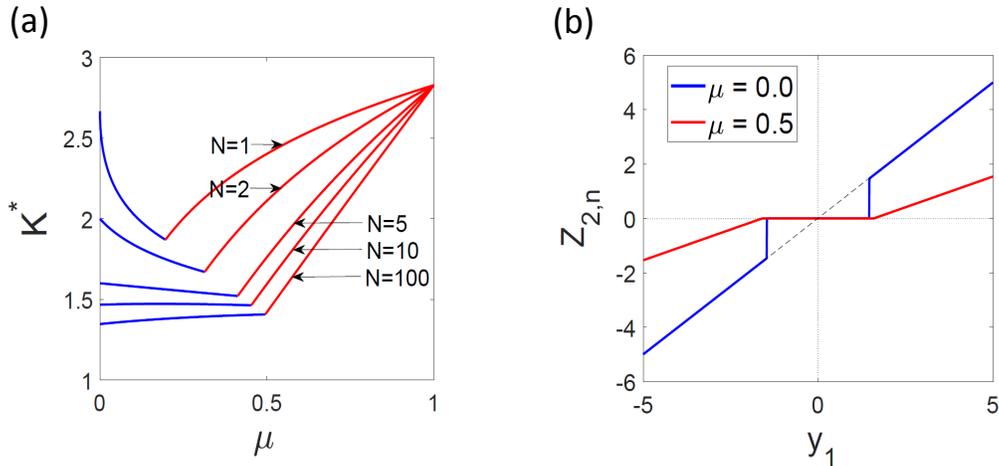


Figure 2.1: The threshold  $K^*(\mu, N)$  and the strategy  $Z_{2,n}(s, y_1; K^*)$  in two liquidity regimes.

**Corollary 7.** *The linear-triggering strategy Eq. (2.10) implies the heuristic learning rule,*

$\hat{\theta}_T = s \cdot (\hat{v}_T - \lambda_1 y_1) \mathbf{1}_{|y_1| > K^*}$ , *which estimates  $\tilde{\theta} = \tilde{v} - p_1$ , with*

$$\hat{v}_T(y_1; \xi_v) = \rho \lambda [y_1 - \text{sign}(y_1)(1+c)\kappa\sigma_u] \mathbf{1}_{|y_1| > \kappa\sigma_u}. \quad (2.13)$$

*Proof.* See Appendix 2.5.4 as well. □

The learning rule  $\hat{v}_T$  looks similar to the MAP estimator  $\hat{v}_{map}$  in Eq. (1.19), except that the horizontal intercept differs by a factor  $(1+c)$ . The learning threshold,  $\kappa\sigma_u \equiv \frac{\rho\lambda_1\sigma_u^2}{\xi_v}$ , is independent of the parameter  $c$ , because parallel shifts of the informed trading strategy do not change the signal-to-noise ratio perceived by arbitrageurs. This learning threshold depends on the parameter  $\rho$ , because more aggressive informed trading (smaller  $\rho$ ) can make arbitrageurs learn faster (smaller  $\kappa\sigma_u$ ). The overall learning rule,  $\hat{\theta}_T(y_1; K^*)$ , is governed by the threshold  $K^*$ , which is the maximum of learning threshold  $\kappa\sigma_u$  and strategic intercept term  $\frac{\rho(1+c)\kappa\sigma_u}{\rho-1}$ . Since this intercept increases (with  $\mu$ ) from 0 to  $2\kappa\sigma_u$ , it must cross  $\kappa\sigma_u$  at some intermediate value of  $\mu$ . This indicates a kink in the equilibrium threshold:

**Corollary 8.** *There are two liquidity regimes separated by the critical liquidity value*

$$\mu_c(N) = \sqrt{N(N+2)^3} - N(N+3) - 1 \in \left[ 3\sqrt{3} - 5, \frac{1}{2} \right]. \quad (2.14)$$

*For  $\mu \in [0, \mu_c]$ ,  $Z_{2,n}(s, y_1; K^*)$  is discontinuous at  $|y_1| = K^* = \kappa\sigma_u$  which decreases with  $\mu$ .*

*For  $\mu \in [\mu_c, 1]$ ,  $Z_{2,n}(s, y_1; K^*)$  is continuous and has  $K^* = \frac{\rho(1+c)}{\rho-1}\kappa\sigma_u$  which increases with  $\mu$ .*

*Proof.* The critical liquidity  $\mu_c$  is set by substituting the expressions of  $\rho$  and  $c$ , Eq. (2.8)

and Eq. (2.9), into the crossover condition  $1 = \frac{\rho(1+c)}{\rho-1}$  or  $1 + \rho c = 0$ . □

Corollary 8 suggests that  $K^* = \kappa\sigma_u$  for any  $\mu < 3\sqrt{3} - 5$  and  $K^* = \frac{\rho(1+c)}{\rho-1}\kappa\sigma_u$  for  $\mu > 0.5$ . The rescaled threshold  $K^*/\sigma_u$  only depends on the liquidity level  $\mu$  and the

competition condition  $N$  (Fig. 2.1). This threshold ensures that the algorithmic strategy  $Z_{2,n}$  only performs momentum trading. Under good liquidity  $\mu \in [0, \mu_c]$ , the equilibrium threshold is set by the learning hurdle of  $\hat{v}_T$ , i.e.,  $K^* = \kappa\sigma_u$ . Traders who use a threshold lower than  $\kappa\sigma_u$  may engage in unjustified trading for a range of states where their estimated signal  $\hat{v}_T$  is zero. Under poor liquidity  $\mu \in [\mu_c, 1]$ , the equilibrium threshold is set by the horizontal intercept of  $\hat{\theta}_T$ , i.e.,  $K^* = \frac{\rho(1+c)}{\rho-1}\kappa\sigma_u$ . Traders who use a threshold lower than this may do contrarian trading for a range of states where their estimated residual signal  $\hat{\theta}_T$  is zero. Arbitrageurs will keep undercutting their thresholds as far as possible<sup>8</sup> until they hit the lower bound  $K^*$  in Eq. (2.11) which excludes contrarian trading or any unjustified trading.

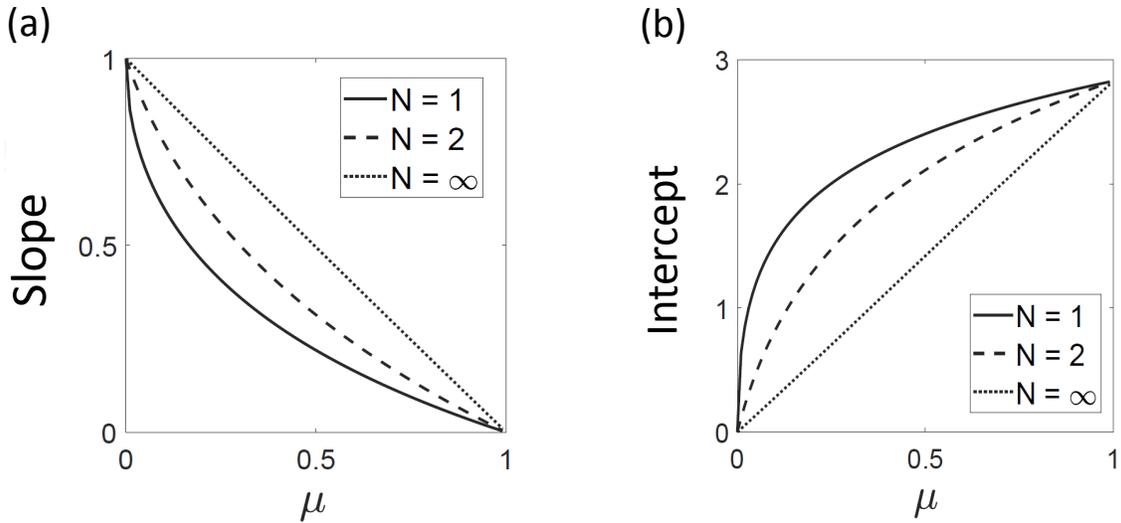


Figure 2.2: The slope and intercept of  $Z_2(y_1) = \sum_{n=1}^N Z_{2,n}(s=1, y_1; K^*)$  as a function of  $\mu$ .

<sup>8</sup>As long as the informed trader's strategy monotonically increases with her signal, it will be profitable for arbitrageurs to undercut the threshold as much as possible.

As shown in Fig. 2.2, the total arbitrage trading  $Z_2(y_1) \equiv \sum_{n=1}^N Z_{2,n}(s=1, y_1; K^*)$  has a slope,  $\frac{N(1-\mu)(\rho-1)}{N+2}$ , which decreases from 1 to 0 as  $\mu$  varies from 0 to 1. Its horizontal intercept,  $\frac{\rho(1+c)}{\rho-1}\kappa\sigma_u$ , increases from 0 to  $2\sqrt{2}\sigma_u$ . At constant market depth, the total arbitrage trading collapses to the 45° line,  $\lim_{\mu \rightarrow 0} Z_2 = y_1 \mathbf{1}_{|y_1| > \kappa\sigma_u}$ , regardless of the number  $N$ . This is an “order-flow mimicking” strategy, since the total quantity traded by arbitrageurs exactly mimics the total order flow they observed earlier. Also, this is like a pool of stop-loss orders which get triggered to execute whenever the price change surpasses  $\lambda_1 \kappa \sigma_u = \frac{4\sqrt{2}}{9} \frac{N+1}{N} \sigma_v$  in either direction. A function of the form,  $F(y) = y \mathbf{1}_{|y| > K}$ , is often called “hard-thresholding” in machine learning. For  $\mu > 0.5$ , arbitrageurs always use the “soft-thresholding” strategy.

After observing the total order flow  $y_1$ , the informed trader can figure out whether arbitrageurs will be triggered or not given their threshold  $K^*$ . This allows him to solve  $X_2(v, y_1, K^*)$  from Eq. (2.12). Using backward induction, he needs to decide  $x_1 = X_1(v, K^*)$  by carefully evaluating how his trade affects the probability of triggering those arbitrageurs who may compete with him:  $Q^*(x_1) = \mathbb{E} [\mathbf{1}_{|\tilde{y}_1| > K^*} | X_1(v, K^*) = x_1]$ .

Let’s look at the strategy of informed trader in different liquidity regimes. If market liquidity at  $t = 2$  is good ( $\mu < \mu_c$ ), her initial strategy  $X_1(s=1, v; K^*)$  is bended toward  $K^*$  to distort arbitrageurs’ learning [Fig. 2.3(a)]. With  $\tilde{x}_1 \approx K^*$  for a range of  $\tilde{v}$ , it will be difficult for arbitrageurs to infer the strength of  $\tilde{v}$  from  $\tilde{y}_1 = \tilde{x}_1 + \tilde{u}_1$ . Their trading decisions are error-prone because they are largely influenced by noise trading  $\tilde{u}_1$ . The nonlinear pure strategy allows the informed trader to hide her signal temporarily and inhibit the response of arbitrageurs. This strategy distorts the learning of arbitrageurs and twists their responses. This effect is strong under good liquidity conditions ( $\mu < \mu_c$ ). If future liquidity is poor

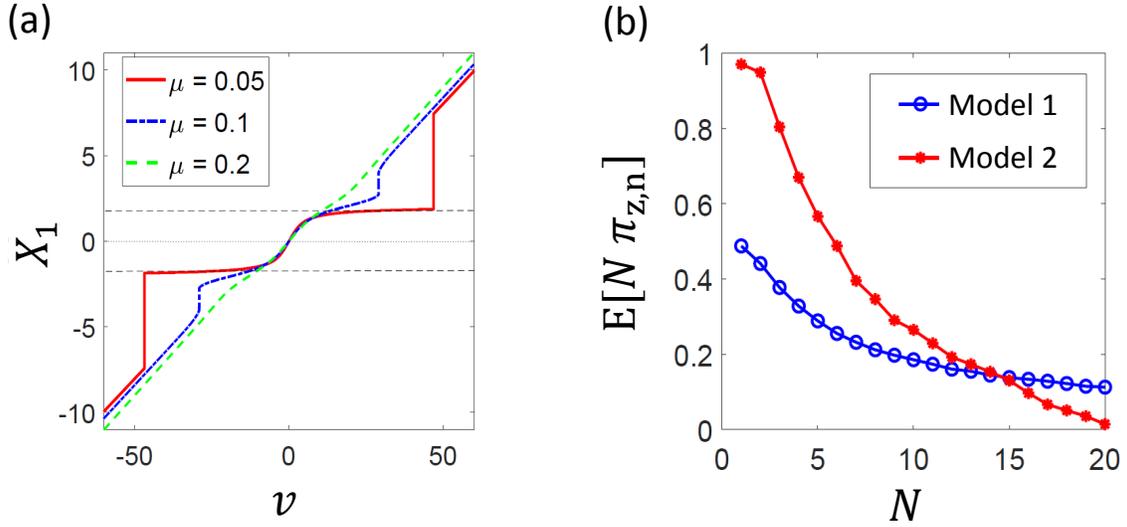


Figure 2.3: (a) the informed trader’s strategy  $X_1(s = 1, v)$  under different  $\mu$ . (b) the total payoffs to arbitrageurs in two models under respective linear-triggering strategies.

( $\mu > \mu_c$ ), the informed trader will trade more at  $t = 1$  and play the game more honestly. Poor liquidity discourages the informed trader’s “manipulative” trading. Fig. 2.3(a) confirms that when the signal is sufficiently strong, the informed trader ultimately shifts his strategy to the conjectured asymptotic line:  $X_1(v) \rightarrow \frac{v}{\rho\lambda_1} \pm c\kappa\sigma_u$ . It is costly to hide a strong signal, especially when future liquidity is poor. That is why  $X_1(v)$  is increasingly aligned with the linear strategy as  $\mu$  increases. Overall, the informed trader induces arbitrageurs to trade more competitively. This disrupts the “cartel” effect we discussed in the previous chapter<sup>9</sup>. Facing the savvy informed trader, arbitrageurs can no longer sustain extra market power nor earn noncompetitive profits at large  $N$  [Fig. 2.3(b)].

<sup>9</sup>If we introduce a Bayesian-rational arbitrageur into the group of arbitrageurs who all play the linear-triggering strategy, this rational trader will disrupt their “tacit collusion” and make the market more competitive. The informed trader here plays a similar role because he disrupts the would-be “collusive” group.

### 2.3.2 Disruptive Strategies and Price Manipulations

In this trading game with linear-triggering strategies, there is an implicit belief in the arbitrageurs' minds that the informed trader will play a monotone strategy which increases with her signal. Numerically, this conjecture is found to hold in the liquidity regime where  $\mu > \mu_\epsilon \approx 0.005$ . However, the conjectured equilibrium becomes unstable when market depth is almost constant ( $\mu \rightarrow 0$ ). If  $\mu$  is arbitrarily close to 0, arbitrageurs' trading will closely mimic the order flow  $y_1$ . This may invite the informed trader to trick them.

Let's consider the limiting case where  $v \rightarrow 0$  and  $\mu \rightarrow 0$ : the informed trader receives a non-directional signal about the asset value and also anticipates that the total order flow from arbitrageurs is  $\lim_{\mu \rightarrow 0} Z_2^{AT}(y_1) = y_1 \mathbf{1}_{|y_1| > \kappa \sigma_u}$ . How will the informed trader trade if he knows that other traders will mimic the observed order flow once they get triggered?

**Corollary 9.** *At  $v = 0$  and as  $\mu \rightarrow 0$ , the informed trader chooses  $x_1$  by solving*

$$\frac{x_1}{2 + \kappa^2} = \frac{\phi(\kappa - x_1) - \phi(\kappa + x_1)}{\operatorname{erf}\left(\frac{\kappa - x_1}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{\kappa + x_1}{\sqrt{2}}\right)}. \quad (2.15)$$

Here, we set  $\sigma_u = 1$  and use  $\phi(\cdot)$  to denote the probability density of a standard Gaussian distribution. The informed trader will first trade a sufficiently large  $x_1$  to trigger arbitrageurs<sup>10</sup> and then trade  $x_2 = -y_1$  to offset their subsequent momentum trading, i.e.,  $\lim_{\mu \rightarrow 0} X_2(v = 0, y_1) = -y_1 = -\lim_{\mu \rightarrow 0} Z_2(y_1)$ . This Bayesian-rational strategy has a terminal position of  $x_1 + x_2 = -u_1$  which is zero on average, with an expected profit of  $\lambda_1 \sigma_u^2$ .

<sup>10</sup>There may exist multiple solutions to Eq. (2.15): one is obviously  $x_1 = 0$  and the other two solutions are  $\pm\infty$  which lead to the global maximum profit. As long as the informed trader chooses a large enough  $x_1$  (instead of  $\pm\infty$ ), the probability of triggering arbitrageurs can be arbitrarily close to one and his expected profit is arbitrarily close to the maximum  $\lambda_1 \sigma_u^2$ .

*Proof.* This rational strategy follows from Eq. (2.10) and Eq. (2.12) by taking both limits  $v \rightarrow 0$  and  $\mu \rightarrow 0$ . Detailed proof can be found in the Appendix 2.5.5.  $\square$

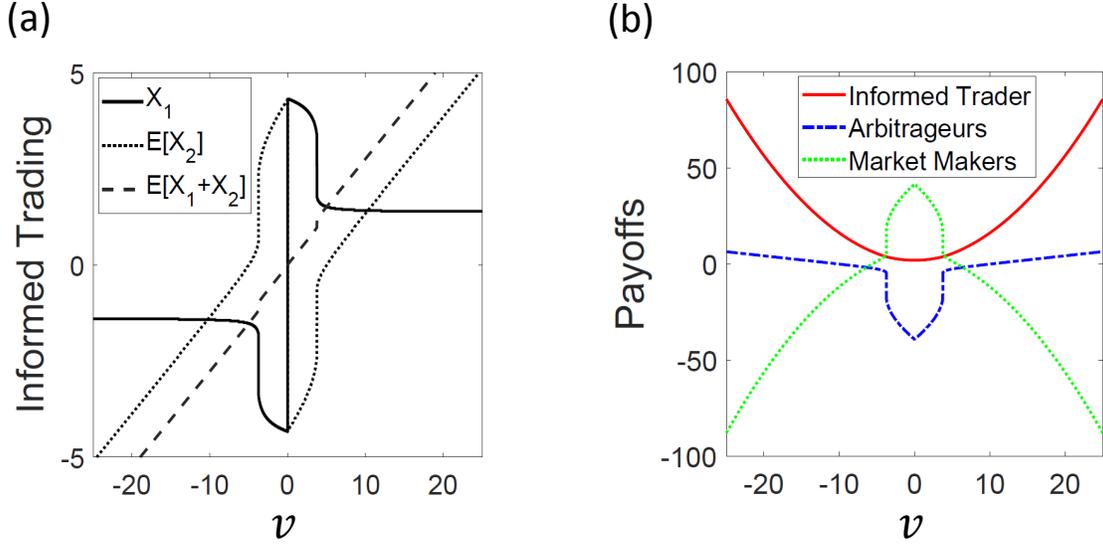


Figure 2.4: (a) the optimal strategy of informed trader under  $\mu = 10^{-4}$ ,  $N = 3$  and  $\xi_v = 3$ . (b) the total payoffs to different groups of traders.

As shown in Fig. 2.4(a), when the private signal  $v$  is very small, the informed trader places a large order  $|x_1| \gg K^* = \kappa\sigma_u$  to trigger arbitrageurs whose trading at  $t = 2$  closely mimics the total order flow observed at  $t = 1$ . This allows the informed trader to liquidate most of her inventory at more favorable prices at  $t = 2$ . The terminal position  $E[\tilde{x}_1 + \tilde{x}_2 | \tilde{v} = v]$  is almost linear with her private signal  $v$ , but her strategy in each period is non-monotone with her signal. Fig. 2.4(b) shows the total payoffs to different groups of traders. Arbitrageurs incur dramatic losses near the origin as they have been fooled by the informed trader who earns a small profit on average. The losses of arbitrageurs mostly benefit market makers.

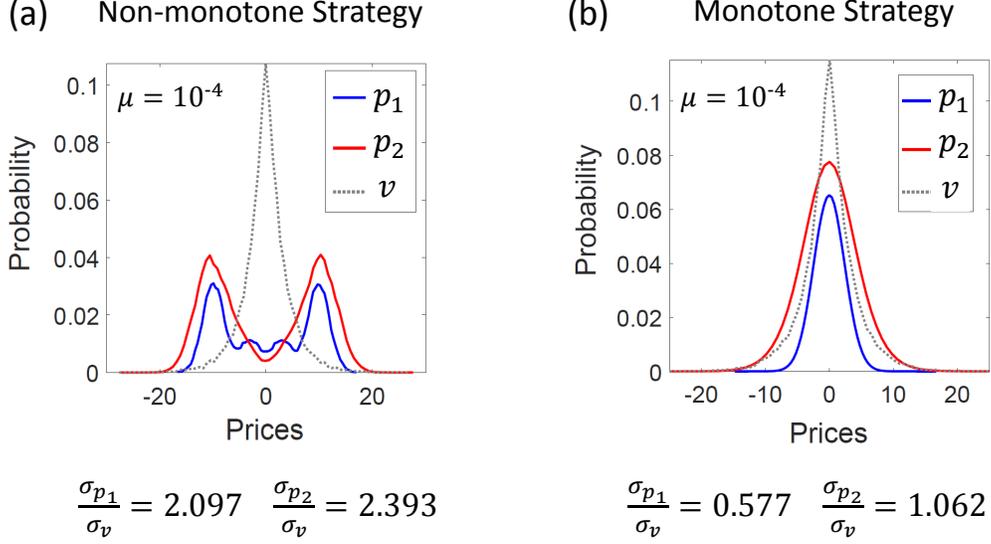


Figure 2.5: The unconditional probability distributions of the prices  $\tilde{p}_1$  and  $\tilde{p}_2$  under the non-monotone strategy versus the monotone strategy, given  $\mu = 10^{-4}$ ,  $N = 3$  and  $\sigma_v = 3\sqrt{2}$ .

The non-monotone strategy seems disruptive and resembles controversial strategies in the real world, including *momentum ignition* and *stop-loss hunting*. These schemes are usually regarded as *trade-based price manipulations* by regulators. If such non-monotone strategies are prohibited (by regulators) in the model, the informed trader at the state  $v = 0$  will not trade at  $t = 1$ . Instead, she will watch the market first and trade at  $t = 2$  against either the noise-driven price changes or the order flows from arbitrageurs who are falsely triggered. Kyle and Viswanathan (103) recommend two economic criteria for regulators to define illegal price manipulations. These are pricing accuracy and market liquidity. Fig. 2.5 compares the (unconditional) probability distributions of prices when the non-monotone strategy is allowed or banned. With the non-monotone strategy in Fig. 2.4(a), price distributions are bimodal in both periods [Fig. 2.5(a)]. Pricing accuracy is poor as prices do not reflect the fundamental value  $\tilde{v}$  (with a unimodal distribution). Price

volatilities are at least twice as large as the fundamental volatility  $\sigma_v$ . If a common investor arrives and trades this asset, she is likely to buy at a much higher ask price or sell at a much lower bid price. The bimodal price pattern reflects a much wider bid-ask spread for common investors. In contrast, if regulators set rules to ban such disruptive strategies, the price distributions become bell-shaped in both periods with reasonable price volatilities and pricing accuracy [Fig. 2.5(b)].

Regulators need to sort out the economic conditions for the *trade-base manipulations*. The results in this chapter prescribe a list of conditions that could be necessary for the non-monotone disruptive strategy.

- (1) Speculators think that market makers set inaccurate prices by using incorrect priors.
- (2) Speculators have fat-tail priors about the fundamental value or trading opportunities.
- (3) There is strategic interplay between the informed trader and those speculators.
- (4) Market depth is not decreasing when the informed trader liquidates her inventory.
- (5) Traders face no trading costs, no inventory costs, nor threat from regulators.
- (6) There is no other informed trader who could interfere with the disruptive strategy.

The (non-monotone) disruptive strategy may fail if any of these conditions is not satisfied. It seems not easy at all, but the key condition is that the total feedback trading from speculators has a slope no less than one. This could happen if speculators underestimate the actual number of speculators ( $N$ ), since each speculator's demand is inversely proportional to the number of competitors (estimated by the speculator). This could also happen in the liquidity regime with  $\mu < 0$ , where the informed trader could dump her early inventory at

a lower cost and speculators may trade more aggressively. In the conjectured equilibrium, the response slope of each speculator is given by  $\frac{(1-\mu)(\rho-1)}{N+2}$ . If all speculators keep using this strategy in the liquidity regime  $\mu < 0$ , the slope of their aggregate response will be greater than one:  $N \frac{(1-\mu)(\rho-1)}{N+2} > 1$ . Over-trading makes speculators susceptible to “disruptive attacks”. For the informed trader, the profits of tricking speculators can be outweighed by the losses if she fails to liquidate the undesirable inventory in the second period.

## 2.4 Conclusion

In this chapter, we consider a model of strategic trading between informed trader and algorithmic arbitrageurs. It is shown that the informed trader will try to distort arbitrageurs’ learning and induce them to trade more aggressively. Under certain market conditions, the informed trader may play a disruptive strategy that resembles real-life controversial practices (e.g., momentum ignition in high frequency trading). Such trading schemes can distort the informational content of prices and destabilize stock prices at the expense of common investors. This model can provide policy implications and empirical predictions.

## 2.5 Appendix

### 2.5.1 Nonlinear Rational-Expectations Equilibrium

For  $s = 1$ , we investigate the *rational-expectations equilibrium* (REE) in the model of savvy informed trader who anticipates arbitrageurs and strategically interacts with them. Based on  $\mathcal{I}_{2,x} = \{v, s, y_1\}$ , the informed trader conjectures her residual demand at  $t = 2$  and solves

$$X_2(v, y_1) = \arg \max_{x_2} \mathbb{E} [(v - P_2(\tilde{y}_1, \tilde{y}_2)) x_2 | \mathcal{I}_{2,x}] = (1 - \mu) \frac{v - \lambda_1 y_1}{2\lambda_1} - \frac{\mathbb{E}[Z_2 | \mathcal{I}_{2,x}]}{2}. \quad (2.16)$$

As the informed trader takes into account the price impact of all arbitrageurs, she will reduce her trading quantity by one half of the total arbitrage trading that she expects at  $t = 2$ .

The information set of arbitrageurs right after  $t = 1$  is  $\mathcal{I}_{2,z} = \{s, y_1\}$ , which is nested into the informed trader's information set  $\mathcal{I}_{2,x} = \{v, s, y_1\}$ . The  $n$ -th arbitrageur's objective is

$$\max_{z_{2,n}} \mathbb{E} [z_{2,n} (\tilde{v} - \lambda_1 \tilde{y}_1 - \lambda_2 [X_2(\tilde{v}, \tilde{y}_1) + z_{2,n} + Z_{2,-n}(\tilde{y}_1) + \tilde{u}_2]) | \mathcal{I}_{2,z}], \quad (2.17)$$

from which she can solve the optimal strategy as below

$$Z_{2,n}(y_1) = (1 - \mu) \frac{\hat{v} - \lambda_1 y_1}{4\lambda_1} - \frac{\mathbb{E}[Z_{2,-n} | \mathcal{I}_{2,z}]}{2} + \frac{\mathbb{E}[\mathbb{E}[Z_2 | \mathcal{I}_{2,x}] | \mathcal{I}_{2,z}]}{4}. \quad (2.18)$$

Arbitrageurs are symmetric in terms of their information and objectives. The  $n$ -th arbitrageur conjectures that the other arbitrageurs will trade  $Z_{2,m} = \eta \cdot (\hat{v} - \lambda_1 y_1)$  for  $m = 1, \dots, N$  and  $m \neq n$ , and she also conjectures the informed trader's conjecture that all arbitrageurs trade symmetrically  $Z_{2,n} = \eta \cdot (\hat{v} - \lambda_1 y_1)$  for  $n = 1, \dots, N$ . So her optimal strategy becomes

$$Z_{2,n}(y_1) = \left( \frac{1 - \mu}{4\lambda_1} - \frac{(N - 1)\eta}{2} + \frac{N\eta}{4} \right) (\hat{v} - \lambda_1 y_1). \quad (2.19)$$

In a symmetric equilibrium, every arbitrageur conjectures in the same way and solves the same problem. This symmetry requires  $\eta = \frac{1 - \mu}{4\lambda_1} - \frac{(N - 1)\eta}{2} + \frac{N\eta}{4}$  that has a unique solution

$\eta = \frac{1-\mu}{(N+2)\lambda_1}$ . Thus the total order flow from arbitrageurs at  $t = 2$  can be written as

$$Z_2 = \sum_{n=1}^N Z_{2,n} = N\eta \cdot (\hat{v} - \lambda_1 y_1) = \frac{N(\hat{v} - \lambda_1 y_1)}{(N+2)\lambda_2}. \quad (2.20)$$

One can prove a simple result that  $Z_{2,n} = E[X_2 | \mathcal{I}_{2,z}]$ , i.e., every arbitrageur expects that the informed trader on average trades the same quantity as she does. By Eq. (2.16) and (2.20),

$$\begin{aligned} E[X_2(\tilde{v}, \tilde{y}_1) | \mathcal{I}_{2,z}] &= \frac{E[\tilde{v} | \mathcal{I}_{2,z}] - \lambda_1 y_1}{2\lambda_2} - \frac{E[E[Z_2 | \mathcal{I}_{2,x}] | \mathcal{I}_{2,z}]}{2} \\ &= \frac{\hat{v} - \lambda_1 y_1}{2\lambda_2} - \frac{Z_2}{2} = \frac{Z_2}{N} = Z_{2,n}. \end{aligned} \quad (2.21)$$

As  $\hat{v} = E[\tilde{v} | \mathcal{I}_{2,z}]$ , we obtain the following

$$Z_{2,n}(y_1) = \eta(\hat{v} - \lambda_1 y_1) = \frac{1-\mu}{(N+2)\lambda_1} (E[\tilde{v} | \mathcal{I}_{2,z}] - \lambda_1 y_1) = E[X_2 | \mathcal{I}_{2,z}], \quad (2.22)$$

$$X_2(v, y_1) = \frac{v - \lambda_1 y_1}{2\lambda_2} - \frac{Z_2}{2} = \frac{v - \lambda_1 y_1}{2\lambda_2} - \frac{N}{N+2} \frac{E[\tilde{v} | \mathcal{I}_{2,z}] - \lambda_1 y_1}{2\lambda_2}. \quad (2.23)$$

One can rewrite the second-period informed trading strategy as

$$X_2(v, y_1) = \frac{v - \lambda_1 y_1}{(N+2)\lambda_2} + \frac{N}{N+2} \cdot \frac{v - \hat{v}}{2\lambda_2}, \quad (2.24)$$

where the first term is proportional to her informational advantage over market makers and the second term is proportional to her residual advantage over arbitrageurs. In the competitive case ( $N \rightarrow \infty$ ), there are two simple results

$$\lim_{N \rightarrow \infty} X_2 = (1-\mu) \frac{v - \hat{v}}{2\lambda_1} \quad \text{and} \quad \lim_{N \rightarrow \infty} Z_2 = (1-\mu) \frac{\hat{v} - \lambda_1 y_1}{\lambda_1}. \quad (2.25)$$

Let  $\hat{v} = E[\tilde{v} | \mathcal{I}_{2,z}] = g(y_1)$ . The informed trader conjectures the average price at  $t = 2$  to be

$$\begin{aligned} E[\tilde{p}_2 | \mathcal{I}_{2,x}] &= E \left[ \lambda_1 \tilde{y}_1 + \lambda_2 \left( X_2 + \sum_{n=1}^N Z_{2,n} + \tilde{u}_2 \right) \middle| \mathcal{I}_{2,x} \right] \\ &= \frac{(N+2)v + Ng(y_1) + 2\lambda_1 y_1}{2(N+2)}, \end{aligned} \quad (2.26)$$

The informed trader's expected profit from her second-period trading is

$$\begin{aligned}\Pi_{2,x}(v, y_1) &= \mathbb{E}[x_2(v - \tilde{p}_2)|\mathcal{I}_{2,x}] \\ &= \frac{1}{\lambda_2} \left( \frac{(N+2)v - Ng(y_1) - 2\lambda_1 y_1}{2(N+2)} \right)^2.\end{aligned}\quad (2.27)$$

The informed trader needs to choose  $x_1$  that maximizes her total expected profits:

$$\begin{aligned}\Pi_x(v) &= \max_{x_1} \mathbb{E}[x_1(v - \lambda_1 \tilde{y}_1) + \Pi_{2,x}(v, \tilde{y}_1)|\mathcal{I}_{1,x}] \\ &= \max_{x_1} x_1(v - \lambda_1 x_1) + \frac{1-\mu}{\lambda_1} \mathbb{E} \left[ \left( \frac{(N+2)v - Ng(\tilde{y}_1) - 2\lambda_1 \tilde{y}_1}{2(N+2)} \right)^2 \middle| \mathcal{I}_{1,x} \right],\end{aligned}\quad (2.28)$$

where  $\mathcal{I}_{1,x} = \{v, s = 1\}$ . As regularity conditions permit, one can interchange expectation and differentiation operations to derive the first order condition (FOC) for  $x_1 = X_1(v)$ :

$$0 = v - 2\lambda_1 x_1 - \frac{1-\mu}{\lambda_1} \mathbb{E} \left[ \frac{(N+2)v - Ng(x_1 + \tilde{u}_1) - 2\lambda_1(x_1 + \tilde{u}_1)}{2(N+2)} \cdot \frac{Ng'(x_1 + \tilde{u}_1) + 2\lambda_1}{N+2} \right].\quad (2.29)$$

When  $s = 1$ , there does not exist a linear REE where the informed trader's strategy  $X_1$  is a linear function of  $v$ . This is proved by contradiction: Suppose  $X_1$  is a linear function of  $v$ , the posterior mean  $g(y_1) = \mathbb{E}[\tilde{v}|\mathcal{I}_{2,z}]$  will be a nonlinear function of  $y_1$ . With a nonlinear  $g(y_1)$ , the FOC Eq. (2.29) does not permit a linear solution to  $X_1(v)$ . Nonlinearity makes Eq. (2.29) and the REE intractable in general.

## 2.5.2 Asymptotic Linearity

Based on the asymptotic conjecture of  $X_1(v) \rightarrow \frac{v}{\rho\lambda_1} + c\kappa\sigma_u$  in the high signal regime, arbitrageurs will find that the posterior distribution of  $x_1$  conditional on  $y_1$  is asymptotically

$$f(x_1|y_1) \rightarrow \frac{\rho\lambda_1}{\xi_v f(y_1) \sqrt{2\pi\sigma_u^2}} \exp \left[ -\frac{(y_1 - x_1)^2}{2\sigma_u^2} - \frac{\rho\lambda_1(x_1 - c\kappa\sigma_u)}{\xi_v} \right]. \quad (2.30)$$

At large order flows, it is deduced that  $E[\tilde{x}_1|y_1] \rightarrow y_1 - \kappa\sigma_u$  and furthermore

$$E[\tilde{v}|\mathcal{I}_{2,z}] \rightarrow \rho\lambda_1[y_1 - (1+c)\kappa\sigma_u]. \quad (2.31)$$

This result makes the informed trader's FOC Eq. (2.29) for  $x_1 = X_1(v)$  linear again:

$$0 = v - 2\lambda_1 x_1 - \frac{1}{\lambda_2} E \left[ \left( \frac{(N+2)v - N\rho\lambda_1[\tilde{y}_1 - (1+c)\kappa\sigma_u] - 2\lambda_1\tilde{y}_1}{2(N+2)} \right) \left( \frac{N\rho\lambda_1 + 2\lambda_1}{N+2} \right) \middle| \mathcal{I}_{1,x} \right]. \quad (2.32)$$

After some calculation with the notation  $\delta \equiv \frac{\lambda_2}{\lambda_1} = \frac{1}{1-\mu}$ , we get

$$0 = v - 2\lambda_1 x_1 - \frac{N\rho + 2}{2\delta(N+2)^2} [(N+2)v - (N\rho + 2)\lambda_1 x_1 + N(1+c)\kappa\rho\lambda_1\sigma_u]. \quad (2.33)$$

This FOC leads to a linear expression of  $x_1$  which conforms to the original linear conjecture:

$$X_1(v) = \frac{(N+2)[2\delta(N+2) - N\rho - 2]}{4\delta(N+2)^2 - (N\rho + 2)^2} \left( \frac{v}{\lambda_1} \right) - \frac{N\rho(N\rho + 2)(1+c)\kappa}{4\delta(N+2)^2 - (N\rho + 2)^2} \sigma_u. \quad (2.34)$$

Matching the first term leads to a quadratic equation for  $\rho$ :

$$-2(\rho - 1)(N\rho + 2) + 2\delta(\rho - 2)(N+2)^2 = 0. \quad (2.35)$$

There are two roots to this equation but only one of them is sensible as it increases with  $\delta$ :

$$\rho(\delta, N) = \frac{N + \delta(N+2)^2 - 2 - (N+2)\sqrt{\delta^2(N+2)^2 - 2\delta(3N+2) + 1}}{2N}. \quad (2.36)$$

Substituting  $\delta = \frac{1}{1-\mu}$  into the above equation leads to

$$\rho(\mu, N) = \frac{2 + 5N + N^2 + 2\mu - N\mu - (N + 2)\sqrt{N^2 + (1 + \mu)^2 + 2N(3\mu - 1)}}{2N(1 - \mu)}. \quad (2.37)$$

For  $N = 0$ , we have  $\rho = \frac{3+\mu}{1+\mu}$  which is identical to the parameter  $\rho$  in the previous model.

It is easy to derive in the monopolistic case ( $N = 1$ ) that  $\rho(\mu, N = 1) = \frac{8+\mu-3\sqrt{\mu(8+\mu)}}{2(1-\mu)}$ .

In the competitive case, we have  $\lim_{N \rightarrow \infty} \rho = 2$ . Intuitively, the informed trader would only trade half of his private signal when he expected a competitive fringe of arbitrageurs.

There are two more useful limits:  $\lim_{\mu \rightarrow 0} \rho = 2 \left(1 + \frac{1}{N}\right)$  and  $\lim_{\mu \rightarrow 1} \rho = 2$ . This equilibrium parameter  $\rho$  decreases with  $\mu$  and  $N$ . It is bounded in the range  $\left[2, \frac{2(N+1)}{N}\right]$ . Now we match the intercept terms and utilize the slope-matching relation to obtain

$$c = -\frac{N(2 + N\rho)}{2\delta(2 + N)^2 - 2(2 + N\rho)} = -\frac{3 + N - \mu - \sqrt{N^2 + (1 + \mu)^2 + 2N(3\mu - 1)}}{1 + N + \mu + \sqrt{N^2 + (1 + \mu)^2 + 2N(3\mu - 1)}} \cdot \frac{N}{2}. \quad (2.38)$$

In the competitive case, we have

$$\lim_{N \rightarrow \infty} c = \lim_{N \rightarrow \infty} \frac{-N(2 + 2N)}{2\delta(2 + N)^2 - 2(2 + 2N)} = -\frac{1}{\delta} = -(1 - \mu). \quad (2.39)$$

There are two more useful limits:  $\lim_{\mu \rightarrow 0} c = -1$  and  $\lim_{\mu \rightarrow 1} c = 0$ .

*Approximation to the rational equilibrium.* The symmetry indicates that  $X_1(-v) = -X_1(v)$ .

If  $X_1(v)$  is monotone, it should cross the origin and be roughly linear in that neighborhood.

With the linearized conjecture  $X_1(v \rightarrow 0) \rightarrow \frac{v}{\alpha\lambda_1}$ , one can use Taylor expansion of Eq. (1.14)

at small  $y_1$  to approximate  $E[\tilde{v}|y_1 \ll \kappa\sigma_u] \approx \alpha\beta\lambda_1 y_1$ , where  $\alpha$  and  $\beta$  are determined by

$$\beta N[\beta N - (N + 2)]\alpha^2 + 2 \left( \frac{(N + 2)^2}{1 - \mu} + 2\beta N - (N + 2) \right) \alpha - 4 \left( \frac{(N + 2)^2}{1 - \mu} - 1 \right) = 0,$$

$$\beta = 1 + \left( \frac{\alpha\lambda_1\sigma_u}{\xi} \right)^2 - \left( \frac{\alpha\lambda_1\sigma_u}{\xi} \right) \frac{e^{-\frac{(\alpha\lambda_1\sigma_u)^2}{2\xi^2}} \sqrt{\frac{2}{\pi}}}{\operatorname{erfc}\left(\frac{\alpha\lambda_1\sigma_u}{\sqrt{2}\xi}\right)}.$$

The first equation is derived from the FOC Eq. (2.29) and the second one is from the Taylor expansion of Eq. (1.14). Given  $\{\mu, N, \xi\}$ , one can numerically find a unique pair of positive solutions to  $\alpha$  and  $\beta$ . With constant depth ( $\mu = 0$ ), the first equation becomes  $\alpha = \frac{2(N+3)}{N+2-N\beta}$  and the total demand from arbitrageurs becomes  $\lim_{\mu \rightarrow 0} Z_2 \approx \lim_{\mu \rightarrow 0} \frac{N(\alpha\beta-1)}{N+2} y_1 = (\alpha - 3)y_1$  for small  $y_1$ . The rational equilibrium is not tractable, but one can approximate the arbitrageurs' rational strategy by smoothly pasting the two regimes of asymptotic linearity. There are different methods to make a smooth transition between two linear segments; for example, any sigmoid functions that approach the Heaviside function may work. Here, I use  $q(y) = \frac{1}{2}\text{erfc}[a(\kappa\sigma_u - y)] + \frac{1}{2}\text{erfc}[a(\kappa\sigma_u + y)]$ , with a tunable parameter  $a > 0$  and approximate the posterior mean estimate of  $\tilde{v}$  by

$$\hat{v}_a(y_1) \approx [1 - q(y_1)]\alpha\beta\lambda_1 y_1 + q(y_1)\rho\lambda_1[y_1 - \text{sign}(y_1)(1 + c)\kappa\sigma_u]. \quad (2.40)$$

Clearly,  $\hat{v}_a \rightarrow \alpha\beta\lambda_1 y_1$  at  $|y_1| \ll \kappa\sigma_u$  and  $\hat{v}_a \rightarrow \rho\lambda_1[y_1 - \text{sign}(y_1)(1 + c)\kappa\sigma_u]$  at  $|y_1| \gg \kappa\sigma_u$ .

The figure below shows numerical approximations to the Bayesian-rational strategy  $Z_{2,n}^o(s = 1, y_1; \xi)$  under different  $\xi$ , compared with the linear-triggering strategy  $Z_{2,n}(s = 1, y_1; K^*)$ .



Figure 2.6: Approximate rational strategies and the linear-triggering strategy (red line).

### 2.5.3 Learning Bias and Strategic Informed Trading

**Corollary 10.** *Arbitrageurs tend to underestimate the private signal  $\tilde{v}$  by a negative amount  $-\rho\kappa\lambda_1\sigma_u < 0$ . Anticipating this estimation bias, the informed trader in the high signal regime will strategically shift her demand downward by an amount of  $c\kappa\sigma_u < 0$  at  $t = 1$  and upward by an amount of  $d\kappa\sigma_u > 0$  at  $t = 2$  where the parameter  $d(\mu, N)$  is given by Eq. (2.45). her average terminal position contains an informational component and a strategic component, that is,  $E[X_1(v) + X_2(v, \tilde{u}_1)] \rightarrow X_{inf}^*(v) + X_{str}^*$ , where  $X_{str}^* = (c + d)\kappa\sigma_u$  and*

$$X_{inf}^* = \frac{N + 1 + \mu + \rho(1 - \mu)}{N + 2} \frac{v}{\rho\lambda_1}, \quad (2.41)$$

Given any  $N > 0$ , the maximum of  $X_{inf}^*(v)$  is at  $\mu_c(N) = \sqrt{N(N + 2)^3} - N(N + 3) - 1$ .

Proof: In the asymptotic rational equilibrium we have shown  $E[\tilde{v}|\mathcal{I}_{2,z}] \rightarrow \rho\lambda_1[y_1 - (1 + c)\kappa\sigma_u]$  and  $y_1 = X_1(v) + \tilde{u}_1 \rightarrow (\rho\lambda_1)^{-1}\tilde{v} + c\kappa\sigma_u + \tilde{u}_1$ . Arbitrageurs tend to underestimate  $\tilde{v}$ ,

$$E[\tilde{v}|\mathcal{I}_{2,z}] - \tilde{v} = -\rho\lambda_1\kappa\sigma_u + \rho\lambda_1\tilde{u}_1 \sim \mathcal{N}[-\rho\lambda_1\kappa\sigma_u, (\rho\lambda_1\sigma_u)^2], \quad (2.42)$$

which has a negative mean  $-\rho\lambda_1\kappa\sigma_u < 0$ . This learning bias of arbitrageurs entices the informed trader to strategically exploit it. This can be seen from her asymptotic strategy:

$$X_2(v, y_1) \rightarrow (1 - \mu) \left[ \frac{v - \lambda_1 y_1}{2\lambda_1} - \frac{N}{N + 2} \frac{(\rho - 1)y_1 - (1 + c)\kappa\rho\sigma_u}{2} \right], \quad (2.43)$$

whose average contains both an informational component and a strategic one:

$$E[X_2|\tilde{v} = v] = \frac{(1 - \mu)(1 - \rho^{-1})}{\lambda_1(N + 2)} v + \frac{(1 - \mu)(N\rho - 2c)}{2(N + 2)} \kappa\sigma_u. \quad (2.44)$$

We define another parameter  $d$  for this strategic shift which decreases with  $\mu$  and  $N$ :

$$d(\mu, N) = \frac{(1 - \mu)(N\rho - 2c)}{2(N + 2)} = \frac{2N(1 - \mu)}{1 + N + \mu + \sqrt{N^2 + (1 + \mu)^2 + 2N(3\mu - 1)}}. \quad (2.45)$$

It has the following limit results:  $\lim_{\mu \rightarrow 0} d = 1$ ,  $\lim_{\mu \rightarrow 1} d = 0$ , and  $\lim_{N \rightarrow \infty} d = 1 - \mu$ . Thus, we have shown that  $X_1 \rightarrow \frac{v}{\rho\lambda_1} + c\kappa\sigma_u$  where  $c < 0$  and  $E[X_2|v] \rightarrow \frac{(1-\mu)(1-\rho^{-1})}{\lambda_1(N+2)}v + d\kappa\sigma_u$  where  $d > 0$ . This shows how the informed trader strategically exploit the arbitrageurs' bias  $\kappa\sigma_u$ .

The asymptotic terminal position of the informed trader can be decomposed into an informational term and a strategic term, that is,  $E[X_1(v) + X_2(v, \tilde{u}_1)] \rightarrow X_{inf}^*(v) + X_{str}^*$  where  $X_{str}^* = (c + d)\kappa\sigma_u \geq 0$ . The information-based target inventory is found to be

$$\begin{aligned} X_{inf}^*(v; \mu, N) &= \frac{v}{\rho\lambda_1} + \frac{(1-\mu)(1-\rho^{-1})}{\lambda_1(N+2)}v = \frac{N+1+\mu+\rho(1-\mu)}{N+2} \cdot \frac{v}{\rho\lambda_1} \\ &= \frac{1+3N+\mu-\sqrt{N^2+(1+\mu)^2+2N(3\mu-1)}}{N\rho} \cdot \frac{v}{2\lambda_1}. \end{aligned} \quad (2.46)$$

which is hump-shaped and reaches its maximum at

$$\mu_c(N) = \sqrt{N(N+2)^3} - N(N+3) - 1. \quad (2.47)$$

For example,  $X_{inf}^*$  has its maximum  $0.5359\frac{v}{\lambda_1}$  at  $N = 1$  and  $\mu_c(N = 1) = 3\sqrt{3} - 5 = 0.196152$ .

The informed trader manages to reach an informational target position roughly equal to  $\frac{v}{2\lambda_1}$ .



Figure 2.7: The information-based target inventory  $X_{inf}^*(v)$  and the strategic position  $X_{str}^*$ .

## 2.5.4 Proof of Proposition 6

The candidate linear-triggering strategy for each arbitrageur along the REE asymptotes is

$$Z_{2,n}(s, y_1; K_n) = sZ^\infty(y_1, K^*)\mathbf{1}_{|y_1| > K_n} = s \frac{(1-\mu)(\rho-1)}{N+2} \left[ y_1 - \text{sign}(y_1) \frac{\rho(1+c)\kappa\sigma_u}{\rho-1} \right] \mathbf{1}_{|y_1| > K_n}. \quad (2.48)$$

For  $s = 1$ , this can be rewritten as

$$\begin{aligned} Z_{2,n}(s, y_1; K_n) &= \frac{\rho\lambda_1[y_1 - \text{sign}(y_1)(1+c)\kappa\sigma_u] - \lambda_1 y_1}{(N+2)\lambda_2} \mathbf{1}_{|y_1| > K_n} \\ &= \eta \cdot [\hat{v}_T(y_1; \xi_v) - \lambda_1 y_1] \mathbf{1}_{|y_1| > K_n}, \end{aligned} \quad (2.49)$$

where  $\eta = \frac{1-\mu}{(N+2)\lambda_1}$  and the implied learning rule for  $\tilde{v}$  is

$$\hat{v}_T(y_1; \xi_v) = \rho\lambda [y_1 - \text{sign}(y_1)(1+c)\kappa\sigma_u] \mathbf{1}_{|y_1| > \kappa\sigma_u}. \quad (2.50)$$

The learning threshold  $\kappa\sigma_u$  here ensures that  $\hat{v}_T$  takes the same sign as  $y_1$ .

Now I prove that in equilibrium every arbitrageur will choose the same threshold

$$K^* = \max \left[ \kappa\sigma_u, \frac{\rho(1+c)\kappa\sigma_u}{\rho-1} \right]. \quad (2.51)$$

Intuitively, any trader choosing  $K_n$  lower than the learning threshold  $\kappa\sigma_u$  may take actions to trade over the states  $|y_1| \in [K_n, \kappa\sigma_u]$  where she actually learns nothing under her learning rule, i.e.,  $\hat{v}_T = 0$  for  $|y_1| \in [K_n, \kappa\sigma_u]$ . To exclude irrational trading when the inferred signal is zero, the equilibrium threshold must have a lower bound  $\kappa\sigma_u$ . On the other hand, any trader choosing  $K_n$  lower than the intercept  $\frac{\rho(1+c)\kappa\sigma_u}{\rho-1}$  may trade against the price trend (contrarian trading) over the states  $|y_1| \in \left[ K_n, \frac{\rho(1+c)\kappa\sigma_u}{\rho-1} \right]$ . This may go against the true (fat-tail) signal and incur losses on average. Therefore, the condition  $K_n \geq \max \left[ \kappa\sigma_u, \frac{\rho(1+c)\kappa\sigma_u}{\rho-1} \right]$  could make

arbitrageurs dedicate to the momentum trading strategy which is desirable in our fat-tail setup. When traders choose thresholds, they actually engage in Bertrand-type competition: each of them will keep undercutting the threshold as long as it is more profitable than the case she follows the common threshold used by other traders. Under this competition, the equilibrium threshold is the boundary  $K^*$  given by Eq. (2.51).

Let's first show that to use any threshold  $K'$  lower than  $K^*$  cannot be an equilibrium. It suffices to show that when everyone else uses  $K_{-n} = K' < K^*$ , it is a profitable deviation for the  $n$ -th trader to choose  $K_n = K^*$ . We need to compare the difference of expected profits:

$$\begin{aligned}
& \mathbb{E} [\tilde{\pi}_{z,n}(\tilde{y}_1; K_n = K^*, K_{-n} = K') - \tilde{\pi}_{z,n}(\tilde{y}_1; K_n = K', K_{-n} = K') | \tilde{y}_1 = y_1] \\
= & \mathbb{E} \left[ \left\{ -\eta \left( \frac{1}{2} - \frac{\lambda_1 \eta (N-2)}{2(1-\mu)} \right) (\hat{v}_T - \lambda_1 y_1)^2 - \eta (\hat{v}_T - \lambda_1 y_1) \frac{\tilde{v} - \hat{v}_T}{2} \right\} \mathbf{1}_{K' < |y_1| < K^*} \middle| y_1 \right] \\
= & -\frac{\eta}{2} \left[ \frac{4\Theta^2}{N+2} + (\hat{v} - \hat{v}_T)\Theta \right] \mathbf{1}_{K' < |y_1| < K^*}, \tag{2.52}
\end{aligned}$$

where  $\Theta = \hat{v}_T - \lambda_1 y_1$  is negative for  $K' < y_1 < K^*$  and positive for  $K' < -y_1 < K^*$ . For the case  $K^* = \kappa\sigma_u$ , we have  $\hat{v}_T = 0$  but  $\hat{v} \geq 0$  for  $|y_1| \in [K', \kappa\sigma_u]$ . It means the last expression of  $\Theta$  is a parabola that opens downward and crosses the origin. Since  $\Theta$  takes the opposite sign of  $y_1$  and  $\hat{v}$  for  $K' < |y_1| < K^*$ , the last expression is strictly positive for  $K' < |y_1| < K^*$ . Similar arguments can be applied to the case  $K^* = \frac{\rho(1+c)\kappa\sigma_u}{\rho-1}$ . Therefore,  $\mathbb{E} [\tilde{\pi}_{z,n}(\tilde{y}_1; K_n = K^*, K_{-n} = K') - \tilde{\pi}_{z,n}(\tilde{y}_1; K_n = K', K_{-n} = K')] > 0$  for  $K' < K^*$ , i.e., any threshold less than  $K^*$  cannot be an equilibrium threshold.

Similarly, any threshold  $K'$  larger than  $K^*$  cannot be an equilibrium threshold either. As before, it suffices to show that the deviation is profitable for any trader by just choosing

$K_n = K^*$  less than  $K'$  used by others. The payoff difference given  $y_1$  is positive as well:

$$\begin{aligned}
& \mathbb{E} [\tilde{\pi}_{z,n}(\tilde{y}_1; K_n = K^*, K_{-n} = K') - \tilde{\pi}_{z,n}(\tilde{y}_1; K_n = K', K_{-n} = K') | \tilde{y}_1 = y_1] \\
&= \mathbb{E} \left[ \eta(\hat{v}_T - \lambda_1 y_1) \mathbf{1}_{K^* < |y_1| < K'} \left[ \frac{\tilde{v} - \lambda_1 y_1}{2} - \frac{\lambda_1 \eta(\hat{v}_T - \lambda_1 y_1)}{1 - \mu} \right] \middle| y_1 \right] \\
&= \frac{\eta}{2} \left[ \frac{N}{N+2} \Theta^2 + (\hat{v} - \hat{v}_T) \Theta \right] \mathbf{1}_{K^* < |y_1| < K'} > 0.
\end{aligned} \tag{2.53}$$

It rules out any threshold larger than  $K^*$  to be an equilibrium. So the only possible equilibrium choice is  $K^*$ . When every trader uses the same threshold  $K^*$ , no one will deviate.

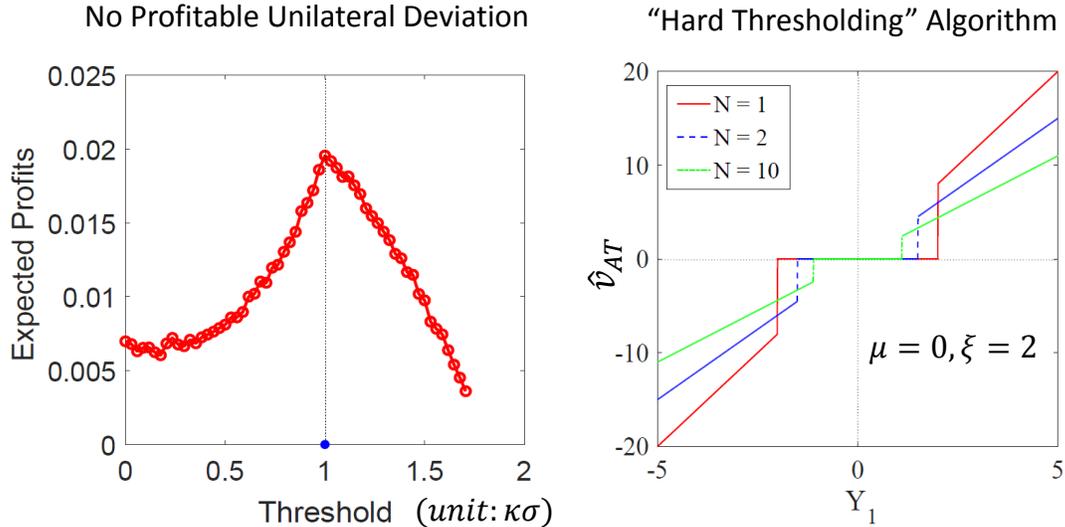


Figure 2.8: Left: Trading profits if someone deviates from  $K^*$ . Right: Learning rule  $\hat{v}_T(y_1)$ .

Now look at the informed trader in this algorithmic trading game. If arbitrageurs all use the same threshold  $K$  (which can be general), the informed trader at  $t = 2$  will trade

$$X_2(v, y_1; K) = (1 - \mu) \frac{v - \lambda_1 y_1}{2\lambda_1} - s \frac{N(1 - \mu)(\rho - 1)}{2(N + 2)} \left[ y_1 - \text{sign}(y_1) \frac{\rho(1 + c)\kappa\sigma_u}{\rho - 1} \right] \mathbf{1}_{|y_1| > K}, \tag{2.54}$$

and pick  $X_1(v, K)$  that maximizes her total payoff. The price at  $t = 2$  can be written as

$$\tilde{p}_2 = \lambda_1 y_1 + \lambda_2 (x_2 + Z_2 \mathbf{1}_{|y_1| > K} + \tilde{u}_2) = \begin{cases} \frac{(N+2)\tilde{v} + 2\lambda_1 y_1 + N\hat{v}_T}{2(N+2)} + \lambda_2 \tilde{u}_2, & \text{if } |y_1| > K \\ \frac{\tilde{v} + \lambda_1 y_1}{2} + \lambda_2 \tilde{u}_2, & \text{if } |y_1| < K, \end{cases} \quad (2.55)$$

depending on whether the arbitrageurs are triggered. It is interesting to notice that the expected price conditional on  $v$  and  $y_1$  is piecewise linear,

$$\mathbb{E}[P_2|v, y_1] = \frac{v + \lambda_1 y_1}{2} + \frac{N\lambda_1[(\rho - 1)y_1 - \text{sign}(y_1)\rho(1 + c)\kappa\sigma_u]}{2(N + 2)} \mathbf{1}_{|y_1| > K}. \quad (2.56)$$

We have  $\lim_{\mu \rightarrow 0} \mathbb{E}[P_2|v, y_1] = \frac{v + \lambda_1 y_1}{2} + \frac{\lambda_1 y_1 \mathbf{1}_{|y_1| > K}}{2} \leq \lambda_1 y_1 + \frac{v}{2}$ , which means the average price change at  $t = 2$  is at most half of the fundamental value when market depth is constant.

The informed trader's expected profit in the second period is also contingent on the state of arbitrageurs:

$$\Pi_{2,x}(v, y_1; K) = \mathbb{E}[x_2(v - \tilde{p}_2)|\mathcal{I}_{2,x}] = \begin{cases} \frac{1}{\lambda_2} \left[ \frac{(N+2)v - N\hat{v}_T(y_1) - 2\lambda_1 y_1}{2(N+2)} \right]^2, & \text{if } |y_1| > K \\ \frac{1}{\lambda_2} \left( \frac{v - \lambda_1 y_1}{2} \right)^2, & \text{if } |y_1| < K. \end{cases} \quad (2.57)$$

Note that her expected profit in the second period is always positive because the informed trader fully anticipates the response of arbitrageurs. The informed trader needs to determine  $x_1 = X_1(v; K)$  that maximizes the total expected profit from both periods. The calculation of her total profit, conditional on the private signal  $v$ , can be decomposed into three terms:

$$\begin{aligned} \Pi_x(v, x_1; K) &= \max_{x_1} \mathbb{E} [\Pi_{1,x} + \Pi_{2,x} \mathbf{1}_{|\tilde{y}_1| < K} + \Pi_{2,x} \mathbf{1}_{|\tilde{y}_1| > K} | \mathcal{I}_{1,x}] \\ &= x_1(v - \lambda_1 x_1) + \mathbb{E} \left[ \frac{(v - \lambda_1(x_1 + \tilde{u}_1))^2}{4\lambda_2} \mathbf{1}_{|x_1 + \tilde{u}_1| < K} \middle| \mathcal{I}_{1,x} \right] \\ &\quad + \mathbb{E} \left[ \frac{[(N+2)v - N\hat{v}_T(x_1 + \tilde{u}_1) - 2\lambda_1(x_1 + \tilde{u}_1)]^2}{4(N+2)^2\lambda_2} \mathbf{1}_{|x_1 + \tilde{u}_1| > K} \middle| \mathcal{I}_{1,x} \right] \end{aligned} \quad (2.58)$$

On one hand, the informed trader may want to trade less to avoid triggering arbitrageurs and take full advantage of her information at  $t = 2$ . On the other hand, it is costly to hide

her private signal if it is strong. This trade-off will reflect in the relative values of  $\Pi_{2,x}^-$  and  $\Pi_{2,x}^+$  which are defined below. Hereafter, I set  $\sigma_u = 1$  for convenience. By direct integration, one can derive their expressions:

$$\begin{aligned}
& \Pi_{2,x}^-(v, x_1; K) \\
& \equiv \mathbb{E}[\Pi_{2,x} \mathbf{1}_{|\tilde{y}_1| < K} | \mathcal{I}_{1,x}] \\
& = \mathbb{E} \left[ \frac{(v - \lambda_1 \tilde{y}_1)^2}{4\lambda_2} \mathbf{1}_{|\tilde{y}_1| < K} \middle| \mathcal{I}_{1,x} \right] \\
& = \frac{(1 - \mu)[2v - \lambda_1(K + x_1)]\phi(K - x_1)}{4} - \frac{(1 - \mu)[2v + \lambda_1(K - x_1)]\phi(K + x_1)}{4} \\
& \quad + \frac{(1 - \mu)[(v - \lambda_1 x_1)^2 + \lambda_1^2]}{8\lambda_1} \left[ \operatorname{erf} \left( \frac{K - x_1}{\sqrt{2}} \right) + \operatorname{erf} \left( \frac{K + x_1}{\sqrt{2}} \right) \right], \tag{2.59}
\end{aligned}$$

$$\begin{aligned}
& \Pi_{2,x}^+(v, x_1; K) \\
& \equiv \mathbb{E}[\Pi_{2,x} \mathbf{1}_{|\tilde{y}_1| > K} | \mathcal{I}_{1,x}] \\
& = \mathbb{E} \left[ \frac{[(N + 2)v - N\hat{v}_T(\tilde{y}_1) - 2\lambda_1 \tilde{y}_1]^2}{4(N + 2)^2\lambda_2} \mathbf{1}_{|\tilde{y}_1| > K} \middle| \mathcal{I}_{1,x} \right] \\
& = \frac{(1 - \mu)(N\rho + 2)}{4(N + 2)^2} [-2Nw\rho\lambda_1 - 2(N + 2)v + \lambda_1(N\rho + 2)(K + x_1)]\phi(K - x_1) \\
& \quad + \frac{(1 - \mu)(N\rho + 2)}{4(N + 2)^2} [-2Nw\rho\lambda_1 + 2(N + 2)v + \lambda_1(N\rho + 2)(K - x_1)]\phi(K + x_1) \\
& \quad + \frac{1 - \mu}{8(N + 2)^2\lambda_1} \left\{ [(N + 2)v + Nw\rho\lambda_1 - (N\rho + 2)\lambda_1 x_1]^2 + \lambda_1^2(N\rho + 2)^2 \right\} \operatorname{erfc} \left( \frac{K - x_1}{\sqrt{2}} \right) \\
& \quad + \frac{1 - \mu}{8(N + 2)^2\lambda_1} \left\{ [(N + 2)v - Nw\rho\lambda_1 - (N\rho + 2)\lambda_1 x_1]^2 + \lambda_1^2(N\rho + 2)^2 \right\} \operatorname{erfc} \left( \frac{K + x_1}{\sqrt{2}} \right). \tag{2.60}
\end{aligned}$$

where  $w = (1 + c)\kappa\sigma_u$  is the horizontal intercept of  $\hat{v}_T(y_1)$  and where  $\phi(K \pm x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(K \pm x)^2}{2}}$  denotes the probability density function of the standard normal distribution (with  $\sigma_u = 1$ ).

Taking the first derivative,  $\frac{d\Pi_x}{dx_1} = 0$ , one can find the FOC for  $X_1(v; K) = x_1$ :

$$\begin{aligned}
0 = & v - 2\lambda_1 x_1 - \frac{(1-\mu)(v - \lambda_1 x_1)}{4} \left[ \operatorname{erf} \left( \frac{K - x_1}{\sqrt{2}} \right) + \operatorname{erf} \left( \frac{K + x_1}{\sqrt{2}} \right) \right] \\
& + \frac{(1-\mu)[(v - \lambda_1 K)^2 + 2\lambda_1^2]}{4\lambda_1} [\phi(K + x_1) - \phi(K - x_1)] + (1-\mu)Kv\phi(K + x_1) \\
& + \frac{(1-\mu)\phi(K - x_1)}{4\lambda_1(N+2)^2} \{ [K\lambda_1(N\rho + 2) - (N+2)v]^2 + 2\lambda_1^2(N\rho + 2)^2 \\
& \quad + \lambda_1 w N \rho [\lambda_1 w N \rho - 2K\lambda_1(N\rho + 2) + 2(N+2)v] \} \\
& - \frac{(1-\mu)\phi(K + x_1)}{4\lambda_1(N+2)^2} \{ [K\lambda_1(N\rho + 2) + (N+2)v]^2 + 2\lambda_1^2(N\rho + 2)^2 \\
& \quad + \lambda_1 w N \rho [\lambda_1 w N \rho - 2K\lambda_1(N\rho + 2) - 2(N+2)v] \} \\
& - \frac{(1-\mu)(N\rho + 2)}{4(N+2)^2} [(N+2)v + Nw\rho\lambda_1 - (N\rho + 2)\lambda_1 x_1] \operatorname{erfc} \left( \frac{K - x_1}{\sqrt{2}} \right) \\
& - \frac{(1-\mu)(N\rho + 2)}{4(N+2)^2} [(N+2)v - Nw\rho\lambda_1 - (N\rho + 2)\lambda_1 x_1] \operatorname{erfc} \left( \frac{K + x_1}{\sqrt{2}} \right). \tag{2.61}
\end{aligned}$$

This FOC equation defines the informed trader's optimal strategy  $X_1 = x_1(v; K)$  at  $t = 1$ .

The unconditional expected total profit of all arbitrageurs  $\Pi_z^{tot} \equiv \mathbb{E} \left[ \sum_{n=1}^N \tilde{\pi}_{z,n}(\tilde{v}, \tilde{u}_1, \tilde{u}_2) \right] =$

$\mathbb{E}[(\tilde{v} - \tilde{p}_2)Z_2 \mathbf{1}_{|\tilde{y}_1| > K}]$ . After solving  $x_1 = X_1(v; K)$  given any  $v$ , one can compute the condi-

tional expected profit:

$$\begin{aligned}
& \mathbb{E} \left[ \sum_{n=1}^N \tilde{\pi}_{z,n}(\tilde{v}, \tilde{u}_1, \tilde{u}_2) \middle| \tilde{v} = v \right] = \mathbb{E} [(\tilde{v} - \lambda_1 \tilde{y}_1 - \lambda_2 (X_2(\tilde{v}, \tilde{y}_1) + Z_2(\tilde{y}_1))) Z_2(\tilde{y}_1) \mathbf{1}_{|\tilde{y}_1| > K} | \tilde{v} = v] \\
= & \frac{N(1-\mu)}{2(N+2)^2} [w\lambda_1\rho(2N\rho + 2 - N) + (\rho - 1)((N+2)v - \lambda_1(N\rho + 2)(x_1 + K))] \phi(K - x_1) \\
& + \frac{N(1-\mu)}{2(N+2)^2} [w\lambda_1\rho(2N\rho + 2 - N) - (\rho - 1)((N+2)v - \lambda_1(N\rho + 2)(x_1 - K))] \phi(K + x_1) \\
& - \frac{N(1-\mu)}{4(N+2)^2} [(N+2)v(\rho(w - x_1) + x_1) + \lambda_1 N \rho^2 (1 + (w - x_1)^2) \\
& \quad - \lambda_1 \rho (N - 2)(x_1^2 - wx_1 + 1) - 2\lambda_1(1 + x_1^2)] \operatorname{erfc} \left( \frac{K - x_1}{\sqrt{2}} \right) \\
& + \frac{N(1-\mu)}{4(N+2)^2} [(N+2)v(\rho(w + x_1) - x_1) - \lambda_1 N \rho^2 (1 + (w + x_1)^2) \\
& \quad + \lambda_1 \rho (N - 2)(x_1^2 + wx_1 + 1) + 2\lambda_1(1 + x_1^2)] \operatorname{erfc} \left( \frac{K + x_1}{\sqrt{2}} \right), \tag{2.62}
\end{aligned}$$

where  $w \equiv (1 + c)\kappa\sigma_u$  and  $\sigma_u = 1$ . Finally, the unconditional total payoff to arbitrageurs is

$$\Pi_z^{tot} = \mathbb{E} \left[ \sum_{n=1}^N \tilde{\pi}_{z,n}(\tilde{v}, \tilde{u}_1, \tilde{u}_2) \right] = \int_{-\infty}^{+\infty} f_L(v) \mathbb{E} \left[ \sum_{n=1}^N \tilde{\pi}_{z,n}(\tilde{v}, \tilde{u}_1, \tilde{u}_2) \middle| \tilde{v} = v \right] dv. \quad (2.63)$$

### 2.5.5 Proof of Corollary 9

Since  $\lim_{\mu \rightarrow 0} c = -1$  and  $\lim_{\mu \rightarrow 0} \rho = 2 + \frac{2}{N}$ , one can derive that for the informed trader

$$\begin{aligned} \lim_{v \rightarrow 0} \lim_{\mu \rightarrow 0} \Pi_x &= \lambda_1 - \frac{3\lambda(1 + x_1^2)}{8} \left[ \operatorname{erf} \left( \frac{K - x_1}{\sqrt{2}} \right) + \operatorname{erf} \left( \frac{K + x_1}{\sqrt{2}} \right) \right] \\ &\quad + \frac{3\lambda_1}{4} \{ [\phi(K - x_1) + \phi(K + x_1)] K + [\phi(K - x_1) - \phi(K + x_1)] x_1 \}, \end{aligned} \quad (2.64)$$

which only depends on  $x_1$ ,  $\lambda_1$ , and  $K$ . The FOC equation in this limiting case becomes

$$\frac{3\lambda}{4} \left[ (2 + K^2) [\phi(K - x_1) - \phi(K + x_1)] - x_1 \left( \operatorname{erf} \left( \frac{K - x_1}{\sqrt{2}} \right) + \operatorname{erf} \left( \frac{K + x_1}{\sqrt{2}} \right) \right) \right] = 0. \quad (2.65)$$

Using the equilibrium threshold  $K^*(\mu = 0) = \kappa$  with  $\sigma_u = 1$ , we can rewrite the FOC as:

$$\frac{x_1}{2 + \kappa^2} = \frac{\phi(\kappa - x_1) - \phi(\kappa + x_1)}{\operatorname{erf} \left( \frac{\kappa - x_1}{\sqrt{2}} \right) + \operatorname{erf} \left( \frac{\kappa + x_1}{\sqrt{2}} \right)}, \quad (2.66)$$

which may have multiple solutions: one is obviously  $x_1 = 0$  and the other two are  $\pm\infty$ .

As long as the informed trader trades a sufficiently large quantity  $x_1 \gg \kappa$  (instead of  $\pm\infty$ ), the probability of triggering arbitrageurs to trade is arbitrarily close to one. In the second period, the informed trader's optimal strategy is found to be  $\lim_{\mu \rightarrow 0} X_2(v = 0, y_1) = -y_1$ , which exactly offsets the total quantity traded by arbitrageurs  $\lim_{\mu \rightarrow 0} Z_2(y_1) = y_1$ . Thus, the terminal position of the informed trader is  $x_1 + x_2 = -u_1$  which is zero on average. The expected profit from this disruptive strategy is found to be  $\Pi_x \approx \lambda_1 \sigma_u^2$ , which is limited by the noise trading volatility in the first period.

## 2.5.6 Economic Conditions for Trade-Based Manipulations

The disruptive strategy by the savvy informed trader involves three key conditions:

- (1) *Speculators think that market makers set inaccurate prices by using incorrect priors.*
- (2) *Speculators have fat-tail priors about the fundamental value or trading opportunities.*
- (3) *There is strategic interaction between the informed trader and those speculators.*

First, if speculators (arbitrageurs) trust in market efficiency, they will not trade in a market where they have no superior information. Moreover, for any trader to play the disruptive strategy, she needs to know that the asset liquidation value will not deviate from its initial price (i.e.  $v \approx 0$ ). The third condition emphasizes the strategic interaction between the informed trader and the group of less-informed speculators. The informed trader can twist her strategy to induce more aggressive trading by speculators.

Now we discuss the second condition. If speculators have the Gaussian prior instead of the fat-tail prior, they can conjecture a linear equilibrium where the informed trader's strategy  $X_1(v)$  is a linear function and arbitrageurs' strategy  $Z_{2,n}(y_1)$  is also linear with  $y_1$  for  $n = 1, \dots, N$ . With the conjecture  $X_1 = \frac{v}{\rho\lambda_1}$ , arbitrageurs' posterior expectation becomes  $\hat{v} = E[\tilde{v}|y_1] = Ay_1$  where  $A \equiv \frac{\rho\lambda_1\sigma_v^2}{\sigma_v^2 + (\rho\lambda_1\sigma_u)^2}$ . Substituting this linear estimator into the informed trader's FOC (still Eq. 2.29) yields

$$0 = v - 2\lambda_1 x_1 - \frac{1 - \mu}{\lambda_1} \left[ \frac{(N + 2)v - (NA + 2\lambda_1)x_1}{2(N + 2)} \cdot \frac{NA + 2\lambda_1}{N + 2} \right], \quad (2.67)$$

from which one can solve

$$x_1 = \frac{2\lambda_1(N + 2)^2 - (1 - \mu)(N + 2)(NA + 2\lambda_1)}{4\lambda_1^2(N + 2)^2 - (1 - \mu)(NA + 2\lambda_1)^2} v. \quad (2.68)$$

Now one can match the above coefficient with the linear conjecture  $X_1 = \frac{v}{\rho\lambda_1}$  to obtain

$$\frac{1}{\rho\lambda_1} = \frac{2\lambda_1(N+2)^2 - (1-\mu)(N+2)(NA(\rho) + 2\lambda_1)}{4\lambda_1^2(N+2)^2 - (1-\mu)(NA(\rho) + 2\lambda_1)^2}. \quad (2.69)$$

One has to solve a quintic<sup>11</sup> equation of  $\rho$  which can be done numerically given  $\{\mu, N, \lambda_1\}$ .

With the Gaussian prior, the optimal strategies for arbitrageurs are always linear, with a response slope less than one. The informed trader's strategy is also linear. As an example, let's consider the simple case of  $\sigma_v = \frac{3}{\sqrt{2}}$ ,  $\sigma_u = 1$ , and  $\mu = 0$  in the limit  $N \rightarrow \infty$ . Eq. (2.69) becomes  $-4 + 2\rho - \frac{18\rho^4}{(2\rho^2+9)^2} = 0$ , where the unique real solution turns out to be  $\rho = 3$ .

Back to the model with fat-tail prior, both Fig. 2.3 and Fig. 2.4 demonstrate the importance of the relative liquidity condition at  $t = 2$ . The informed trader's strategy  $X_1(v)$  is always a monotone and increasing function of  $v$  in the liquidity regime  $\mu > \mu_\epsilon$  where  $\mu_\epsilon \approx 0.005$  according to numerical experiments. The presence of trading costs and/or inventory holding costs will erode the finite profits earned by the disruptive strategy. Potential punishment by regulators plays a similar role to discourage disruptive trading. Moreover, if there are two informed traders in the market and both of them know  $v \approx 0$ , then neither of them would play the non-monotone disruptive strategy. This is intuitive: if one of them had engaged in aggressive trading (to entice other speculators), then the other informed trader would trade against her in the second stage so that the initial instigator could not fully liquidate the inventory. The above arguments explain the other conditions in the main text.

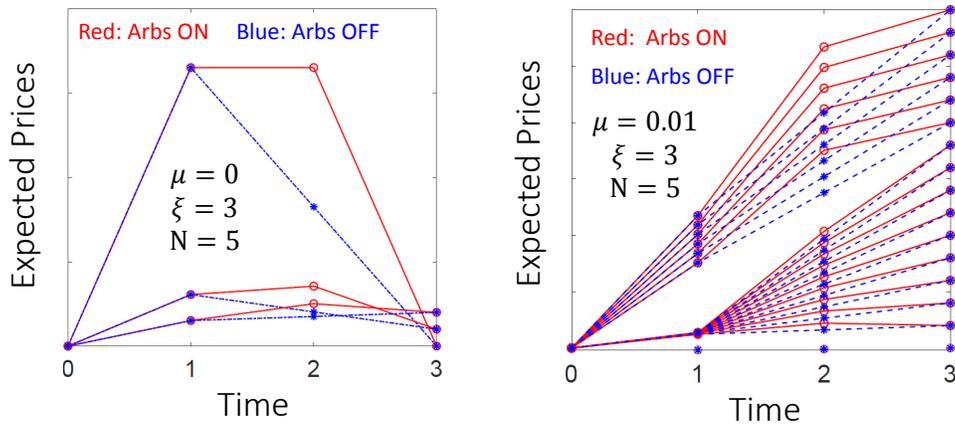
- (4) *Market depth is not decreasing when the informed trader liquidates her inventory.*
- (5) *Traders face no trading costs, no inventory costs, nor threat from regulators.*
- (6) *There is no other informed trader who could interfere with the disruptive trading scheme.*

---

<sup>11</sup>A quintic equation is defined by a polynomial of degree five.

In fact, if speculators (arbitrageurs) restrict themselves to convex strategies, their linear-triggering strategy at  $\mu = 0$  will be exactly linear without any kinks:  $\lim_{\mu \rightarrow 0} Z_{2,n}^{conv}(y_1) = \frac{y_1}{N}$ . In this case, their total order flow is  $\lim_{\mu \rightarrow 0} Z_2^{conv}(y_1) = y_1$  and the best response of informed trader at  $t = 1$  becomes monotone again.

Finally, I report the expected price trajectories contingent on whether the speculators with linear-triggering strategies are triggered or not. The figure below shows the sample path of prices which can overshoot or undershoot depending on whether speculators are triggered. The price pattern at  $\mu = 0$  shown in the left panel exhibits the “bubble and crash” of asset prices. The clustering of prices in the right panel illustrates how the informed trader hides her private signal in the first period to inhibit the inference and response of speculators.





# Chapter 3

## What if the Long Forward Rate is Flat?

### 3.1 Introduction

This chapter studies several fundamental issues regarding long-run asset pricing and interest rate modeling. Hansen and Scheinkman (70) employ the advanced semigroup approach to develop the permanent-transitory decomposition of the pricing kernel in a frictionless, Markovian economy. Their result is of great interest for the long-term asset valuation and risk management, with many implications for macro-finance studies as well. The present article employs the martingale approach to give a simple proof for the Hansen-Scheinkman decomposition under the economic condition that the long forward rate is constant. When the long-maturity bond is taken as the numeraire, an explicit expression is derived for the permanent martingale component as it defines the change of probabilities to the long forward measure. This permanent martingale is shown to covary with the transitory component, consistent with empirical findings. This work further identifies a strong restriction in a class of term structure models and compares different approaches of interest rate modeling.

In financial markets that admit no arbitrage, the price of an asset today is determined by taking the expectation of future cash flows discounted by a strictly positive *stochastic discount factor* (SDF) which is also known as the *pricing kernel* (118, 49). The concept of risk-neutral probability is introduced by Cox and Ross (36) and is further developed by Harrison and Kreps (73). The SDF for risk-neutral pricing is a product of the risk-free discounting factor and a positive martingale component that encodes the risk premium and defines a change of measure from the physical probability to the risk neutral probability (under which future cash flows can be discounted by the risk-free rate). This SDF decomposition is widely used but not unique. For example, one can also choose the  $T$ -maturity zero-coupon bond as the numeraire so that the SDF contains a new martingale component which defines the change of probabilities to the  $T$ -forward measure absolutely continuous with respect to the risk-neutral measure. This approach was pioneered by Jamshidian (86) as a useful means for bond option pricing. Under the forward measure, forward prices are martingales, as also remarked and studied by Geman (62).

Cash flows can occur over a wide range of investment horizons. The dynamics of risk factors and the properties of risk-return tradeoffs may differ across alternative time scales. For asset pricing in the long horizon, it seems to be appropriate and convenient to introduce an alternative SDF decomposition by using the long bond (i.e., zero-coupon bond with extremely long maturity) as the numeraire. Alvarez and Jermann (5) explore this topic in discrete time and find that the pricing kernel has a very volatile permanent martingale component. Hansen and Scheinkman (70) establish a systematic development for the long-run SDF factorization in a general continuous-time Markovian economy. Using

the approach of semigroup pricing operator<sup>1</sup>, Hansen and Scheinkman (70) show a multiplicative decomposition of SDF into three components: an exponential term determined by the principal eigenvalue, a transient eigenfunction term, and a permanent martingale term. The principal eigenvalue encodes the risk adjustment, the martingale component alters the probability measure to capture long-run approximations, and the principal eigenfunction gives the long-run state dependence. Further theoretical developments in Ref. (68, 71, 17, 18, 113, and 114). Among those, Qin and Linetsky (114) take the martingale approach and make an excellent extension of the long-term SDF factorization to general semi-martingale environments. Related empirical studies can be found in Ref. (69, 9, 8, 115, 11, 10, and 30). Among those, Bakshi and Chabi-Yo (9) derive and test new variance bounds closely related to the results of Alvarez and Jermann (5). More recently, Christensen (30) provides a nonparametric approach to reconstruct the time series of the estimated permanent and transitory components.

A further strand of literature closely related to the one outlined above is on the recovery theory proposed by Ross (119). It is a theoretical procedure based on the Perron-Frobenius theorem that permits the recovery of physical probabilities from prices of contingent claims in a finite-state discrete-time Markov environment. Martin and Ross (110) also discuss identification of recovered probability measure with the long forward measure. Carr and Yu (24) extend Ross' arguments to one-dimensional diffusion processes that are constrained on finite intervals with regular boundary conditions. This setup allows the use of Sturm-Liouville theory to single out a unique positive eigenfunction and achieve the recovery result

---

<sup>1</sup>When the economy is driven by time-homogeneous Markovian state variables, the pricing operators indexed by time form a semigroup operator. This semigroup approach was introduced by Garman (60).

similar to Ross (119). Further extensions are made by Dubynskiy and Goldstein (44) and Walden (130) in more general diffusion models. Borovicka et al. (18) remark that in Ross' recovery the key assumption of transition-independent pricing kernel<sup>2</sup> restricts the permanent martingale component (in the Hansen-Scheinkman decomposition) to be degenerate. This limitation has quite different economic implications that are not fully clear in the literature.

The work in this chapter complements the existing literature following Alvarez and Jermann (5) and Hansen and Scheinkman (70). Here, the no-arbitrage martingale approach is taken in a general Markovian diffusion setup under the key condition that the long forward rate is constant. A constant long forward rate is plausible in a normal economy according to Dybvig, Ingersoll, and Ross (48). If so, the price behavior of long-maturity bonds is found to exhibit separable dependence on time and state (Proposition 7). This property can be utilized to give a simpler proof (Proposition 8) of the Hansen-Scheinkman decomposition, i.e., the pricing kernel can be expressed as a product of the transient and permanent components, where the transient one encodes the same information of long bonds and the permanent one is a martingale defining the transition to the long forward measure. In fact, the long forward rate corresponds to the principal eigenvalue of the pricing operator, whereas the associated eigenfunction is determined by the state-dependence of long bonds (Corollary 11). When the Markovian economy is recurrent, the long forward rate must be constant because it can never fall by the arguments of Dybvig et al. (48) and it cannot rise either by the recurrent property. In this case, one can find a pair of principal eigenvalue and eigenfunction of the pricing operator (Corollary 12) as in Hansen and Scheinkman (70).

---

<sup>2</sup>Pricing kernels are generally transition-dependent, for example, in models with recursive preferences as analyzed by Hansen and Scheinkman (71).

Next, analyzing the forward measure and its Radon-Nikodym derivative yields an analytical expression for the permanent martingale term whose Girsanov kernel is the sum of the market prices of risk and the long bond return volatility (Proposition 9). This result implies the covariance between permanent and transitory components (Corollary 13). In general, the permanent martingale is not degenerate nor orthogonal to the transitory component, consistent with the empirical results reported by Bakshi et al. (10) and Christensen (30).

An important observation in this chapter is that the long-run SDF decomposition can be feasible but may not be unique when the economy does not have an equivalent martingale measure. Heston, Loewenstein, and Willard (77) point out that there may exist multiple solutions to the asset pricing partial differential equations when certain theoretical conditions fail to hold. For example, an interest rate model developed by Cox, Ingersoll, and Ross (34) may exclude the equivalent martingale measure under certain parameters. This indicates bubbles in bond prices and potential arbitrage opportunities. In this stationary economy, a unique long-forward rate can be found, but the long-run SDF decomposition is not unique as it depends on how bonds are valued.

The Girsanov kernel for the long forward measure uncovers a strong restriction in a family of interest rate models which implicitly assume a degenerate permanent martingale. This restriction means that the market prices of risk have to cancel with the long bond return volatility in those models (Proposition 10). Examples include the quadratic term structure model proposed by Constantinides (33) and models generated by the probabilistic potential approach of Rogers (117). A remedy is suggested to avoid this restriction issue. It is recognized that pricing kernels in models such as Ross' Recovery and Lucas' Tree are

transition-independent. Nonetheless, the general equilibrium production economy developed by Cox, Ingersoll, and Ross (35) naturally produces a transition-dependent SDF. This SDF has a non-degenerate permanent martingale term and encodes flexible risk premiums (Proposition 11), showing the advantage of this framework for interest rate modeling.

## 3.2 Long-Run Pricing Kernel Factorization

This section gives the general setup and key results. If the long forward rate is flat, then long-maturity bonds could exhibit almost separable dependence on time and state. This feature inspires a simple proof for the Hansen-Scheinkman decomposition which was originally developed using the semigroup approach. Several examples in this section will show how to find the long forward measure without solving the eigenfunction problem.

### 3.2.1 General Setup and Assumptions

I focus on bond pricing in this chapter. The state variables considered here are any of those that affect the bond market. The following standard assumptions are made about the economy:

(A1) *The economy is stationary and Markovian.*

(A2) *The market is frictionless.*

(A3) *The market admits no arbitrage.*

(A4) *The pricing rule is time consistent.*

As for (A1), the state of economy is characterized by a  $k \times 1$  vector of state variables,  $X_t$ ,

which are defined on the probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$  and take values on a state space  $\mathbb{R}^k$ .

The sample paths of  $X$  are continuous from the right with left limits. They are assumed to be stationary and ergodic, satisfying the general time-homogeneous Itô diffusion process:

$$dX_t = \mu_X(X_t)dt + \Sigma_X(X_t)dW_t, \quad (3.1)$$

where  $W_t$  represents a  $d$ -dimensional  $\mathcal{F}_t$ -measurable Brownian motion. The drift term  $\mu_X(\cdot)$  and the diffusion term  $\Sigma_X(\cdot)$  satisfy the usual Lipschitz continuity condition that ensures the existence of a unique strong solution  $X$  to Eq. (3.1). It is further assumed that  $\Sigma_X^\top(x)\Sigma_X(x)$  is nonsingular and invertible so that the Brownian increment can be inferred from the sample path of  $X$ . As a direct implication of **(A2)** and **(A3)**, arbitrage opportunities are ruled out in a frictionless economy if and only if there exists a strictly positive stochastic discount factor (SDF) process; see Harrison and Kreps (73), denoted by  $\{M_t\}_{t \geq 0}$  with  $M_0 = 1$ . These conditions also imply the existence of a risk-neutral measure  $\mathbb{Q}$  equivalent to the physical measure  $\mathbb{P}$  so that assets in this economy can be priced in the risk-neutral world by discounting their future cash flows using the (locally) risk-free interest rate,  $r(X_t)$ . The time consistency of the pricing rule **(A4)** means that asset pricing is independent of the calendar time at which the asset is valued. Mathematically, this property can be expressed as  $M_{t+s}(\omega) = M_t(\omega)M_s(\Theta_t(\omega))$ , where  $\Theta_s : \Omega \rightarrow \Omega$  defines the time-shift operator so that  $X_s(\Theta_t(\omega)) = X_{t+s}(\omega)$ . The time- $t$  price of a contractual payoff  $\psi(X_T)$  at time  $T \geq t$  is

$$\mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(X_s)ds} \psi(X_T) \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}} \left[ \frac{M_T}{M_t} \psi(X_T) \middle| \mathcal{F}_t \right], \quad (3.2)$$

where the last step follows from the Markovian assumption **(A1)**. Expression (3.2) allows us to define a pricing operator  $\mathcal{P}_t$  by the pricing rule below:

$$\mathcal{P}_t \psi(x) = \mathbb{E}^{\mathbb{P}} [M_t \psi(X_t) | X_0 = x], \quad (3.3)$$

which sets the time-zero price of a payoff  $\psi(X_t)$  realized at time  $t$  as a function of  $X_0 = x$ . As remarked by Garman (60), the temporal consistency of pricing rule (A4) implies that the family of linear pricing operators  $\{\mathcal{P}_t : t \geq 0\}$  forms a semigroup:  $\mathcal{P}_0 = \mathbb{I}$  and  $\mathcal{P}_{t+s}\psi(x) = \mathcal{P}_t\mathcal{P}_s\psi(x)$ . This forms the basis of semigroup arguments of Hansen and Scheinkman (70).

### 3.2.2 Long-run SDF Factorization

The long-maturity discount bond price is key to the long-run SDF decomposition. Clearly, the time- $t$  price of a default-free zero-coupon bond that pays one dollar at time  $T$  is

$$\mathcal{P}_{T-t}1 = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(X_s)ds} | \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[M_{T-t} | X_t] = P(T-t, X_t), \quad (3.4)$$

which only depends on its remaining maturity  $(T-t)$  and the state variables at time  $t$ . If Assumption (A4) is relaxed (e.g., if investors have time-varying preference), the bond price may depend on the calendar time of valuation. We write  $P(t, T, X_t)$  in general.

**Definition 1.** It is standard to define the forward rate as

$$f(t, T, X_t) = -\frac{\partial \log P(t, T, X_t)}{\partial T}, \quad (3.5)$$

and the zero coupon rate (or yield) as

$$z(t, T, X_t) = -\frac{\log P(t, T, X_t)}{T-t}. \quad (3.6)$$

Let's further define the long forward rate and the long zero coupon rate, respectively, by

$$f_l(t, X_t) = \lim_{T \rightarrow \infty} f(t, T, X_t), \quad z_l(t, X_t) = \lim_{T \rightarrow \infty} z(t, T, X_t). \quad (3.7)$$

**Lemma 1.** *If the (nonzero) long forward rate exists, then the long zero coupon rate also exists and equals the long forward rate state by state at any point of time:  $f_l(t, X_t) = z_l(t, X_t)$ .*

*Proof.* Dybvig et al. (48) proved the above statement in discrete time. For the continuous-time setup, a simple relationship exists between the forward rate and the zero-coupon rate:

$$z(t, T, X_t) = \frac{1}{T-t} \int_t^T f(t, s, X_t) ds. \quad (3.8)$$

For an arbitrary state vector  $X_t = x \in \mathbb{R}^k$ , one can take limits on both sides and derive that

$$z_l(t, x) = \lim_{T \rightarrow \infty} z(t, T, x) = \lim_{T \rightarrow \infty} \frac{\int_t^T f(t, s, x) ds}{T-t} = \lim_{T \rightarrow \infty} f(t, T, x) = f_l(t, x). \quad (3.9)$$

The L'Hôpital's rule is used since both  $\int_t^T f(t, s, x) ds$  and  $(T-t)$  explode as  $T \rightarrow \infty$ .  $\square$

In a normal economy, the discount bond price approaches zero when its maturity goes to infinity, i.e.,  $P(t, T, X_t) \rightarrow 0$  as  $T \rightarrow \infty$ . The long forward rate actually sets the pace of this asymptotic behavior. In fact, Lemma 1 imposes simple constraints on the price behavior of long-maturity discount bonds.

**Proposition 7.** *If the long forward rate is constant (denoted by  $\rho$ ), then the logarithm of discount bond price can be decomposed into three terms*

$$\log P(T-t, X_t) = -\rho \cdot (T-t) - g(X_t) + \varepsilon(T-t, X_t), \quad (3.10)$$

where  $g(x) \equiv \lim_{T \rightarrow \infty} (z(T-t, x) - \rho) \cdot (T-t)$  is a limiting function of  $X_t = x$  and the residual term  $\varepsilon(T-t, X_t)$  is vanishing at infinitely long maturity:  $\lim_{T \rightarrow \infty} \varepsilon(T-t, X_t) = 0$ .

*Proof.* See Appendix 3.6.1  $\square$

Proposition 7 suggests that the discount bond price exhibits separable dependence on time and state variables when its maturity is infinitely long. This property enables us to introduce the concept of *long bond* by eliminating the transient dependencies of discount bonds.

**Definition 2.** The price function below will be referred to as the *long bond*:

$$P_\infty(T-t, X_t) \equiv e^{-\rho(T-t)-g(X_t)}. \quad (3.11)$$

**Condition 1.** The residual term  $\varepsilon(T-t, x)$  is non-decreasing with  $T$  given any  $x$ .

**Condition 2.** The residual term  $\varepsilon(T-t, x) \leq K(x)$  for any  $x$  where  $E^{\mathbb{Q}}[e^{K(X_t)-g(X_t)}] < \infty$ .

**Proposition 8.** *If the long forward rate is constant (denoted  $\rho$ ), then the stochastic discount factor can be expressed in a multiplicative form under Condition 1 or 2:*

$$M_t = e^{-\rho t} \frac{\phi(X_0)}{\phi(X_t)} \xi_t^P, \quad (3.12)$$

where  $\phi(X_t) = e^{-g(X_t)} > 0$  and  $\xi_t^P$  is a local martingale with respect to  $\mathbb{P}$  satisfying  $\xi_0^P = 1$ .

When  $\xi_t^P$  is a strictly positive martingale, it defines a change of measure from the physical probability  $\mathbb{P}$  to the long forward measure  $\mathbb{Q}_L$  where the long bond serves as the numeraire.

*Proof.* Since the zero-coupon bond price is a  $\mathbb{Q}$ -martingale, we have for any  $t \in [0, T]$ ,

$$P(T, X_0) = E^{\mathbb{Q}}[P(T-t, X_t)|X_0] = E^{\mathbb{P}}[M_t P(T-t, X_t)|X_0]. \quad (3.13)$$

If the long forward rate has a constant limit  $\rho$ , then the discount bond prices can be decomposed as Eq. (3.10) in Proposition 7. This result can be used to rewrite Eq. (3.13)

as

$$\frac{P(T, X_0)}{P_\infty(T, X_0)} = e^{\varepsilon(T, X_0)} = E^{\mathbb{P}} \left[ M_t \frac{P(T-t, X_t)}{P_\infty(T, X_0)} \middle| X_0 \right] = E^{\mathbb{P}} \left[ e^{\rho t} M_t \frac{\phi(X_t)}{\phi(X_0)} e^{\varepsilon(T-t, X_t)} \middle| X_0 \right]. \quad (3.14)$$

As  $T \rightarrow \infty$ , both  $\varepsilon(T-t, X_t)$  and  $\varepsilon(T, X_0)$  shrink to zero. Under Condition 1, the term  $e^{\varepsilon(T-t, X_t)}$  is non-decreasing with its maturity so that one can use the *Monotone Convergence Theorem* to exchange limit and expectation. Under Condition 2, the term  $\phi(X_t)e^{\varepsilon(T-t, X_t)}$  in Eq. (3.14) is bounded by an integrable function so that one can apply the *Dominated Convergence Theorem* to exchange limit and expectation. Thus, under Condition 1 or 2, the following result holds after taking the limit  $T \rightarrow \infty$  on both sides of Eq. (3.14):

$$\lim_{T \rightarrow \infty} e^{\varepsilon(T, X_0)} = 1 = \mathbb{E}^{\mathbb{P}} \left[ \lim_{T \rightarrow \infty} e^{\rho t} M_t \frac{\phi(X_t)}{\phi(X_0)} e^{\varepsilon(T-t, X_t)} \middle| X_0 \right] = \mathbb{E}^{\mathbb{P}} \left[ e^{\rho t} M_t \frac{\phi(X_t)}{\phi(X_0)} \middle| X_0 \right]. \quad (3.15)$$

Define  $\xi_t^P \equiv e^{\rho t} M_t \frac{\phi(X_t)}{\phi(X_0)}$  which starts from the unity ( $\xi_{t=0}^P = 1$ ). By Eq. (3.15),  $\xi^P$  is a diffusion process with zero drift and hence it is a local martingale. When  $\xi_t^P$  is a strictly positive martingale, it will define a new measure  $\mathbb{Q}_L$  equivalent to  $\mathbb{P}$  on each  $\mathcal{F}_t$ . Under this measure  $\mathbb{Q}_L$ , the long bond price effectively serves as the numeraire and the pricing rule reads:

$$\mathcal{P}_{T-t}\psi(x) = \mathbb{E}^{\mathbb{Q}_L} \left[ \frac{M_T}{M_t} \frac{\xi_t^P}{\xi_T^P} \psi(X_T) \middle| X_t = x \right] = \mathbb{E}^{\mathbb{Q}_L} \left[ e^{-\rho(T-t)} \frac{\phi(X_t)}{\phi(X_T)} \psi(X_T) \middle| X_t = x \right]. \quad (3.16)$$

Hence, the new measure  $\mathbb{Q}_L$  will be referred to as the *long forward measure* henceforth.  $\square$

The above SDF factorization is derived by the arbitrage-free martingale approach. This contrasts with the semigroup operator approach by Hansen and Scheinkman (70) who show that the factorization result could be achieved by solving the eigenfunction problem of pricing operator. We demonstrate the equivalence of these two approaches:

**Corollary 11.** *If the long forward rate is constant, then  $\rho$  and  $\phi(x) = e^{-g(x)} > 0$  defines a pair of eigenvalue and eigenfunction for the pricing operator under any initial state  $X_0 = x$ :*

$$\mathcal{P}_t \phi(x) = e^{-\rho t} \phi(x). \quad (3.17)$$

*Proof.* Under the long forward measure  $\mathbb{Q}_L$ , the time-0 price of a time- $t$  payoff  $\psi(X_t)$  is

$$\mathcal{P}_t\psi(x) = \mathbb{E}^{\mathbb{Q}_L} \left[ e^{-\rho t} \frac{\phi(X_0)}{\phi(X_t)} \psi(X_t) \middle| X_0 = x \right]. \quad (3.18)$$

Consider a special contract that pays  $\psi(X_t) = \phi(X_t) > 0$  at time  $t$ . Its time-0 price becomes

$$\mathcal{P}_t\phi(x) = \mathbb{E}^{\mathbb{Q}_L} \left[ e^{-\rho t} \frac{\phi(X_0)}{\phi(X_t)} \phi(X_t) \middle| X_0 = x \right] = e^{-\rho t} \phi(x), \quad (3.19)$$

which holds for any  $X_0 = x$ . The infinitesimal generator  $\mathcal{A}$  of the operator  $\mathcal{P}_t = e^{t\mathcal{A}}$  satisfies

$$\mathcal{A}\phi = -\rho\phi. \quad (3.20)$$

Thus,  $\phi(x) = e^{-g(x)}$  is a positive eigenfunction of  $\mathcal{A}$  and the associated eigenvalue is  $-\rho$ .  $\square$

### 3.2.3 When will the Long Forward Rate be Flat?

The development so far hinges on the economic condition that the long forward rate has a constant limit. An interesting finding by Dybvig et al. (48) in discrete time is that *the long forward rates never fall*. This result also holds in general continuous-time setup, as proved by Hubalek, Klein, and Teichmayn (83).

**Lemma 2.** *(The long forward rates can never fall.) Suppose that the long forward rate exists at time  $s$  and  $t$  with  $s < t$ , then  $f_l(s, X_s) \leq f_l(t, X_t)$  almost surely.*

*Proof.* See Appendix 3.6.2. The proof is adapted from Ref. (83).  $\square$

A sufficient condition to make the long forward rate constant is that the state of economy is recurrent: given that the long forward rates cannot fall, if they must return again and again to the current level, these rates cannot rise either. A constant long forward rate is unique.

**Corollary 12.** *If the economy is driven by recurrent Markov processes, the long forward rate cannot rise and has to be constant over time. In this case, both Propositions 7 and 8 hold.*

At first glance, it seems to be a strong assumption that the entire state of the economy will return again and again to any realized state. There may be some weaker condition than this assumption. Dybvig et al. (48) wrote that: *“On economic grounds, it is hard to imagine that we will ever learn enough about how the economy works to ensure that the long discount rate can never subsequently return to today’s value. Given that the long discount rate cannot fall, this is enough to prove it is constant. Therefore, we should think of the ordinary situation as one in which the long discount rate is constant.”*

The arguments can be strengthened: First, the forward rate derived from the discount bonds should not depend on every state variable of the entire economy. Empirical studies suggest that the bond market is priced by only a few risk factors that are largely uncorrelated with the equity risk factors; see Fama and French (50). Second, the state variables (e.g., term premia) that influence the forward rates are usually stationary or mean-reverting, making the interest rate process stationary or mean-reverting as well. If those bond-market state variables repeatedly revisit their current levels, the forward rate cannot rise and has to be constant according to Corollary 12. The assumption of stationarity, recurrency, or ergodicity have usually been made in the literature following Hansen and Scheinkman (70) to prove the existence of the principal eigenvalue and eigenfunction for the SDF factorization. Why are those assumptions crucial for the SDF decomposition? The answer is clear from Corollary 12, because all those assumptions imply that the long forward rate is flat which is the central condition for Proposition 8.

### 3.2.4 How to Find the Long Forward Rate?

To factorize the pricing kernel as in Eq. (3.12), one just needs to know the long forward rate  $\rho$  and the function  $g(X_t)$  which governs the price volatility of long bond. The decomposition in Eq. (3.10) is quite useful if one can guess the functional form of  $g(x)$ . The long bond price  $P_\infty(T-t, X_t) = e^{-\rho(T-t)-g(X_t)}$  should be an asymptotic solution to the general bond pricing equation. This point is illustrated through a few examples below.

*Example 2.1.* Consider the single-factor model in Cox et al. (34) where the risk-free interest rate  $r_t = X_t$  follows a square-root diffusion process under risk-neutral measure  $\mathbb{Q}$ :

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t^{\mathbb{Q}}, \quad (3.21)$$

with  $\kappa > 0$ ,  $\theta > 0$ , and  $\sigma > 0$ . The Feller condition  $2\kappa\theta > \sigma^2$  ensures the strict positivity of interest rate. The above Itô process is stationary, recurrent and mean-reverting. So the long forward rate in this economy is constant by Corollary 12. Applying the Feynman-Kac theorem to the risk-neutral pricing yields the asset pricing equation:

$$\frac{1}{2}\sigma^2x\frac{\partial^2P}{\partial x^2} + \kappa(\theta - x)\frac{\partial P}{\partial x} + \frac{\partial P}{\partial t} = xP, \quad (3.22)$$

The CIR model has an affine term structure so that one can guess  $g(X_t)$  is a linear function of  $X_t$ . Plugging the long bond  $P_\infty(T-t, x) = e^{-\rho(T-t)-\gamma x}$  into Eq. (3.22) yields

$$\frac{1}{2}\gamma^2\sigma^2x - \gamma\kappa(\theta - x) + \rho = x, \quad (3.23)$$

which holds if and only if  $\rho$  and  $\gamma$  solve

$$\rho = \gamma\kappa\theta \quad (3.24)$$

$$\frac{1}{2}\gamma^2\sigma^2 + \gamma\kappa = 1. \quad (3.25)$$

The admissible (positive) solution is

$$\gamma = \frac{\sqrt{\kappa^2 + 2\sigma^2} - \kappa}{\sigma^2} = \frac{2}{\sqrt{\kappa^2 + 2\sigma^2} + \kappa} \quad (3.26)$$

$$\rho = \gamma\kappa\theta = \frac{\kappa\theta}{\sigma^2}(\sqrt{\kappa^2 + 2\sigma^2} - \kappa). \quad (3.27)$$

It is not necessary to solve the entire partial differential equation to obtain  $\rho$  and  $\gamma$ , although it can be a good exercise to verify the solution when a closed-form bond price formula is available. The semigroup approach of Hansen and Scheinkman (70) requires finding the principal eigenvalue and eigenfunction of the semigroup pricing operator. In this example, it amounts to dealing with the following Sturm-Liouville ordinary differential equation (ODE):

$$\mathcal{A}\phi(x) = \frac{1}{2}\sigma^2 x \frac{\partial^2 \phi}{\partial x^2} + \kappa(\theta - x) \frac{\partial \phi}{\partial x} - x\phi = -\rho\phi. \quad (3.28)$$

This is the confluent hypergeometric equation whose solutions can be expressed in terms of Kummer and Tricomi functions; see Qin and Linetsky (113). It takes some more efforts to identify that the principal eigenvalue of the CIR pricing operator is indeed given by Eq. (3.27) and the principal (strictly positive) eigenfunction is indeed  $\phi(x) = e^{-\gamma X_t}$  where  $\gamma$  is given by Eq. (3.26). This example shows the technical convenience brought by the long bond formula.

The same approach can be applied to the more general multi-factor affine term structure model<sup>3</sup> where the functional form of  $g(X_t)$  is known to be an affine function of the factors  $X_t$ . Similarly, it can be applied to quadratic-Gaussian term structure models<sup>4</sup> where the discount bond prices are exponential-quadratic and bond yields are quadratic functions of Gaussian factors. In all such models, if the long forward rate is a positive constant, then

---

<sup>3</sup>See Ref. 109, 47, and 37.

<sup>4</sup>See Ref. 33 and 3.

all one needs to solve is a system of algebraic equations, instead of a system of Riccati ODEs or Sturm-Liouville equations. It is worth noting that the proof of Proposition 7 does not require that the underlying state variables evolve as the Itô diffusion process. 7 can also be applicable for jump-diffusion processes. This is illustrated by the next example.

*Example 2.2.* The interest rate  $r_t = X_t$  is assumed to follow a square-root diffusion process augmented with a Poisson jump process<sup>5</sup> under the risk-neutral measure  $\mathbb{Q}$ :

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t^{\mathbb{Q}} + dJ_t. \quad (3.29)$$

Here,  $J_t$  denotes the compound Poisson process with a jump arrival rate  $\lambda > 0$  and exponentially distributed jump sizes with mean size  $\mu > 0$ . This model is still stationary and recurrent, implying that the long forward rate is flat. Again, substituting the long bond price  $P_{\infty}(T - t, x) = e^{-\rho(T-t) - \gamma x}$  into the bond pricing equation

$$\frac{1}{2}\sigma^2 x \frac{\partial^2 P}{\partial x^2} + \kappa(\theta - x) \frac{\partial P}{\partial x} + \frac{\partial P}{\partial t} + \int_0^{\infty} [P(\cdot, x + y) - P(\cdot, x)] \frac{\lambda}{\mu} e^{-y/\mu} dy = xP, \quad (3.30)$$

one can derive a simple algebraic equation

$$\frac{1}{2}\gamma^2 \sigma^2 x - \gamma \kappa(\theta - x) + \rho - \frac{\lambda \mu \gamma}{1 + \mu \gamma} = x. \quad (3.31)$$

The admissible (positive) solution turns out to be

$$\gamma = \frac{\sqrt{\kappa^2 + 2\sigma^2} - \kappa}{\sigma^2} \quad (3.32)$$

$$\rho = \gamma \kappa \theta + \frac{\lambda \mu \gamma}{1 + \mu \gamma} = \frac{\kappa \theta}{\sigma^2} (\sqrt{\kappa^2 + 2\sigma^2} - \kappa) + \frac{\lambda \mu \gamma}{1 + \mu \gamma}. \quad (3.33)$$

The parameter  $\gamma$  is the same as Eq. (3.26) in the CIR model, whereas the long forward rate  $\rho$  now is equal to the CIR long rate in Eq. (3.27) plus an additional term,  $\frac{\lambda \mu \gamma}{1 + \mu \gamma}$ , due to the

---

<sup>5</sup>See Ref. 46 and 51 for this example.

jump component. The principal eigenvalue in this example is larger than that of the original CIR model. The extra premium  $\frac{\lambda\mu\gamma}{1+\mu\gamma}$  compensates investors for their exposure to the jump risk which increases with the jump arrival rate  $\lambda$  as well as the mean jump size  $\mu$ .

The eigenfunction problem for a jump-diffusion pricing operator is usually intractable. The asymptotic analysis based on the long bond formula somehow circumvents the difficulty. This approach can be extended to general multi-factor affine jump-diffusion models. It is, however, not restricted to affine term structure models, as shown by the next example.

*Example 2.3.* Merton (111) derived the following quadratic drift model for the interest rate:

$$dX_t = \kappa(\theta - X_t)X_t dt + \sigma X_t dW_t^{\mathbb{Q}}. \quad (3.34)$$

When the Feller's condition holds ( $2\kappa\theta/\sigma^2 > 1$ ), both the origin and infinity are inaccessible, and the process is recurrent with a stationary Gamma distribution. The pricing equation is

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + \kappa(\theta - x)x \frac{\partial P}{\partial x} + \frac{\partial P}{\partial t} = xP. \quad (3.35)$$

Substituting the long bond price,  $P_{\infty}(T-t, x) = e^{-\rho(T-t)}x^{-\gamma}$ , into the above equation yields

$$\frac{1}{2}\sigma^2\gamma(\gamma+1)x^{-\gamma} - \gamma\kappa(\theta-x)x^{-\gamma} + \rho x^{-\gamma} = x^{-\gamma+1}. \quad (3.36)$$

The solution is found to be  $\gamma = 1/\kappa$  and

$$\rho = \gamma\kappa\theta - \frac{1}{2}\sigma^2\gamma(\gamma+1) = \theta - \frac{\sigma^2(1+\kappa)}{2\kappa^2}. \quad (3.37)$$

It can take a lot of algebra to verify that  $\phi(x) = x^{1/\kappa}$  is indeed the principal eigenfunction.

### 3.3 Long Forward Measure and SDF Decomposition

This section discusses the long forward measure and the associated SDF decomposition. An explicit expression for the permanent martingale component is provided. This result reveals an implicit restriction on the market prices of risk in a popular approach for interest rate modeling. This restriction is due to the degeneracy of permanent martingale component.

#### 3.3.1 The $T$ -Forward Measure

The  $T$ -forward measure, denoted by  $\mathbb{Q}_T$ , refers to a change of numeraire where the new unit of account is the  $T$ -maturity discount bond. The change from the risk-neutral measure  $\mathbb{Q}$  to the forward measure  $\mathbb{Q}_T$  is defined by the Radon-Nikodym derivative below

$$\eta_t^T \equiv e^{-\int_t^T r_s ds} \frac{P(t, T, X_t)}{P(0, T, X_0)} = \left( \frac{\mathbb{Q}_T}{\mathbb{Q}} \right)_{\mathcal{F}_t}, \quad (3.38)$$

where  $P(t, T, X_t) = \mathbb{E}_t^{\mathbb{P}}[M_T/M_t]$  is the pure discount bond price. Under  $\mathbb{Q}$ , this solves

$$\frac{dP(t, T, X_t)}{P(t, T, X_t)} = r_t(X_t)dt - \sigma_P(t, T, X_t)^\top dW_t^{\mathbb{Q}}, \quad (3.39)$$

where the  $d \times 1$  vector  $\sigma_P(t, T, X_t)$  represents the instantaneous volatility at time  $t$  of the discount bond return. By Itô's lemma, it is easy to deduce that  $\frac{d\eta_t^T}{\eta_t^T} = -\sigma_P(t, T, X_t)^\top dW_t^{\mathbb{Q}}$ . So the Girsanov kernel for the measure change from  $\mathbb{Q}$  to  $\mathbb{Q}_T$  is set by the volatility  $\sigma_P(t, T, X_t)$ .

The Radon-Nikodym derivative for the change of measure from  $\mathbb{P}$  to  $\mathbb{Q}_T$  on  $\mathcal{F}_t$  is

$$\xi_t^T \equiv \frac{M_T}{\mathbb{E}_t[M_T]} = \frac{1}{P(t, T, X_t)} \frac{M_T}{M_t} = \left( \frac{\mathbb{Q}_T}{\mathbb{P}} \right)_{\mathcal{F}_t}, \quad (3.40)$$

which satisfies  $\xi_t^T > 0$  and  $E_t^{\mathbb{P}}[\xi_t^T] = 1$ . Under the  $T$ -forward measure, the time- $t$  price of a state-contingent payoff  $\psi(X_T)$  at maturity  $T$  is given by

$$\mathcal{P}_{T-t}\psi(X_t) = E^{\mathbb{P}} \left[ \frac{M_T}{M_t} \psi(X_T) \middle| \mathcal{F}_t \right] = P(t, T, X_t) E^{\mathbb{P}} [\xi_t^T \psi(X_T) | \mathcal{F}_t] = P(t, T, X_t) E^{\mathbb{Q}^T} [\psi(X_T)], \quad (3.41)$$

from which one can see the  $T$ -bond  $P(t, T, X_t)$  serves as the new numeraire. Let  $\vartheta_t(X_t)$  be the  $d \times 1$  vector of market prices of risk associated with the  $d \times 1$  vector of Brownian motions  $W_t$ . Then,  $-\vartheta_t$  is the Girsanov kernel for transition from  $\mathbb{P}$  to  $\mathbb{Q}$ , i.e.,

$$\frac{dM_t}{M_t} = -r_t dt - \vartheta_t^\top dW_t^{\mathbb{P}}. \quad (3.42)$$

Applying Itô's lemma to Eq. (3.40) yields

$$\frac{d\xi_t^T}{\xi_t^T} = [\vartheta_t(X_t) + \sigma_P(t, T, X_t)]^\top dW_t^{\mathbb{P}} = \tilde{\vartheta}(t, T, X_t)^\top dW_t^{\mathbb{P}}, \quad (3.43)$$

where the  $d \times 1$  kernel vector,  $\tilde{\vartheta}(t, T, X_t) \equiv \vartheta_t(X_t) + \sigma_P(t, T, X_t)$ , can be interpreted as the price of Brownian motion risk under the new numeraire. Eq. (3.43) can also be written as

$$\xi_t^T \equiv \exp \left[ \int_t^T \tilde{\vartheta}(s, T, X_s)^\top dW_s - \frac{1}{2} \int_t^T \tilde{\vartheta}(s, T, X_s)^\top \tilde{\vartheta}(s, T, X_s) ds \right], \quad (3.44)$$

which can be a strictly positive martingale under regularity conditions. For more discussions about the forward measure, see Musiela and Rutkowski (112).

### 3.3.2 The Long Forward Measure

The measure  $\mathbb{Q}_L$  in Proposition 8 is referred to as the long forward measure since the long bond price  $P_\infty$  serves as the numeraire under  $\mathbb{Q}_L$ . In fact, the long-forward measure is the limit of  $T$ -forward measure with  $T \rightarrow \infty$ . The results in the previous subsection can be used to prove the result below:

**Proposition 9.** *If the long forward rate is constant and the market prices of Brownian motion risk (denoted by  $\vartheta_t$ ) exist, then the pricing kernel can be factorized as  $M_t = e^{-\rho t} \frac{\phi(X_0)}{\phi(X_t)} \xi_t^P$ , where  $\phi(x) = e^{-g(x)}$  and the permanent (local) martingale  $\xi_t^P$  solves the Itô process*

$$\frac{d\xi_t^P}{\xi_t^P} = -[\vartheta_t + \sigma_\infty(X_t)]^\top dW_t. \quad (3.45)$$

Here,  $\sigma_\infty = \Sigma_X \frac{\partial g}{\partial X}$  represents the long bond return volatility.

*Proof.* See Appendix 3.6.3. □

Now let's define  $\pi_t \equiv e^{-\rho t} \frac{\phi(X_0)}{\phi(X_t)}$  as the transient component of the pricing kernel. Both the permanent and transient components,  $\pi_t$  and  $\xi_t^P$ , are correlated with each other in general.

**Corollary 13.** *By Itô's lemma, the covariance between the permanent and transient components is typically stochastic, depending on the state of the economy:*

$$\text{Cov} \left( \frac{d\pi_t}{\pi_t}, \frac{d\xi_t^P}{\xi_t^P} \right) = -\sigma_\infty(X_t)^\top [\vartheta_t(X_t) + \sigma_\infty(X_t)] dt. \quad (3.46)$$

Intuitively, the long bond yields fluctuate with business conditions. This affects the discounting rate for valuing other risky assets. Eq. (3.46) suggests that the covariance can vanish for either  $\sigma_\infty(X_t) = 0$  or  $\vartheta_t + \sigma_\infty(X_t) = 0$ . For illustration, the CIR model is revisited:

*Example 2.1 (continued).* For the one-factor CIR model ( $r_t = X_t$ ), the long bond price is  $P_\infty(T - t, X_t) = \exp[-\rho(T - t) - \gamma X_t]$ , whose return volatility is  $\sigma_\infty(X_t) = \gamma\sigma\sqrt{X_t}$ . With the linear risk premium as in Cox et al. (34), the market price of risk is  $\vartheta_t = \lambda\sqrt{X_t}$ , where  $\lambda$  is constant. Since parameters such as  $\kappa$  and  $\theta$  are defined under the risk neutral measure  $\mathbb{Q}$ , the drift of long bond return process under  $\mathbb{P}$  is

$$\mu_\infty(X_t) = r_t - \sigma_\infty(X_t)\vartheta_t = (1 - \lambda\gamma\sigma)X_t. \quad (3.47)$$

The pricing kernel in the CIR model thus takes the following factorization:

$$M_t^{CIR} = \pi_t \cdot \xi_t^P = e^{-\rho t} e^{\gamma(X_t - X_0)} \xi_t^P, \quad (3.48)$$

where the Girsanov kernel of permanent martingale  $\xi_t^P$  is  $-[\vartheta_t + \sigma_\infty(X_t)] = -(\lambda + \gamma\sigma)\sqrt{X_t}$ .

The covariance between the transient and permanent components is

$$Cov\left(\frac{d\pi_t}{\pi_t}, \frac{d\xi_t^P}{\xi_t^P}\right) = -\gamma\sigma(\lambda + \gamma\sigma)X_t dt, \quad (3.49)$$

the sign of which is the opposite sign of  $(\lambda + \gamma\sigma)$  if the Feller condition holds ( $P(X_t > 0) = 1$ ).

Unless investors' preferences are restricted to satisfy  $\lambda + \gamma\sigma = 0$ , the orthogonal situation

$\frac{d\pi_t}{\pi_t} \cdot \frac{d\xi_t^P}{\xi_t^P} = 0$  should rarely hold. Of course, it holds in the trivial case of a constant interest

rate (i.e.,  $\sigma = 0$ ), and the corresponding market price of risk vanishes ( $\vartheta_t = 0$ ) in the ab-

sence of any interest rate risk. Constant interest rates imply that the permanent martingale

is degenerate ( $\xi_t^T = 1$ ) and the pricing kernel is simply an exponential function of time.

For  $\lambda \neq -\gamma\sigma$ ,  $\frac{d\pi_t}{\pi_t}$  and  $\frac{d\xi_t^P}{\xi_t^P}$  are perfectly correlated (if  $\lambda < -\gamma\sigma$ ) or perfectly anti-correlated

( $\lambda > -\gamma\sigma$ ). Thus, the instantaneous correlation coefficient between the increments of per-

manent and transient components is either 1, 0, or  $-1$ . These extreme correlation values are

resulted from the single-factor assumption. As shown in the next example, a multi-factor

CIR model can generate a wide range of values for this correlation coefficient.

*Example 3.1.* A two-factor CIR model was studied by Longstaff and Schwartz (109). Let's

consider a multi-factor CIR model for the interest rate given by  $r_t = \sum_{i=1}^k X_i(t)$ , where

$$dX_i = \kappa_i(\theta_i - X_i)dt + \sigma_i\sqrt{X_i}dW_i^{\mathbb{Q}}, \quad (3.50)$$

for positive constants  $\kappa_i$ ,  $\theta_i$ , and  $\sigma_i$ , with the  $W_i^{\mathbb{Q}}$  being independent Brownian motions under

the risk neutral measure  $\mathbb{Q}$ . Assume that the market price of risk  $\vartheta_t$  associated with the  $i$ -th Brownian motion  $W_i^{\mathbb{Q}}$  is equal to  $\lambda_i \sqrt{X_i(t)}$  with constant  $\lambda_i$  for  $i = 1, \dots, k$ . The long bond price now takes the form  $P_\infty(t, T) = \exp[-\rho(T - t) - \gamma^\top X_t]$ , where it is easy to solve that  $\rho = \sum_{i=1}^k \gamma_i \kappa_i \theta_i$  and  $\gamma = (\gamma_1, \dots, \gamma_k)^\top$  with  $\gamma_i = (\sqrt{\kappa_i^2 + 2\sigma_i^2} - \kappa_i)/\sigma_i^2$  for  $i = 1, \dots, k$ . The correlation coefficient between the transient and permanent components is found to be

$$\text{Corr} \left[ \frac{d\pi_t}{\pi_t}, \frac{d\xi_t^P}{\xi_t^P} \right] = -\frac{\sigma_\infty^\top (\vartheta_t + \sigma_\infty)}{\sqrt{\|\sigma_\infty\|_2} \sqrt{\|\vartheta_t + \sigma_\infty\|_2}} = -\frac{\sum_i \gamma_i \sigma_i (\lambda_i + \gamma_i \sigma_i) X_i(t)}{\sqrt{\sum_i \gamma_i^2 \sigma_i^2 X_i(t)} \sqrt{\sum_i (\lambda_i + \gamma_i \sigma_i)^2 X_i(t)}},$$

which can take any value in  $[-1, 1]$ , depending on the values of  $X_1(t), \dots, X_k(t)$  and  $(\lambda_1, \dots, \lambda_k)$ .

In general, the permanent martingale component in the bond market is *not* orthogonal to the transitory component, i.e.,  $Cov \left( \frac{d\pi_t}{\pi_t}, \frac{d\xi_t^P}{\xi_t^P} \right) \neq 0$ . This can be easily observed from Eq. (3.46). Of course, there is also a large volatile permanent martingale that links to the state variables (risk factors) for the equity market. Since the correlation between bond and equity is weak, the correlation between the entire permanent martingale component and the transitory component is mostly attributable to their common risk factors in the bond market. As we stressed before, the permanent martingale attributable to the bond market is non-degenerate unless the condition  $\vartheta_t + \sigma_\infty(X_t) = 0$  holds. But there is no economic reason to believe that investors would restrict their preferences to meet  $\vartheta_t = -\sigma_\infty(X_t)$ . Therefore, the long-run SDF factorization is unlikely to be an orthogonal decomposition if it contains a nontrivial martingale component. This is consistent with empirical results in Ref. (5, 9, 10, and 30).

### 3.3.3 Long-Forward Rate under Bond Price Bubbles

In an arbitrage-free market with a strictly positive SDF, the principal eigenvalue (i.e., the long-forward rate) in the Hansen-Scheinkman decomposition is usually associated with a unique eigenfunction (related to the state-dependence of long bonds). We will explore a stationary economy that does not have the *equivalent martingale measure* (EMM) and thus permits multiple solutions to discount bond pricing. In this economy, the constant long-forward rate is associated with multiple eigenfunctions. This result sheds light on the applicability of Hansen-Scheinkman factorization in the presence of asset price bubbles.

Let's review the arbitrage example that was first conjectured in Section 5 of Cox et al. (34) and later clarified by Heston et al. (77). Consider a single-factor CIR economy where the interest rate process under the physical measure  $\mathbb{P}$  is

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^{\mathbb{P}}, \quad (3.51)$$

with  $\kappa > 0$  and  $\theta > 0$ . When the Feller condition holds (i.e.,  $2\kappa\theta > \sigma^2$ ), the upward drift is large enough to ensure that  $r$  cannot hit the origin in finite time under  $\mathbb{P}$ . This inequality also prevents the (local) market price of risk from becoming infinite and thus rules out instantaneously profitable arbitrage opportunities. With a linear risk premium  $\psi_0 + \psi_1 r$ , the CIR general equilibrium<sup>6</sup> implies the bond pricing equation below

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} + \kappa(\theta - r) \frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = (\psi_0 + \psi_1 r)P_r, \quad (3.52)$$

with the boundary condition  $P(T, T, r) = 1$ . We denote  $\kappa' = \kappa + \psi_1$ ,  $\theta' = \frac{\kappa\theta - \psi_0}{\kappa + \psi_1}$ , and

---

<sup>6</sup>See Theorem 2 of Cox et al. (35) and Section 5 of Cox et al. (34). Another popular approach for bond pricing is to invoke the no-arbitrage arguments, as developed in Ref. 127, 42, 116, and 19.

$\eta = \sqrt{(\kappa + \psi_1)^2 + 2\sigma^2}$ . Then Eq. (3.52) can be rewritten as

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} + \kappa'(\theta' - r) \frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0. \quad (3.53)$$

By Feynman-Kac formula, a solution to Eq. (3.53) can be written as a conditional expectation  $P(t, T, r) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r_s ds} \mid r_t = r \right]$ , under the new probability measure  $\mathbb{Q}$  such that  $r$  is an Itô process driven by the equation

$$dr_t = \kappa'(\theta' - r_t)dt + \sigma\sqrt{r_t}dW_t^{\mathbb{Q}}. \quad (3.54)$$

The key point is that  $\mathbb{Q}$  is not necessarily equivalent with the physical probability  $\mathbb{P}$ , as they may not agree on zero probability events. The standard CIR solution to the discount bond price is given by  $P_1(t, T, r) = A(t, T) \exp[-rB(t, T)]$ , where

$$A(t, T) = \left[ \frac{2\eta e^{(\kappa' + \eta)(T-t)/2}}{2\eta + (\kappa' + \eta)(e^{\eta(T-t)} - 1)} \right]^{\frac{2\kappa'\theta'}{\sigma^2}}, \quad (3.55)$$

$$B(t, T) = \frac{2(e^{\eta(T-t)} - 1)}{2\eta + (\kappa' + \eta)(e^{\eta(T-t)} - 1)}. \quad (3.56)$$

Heston et al. (77) noticed that some parameter values may give rise to arbitrage opportunities as there may exist multiple PDE solutions that permit money market bubbles. The following lemma is taken from Example 1.1 from their paper:

**Lemma 3.** *For  $\psi_0 > \frac{2\kappa\theta - \sigma^2}{2}$  such that  $2\kappa'\theta' < \sigma^2$ , the original CIR solution is not the cheapest nonnegative solution. The cheaper solution is*

$$P_2(t, T, r) = A(t, T)e^{-rB(t, T)} \left[ 1 - \frac{\Gamma(\nu, r\zeta(t, T))}{\Gamma(\nu, 0)} \right], \quad (3.57)$$

where  $\nu = 1 - 2\kappa'\theta'/\sigma^2$  is a dimensionless positive parameter,  $\Gamma(\nu, x) = \int_x^\infty e^{-y}y^{\nu-1}dy$  is the incomplete gamma function, and

$$\zeta(t, T) = \frac{2}{\sigma^2} \frac{e^{-\kappa'(T-t)} A(t, T)^{\frac{\sigma^2}{\kappa'\theta'}}}{\int_t^T e^{-\kappa'(T-s)} A(s, T)^{\frac{\sigma^2}{\kappa'\theta'}} ds}. \quad (3.58)$$

This cheaper solution  $P_2(t, T, r)$  is strictly less than the standard CIR solution  $P_1(t, T, r)$  prior to maturity, but it agrees with  $P_1(t, T, r)$  at maturity for  $r > 0$ .

Several important implications can be drawn from this example:

- There is no equivalent (local) martingale measure in this economy when  $2\kappa'\theta' < \sigma^2$ , because if there were an EMM (say  $\mathbb{Q}$ ), the locally risk-free rate would have a weak drift  $\kappa'(\theta' - r)$  and hit zero with positive probability under  $\mathbb{Q}$  which disagrees with  $\mathbb{P}$  regarding zero probability events. As Heston et al. (77) explained, the bond price difference  $P_1 - P_2$  is the price of a claim that pays 1 dollar if the interest rate hits zero prior to maturity<sup>7</sup>.

- If the bond price is given by the standard CIR solution  $P_1(t, T, r)$ , there will be an arbitrage opportunity which is exploitable by short selling  $P_1$  and going long the replicating strategy for  $P_2$ <sup>8</sup>. This strategy essentially sells the claim that pays in the event the interest rate hits zero under  $\mathbb{Q}$ . When  $r$  is very close to zero, the strategy may suffer temporary losses as  $P_2 - P_1 < 0$ . However, these losses are lower bounded because  $P_2 - P_1 \geq -1$ . This arbitrage strategy can be feasible at a large scale if investors adjust the wealth constraints for this strategy; see Heston et al. (77) as well as Delbaen and Schachermayer (39).

- The pricing equation permits a class of solutions, indexed by  $w \in [0, 1]$  and given by

$$P_{1+w}(t, T, r) = A(t, T)e^{-rB(t, T)} \left[ 1 - w \frac{\Gamma(\nu, r\zeta(t, T))}{\Gamma(\nu, 0)} \right]. \quad (3.59)$$

For any  $w < 1$ , the replicating cost of  $P_{1+w}$  exceeds the replicating cost of  $P_2$ . This violates

---

<sup>7</sup>Although the origin is inaccessible under  $\mathbb{P}$ , the measure  $\mathbb{Q}$  would assign this a positive probability and price this event.  $P_2(t, T, r)$  becomes zero if the interest rate hits the origin prior to maturity.

<sup>8</sup>There exist replicating portfolios for the discount bond price given by either solution when  $2\kappa\theta > \sigma^2$ . It indicates an arbitrage opportunity as two replicating portfolios with the same payoff have different costs.

the law of one price as their payouts are the same at maturity. It leads to the bond price bubble  $P_{1+w} - P_2 = wA(T-t)e^{-B(T-t)r} \frac{\Gamma(\nu, r\zeta(t, T))}{\Gamma(\nu, 0)}$ , which is nonnegative and bounded above by  $w$ . Moreover, the long forward rate is the same for this class of bond price solutions:

$$\rho = \lim_{T \rightarrow \infty} -\frac{\partial \log P_{1+w}(t, T, r)}{\partial T} = \frac{\kappa' \theta' (\eta - \kappa')}{\sigma^2} = \frac{\kappa \theta - \psi_0}{\sigma^2} \left( \sqrt{(\kappa + \psi_1)^2 + 2\sigma^2} - \kappa - \psi_1 \right). \quad (3.60)$$

Thus, a unique long forward rate exists in this economy where there is no EMM however.

It seems sensible to price discount bonds by the lowest cost of their replicating portfolios, i.e.,  $P_2(t, T, r)$  given by Eq. (3.57). With the linear risk premium  $\psi_0 + \psi_1 r$ , the instantaneous market price of risk (or Sharpe ratio) is given by  $\vartheta \equiv \frac{\psi_0 + \psi_1 r}{\sigma \sqrt{r}}$ . When  $2\kappa\theta \geq \sigma^2$ , it is finite almost surely and rules out any instantaneously profitable arbitrages. The market price of risk  $\vartheta$  defines an exponential local martingale,  $\xi_t = \exp \left[ -\frac{1}{2} \int_0^t |\vartheta(r_s)|^2 ds - \int_0^t \vartheta(r_s) dW_s \right]$ , which cannot be a strictly positive martingale if  $2\kappa'\theta' < \sigma^2$ . In other words,  $M_t = e^{-\int_0^t r_s ds} \xi_t$  is not an equivalent martingale measure in this economy if  $2\kappa'\theta' < \sigma^2$ . Can we still factorize  $M_t$  in the multiplicative form? It is easy to verify the following limit result

$$\lim_{T \rightarrow \infty} \zeta(t, T) = \lim_{T \rightarrow \infty} \frac{2}{\sigma^2} \left[ \int_t^T e^{\kappa'(s-t)} \left( \frac{A(s, T)}{A(t, T)} \right)^{\frac{\sigma^2}{\kappa'\theta'}} ds \right]^{-1} = \frac{2}{\sigma^2 \int_t^\infty e^{\kappa'(s-t)} e^{-(\kappa' - \eta)(s-t)} ds} = \frac{2\eta}{\sigma^2}. \quad (3.61)$$

The asymptotic behavior of discount bond becomes  $P_2(t, T, r) \rightarrow e^{-\rho(T-t) - g(r_t)}$ , where

$$g(r_t) = \gamma r_t - \log \left[ 1 - \frac{\Gamma(\nu, \frac{2\eta}{\sigma^2} r_t)}{\Gamma(\nu, 0)} \right], \quad \text{and} \quad \gamma = \frac{\eta - \kappa'}{\sigma^2} = \frac{\sqrt{(\kappa + \psi_1)^2 + 2\sigma^2} - \kappa - \psi_1}{\sigma^2}. \quad (3.62)$$

Proposition 8 suggests the following factorization:

$$M_t = e^{-\rho t} \frac{e^{\gamma r_0} \left[ \Gamma(\nu, 0) - \Gamma(\nu, \frac{2\eta}{\sigma^2} r_0) \right]}{e^{\gamma r_t} \left[ \Gamma(\nu, 0) - \Gamma(\nu, \frac{2\eta}{\sigma^2} r_t) \right]} \xi_t^P. \quad (3.63)$$

By Proposition 9, we have  $d\xi_t^P = -[\vartheta_t + \sigma_\infty(r_t)]\xi_t^P dW_t^{\mathbb{P}}$ , where

$$\vartheta_t + \sigma_\infty(r_t) = \frac{\psi_0 + \psi_1 r_t}{\sigma \sqrt{r_t}} + \gamma \sigma \sqrt{r_t} - \frac{\sigma \left(\frac{2\eta r_t}{\sigma^2}\right)^\nu \exp\left(\frac{-2\eta r_t}{\sigma^2}\right)}{\sqrt{r_t} [\Gamma(\nu, 0) - \Gamma(\nu, 2\eta r_t/\sigma^2)]}. \quad (3.64)$$

This example of CIR model shows that the long-run SDF decomposition can be feasible even when the economy does not have an equivalent martingale measure. In this case, the long forward rate is unique, but the SDF decomposition is not necessarily unique.

### 3.3.4 The Markov Potential Approach for Term Structure Models

There has been an alternative approach preceded by Constantinides (33) and formalized by Rogers (117) to model interest rates. This approach directly specifies the pricing kernel by modeling it as the probabilistic potential. It has become popular partially because it can generate a variety of models with positive interest rates. Nonetheless, there is a strong restriction in this approach, which can limit its use in model calibration with market data.

**Definition 3.** A random process  $Z_t$  is called a probabilistic potential if  $Z_t$  is a non-negative supermartingale for any  $t$  and  $E[Z_t] \rightarrow 0$  as  $t \rightarrow \infty$ .

If interest rates are strictly positive and the economic condition  $\lim_{T \rightarrow \infty} P(t, T, X_t) = 0$  holds, then the pricing kernel  $M_t$  is a probabilistic potential. In other words, any pricing kernel generating a positive term structure is a potential; see Chapter 28 in Bjork (15). How to construct a potential that qualifies as a pricing kernel for arbitrage-free bond pricing? Rogers (117) proposed a systematic procedure of constructing such potentials in terms of the resolvent of time-homogeneous Markov process.

Without loss of generality, consider an economy where the state variables  $X_t$  follow the multi-dimensional Itô diffusion process under the physical measure  $\mathbb{P}$ :

$$dX_t = \mu_X(X_t)dt + \Sigma_X(X_t)dW_t, \quad (3.65)$$

with the infinitesimal generator (taking a scalar  $X$  for example)

$$\mathcal{G} = \mu_X(x) \frac{\partial}{\partial x} + \frac{1}{2} \Sigma_X(x)^\top \Sigma_X(x) \frac{\partial^2}{\partial x^2}. \quad (3.66)$$

We shall focus on the intuition and skip technical discussions on regularity conditions which can be found in Rogers (117). For any nonnegative  $\alpha$ , the *resolvent*  $R_\alpha$  is an operator that maps a bounded measurable real valued function  $q$  into the real valued function  $R_\alpha q$ . It is defined by the expectation of an integral of  $q$  discounted by  $\alpha$  under the physical measure:

$$R_\alpha q(x) = \mathbb{E}^\mathbb{P} \left[ \int_0^\infty e^{-\alpha s} q(X_s) ds \middle| X_0 = x \right]. \quad (3.67)$$

It can be proved that for any bounded nonnegative function  $q(x)$ , the process  $Z_t = e^{-\alpha t} R_\alpha q(X_t)$  is a potential (15). If  $Z_t$  is chosen as the pricing kernel, the discount bond price is

$$P(t, T, X_t) = \frac{\mathbb{E}^\mathbb{P}[Z_T | \mathcal{F}_t]}{Z_t} = e^{-\alpha(T-t)} \frac{\mathbb{E}^\mathbb{P}[R_\alpha q(X_T) | \mathcal{F}_t]}{R_\alpha q(X_t)} \quad (3.68)$$

and the risk-free interest rate is given by  $r_t = \frac{q(X_t)}{R_\alpha q(X_t)}$ . There is a useful relationship,  $R_\alpha = (\alpha - \mathcal{G})^{-1}$ , that can help us derive the interest rate. If one defines  $f(x) = R_\alpha q(x)$ , then

$$r_t = \frac{q(X_t)}{R_\alpha q(X_t)} = \frac{(\alpha - \mathcal{G})f(X_t)}{f(X_t)} = \alpha - \frac{\mathcal{G}f(X_t)}{f(X_t)}. \quad (3.69)$$

This is easy to handle when the infinitesimal generator  $\mathcal{G}$  is given.

Rogers (117) proposed to construct the pricing kernel for positive interest rates in the following procedure. The first step is to specify a Markov process  $X$ , a real positive constant

$\alpha$ , and a nonnegative function  $f$ . Define  $q(x) = (\alpha - \mathcal{G})f(x)$  and choose parameters to ensure that  $q(x)$  is a nonnegative function. Since  $f(x) = R_\alpha q(x)$ , the interest rate is determined by Eq. (3.69). Below is a simple example as discussed in both Rogers (117) and Bjork (15):

*Example 3.2.* Consider an economy driven by a single-factor Ornstein–Uhlenbeck process:

$dX_t = -\beta X_t dt + \sigma dW_t$ . With  $f(x) = e^{\gamma x}$  for some constant  $\gamma$ , one can construct the potential

$$Z_t = e^{-\alpha t} f(X_t) = e^{-\alpha t + \gamma X_t} \quad (3.70)$$

as the pricing kernel. This generates the interest rate

$$r_t = \alpha - \frac{1}{2}\gamma^2\sigma^2 + \beta\gamma X_t, \quad (3.71)$$

which is a one-factor affine Gaussian model, similar to Vasicek (127). In this economy, the discount bond price is given by

$$P(t, T, X_t) = \frac{\mathbb{E}^\mathbb{P}[Z_T | \mathcal{F}_t]}{Z_t} = e^{-\alpha(T-t)} \mathbb{E}_t^\mathbb{P}[e^{\gamma(X_T - X_t)}]. \quad (3.72)$$

One can write  $X_T - X_t = \int_t^T dX_s = -\beta \int_t^T X_s ds + \sigma \int_t^T dW_s$  so that Eq. (3.72) becomes

$$P(t, T, X_t) = e^{-(\alpha - \frac{1}{2}\gamma^2\sigma^2)(T-t)} \mathbb{E}_t^\mathbb{P} \left[ e^{-\beta\gamma \int_t^T X_s ds} \exp \left( -\frac{1}{2} \int_t^T \gamma^2\sigma^2 ds + \gamma\sigma \int_t^T dW_s \right) \right]. \quad (3.73)$$

The exponential martingale in the expectation of Eq. (3.73) defines a change of measure from  $\mathbb{P}$  to a new measure  $\tilde{\mathbb{Q}}$  via the Radon-Nikodym derivative

$$\left( \frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \right)_{\mathcal{F}_t} = \exp \left( -\frac{1}{2} \int_t^T \gamma^2\sigma^2 ds + \gamma\sigma \int_t^T dW_s \right). \quad (3.74)$$

By Eq. (3.71), the bond price Eq. (3.73) can be rewritten as

$$P(t, T) = e^{-(\alpha - \frac{1}{2}\gamma^2\sigma^2)(T-t)} \mathbb{E}_t^{\tilde{\mathbb{Q}}} \left[ e^{-\beta\gamma \int_t^T X_s ds} \right] = \mathbb{E}^{\tilde{\mathbb{Q}}} \left[ e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]. \quad (3.75)$$

Alternatively, the discount bond price under the risk-neutral measure  $\mathbb{Q}$  should be

$$P(t, T) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right]. \quad (3.76)$$

Comparing Eq. (3.75) and Eq. (3.76), one can see that  $\mathbb{Q}$  and  $\tilde{\mathbb{Q}}$  coincide. Thus the market price of risk should be  $\vartheta = -\gamma\sigma$ . Moreover, the long-term bond price can be easily found,  $P_\infty(t, T) = \exp[-\alpha(T-t) - \gamma X_t]$ . The long forward rate is set by the constant parameter  $\alpha > 0$  and the long bond return volatility is  $\sigma_\infty = \gamma\sigma$ . In this example, one can easily see that the market price of risk exactly cancels out with the long bond volatility, i.e.,  $\vartheta + \sigma_\infty = 0$ .

In fact,  $\vartheta_t + \sigma_\infty = 0$  is the general restriction on any interest rate models constructed by Rogers' approach. It is difficult to justify that the market prices of risk have to equal the long bond return volatility. Practitioners may find it difficult to calibrate this type of model since its risk premium is overly restricted. Traders may incur utility loss if they are unaware of the restriction in this modeling approach.

**Proposition 10.** *Interest rate models constructed by the probabilistic potential approach are subject to an implicit restriction on the market prices of risk. Specifically, if one assumes the pricing kernel is of the form  $M_t = e^{-\alpha t + g(X_t)}$  where  $X_t$  is a Markov homogeneous process with stationary limiting distribution, then the long bond price takes the form  $P_\infty(t, T, X_t) = e^{-\alpha(T-t) - g(X_t)}$ . The market prices of risk are restricted to be*

$$\vartheta_t(X_t) = -\sigma_\infty(X_t) = -g'(X_t)^\top \Sigma_X(X_t). \quad (3.77)$$

*Proof.* See Appendix 3.6.4. □

*Example 3.3.* Squared-autoregressive-independent-variable nominal term structure (SAINTS) model by Constantinides (33) is probably the earliest model that directly constructs  $M_t$  as a potential which generates a quadratic-Gaussian interest rate process. Specifically,

$$M_t = \exp \left\{ - \left( \rho + \frac{\sigma_0^2}{2} \right) t + x_0(t) + \sum_{i=1}^N (x_i(t) - \alpha_i)^2 \right\}, \quad (3.78)$$

where  $x_i(t)$  for  $i = 0, \dots, N$  are Ornstein-Uhlenbeck processes defined by

$$dx_i(t) = -\lambda_i x_i(t) dt + \sigma_i dW_i(t). \quad (3.79)$$

Here,  $W_0(t), W_1(t), \dots, W_N(t)$  are independent Brownian motions, and  $\rho > 0, \sigma_0 \geq 0, \sigma_i > 0, \alpha_i, \lambda_0 = 0, \lambda_i > 0$  (for  $i = 1, \dots, N$ ) are all constants. Note that only  $x_0(t)$  is not stationary because of  $\lambda_0 = 0$ , but one can denote  $\tilde{x}_0(t) \equiv -\frac{\sigma_0^2}{2}t + x_0(t)$  to define an exponential martingale  $e^{\tilde{x}_0(t)}$  orthogonal to any other state variables. The mere presence of  $\tilde{x}_0(t)$  in  $M_t$  will not affect interest rate dynamics nor bond prices. Direct application of Proposition 10 to the Constantinides model leads to the finding that the long forward rate is given by  $\rho$ . Moreover, the market prices of risk associated with the 0-th Brownian motion and the  $i$ -th Brownian motion are  $-\sigma_0$  and  $-2\sigma_i[x_i(t) - \alpha_i]$ , respectively. These exactly cancel out with the long bond return volatility. The long bond price takes the form:

$$P_\infty(t, T, X_t) = \exp \left[ -\rho(T - t) - \sum_{i=1}^N (x_i(t) - \alpha_i)^2 \right]. \quad (3.80)$$

This expression can also be obtained by examining the closed-form solution to  $P(t, T, X_t)$  in Constantinides (33) and taking the limit  $T \rightarrow \infty$ . Applying the general result of Eq. (3.123), one can also obtain the interest rate process derived in Ref. (33).

As stated in Proposition 10, all interest rate models constructed from the potential approach is subject to the strong restriction that the market prices of risk exactly cancel with the long bond return volatility. Investors may incur utility losses by using a restricted model.

Proposition 9 provides a remedy to this problem: one can first construct a potential  $Z_t = e^{-\alpha t - g(X_t)}$  and then model the pricing kernel as  $M_t = Z_t \xi_t^P$  with a non-trivial martingale component  $\xi_t^P$ . This modification relaxes the constraint on market prices of risk and permits more freedom in model calibration. For example, Filipovic et al. (52) develop a linear-rational term structure model following Roger's potential approach. The initial specification is too restrictive to match observed dynamics of bond risk premiums. For this reason, the authors incorporate a martingale component which gives them the freedom to model additional unspanned factors. Such unspanned risk factors only affect the risk premium as they enter through the equivalent martingale measure and they affect the distribution of future bond prices under the physical measure  $\mathbb{P}$  but not under the risk-neutral measure  $\mathbb{Q}$ . These unspanned factors are similar to the unspanned components of the macro risk factors in the term structure model of Joslin et al. (92).

## 3.4 Path-Dependence of Pricing Kernels

This section will discuss the path dependence of equilibrium asset pricing models. Pricing kernels are often transition-independent in an endowment economy such as Lucas' tree model. Nonetheless, the production economy (e.g., the Cox-Ingersoll-Ross model) naturally exhibits transition dependence. This subtle difference can generate quite different implications for term structure models developed by the equilibrium approach. For the log-utility case, one can modify the Lucas' tree model to restore the path-dependent dynamics for asset pricing.

### 3.4.1 Ross' Recovery

Recently, Ross (119) proposes a recovery theorem and claims that this theory allows for the entire recovery of both the pricing kernel and real probabilities from option prices. Under his model assumptions, option prices can help forecast not only the average return but also the entire return distribution if the state variable is Markovian and determines the aggregate consumption. This result is quite surprising. In the log-normal Black-Scholes world the stock returns are normally distributed but its drift is not directly observable nor priced in options. How could one infer the distribution of stock returns from option prices which contain little information about the drift of returns? Under the risk-neutral measure, the expected return on all assets is the risk free rate. How could one know the risk adjustment and use risk-neutral prices to estimate natural probabilities? In models with a representative agent, this is equivalent to knowing both the agent's risk aversion and subjective probability measure. Both types of information are not directly observable however.

Here is the setup of Ross' recovery theorem (119): consider a representative agent (one-

period) model in an economy described by a finite time-homogeneous Markov process and indexed by  $X_i$ . As a key assumption, Ross defines that a pricing kernel is called *transition independent* if there is a positive function of the states,  $\phi(X) = U'(c(X))$ , and a positive constant  $\delta$  such that for any transition from  $X_i$  to  $X_j$ , the pricing kernel has the form

$$M(X_i, X_j) = \delta \frac{\phi(X_j)}{\phi(X_i)}. \quad (3.81)$$

This allows us to express the Arrow-Debreu state price,  $p(X_i, X_j)$ , of the contingent claim which pays one dollar when the next state is  $X_j$  given the current state  $X_i$  with transition probability  $f(X_i, X_j)$ :

$$p(X_i, X_j) = M(X_i, X_j)f(X_i, X_j) = \delta \frac{U'(c(X_j))}{U'(c(X_i))} f(X_i, X_j) = \delta \frac{\phi(X_j)}{\phi(X_i)} f(X_i, X_j). \quad (3.82)$$

or more concisely,  $U'_i p_{ij} = \delta U'_j f_{ij}$ , where  $U'_i = U'(c(X_i)) \equiv \phi(X_i) > 0$ . In the matrix form, the above equation can be rewritten as  $DP = \delta FD$  for  $F = \frac{1}{\delta} DPD^{-1}$ , where  $P$  is the  $N \times N$  matrix of Arrow-Debreu state prices  $[p_{ij}]$ ,  $F$  is  $N \times N$  matrix of the natural transition probability,  $[f_{ij}]$ , and  $D$  is the diagonal matrix with the marginal utility across all states, i.e.  $D = \text{diag}(U'_1, U'_2, \dots, U'_N)$ . Since  $F$  is a positive matrix whose rows are transition probabilities, the sum of each row elements has to be one, that is,  $F\vec{e} = \vec{e}$ , where  $\vec{e}$  denotes a  $N \times 1$  vector with ones in all the entries. Then we have  $F\vec{e} = \frac{1}{\delta} DPD^{-1}\vec{e} = \vec{e}$ , which is equivalent to  $PD^{-1}\vec{e} = \delta D^{-1}\vec{e}$ . If we denote  $z \equiv D^{-1}\vec{e}$ , this equality simply becomes  $Pz = \delta z$ . By no-arbitrage condition, the matrix  $P$  is nonnegative. If  $P$  is irreducible, the Perron-Frobenius Theorem states that any nonnegative irreducible matrix has a unique positive characteristic vector,  $z$ , and an associated positive characteristic root,  $\lambda$ . Applying the Perron-Frobenius theorem to  $Pz = \delta z$  leads to the major claim in Ref. (119): *If there is no arbitrage, the pricing matrix is irreducible, and it is generated by a transition independent*

kernel, then there exists a unique (positive) solution to the problem of finding the physical probability transition matrix,  $F$ , the discount rate,  $\delta$ , and the pricing kernel,  $M$ .

As Borovicka et al. (18) point out, what is recovered by Ross' theorem via the Perron-Frobenius theory is not the physical probability but the long forward measure. The key assumption in Ross (119) is the transition-independent pricing kernel. According to Proposition 10, the property of transition independence imposes a strong restriction on the market prices of risk and precludes a nontrivial martingale component in the pricing kernel. To discuss continuous-time economy, the concept of path dependence needs to be extended:

**Definition 4.** In continuous time models, the pricing kernel  $M_t$  is called transition (or path) independent if its ratio at two time points can be expressed in the following form:

$$\frac{M_T}{M_t} = \frac{e^{-\delta T} \phi(X_T)}{e^{-\delta t} \phi(X_t)}, \quad (3.83)$$

i.e., the pricing kernel is of the Rogers' potential form  $M_t = e^{-\alpha t + g(X_t)}$  with  $g(x) \equiv \log \phi(x)$ .

### 3.4.2 Lucas' Tree Model of an Endowment Economy

Now let's discuss the classic Lucas' tree model. Consider a representative agent with time-additive CRRA utility who consumes the perishable fruit/good dropping off from the Lucas' tree. This consumption good is given by  $D_t = \phi(X_t)$ , which is a function of the single state variable  $X_t$  that solves the stochastic differential equation:

$$dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dW_t. \quad (3.84)$$

Using the Cox-Huang martingale approach, one can find the pricing kernel

$$M_t = e^{-\delta t} \frac{U'(D_t)}{U'(D_0)} = e^{-\delta t} \left[ \frac{\phi(X_t)}{\phi(X_0)} \right]^{-\gamma}, \quad (3.85)$$

where  $\delta$  is the time-discount parameter and  $\gamma$  is the representative agent's risk aversion. Obviously, the pricing kernel is transition-independent. Variations of Lucas' tree model are usually featured with such transition-independent pricing kernels<sup>9</sup>, and the risk premiums in such models are restricted according to Proposition 10.

If the perishable good follows  $D(X_t) = e^{X_t}$ , the representative agent with CRRA utility will consume  $c(X_t) = D(X_t) = e^{X_t}$  to clear the consumption market. It is straightforward to deduce the pricing kernel  $M_t = e^{-\delta t - \gamma(X_t - X_0)}$  and the market price of risk  $\vartheta_t = \gamma\sigma(X_t)$ . The default-free zero coupon bond price in this economy is given by

$$P(t, T, X_t) = \frac{\mathbb{E}_t^{\mathbb{P}}[M_T]}{M_t} = e^{-\delta(T-t)} \mathbb{E}_t^{\mathbb{P}}[e^{\gamma(X_t - X_T)}] = e^{-\delta(T-t) + \gamma X_t} \mathbb{E}_t^{\mathbb{P}}[e^{-\gamma X_T}], \quad (3.86)$$

where the term  $\mathbb{E}_t^{\mathbb{P}}[e^{-\gamma X_T}]$  can be computed if the (conditional) moment generating function of  $X_T$  given  $X_t$  is analytically known. Under regularity conditions<sup>10</sup>, the state variable has a limiting stationary distribution such that  $\lim_{T \rightarrow \infty} \mathbb{E}_t^{\mathbb{P}}[e^{-\gamma X_T}]$  is some constant. If so, the long bond price is proportional to  $e^{-\delta(T-t) + \gamma X_t}$  and the long forward rate is equal to the time discount factor  $\delta$ . For example, suppose  $X$  follows the Ornstein–Uhlenbeck process,  $dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$ , as in Vasicek (127). Then the interest rate process is:

$$r_t = \delta + \gamma\kappa(\theta - X_t) - \frac{1}{2}\gamma^2\sigma^2, \quad (3.87)$$

which is countercyclical with  $X_t$ . In good states (high  $X_t$ ), the marginal utility  $e^{-\gamma X_t}$  and risk-free rate  $r_t$  are both low, asking for high bond prices. The market price of risk is  $\vartheta = \gamma\sigma$

---

<sup>9</sup>For example, consumption-based asset pricing models are usually driven by exogenous dividend processes just like the Lucas' tree. In an endowment economy with heterogeneous agents, if the Arrow-Debreu equilibrium allocation exists, then it is also Pareto optimal; one can effectively derive the pricing kernel from the social planner's utility gradient which is a smooth function of the aggregate endowment.

<sup>10</sup>See Ref. 94 for further reference.

and the volatility of long bond returns is  $\sigma_\infty = -\gamma\sigma = -\vartheta$ . This economic model is similar to Example 3.1 where  $X_t$  is pro-cyclical with  $r_t$ . In general, Rogers' potential approach for term structure modeling can be supported by an endowment economy where a representative agent consumes the exogenously given dividends.

Clearly, Lucas' tree model precludes the permanent martingale component. As a remedy, the pricing kernel can be augmented with an exponential martingale, i.e.,  $M_t = e^{-\rho t + g(X_t)} \xi_t^P$ , where the martingale component satisfies  $\frac{d\xi_t^P}{\xi_t^P} = -[\vartheta_t + \sigma_\infty]dW_t$  with  $\sigma_\infty(X_t) = g'(X_t)\sigma_X(X_t)$ . By Itô's lemma, we have  $\frac{dM_t}{M_t} = -r_t dt - \vartheta_t dW_t$ , where the interest rate process now becomes<sup>11</sup>

$$r_t = \rho - g'(X_t)\mu_X(X_t) - \frac{1}{2}[g''(X_t) - g'(X_t)^2]\sigma_X(X_t)^2 + \vartheta_t g'(X_t)\sigma_X(X_t). \quad (3.88)$$

The introduction of a non-degenerate permanent martingale makes the pricing kernel path-dependent and allows risk premiums to be flexible for model calibration. Next subsection will demonstrate that the pricing kernel,  $M_t = e^{-\rho t + g(X_t)} \xi_t^P$ , can be supported by general equilibrium of a production economy.

### 3.4.3 Cox-Ingersoll-Ross Equilibrium of a Production Economy

Cox et al. (34 and 35) develop a continuous-time general equilibrium model of the production economy to examine the behavior of asset prices which are endogenously determined. For illustration, let's work out a simplified CIR model<sup>12</sup> where a single representative agent has

---

<sup>11</sup>If  $\xi_t^P$  is degenerate, then  $\vartheta_t = -g'(X_t)\sigma_X(X_t)$  and  $r_t = \rho - g'(X_t)\mu_X(X_t) - \frac{1}{2}[g''(X_t) + g'(X_t)^2]\sigma_X(X_t)^2$ .

<sup>12</sup>A finite-time version of this example is provided in Chapter 10 of Ref. 45.

the following logarithmic utility function,

$$U(C) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \log(C_t) dt \right]. \quad (3.89)$$

The agent has access to a production technology with the optimal capital-stock process  $K$  (starting from  $K_0 > 0$ ) defined by the Itô process:

$$dK_t = (aK_t X_t - D_t) dt + \epsilon K_t \sqrt{X_t} dW_t, \quad (3.90)$$

which is driven by a single Brownian motion  $W_t$  under  $\mathbb{P}$ . The state variable  $X$  can be thought of as a shock process that influence the productivity of capital  $K$ . This state variable evolves as a square-root diffusion process:

$$dX_t = (b - \kappa X_t) dt + \sigma \sqrt{X_t} dW_t, \quad (3.91)$$

with strictly positive scalars  $a$ ,  $b$ ,  $\kappa$ ,  $\sigma$ , and  $\epsilon$  that satisfy  $2b > \sigma^2$  and  $a > \epsilon^2$ . The optimal consumption plan can be found from the first order condition,  $C_t^* = D_t = \delta K_t$ . This implies

$$\frac{dK_t}{K_t} = \frac{dD_t}{D_t} = (aX_t - \delta) dt + \epsilon \sqrt{X_t} dW_t. \quad (3.92)$$

The pricing kernel is  $M_t = \frac{e^{-\delta t}}{D_t} = \frac{e^{-\delta t}}{\delta K_t}$  which, by Itô's lemma, can be written as

$$\frac{dM_t}{M_t} = (\epsilon^2 - a) X_t dt - \epsilon \sqrt{X_t} dW_t = -r_t dt - \frac{\epsilon \sqrt{r_t}}{\sqrt{a - \epsilon^2}} dW_t. \quad (3.93)$$

The interest rate is  $r_t = (a - \epsilon^2) X_t$ , which solves another square-root diffusion process:

$$dr_t = \kappa(r^* - r_t) dt + \sigma_r \sqrt{r_t} dW_t, \quad (3.94)$$

where  $r^* = b(a - \epsilon^2)/\kappa$  and  $\sigma_r = \sigma \sqrt{a - \epsilon^2}$ . This establishes the one-factor CIR model. One can also take  $r_t$  as the state variable in this economy. The market price of risk

$$\vartheta_t = \frac{\epsilon \sqrt{r_t}}{\sqrt{a - \epsilon^2}} = \epsilon \sqrt{X_t} > 0, \quad (3.95)$$

defines the Girsanov kernel for risk-neutral measure  $\mathbb{Q}$  under which the interest rate solves

$$dr_t = [b(a - \epsilon^2) - (\sigma\epsilon + \kappa)r_t]dt + \sigma_r\sqrt{r_t}dW_t^{\mathbb{Q}}. \quad (3.96)$$

When the Feller condition holds (i.e., the origin is inaccessible) under both  $\mathbb{P}$  and  $\mathbb{Q}$ , one can find the long forward rate in this economy (as demonstrated in Example 2.1),

$$\rho = \frac{2b(a - \epsilon^2)}{\sqrt{(\sigma\epsilon + \kappa)^2 + 2\sigma^2(a - \epsilon^2)} + (\sigma\epsilon + \kappa)} > 0. \quad (3.97)$$

The long bond return volatility is

$$\sigma_\infty = \frac{2\sigma\sqrt{(a - \epsilon^2)r_t}}{\sqrt{(\sigma\epsilon + \kappa)^2 + 2\sigma^2(a - \epsilon^2)} + (\sigma\epsilon + \kappa)} > 0. \quad (3.98)$$

It is obvious that  $\vartheta_t + \sigma_\infty > 0$  and the permanent martingale component is non-degenerate.

**Proposition 11.** *The pricing kernel  $M_t$  in this CIR economy is transition-dependent, as it cannot be expressed in the potential form,  $e^{-\rho t + g(X_t)}$ , for some well-behaved function  $g(\cdot)$ .*

*Proof.* Suppose we could write  $M_t = e^{-\rho t + g(X_t)}$  for some  $g(\cdot)$ . By Itô's lemma,  $g'(X_t)\sigma\sqrt{X_t} = -\epsilon\sqrt{X_t}$ , which implies  $g'(X_t) = -\frac{\epsilon}{\sigma}$ . By Eq. (3.123), the interest rate is

$$r_t = \rho - g'(X_t)(b - \kappa X_t) - \frac{1}{2}[g'(X_t)^2 + g''(X_t)]\sigma^2 X_t = \rho + \frac{\epsilon}{\sigma}(b - \kappa X_t) - \frac{1}{2}\epsilon^2 X_t, \quad (3.99)$$

which conflicts with the equilibrium result  $r_t = (a - \epsilon^2)X_t$  because  $\rho + \frac{\epsilon b}{\sigma} > 0$ .  $\square$

With a transition-dependent pricing kernel, the CIR model does not suffer from the restriction stated in Proposition 10. After some algebra, one can rewrite Eq. (3.92) as

$$\frac{dK_t}{K_t} = \frac{dD_t}{D_t} = \left( \frac{a}{a - \epsilon^2} r_t - \delta \right) dt + \sqrt{\frac{\epsilon^2 r_t}{a - \epsilon^2}} dW_t, \quad (3.100)$$

where the expected growth rate of the stock-capital is always less than the risk-free rate  $r_t$ .

Why does the CIR model admit a transition-dependent pricing kernel? To see this, we consider a simple Lucas' tree model where the aggregate dividend grows as the geometric Brownian motion,  $\frac{dD_t}{D_t} = \mu dt + \sigma dW_t$ , with constant  $\mu$  and  $\sigma$ . The pricing kernel here is  $M_t = e^{-\delta t} D_t^{-1}$  which is another geometric Brownian motion  $\frac{dM_t}{M_t} = -r dt - \sigma dW_t$ , with a constant risk-free rate  $r = \delta + \mu - \sigma^2$ . One can rewrite  $M_t = e^{-rt - X_t}$  where  $dX_t = \frac{\sigma^2}{2} dt + \sigma dW_t$  defines an exponential martingale  $e^{-X_t}$  that does not affect the term structure. Either  $X_t$  or  $D_t$  can be chosen as the state variable in this model, since either of them contains all the information that a representative agent needs to forecast the future state of the economy.

Can we take  $D_t$  as the only state variable in the CIR model? The answer is no. Knowing  $D_t$  alone is not enough for the representative agent to forecast the future state even over an infinitesimal period because both the drift and diffusion terms of  $D_t$  in Eq. (3.92) are stochastically driven by the underlying variable  $X_t$ . Forecasting the next state of the economy (at  $t + dt$ ) requires the knowledge of both  $D_t$  and  $X_t$ . Since the sample path of  $D$  is determined by the sample path of  $X$ , knowing the sample path of  $X$  up to time  $t$  is sufficient to reconstruct  $D_t$ . Therefore, the state of the economy is characterized by the full history of  $X$  up to time  $t$ . Such history-dependence accounts for the transition-dependent property of pricing kernel in the CIR model<sup>13</sup>. This property suggests that the CIR general equilibrium is an appropriate economic framework for term structure modeling<sup>14</sup>. One can modify the Lucas' model to remedy the restriction issue in Proposition 10.

---

<sup>13</sup>If the pricing kernel can be expressed as a potential,  $M_t = e^{-\rho t + g(X_t)}$ , then it means that the state of the economy is always determined by  $X_t$ , and there is no need to know the full history of  $X$ .

<sup>14</sup>Heath, Jarrow, and Morton (74) (HJM) developed a term structure modeling framework based on specifying forward rates volatilities and the market price of risk. Jin and Glasserman (91) show that every HJM arbitrage-free model can be supported by a production economy equilibrium of Cox-Ingersoll-Ross (35).

**Corollary 14.** *For the log-utility case, the equilibrium Lucas model of an endowment economy can be modified to coincide with the equilibrium CIR model of a production economy.*

*Proof.* See Appendix 3.6.5 □

This result indicates the potential equivalence of these equilibrium models. However, it may not hold beyond logarithmic utility. The advantages of CIR general equilibrium are obvious. First, it allows the pricing kernel to be path dependent without losing internal consistency. Second, it provides a detailed equilibrium picture showing how the underlying economic variables affect the term structure. Third, it can give an equilibrium support to models constructed from the arbitrage approach.

### 3.4.4 Extension and Future Research

So far we have focused on the bond market and term structure models. The analysis can be easily extended to the equity market. For notational convenience, let's keep using the  $k \times 1$  vector  $X_t$  to denote a *fundamental* group of state variables which determine the term structure of interest rates and thus affect the prices of all traded securities in the economy. The word "fundamental" is in the sense this group of variables influences all financial assets through the stochastic discount factor. Fixed-income securities and derivatives (including bonds across different maturities, discount or coupon bonds, and bond derivatives) are solely priced by these fundamental state variables. As before, the long forward rate is constant if the economy is driven by recurrent Markovian processes. The state space can be expanded by including the  $m \times 1$  vector  $Y_t$  which denotes the state variables that solely affect the pricing of other risky securities (equities). Both  $X_t$  and  $Y_t$  are driven by a vector  $d \times 1$

Brownian motions. It is convenient to think of  $X_t$  as bond-market risk factors and  $Y_t$  as equity-market risk factors. By construction,  $X_t$  and  $Y_t$  can be orthogonal to each other: if the  $i$ -th state variable in  $Y_t$ , say  $Y_t^{(i)}$ , is correlated with the  $j$ -th state variable in  $X_t$ , say  $X_t^{(j)}$ , then one can decompose and carve out the correlated component in  $Y_t^{(i)}$  and move it from the  $Y_t$  group to the  $X_t$  group. The following definition can be helpful.

**Definition 5.** The pricing kernel for the overall market is said to be orthogonally factorized if  $M_t = M_t^X M_t^Y$  and  $dM_t^X \cdot dM_t^Y = 0$  such that

$$\frac{dM_t}{M_t} = \frac{dM_t^X}{M_t^X} + \frac{dM_t^Y}{M_t^Y}, \quad (3.101)$$

The time- $t$  value of any  $X$ -state-contingent claim  $\psi(X_T)$  is independent of the  $Y$ -state space:

$$V_t[\psi(X_T)] = \mathbb{E}^{\mathbb{P}} \left[ \frac{M_T}{M_t} \psi(X_T) | \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}} \left[ \frac{M_T^X}{M_t^X} \frac{M_T^Y}{M_t^Y} \psi(X_T) | \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}} \left[ \frac{M_T^X}{M_t^X} \psi(X_T) | \mathcal{F}_t \right]. \quad (3.102)$$

For example, the discount bond prices solely depend on  $X$  state variables.

The above extension can be useful for asset pricing problems as the pricing kernel in general is transition-dependent. Stochastic volatility models exhibit such transition dependence. These include the option-pricing model by Heston (76) and also the recent development by Christoffersen, Heston, and Jacobs (31) who propose a GARCH option model with a variance-dependent pricing kernel. If we make a transition from the physical probability to the long forward measure, equity prices will be adjusted by the long bond volatility. The advantage of long forward measure is the constant long rate, similar to the risk-neutral measure in Black-Scholes model where the interest rate is constant. This can be convenient sometimes.

## 3.5 Conclusion

This article applies the martingale approach to explore several issues regarding the pricing kernel decomposition when the long bond is taken as numeraire. The central condition in my development is that the long forward rate is constant, a plausible condition in a normal economy. This condition imposes constraints on the asymptotic behaviors of long bonds and leads to a simple proof for the Hansen-Scheinkman decomposition, i.e., the pricing kernel can be expressed as a product of a transitory component and a permanent martingale one.

When the state variables are stationary and recurrent, the long forward rate has to be flat and the decomposition result immediately follows from the martingale arguments. This long-run factorization may not be unique if the economy does not permit an equivalent martingale measure. Examination of the long forward measure produces an explicit expression for the permanent martingale component. This demonstrates how the transitory and permanent components comove with each other over time. More importantly, this result reveals an implicit restriction in a popular approach of modeling interest rates. This approach directly specifies the pricing kernel as a probabilistic Markov potential, which neglects the permanent martingale in the pricing kernel and thus imposes problematic constraints on the market prices of risk which can distort model calibration.

Finally, a comparison is made between the endowment economy (e.g., Lucas' tree model) and the production economy to illustrate the origin of path-dependence of pricing kernels. To restore path dependence, a simple remedy is suggested for equilibrium models of the endowment economy for the log-utility case. This work shows the advantages of Cox-Ingersoll-Ross general equilibrium for modeling interest rates and asset prices.

## 3.6 Appendix

### 3.6.1 Proof of Proposition 7

If  $\lim_{T \rightarrow \infty} f(t, T, X_t) = \rho$ , then  $\lim_{T \rightarrow \infty} z(t, T, X_t) = \rho$  by Lemma 1. Under (A4), we can write  $z(t, T, X_t) = z(T - t, X_t)$  and introduce  $\tau \equiv T - t \geq 0$  to define a new function

$$h(\tau, X_t) \equiv [z(\tau, X_t) - \rho] \cdot \tau, \quad (3.103)$$

which allows us to express

$$P(T - t, X_t) = P(\tau, X_t) = \exp[-z(\tau, X_t) \cdot \tau] = \exp[-\rho\tau - h(\tau, X_t)] \quad (3.104)$$

Therefore, the forward rate can be written as

$$f(T - t, X_t) = -\frac{\partial \log P(T - t, X_t)}{\partial T} = \rho + \frac{\partial h(\tau, X_t)}{\partial \tau}. \quad (3.105)$$

Since  $\lim_{T \rightarrow \infty} f(T - t, X_t) = \lim_{\tau \rightarrow \infty} f(\tau, X_t) = \rho$ , the above expression implies that

$$\lim_{\tau \rightarrow \infty} \frac{\partial h(\tau, X_t)}{\partial \tau} = 0. \quad (3.106)$$

Since  $\lim_{T \rightarrow \infty} z(T - t, X_t) = \rho$  and  $\frac{\partial h(\tau, X_t)}{\partial \tau} = z(\tau, X_t) - \rho + \frac{\partial z(\tau, X_t)}{\partial \tau} \tau$ , the above result indicates that

$$\lim_{\tau \rightarrow \infty} \frac{\partial z(\tau, X_t)}{\partial \tau} \tau = 0. \quad (3.107)$$

For an arbitrary state  $X_t = x$ , we are interested in the asymptotic behavior of  $h(\tau, x) = [z(\tau, x) - \rho] \cdot \tau$  which however is a type of “ $0 \cdot \infty$ ” limit. Applying the L'Hôpital's rule yields

$$\lim_{\tau \rightarrow \infty} h(\tau, x) = \lim_{\tau \rightarrow \infty} \frac{\frac{\partial [z(\tau, x) - \rho]}{\partial \tau}}{\frac{\partial \tau^{-1}}{\partial \tau}} = - \lim_{\tau \rightarrow \infty} \frac{\partial z(\tau, x)}{\partial \tau} \tau^2, \quad (3.108)$$

which is again a type of “ $0 \cdot \infty$ ” limit by Eq. (3.107). Repeating the L’Hôpital’s rule yields

$$\lim_{\tau \rightarrow \infty} h(\tau, x) = - \lim_{\tau \rightarrow \infty} \frac{\frac{\partial}{\partial \tau} \left[ \frac{\partial z(\tau, x)}{\partial \tau} \tau \right]}{\frac{\partial \tau^{-1}}{\partial \tau}} = \lim_{\tau \rightarrow \infty} \left( \frac{\partial^2 z}{\partial \tau^2} \tau + \frac{\partial z}{\partial \tau} \right) \tau^2. \quad (3.109)$$

Combining Eq. (3.108) and Eq. (3.109) leads to

$$\lim_{\tau \rightarrow \infty} \left( \frac{\partial^2 z}{\partial \tau^2} \tau + 2 \frac{\partial z}{\partial \tau} \right) \tau^2 = 0. \quad (3.110)$$

This condition suggests that  $(\tau, x)$  has to satisfy the form below in general:

$$z(\tau, x) = \frac{g(x)}{\tau} + \rho + w(\tau, x), \quad (3.111)$$

where  $w(\tau, x)$  is a residual term decays to zero at a speed faster than  $\tau^{-1}$ . This follows from the general solution to ordinary differential equation  $\left( \frac{\partial^2 z}{\partial \tau^2} \tau + 2 \frac{\partial z}{\partial \tau} \right) \tau^2 = 0$  in Eq. (3.110).

As a result, we have

$$\lim_{\tau \rightarrow \infty} h(\tau, x) = \lim_{\tau \rightarrow \infty} [g(x) + w(\tau, x)\tau] = g(x). \quad (3.112)$$

If we define  $\varepsilon(\tau, X_t) \equiv g(X_t) - h(\tau, X_t) = w(\tau, X_t) \cdot \tau$ , then  $\lim_{T \rightarrow \infty} \varepsilon(T - t, X_t) = 0$ . This leads to Eq. (3.10) in Proposition 7:

$$\begin{aligned} \log P(T - t, X_t) &= -\rho \cdot (T - t) - h(T - t, X_t) \\ &= -\rho \cdot (T - t) - g(X_t) + \varepsilon(T - t, X_t). \end{aligned} \quad (3.113)$$

### 3.6.2 Proof of Lemma 2

Given that the long forward rates  $f_l(s, X_s)$  and  $f_l(t, X_t)$  exist for  $s < t$ , we can define  $q(s) = \lim_{T \rightarrow \infty} P(s, T, X_s)^{\frac{1}{T}} = e^{-f_l(s, X_s)}$  and  $q(t) = \lim_{T \rightarrow \infty} P(t, T, X_t)^{\frac{1}{T}} = e^{-f_l(t, X_t)}$  where

$q(\cdot)$  is a monotonic transformation. Under the  $t$ -forward measure  $\mathbb{Q}^t$ , the following holds

$$\frac{P(s, T, X_s)}{P(s, t, X_s)} = \mathbb{E}^{\mathbb{Q}^t} [P(t, T, X_t) | \mathcal{F}_s], \quad \text{such that } q(s) = \lim_{T \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^t} [P(t, T, X_t) | \mathcal{F}_s]^{\frac{1}{T}}. \quad (3.114)$$

We introduce an arbitrary bounded random variable  $Z \geq 0$  with  $\mathbb{E}^{\mathbb{Q}^t} [Z] = 1$  to prove that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^t} [Zq(t)] &= \mathbb{E}^{\mathbb{Q}^t} \left[ \liminf_{T \rightarrow \infty} ZP(t, T, X_t)^{\frac{1}{T}} \right] \\ &\leq \mathbb{E}^{\mathbb{Q}^t} \left[ \liminf_{T \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^t} [ZP(t, T, X_t)^{\frac{1}{T}} | \mathcal{F}_s] \right] \\ &\leq \mathbb{E}^{\mathbb{Q}^t} \left[ \liminf_{T \rightarrow \infty} \mathbb{E}^{\mathbb{Q}^t} [Z^{\frac{T}{T-1}} | \mathcal{F}_s]^{\frac{T-1}{T}} \mathbb{E}^{\mathbb{Q}^t} [P(t, T, X_t) | \mathcal{F}_s]^{\frac{1}{T}} \right] \\ &= \mathbb{E}^{\mathbb{Q}^t} [Zq(s)], \end{aligned} \quad (3.115)$$

where the first inequality follows from the Fatou's lemma, the second inequality is due to Hölder's inequality and the last step is obtained by the dominated convergence theorem. As the random variable  $Z \geq 0$  is arbitrary with  $\mathbb{E}^{\mathbb{Q}^t} [Z] = 1$ , it must be that  $q(t) \leq q(s)$  which is equivalent to  $f_l(s, X_s) \leq f_l(t, X_t)$  for arbitrary  $s < t$ .

### 3.6.3 Proof of Proposition 9

Let's take the limit  $T \rightarrow \infty$  to pass into the long forward measure  $\mathbb{Q}_L$ , which is conceptually the large  $T$  limit of  $\mathbb{Q}_T$ . Consider two arbitrary time points  $s$  and  $t$ , with  $0 \leq s \leq t < \infty$ . The notation will be abused a little bit by replacing the superscript  $T$  with the symbol  $\infty$  to indicate the long forward limit. Using Eq. (3.40), one can evaluate the following ratio:

$$\frac{\xi_t^\infty}{\xi_s^\infty} = \lim_{T \rightarrow \infty} \frac{P(s, T, X_s) M_s}{P(t, T, X_t) M_t} = \lim_{T \rightarrow \infty} \frac{e^{-\rho(T-s)-g(X_s)-\epsilon(T-s, X_s)} M_s}{e^{-\rho(T-t)-g(X_t)-\epsilon(T-t, X_t)} M_t} = e^{-\rho(t-s)} \frac{\phi(X_s) M_s}{\phi(X_t) M_t}. \quad (3.116)$$

For the long bond price process  $P_\infty(t, T, X_t) = e^{-\rho(T-t)-g(X_t)}$ , the long bond return volatility is independent of  $t$  and given by

$$\lim_{T \rightarrow \infty} \sigma_P(t, T, X_t) = \Sigma_X(X_t) \frac{\partial g(X_t)}{\partial X} \equiv \sigma_\infty(X_t). \quad (3.117)$$

where the  $d \times k$  matrix  $\Sigma_X$  is defined in  $dX_t = \mu_X(X_t)dt + \Sigma_X(X_t)dW_t$ . Direct examination of the ratio  $\frac{\xi_t^\infty}{\xi_s^\infty}$  by using Eq. (3.44) in the limit  $T \rightarrow \infty$  yields

$$\frac{\xi_t^\infty}{\xi_s^\infty} = \exp \left( \int_s^t [\vartheta_u(X_u) + \sigma_\infty(X_u)]^\top dW_u + \frac{1}{2} \int_s^t [\vartheta_u(X_u) + \sigma_\infty(X_u)]^\top [\vartheta_u(X_u) + \sigma_\infty(X_u)] du \right). \quad (3.118)$$

Now combining Eq. (3.116) with Eq. (3.118) and taking  $s = 0$  leads to

$$M_t = e^{-\rho t} \frac{\phi(X_0)}{\phi(X_t)} \exp \left( - \int_0^t [\vartheta_u + \sigma_\infty(X_u)]^\top dW_u - \frac{1}{2} \int_0^t [\vartheta_u + \sigma_\infty(X_u)]^\top [\vartheta_u + \sigma_\infty(X_u)] du \right). \quad (3.119)$$

Comparing Eq. (3.119) with Eq. (3.12), one can see that the permanent (local) martingale component  $\xi_t^P$  in the factorization  $M_t = e^{-\rho t} \frac{\phi(X_0)}{\phi(X_t)} \xi_t^P$  is given by

$$\xi_t^P = \exp \left\{ - \int_0^t [\vartheta_u + \sigma_\infty(X_u)]^\top dW_u - \frac{1}{2} \int_0^t [\vartheta_u + \sigma_\infty(X_u)]^\top [\vartheta_u + \sigma_\infty(X_u)] du \right\}. \quad (3.120)$$

In other words,  $\xi_t^P$  solves the Itô diffusion process

$$\frac{d\xi_t^P}{\xi_t^P} = -[\vartheta_t + \sigma_\infty(X_t)]^\top dW_t, \quad (3.121)$$

where  $\sigma_\infty = \Sigma_X \frac{\partial g}{\partial X}$ . This expression reveals the economic content of the permanent martingale component. Its Girsanov kernel is the sum of long bond return volatility and market price of risk. To change from the physical probability to the long forward measure, all the adjustment one needs is to add the long bond volatility to the market prices of risk.

### 3.6.4 Proof of Proposition 10

Suppose the SDF (or pricing kernel) is constructed from a potential, say  $M_t = e^{-\alpha t} f(X_t)$ .

A smooth nonnegative function  $f(x)$  can always be written as  $f(x) = e^{g(x)}$  for some other smooth function  $g(x)$ . Using Itô's lemma, one can derive that

$$\frac{dM_t}{M_t} = -r_t(X_t)dt + g'(X_t)^\top \Sigma_X(X_t) dW_t, \quad (3.122)$$

where  $g'(x) = \frac{dg(x)}{dx}$  and the interest rate is given by

$$r_t = \alpha - g'(X_t)^\top \mu_X - \frac{1}{2} g'(X_t)^\top \Sigma_X \Sigma_X^\top g'(X_t) - \frac{1}{2} \text{tr}[\Sigma_X \Sigma_X^\top g''(X_t)]. \quad (3.123)$$

The market price of risk is  $\vartheta_t(X_t) = -g'(X_t)^\top \Sigma_X$  which defines the Girsanov kernel of  $\frac{d\mathbb{Q}}{d\mathbb{P}}$ .

With  $M_t = e^{-\alpha t + g(X_t)}$ , the discount bond price is found to be

$$P(t, T, X_t) = \frac{\mathbb{E}^\mathbb{P}[M_T | \mathcal{F}_t]}{M_t} = e^{-\alpha(T-t)} \mathbb{E}_t^\mathbb{P}[e^{g(X_T) - g(X_t)}] = e^{-\alpha(T-t)} \mathbb{E}_t^\mathbb{P}[e^{\int_t^T dg(X_s)}], \quad (3.124)$$

where by Itô's lemma,

$$dg(X_t) = \left\{ g'(X_t)^\top \mu_X + \frac{1}{2} \text{tr}[\Sigma_X \Sigma_X^\top g''(X_t)] \right\} dt + g'(X_t)^\top \Sigma_X dW_t. \quad (3.125)$$

Substituting the above expression into Eq. (3.124) and using Eq. (3.123), one can derive

that

$$P(t, T, X_t) = \mathbb{E}^\mathbb{P} \left[ e^{-\int_t^T r_s ds} \frac{d\mathbb{Q}}{d\mathbb{P}} \Big| \mathcal{F}_t \right] = \mathbb{E}^\mathbb{Q} \left[ e^{-\int_t^T r_s ds} \Big| \mathcal{F}_t \right]. \quad (3.126)$$

If the Markov process  $X_t$  has a stationary limiting distribution independent of the initial states, then at the long maturity limit the discount bond price satisfies (up to some constant)

$$P(t, T, X_t) = e^{-\alpha(T-t) - g(X_t)} \mathbb{E}^\mathbb{P} [e^{g(X_T)} | \mathcal{F}_t] \rightarrow e^{-\alpha(T-t) - g(X_t)} \mathbb{E}^\mathbb{P} [e^{g(X_T)}] = \text{const} \cdot e^{-\alpha(T-t) - g(X_t)}. \quad (3.127)$$

For this long bond  $P_\infty(t, T, X_t) = e^{-\alpha(T-t)-g(X_t)}$ , we can use Itô's lemma to show that

$$\frac{dP_\infty(t, T, X_t)}{P_\infty(t, T, X_t)} = \mu_\infty(X_t)dt - \sigma_\infty(X_t)dW_t = [r_t + g'(X_t)^\top \Sigma_X \Sigma_X^\top g'(X_t)] dt - g'(X_t)^\top \Sigma_X dW_t, \quad (3.128)$$

which implies  $\mu_\infty(X_t) - r_t = g'(X_t)^\top \Sigma_X \Sigma_X^\top g'(X_t) = -\sigma_\infty(X_t) \cdot \vartheta_t(X_t)$ . In this general setup, it becomes obvious that  $\sigma_\infty(X_t) = g'(X_t)^\top \Sigma_X = -\vartheta_t(X_t)$ .

### 3.6.5 Proof of Corollary 14

The pricing kernel can be derived from the first order condition for the representative agent's problem. This gives  $M_t = e^{-\delta t} U'(D_t) = e^{-\delta t + g(D_t)}$ . We can assume that the exogenous aggregate dividend evolves as a general Itô diffusion process:

$$\frac{dD_t}{D_t} = \mu_D(X_t)dt + \sigma_D(X_t)dW_t, \quad (3.129)$$

where the state variable  $X_t$  is self-governed in the sense that  $dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dW_t$ .

In this general setup, the pricing kernel can depend on sample paths and this Lucas-type model can also be supported by a CIR-type general equilibrium. By Itô's lemma,

$$\frac{dM_t}{M_t} = \left\{ -\delta + g'(D_t)D_t\mu_D(X_t) + \frac{1}{2}[g'(D_t)^2 + g''(D_t)]D_t^2\sigma_D^2(X_t) \right\} dt + g'(D_t)D_t\sigma_D(X_t)dW_t. \quad (3.130)$$

The market price of risk for the representative CRRA-utility agent is

$$\vartheta_t = -g'(D_t)D_t\sigma_D(X_t) = -\frac{U''(D_t)}{U'(D_t)}D_t\sigma_D(X_t) = \gamma\sigma_D(X_t). \quad (3.131)$$

which is the product of the investors' relative risk aversion and the instantaneous volatility of the aggregate dividend. In this case, Eq. (3.130) becomes

$$\frac{dM_t}{M_t} = - \left[ \delta + \gamma\mu_D(X_t) - \frac{1}{2}\gamma(\gamma + 1)\sigma_D^2(X_t) \right] dt - \gamma\sigma_D(X_t)dW_t, \quad (3.132)$$

suggesting that the interest rate is  $r_t = \delta + \gamma\mu_D(X_t) - \frac{1}{2}\gamma(\gamma + 1)\sigma_D^2(X_t)$  which only depends on  $X_t$ . For logarithmic utility, we have  $g(D_t) = -\log(D_t)$  and Eq. (3.130) is reduced to be

$$\frac{dM_t}{M_t} = - [\delta + \mu_D(X_t) - \sigma_D^2(X_t)] dt - \sigma_D(X_t)dW_t. \quad (3.133)$$

Now the interest rate is  $r_t = \delta + \mu_D(X_t) - \sigma_D^2(X_t)$  and the market price of risk is  $\vartheta_t = \sigma_D(X_t)$ .

The specification of  $\mu_D = aX_t - \delta$  and  $\sigma_D = \epsilon\sqrt{X_t}$  reproduces the pricing kernel Eq. (3.93)

and the interest rate  $r_t = (a - \epsilon^2)X_t$  in the CIR model. Securities whose contractual payoffs

$\Psi(X_t, T)$  do not explicitly depend on  $D$  are priced by the valuation equation:

$$\frac{1}{2}\sigma_X^2 \frac{\partial^2 V}{\partial x^2} + (\mu_X - \sigma_X \sigma_D) \frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} = r_t V, \quad (3.134)$$

with the boundary condition  $V(X_t, T) = \Psi(X_t, T)$ . It is exactly the CIR valuation equation.

# Bibliography

- [1] Dilip Abreu and Markus K Brunnermeier. Synchronization risk and delayed arbitrage. *Journal of Financial Economics*, 66(2-3):341–360, 2002.
- [2] Dilip Abreu and Markus K Brunnermeier. Bubbles and crashes. *Econometrica*, 71(1):173–204, 2003.
- [3] Dong-Hyun Ahn, Robert F Dittmar, and A Ronald Gallant. Quadratic term structure models: Theory and evidence. *Review of Financial Studies*, 15(1):243–288, 2002.
- [4] Franklin Allen and Douglas Gale. Stock-price manipulation. *Review of Financial Studies*, 5(3):503–529, 1992.
- [5] Fernando Alvarez and Urban J Jermann. Using asset prices to measure the persistence of the marginal utility of wealth. *Econometrica*, 73(6):1977–2016, 2005.
- [6] Kerry Back. Insider trading in continuous time. *Review of Financial Studies*, 5(3):387–409, 1992.
- [7] Kerry Back, C Henry Cao, and Gregory A Willard. Imperfect competition among informed traders. *Journal of Finance*, 55(5):2117–2155, 2000.

- [8] David Backus, Mikhail Chernov, and Stanley Zin. Sources of entropy in representative agent models. *Journal of Finance*, 69(1):51–99, 2014.
- [9] Gurdip Bakshi and Fousseni Chabi-Yo. Variance bounds on the permanent and transitory components of stochastic discount factors. *Journal of Financial Economics*, 105(1):191–208, 2012.
- [10] Gurdip Bakshi, Fousseni Chabi-Yo, and Xiaohui Gao. A recovery that we can trust? deducing and testing the restrictions of the recovery theorem. *Review of Financial Studies*, 31(2):532–555, 2017.
- [11] Ravi Bansal, Dana Kiku, and Amir Yaron. Risks for the long run: Estimation with time aggregation. *Journal of Monetary Economics*, 82:52–69, 2016.
- [12] Nicholas Barberis, Robin Greenwood, Lawrence Jin, and Andrei Shleifer. X-capm: An extrapolative capital asset pricing model. *Journal of Financial Economics*, 115(1):1–24, 2015.
- [13] Nicholas Barberis, Robin Greenwood, Lawrence Jin, and Andrei Shleifer. Extrapolation and bubbles. *Journal of Financial Economics*, 2018.
- [14] Nicholas Barberis and Richard Thaler. A survey of behavioral finance. *Handbook of the Economics of Finance*, 1:1053–1128, 2003.
- [15] Tomas Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, 2009.
- [16] Oleg Bondarenko. Statistical arbitrage and securities prices. *Review of Financial Studies*, 16(3):875–919, 2003.

- [17] Jaroslav Borovička and Lars Peter Hansen. Examining macroeconomic models through the lens of asset pricing. *Journal of Econometrics*, 183(1):67–90, 2014.
- [18] Jaroslav Borovička, Lars Peter Hansen, and José A Scheinkman. Misspecified recovery. *Journal of Finance*, 71(6):2493–2544, 2016.
- [19] Michael J Brennan and Eduardo S Schwartz. A continuous time approach to the pricing of bonds. *Journal of Banking and Finance*, 3(2):133–155, 1979.
- [20] Markus K Brunnermeier. Information leakage and market efficiency. *Review of Financial Studies*, 18(2):417–457, 2005.
- [21] Markus K Brunnermeier and Lasse Heje Pedersen. Predatory trading. *Journal of Finance*, 60(4):1825–1863, 2005.
- [22] John Y Campbell, Andrew W Lo, and Archie Craig MacKinlay. *The Econometrics of Financial Markets*, volume 2. princeton University press Princeton, NJ, 1997.
- [23] Mark M Carhart. On persistence in mutual fund performance. *Journal of Finance*, 52(1):57–82, 1997.
- [24] Peter Carr and Jiming Yu. Risk, return, and ross recovery. *Journal of Derivatives*, 20(1):38, 2012.
- [25] Archishman Chakraborty and Bilge Yilmaz. Informed manipulation. *Journal of Economic Theory*, 114(1):132, 2004.
- [26] Archishman Chakraborty and Bilge Yilmaz. Manipulation in market order models. *Journal of Financial Markets*, 7(2):187–206, 2004.

- [27] Archishman Chakraborty and Bilge Yilmaz. Microstructure bluffing with nested information. *American Economic Review*, 98(2):280–84, 2008.
- [28] Louis KC Chan, Narasimhan Jegadeesh, and Josef Lakonishok. Momentum strategies. *Journal of Finance*, 51(5):1681–1713, 1996.
- [29] Alexander M Chincó, Adam D Clark-Joseph, and Mao Ye. Sparse signals in the cross-section of returns. Technical report, National Bureau of Economic Research, 2017.
- [30] Timothy M Christensen. Nonparametric stochastic discount factor decomposition. *Econometrica*, 85(5):1501–1536, 2017.
- [31] Peter Christoffersen, Steven Heston, and Kris Jacobs. Capturing option anomalies with a variance-dependent pricing kernel. *Review of Financial Studies*, 26(8):1963–2006, 2013.
- [32] Pierre Collin-Dufresne and Vyacheslav Fos. Insider trading, stochastic liquidity, and equilibrium prices. *Econometrica*, 84(4):1441–1475, 2016.
- [33] George M Constantinides. A theory of the nominal term structure of interest rates. *Review of Financial Studies*, 5(4):531–552, 1992.
- [34] John C Cox, Jonathan E Ingersoll, and Stephen A Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–408, 1985.
- [35] John C Cox, Jonathan E Ingersoll, and Stephen A Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, pages 363–384, 1985.

- [36] John C Cox and Stephen A Ross. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3(1-2):145–166, 1976.
- [37] Qiang Dai and Kenneth J Singleton. Specification analysis of affine term structure models. *Journal of Finance*, 55(5):1943–1978, 2000.
- [38] Kent Daniel, David Hirshleifer, and Avanidhar Subrahmanyam. Investor psychology and security market under-and overreactions. *Journal of Finance*, 53(6):1839–1885, 1998.
- [39] Freddy Delbaen and Walter Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300(1):463–520, 1994.
- [40] J Bradford DeLong, Andrei Shleifer, Lawrence H Summers, and Robert J Waldmann. Positive feedback investment strategies and destabilizing rational speculation. *Journal of Finance*, 45(2):379–395, 1990.
- [41] Jérôme Detemple and Marcel Rindisbacher. Dynamic asset allocation: Portfolio decomposition formula and applications. *Review of Financial Studies*, 23(1):25–100, 2009.
- [42] L Uri Dothan. On the term structure of interest rates. *Journal of Financial Economics*, 6(1):59–69, 1978.
- [43] James Dow and Sergio Ribeiro da Costa Werlang. Uncertainty aversion, risk aversion, and the optimal choice of portfolio. *Econometrica*, pages 197–204, 1992.
- [44] Sergey Dubynskiy and Robert S Goldstein. Recovering drifts and preference parameters from financial derivatives. 2013.

- [45] Darrell Duffie. *Dynamic asset pricing theory*. Princeton University Press, 2010.
- [46] Darrell Duffie and Nicolae Garleanu. Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, pages 41–59, 2001.
- [47] Darrell Duffie and Rui Kan. A yield-factor model of interest rates. *Mathematical finance*, 6(4):379–406, 1996.
- [48] Philip H Dybvig, Jonathan E Ingersoll, and Stephen A Ross. Long forward and zero-coupon rates can never fall. *Journal of Business*, pages 1–25, 1996.
- [49] Philip H Dybvig and Stephen A Ross. Arbitrage. In *Finance*, pages 57–71. Springer, 1989.
- [50] Eugene F Fama and Kenneth R French. Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, 33(1):3–56, 1993.
- [51] Damir Filipović. A general characterization of one factor affine term structure models. *Finance and Stochastics*, 5(3):389–412, 2001.
- [52] Damir Filipović, Martin Larsson, and Anders B Trolle. Linear-rational term structure models. *Journal of Finance*, 72(2):655–704, 2017.
- [53] F Douglas Foster and S Viswanathan. Strategic trading with asymmetrically informed traders and long-lived information. *Journal of Financial and Quantitative Analysis*, 29(4):499–518, 1994.
- [54] F Douglas Foster and S Viswanathan. Strategic trading when agents forecast the forecasts of others. *Journal of Finance*, 51(4):1437–1478, 1996.

- [55] Merritt B Fox, Lawrence R Glosten, and Gabriel V Rauterberg. Stock market manipulation and its regulation. *Yale J. on Reg.*, 35:67, 2018.
- [56] Joachim Freyberger, Andreas Neuhierl, and Michael Weber. Dissecting characteristics nonparametrically. Technical report, National Bureau of Economic Research, 2017.
- [57] Xavier Gabaix. A sparsity-based model of bounded rationality. *Quarterly Journal of Economics*, 129(4):1661–1710, 2014.
- [58] Xavier Gabaix, Parameswaran Gopikrishnan, Vasiliki Plerou, and H Eugene Stanley. Institutional investors and stock market volatility. *Quarterly Journal of Economics*, 121(2):461–504, 2006.
- [59] Xavier Gabaix, Arvind Krishnamurthy, and Olivier Vigneron. Limits of arbitrage: theory and evidence from the mortgage-backed securities market. *Journal of Finance*, 62(2):557–595, 2007.
- [60] Mark B Garman. Towards a semigroup pricing theory. *Journal of Finance*, 40(3):847–861, 1985.
- [61] Evan Gatev, William N Goetzmann, and K Geert Rouwenhorst. Pairs trading: Performance of a relative-value arbitrage rule. *The Review of Financial Studies*, 19(3):797–827, 2006.
- [62] Helyette Geman. The importance of the forward neutral probability in a stochastic approach of interest rates. Technical report, Working paper, ESSEC, 1989.

- [63] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with a unique set of priors. *Journal of Mathematical Economics*, 18(1), 1989.
- [64] Itay Goldstein and Alexander Guembel. Manipulation and the allocational role of prices. *Review of Economic Studies*, 75(1):133–164, 2008.
- [65] Richard C Grinold and Ronald N Kahn. Active portfolio management. 2000.
- [66] Denis Gromb and Dimitri Vayanos. Limits of arbitrage. *Annual Review of Financial Economics*, 2(1):251–275, 2010.
- [67] Jungsuk Han and Albert S Kyle. Speculative equilibrium with differences in higher-order beliefs. *Management Science*, 2017.
- [68] Lars Peter Hansen. Dynamic valuation decomposition within stochastic economies. *Econometrica*, 80(3):911–967, 2012.
- [69] Lars Peter Hansen, John C Heaton, and Nan Li. Consumption strikes back? measuring long-run risk. *Journal of Political Economy*, 116(2):260–302, 2008.
- [70] Lars Peter Hansen and José A Scheinkman. Long-term risk: An operator approach. *Econometrica*, 77(1):177–234, 2009.
- [71] Lars Peter Hansen and José A Scheinkman. Recursive utility in a markov environment with stochastic growth. *Proceedings of the National Academy of Sciences*, 109(30):11967–11972, 2012.
- [72] LarsPeter Hansen and Thomas J Sargent. Robust control and model uncertainty. *American Economic Review*, 91(2):60–66, 2001.

- [73] J Michael Harrison and David M Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–408, 1979.
- [74] David Heath, Robert Jarrow, and Andrew Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica: Journal of the Econometric Society*, pages 77–105, 1992.
- [75] Terrence Hendershott, Charles M Jones, and Albert J Menkveld. Does algorithmic trading improve liquidity? *Journal of Finance*, 66(1):1–33, 2011.
- [76] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.
- [77] Steven L Heston, Mark Loewenstein, and Gregory A Willard. Options and bubbles. *Review of Financial Studies*, 20(2):359–390, 2006.
- [78] Steve Hogan, Robert Jarrow, Melvyn Teo, and Mitch Warachka. Testing market efficiency using statistical arbitrage with applications to momentum and value strategies. *Journal of Financial Economics*, 73(3):525–565, 2004.
- [79] Craig W Holden and Avanidhar Subrahmanyam. Long-lived private information and imperfect competition. *Journal of Finance*, 47(1):247–270, 1992.
- [80] Harrison Hong and Jeremy C Stein. A unified theory of underreaction, momentum trading, and overreaction in asset markets. *Journal of Finance*, 54(6):2143–2184, 1999.

- [81] Harrison Hong and Jeremy C Stein. Differences of opinion, short-sales constraints, and market crashes. *Review of Financial Studies*, 16(2):487–525, 2003.
- [82] Jing-Zhi Huang and Zhan Shi. Determinants of bond risk premia. Technical report, Citeseer, 2011.
- [83] Friedrich Hubalek, Irene Klein, and Josef Teichmayn. A general proof of the dybvig-ingersoll-ross theorem: Long forward rates can never fall. *Mathematical Finance*, 12(4):447–451, 2002.
- [84] Gur Huberman and Werner Stanzl. Price manipulation and quasi-arbitrage. *Econometrica*, 72(4):1247–1275, 2004.
- [85] Steven Huddart, John S Hughes, and Carolyn B Levine. Public disclosure and dissimulation of insider trades. *Econometrica*, 69(3):665–681, 2001.
- [86] Farshid Jamshidian. An exact bond option formula. *Journal of Finance*, 44(1):205–209, 1989.
- [87] Robert A Jarrow. Market manipulation, bubbles, corners, and short squeezes. *Journal of Financial and Quantitative Analysis*, 27(3):311–336, 1992.
- [88] Robert A Jarrow. Asset price bubbles. *Annual Review of Financial Economics*, 7:201–218, 2015.
- [89] Narasimhan Jegadeesh and Sheridan Titman. Returns to buying winners and selling losers: Implications for stock market efficiency. *Journal of Finance*, 48(1):65–91, 1993.

- [90] Guolin Jiang, Paul G Mahoney, and Jianping Mei. Market manipulation: A comprehensive study of stock pools. *Journal of Financial Economics*, 77(1):147–170, 2005.
- [91] Yan Jin and Paul Glasserman. Equilibrium positive interest rates: a unified view. *Review of Financial Studies*, 14(1):187–214, 2001.
- [92] Scott Joslin, Marcel Pribsch, and Kenneth J Singleton. Risk premiums in dynamic term structure models with unspanned macro risks. *Journal of Finance*, 69(3):1197–1233, 2014.
- [93] Daniel Kahneman, Jack L Knetsch, and Richard H Thaler. Anomalies: The endowment effect, loss aversion, and status quo bias. *Journal of Economic perspectives*, 5(1):193–206, 1991.
- [94] Hossein B Kazemi. An intemporal model of asset prices in a markov economy with a limiting stationary distribution. *Review of Financial Studies*, 5(1):85–104, 1992.
- [95] Asim Ijaz Khwaja and Atif Mian. Unchecked intermediaries: Price manipulation in an emerging stock market. *Journal of Financial Economics*, 78(1):203–241, 2005.
- [96] Andrei Kirilenko, Albert S Kyle, Mehrdad Samadi, and Tugkan Tuzun. The flash crash: High-frequency trading in an electronic market. *Journal of Finance*, 72(3):967–998, 2017.
- [97] Peter Kondor. Risk in dynamic arbitrage: the price effects of convergence trading. *Journal of Finance*, 64(2):631–655, 2009.

- [98] Robert A Korajczyk and Dermot Murphy. High frequency market making to large institutional trades. *Review of Financial Studies, Forthcoming*, 2018.
- [99] Serhiy Kozak, Stefan Nagel, and Shrihari Santosh. Shrinking the cross section. Technical report, National Bureau of Economic Research, 2017.
- [100] Praveen Kumar and Duane J Seppi. Futures manipulation with “cash settlement”. *Journal of Finance*, 47(4):1485–1502, 1992.
- [101] Albert S Kyle. Continuous auctions and insider trading. *Econometrica: Journal of the Econometric Society*, pages 1315–1335, 1985.
- [102] Albert S Kyle and Anna A Obizhaeva. Market microstructure invariance: Empirical hypotheses. *Econometrica*, 84(4):1345–1404, 2016.
- [103] Albert S Kyle and S Viswanathan. How to define illegal price manipulation. *American Economic Review*, 98(2):274–79, 2008.
- [104] Albert S Kyle and Wei Xiong. Contagion as a wealth effect. *Journal of Finance*, 56(4):1401–1440, 2001.
- [105] Charles MC Lee and Bhaskaran Swaminathan. Price momentum and trading volume. *Journal of Finance*, 55(5):2017–2069, 2000.
- [106] Bruce N Lehmann. Fads, martingales, and market efficiency. *Quarterly Journal of Economics*, 105(1):1–28, 1990.
- [107] Michael Lewis. *Flash Boys: a Wall Street Revolt*. WW Norton & Company, 2014.

- [108] Thomas F Loeb. Trading cost: the critical link between investment information and results. *Financial Analysts Journal*, pages 39–44, 1983.
- [109] Francis A Longstaff and Eduardo S Schwartz. Interest rate volatility and the term structure: A two-factor general equilibrium model. *Journal of Finance*, 47(4):1259–1282, 1992.
- [110] Ian Martin and Steve Ross. The long bond. *London School of Economics and MIT (documento no publicado)*, 2013.
- [111] Robert C Merton. An asymptotic theory of growth under uncertainty. *Review of Economic Studies*, 42(3):375–393, 1975.
- [112] Marek Musiela and Marek Rutkowski. Continuous-time term structure models: Forward measure approach. *Finance and Stochastics*, 1(4):261–291, 1997.
- [113] Likuan Qin and Vadim Linetsky. Positive eigenfunctions of markovian pricing operators: Hansen-scheinkman factorization, ross recovery, and long-term pricing. *Operations Research*, 64(1):99–117, 2016.
- [114] Likuan Qin and Vadim Linetsky. Long-term risk: A martingale approach. *Econometrica*, 85(1):299–312, 2017.
- [115] Likuan Qin, Vadim Linetsky, and Yutian Nie. Long forward probabilities, recovery and the term structure of bond risk premiums. *Available at SSRN 2721366*, 2016.
- [116] Scott F Richard. An arbitrage model of the term structure of interest rates. *Journal of Financial Economics*, 6(1):33–57, 1978.

- [117] Leonard CG Rogers. The potential approach to the term structure of interest rates and foreign exchange rates. *Mathematical Finance*, 7(2):157–176, 1997.
- [118] Stephen A Ross. A simple approach to the valuation of risky streams. *Journal of Business*, pages 453–475, 1978.
- [119] Steve A. Ross. The recovery theorem. *Journal of Finance*, 70(2):615–648, 2015.
- [120] William Samuelson and Richard Zeckhauser. Status quo bias in decision making. *Journal of risk and uncertainty*, 1(1):7–59, 1988.
- [121] Jose A Scheinkman and Wei Xiong. Overconfidence and speculative bubbles. *Journal of Political Economy*, 111(6):1183–1220, 2003.
- [122] Andrei Shleifer and Robert W Vishny. The limits of arbitrage. *Journal of Finance*, 52(1):35–55, 1997.
- [123] A Christian Silva, Richard E Prange, and Victor M Yakovenko. Exponential distribution of financial returns at mesoscopic time lags: a new stylized fact. *Physica A*, 344(1-2):227–235, 2004.
- [124] Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 267–288, 1996.
- [125] Jos van Bommel. Rumors. *Journal of Finance*, 58(4):1499–1520, 2003.
- [126] Vincent van Kervel and Albert J Menkveld. High-frequency trading around large institutional orders. *Journal of Finance, Forthcoming*, 2017.

- [127] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, 1977.
- [128] Dimitri Vayanos. Strategic trading and welfare in a dynamic market. *Review of Economic Studies*, 66(2):219–254, 1999.
- [129] Dimitri Vayanos. Strategic trading in a dynamic noisy market. *Journal of Finance*, 56(1):131–171, 2001.
- [130] Johan Walden. Recovery with unbounded diffusion processes. *Review of Finance*, 21(4):1403–1444, 2017.
- [131] Wei Xiong. Convergence trading with wealth effects: an amplification mechanism in financial markets. *Journal of Financial Economics*, 62(2):247–292, 2001.
- [132] Liyan Yang and Haoxiang Zhu. Back-running: Seeking and hiding fundamental information in order flows. *Working paper, SSRN*, 2017.