ABSTRACT

Title of dissertation:	ESSAYS ON AUCTION DESIGN
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This dissertation studies the design of auction markets where bidders are uncertain of their own values at the time of bidding. A bidder's value may depend on other bidders' private information, on total quantity of items allocated in the auction, or on the auctioneer's private information.

Chapter 1 provides a brief introduction to auction theory and summarizes the main contribution of each following chapter. Chapter 2 of this dissertation extends the theoretical study of position auctions to an interdependent values model in which each bidder's value depends on its opponents' information as well as its own information. I characterize the equilibria of three standard position auctions under this information structure, including the Generalized Second Price (GSP) auctions, Vickrey-Clarke-Groves (VCG) auctions, and the Generalized English Auctions (GEA). I first show that both GSP and VCG auctions are neither efficient nor optimal under interdependent values. Then I propose a modification of these two auctions by allowing bidders to condition their bids on positions to implement efficiency. I show that the modified auctions proposed in this chapter are not only

efficient, but also maximize the search engine's revenue.

While the uncertainty of each bidder about its own value comes from the presence of common component in bidders ex-post values in an interdependent values model, bidders can be uncertain about their values when their values depend on the entire allocation of the auction and when their values depend on the auctioneer's private information. Chapter 3 of this dissertation studies the design of efficient auctions and optimal auctions in a license auction market where bidders care about the total quantity of items allocated in the auction. I show that the standard uniform-price auction and the ascending clock auction are inefficient when the total supply needs to be endogenously determined within the auction. Then I construct a multi-dimensional uniform-price auction and a Walrasian clock auction that can implement efficiency in a dominant strategy equilibrium under surplus-maximizing reserve prices and achieve optimal revenue under revenue-maximizing reserve prices.

Chapter 4 of this dissertation analyzes an auctioneer's optimal information provision strategy in a procurement auction in which the auctioneer has private preference over bidders' non-price characteristics and bidders invest in cost-reducing investments before entering the auction. I show that providing more information about the auctioneer's valuation over bidders' non-price characteristics encourages those favored bidders to invest more and expand the distribution of values in the auction. Concealment is the optimal information provision policy when there are two suppliers.

ESSAYS ON AUCTION DESIGN

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2018

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Acknowledgments

I owe my gratitude to all the people who made this dissertation possible. It is my great pleasure to thank them here.

First, I would like to thank my primary advisor, Professor Lawrence Ausubel for his continuous support, guidance and encouragement over the past few years. It has been my honor to learn from and be guided by a leading scholar in auction theory. I am indebted to him for inspiring my interests in auction theory, giving many thoughtful comments on my work, and providing me a lot of encouragement and support during my years in graduate school. It would be impossible for me to have finished this dissertation without his support.

I would also like to thank Professor Daniel Vincent for his guidance and many helpful comments on my thesis. I am indebted to him for many hours of discussion about formulating research ideas, constructing models, and fining details of analyses and proofs. His expertise in mechanism design, microeconomic theory, and industrial organization theory has inspired my interests in these fields and expanded my research ideas.

I want to thank Professor Emel Filiz-Ozbay for her continuous support and helpful suggestions. My thesis has been greatly benefited from the many hours of discussion with her. Her guidance has deepened my understanding and broaden my interests in microeconomic theory.

I want to thank Professor Ginger Zhe Jin and Professor Mohammad Hajiaghayi for agreeing to serve on my thesis committee and for sparing their invaluable time

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List of Abbreviations

BNE   Bayesian Nash Equilibrium

CTR   Click Through Rate

GEA   Generalized English Auction
GSP   Generalized Second Price

IC   Incentive Compatible
IR   Individually Rational

NE   Nash Equilibrium

PBE   Perfect Bayesian Equilibrium

SPE   Subgame Perfect Equilibrium

VCG   Vickrey-Clarke-Groves

Chapter 1:   Introduction

1.1   Introduction

Auctions have been used since antiquity for selling a variety of objects. The earliest record of auction appears as early as 500 B.C. (Krishna 2002 [1]). Some most commonly used auctions include auctions for selling treasury bills, mineral rights, art and antiques, agricultural produce and livestock, used cars, luxury wines, etc. In addition to selling items, auctions can also be used for purchasing a variety of items. For example, government usually use procurement auctions to allocate construction contracts, and firms often use auctions for buying inputs, subcontracting tasks, and acquiring another firm in takeover battles. In most recent decades, auctions of rights to use the electromagnetic spectrum for telecommunication are used widely in many countries.

Since the auctioneer is uncertain about bidders' valuations of the item being sold, a common aspect of auctions is that they elicit information, in the form of bids, from potential buyers regarding their willingness to pay or from potential sellers regarding their willingness to sell, and the outcome - that is, who wins what and pays how much - is determined solely on the basis of the received information (Krishna 2002 [1]). Therefore, auctions provide a simple and well-defined economic

environment of incomplete information game and a valuable testing-ground for economic theory. In the benchmark model of independent private values, each bidder knows the value of object to itself at the time of bidding. The auctioneer does not have private information, and other bidder's information does not affect a particular bidder's value.

The design of auction depends on specifics of the auction market, including nature and quantity of items, information structure among bidders, existence of budget constraints, fairness considerations, etc. Depending on the nature of items being sold, auctions can be categorized into single-unit auctions, multiple identical unit auctions, and multiple heterogeneous unit auctions. The most commonly used single-unit auction formats include first-price sealed-bid auctions, second-price sealed-bid auctions, open ascending price (English) auctions, and open descending price (Dutch) auctions. Under the context of multiple identical unit auctions, some commonly used sealed-bid auction formats include discriminatory (pay-as-bid) auctions, uniform-price auctions, Vickrey auctions, and dynamic aution formats include Dutch auctions, English auctions, and Ausubel auctions. There is a growing recent literature on auctions for multiple heterogenous items, and some auction formats that have been designed by auction theorists include simultaneous ascending auctions, ascending proxy auctions, and combinatorial auctions. This dissertation studies auction design in three different types of auction markets: multiple heterogenous items, multiple identical items, and single item auctions.

From the perspective of an auction designer, some of the most important criteria in auction design are listed below:

1. Efficiency: The objects end up in the hands of the bidders who value them most, i.e., the total surplus generated in the auction is maximized.

2. Revenue: The auction raises high expected revenue for the seller.

3. Simplicity: The allocation and payment rules are straightforward and easy to understand for the bidders.

4. Transparency: The allocation and payment rules are transparent to all participants.

5. Speed of the auction: The auction can conclude in a short period of time.

6. Facilitate bidder participation: The auction can attract sufficient number of bidders to participate.

7. Collusion-proof: The auction is not highly vulnerable to collusion.

This dissertation explores auction design in three different markets using efficiency and revenue maximization as the two main objectives. Each chapter examines an important departure from the standard pure private value model and explores how to use auction design to implement the goals of efficiency and revenue-maximization under this departure. The departure from pure private value model comes from the fact that each bidder's value depends on other bidders' private information in Chapter 2, on total supply in the auction in Chapter 3, and on the auctioneer's private information in Chapter 4. In the following subsection, I will provide a brief overview of motivation and a summary of main results and contributions of each chapter.

1.2   Summary of Contributions

Existing theoretical literature models position auctions either under complete information or with incomplete information assuming pure private values. Since advertisers bidding for the same keyword in sponsored search auctions are often oligopoly competitors operating in the same industry, each bidders value from receiving a click of its online advertisement can depend on other bidders private information as well as its own information, which is better described by the interdependent values model introduced by Milgrom and Weber (1982) [2] for single-unit auctions.

Chapter 2 of this dissertation extends the theoretical study of position auctions to an interdependent values model in which each bidder's value depends on its opponents' information as well as its own information. Position auctions are used by major search engines to allocate multiple advertising positions on search result pages. In this chapter, I examine efficiency and revenues of three position auction formats: Generalized Second-Price (GSP) auctions, VCG-like auctions and Generalized English Auctions (GEA). I find that both the GSP auction and the VCG-like auction with one-dimensional bidding language can be inefficient under interdependent values, which contrasts previous literature that favors the GSP auction for its simplicity. I next show this inefficiency problem can be fully resolved by adopting a multi-dimensional bidding language that allows bidders to bid differently across positions. Moreover, the dynamic GEA that implicitly adopts a multi-dimensional bidding language always implements efficiency in an ex-post equilibrium. Then I

provide a revenue ranking of the three efficient position auctions and character-

ize the optimal position auction under interdependent values. I find that under

independent signals and a set of regularity conditions, the three efficient position

auctions also implement the optimal revenue subject to no reserve price. The main

results of this chapter imply that there is a trade-off between simplicity versus effi-

ciency and revenue in auction design: using a simple bidding language can come at

a loss of efficiency and revenue. This trade-off depends critically on the information

structure.

In auctions for selling operating licenses in some downstream market, the

quantity of licenses allocated in the auction determines the structure of downstream

market and therefore enters each bidder's value in the auction. How to design an

auction to maximize total surplus of auction participants is an interesting problem,

as there exists a trade-off of selling more licenses and preserving the values of winning

a license for the winners.

Chapter 3 of this dissertation studies the design of efficient auctions and opti-

mal auctions in a license auction market where bidders care about the total quan-

tity of items allocated in the auction. I first characterize the VCG mechanism in

this environment and show that a sequence of reserve prices that specify minimum

acceptable bid for every additional unit to be allocated are needed to determine

supply endogenously in any efficient auction. Then I characterize the equilibria of

the uniform-price auction and the ascending clock auction after introducing such

reserve prices and show that both auctions are inefficient under any reserve prices.

I next construct a multi-dimensional uniform-price auction that allows bidders to

condition their bids on total supply to implement efficiency. On the other hand, I show that a Walrasian clock auction can implement the efficient outcome in a dominant strategy equilibrium through a tatonnement process. I next characterize the optimal auction under the presence of quantity externalities and show that the optimal reserve price is strictly higher than the efficient reserve price for every additional unit. Moreover, both the efficient and the optimal reserve prices under the presence of quantity externalities are higher than their counterparts in markets without quantity externalities.

In the standard model of procurement auctions, the suppliers provide identical products and are exogenously differentiated in costs. In practice, suppliers are often horizontally differentiated in non-price characteristics and can engage in pre-auction cost-reducing investments. When the auctioneer has private valuation over bidders' non-price characteristics, whether to disclose this valuation will affect bidders' investment incentives and endogenously determine the profile of bidders' values in the auction.

Chapter 4 of this dissertation analyzes an auctioneer's optimal information provision strategy in a procurement auction in which bidders invest in cost-reducing investments before entering the auction. In this chapter, I analyze the equilibrium investment strategies of bidders under three different information provision schemes: public disclosure, private disclosure, and concealment of preferences over bidders. I find that pre-auction investments are strategic substitutes among bidders, and providing more information about the auctioneer's preference encourages those more favored bidders to invest more, which results in a more dispersed distribution of

costs among bidders in the auction. Then I compare the expected revenues in a second-score auction under these three information provision schemes. I find that concealment is the optimal information scheme when there are only two bidders, while the revenue ranking can be reversed when the number of bidders is sufficiently large.

Chapter 2:   Position Auctions with Interdependent Values

2.1   Introduction

Position auctions are used by many search engines to allocate a list of advertising positions on search result pages. When an Internet user enters a keyword or phrase on a search engine, the list of advertisements generated by that search is the result of a position auction. Because of consumers' sequential search habits[1], advertising links placed on the top of web page receive more clicks than those placed on the bottom of web page (Brooks 2004 [5]), representing a typical set of vertically differentiated items. Each advertising link's click probability can be measured by click-through-rate (CTR), which is given by the average number of clicks the link receives per unit time.

There are three different designs of position auction that have been analyzed in the literature, including the Generalized Second Price (GSP) auction, the Vickrey-Clarke-Groves (VCG) auction, and the Generalized English Auction (Edelman et

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[1]Consumers tend to search from top to bottom when reading a list and may end search at any time, so the top links are more likely to be clicked than the bottom links. This search behavior can be viewed as a rule of thumb, or as a rational behavior given positive search cost and correct expectation about advertisers' relevance ((Athey and Ellison 2011 [3]); (Chen and He 2011 [4])).

al. 2007 [6]). Variants of the Generalized Second Price auction have been extensively used by major search engines including Google and Yahoo![2]. In the standard model of GSP auction with pay-per-click payment rule described in Edelman et al. (2007) [6], advertisers submit one-dimensional per-click bids that can be applied to any position. The positions are allocated according to the ranking of bids, and each bidder who wins a position pays the bid of the bidder who is placed one position below for each click. The total revenue generated by an advertising position depends on the advertiser's per-click payment and the click-through-rate (CTR) of the position.

The GSP auction[3] is favored by many previous studies for its efficiency and revenue properties under complete information as well as its simplicity in bidding language and payment rule: one-dimensional bids are used to determine the allocation of multiple positions, and payment for each position depends only on the highest losing bid for that specific position. Besides its simplicity, the GSP auction always implements the efficient allocation and yields weakly higher revenue than

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[2]The Generalized Second Price auction has several variations in its form. One important variation is to adopt a vector of "quality scores" computed based on click-through-rate history to adjust bids and rank advertisers in the order of adjusted bids instead of raw bids. Another variation is to adopt a pay-per-impression scheme instead of a pay-per-click scheme. Under the pay-per-click scheme, an advertiser will be charged every time a user clicks on its advertisement. Under the pay-per-impression scheme, an advertiser will be charged every time a user sees the search result page that contains its advertisement regardless of whether the user clicks on the advertisement or not. Google currently uses the pay-per-click GSP auction with quality scores.

[3]Following Edelman et al. (2007) [6], the GSP auction in this paper refers to the auction with pay-per-click GSP payment rule and leaves aside the quality scores.

the VCG auction under complete information (Edelman et al. 2007 [6]; Milgrom 2010 [7]). These desirable properties of the GSP auction depend critically on the complete information assumption. The justification of complete information in the literature comes from the claim that advertisers can learn about their own values as well as each other's values from information revealed in previous rounds of auction. However, this claim implicitly assumes that advertisers' values do not evolve over time intervals between bidding, so the information revealed in previous rounds fully reveals advertisers' values for the current round.

In practice, considerable uncertainty exists in the environment of sponsored search auctions. For example, consider the keyword "iphone." Each advertiser's value from receiving a click of the online advertisement depends on how likely consumers are going to purchase a new iphone upon click, which can be affected significantly by product upgrading and new releases in the iphone market. Consider the time when Apple releases a new version of iphone, then each advertiser's value changes continuously over time after the first day of release, and it is not practical to precisely predict consumer demand or advertisers' values in advance. Consider another keyword for example, "hotel in DC," then each advertiser's value per click depends on how likely consumers are going to book a hotel after clicking on its advertisement, which can be affected by a variety of factors including weather, day of the week, time of the year, special events in DC, etc. Therefore, advertisers' values evolve continuously for many keywords that are related to markets with frequent demand shocks. The evolution in advertisers' values as result of shocks is also pointed out in Fershtman and Pavan (2016) [8] and Abhishek and Hosanagar

(2012) [9]. On the other hand, it is not practical for advertisers to update bids in a continuous manner given the fact that each advertiser is interested in a large set of keywords. The stochasticity in consumer demand and the existence of time interval between bid updating imply that information revealed from previous rounds in the auction does not provide complete information about an advertiser's own value or its opponents values at the current round of bidding.

Another important fact not captured by prior literature is that advertisers bidding for a position under the same keyword are often oligopoly competitors operating in the same industry. Compared to traditional advertising, a main advantage of sponsored advertising is that it allows advertisers to effectively target consumers. This advantage naturally comes with the fact that advertisers under the same keyword are selling identical or imperfectly substitutable products or services in the market related to the search keyword. Since each advertiser's value per click evolves continuously under demand shocks, it is reasonable to assume that there is some common component in bidders' ex-post values that is driven by demand shocks in the same market. For example, when Apple releases a new version of iphone, consumers are more likely to buy a new iphone after click on an advertisement, and all advertisers are subject to the same demand shock. While consumer demand cannot be precisely predicted, each advertiser can still have some imprecise estimation of its value per click. Suppose a given advertiser receives a private signal that contains information about how likely consumers are going to purchase a new iphone after the release. Then the private signals of other advertisers would be informative about the first advertiser's ex-post value per click, given the fact that consumer demand

drives a common component in all advertisers' ex-post values. Since the advertisers' private signals contain information about consumer demand in the same market, it is also reasonable to assume the signals are affiliated in distribution. Therefore, the information structure in position auctions is better described by the interdependent values model introduced by Milgrom and Weber (1982) [2] for single-unit auctions, in which one bidder's value can depend on other bidders' private information, and bidders' private information are affiliated. However, the performance of the GSP auction, the VCG auction and the Generalized English Auction is not well-understood when bidders have interdependent values. This paper fills the gap in the literature and extends the study of position auctions into a broader class of information structure.

In this paper, I model a single-round position auction under the symmetric interdependent values setup in Milgrom and Weber (1982) [2]. Since each bidder's ex-post value depends on its opponents' private information, bidders can be uncertain about their ex-post values at the time of bidding, and a generalized version of the "winner's curse" in Milgrom and Weber (1982) [2] is present: the expected value per click conditional on winning a superior position is lower than that conditional on winning an inferior position. Winning a top position conveys some bad news, as it implies overestimation of ex-post value from receiving a click. The main analysis of this paper explores how the incomplete information and the presence of the generalized "winner's curse" under interdependent values affect efficiency and revenues of GSP auctions, VCG-like auctions[4], and Generalized English Auctions.

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[4]Although the VCG mechanism is not defined under interdependent values, I define a VCG-

Under this symmetric interdependent values model, I first show that both the GSP auction and the VCG-like auction can be inefficient, which contrasts with previous literature that favors the GSP auction for its simplicity. Then I propose a modification of the GSP auction and the VCG-like auction by adopting a multi-dimensional bidding language that allows each bidder to bid differently across positions (i.e., using a multi-dimensional bidding language) to improve efficiency. I call these two modified auctions K-dimensional GSP auctions and K-dimensional VCG auctions, respectively. I characterize the unique symmetric Bayesian Nash equilibrium in these two modified auctions and show that efficiency can be fully implemented in both auctions after adopting this multi-dimensional bidding language. On the other hand, the Generalized English Auction that implicitly adopts a multi-dimensional bidding language always implements the efficient allocation in an ex-post equilibrium. Moreover, the K-dimensional GSP auction and the K-dimensional VCG auction are always revenue equivalent, while the dynamic Generalized English Auction yields higher revenue under affiliated signals. In the special case of independent signals, all three efficient auctions are revenue equivalent. I also characterize the optimal position auction that generates the highest expected revenue subject to no reserve price as a direct revelation mechanism and show that under certain regularity conditions, this optimal auction is equivalent to the Generalized-VCG mechanism that assigns all positions efficiently in an ex-post equilibrium. When bidders have in-

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like auction called the *one-dimensional VCG auction* that adopts a VCG-like payment rule under interdependent values. This one-dimensional VCG auction is analogous to the second-price auction in Milgrom and Weber (1982) [2]'s study of single-unit auctions under interdependent values.

dependent signals, the optimal revenue subject to no reserve price can be practically implemented by the K-dimensional GSP auction, the K-dimensional VCG auction, and the Generalized English Auction under mild regularity conditions. Therefore, when bidders have interdependent values, modifying the bidding language from one-dimensional to multi-dimensional in position auctions not only improves efficiency, but also improves revenue under certain conditions.

The inefficiency of the GSP auction and the VCG-like auction comes from the fact that both auctions use a simple one-dimensional bidding language that restricts bidders to submit the same bid for all positions, while the expected payoff of winning a superior position can be lower than that of an inferior position in both auctions under interdependent values. As long as there are more than one positions, the bid-shading incentive is stronger for bidders with higher signals, since they are more likely to win the superior position, while bidders who receive lower signals are more concerned with winning any position instead of winning a top position. The differentiated bid-shading incentives across bidders drive the inefficiency in both GSP auctions and VCG-like auctions. By allowing bidders to submit multi-dimensional bids that express willingness to pay per click separately for each position, bidders can easily incorporate the difference in expected payoffs from winning different positions into their bids. Therefore, the differentiated bid-shading incentives across bidders are replaced by each bidder's differentiated bid-shading incentives across positions. This explains the efficiency of K-dimensional GSP and VCG auctions. Similarly, in the dynamic Generalized English Auction, bidders not only update beliefs about their expected values per click from the history of drop-out prices, but also update

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beliefs about which position they are going to win by dropping out at the current clock price during the dynamic process. The Generalized English Auction implicitly adopts a multi-dimensional bidding language, which is the main force that drives its efficiency.

Compared to the two static auctions, the dynamic Generalized English Auction not only has an advantage in terms of naturally adopting a multi-dimensional bidding language, but also generates higher expected revenues under affiliated signals. This revenue dominance result comes from the fact that dynamic auctions outperform static auctions in revenue by eliciting more information about bidders' signals through the drop-out process. On the other hand, the revenue equivalence between the K-dimensional GSP and the K-dimensional VCG auctions comes from the fact that both auctions use variations of a "second-price" payment rule in which a given bidder's bid only affects its allocation but not its payment. Bidders are able to incorporate the difference in payment rules into their bidding strategies, which drives the revenue equivalence result between these two static auctions under the general assumption of affiliated signals. The revenue equivalence of all three auctions under independent signals is consistent with the well-known revenue equivalence theorem. Under independent signals, the expected revenue of the three efficient position auctions is also equivalent to the optimal revenue implementable in any Bayesian incentive compatible and individually rational mechanism subject to no reserve price under certain regularity conditions. This result comes from the fact that when the rank ordering of bidders' values is aligned with the rank ordering of bidders' marginal revenues given any realization of signals, the auctioneer's

objective of maximizing total surplus is aligned with the objective of maximizing revenue.

The main contribution of this paper is the follows. First, I extend the study of position auctions into interdependent values and prove the inefficiency of standard auction formats that include GSP auctions and VCG-like auctions. Second, I identify the source of this inefficiency and propose a modification of standard position auction formats that implements full efficiency. Third, I provide a comparison across three different position auction formats in both efficiency and revenue and provide a discussion on the optimal auction under this broad class of information structure. This paper not only provides guidance on the design of sponsored search auctions used by search engines, but also provides implications on allocating sponsored advertisement space in a wide range of two-sided platforms, such as Facebook, Amazon, and Yelp. The main results of this paper imply that there is a trade-off between simplicity versus efficiency and revenue in auction design: simplicity can come at a loss of efficiency and revenue. This trade-off depends critically on the information structure.

2.2   Related Literature

This paper is related to the earliest position auction literature including Edelman et al. (2007) [6] and Varian (2007) [10]. Edelman et al. (2007) [6] characterize the set of locally-envy free equilibria of the GSP auction under complete information, and show that the GSP auction has a locally-envy free equilibrium that yields

the same payoff outcome as the dominant strategy equilibrium of the VCG auction. Moreover, this equilibrium gives the bidder-optimal payoff among all locally-envy free equilibria. In a complementary article, Varian (2007) [10] characterizes the entire set of Nash equilibria in the GSP auction under complete information. Milgrom (2010) [7] shows that the GSP auction can be viewed as a simplified mechanism that restricts each bidder to submit the same bid for all positions. This simplification in bidding language eliminates the lowest revenue equilibrium and leaves only higher revenue equilibria under complete information. Dutting et al. (2011) [11] points out that Milgrom (2010) [7]'s result depends critically on the complete information assumption. This paper provides theoretical support for Dutting et al. (2011) [11]'s discussion of the trade-off between simplicity and expressiveness in mechanism design by showing that the GSP auction with one-dimensional bidding language can be suboptimal in both efficiency and revenue under interdependent values, in sharp contrast to the results in Edelman et al. (2007) [6], Varian (2007) [10] and Milgrom (2010) [7] that favor the GSP auction under complete information. Moreover, the cost of conciseness in the design of GSP auction is also pointed out in the computer science literature. Abrams et al. (2007) [12] show that an equilibrium can fail to exist in the GSP auction with pay-per-click payment scheme when each bidder has a vector of different values for obtaining different slots. Benisch et al. (2008) [13] show that the GSP auction can be arbitrarily inefficient under some distributions of the advertisers' preferences when advertisers have private information and describe a technique that computes an upper bound on the expected efficiency of the GSP auction for a known distribution of advertisers' preferences. This paper comple-

ments these computer science studies by providing some insights on the trade-off between simplicity and efficiency from an economic perspective.

In an incomplete information setting, Edelman et al. (2007) [6] model an ascending auction called the Generalized English Auction (GEA) that implements the same payoff outcome as the dominant strategy equilibrium of the VCG auction under independent private values. However, the GEA is the dynamic format of the VCG auction, rather than the dynamic format of the GSP auction. Little was known about equilibria of the GSP auction under incomplete information until Gomes and Sweeney (2014) [14] first characterized the Bayesian Nash Equilibrium of the GSP auction in an independent private values model and showed this unique equilibrium can be inefficient under some click-through rate profiles. This paper extends Gomes and Sweeney (2014) [14]'s study by introducing interdependent values into the model, identifying the source of inefficiency in the GSP auction, as well as comparing the performance of the GSP auction to other auction formats. An implication of Gomes and Sweeney (2014) [14]'s inefficiency result is that the VCG auction performs better than the GSP auction under incomplete information when bidders have independent private values. Ashlagi (2007) [15] points out that the VCG auction is the unique truth-revealing position auction under an anonymous allocation rule with symmetric independent private values. Under complete information, Varian and Harris (2014) [16] show that the VCG auction performs better than the GSP auction under "broad match" of keywords and under unknown click-through rates. This paper extends the comparison between GSP auctions and VCG auctions to interdependent values.

This paper is closely related to the literature on auctions and mechanism design under interdependent values. Milgrom and Weber (1982) [2] characterize the equilibria of second-price auctions, first-price auctions and English auctions and compare the expected revenues of these auctions under symmetric interdependent values. A number of other articles examine the existence of efficient mechanisms under interdependent values without symmetry assumption (Jehiel and Moldovanu, 2001 [17]; Dasgupta and Maskin, 2000 [18]; Perry and Reny, 2002 [19]; Ausubel, 1999 [20]; Ausubel and Cramton, 2004 [21]). This paper extends the literature on auction design under interdependent values into multi-unit position auctions.

This paper complements the recent position auctions literature[5] that introduces some realistic assumptions into Edelman et al. (2007) [6]'s model. Some studies endogenize advertisers' values by incorporating consumer search into the model and show that firms are ranked in the order of relevance and consumers search sequentially in equilibrium (Athey and Ellison, 2011 [3]; Chen and He, 2011 [4]; Kominers, 2009 [23]). Several other studies introduce allocative externalities among bidders by allowing click-through rate of each position to depend on the allocation of advertisers[6] (Deng and Yu, 2009 [32]; Farboodi and Jafaian, 2013 [33]; Hummel and McAfee, 2014 [34]; Izmalkov et al., 2016 [35]; Lu and Riis, 2016 [36]). There are also

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[5]Most recent advances in this literature are summarized in Qin et al. (2015) [22].

[6]There is a similar line of research in the computer science literature (Aggarwal et al., 2008 [24]; Constantin et al., 2011 [25]; Ghosh and Mahdian, 2008 [26]; Kempe and Mahdian, 2008 [27]). Other computer science literature on similar topics such as algorithm design in adword auctions, forward-looking bidders and prophet inequality include Mehta et al. (2005) [28], Bu et al. (2007) [29], and Alaei et al. (2012 [30]; 2013 [31]).

studies quantify the efficiency loss that may arise in the GSP auction under different modeling assumptions, including correlated private values, allocative externalities, uncertain click-through rate profiles, etc. (Lucier and Leme, 2011 [37]; Roughgarden and Tardos, 2015 [38]; Caragiannis et al., 2015 [39]). This paper differs from the aforementioned studies by keeping Edelman et al. (2007) [6]'s assumption of exogenous click-through rates while introducing informational interdependency in bidders' values, which to my knowledge has not been done by previous studies.

Finally, this study is related to the strand of literature on mechanism design. Myerson (1981) [40] characterizes the optimal mechanism for single-unit auctions with independent private values. Ausubel and Cramton (1999) [41] find that in auction markets with perfect resale, it is optimal to allocate items efficiently. Edelman and Schwarz (2010) [42] generalize Myerson (1981) [40]'s optimal mechanism design to position auctions with independent private values and show that this optimal revenue can be implemented by a Generalized English Auction with an optimal reserve price. Roughgarden and Talgam-Cohen (2013) [43] and [44]Li (2017) extend the characterization of optimal single-unit auction to interdependent values. Ulku (2013) [45] characterize the optimal mechanism for allocating a set of heterogamous items under interdependent values. The last part of this paper provides a corollary of Ulku (2013) [45] under the special environment of position auctions.

2.3 Model

A search engine wishes to sell K positions to $N > K$ bidders[7], each with single-unit demand for an advertising position on the search result page of the same keyword. Bidders are indexed by $i \in \{1, 2, \cdots, N\}$. Positions are indexed by $k \in \{1, 2, \cdots, K\}$ according to their ranks on the web page and are vertically differentiated in their commonly known qualities measured by click-through-rates (CTR): $(\alpha_1, \alpha_2, \cdots, \alpha_K)$[8], in which $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_K$. Each bidder receives a private signal $x_i \in [0, \bar{x}]$ that affects her value from getting a click of her advertisement. The signals are distributed over $[0, \bar{x}]^N$ according to a commonly-known joint distribution function $F(x_1, x_2, \cdots, x_N)$, with density $f(x_1, x_2, \cdots, x_N)$. The value per click[9] of each bidder $v_i(.,.)$ depends on her private signal x_i as well as her opponents' signals $x_{-i} \in [0, \bar{x}]^{N-1}$.

For every bidder i, $v_i(.,.)$ satisfies the following assumptions[10]:

A1 (*Value Symmetry*): For all bidder i, there is a function $v_i : [0, \bar{x}]^N \to \mathbb{R}$ such that each bidder i's ex-post value per click is given by $v_i(x_i, x_{-i})$, given any signal

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[7]In this paper, I use masculine pronoun for the auctioneer (search engine) and feminine pronouns for the bidders (advertisers).

[8]Following Edelman et al. (2007) [6], the CTR of position k is measured by the expected number of clicks per period received by the advertiser whose advertisement is placed on position k. The CTR of each position does not depend on the identity of bidder placed on that position or any other position.

[9]Following Edelman et al. (2007) [6], I assume each bidder's value from getting a click does not depend on the position of her advertisement.

[10]Assumptions A1-A5 follows from Milgrom and Weber (1982) [2].

profile $(x_1, x_2, \cdots, x_N)$. The value function $v_i(x_i, x_{-i})$ is symmetric across bidders. Moreover, the function $v_i(x_i, x_{-i})$ is symmetric in its last $N-1$ arguments, which implies that each bidder's value $v_i(x_i, x_{-i})$ is preserved under any permutation of opponents' signals x_{-i}.

A2 (*Value Monotonicity*): For all i, $v_i(x_i, x_{-i})$ is nonnegative, continuous and strictly increasing in x_i, nondecreasing in every component of x_{-i}:

$$\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} > 0, \quad \frac{\partial v_i(x_i, x_{-i})}{\partial x_j} \geq 0, \quad \forall j \neq i \tag{2.1}$$

Bidders have non-trivially interdependent values if $\frac{\partial v_i(x_i, x_{-i})}{\partial x_j} \neq 0$ for $j \neq i$.

A3 (*Single-crossing Condition*): For all i, for all $j \neq i$, for all signals $(x_1, x_2, \cdots, x_N)$,

$$\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \geq \frac{\partial v_j(x_j, x_{-j})}{\partial x_i} \tag{2.2}$$

Under assumptions **A1**-**A3**, the ranking of signals is aligned with the ranking of values. The bidder who receives the k-th highest signal also has the k-th highest ex-post value.

I assume the joint density function $f(x_1, x_2, \cdots, x_N)$ satisfies the following assumptions:

A4 (*Signal Symmetry*): $f(x_1, x_2, \cdots, x_N)$ is a symmetric function of its arguments.

A5 (*Signal Affiliation*): The variables $x_1, x_2, \cdots, x_N$ are affiliated. For all $x, x' \in [0, \bar{x}]^N$,

$$f(x \vee x')f(x \wedge x') \geq f(x)f(x') \tag{2.3}$$

This model introduces the information structure of interdependent values into

position auctions. I restrict attention to symmetric equilibria in this paper[11]. Given symmetry of the model, it suffices to study the equilibrium bidding strategy of an arbitrary bidder i. A critical object in Milgrom and Weber (1982) [2] is the first order statistic Y_1, which is the random variable denoting the highest signal received by bidder i's opponents. The following definition generalizes the first-order statistic notion to position auctions:

Definition 2.1. *For any arbitrary bidder i, let X be the random variable representing bidder i's own signal x_i. For all $k \in [1, K]$, let Y_k be the k-th order statistic representing the k-th highest signal received by bidder i's opponents. Let $G_k(y_k|x_i)$ be the conditional distribution of statistic Y_k given $X = x_i$, and let $g_k(y_k|x_i)$ be the associated density function. Let $v^k(x_i, y_k)$ be bidder i's expected value of a click conditional on i's signal x_i and the k-th order statistic that takes value y_k:*

$$v^k(x_i, y_k) = E\big[v(x_i, x_{-i})\big|X = x_i, Y_k = y_k\big] \tag{2.4}$$

For an arbitrary bidder i, the realization of Y_k is the minimum value that the signal of bidder i can take such that bidder i should win a position no lower than the k-th highest position in any efficient allocation.

Definition 2.2. *A position auction is efficient if it always assigns positions in the rank ordering of bidders' ex-post values, given any number of positions K, with*

any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$. *Under assumptions* **A1**-**A3**, *a position auction is efficient if it always assigns positions in the rank ordering of bidder's private signals.*

The "winner's curse" concept in Milgrom and Weber (1982) [2] can be extended to this model of position auctions with interdependent values in the following sense: at any monotonic bidding equilibrium, winning a higher ranked position conveys worse information about bidder i's expected value than winning a lower ranked position. For all $k, j \in \{1, 2, \cdots, K\}$, if $k < j$, then $v^k(x_i, x_i) \leq v^j(x_i, x_i)$. The inequality is strict under non-trivially interdependent values.

2.4   Inefficiency of One-dimensional Position Auctions

A unique feature of position auctions is that each bidder's value from getting a click does not depend on the position of her advertisement[12]. Based on this assumption, the commonly-used GSP auction adopts a simple bidding language that only requires each bidder to submit a one-dimensional bid based on her value per click from any position and computes her bid profile by scaling her bid by the click-through rates of the K positions, instead of asking each bidder to bid for each position separately.

In this section, I analyze the efficiency of GSP auctions and VCG-like auctions with this one-dimensional bidding language and show that both auctions can be

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[12]Goldman and Rao (2014) [46] use experimental data to test this assumption and get supportive result.

inefficient when there are at least two positions under certain CTR profiles. I begin the analysis by characterizing the allocation rule and payment rule in GSP auctions and VCG-like auctions.

2.4.1 One-dimensional Position Auctions

A position auction $(\tilde{\mu}, \tilde{p})$ that adopts one-dimensional bids $(b_1, b_2, \cdots, b_N) \in \mathbb{R}^N$, in which $b_i \in \mathbb{R}$ represents bidder i's bid per click for any position, is called a *one-dimensional position auction*. The allocation rule $\tilde{\mu}_i(b_1, b_2, \cdots, b_N) = \left(\tilde{\mu}_i^{(1)}(b_1, b_2, \cdots, b_N), \tilde{\mu}_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, \tilde{\mu}_i^{(K)}(b_1, b_2, \cdots, b_N) \right)$ is a vector of K functions, in which $\tilde{\mu}_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^{\mathbb{N}} \to [0, 1]$ maps a profile of bids $(b_1, b_2, \cdots, b_N)$ to the probability that bidder i wins position k. The payment rule $\tilde{p}_i(b_1, b_2, \cdots, b_N) = \left(\tilde{p}_i^{(1)}(b_1, b_2, \cdots, b_N), \tilde{p}_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, \tilde{p}_i^{(K)}(b_1, b_2, \cdots, b_N) \right)$ is a vector of K functions, in which $\hat{p}_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ maps a profile of bids to the payment of bidder i for position k.

For an arbitrary bidder i, given her opponents' bids b_{-i}, define $\hat{b}^k(b_{-i})$ as the k-th highest bid in b_{-i}, which implies $\hat{b}^1(b_{-i}) \geq \hat{b}^2(b_{-i}) \geq \cdots \geq \hat{b}^K(b_{-i})$. For any $k \geq 1$, if there are $n \geq 2$ equivalent k-th highest bids in b_{-i}, then $\hat{b}^k(b_{-i})$, $\hat{b}^{k+1}(b_{-i})$, ..., $\hat{b}^{k+n-1}(b_{-i})$ are assigned randomly for those n equivalent bids. A one-dimensional position auction is *assortative* if it assigns the k-th highest position to the bidder who submits the k-th highest bid.

Definition 2.3. *In a one-dimensional position auction $(\tilde{\mu}, \tilde{p})$, the allocation rule $\tilde{\mu}$ is assortative if for all $k \in \{1, 2, \cdots, K\}$,*

$$\tilde{\mu}_i^{(k)}(b_i, b_{-i}) = \begin{cases} 1 & if \quad \hat{b}^k(b_{-i}) \le b_i < \hat{b}^{k-1}(b_{-i}) \\ \\ 0 & else \end{cases} \tag{2.5}$$

Any tie is broken randomly.

Definition 2.4. *The one-dimensional GSP auction is characterized by the one-dimensional assortative allocation rule and the GSP payment rule given below. For all $k \in \{1, 2, \cdots, K\}$,*

$$\tilde{p}_i^{G,(k)}(b_i, b_{-i}) = \begin{cases} \alpha_k \hat{b}^k(b_{-i}) & if \quad \hat{b}^k(b_{-i}) \le b_i < \hat{b}^{k-1}(b_{-i}) \\ \\ 0 & else \end{cases} \tag{2.6}$$

Next, I define a VCG-like position auction format called *one-dimensional VCG auction* that is analogous to the second-price auction under the context of interdependent values single-unit auction in Milgrom and Weber (1982) [2].

Definition 2.5. *The one-dimensional VCG auction is characterized by the one-dimensional assortative allocation rule and a VCG-like payment rule given below. For all $k \in \{1, 2, \cdots, K\}$,*

$$\tilde{p}_i^{V,(k)}(b_i, b_{-i}) = \begin{cases} \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1})\hat{b}^j(b_{-i}) & if \quad \hat{b}^k(b_{-i}) \le b_i < \hat{b}^{k-1}(b_{-i}) \\ \\ 0 & else \end{cases} \tag{2.7}$$

Although the single-unit second-price auction analyzed by Milgrom and Weber (1982) [2] admits a Bayesian equilibrium that always implements efficient allocation under assumptions **A1**-**A3**, I will show that an analogous result does not exist in the one-dimensional VCG auction with multiple positions and non-trivially interdependent values.

2.4.2 Characterization of Equilibrium

I start the efficiency analysis by providing a necessary and sufficient condition for existence of an efficient Bayesian equilibrium in any one-dimensional assortative position auction.

Lemma 2.1. *A one-dimensional position auction $(\tilde{\mu}, \tilde{p})$ with assortative allocation rule is efficient if and only if there exists a symmetric equilibrium in which each bidder's bidding strategy $\beta(x_i)$ is strictly increasing in x_i, for any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.*

Proof. See Appendix. □

Next, I develop the main result of section 2.4: both the one-dimensional GSP auction and the one-dimensional VCG auction can be inefficient when bidders have interdependent values. Note that it is sufficient to show inefficiency can occur with $K = 2$ positions. For both of the one-dimensional GSP auction (G) and the one-dimensional VCG auction (V), I first provide a necessary condition for any monotonic bidding strategy $\beta^L(x_i)$ to be a Bayesian equilibrium of the auction $L \in \{G, V\}$ with two positions, and then finish the analysis by showing that the unique $\beta^L(x_i)$ characterized by this equilibrium condition cannot be monotonic under some CTR profiles.

Lemma 2.2. *In the one-dimensional GSP auction with two positions, if an efficient Bayesian equilibrium bidding strategy $\beta^G(x_i)$ exists, then $\beta^G(x_i)$ is characterized as below:*

For all $x_i \in [0, \bar{x}]$, $\beta^G(x_i)$ satisfies the Volterra equation

$$\beta^G(x_i) = \frac{g_1(x_i|x_i)\left[(\alpha_1 - \alpha_2)v^1(x_i, x_i) + \alpha_2 \int_0^{x_i} \beta^G(y_2)g_{2|1}(y_2|x_i, x_i)dy_2\right] + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{\alpha_1 g_1(x_i|x_i) + \alpha_2 g_2(x_i|x_i)}$$

$$(2.8)$$

Proof. See Appendix. $\qquad\square$

Lemma 2.3. *In the one-dimensional VCG auction with two positions, if an efficient Bayesian equilibrium bidding strategy $\beta^V(x_i)$ exists, then $\beta^V(x_i)$ is characterized as below:*

For all $x_i \in [0, \bar{x}]$,

$$\beta^V(x_i) = \frac{g_1(x_i|x_i)(\alpha_1 - \alpha_2)v^1(x_i, x_i) + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2} \qquad (2.9)$$

Proof. See Appendix. $\qquad\square$

To better understand the characterization of equilibria in Lemma 2.2 and Lemma 2.3, let $\Pi_1^L(x_i, y_1, y_2)$ and $\Pi_2^L(x_i, y_1, y_2)$ denote the expected payoffs from winning position 1 and 2 in auction $L \in \{G, V\}$ respectively, given the realizations of $X = x_i, Y_1 = y_1, Y_2 = y_2$. The equilibrium bidding strategy $\beta^G(x_i)$ characterized in Lemma 2.2 is derived from the following equilibrium condition:

$$g_1(x_i|x_i) \underbrace{\left[(\alpha_1 - \alpha_2)v^1(x_i, x_i) - \alpha_1\beta^G(x_i) + \alpha_2 \int_0^{x_i} \beta^G(y_2)g_{2|1}(y_2|x_i, x_i)dy_2\right]}_{E[\Pi_1^G - \Pi_2^G|X=x_i, Y_1=x_i]}$$

$$+ g_2(x_i|x_i) \underbrace{\left[\alpha_2 v^2(x_i, x_i) - \alpha_2\beta^G(x_i)\right]}_{E[\Pi_2^G|X=x_i, Y_2=x_i]} = 0$$

$$(2.10)$$

Similarly, the equilibrium bidding strategy $\beta^V(x_i)$ characterized in Lemma 2.3

is derived from the following equilibrium condition:

$$g_1(x_i|x_i) \underbrace{\left[(\alpha_1 - \alpha_2)(v^1(x_i, x_i) - \beta^V(x_i)) \right]}_{E[\Pi_1^V - \Pi_2^V | X = x_i, Y_1 = x_i]} + g_2(x_i|x_i) \underbrace{\left[\alpha_2(v^2(x_i, x_i) - \beta^V(x_i)) \right]}_{E[\Pi_2^V | X = x_i, Y_2 = x_i]} = 0$$

(2.11)

Note that in the special case of independent private values where $v^k(x_i, x_i) = x_i$ for all k, the equilibrium of the one-dimensional VCG auction is given by $\beta^V(x_i) = x_i$, consistent with the dominant strategy equilibrium in the VCG auction under independent private values. In the special case of $\alpha_2 = 0$, the equilibrium $\beta^V(x_i) = v^1(x_i, x_i)$ is consistent with the symmetric equilibrium of the second-price auction in Milgrom and Weber (1982) [2]. In the special case of $\alpha_1 = \alpha_2$, the equilibrium $\beta^V(x_i) = v^2(x_i, x_i)$ is consistent with the equilibrium of the uniform-price auction with single-unit demands in Ausubel et al. (2014) [47].

Equations (2.10) and (2.11) imply that in both one-dimensional GSP auctions and one-dimensional VCG auctions with two positions, for an arbitrary bidder i with signal x_i, the net impact of winning position 1 instead of position 2 on the margin of $Y_1 = x_i$ and winning position 2 instead of nothing on the margin of $Y_2 = x_i$ weighted by corresponding probability masses must equal zero at any efficient equilibrium. For all $x_i \in [0, \bar{x}]$,

$$g_1(x_i|x_i)E\left[\Pi_1^L - \Pi_2^L \Big| X = x_i, Y_1 = x_i \right] + g_2(x_i|x_i)E\left[\Pi_2^L \Big| X = x_i, Y_2 = x_i \right] = 0, \quad L \in \{G, V\}$$

(2.12)

The intuition behind this equilibrium condition is that in a one-dimensional assortative position auction, for any bidder i, increasing bid increases the probability of winning position 1 instead of position 2 and the probability of winning position 2

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instead of nothing at the same time, so each bidder's optimal strategy $\beta^L(x_i)$ must balance the trade-offs between every pair of adjacent positions at corresponding margins. I next show that the unique $\beta^L(x_i)$ satisfying this equilibrium condition cannot be monotonic under some CTR profile, for both $L = G, V$.

2.4.3 Efficiency Analysis

The following two propositions present the main result of section 2.4:

Proposition 2.1. *For any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1**-**A3**, there exists some number of positions K with some CTR profile under which no efficient Bayesian equilibrium exists in the one-dimensional GSP auction.*

Proof. See Appendix. □

Proposition 2.2. *For any non-trivially interdependent value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1**-**A3** and $\frac{\partial v_i(x_i, x_{-i})}{\partial x_j} \neq 0$ for $j \neq i$, there exists some number of positions K with some CTR profile under which no efficient Bayesian equilibrium exists in the one-dimensional VCG auction.*

Proof. See Appendix. □

The intuition behind Proposition 2.1 and Proposition 2.2 is that in both one-dimensional GSP auctions and one-dimensional VCG auctions with two positions, there exists some CTR profile under which the superior position is less desirable than the inferior position given expected payoffs, which leads to differential bid-shading incentives across bidders and results in non-existence of monotonic equilibrium bidding strategy. The following analysis elaborates this intuition in each auction.

The source of inefficiency of one-dimensional GSP auctions comes from the single-dimensionality of its bidding language and its payment rule. In a one-dimensional GSP auction with two positions, when the click rate of the second position is close to that of the first position, a bidder receives similar number of clicks from winning either position but pays a much higher price for each click from winning the first position than winning the second position conditional on $Y_1 = x_i$:

$$\lim_{\alpha_2 \to \alpha_1} E\left[\Pi_1^G - \Pi_2^G \Big| X = x_i, Y_1 = x_i\right] = \alpha_1 \int_0^{x_i} \left(\beta^G(y_2) - \beta^G(x_i)\right) g_{2|1}(y_2|x_i, x_i) dy_2 < 0$$

(2.13)

The inequality follows from the fact that $\beta^G(y_2) < \beta^G(x_i)$ for all $y_2 \in [0, x_i)$ given any strictly increasing function $\beta^G(.)$. Therefore, at any monotonic equilibrium, the second position is more desirable than the first position conditional on $Y_1 = x_i$ for any bidder i when α_2 is sufficiently close to α_1. Under the one-dimensional bidding language, each bidder is forced to submit the same bid for both positions, so the equilibrium bid must balance net trade-offs between all pairs of adjacent positions weighted by corresponding probability masses $g_k(x_i|x_i)$ that varies with signal x_i, as shown in equation (2.12). Because the weight attached to $E[\Pi_1^G - \Pi_2^G|X = x_i, Y_1 = x_i]$ is higher for bidders with higher signals x_i compared to those with lower signals, the bid-shading incentive is stronger for the former. This differentiated bid-shading incentive across bidders' signals can lead to violation of monotonicity of the unique equilibrium bidding strategy $\beta^G(x_i)$ characterized in Lemma 2.2. Therefore, a symmetric and strictly increasing equilibrium bidding strategy does not exist under certain CTR profiles. According to Lemma 2.1, there exists no efficient equilibrium

in the one-dimensional GSP auction under some CTR profile.

The result of Proposition 2.1 is consistent with the main result of Gomes and Sweeney (2014) [14], who solve the Volterra equation that characterizes the equilibrium of the one-dimensional GSP auction under independent private values and show it can be non-monotonic under some CTR profile. They also show non-monotonicity tends to occur when the click-through rate of the inferior position is close to that of the superior position when there are two positions. Proposition 2.1 introduces interdependent values into their model and identifies an additional source of non-existence of monotonic equilibrium: the non-existence of monotonic equilibrium not only comes from the GSP payment rule, but also comes from the single-dimensionality of bidding language in the GSP auction. Proposition 2.2 provides further support for this argument by showing that with the one-dimensional bidding language, modifying the GSP payment rule to the more complicated VCG-like payment rule does not resolve the inefficiency problem under interdependent values, as differentiated bid-shading incentives across bidders still exist in the one-dimensional VCG auction.

The source of inefficiency of one-dimensional VCG auctions comes from its one-dimensional bidding language and the presence of the generalized "winner's curse" under interdependent values. Similar to the one-dimensional GSP auction, the weight attached to trade-offs between each pair of adjacent positions $g_k(x_i|x_i)$ varies in x_i in the one-dimensional VCG auction. Under the VCG-like payment rule, it is optimal for each bidder to bid her true expected value per click conditional on $Y_2 = x_i$ if the probability of winning the first position is zero so that

only the trade-off between winning the second position and nothing needs to be considered. However, for any bidder who receives a signal $x_i > 0$, there is positive probability of winning the first position at any monotonic equilibrium. With non-trivially interdependent values, the expected value $v^k(x_i, x_i)$ differs across positions, with $v^1(x_i, x_i) < v^2(x_i, x_i)$ under the generalized "winner's curse." Therefore, every bidder with $x_i > 0$ shades bid below $v^2(x_i, x_i)$. Bidders with higher signals have stronger bid-shading incentive, since they need to weigh the impact of the generalized "winner's curse" more significantly given that they are more likely to win the first position when other bidders bid monotonically. This differentiated bid-shading incentive can lead to non-monotonicity of the unique equilibrium bidding strategy $\beta^V(x_i)$ characterized in Lemma 2.3, which implies that the one-dimensional VCG auction must also be inefficient under some CTR profile when there are two positions. Moreover, the non-existence of monotonic equilibrium in the one-dimensional VCG auction also tend to occur when α_2 is close to α_1, as the bid-shading incentive under the generalized "winner's curse" is amplified when the quality of the superior position is not significantly better than the quality of the inferior position.

To summarize this section, it can be concluded that a common source of inefficiency of the one-dimensional GSP auction and the one-dimensional VCG auction comes from the fact that both auctions use a simple one-dimensional bidding language to determine the allocation of multiple differentiated positions. Restricting bidders to one-dimensional bids requires the equilibrium bid to balance the net trade-offs between all pairs of adjacent positions on different margins, which is impossible for any monotonic bidding strategy under certain CTR profiles. It is natural to

conjecture that allowing bidders to submit different bids for each position such that the equilibrium bid for each position k balances only the trade-off between position k and position $k+1$ conditional on $Y_k = x_i$ may resolve the inefficiency problem. The next section confirms this conjecture.

2.5 Efficiency of K-dimensional Position Auctions

In this section, I propose a modification of the one-dimensional GSP auction and the one-dimensional VCG auction by allowing each bidder to submit K separate bids (i.e., a K-dimensional bid) and show that both auctions have unique efficient Bayesian equilibria given any number of positions K with any CTR profile after this modification. Moreover, the Generalized English Auction that implicitly adopts a K-dimensional bidding language has a unique efficient ex-post equilibrium. The main result of this section shows that adopting a multi-dimensional bidding language can fully implement efficiency in position auctions under interdependent values.

2.5.1 K-dimensional Position Auctions

I first construct a class of position auctions that adopts a K-dimensional bidding language and a K-dimensional assortative allocation rule that corresponds to the assortative allocation rule in one-dimensional position auctions. A position auction (μ, p) that adopts K-dimensional bids $(b_1, b_2, \cdots, b_N) \in \mathbb{R}^K \times \mathbb{R}^N$, in which $b_i \in \mathbb{R}^K$ represents bidder i's bid per click for every position $k \in \{1, 2, \cdots, K\}$, is called a *K-dimensional position auction*. The allocation rule $\mu_i(b_1, b_2, \cdots, b_N) =$

$\left(\mu_i^{(1)}(b_1, b_2, \cdots, b_N), \mu_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, \mu_i^{(K)}(b_1, b_2, \cdots, b_N) \right)$ is a vector of K

functions, in which $\mu_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^K \times \mathbb{R}^N \to [0, 1]$ maps a profile of bids

$(b_1, b_2, \cdots, b_N)$ to the probability that bidder i wins position k. The payment rule

$p_i(b_1, b_2, \cdots, b_N) = \left(p_i^{(1)}(b_1, b_2, \cdots, b_N), p_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, p_i^{(K)}(b_1, b_2, \cdots, b_N) \right)$

is a vector of K functions, in which $p_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^K \times \mathbb{R}^N \to \mathbb{R}$ maps a profile

of bids to the payment of bidder i for position k.

For any position k, define $S_k(b_1, b_2, \cdots, b_N)$ as the set of bidders who should

win some position strictly above the k-th highest position at bidding profile $(b_1, b_2, \cdots, b_N)$

according to the allocation rule of the auction:

$$S_k(b_1, b_2, \cdots, b_N) = \left\{ j \in \{1, 2, \cdots, N\} \,\middle|\, \exists \, k' < k \text{ s.t. } \mu_j^{(k')}(b_j, b_{-j}) = 1 \right\} \quad (2.14)$$

For any arbitrary bidder i, given any profile of K-dimensional bids (b_i, b_{-i}), define

$\max \left\{ b_{-i/S_k(b_i, b_{-i})}^k \right\}$ as the highest bid for position k among bidder i's opponents who

do not win any position above k. A K-dimensional position auction is *assortative* if

its allocation rule is characterized by the following definition:

Definition 2.6. *In a K-dimensional position auction* (μ, p), *the allocation rule* μ *is*

assortative if for all $k \in \{1, 2, \cdots, K\}$,

$$\mu_i^{(k)}(b_i, b_{-i}) = \begin{cases} 1 & \text{if} \quad i \notin S_k, \ \max \left\{ b_{-i/S_k(b_i, b_{-i})}^k \right\} \leq b_i^k \\ 0 & \text{else} \end{cases} \quad (2.15)$$

Any tie is broken randomly.

In an assortative K-dimensional position auction (μ, p), each bidder submits a

vector of K bids $(b_i^1, b_i^2, \cdots, b_i^K)$ simultaneously in a sealed-bid format. The auction-

eer collects all bids at once and assigns the first position to the bidder who submits the highest bid for position 1, the second position to the bidder who submits the highest bid for position 2, among those who do not win position 1, etc. Once a bidder is assigned a position k, her bids for lower positions b_i^j with $j > k$ will not be considered in the allocation of lower positions and will be equated to zero.

I next construct two assortative K-dimensional position auctions that can be viewed as modified one-dimensional GSP auction and modified one-dimensional VCG auction, respectively. I call these auctions K-dimensional GSP auction and K-dimensional VCG auction.

Definition 2.7. *The K-dimensional GSP auction is characterized by the assortative K-dimensional allocation rule and the following payment rule: for all $k \in \{1, 2, \cdots, K\}$,*

$$
p_i^{G,(k)}(b_i, b_{-i}) = \begin{cases} \alpha_k \max \left\{ b_{-i/S_k(b_i, b_{-i})}^k \right\} & \text{if} \quad i \notin S_k, \ \max \left\{ b_{-i/S_k(b_i, b_{-i})}^k \right\} \leq b_i^k \\ 0 & \text{else} \end{cases}
$$

$$(2.16)$$

Definition 2.8. *The K-dimensional VCG auction can be characterized by the assortative K-dimensional allocation rule and the following payment rule: for all $k \in \{1, 2, \cdots, K\}$,*

$$
p_i^{V,(k)}(b_i, b_{-i}) = \begin{cases} \sum_{j=k}^{K} (\alpha_j - \alpha_{j+1}) \max \left\{ b_{-i/S_j(b_i, b_{-i})}^j \right\} & \text{if} \quad i \notin S_k, \ \max \left\{ b_{-i/S_k(b_i, b_{-i})}^k \right\} \leq b_i^k \\ 0 & \text{else} \end{cases}
$$

$$(2.17)$$

In addition to the class of static K-dimensional position auctions proposed

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above, the dynamic Generalized English Auction (GEA) in Edelman et al. (2007) [6] also adopts a K-dimensional bidding language. The rule of the Generalized English Auction is given as follows. There is a continuously ascending clock showing the current price. Initially, all advertisers are in the auction. An advertiser can drop out at any time, and her bid is the price on the clock when she drops out. The auction ends when there is only one bidder left. This last bidder wins the first position, and her per click payment equals to the next-to-last bidder's drop-out price. The next-to-last bidder wins the second position, and her per click payment equals to the third highest bid, etc. Any tie is broken randomly when bidders drop out simultaneously. All drop-out prices are observable, so each bidder's bidding strategy will be different every time a bidder drops out. This dynamic process implicitly allows for a K-dimensional bidding language.

2.5.2 Characterization of Equilibria and Efficiency Analysis

To begin the efficiency analysis of K-dimensional position auctions, I first provide a necessary and sufficient condition for any K-dimensional assortative position auction to be efficient:

Lemma 2.4. *A K-dimensional position auction* (μ, p) *with assortative allocation rule is efficient if and only if given any number of positions* K, *there exists a symmetric equilibrium in which each bidder's bidding strategy* $\big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ *satisfies* $\beta_k'(x_i) > 0$ *for every position* $k \in \{1, 2, \cdots, K\}$, *under any CTR profile* $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.

Proof. See Appendix. □

Next, I develop the main result of section 2.5: the K-dimensional GSP auction, the K-dimensional VCG auction, and the Generalized English Auction are always efficient given any value function satisfying assumptions **A1**-**A3**, for any number of positions K, with any CTR profile. I first characterize the unique symmetric equilibria of the K-dimensional GSP Auction, the K-dimensional VCG auction and the Generalized English Auction. It will be shown that the equilibria of all three auctions satisfy the necessary and sufficient condition in Lemma 2.4.

2.5.2.1 Equilibrium of K-dimensional GSP Auction

The unique symmetric Bayesian equilibrium bidding strategy in the K-dimensional GSP auction is given in Proposition 2.3:

Proposition 2.3. *Define the K-dimensional bidding strategy* $\beta(x_i) = \big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ *as follows:*

$$\beta_K(x_i) = v^K(x_i, x_i) \tag{2.18}$$

for the last position K.

$$\beta_k(x_i) = v^k(x_i, x_i) - \frac{\alpha_{k+1}}{\alpha_k}\left[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1})dG_{k+1}\big(y_{k+1}\big|X = x_i, Y_k = x_i\big)\right] \tag{2.19}$$

for any position $k \in \{1, 2, \cdots, K-1\}$.

Let $b_i^* = \beta(x_i) = \big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ *for each bidder i, then the n-tuple of strategies* $(b_1^*, b_2^*, \cdots, b_N^*)$ *is a Bayesian Nash equilibrium of the K-dimensional GSP auction.*

Proof. See Appendix. □

Proposition 2.3 shows that the equilibrium bid for the last position K in the K-dimensional GSP auction is the expected value per click conditional on receiving a signal just high enough to win the last position, $Y_k = x_i$. On the other hand, the equilibrium bid for any position above the last position in the K-dimensional GSP auction is the expected value per click subtracted by the expected payoff from winning the next position divided by α_k, conditional on $Y_k = x_i$. The subtracted term can be interpreted as the per-click opportunity cost of winning position k. Since $\beta_K(x_i) = v^K(x_i, x_i)$ is strictly increasing in x_i, and

$$\frac{d\beta_k(x_i)}{dx_i} = \left(1 - \frac{\alpha_{k+1}}{\alpha_k}\right)\frac{\partial v^k(x_i, x_i)}{\partial x_i} + \frac{\alpha_{k+1}}{\alpha_k}\beta_{k+1}(x_i)g_{k+1}(x_i|x_i, x_i) > 0, \quad \forall k \in \{1, 2, \cdots, K-1\}$$

$$(2.20)$$

The symmetric equilibrium bidding strategy $\beta_k(x_i)$ for every position k is strictly increasing in x_i. According to Lemma 2.4, the K-dimensional GSP auction is always efficient.

Corollary 2.1. *The K-dimensional GSP auction always implements the ex-post efficient allocation in a symmetric Bayesian equilibrium given any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1**-**A3**, for any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.*

To better understand the equilibrium characterized in Proposition 2.3, let $\Pi_k^G(x_i, y_1, \cdots, y_{N-1})$ denote the payoff of winning position k given realizations $X = x_i, Y_1 = y_1, \cdots, Y_{N-1} = y_{N-1}$ in the K-dimensional GSP auction. The equilibrium

bidding strategy $\left(\beta_1(x_i), \cdots, \beta_K(x_i)\right)$ solves

$$\underbrace{\alpha_k \left[v^k(x_i, x_i) - \beta_k(x_i)\right]}_{E[\Pi_k^G | X = x_i, Y_k = x_i]} = \underbrace{\alpha_{k+1} \left[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1|k} \left(y_k | x_i, x_i\right)\right]}_{E[\Pi_{k+1}^G | X = x_i, Y_k = x_i]},$$

$$\forall k \in \{1, 2, \cdots, K\}$$
$$(2.21)$$

which implies that at the symmetric equilibrium of the K-dimensional GSP auction, each bidder should be indifferent between winning position k and position $k + 1$ conditional on $Y_k = x_i$, at which value her signal is just high enough to win position k, for any position $k \in \{1, 2, \cdots, K\}$.

2.5.2.2   Equilibrium of K-dimensional VCG Auction

The unique symmetric Bayesian equilibrium bidding strategy in the K-dimensional VCG auction is given by Proposition 2.4:

Proposition 2.4. *Let* $\beta_k(x_i) = v^k(x_i, x_i)$ *for all* $k \in \{1, 2, \cdots, K\}$. *Let* $b_i^* = \beta(x_i) = (\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i))$, *then the n-tuple of strategies* $(b_1^*, b_2^*, \cdots, b_N^*)$ *is a Bayesian Nash equilibrium of the K-dimensional VCG auction.*

Proof. See Appendix. □

Since $v^k(x_i, x_i)$ is strictly increasing in x_i for all $k \in \{1, 2, \cdots, K\}$, the K-dimensional VCG auction is always efficient.

Corollary 2.2. *The K-dimensional VCG auction always implements the ex-post efficient allocation in a symmetric Bayesian equilibrium given any value function*

$v_i(x_i, x_{-i})$ satisfying assumptions **A1**-**A3**, for any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.

To better understand the equilibrium bidding strategy characterized in Proposition 2.4, let $\Pi_k^V(x_i, y_1, \cdots, y_{N-1})$ denote the payoff of winning position k given realizations $X = x_i, Y_1 = y_1, \cdots, Y_{N-1} = y_{N-1}$ in the K-dimensional VCG auction. The equilibrium bidding strategy $\big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ in the K-dimensional VCG auction solves

$$\underbrace{(\alpha_k - \alpha_{k+1})\Big[v^k(x_i, x_i) - \beta_k(x_i)\Big]}_{E[\Pi_k^V - \Pi_{k+1}^V | X = x_i, Y_k = x_i]} = 0, \quad \forall k \in \{1, 2, \cdots, K\} \qquad (2.22)$$

which implies that at the equilibrium of K-dimensional VCG auction, each bidder with signal x_i is indifferent between winning position k and position $k + 1$ when $Y_k = x_i$, for all position k. Comparing equation (2.21) and equation (2.22), it follows that the equilibria of K-dimensional GSP auction and K-dimensional VCG auction can be characterized by the same condition:

$$E\Big[\Pi_k^L - \Pi_{k+1}^L \Big| X = x_i, Y_k = x_i\Big] = 0, \quad \forall \, k \in \{1, 2, \cdots, K\}, \quad \forall \, L \in \{G, V\} \ (2.23)$$

Equation (2.23) shows that with K-dimensional bidding language, each bidder submits K separate bids such that the bid for position k balances only the trade-off between position k and position $k + 1$ conditional on $Y_k = x_i$, in contrast to the equilibrium condition in one-dimensional position auctions characterized by equation (2.12). The differentiated bid-shading incentive across bidders' signals in the one-dimensional auctions is replaced by the differentiated bid-shading incentive across positions in the K-dimensional auctions, which resolves the inefficiency

problem.

The next example provides an illustration of the Bayesian equilibrium bidding strategies in the K-dimensional VCG auction and K-dimensional GSP auction.

Example 2.1. *Consider the K-dimensional VCG auction and K-dimensional GSP auction with $K = 2$ positions and $N = 3$ bidders, with click-through-rates normalized to $(1, \alpha_2)$. The bidders' private signals are independently and identically drawn from the uniform distribution on $[0, 1]$. Bidder i's value per click v_i is a function of her own signal x_i and her opponents' signals x_j, x_k:*

$$v_i(x_i, x_j, x_k) = \lambda x_i + \frac{1 - \lambda}{2}(x_j + x_k), \quad \lambda \in \left[\frac{1}{3}, 1\right] \tag{2.24}$$

Figure 1 plots the equilibrium bidding strategy $\left(\beta_1^V(x), \beta_2^V(x)\right)$ in the K-dimensional VCG auction and $\left(\beta_1^G(x), \beta_2^G(x)\right)$ in the K-dimensional GSP auction, under different values of $\lambda \in \{1, \frac{1}{2}, \frac{1}{3}\}$ and $\alpha_2 \in \{0.75, 0.25\}$.

Figure 2.1 provides two main insights. First, comparing the equilibria under values of $\lambda = 1, \frac{1}{2}, \frac{1}{3}$ given the same α_2 illustrates the impact of increasing degree of interdependency among bidders' values on the equilibria of the two auctions. Since $\beta_1^L(x_i) \le \beta_2^L(x_i)$ for both auctions $L \in \{G, V\}$ under any α_2, the equilibrium bid of any bidder for position 1 is weakly lower than that for position 2 in both auctions. The difference $\left(\beta_2^L(x_i) - \beta_1^L(x_i)\right)$ is increasing in x_i in the K-dimensional GSP auction, while stays constant in x_i in the K-dimensional VCG auction. Moreover, $\left(\beta_2^L(x_i) - \beta_1^L(x_i)\right)$ is greater in both auctions when λ is lower, which means the degree of bid-shading for position 1 is more significant in both auctions when the
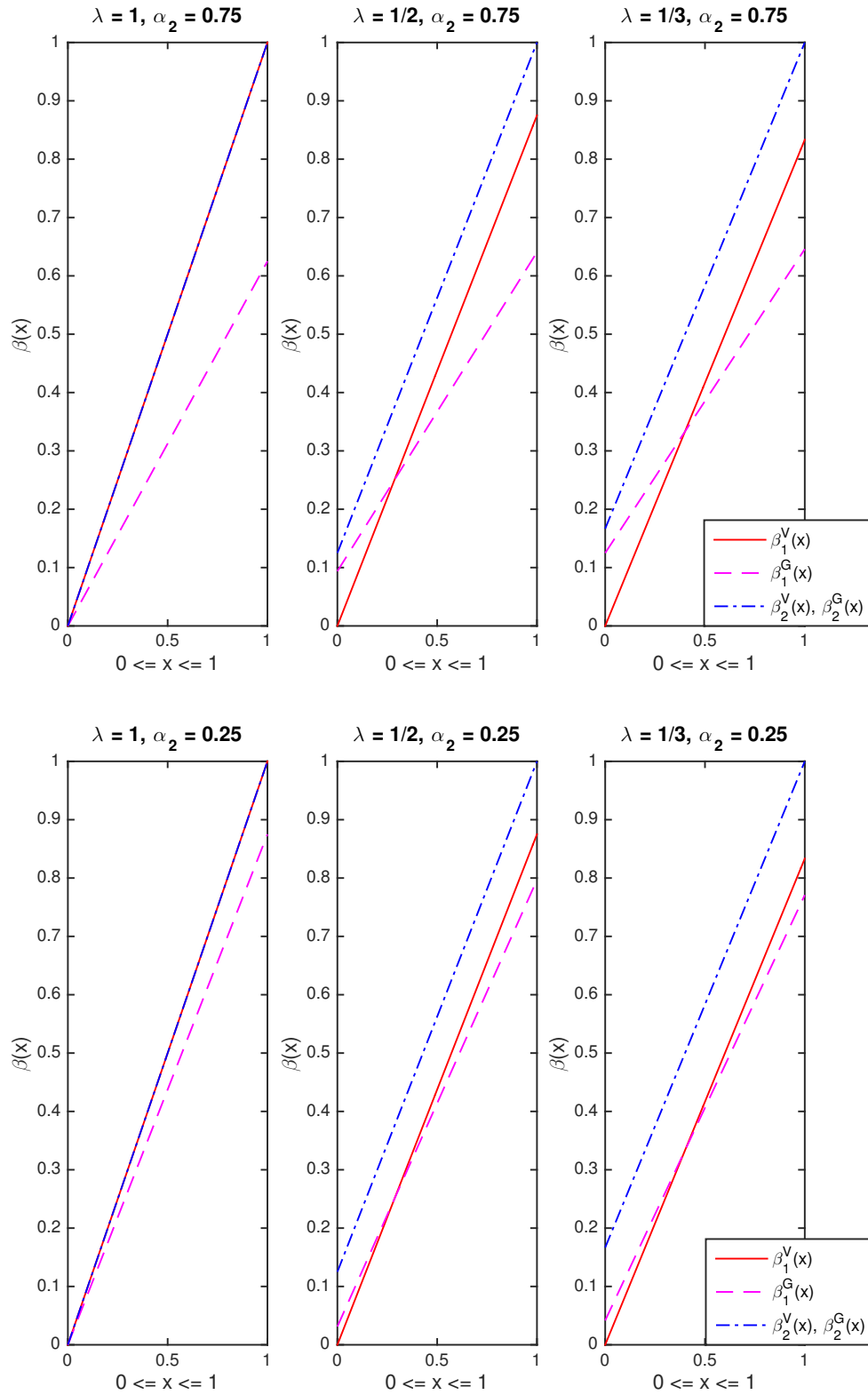
Figure 2.1: Equilibrium Bidding Strategies in K-dimensional VCG and GSP Auction

degree of interdependency in values is stronger and the impact of the generalized

"winner's curse" is more significant.

Second, comparing the equilibria under $\alpha_2 = 0.75$ to $\alpha_2 = 0.25$ under the same

value of λ shows the impact of increasing difference in click-through rates between

the superior position and the inferior position on the equilibria of the two auctions.

It can be shown that under the same λ, $\left(\beta_2^G(x_i) - \beta_1^G(x_i)\right)$ increases in α_2 as well as

in x_i in the K-dimensional GSP auction, while $\left(\beta_2^V(x_i) - \beta_1^V(x_i)\right)$ remains unaffected

by α_2 and stays constant in x_i in the K-dimensional VCG auction. Therefore, the

bid-shading incentive for position 1 is greater when the click-through rates of two

positions are closer in the K-dimensional GSP auction, while the equilibrium bids

are unaffected by click-through rates in the K-dimensional VCG auction.

2.5.2.3 Equilibrium of Generalized English Auction

The next result of this section characterizes the unique symmetric equilibrium

of the Generalized English Auction (GEA) under interdependent values and shows

this dynamic auction that implicitly adopts a K-dimensional bidding language is

also efficient.

At any time in the auction, let n denote the number of bidders who are still

active in the auction, and $(N - n)$ denote the number of bidders who have dropped

out. Let $(p_N, p_{N-1} \cdots, p_{n+1})$ denote the drop-out prices of the $(N - n)$ bidders,

in which p_N is the bid of the first drop out bidder, and p_{n+1} is the bid of the last

drop out bidder at current time, so $p_N \leq p_{N-1} \leq \cdots \leq p_{n+1}$. When there are n

remaining bidders in the auction, the equilibrium strategy for bidder i specifies her optimal drop out price given her private signal x_i and a history of drop out prices $(p_N, p_{N-1}, \cdots, p_{n+1})$. Define

$$v^{(k)}(x_i, y_k, y_{k+1}, \cdots, y_{N-1}) = E\left[v_i \middle| X = x_i, Y_k = y_k, Y_{k+1} = y_{k+1}, \cdots, Y_{N-1} = y_{N-1}\right]$$

(2.25)

as bidder i's expected value conditional on her own signal $X = x_i$ and the realization of all of the $(N-k)$ lowest signals among opponents' signals, $Y_k = y_k, Y_{k+1} = y_{k+1}, \cdots, Y_{N-1} = y_{N-1}$. The unique symmetric equilibrium of the GEA under interdependent values is characterized in Proposition 2.5:

Proposition 2.5. *Define strategy $b^* = (b_N^*, b_{N-1}^*, \cdots, b_2^*)$ as follows:*

$$b_N^*(x_i) = v^{(K)}(x_i, \underbrace{x_i, \cdots, x_i}_{(N-K)})$$

$$b_n^*(x_i | p_N, \cdots, p_{n+1}) =$$

$$\begin{cases} v^{(K)}(x_i, \underbrace{x_i, \cdots, x_i}_{(n-K)}, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\ lowest\ signals}) & if \quad (K+1) \leq n \leq (N-1) \\[3em] v^{(n-1)}(x_i, x_i, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\ lowest\ signals}) - \frac{\alpha_n}{\alpha_{n-1}}\left[v^{(n-1)}(x_i, x_i, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\ lowest\ signals}) - p_{n+1}\right] & if \quad n \leq K \end{cases}$$

(2.26)

in which $y_n, y_{n+1}, \cdots, y_{N-1}$ are calculated from

$$b_N^*(y_{N-1}) = p_N$$

$$b_{N-1}^*(y_{N-2} | p_N) = p_{N-1}$$

$$\cdots$$

$$b_{n+1}^*(y_n | p_N, \cdots, p_{n+2}) = p_{n+1}$$

(2.27)

The N-tupple bidding strategy $(b^*, ..., b^*)$ is an ex-post equilibrium of the Generalized English Auction under interdependent values.

Proof. See Appendix. □

Since the equilibrium bidding strategy $b_n^*(x_i)$ at any stage of the GEA is increasing in x_i, the GEA is also efficient.

Corollary 2.3. *The Generalized English Auction always implements the ex-post efficient allocation in an ex-post equilibrium, given any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1**-**A3**, for any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.*

To better understand the equilibrium of GEA, let $\Pi_k^E(x_i, y_1, y_2, \cdots, y_{N-1})$ be the payoff from winning position k conditional on $X = x_i, Y_1 = y_1, \cdots, Y_{N-1} = y_{N-1}$. The equilibrium condition of GEA characterized in Proposition 2.5 can be interpreted as

$$E\left[\Pi_K^E \middle| X = x_i, Y_K = x_i, \cdots, Y_N = x_i\right] = 0, \text{ if } n = N$$

$$E\left[\Pi_K^E \middle| X = x_i, Y_K = x_i, \cdots, Y_{n-1} = x_i, Y_n = y_n, \cdots, Y_N = y_N\right] = 0, \text{ if } K + 1 \le n \le N - 1$$

$$E\left[\Pi_k^E - \Pi_{k+1}^E \middle| X = x_i, Y_k = x_i, Y_{k+1} = y_{k+1}, \cdots, Y_N = y_N\right] = 0, \text{ if } n = k + 1 \le K$$

$$(2.28)$$

which implies that the optimal drop-out price at any time of the auction must balance the trade-off between winning position k and position $k + 1$ conditional on $Y_k = x_i$, given the profile of revealed signals from the history of drop-out prices. When there are more bidders than positions left in the auction, each bidder's optimal drop-out strategy specifies the price at which she is indifferent between winning the

lowest position and winning nothing. When there are (weakly) fewer bidders than positions left in the auction, each bidder's optimal drop-out strategy specifies the price at which she is indifferent between winning the next position higher than the current lowest position and winning the current lowest position at the most recent drop-out price. Comparing the characterization of equilibrium in (2.28) to the characterization of equilibrium in (2.23), it can be shown that the equilibrium condition of GEA is similar to the equilibrium condition of the K-dimensional GSP auction and the K-dimensional VCG auction, while the only difference comes from that each remaining bidder can update her belief from revealed signals of drop-out bidders in GEA.

2.6 Revenue of K-dimensional Position Auctions

In this section, I compare the expected revenues of the three efficient K-dimensional position auctions analyzed in section 2.5 and characterize the optimal design of position auction under interdependent values as a direct revelation mechanism. Then I compare the expected revenues of the three efficient position auctions to the optimal revenue subject to no reserve price.

2.6.1 Revenue Ranking

The following proposition gives the revenue ranking of the K-dimensional GSP auction, the K-dimensional VCG auction and the GEA.

Proposition 2.6. *The expected revenue of the Generalized English Auction is higher*

*than the expected revenue of the K-dimensional VCG auction, which in turn equals to the expected revenue of the K-dimensional GSP auction, for any value function $v_i(x_i, x_{-i})$ and distribution of signals $F(x_1, x_2, \cdots, x_N)$ satisfying assumptions **A1**-**A5**.*

$$R^{GEA} \geq R^{K-VCG} = R^{K-GSP} \tag{2.29}$$

Proof. See Appendix. □

The intuition behind revenue equivalence of the K-dimensional GSP auction and the K-dimensional VCG auction is the following. Both auctions are sealed-bid auctions, so no information is elicited before final allocation and payments are determined. Both auctions adopt the same K-dimensional assortative allocation rule and some variation of a "second-price" payment rule under which each bidder's bid only affect her allocation but not her payment. In the proof of Proposition 2.6, it is shown that although each bidder's payment in the two auctions depends on her opponents' bids in different ways, bidders are able to incorporate different payment rules into their bidding strategies so that the expected payment for a bidder with the same signal x_i is the same in the two auctions.

The intuition behind the revenue ranking of the GEA and the K-dimensional VCG auction comes from the Linkage Principle in Milgrom and Weber (1982) [2]. With affiliated signals, the dynamic auction performs better than static auctions since part of the signals are elicited during the drop-out process. On the other hand, with independent signals, the GEA is revenue equivalent to the other two static K-dimensional position auctions, which gives the following corollary:

Corollary 2.4. *When bidders' signals are independently and identically distributed, the Generalized English Auction, the K-dimensional VCG auction and the K-dimensional GSP auction yield the same expected revenue, for any value function $v_i(x_i, x_{-i})$ that satisfies assumptions **A1**-**A3**.*

Corollary 2.4 is consistent with the Revenue Equivalence Theorem in auction theory. When bidders have independent signals, the K-dimensional GSP auction, the K-dimensional VCG auction and the Generalized English Auction always implement the same allocation and yield zero expected payoff to the bidder with lowest signal. The revenue equivalence follows as a result.

2.6.2 Revenue Comparison with the Optimal Position Auction

I next characterize the optimal position auction under interdependent values subject to no reserve price as a corollary of Ulku (2013) [45]'s result[13] and then compare expected revenues of the K-dimensional GSP auction, the K-dimensional VCG auction and the Generalized English Auction to the optimal revenue implementable in position auctions subject to no reserve price.

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[13]Ulku (2013) [45] characterizes the optimal mechanism for allocating a set of heterogeneous items under interdependent values. This paper provides a corollary of Ulku (2013) [45] in the special environment of position auctions and provides a discussion on the connection between efficient and optimal mechanisms under this context.

2.6.2.1 Mechanism Design and Solution Concepts

Under the revelation principle, I characterize the optimal position auction as a direct mechanism, in which bidders report private signals directly, while the value function $v(x_i, x_{-i})$ and signal distribution $F(x_1, x_2, \cdots, x_N)$ are common knowledge. To make the expected revenue of the optimal position auction comparable to expected revenues of the three practical auctions analyzed in section 5, I restrict attention to the optimal position auction subject to no reserve price. A position auction mechanism (μ, p) consists of an allocation rule $\mu_i(x)$ and a payment rule $p_i(x)$ for every bidder i, where $\mu_i(x) = \left(\mu_i^{(1)}(x), \mu_i^{(2)}(x), \cdots, \mu_i^{(K)}(x) \right)$ is the vector of probabilities that bidder i wins position $1, 2, \cdots, K$ given reported signals $x \in [0, \bar{x}]^N$, and $p_i(x)$ is the expected payment of bidder i given reported signals $x \in [0, \bar{x}]^N$. In a deterministic mechanism, $\mu_i^{(k)}(x) \in \{0, 1\}$ for all k and $p_i(x)$ is the actual payment.

Given a CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$ and allocation rule μ, the expected click-through rate q_i assigned to bidder i under report $x = (x_1, x_2, \cdots, x_N)$ is given by

$$q_i(x) = \sum_{k=1}^{K} \alpha_k \mu_i^{(k)}(x) \tag{2.30}$$

For notational simplicity, I use the expected click-through rates (CTR) $\left\{ q_i(x) \right\}_{i=1}^{N}$ instead of N vectors of expected probabilities $\left\{ \left(\mu_i^{(1)}(x), \mu_i^{(2)}(x), \cdots, \mu_i^{(K)}(x) \right) \right\}_{i=1}^{N}$ as the allocation rule in the analysis. I use (q, p) and (μ, p) to refer to the same mechanism interchangeably if $q_i(x) = \sum_{k=1}^{K} \alpha_k \mu_i^{(k)}(x)$. The feasibility condition of the allocation rule in a position auction mechanism is defined below:

Definition 2.9. *An allocation rule in the form of $\mu(x)$ is feasible if*

$$0 \leq \sum_{i=1}^{N} \mu_i^{(k)}(x) \leq 1, \quad \forall \; k, \quad and \quad 0 \leq \sum_{k=1}^{K} \mu_i^{(k)}(x) \leq 1, \quad \forall \; i \qquad (2.31)$$

An allocation rule in the form of $q(x)$ is feasible if $q_i(x) = \sum_{k=1}^{K} \alpha_k \mu_i^{(k)}(x)$ for all i for some allocation rule $\mu(x)$ satisfying condition (2.31).

For any bidder i with signal x_i, the interim utility $U_i(x_i)$ is given by

$$U_i(x_i) = \int_{x_{-i}} \left[q_i(x_i, x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i, x_{-i}) \right] f_{-i|i}(x_{-i}|x_i) dx_{-i} \qquad (2.32)$$

where $u_i(x_i, x_{-i}) = q_i(x_i, x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i, x_{-i})$ is the ex-post utility of bidder i given the signal profile (x_i, x_{-i}). I now give the definition of two solution concepts:

Definition 2.10. *A position auction mechanism (q, p) is Bayesian incentive compatible (IC) and individually rational (IR) if for every bidder i, for any true signal x_i and any report x_i',*

$$U_i(x_i) \geq \int_{x_{-i}} \left[q_i(x_i', x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i', x_{-i}) \right] f_{-i|i}(x_{-i}|x_i) dx_{-i}$$
$$U_i(x_i) \geq 0 \qquad (2.33)$$

Definition 2.11. *A position auction mechanism (q, p) is ex-post incentive compatible (IC) and individually rational (IR) if for every bidder i, for any true signal profile (x_i, x_{-i}) and any report x_i',*

$$u_i(x_i, x_{-i}) \geq q_i(x_i', x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i', x_{-i})$$
$$u_i(x_i, x_{-i}) \geq 0 \qquad (2.34)$$

In the following analysis, I characterize the optimal position auction subject to no reserve price among all ex-post IC and IR mechanisms under affiliated signals,

and show that under certain regularity conditions, this optimal auction is equivalent to the Generalized-VCG mechanism that assigns all positions efficiently, Moreover, the optimal auction yields higher revenue than the GEA, the K-dimensional GSP auction and the K-dimensional VCG auction. Then I show that in the special case of independent signals, this mechanism is also optimal subject to no reserve price among all Bayesian IC and IR mechanisms, and implements equivalent revenue as the GEA, the K-dimensional GSP auction, and the K-dimensional VCG auction under the same set of regularity conditions.

2.6.2.2 Characterization of the Optimal Position Auction

I first characterize the optimal mechanism subject to no reserve price among ex-post IC and IR mechanisms under interdependent values with affiliated signals.

Given any profile of signals x, define bidder i's marginal revenue $MR_i(x_i, x_{-i})$ as

$$MR_i(x_i, x_{-i}) = v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i | x_{-i})}{f_i(x_i | x_{-i})} \times \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \qquad (2.35)$$

For any bidder i, given a vector of opponents' reported signals x_{-i}, define $\hat{X}^k(x_{-i})$ as the minimum value that bidder i's signal can take such that bidder i has the k-th highest MR_i among all bidders:

$$\hat{X}^k(x_{-i}) = \inf \left\{ x_i \mid MR_i(x_i, x_{-i}) \geq kmax_{j \neq i} \{ MR_j(x_j, x_i, x_{-ij}) \} \right\} \qquad (2.36)$$

in which $kmax_{j \neq i} \{ MR_j(x_j, x_i, x_{-ij}) \}$ is value of the k-th highest marginal revenue among bidder i's opponents given report x, and x_{-ij} is the vector of signals reported by bidders other than i and j. The following two regularity conditions are provided

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such that the optimal position auction subject to no reserve price assigns positions in the rank ordering of $MR_i(x)$ given report x.

R1 (*Value Regularity*): Given any profile of signals x, for any two bidders i, j,

$$\text{If} \quad x_i > x_j, \quad \text{then} \quad v_i(x_i, x_j, x_{-ij}) > v_j(x_j, x_i, x_{-ij}) \tag{2.37}$$

Note that **R1** is directly implied by assumptions **A1**-**A3**.

R2 (*MR Monotonicity*): Given any report of signals x, for all bidder i,

$$\frac{\partial MR_i(x_i, x_{-i})}{\partial x_i} > 0, \quad \forall x_{-i} \tag{2.38}$$

I next provide a corollary of Ulku (2013) [45] by characterizing the optimal position auction subject to no reserve price among all ex-post IC and IR mechanisms under **R1** and **R2**.

Corollary 2.5. *Under regularity conditions **R1** and **R2**, suppose the expected CTR is given by*

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{X}^k(x_{-i}) \le x_i < \hat{X}^{k-1}(x_{-i}) \\ \\ 0 & \text{if} \quad x_i < \hat{X}^K(x_{-i}) \end{cases} \tag{2.39}$$

Any tie is broken randomly. Suppose also that the payment rule is given by

$$p_i^*(x_i, x_{-i}) = q_i(x_i, x_{-i})v_i(x_i, x_{-i}) - \int_0^{x_i} q_i(s, x_{-i})\frac{\partial v_i(s, x_{-i})}{\partial s}ds \tag{2.40}$$

Then (q^*, p^*) is an optimal position auction among all the ex-post IC and IR mechanisms subject to no reserve price.

Proof. See Appendix. □

Note that in the special case of independent signals, each bidder's marginal revenue is given by

$$MR_i(x_i, x_{-i}) = v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i)}{f_i(x_i)} \times \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \tag{2.41}$$

as $F_i(x_i|x_{-i}) = F_i(x_i)$ and $f_i(x_i|x_{-i}) = f_i(x_i)$ under independent signals. The next proposition shows that under **R1** and **R2**, conditional on having no reserve price, the optimal position auction (q^*, p^*) characterized in corollary 2.6 is also optimal among all Bayesian IC and IR mechanisms when signals are independent.

Proposition 2.7. *Under regularity conditions **R1** and **R2**, if signals are independent, then (q^*, p^*) is an optimal position auction among all the Bayesian IC and IR mechanisms subject to no reserve price.*

Proof. See Appendix. □

Since all ex-post IC and IR mechanisms are also Bayesian IC and IR mechanisms, the optimality of (q^*, p^*) under independent signals is stronger.

2.6.2.3 Revenue Comparison

I next show the optimal position auction subject to no reserve price characterized in Corollary 2.7 is equivalent to the Generalized-VCG mechanism proposed by Ausubel (1999) [20], then compare the expected revenue of the Generalized-VCG mechanism to the expected revenues of the GEA, the K-dimensional GSP auction and the K-dimensional VCG auction.

For an arbitrary bidder i, given a vector of opponents' bids x_{-i}, let $\hat{x}^k(x_{-i})$ be the minimum value that bidder i's signal can take such that bidder i has at least

the k-th highest value among all bidders:

$$\hat{x}^k(x_{-i}) = \inf \left\{ x_i \ \Big| \ v_i(x_i, x_{-i}) \geq kmax_{j \neq i}\{v_j(x_j, x_i, x_{-ij})\} \right\} \qquad (2.42)$$

in which $kmax_{j \neq i}\{v_j(x_j, x_i, x_{-ij})\}$ is the k-th highest value received by bidder i's opponents given report x, and x_{-ij} is the vector of signals reported by bidders other than i and j.

Define the Generalized-VCG mechanism in the context of position auctions as follows:

$$q_i^V(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{x}^k(x_{-i}) \leq x_i < \hat{x}^{k-1}(x_{-i}) \\ \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases}$$

$$p_i^V(x_i, x_{-i}) = \begin{cases} \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1})v_i(\hat{x}^j(x_{-i}), x_{-i}) & \text{if} \quad \hat{x}^k(x_{-i}) \leq x_i < \hat{x}^{k-1}(x_{-i}) \\ \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases}$$

$$(2.43)$$

Any tie is broken randomly.

Proposition 2.8 shows that the optimal position auction subject to no reserve price (q^*, p^*) among all ex-post IC and IR mechanisms is equivalent to the Generalized-VCG mechanism when an additional regularity condition described below is satisfied:

R3 (*MR regularity*): For all i, j, given any report x,

$$\text{if} \quad x_i > x_j, \quad \text{then} \quad MR_i(x_i, x_j, x_{-ij}) > MR_j(x_j, x_i, x_{-ij}) \qquad (2.44)$$

Proposition 2.8. *Under regularity conditions **R1**, **R2** and **R3**, the optimal position auction subject to no reserve price among all ex-post IC and IR mechanisms is*

equivalent to the Generalized-VCG mechanism that assigns all positions efficiently. This optimal revenue is weakly higher than the expected revenue of the GEA, which is in turn weakly higher than the expected revenues of the K-dimensional GSP auction and the K-dimensional VCG auction.

Proof. See Appendix. □

The intuition behind the revenue ranking in Proposition 2.8 comes from that in the Generalized-VCG mechanism, the payment of each bidder depends on the entire reported signal profile from opposing bidders, while the payment of each bidder only depends on a subset of opponents' signals in the GEA, and depends on none of opponents' signals in the K-dimensional GSP auction and the K-dimensional VCG auction. Under the logic of the Linkage Principle, when signals are affiliated, the expected revenue of an auction is greater when each bidder's payment depends on more of its opponents' signals. Therefore, the Generalized-VCG mechanism dominates the K-dimensional GSP auction and the K-dimensional VCG auction in both revenue and incentive compatibility, as the latter two are Bayesian incentive compatible but not ex-post incentive compatible.

On the other hand, revenue equivalence holds among the Generalized-VCG mechanism, the GEA, the K-dimensional GSP auction and the K-dimensional VCG auction under independent signals:

Corollary 2.6. *When bidders have independent signals, under regularity conditions* **R1**, **R2** *and* **R3**, *the optimal position auction subject to no reserve price among all Bayesian IC and IR mechanisms is equivalent to the Generalized-VCG mecha-*

nism that assigns all positions efficiently. Moreover, this optimal revenue can be practically implemented by the GEA, the K-dimensional GSP auction, and the K-dimensional VCG auction.

The main insight from Corollary 2.6 is that under independent signals and regularity conditions **R1**, **R2** and **R3**, the three K-dimensional position auctions analyzed in section 2.5 dominates the one-dimensional position auctions analyzed in section 2.4 in both efficiency and revenue.

2.7   Conclusions

Given the performance of GSP auction under complete information analyzed in previous literature, it is an important question to ask whether the desirable properties of GSP auction are preserved under other information structures. This paper shows that efficiency is not preserved in the GSP auction when bidders have interdependent values, which is a more realistic information structure given the oligopolistic competition feature among advertisers bidding for the same keyword and the uncertainty in consumer demand that all advertisers face as a result of continuous demand shocks. This inefficiency result extends Gomes and Sweeney (2014) [14]'s result into a broader range of information structure and provides a sharp contrast to previous studies that favor the GSP auction under complete information, implying that the GSP auction can be a suboptimal mechanism when the information structure deviates from complete information.

In addition to proving inefficiency of the GSP auction, this paper proves that

the VCG auction can also be inefficient under interdependent values and shows that the inefficiency of both GSP auction and VCG auction comes from the one-dimensional bidding language. The one-dimensional bidding language restricts each bidder to submit the same bid for all positions and forces each bidder to balance the net trade-offs between all pairs of adjacent positions on corresponding margins in equilibrium. Since a superior position can be less desirable than an inferior position under some CTR profile given the expected payoffs in the GSP auction, bidders with higher signals have stronger bid-shading incentives than bidders with lower signals. As a result, bidders with higher values may lose to bidders with lower values under this differentiated bid-shading incentive. When bidders' values are non-trivially interdependent, winning a superior position conveys worse news about expected value than winning an inferior position under the generalized "winner's curse," which causes differential bid-shading incentive across bidders and leads to inefficiency in the one-dimensional VCG auction as well. On the other hand, the dynamic Generalized English Auction that implicitly adopts a multi-dimensional bidding language does not have this inefficiency problem, as bidders are able to incorporate the differential bid-shading incentives into their bidding strategies across positions.

The main conclusion of this paper is that when bidders have interdependent values, adopting a multi-dimensional bidding language that allows bidders to bid differently across positions not only improves efficiency, but also improves revenue. This conclusion implies that there exists a trade-off between simplicity versus efficiency and revenue in auction design. Moreover, comparing the equilibrium of

one-dimensional auction to its K-dimensional counterparts shows that the complexity in bidding strategy is reduced under K-dimensional auctions, which implies that there also exists a trade-off between simplicity in auction design and simplicity in bidding strategy. This insight can also be implied from the observation that under both one-dimensional and K-dimensional bidding language, the equilibrium bidding strategy in the VCG auction is much simpler than the equilibrium bidding strategy in the GSP auction, although the VCG payment rule is more complicated compared to the GSP payment rule. These implications provide some insights to the discussion on the cost of simplicity in mechanism design in both economics and computer science literature.

This paper provides some guidance on the design of auctions for allocating sponsored advertising spaces on a wide range of online platforms, including search engines such as Google and Yahoo!, online shopping platforms such as Amazon and eBay, online rating and booking platforms such as Yelp and TripAdvisor, and social media such as Facebook, Twitter, and Instagram. All of these two-sided platforms share the common characteristics that advertisers competing for the same advertising space are likely selling substitutable products or services and therefore are subject to demand shocks in the same market. When interdependency is likely to present in bidders' values, it may worth to use the more complicated, multi-dimensional bidding language in order to guarantee efficiency and improve revenue.

This paper points to two future research directions. First, this paper follows previous literature on position auctions and assumes bidders have single-unit demands. However, bidders may have multi-unit demands in real position auctions.

For example, an advertiser may demand consecutive slots on the first search result page or demand a slot on each of the first three search result pages under a keyword. One natural extension of this paper is to allow bidders to have multi-unit demands and explore how introducing multi-unit demands affects the efficiency and revenue properties of the auctions studied in this paper. Second, it would be interesting to conduct an experimental study to test the theoretical predictions in this paper and quantify the change in efficiency and revenue that results from modifying the bidding language from one-dimensional to K-dimensional in position auctions.

Chapter 3: Auctions with Quantity Externalities and Endogenous Supply

3.1   Introduction

Auctions are used to sell operating permits in many industries, including telecommunication, energy and electricity power. A common characteristic in these industries is that, the total number of licenses allocated in the auction will determine the total number of competitors in the downstream market associated to the auction and therefore enter each bidder's value of obtaining a license. Moreover, each bidder's value of obtaining a license is decreasing in the total number of licenses allocated in the auction. For example, consider auctions for allocating operating permit in a regulated industry where each firm must acquire a license to enter the market. The number of licenses allocated in the auction determines the number of competitors in the downstream market, so each bidder's value of winning a license depends on how many licenses are allocated in total. Selling more licenses will lead to more intensive competition in the downstream market and will reduces the value of winning a license to each bidder. By winning a license, each bidder may impose some negative externalities on other winning bidders. The negative externalities on

other winning bidders comes from the greater quantity of licenses supplied in the auction and therefore is called quantity externalities.

Under the presence of quantity externalities, there exists a trade-off between selling more licenses and preserving bidders' values from winning each license from the perspective of surplus maximization[1] of all participants in auction. Selling all the licenses up to the capacity constraint may not be surplus-maximizing, as the winners' values are decreasing in total supply. Given the presence of this trade-off, how to design an auction to determine both total supply and allocation to maximize producer surplus is an interesting problem for practitioners. In this paper, I analyze a license auction in which each bidder's value of obtaining a license is a function of its own private cost and total number of licenses allocated in the auction. The main objective of this paper is to design an efficient auction that determines both total supply and allocation to maximize producer surplus based on the bidding profile. I also provide a discussion of auction design under other objectives, including maximizing revenue and maximizing a weighted average of consumer surplus and producer surplus.

I first characterize the VCG mechanism that requires each bidder to report

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[1]This paper focuses on maximizing total surplus generated in the auction market, which is equivalent to maximizing producer surplus in the downstream market. Since the auctioneer does not have value over licenses, the term "total surplus" in the auction market is equivalent to "total producer surplus" in the downstream market. I will refer to it as "producer surplus" in the remaining of this chapter. The term "efficiency" refers to maximizing producer surplus in the downstream market. I will include a discussion on auction design when consumer surplus in downstream market is also considered in the end of this chapter.

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private cost directly and shows that a sequence of reserve prices are needed to determine supply endogenously in any surplus-maximizing auction. That is, there is a minimum acceptable bid for every additional unit to be sold in the auction. A k-th unit will not be sold if the price of the k-th unit fails to meet the minimum acceptable bid for that unit. Then I show that uniform-price auctions and ascending clock auctions are inefficient after introducing such reserve price. The inefficiency in both auctions come from that introducing a sequence of reserve prices to endogenously determine supply will differentiate the expected values conditional on winning among bidders, as a high-cost bidder will win a license only when the competition in auction is weak, auction clearing price is low and total supply is low. Therefore, a higher cost bidder will have a higher expected value conditional on winning compared to a lower cost bidder. Moreover, there exists a continuum of bidders whose optimal bids massed at the point of every reserve price, leading to a pooling equilibrium. After showing the inefficiency of uniform-price auctions and ascending clock auctions, I construct a multi-dimensional uniform-price auction that allows bidders to condition their bids on total supply and show this auction can implement the efficient allocation in a dominant strategy equilibrium. Moreover, I construct a Walrasian Clock auction that can dynamically implement the efficient allocation in a dominant strategy equilibrium. I also characterize the revenue-maximizing mechanism and compare the optimal reserve prices to the efficient reserve prices.

The main contribution of this paper is the follows. First, I characterize the feature of efficient auctions and optimal auctions under the presence of quantity externalities. Second, I prove the inefficiency of uniform-price auctions and ascending

clock auctions after introducing such reserve prices and construct two alternative auctions to implement the efficient allocation. Third, I provide a comparison between optimal reserve prices and efficient reserve prices and provide some guidance to the design of practical license auctions that may involve quantity externalities among bidders.

3.2   Related Literature

This paper is related to the strand of literature on auctions with allocative externalities, in which each bidder cares about other bidders' allocation. Jehiel et al. (1996) [48] construct a revenue-maximizing auction mechanism when a sale creates negative externalities on losing bidders, and the magnitude of those externalities depends on the identity of winner. In a subsequent paper, Jehiel et al. (1999) [49] characterize the optimal multi-dimensional mechanism under the setting when each buyer's multi-dimensional type specifies the payoffs for every possible allocation in the auction. Varma (2002) [50] analyzes equilibrium bidding strategy in the open ascending-bid auction with identity-dependent externalities and shows that ascending clock auction yields higher revenue compared to sealed-bid auctions, since bidders can better avoid pay-off reducing externalities in a dynamic auction that reveals more information about the identity of potential winner. This paper considers a different type of allocative externalities that is caused by implementing a different market structure through auction in the post-auction market.

The interplay between license auctions and post-auction market competition

has been extensively studied in the literature. Jehiel and Moldovanu (2000a) [51] derive equilibria for a second-price auction in which the payoff to each losing bidder is a function of the winner's type and its own type and point out various effect caused by both positive and negative externalities. Moldovanu and Sela (2003) [52] study an auction for allocating a cost-reducing patent, in which each firm's value of obtaining the patent depends on other firms' pre-auction production costs. They show that standard auctions, including first-price auctions and second-price auctions, lead to inefficient allocation when firms do not know each other's production cost at the time of bidding. Zhong (2005) [53] analyzes a license auction in which potential firms first compete for one license then the license winner competes with one incumbent in the market. He identifies the impact of disclosing the winning bid to the incumbent after auction on both Cournot and Bertrand markets and shows that price disclosure will increase revenue in Cournot market while decrease revenue in Bertrand markets. Georee (2003) [54] studies bidders' incentive of signaling through bidding for gaining advantage in post-auction competition in different auction formats. He shows that the equilibrium bidding functions are biased upwards in second-price auctions as bidders wish to exaggerate their competitiveness in the downstream market, while this signaling phenomenon is less prominent in first-price auctions and English auctions as the winner incurs the cost of her signaling choice. All of these papers assume the total quantity of license is fixed to be one, so the impact of auction on post-auction market comes from the dependency of post-auction market outcome on the type of winning bidder. This paper complements these papers by analyzing a license auction with post-auction market competition, where

the impact of auction on post-auction comes from the dependency of post-auction market structure on the quantity of licenses allocated in the auction.

The presence of quantity externalities in auction markets has also been discussed in several previous papers. Katz and Shapiro (1986) [55] characterize an upstream research lab's optimal pricing strategy when selling licenses to downstream oligopolistic firms when each firm's willingness to pay for a license depends only upon how many of its rivals are obtaining licenses. They analyze a class of licensing mechanisms in which the licensor announces it will sell no more than some fixed number of licenses, each subject to a minimum bid, and characterize the optimal licensing strategy among this class of mechanisms. In Katz and Shapiro (1986) [55], firms are identical and do not have private information, so each firm's value of a license depends only on the quantity supplied in the auction. Jehiel and Moldovanu (2000b) [56] analyze the interplay between license auctions and market structure in a model with multiple incumbents and multiple potential entrants. In their model, each firm's value of a license depends on the number of incumbents, the number of new entrants, and whether the firm is an incumbent or an entrant. They focus on how auction format affects the incumbents' incentives to preempt entry by bidding for new licenses and show that the relation between number of available licenses and the number of incumbents plays a major role. Rodriguez (1997) [57] studies entry preemption in sequential license auctions and also shows that entry preemption in equilibrium depends critically on the number of incumbents. This paper is closely related to these studies by also analyzing a license auction where each firm cares about how many licenses are allocated in the auction, while extending the afore-

mentioned studies by allowing each firm's value of getting an license depends not only on the quantity supplied in the auction, but also on the firm's private production costs. It will be shown that the optimal mechanism uses a sequence of reserve prices to determine supply endogenously, which falls within the class of licensing mechanisms discussed in Katz and Shapiro (1986) [55]. Gebhardt and Wambach (2008) [58] also considers a license auction in which each winner's payoff depends on the total quantity supplied and the bidder's private cost. They propose a jumping English auction that maximizes the sum of consumer surplus and producer surplus by choosing both supply and allocation within the auction. This paper considers a similar model with the objective of maximizing producer surplus and constructs a Walrasian clock auction to implement the efficient allocation. Ranger (2004) [59] studies a capacity auction that allocates capacity constraints to bidders who compete in a Cournot game in a downstream market. He constructs a modified version of Ausubel and Milgrom (2002) [60]'s generalized ascending proxy auction that allows bidders to bid over entire allocations and shows this auction can implement the efficient allocation. This paper is also closely related to Ranger (2004) [59] but considers another type of quantity externalities that comes from the market structure instead of capacity constraints imposed on each oligopolistic firm.

Finally, this paper is related to the literature on auctions with endogenous supply. Hansen (1988) [61] studies a procurement auction in which sellers competing to sell to a market with negatively sloped demand curve, and the total quantity to be supplied in the procured contract depends on the final price in the auction. They show that an open auction yields higher revenue than a sealed-bid auction. Ozcan

(2004) [62] models a two-stage sequential auction that mimics the license auction for the Turkish Global Mobile Telecommunications in 2000: the first license is sold through a standard first-price auction, then the auctioneer uses the price of the first license to be the reserve price for the second license in a subsequent auction. The main result in Ozcan (2004) [62] shows that this auction yields less revenue than a sealed-bid second-price auction for selling a monopoly license. Lengwiler (1999) [63] analyzes a multi-unit auction when the auctioneer can produce arbitrary quantities at constant unit cost and can adjust supply as a function of bidding. He shows that both pay-as-bid auctions and uniform-price auctions are inefficient. Izmalkov et al. (2016) [35] considers a position auction model in which the click-through rate of each advertiser depends on both the ranking of advertisement and the total number of advertisements placed on the website. They constructed both the efficient auction and the optimal auction as direct revelation mechanisms and shows that supply should be determined endogenously in both auctions. This paper is closely related to this strand of literature and incorporate the quantity externalities within bidders' values of obtaining a license. I construct the efficient and optimal auction as both direct and indirect revelation mechanisms and characterize the corresponding efficient and optimal reserve prices that should be used to determine supply endogenously.

3.3   Model

An auctioneer wishes to sell up to K identical licenses through auction. There is a set of N firms, each demanding one unit of the licenses to enter a regulated

market. There are two periods. In the first period, Firms bid for licenses in an auction. In the second period, winning firms operate in some downstream market of an indivisible good. Each firm's value of obtaining a license in the auction depends on its expected profit in the post-auction downstream market. Suppose all firms have identical capacity constraint and each firm with a license will produce 1 unit of an indivisible good in the downstream market. The market inverse demand schedule is $P(n)$, in which n is the total supply that equals to the number of winning firms in the auction. For simplicity, assume the demand schedule $P(n)$ in this discrete model has similar property to a linear demand curve: $P(n) - P(n+1) = \delta > 0$ for all n[2]. Therefore, $P(1) > P(2) > P(3) > \cdots > P(K)$.

The firms are indexed by $i \in \{1, 2, \cdots, N\}$. Each firm has a private cost c_i of producing the indivisible good in the downstream market. The firms' costs are independently and identically distributed over $[\underline{c}, \bar{c}]^N$ according to distribution function $F(c)$, with density $f(c)$. Therefore, each firm's value of obtaining a license given n licenses are sold in the auction is given by

$$\pi(c_i, n) = P(n) - c_i \qquad (3.1)$$

The auctioneer can freely choose to sell any number of licenses up to K licenses. Since $P(n)$ is lower when there are more firms winning a license, each firm's value of obtaining a license is endogenously determined by the number of licenses allocated in the auction. For simplicity, assume $P(K) \geq \bar{c}$, so that every bidder has positive

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[2]The results of this paper will not change significantly if I relax this assumption and only assume $n[P(n) - P(n+1)]$ is increasing in n. For notational simplicity, I assume $P(n) - P(n+1) = \delta$ for all n.

value of winning one license when no more than K licenses are allocated in the auction.

It is well-known that the uniform-price auction implements the efficient allocation in multi-unit auctions with fixed supply and single-unit demands. However, how to design an auction to implement the efficient allocation that maximizes producer surplus in the downstream market is a non-trivial problem, as both the efficient number of licenses to allocate in the auction and the associated set of winning bidders depend on bidders' private costs $(c_1, c_2, \cdots, c_N)$. Given any vector of bidders' costs $c \equiv (c_1, c_2, \cdots, c_N)$, let $c_{(n)}$ denote the n-th lowest value among c_i and break ties evenly, then $c_{(1)} \leq c_{(2)} \leq c_{(3)} \leq \cdots \leq c_{(N)}$. The producer surplus determined in the auction by selling n licenses to a set of $S_n \subset \{1, 2, \cdots, N\}$ bidders is given by

$$PS(c, n, S_n) = n \times P(n) - \sum_{i \in S_n} c_i \tag{3.2}$$

Given any level of total supply n, the producer surplus of selling n licenses is maximized by selling to the n lowest-cost bidders, and the maximized total surplus of selling n licenses is given by

$$PS^*(c, n) = n \times P(n) - \sum_{i=1}^{n} c_{(i)} \tag{3.3}$$

The producer surplus is maximized when the total supply in the auction $n^* = \text{argmax}_n TS^*(c, n)$. To find the efficient supply level, observe that given a vector of costs c, the marginal benefit contributed to the producer surplus by selling the n-th license is given by

$$MB(c, n) = P(n) - c_{(n)} \tag{3.4}$$

since the bidder with the n-th lowest type would not have won a license when the total supply is $(n-1)$. On the other hand, selling one more license reduces the value of winning for the $(n-1)$ lowest-cost bidders who would still have won a license at the total supply of $(n-1)$, which implies that the marginal cost of selling the n-th license is given by

$$MC(n) = (n-1)[P(n-1) - P(n)] = (n-1)\delta \qquad (3.5)$$

The n-th license should be sold in an efficient auction if and only if $MB(c, n) > MC(n)$, which depends on the realization of $c_{(n)}$. Since

$$MB(c, n) > MB(c, n+1) \quad \text{for all} \quad n$$
$$\qquad (3.6)$$
$$MC(n) < MC(n+1) \quad \text{for all} \quad n$$

The maximized producer surplus $PS^*(c, n)$ with supply n is maximized at $n = n^*$ such that

$$MB(c, n^*) \geq MC(n^*)$$
$$\qquad (3.7)$$
$$MB(c, n^* + 1) < MC(n^* + 1)$$

Definition 3.1. *A license auction is efficient if for any realization of bidders' costs $c \equiv (c_1, c_2, \cdots, c_N)$, it always allocate to the lowest-cost n^* bidders, in which*

$$n^* = \max\{n | MB(c, n) \geq MC(n), n \leq K\} \qquad (3.8)$$

To make the problem more interesting, assume that for all $n \geq 2$, there exists an interval $(c_n^e, \bar{c}]$ such that

$$P(n) - c_i < (n-1)\delta \quad \text{for all} \quad c_i \in (c_n^e, \bar{c}] \qquad (3.9)$$

That is, for all possible supply level $n \geq 2$, there exists a continuum of types $(c_n^e, \bar{c}]$ such that if $c_{(n)}$ falls into this interval, it is not efficient to allocate the n-th license. $c_n^e = P(n) - (n-1)\delta$ is the maximum acceptable value that the n-th lowest cost can take for n licenses to be allocated. I call each c_n^e the "efficient reserve cost" for the n-th unit.

Corollary 3.1. *The efficient reserve costs $(c_1^e, c_2^e, \cdots, c_K^e)$ in which c_n^e specifies the highest possible value that the n-th lowest bidder's cost $c_{(n)}$ can take for an n-th license to be sold in an efficient auction are characterized below:*

$$c_1^e = \bar{c}$$
$$c_n^e = P(n) - (n-1)\delta \quad \text{for all} \quad n \in \{2, \cdots, K\}$$

(3.10)

3.4   An Efficient Direct Mechanism

In this section, I show that an efficient direct revelation mechanism exists in this environment: the VCG mechanism implements the efficient allocation in a dominant strategy equilibrium. Since each bidder's value of obtaining a license depends only on its own private cost but not on other bidders' costs, the VCG mechanism can be directly applied. The VCG mechanism is defined as follows:

Definition 3.2. *In a VCG mechanism, each bidder is asked to report his private cost c_i. Given any profile of reported costs $\hat{c}$, rank the reports in an ascending order: $\hat{c}_{(1)} \leq \hat{c}_{(2)} \leq \cdots \leq \hat{c}_{(N)}$, then allocate licenses according to the following algorithm:*

(R1) Allocate one license to the bidder with the lowest reported cost $\hat{c}_{(1)}$ and continue to (R2).

(R2) *Allocate one license to the bidder with the second lowest reported cost $\hat{c}_{(2)}$ and continue to (R3) if $P(2) - \hat{c}_{(2)} \geq \delta$. Restrict total supply to one license otherwise.*

(R3) *Allocate one license to the bidder with the third lowest reported cost $\hat{c}_{(3)}$ and continue to (R4) if $P(3) - \hat{c}_{(3)} \geq 2\delta$. Restrict total supply to two licenses otherwise.*

. . .

(RK) *Allocate one license to the bidder with the K-th lowest reported cost $\hat{c}_{(K)}$ if $P(K) - \hat{c}_{(K)} \geq (K - 1)\delta$. Restrict total supply to $(K - 1)$ licenses otherwise.*

For any total supply n, the payment of each bidder who wins one out of n licenses is defined below:

$$p^n(\hat{c}) = \max\left\{P(n) - \hat{c}_{(n+1)}, (n - 1)\delta\right\}$$

The payment rule in the VCG mechanism for any total supply $n > 1$ depends on what is the social opportunity cost of providing one out of n licenses to bidder i whose reported cost ranks in the lowest n reported costs among all bidders. The fact that n is the efficient supply given report $\hat{c}$ implies

$$P(n) - \hat{c}_{(n)} \geq (n - 1)\delta,$$

$$P(n - 1) - \hat{c}_{(n)} > (n - 2)\delta, \qquad (3.11)$$

$$P(n - 1) - \hat{c}_{(n-1)} > (n - 2)\delta$$

Therefore, if any of the n lowest reported cost bidders is absent in the auction, it is still efficient to sell at least $(n - 1)$ licenses, as the marginal benefit of selling the $(n-1)$-th license after removing any one of the lowest n cost bidders will still exceed

the marginal cost. On the other hand,

$$P(n+1) - \hat{c}_{(n+1)} < n\delta,$$

$$P(n+1) - \hat{c}_{(n+2)} < n\delta \tag{3.12}$$

Therefore, if any of the n lowest reported cost bidders is absent in the auction, it is still efficient to sell no more than n licenses, since the marginal benefit of selling the $(n+1)$-th license after removing any one of the n lowest cost bidders will still be lower than the marginal cost.

In summary, it is never efficient to allocate fewer than $(n-1)$ licenses or more than n licenses if one of the winning bidders does not participate in the mechanism. Whether to allocate n licenses or $(n-1)$ licenses when one of the winning bidders is absent depends on the lowest reported cost among losing bidders $\hat{c}_{n+1}$.

If $P(n) - \hat{c}_{(n+1)} \geq (n-1)\delta$, then it is still efficient to sell n licenses when any of the n lowest cost winning bidders is absent, and the total externalities a winning bidder imposes on its opponents is the value of winning one out of n licenses by the lowest cost rejected bidder.

If $P(n) - \hat{c}_{(n+1)} < (n-1)\delta$, then it is efficient to restrict supply to $(n-1)$ licenses when any of the n lowest cost winning bidders is absent, and the total externalities a winning bidder imposes on its opponents is the change in values of obtaining a license by all the other $(n-1)$ winners when one fewer license is sold.

Corollary 3.2. *Truth-telling is a dominant strategy equilibrium in the VCG mechanism.*

Proof. See Appendix. $\square$

3.5 Practical Implementation of Efficient Auctions

The VCG mechanism implies that in any indirect mechanism where bidders submit bids rather than report costs directly to the auctioneer, the efficient level of supply should be determined based on the bidding profile. A natural instrument to implement this goal is to use a sequence of reserve prices to endogenously determine the desired level of supply: For every possible level of supply $n = 1, 2, \cdots, K$, set a reserve price $r_n \geq 0$ such that the n-th license will be allocated if and only if the n-th highest bid is no lower than r_n.

In any efficient auction, since the seller has no value over licenses and the marginal cost of selling the first license is zero, the reserve price of the first unit r_1 should always be set to zero. On the other hand, since the marginal cost of selling an additional license is strictly positive and strictly increasing in the number of licenses already sold, the reserve price of each additional unit must be positive and strictly higher than that of the previous unit, i.e., $0 < r_n < r_{n+1}$ for all $n \in \{2, 3, \cdots, K\}$. According to the assumption in section 3.3, since for every possible supply level $n \in \{2, 3, \cdots, K\}$, there exists a continnum of costs $(c_n^e, \bar{c}]$ such that it is not efficient to allocate the n-th license when $c_{(n)} \in (c_n^e, \bar{c}]$, we must have r_n binding for all types in $(c_n^e, \bar{c}]$, for all $n \in \{2, 3, \cdots, K\}$.

Lemma 3.1. *In any efficient auction that adopts a sequence of reserve prices* $(r_1, r_2, \cdots, r_K)$ *where* r_n *specifies the minimum acceptable bid for the n-th license to*

be allocated, the reserve prices must satisfy

$$0 = r_1 < r_2 < r_3 < \cdots < r_K \tag{3.13}$$

and $(r_2, r_3, \cdots, r_K)$ are binding for a continuum of types.

In this section, I analyze the equilibria of uniform-price auctions and ascending clock auctions after introducing a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ that specifies the minimum acceptable bid for every additional unit. I prove that inefficiency occurs in both auctions under any reserve prices that satisfy $r_1 < r_2 < \cdots < r_K$, in which at least one r_n is binding for some types. Furthermore, the nature of inefficiency is similar across uniform-price and ascending clock auctions: pooling occurs for a positive measure of types among bidders. This inefficiency result implies that equilibrium bidding strategies change dramatically in multi-unit auctions and ascending clock auctions with reserve prices when bidders care about the total quantity of items allocated in the auction. Introducing an effective reserve price for every additional unit to be sold in the auction will distort efficiency in both uniform-price auctions and ascending clock auctions.

I next construct two alternative auction designs to maximize producer surplus with efficient reserve prices that corresponds to the efficient reserve costs characterized in Corollary 3.1, including a multi-dimensional uniform-price auction that allows bidders to condition their bids on total supply in the auction, and a Walrasian Clock auction that adjusts the clock price based on whether there is excess supply in the auction at each efficient reserve price. I will show that both auctions can implement the efficient allocation in a dominant strategy equilibrium.

3.5.1 Uniform-Price Auction with Endogenous Supply

Consider a uniform-price auction with endogenous supply up to K licenses defined below:

Definition 3.3. *In a uniform-price auction with endogenous supply, the auctioneer announces a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ at the beginning of auction, where $r_1 < r_2 < r_3 < \cdots < r_K$. For any feasible level of supply $n \in \{1, 2, \cdots, K\}$, r_n denotes the minimum acceptable bid for the n-th unit: an n-th license is allocated only if the n-th highest bid is no lower than r_n. After observing the reserve prices, all bidders submit sealed bids simultaneously. The auctioneer ranks all bids from top to bottom. Let $b_{(n)}$ denote the n-th highest bid in the bidding profile and break ties randomly, then $b_{(1)} \geq b_{(2)} \geq \cdots \geq b_{(N)}$. The licenses are allocated according to the following algorithm:*

(R1) Allocate a license to the bidder who submits the highest bid $b_{(1)}$ and proceed to step (R2).

(R2) If $b_{(2)} \geq r_2$, allocate a second license to the bidder who submits the second highest bid $b_{(2)}$ and proceed to step (R3). If $b_{(2)} < r_2$, restrict supply to one license and charge the winner $\max\{r_1, b_{(2)}\}$.

. . .

(Rn) If $b_{(n)} \geq r_n$, allocate an n-th license to the bidder who submits the n-th highest bid $b_{(n)}$ and proceed to step (Rn+1). If $b_{(n)} < r_n$, restrict supply to $(n-1)$ licenses and charge all winners a uniform price of $\max\{r_{n-1}, b_{(n)}\}$.

. . .

(RK) *If $b_{(K)} \geq r_K$, allocate a K-th license to the bidder who submits the K-th highest bid $b_{(K)}$. If $b_{(K)} < r_K$, restrict supply to $(K-1)$ licenses and charge all winners a uniform price of $\max\{r_{K-1}, b_{(K)}\}$.*

I will next show that the uniform-price auction is inefficient after introducing any supply-determining reserve prices $(r_1, r_2, \cdots, r_K)$ that satisfies $r_1 < r_2 < \cdots < r_K$, and at least one r_n is binding for a continuum of types. Note that it is sufficient to prove inefficiency occurs when the total supply is at most 2 licenses. The next lemma shows that when the total supply is at most $K = 2$ licenses, there exists no symmetric separating monotonic equilibrium in the uniform-price auction with endogenous supply.

Lemma 3.2. *In a uniform-price auction with endogenous supply of up to $K = 2$ licenses and reserve prices (r_1, r_2) where $r_1 = 0, r_2 > 0$, and $r_2 > P(2) - \bar{c}$, there exists no symmetric separating monotonic equilibrium.*

Proof. See Appendix. □

The result of Lemma 3.2 comes from the fact that bidders with different costs have different expected value conditional on winning. With $r_2 > 0$, any bid $b_i < r_2$ only affects the probability of winning when the total supply is 1 and does not affect the probability of winning when the total supply is 2. Suppose a monotonic separating equilibrium exists. Conditional on the total supply being restricted to 1, it is optimal to bid the true value of obtaining a license $P(1) - c_i$ when $c_i > P(1) - r_2$. On the other hand, any bid $b_i \geq r_2$ only affects the probability of winning when the total supply is 2 and does not affect the probability of winning when total supply

is 1, as that event happens only when the lowest-cost opponent bids below r_2 and does not depend on bidder i's bid. Conditional on the total supply being equal to 2, it is optimal to bid the true value of obtaining a license $P(2) - c_i$ for a bidder with $c_i \leq P(2) - r_2$. Since $P(2) - r_2 < \min\left\{P(1) - r_2, \bar{c}\right\}$, no separating equilibrium bidding strategy for bidders with costs $c_i \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\}\right]$ can make $\beta(c_i)$ satisfy monotonicity condition, which contradicts the assumption that a monotonic separating equilibrium exists.

The intuition of this result comes from that high-cost bidders will win a license only if total supply is 1 and therefore bid more aggressively than low-cost bidders who are more likely to win a license but must take into account that winning when total supply being equals to 2 is also possible.

The next lemma examines the equilibrium bidding strategy for bidders with costs $c_i \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\}\right]$ and shows that all but two possible bids are dominated for these bidders. Therefore, pooling must occur for these bidders in any symmetric pure strategy equilibrium.

Lemma 3.3. *In a uniform-price auction with total supply $K \leq 2$ and reserve prices (r_1, r_2) where $r_1 = 0, r_2 > 0$ and $r_2 > P(2) - \bar{c}$, for bidders with costs $c_i \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\}\right]$, any bid $b_i > r_2$ and $b_i < r_2 - \epsilon$ for some arbitrarily small ϵ are dominated. Pooling must occur in any symmetric pure strategy equilibrium.*

Proof. See Appendix. □

The result of Lemma 3.3 comes from that for bidders with costs $c_i \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\}\right]$, winning is desirable only if the total supply is 1, while winning

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when total supply is 2 yields a negative payoff. Since increasing bid when $b_i \geq r_2$ only increases probability of winning when total supply is 2 and does not affect probability of winning when total supply is 1, bidding strictly higher than r_2 is dominated by bidding r_2. Since decreasing bid when $b_i < r_2$ only decreases probability of winning when total supply is 1 and does not affect probability of winning when total supply is 2, bidding strictly lower than $r_2 - \epsilon$ is dominated by bidding $r_2 - \epsilon$, in which ϵ is arbitrarily small.

The next lemma builds upon the results of Lemma 3.2 and Lemma 3.3 and shows that there exists some $\hat{c} \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\}\right]$ such that all bidders with costs $c_i \in \left(P(2) - r_2, \hat{c}\right]$ bid r_2 and all bidders with costs $c_i \in \left[\hat{c}, \min\{P(1) - r_2, \bar{c}\}\right]$ in equilibrium.

Lemma 3.4. *In a uniform-price auction with endogenous supply of up to $K = 2$ licenses and reserve prices (r_1, r_2) where $r_1 = 0, r_2 > 0$ and $r_2 > P(2) - \bar{c}$, pooling occurs for bidders with costs $c_i \in \left[P(2) - r_2, \min\{P(1) - r_2, \bar{c}\}\right]$. There exists some type $\hat{c} \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\}\right]$ s.t. a bidder with cost $\hat{c}$ is indifferent between bidding r_2 and $r_2 - \epsilon$ for some arbitrarily small ϵ. The Bayesian equilibrium bidding strategy is characterized below:*

When $r_2 > P(1) - \bar{c}$:

$$\beta(c_i) = \begin{cases} P(1) - c_i & for \quad c_i \in \left(P(1) - r_2, \bar{c}\right] \\[2ex] r_2 - \epsilon & for \quad c_i \in \left[\hat{c}, P(1) - r_2\right] \\[2ex] r_2 & for \quad c_i \in \left[P(2) - r_2, \hat{c}\right] \\[2ex] P(2) - c_i & for \quad c_i \in \left[\underline{c}, P(2) - r_2\right] \end{cases} \tag{3.14}$$

When $P(1) - \bar{c} > r_2 > P(2) - \bar{c}$:

$$\beta(c_i) = \begin{cases} r_2 - \epsilon & for \quad c_i \in \left[\hat{c}, \bar{c}\right] \\[2ex] r_2 & for \quad c_i \in \left[P(2) - r_2, \hat{c}\right] \\[2ex] P(2) - c_i & for \quad c_i \in \left[\underline{c}, P(2) - r_2\right] \end{cases} \tag{3.15}$$

Proof. See Appendix. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following proposition generalizes the results from Lemma 3.2 to Lemma 3.4 into uniform-price auctions with endogenous supply of at most K licenses for any integer $K \geq 2$.

Proposition 3.1. *In a uniform-price auction with endogenous supply of up to K licenses and reserve prices $(r_1, r_2, \cdots, r_K)$ where $r_1 < r_2 < r_3 < \cdots < r_K$, if $r_{\tilde{n}} > P(\tilde{n}) - \bar{c}$ for some $\tilde{n}$, then pooling occurs for bidders with costs $c_i \in \left[P(n) - r_n, P(n-1) - r_n\right]$ for all $n \in \{\tilde{n}, \tilde{n}+1, \cdots, K\}$[3].*

Suppose $\tilde{n} = 2$, then the unique symmetric Bayesian pure strategy equilibrium bidding strategy in a uniform-price auction with endogenous supply of up to K

---

[3]If $\tilde{n} = 1$ or 2, then pooling occurs for bidders with $c_i \in \left[P(2) - r_2, \min\{P(1) - r_1, \bar{c}\}\right]$ and $c_i \in \left[P(n) - r_n, P(n-1) - r_n\right]$ for all $n \in \{3, 4, \cdots, K\}$

licenses is characterized below:

When $r_2 > P(1) - \bar{c}$:

$$\beta(c_i) = \begin{cases} P(1) - c_i & for \quad c_i \in \left(P(1) - r_2, \bar{c}\right] \\[2ex] r_2 - \epsilon & for \quad c_i \in \left[\hat{c}_2, P(1) - r_2\right] \\[2ex] r_2 & for \quad c_i \in \left[P(2) - r_2, \hat{c}_2\right] \\[2ex] P(2) - c_i & for \quad c_i \in \left(P(2) - r_3, P(2) - r_2\right] \\[2ex] r_3 - \epsilon & for \quad c_i \in \left[\hat{c}_3, P(2) - r_3\right] \\[2ex] r_3 & for \quad c_i \in \left[P(3) - r_3, \hat{c}_3\right] \\[2ex] P(3) - c_i & for \quad c_i \in \left(P(3) - r_4, P(3) - r_3\right] \\[2ex] \dots & \\[2ex] P(K) - c_i & for \quad c_i \in \left[\underline{c}, P(K) - r_K\right] \end{cases} \qquad (3.16)$$

When $P(1) - \bar{c} > r_2 > P(2) - \bar{c}$:

$$\beta(c_i) = \begin{cases} r_2 - \epsilon & for \quad c_i \in \left[\hat{c}_2, \bar{c}\right] \\[2ex] r_2 & for \quad c_i \in \left[P(2) - r_2, \hat{c}_2\right] \\[2ex] P(2) - c_i & for \quad c_i \in \left(P(2) - r_3, P(2) - r_2\right] \\[2ex] r_3 - \epsilon & for \quad c_i \in \left[\hat{c}_3, P(2) - r_3\right] \\[2ex] r_3 & for \quad c_i \in \left[P(3) - r_3, \hat{c}_3\right] \\[2ex] P(3) - c_i & for \quad c_i \in \left(P(3) - r_4, P(3) - r_3\right] \\[2ex] \dots & \\[2ex] P(K) - c_i & for \quad c_i \in \left[\underline{c}, P(K) - r_K\right] \end{cases} \tag{3.17}$$

where $\hat{c}_n$ is the type that is indifferent between bidding r_n and $r_n - \epsilon$.

Therefore, pooling occurs for a positive measure of types in the unique symmetric pure strategy Bayesian equilibrium given any reserve prices that satisfy $r_1 < r_2 < \cdots < r_K$ and at least one r_n is binding for a continnum of bidders. According to Lemma 3.1, any efficient reserve prices must satisfy $0 = r_1 < r_2 < \cdots < r_K$ and at least one r_n binding for some bidders, which implies that a uniform-price auction with endogenous supply is always inefficient under any reserve prices.

Corollary 3.3. *A uniform-price auction with endogenous supply of up to K licenses is inefficient given any reserve prices $(r_1, r_2, \cdots, r_K)$, for any possible supply level $K \geq 2$.*

3.5.2 Ascending Clock Auction with Endogenous Supply

In this section, I analyze the ascending clock auction with a sequence of posted reserve prices that endogenously determine a total supply of at most K licenses and show that this auction also fails to implement efficiency with any reserve prices that satisfy $r_1 < r_2 < \cdots < r_K$ and at least one r_n is binding for some bidders.

Consider an ascending clock auction defined as follows:

Definition 3.4. *In an ascending clock auction with endogenous supply, there is a continuously ascending clock showing the current price. All bidders are in the auction at beginning. A bidder can drop out at any time, and his bid is the clock price when he drops out. A sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ where $0 = r_1 < r_2 < \cdots r_K$ is announced before auction starts, in which each r_n represents the minimum acceptable bid for the n-th unit to be allocated.*

At any time in the auction, let k denote the number of bidders who are still active in the auction, and $(N - k)$ denote the number of bidders who have dropped out. Let $(p_N, p_{N-1}, \cdots, p_{k+1})$ denote the drop-out prices of the $(N - k)$ bidders, in which p_N is the bid of the first drop-out bidder, and p_{k+1} is the bid of the last drop-out bidder at that time, so $p_N \leq p_{N-1} \leq \cdots \leq p_{k+1}$.

(RK): Set the total supply equals to the maximum possible supply K at the beginning. Keep the total supply to be K until the $(N - K + 1)$-th bidder drops out at price p_K, which leaves $k = (K - 1)$ bidders in the auction.

- *If $p_K \geq r_K$, then stop the auction and sell one license to each of the K bidders, including the bidder who drops out at p_K. Charge all bidders a uniform price of*

$\max\{r_K, p_{K+1}\}$, *in which p_{K+1} is the drop-out price of the $(N-K)$-th drop-out bidder.*

- *If $p_K < r_K$, then proceed to (RK-1) and decrease the total supply to $K-1$.*

(RK-1): *Set total supply to be $K-1$ and keep increasing clock price. Keep the total supply to be $K-1$ until the $(N-K+2)$-th bidder drops out at some higher price $p_{K-1} > p_K$, which reduces the number of active bidders from $(K-1)$ to $(K-2)$ in the auction.*

- *If $p_{K-1} \geq r_{K-1}$, then stop the auction and sell one license to each of the $K-1$ bidders, including the bidder who drops out at p_{K-1}. Charge all bidders a uniform price of $\max\{r_{K-1}, p_K\}$.*

- *If $p_{K-1} < r_{K-1}$, then proceed to (RK-2) and decrease the total supply to $K-2$.*

Repeat this algorithm and keep decreasing supply until reaching some integer n such that $p_n \geq r_n$ at the time when the number of active bidders k drops from n to $(n-1)$. Then allocate a license to each of the n bidders who are still in the auction at the price of $p_n - \epsilon$ and charge all bidders a uniform price of $\max\{r_n, p_{n+1}\}$. If $p_2 < r_2$ at the time when the number of active bidders drops from 2 to 1 in the auction, allocate 1 license to the only active bidder and charge a price of p_2.

The following proposition shows that the ascending clock auction with endogenous supply is also inefficient.

Proposition 3.2. *Let $p^k(c_i|p)$ denote the optimal drop-out price for bidder i when there are k bidders left in the auction and the current price is p. Suppose $r_2 >$*

$P(2) - \bar{c}$. *A symmetric pure strategy Bayesian equilibrium in the ascending auction with endogenous supply is given below:*

When the current clock price $p \in [0, r_2)$,

- *If $P(1) - r_2 < \bar{c}$, then*

$$
p^k(c_i | p < r_2) = \begin{cases} P(1) - c_i & for \quad c_i \in \left(P(1) - r_2, \bar{c}\right] \cup \left[\underline{c}, P(2) - r_2\right] \\ r_2 - \epsilon & for \quad c_i \in \left[\hat{c}_2^k, P(1) - r_2\right] \\ r_2 & for \quad c_i \in \left[P(2) - r_2, \hat{c}_2^k\right] \end{cases}
$$

(3.18)

in which $\hat{c}_2^k$ is the value of c_i such that a bidder with cost $\hat{c}_2^k$ is indifferent between dropping out at $r_2 - \epsilon$ and r_2 given that there are $k \geq 2$ bidders left in the auction and current clock price $p < r_2$.

- *If $P(1) - r_2 \geq \bar{c}$, then*

$$
p^k(c_i | p < r_2) = \begin{cases} P(1) - c_i & for \quad c_i \in \left[\underline{c}, P(2) - r_2\right] \\ r_2 - \epsilon & for \quad c_i \in \left[\hat{c}_2^k, \bar{c}\right] \\ r_2 & for \quad c_i \in \left[P(2) - r_2, \hat{c}_2^k\right] \end{cases}
$$

(3.19)

For all $n \in \{2, 3, \cdots, K - 1\}$, when the current clock price $p \in [r_n, r_{n+1})$,

$$
p^k(c_i | r_n \leq p < r_{n+1}) = \begin{cases} P(n) - c_i & for \quad c_i \in \left[P(n) - r_{n+1}, P(n) - r_n\right] \cup \left[\underline{c}, P(n+1) - r_{n+1}\right] \\ r_{n+1} - \epsilon & for \quad c_i \in \left[\hat{c}_{n+1}^k, P(n) - r_{n+1}\right] \\ r_{n+1} & for \quad c_i \in \left[P(n+1) - r_{n+1}, \hat{c}_{n+1}^k\right] \end{cases}
$$

(3.20)

in which $\hat{c}^k_{n+1}$ is the value of c_i at which a bidder is indifferent between dropping out at $r_{n+1} - \epsilon$ and r_{n+1} given that there are k bidders left in the auction under current clock price $p \in [r_n, r_{n+1})$.

When the current clock price $p \geq r_K$,

$$p^k(c_i|p \geq r_K) = P(K) - c_i \quad for \quad c_i \in [\underline{c}, P(K) - r_k] \qquad (3.21)$$

Proof. See Appendix. □

Proposition 3.2 implies that the ascending clock auction is inefficient given any sequence of reserve prices $r_1 < r_2 < r_3 < \cdots < r_K$ where at least one r_n is binding for some bidders. Therefore, it is impossible to implement efficiency using an ascending clock auction. The pooling equilibrium implies that there is positive probability that more than n bidders left in the auction when $p < r_n - \epsilon$, and all remaining bidders drop out simultaneously when the clock price reaches $p = r_n - \epsilon$ for every possible level of supply n where r_n is binding.

Corollary 3.4. *The ascending clock auction with endogeneous supply of up to K licenses is inefficient given any sequence of reserve prices $(r_1, r_2, \cdots, r_K)$, for any potential supply level $K \geq 2$.*

The inefficiency of both uniform-price auctions and ascending clock auctions comes from that simply introducing a sequence of reserve prices to determine total supply makes the total supply depends on the market clearing price in the auction: the total supply is greater when the market clearing price is higher. Since bidders' values of winning a license also depend on the total supply, each bidder's bidding

strategy will be determined by his expectation of total supply at the market clearing price conditional on winning a license at that price.

In the uniform-price auction, bidders with high costs bid more aggressively, since they will win a license only when their opponents also have high costs so that the total supply is low and value of winning a license is high. On the other hand, bidders with low costs bid less aggressively, since they need to consider the situation of winning when their opponents also have low costs so that the total supply is high and value of winning a license is low. For example, a bidder with cost $c_i > P(1) - r_1$ can win only if all opponents have costs above c_i and the total supply is 1, while a bidder with costs close to $\underline{c}$ may win a license under any possible level of total supply $n \in \{1, 2, \cdots, K\}$, so the expected value conditional on winning is much lower for bidders with low costs compared to bidders with high costs. Furthermore, for each threshold price r_n at which the total supply increases from $(n-1)$ to n if the market clearing price reaches r_n, there exists a positive measure of costs s.t. bidders with costs falls in this range will gain a positive payoff from winning when total supply is $(n-1)$ and gain a negative payoff from winning when total supply is n. Pooling is the only possible equilibrium for these bidders.

In the ascending clock auction, bidders' optimal drop-out strategy depends on the current clock price. For all $n \in \{2, 3, \cdots, K\}$, the total supply increases from $(n-1)$ to n and each bidder's value of winning drops from $P(n-1) - c_i$ to $P(n) - c_i$ every time the clock price reaches a threshold level r_n. At any clock price, each bidder goes through a mental calculation of thinking about whether winning yields a positive payoff if the auction is about to end at that price. A bidder will drop out

only if the payoff of winning a license with payment and supply level determined by the current clock price becomes non-positive. When the clock price is low, the value of winning a license is high if the auction ends at the current clock price as the supply will be low, so bidders are willing to stay longer in the auction. When the clock price is very close to r_n, there is a continuum of types $c_i \in \left[P(n) - r_n, P(n-1) - r_n\right]$ s.t. bidders with those types no longer want to stay in the auction if the total supply increases to n from $(n-1)$. Those bidders with costs close to $P(n-1) - r_n$ will drop out at $r_n - \epsilon$ to avoid the loss from winning one out of n licenses after price reaches r_n. On the other hand, bidders with costs close to $P(n) - r_n$ will drop out at r_n given that there is positive probability that more than $(k-n)$ bidders drop out at $r_n - \epsilon$ so that the auction ends before price reaches r_n with a total supply of $(n-1)$ licenses, giving them a strictly positive payoff. Therefore, there always exists some type $\hat{c}_n^k$ such that a bidder with cost $\hat{c}_n^k$ is indifferent between dropping out at r_n and $r_n - \epsilon$. All bidders with costs $c_i \in [\hat{c}_n^k, P(n-1) - r_n]$ drops out simultaneously at $r_n - \epsilon$, and all bidders with costs $c_i \in [P(n) - r_n, \hat{c}_n^k]$ drops out simultaneously at r_n.

3.5.3 Multi-dimensional Uniform-price Auction

The inefficiency result of the standard uniform-price auction and ascending clock auction comes from the fact that each bidder is uncertain about his ex-post value of winning a license when the final supply is endogenously determined in the auction. In this section, I construct a multi-dimensional uniform-price auction that

allows bidders to condition their bids on the total supply in the auction. I will show that this auction can implement the efficient allocation in a dominant strategy equilibrium with a sequence of reserve prices that corresponds to the efficient reserve costs defined in Corollary 3.1.

Consider a multi-dimensional uniform-price auction defined as follows.

Definition 3.5. *In a multi-dimensional uniform-price auction, the auctioneer announces a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ where $r_1 < r_2 < r_3 \cdots < r_K$ at the beginning of the auction. Each bidder submits a vector of bids $(b_i^1, b_i^2, \cdots, b_i^K)$, in which b_i^n denotes the bid conditional on total supply to be $S = n$. For each possible supply level n, let S_n denote the set of winners in round (Rn) or before (Rn). The allocation rule and payment rule are described as follows.*

(R1) Rank all bids conditional on $S = 1$. If the highest bid among $\{b_i^1\}_{i \in N}$ is greater than r_1, allocate one license to the bidder who submits the highest bid among $\{b_i^1\}_{i \in N}$. Otherwise, do not sell any license.

(R2) Rank all bids conditional on $S = 2$ among the remaining bidders $N \setminus S_1$. If the highest bid among $\{b_i^2\}_{i \in N \setminus S_1}$ is greater than r_2, then allocate a second license to that bidder and continue to (R3). Otherwise, restrict supply to be 1 and charge the winner the greater of the highest losing bid among $\{b_i^1\}_{i \in N}$ and r_1.

(R3) Ranks all bids conditional on $S = 3$ among the remaining bidders $N \setminus S_2$. If the highest bid among $\{b_i^3\}_{i \in N \setminus S_2}$ is greater than r_3, then allocate a third license to that bidder and continue to (R3). Otherwise, restrict supply to be 2 and charge both winners the greater of the highest losing bid among $\{b_i^2\}_{i \in N \setminus S_1}$ and r_2.

. . .

(RK) *Rank all bids conditional on $S = K$ among the remaining bidders $N \setminus S_{K-1}$. If the highest bid among $\{b_i^K\}_{i \in N \setminus S_{K-1}}$ is greater than r_K, then allocate a K-th license to that bidder. Otherwise, restrict supply to be $(K - 1)$ and charge all winners the greater of the highest losing bid among $\{b_i^{K-1}\}_{i \in N \setminus S_{K-2}}$ and r_{K-1}.*

The next proposition characterizes the equilibrium in the multi-dimensional uniform-price auction with reserve price $r_1 < r_2 < r_3 < \cdots < r_K$.

Proposition 3.3. *In a multi-dimensional uniform-price auction with reserve prices $(r_1, r_2, \cdots, r_K)$, in which $r_1 < r_2 < r_3 < \cdots < r_K$, a symmetric dominant strategy equilibrium is characterized below:*

For all $n \in \{1, 2, \cdots, K\}$,

$$\beta_n(c_i) = \begin{cases} P(n) - c_i, & for \quad c_i \in \left[\underline{c}, P(n) - r_n\right] \\ 0, & for \quad c_i \in \left(P(n) - r_n, \bar{c}\right] \end{cases} \tag{3.22}$$

Proof. See Appendix. $\square$

Corollary 3.5. *With a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ where $r_n = (n - 1)\delta$ for all n, the multi-dimensional uniform-price auction implements the efficient allocation in a dominant strategy equilibrium and is outcome equivalent to the VCG mechanism.*

The efficiency of the multi-dimensional uniform-price auction comes from the fact that by allowing bidders to submit different bids conditional on different supply, each bidder can easily incorporate the difference in expected values conditional on

winning at different supply levels into their bids. For every possible supply level n, conditional on the total supply to be $S = n$, it is a dominant strategy for each bidder to bid $P(n) - c_i$ if $P(n) - c_i \geq r_n$ and bid 0 if $P(n) - c_i < r_n$. Conditional on bidding $b_i^n > r_n$, the probability for total supply to be $S = n$ does not depend on each bidder's own bids, but only depends on each bidder's opponents' bids. Therefore, the fact that supply is endogenously determined within the auction does not distort each bidder's incentive to bid their true values conditional on each supply level in the multi-dimensional uniform-price auction. Given that it is a dominant strategy for each bidder to bid true value conditional on $S = n$, the efficient reserve prices should be $r_n^e = P(n) - c_n^e = (n - 1)\delta$ for all n. Under this sequence of reserve prices, the multi-dimensional uniform-price auction always implements the efficient allocation.

3.5.4 Walrasian Clock Auction

In this subsection, I construct a Walrasian Clock auction in which the clock price can either go up or go down according to whether there is excess demand or excess supply in the auction. I will show that efficiency can be implemented by this auction through a tatonnement process.

Consider a Walrasian clock auction with endogenous supply of up to K licenses defined as follows:

Definition 3.6. *In a Walrasian clock auction, there is a clock showing the current price. At any time of the auction, each bidder states whether he is "in" or "out"*

of the auction given the current clock price. Denote the number of active bidders at clock price p as $k(p)$, then $k(p)$ represents the aggregate demand at price p.

At the beginning of auction, the auctioneer announces a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ where $r_1 < r_2 < \cdots < r_K$. Starting from supply level $n = K$, run the following algorithm:

(RK) *The auctioneer sets the total supply equals to K and set the clock price p equals to r_K. Each bidder states whether he is "in" or "out" of the auction. Compare the total demand $k(r_K)$ to total supply K:*

- *If $k(r_K) \geq K$, there is excess demand at the clock price r_K. The auctioneer will announce supply to be fixed at K and run an ascending clock auction from clock price r_K.*

- *If $k(r_K) < K$, there is excess supply at r_K. The auctioneer will reduce supply to be $K - 1$ and reducing the clock price to r_{K-1}. The auction continues to (RK-1).*

(RK-1) *The auctioneer sets the total supply equals to $K - 1$ and set the clock price p equals to r_{K-1}. each bidder states whether he is "in" or "out" of the auction. The bidders who stated "in" the auction at price of r_K in the previous round (RK) are required to remain in the auction at price r_{K-1}, so we must have $k(r_{K-1}) \geq k(r_K)$. Compare the total demand $k(r_{K-1})$ to total supply $(K - 1)$:*

- *If $k(r_{K-1}) \geq K - 1$, there is excess demand at the clock price r_{K-1}. The auctioneer will announce supply to be fixed at $(K - 1)$ and then run an ascending clock auction from clock price r_{K-1}.*

- If $k(r_{K-1}) < K - 1$, there is excess supply at r_{K-1}. The auctioneer will reduce supply to be $K - 2$ and reducing the clock price to r_{K-2}. The auction continues to (RK-2).

$\ldots$

(Rn) *The auctioneer sets the total supply equals to n and set the clock price p equals to r_n. Each bidder states whether he is "in" or "out" of the auction. Those who stated "in" in round (Rn+1) must remain in round (Rn). Compare the total demand $k(r_n)$ to total supply n:*

- If $k(r_n) \geq n$, there is excess demand at the clock price r_n. The auctioneer will announce supply to be fixed at n and then run an ascending clock auction from clock price r_n.

- If $k(r_n) < n$, there is excess supply at r_n. The auctioneer will reduce supply to be $(n - 1)$ and reducing the clock price to r_{n-1}. The auction continues to (Rn-1).

Repeat this algorithm until getting $k(r_n) \geq n$ for some integer n and run an ascending clock auction of n items starting from price $p = r_n$. In round (R1), if $k(r_1) > 1$, then run an ascending clock auction that starts from $p = r_1$. If $k(r_1) = 1$, then allocate 1 license to the only active bidder at r_1. If $k(r_1) = 0$, then the seller does not sell any license.

The next proposition characterizes the equilibrium in the Walrasian clock auction:

Proposition 3.4. *For all possible supply levels $n \in \{K, K-1, \cdots, 2, 1\}$, the dominant strategy equilibrium in round (Rn) is characterized as follows.*

1. *When clock price $p = r_n$ at the beginning of (Rn), it is a dominant strategy equilibrium for each bidder i to state "in" if $P(n) - c_i \geq r_n$ and state "out" if $P(n) - c_i < r_n$. Only bidders with costs $c_i \leq P(n) - r_n$ will be in the auction.*

2. *If $k(r_n) \geq n$ and the auction transforms into an ascending clock auction with n items, it is a dominant strategy equilibrium for each bidder who stated "in" at price r_n to drop out at his true value of winning one out of n licenses, $\beta(c_i) = P(n) - c_i$.*

3. *If $k(r_n) < n$ and the auction proceeds to round (Rn-1), the equilibrium strategy in (Rn) with all n replaced by $(n-1)$ is a dominant strategy equilibrium in round (Rn-1).*

Proof. See Appendix. □

Given the equilibrium characterized above, it is straightforward to see that with reserve prices $r_n^e = P(n) - c_n^e = (n-1)\delta$ for all n, the Walrasian clock auction implements the VCG outcome in a dominant strategy equilibrium:

Corollary 3.6. *With a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ where $r_n = (n-1)\delta$ for all n, the Walrasian clock auction dynamically implements the efficient allocation in a dominant strategy equilibrium and is outcome equivalent to the VCG mechanism and the multi-dimensional uniform-price auction.*

3.6 An Optimal Direct Mechanism

In this section, I follow Myerson (1981) [40]'s optimal auction design approach and characterize the optimal auction under quantity externalities as a direct revelation mechanism. I will show that the optimal auction can also be implemented by introducing a sequence of reserve costs to determine supply. I next compare the optimal reserve costs to the efficient reserve costs as well as to the optimal reserve costs in standard auctions without quantity externalities.

3.6.1 Mechanism Design and Solution Concepts

In a direct mechanism, bidders report their private costs c_i directly. An auction mechanism (μ, t) consists of an allocation rule $\mu_i(c)$ and a payment rule $t_i(c)$ for every bidder i, in which $\mu_i = \left(\mu_i^{(1)}(c), \mu_i^{(2)}(c), \cdots, \mu_i^{(K)}(c) \right)$ is the vector of joint probabilities that bidder i wins a license when a total of $n \in \{1, 2, \cdots, K\}$ licenses are allocated in the auction given reported costs $c \in [\underline{c}, \bar{c}]$, and $t_i(c)$ is the expected payment of bidder i given reported costs $c \in [\underline{c}, \bar{c}]$. It is straightforward to define the feasibility constraint in any direct mechanism as follows:

Definition 3.7. *An allocation rule μ is feasible if for any supply level $n \in \{1, 2, \cdots, K\}$,*

(1)

$$0 \leq \mu_i^{(n)} \leq 1, \quad \forall i, \quad and \quad \sum_i \mu_i^{(n)} \leq n \tag{3.23}$$

(2) If $\mu_i^{(n)}(c) > 0$ for some i, then $\mu_j^{(n')}(c) = 0$, for all $n' \neq n$, for all j.

That is, given any level of supply n, each bidder's probability of winning must

fall in $[0, 1]$, the sum of winning probabilities across bidders does not exceed the number of items to be allocated, and the total supply must be unique.

For a bidder i with cost c_i, the interim utility $U_i(c_i)$ is given by

$$U_i(c_i) = \int_{c_{-i}} \underbrace{\left[\sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \left[P(n) - c_i \right] - t_i(c_i, c_{-i}) \right]}_{u_i(c_i, c_{-i})} f_{-i}(c_{-i}) dc_{-i} \qquad (3.24)$$

in which $u_i(c_i, c_{-i}) = \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \left[P(n) - c_i \right] - t_i(c_i, c_{-i})$ is the ex-post utility of bidder i given reports (c_i, c_{-i}). A direct auction mechanism (μ, t) satisfies incentive compatibility condition if the following definition holds:

Definition 3.8. *A direct auction mechanism (μ, t) is Bayesian incentive compatible (IC) and individually rational (IR) if for every bidder i, for any value of true cost c_i and any reported cost c_i',*

$$U_i(c_i) \geq \int_{c_{-i}} \left[\sum_{n=1}^{K} \mu_i^{(n)}(c_i', c_{-i}) \left[P(n) - c_i \right] - t_i(c_i', c_{-i}) \right] f_{-i}(c_{-i}) dc_{-i}$$
$$(3.25)$$
$$U_i(c_i) \geq 0$$

In the following analysis, I characterize the optimal auction mechanism under quantity externalities among all Bayesian IC and IR mechanisms subject to the feasibility constraint.

3.6.2 Characterization of the Optimal Auction Mechanism

For any possible supply level $n \in \{1, 2, \cdots, K\}$, define bidder i's marginal revenue function conditional on total supply equals n as

$$MR(c_i, n) = P(n) - c_i - \frac{F_i(c_i)}{f_i(c_i)} \qquad (3.26)$$

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I also assume that the marginal revenue functions are regular: For any bidder i, for any $n \in \{1, 2, \cdots, K\}$,

$$\frac{\partial MR(c_i, n)}{\partial c_i} < 0 \tag{3.27}$$

The next lemma characterizes any Bayesian IC and IR mechanism in this model:

Lemma 3.5. *A mechanism (μ, t) is Bayesian IC and IR if for every bidder i, the following conditions hold:*

(1) For any $c_i, c_i' \in [\underline{c}, \bar{c}]$, if $c_i' \geq c_i$, then

$$\int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i', c_{-i}) f_{-i}(c_{-i}) dc_{-i} \leq \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} \tag{3.28}$$

(2)

$$U_i(c_i) = U_i(\bar{c}) + \int_{c_i}^{\bar{c}} \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(s, c_{-i}) ds f_{-i}(c_{-i}) dc_{-i} \tag{3.29}$$

(3)

$$U_i(\bar{c}) \geq 0 \tag{3.30}$$

Proof. See Appendix. □

The next lemma characterizes the seller's ex-ante expected revenue in any Bayesian IC and IR mechanism:

Lemma 3.6. *For any Beyesian IC and IR mechanism that satisfies the conditions in Lemma 3.5, the ex-ante expected revenue is given by*

$$ER = \sum_i \int_c \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \times \left\{ P(n) - c_i - \frac{F_i(c_i)}{f_i(c_i)} \right\} \right\} f(c) dc - \sum_i U_i(\bar{c}) \tag{3.31}$$

Proof. See Appendix. □

Let $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(K)}$ denote the realizations of the lowest, the second lowest, ..., the K-th lowest cost among the N bidders. The result of Lemma 3.6 implies that given any possible supply level of n, the optimal auction assigns n licenses to the n lowest-costs bidders only if the n-th lowest-cost bidder's type $c_{(n)}$ satisfies

$$P(n) - c_{(n)} - \frac{F_i(c_{(n)})}{f_i(c_{(n)})} \geq 0 \tag{3.32}$$

However, as the seller needs to optimize over total supply given the presence of quantity externalities, $P(n) - c_{(n)} - \frac{F_i(c_{(n)})}{f_i(c_{(n)})} \geq 0$ is a necessary but not sufficient condition for total supply to be n. The following analysis characterizes the optimal reserve costs in this model. I first consider the total expected revenue from selling a fixed number of n licenses in an optimal auction subject to no reserve price, and then choose reserve prices that select the total supply to maximize revenue.

Given any cost profile c, the ex-post revenue of selling n licenses in an optimal auction without any reserve price is given by

$$R(c, n) = \sum_{i=1}^{n} \left\{ P(n) - c_{(i)} - \frac{F_i(c_{(i)})}{f_i(c_{(i)})} \right\} \tag{3.33}$$

Compare $R(c, n)$ and $R(c, n-1)$ for any n, given a profile of reported costs c, the marginal increment in total revenue by selling an n-th license equals to the marginal revenue of the bidder with the n-th lowest cost:

$$MR(c_{(n)}, n) = P(n) - c_{(n)} - \frac{F_i(c_{(n)})}{f_i(c_{(n)})} \tag{3.34}$$

The marginal decrement in total revenue by selling an n-th license is the loss of $P(n-1) - P(n) = \delta$ in $\pi(c_{(i)}, n-1)$ for all the $(n-1)$ lowest-cost bidders,

$$MC(n) = (n-1)\delta \tag{3.35}$$

The n-th license should be sold in the optimal auction if and only if $MR(c_{(n)}, n) \geq MC(n)$ given the realization of costs c.

Since

$$MR(c_{(n)}, n) > MR(c_{(n+1)}, n+1) \quad \text{for all} \quad n$$
$$MC(n) < MC(n+1) \quad \text{for all} \quad n$$

(3.36)

The total revenue is maximized at the supply level n^* s.t.

$$MR(c_{(n^*)}, n^*) \geq MC(n^*)$$
$$MR(c_{(n^*+1)}, n^* + 1) < MC(n^* + 1)$$

(3.37)

Therefore, the optimal mechanism can be constructed as follows. Each bidder is asked to report his private cost c_i. Given any profile of reported costs $\hat{c}$, rank the reported costs in an ascending order: $\hat{c}_{(1)} \leq \hat{c}_{(2)} \leq \cdots \leq \hat{c}_{(N)}$, then allocate licenses according to the following algorithm:

(R1) allocate one license to the bidder with the lowest reported cost $\hat{c}_{(1)}$ if $P(1) - \hat{c}_{(1)} - \frac{F_i(\hat{c}_{(1)})}{f_i(\hat{c}_{(1)})} \geq 0$ and continue to (R2); stop the algorithm and sell zero license otherwise;

(R2) allocate one license to the bidder with the second lowest reported cost $\hat{c}_{(2)}$ if $P(2) - \hat{c}_{(2)} - \frac{F_i(\hat{c}_{(2)})}{f_i(\hat{c}_{(2)})} \geq \delta$ and continue to (R3); stop the algorithm and sell one license otherwise;

...

(RK) allocate one license to the bidder with the K-th lowest reported cost $\hat{c}_{(K)}$ if $P(K) - \hat{c}_{(K)} - \frac{F_i(\hat{c}_{(K)})}{f_i(\hat{c}_{(K)})} \geq (K-1)\delta$; sell $(K-1)$ license otherwise.

The following proposition gives the formal definition of the optimal direct mechanism:

Proposition 3.5. *Suppose the marginal revenue functions $MR(c_i, n)$ satisfy the regularity condition. For any vector of costs $c \in \mathbb{R}^N$ with any $N \geq n$, define $C^n(c)$ as the n-th lowest value among components of c. Consider the following mechanism (μ^*, t^*):*

$$
\mu_i^{*(n)}(c_i, c_{-i}) =
\begin{cases}
1 & if \quad c_i \leq C^n(c_{-i}), \\[2mm]
 & \quad MR\big(C^n(c_i, c_{-i}), n\big) - (n-1)\delta \geq 0, \\[2mm]
 & \quad and \quad MR\big(C^{n+1}(c_i, c_{-i}), n+1\big) - n\delta < 0 \\[2mm]
0 & else
\end{cases}
\tag{3.38}
$$

for all $n \in \{1, 2, \cdots, K-1\}$, and

$$
\mu_i^{*(K)}(c_i, c_{-i}) =
\begin{cases}
1 & if \quad c_i \leq C^K(c_{-i}), \\[2mm]
 & \quad MR\big(C^K(c_i, c_{-i}), K\big) - (K-1)\delta \geq 0, \\[2mm]
0 & else
\end{cases}
\tag{3.39}
$$

for $n = K$. Any tie is broken randomly. The payment rule is given by

$$
t_i^*(c_i, c_{-i}) = \sum_{n=1}^{K} \mu_i^{*(n)}(c_i, c_{-i})\big[P(n) - c_i\big] - \int_{c_i}^{\bar{c}} \sum_{n=1}^{K} \mu_i^{*(n)}(s, c_{-i}) ds
\tag{3.40}
$$

Then (μ^*, t^*) is an optimal auction among all Bayesian IC and IR mechanisms.

Corollary 3.7. *Let c_n^* denote the optimal reserve costs conditional on selling n licenses for all $n \in \{1, 2, \cdots, K\}$. Then the optimal reserve costs $(c_1^*, c_2^*, \cdots, c_K^*)$ are given by*

$$
P(n) - c_n^* - \frac{F_i(c_n^*)}{f_i(c_n^*)} = (n-1)\delta, \quad for \ all \quad n
\tag{3.41}
$$

Since $MR(c_i, n)$ is strictly decreasing in c_i, this directly implies that given any supply level of $n \geq 2$, the optimal reserve type c_n^* under the presence of quantity externalities is strictly lower than the optimal reserve type c_n' without the presence of quantity externalities, in which c_n' is given by

$$P(n) - c_n' - \frac{F_i(c_n')}{f_i(c_n')} = 0 \quad \text{for all} \quad n \tag{3.42}$$

Corollary 3.8. *For any $n \in \{2, 3, \cdots, K\}$, the optimal reserve cost c_n^* under the presence of quantity externalities is strictly lower than the optimal reserve cost c_n' without the presence of quantity externalities.*

The next corollary compares the optimal reserve costs $(c_1^*, c_2^*, \cdots, c_K^*)$ to the efficient reserve costs $(c_1^e, c_2^e, \cdots, c_K^e)$:

Corollary 3.9. *For any $n \in \{1, 2, \cdots, K\}$, the optimal reserve cost c_n^* is strictly lower than the efficient reserve cost c_n^e.*

Moreover, since the optimal reserve prices $(r_1^*, r_2^*, \cdots, r_K^*)$ also satisfy $r^*1 < r_2^* < \cdots < r_K^*$, according to Lemma 3.1, Proposition 3.1, Proposition 3.2, Proposition 3.4, and Proposition 3.3, it is impossible to implement the optimal revenue in standard uniform-price auctions or ascending clock auctions, while the multi-dimensional uniform-price auction and the Walrasian clock auction can implement the optimal revenue using a sequence of reserve prices that correspond to the optimal reserve costs characterized in Corollary 3.7.

Corollary 3.10. *With a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ where r^*n satisfies $r_n^* = P(n) - c_n^*$, in which $P(n) - c_n^* - \frac{F_i(c_n^*)}{f_i(c_n^*)} = (n-1)\delta$ for all n, the multi-*

dimensional uniform-price auction and the Walrasian clock auction can implement the optimal revenue.

3.7   Auction Design under Consideration of Consumer Surplus

The analysis in previous sections on auction design under the objective of maximizing producer surplus can be extended to the case when consumer surplus is also considered. In this section, I will briefly discuss the efficient supply level and characterize efficient reserve prices when the auctioneer cares about consumer surplus.

Assuming there are at least K consumers in the downstream market. Each has a positive value of obtaining the product provided by suppliers who win a license and enter the industry. The value profile of the K highest value bidders is given by $v_{(1)} \geq v_{(2)} \geq \cdots \geq v_{(K)}$, and assume $v_{(K)} > c_{(K)}$. Assuming the downstream market is allocative efficient. For every supply level n selected in the auction, the n consumers with highest values will each make a transaction at market price $P(n)$. Then the maximized consumer surplus given the value profile v and supply level n is given by

$$CS^*(v, n) = \sum_{i=1}^{n} v_{(i)} - P(n) \times n \qquad (3.43)$$

The consumer surplus in the downstream market will be increased by $v_{(n)} - P(n) + (n-1)\delta > 0$ when an n-th license is allocated. Therefore, the auctioneer should always allocate all K licenses when the objective is to maximize consumer surplus in the downstream market. The consumer surplus maximizing reserve price is given

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by $0 = r_1 = r_2 = \cdots = r_K$. Since the supply will not be determined endogenously within the auction, both the standard uniform-price auction and the ascending clock auction can implement the consumer surplus-maximizing allocation in a dominant strategy equilibrium.

The maximized total surplus in the auction at supply level n is given by

$$TS^*(v, c, n) = \sum_{i=1}^{n} v_{(i)} - \sum_{i=1}^{n} c_{(i)} \tag{3.44}$$

Since $v_{(n)} > c_{(n)}$ for all n, the auctioneer should always allocate all K licenses when the objective is to maximize total surplus in the downstream market. The total surplus maximizing reserve price is also given by $0 = r_1 = r_2 = \cdots = r_K$. The supply will be fixed at K in any total surplus-maximizing auction as well, and the uniform-price auction and the ascending clock auction can both implement the total surplus-maximizing outcome.

Now suppose the auctioneer cares about a weighted average of consumer surplus and producer surplus. The maximized weighted surplus under supply level n is given by

$$WS^*(v, c, n) = \rho CS^*(v, n) + (1 - \rho) PS^*(c, n)$$
$$= \rho \Big[\sum_{i=1}^{n} v_{(i)} - P(n) \times n \Big] + (1 - \rho) \Big[P(n) \times n - \sum_{i=1}^{n} c_{(i)} \Big] \tag{3.45}$$

The marginal benefit of selling an n-th license is

$$MB(n, v, c) = \rho \Big[v_{(n)} - P(n) + \delta(n - 1) \Big] + (1 - \rho) \Big[P(n) - c_{(n)} \Big] \tag{3.46}$$

The marginal cost of selling an n-th license is

$$MC(n) = (1 - \rho)\delta(n - 1) \tag{3.47}$$

Setting marginal benefit to marginal cost will give the efficient reserve cost under the objective of maximizing weighted surplus with parameter ρ:

$$c'_n = \max\left\{\frac{1}{1-\rho}\left[\rho v_{(n)} + (1-2\rho)[p(n) - \delta(n-1)]\right], \bar{c}\right\} \qquad (3.48)$$

in which $\frac{1}{1-\rho}\left[\rho v_{(n)} + (1-2\rho)[p(n) - \delta(n-1)]\right]$ is decreasing in n.

When $1 \geq \rho \geq 0.5$, the auctioneer cares more about the consumer surplus than the producer surplus, the weighted surplus-maximizing reserve cost $c'_n = \bar{c}$ for all n, so the weighted surplus-maximizing reserve price is given by $0 = r'_1 = r'_2 = \cdots = r'_K$, and the auction will allocate K licenses for certain.

When $0.5 > \rho > 0$, the auctioneer cares more about producer surplus than consumer surplus. There may exist some $n' \leq K$ s.t. $c'_n < \bar{c}$ for all $n \geq n'$. n' is the smallest integer that satisfies $\frac{1}{1-\rho}\left[\rho v_{(n)} + (1-2\rho)[p(n) - \delta(n-1)]\right] < \bar{c}$. The value of n' depend on ρ, value profile v and the cost profile c. The weighted surplus-maximizing reserve costs will be $\bar{c} = c'_1 = c'_2 = \cdots = c'_{n'-1} > c'_{n'} > c'_{n'+1} > \cdots > c'_K$. The weighted surplus-maximizing reserve price will be in the form of $0 = r'_1 = r'_2 = \cdots = r'_{n'-1} < r'_{n'} < \cdots < r'_K$. Both the multi-dimensional uniform-price auction and the Walrasian clock auction with the weighted surplus-maximizing reserve prices can implement the weighted surplus-maximizing allocation in a dominant strategy equilibrium.

When $\rho = 0$, the auctioneer only cares about producer surplus, and $0 = r^e_1 < r^e_2 < \cdots < r^e_K$ as shown in the previous analysis.

3.8 Conclusions

This paper characterizes the design of efficient and optimal mechanisms in a license auction that allocates operating permits in a regulated industry. I first show that a sequence of reserve prices that specify minimum acceptable bids are needed to determine supply endogenously in any efficient auction. Then I characterize the equilibria of uniform-price auctions and ascending clock auctions after introducing such reserve prices and show that both auctions are inefficient with any reserve prices. I next construct a multi-dimensional uniform-price auction that can implement the efficient allocation using reserve prices that correspond to the efficient reserve costs in a dominant strategy equilibrium. Then I construct a Walrasian clock auction that can dynamically implement the same efficient outcome as the multi-dimensional uniform-price auction under the same reserve prices. I also characterize the optimal auction and the corresponding optimal reserve prices. In the end of this chapter, I provide a discussion on auction design when the auctioneer cares about consumer surplus. I show that the multi-dimensional uniform-price auction and the Walrasian Clock auction can implement the optimal revenue with appropriately chosen optimal reserve prices and implement the weighted surplus-maximizing outcome with corresponding reserve prices when consumer surplus is also considered in the auction.

One implication of this paper is that in auctions that determine structure of some downstream markets, the supply should always be endogenously determined within the auction. Moreover, the standard multi-unit auctions such as the

uniform-price auction and the ascending clock auction are not well-performed with endogenous supply, since introducing any binding reserve prices that can endogenously determine supply will result in pooling equilibrium for both auctions. This result implies that the standard uniform-price auctions and ascending clock auctions can be suboptimal when each bidder care about the total quantity of items allocated in the auction, and more complicated auction designs are needed from both surplus-maximizing and revenue-maximizing perspectives.

Another implication of this paper is that both the efficient and the optimal reserve prices are higher than their counterparts in markets without quantity externalities. Therefore, auction practitioners may want to consider increasing reserve prices from both surplus-maximizing and revenue-maximizing perspectives after taking quantity externalities into account.

Chapter 4: Information Provision in Procurement Auctions with Endogenous Investments

4.1 Introduction

Auctions are used in procurement settings to allocate contracts to suppliers in a variety of markets such as electricity, government securities, and construction rights. In a benchmark model of single-unit procurement auctions, bidders sell identical products with exogenously differentiated production costs. However, many practical procurement markets have two departures from the standard model. First, the suppliers can be horizontally differentiated in their non-price characteristics, and the auctioneer often has preference over non-price characteristics of the product. Second, bidders can often engage in pre-auction cost-reducing investments. This study is motivated by these two distinctive features in many procurement markets.

Existence of product differentiation is common in procurement auctions. Examples of non-price attributes that the auctioneer might care about include product design, input materials, time of completion, reputation of the supplier, etc. (Asker and Cantillon 2008 [64]). Empirical evidence of product differentiation in procurement auctions is also documented in newspapers and previous studies. For example,

when two aircraft manufacturing companies Airbus and Boeing competed for a contract from Iberia Airlines, their bids were evaluated together with their product characteristics in the procurement auction. According to the *Wall Street Jounral* on March 10, 2003, Iberia has privately known preferences on several characteristics such as fleet composition of the potential suppliers' products, as it will affect future maintenance cost (Thomas and Wilson 2012 [65]). Under the presence of product differentiation, a supplier's value in the auction not only depends on its production cost but also depends on the auctioneer's privately known preference.

Pre-auction investments among bidders are also common in procurements. For example, prior to bidding for a road construction contract, suppliers can invest in machinery and other equipments to reduce cost. Empirical evidence of pre-auction investments can also be found in previous studies. For example, defense contractors invest substantial resources in R & D before bidding for a government contract (Lichtenberg 1986 [66]; Li et al. 2006 [67]).

Under these two departures from the standard procurement auction model, suppliers face a trade-off between higher sunk investment costs and higher expected return in the auction, and the auctioneer's information provision policy can affect the suppliers' investment strategies. Since each supplier will choose the investment level at which the marginal expected return to investment in the auction equals to the marginal investment cost, and the expected return to investment depends on the auctioneer's valuation over the supplier.

It is well understood how to design an optimal auction mechanism that maximizes the auctioneer's expected revenue given homogenous bidders who enter the

auction with private exogenous monetary types (Myerson 1981 [40]). Some studies have explored pre-auction investment incentives with homogenous products (Piccione and Tan 1996 [68]; Bag 1997 [69]; Arozamena and Cantillon 2004 [70]). However, no study has examined suppliers' investment incentives on cost reduction when product differentiation presents. The objective of this study is to investigate the impact of the auctioneer's information provision policy on suppliers' pre-auction investment incentives and the auctioneer's expected revenue when product differentiation presents among suppliers.

In this chapter, I assume that the auctioneer can commit to one of the following three information disclosure policies: publicly disclose her private valuations over all suppliers' products; privately disclose her valuation over each supplier's product; or completely conceal her valuations. Then I analyze equilibrium investment strategy of suppliers before entering a second-score sealed-bid procurement auction and compare the expected revenues of auction under these three information provision schemes. The main result of this chapter shows that pre-auction investments are strategic substitutes among bidders, and providing more information about the auctioneer's preference encourages those more favored bidders to invest more, which increases cost differentiation among bidders. The main analysis focuses on the case when there are only two bidders and shows that disclosing more information will reduce expected revenue by discouraging the lower quality bidder from investment and giving higher informational rent to the higher quality bidder. I also provide a discussion of the general case when there are more than 2 bidders and show that disclosing more information will increase expected revenue by promoting competition among higher

quality bidders when the number of bidders is sufficiently large.

4.2   Related Literature

This paper is connected to the literature on procurement auctions with differentiated products. Asker and Cantillon (2008) [64] provide a systematic analysis of equilibrium behavior in scoring auctions when suppliers have multi-dimensional types. Thomas and Wilson (2012) [65] experimentally compare first-price auctions and multilateral negotiations when horizontal product differentiation is introduced into a procurement auction. The major difference between this chapter and the previous studies on scoring auctions is that the existing literature on scoring auctions takes product characteristics and cost as different dimensions of each bidder's exogenously given multi-dimensional type, while this chapter models product differentiation as assigning each seller a subjective "quality" privately known to the auctioneer and assumes each bidder's cost is endogenously determined by investment.

This chapter is also related to the literature studying optimal information release of the auctioneer when the auctioneer owns private information that enters bidders' valuations. Milgrom and Weber (1982) [2] analyze the optimal release of information in an auction with affiliated values and find that it is optimal for the auctioneer to publicly announce her private information. On the other hand, Ganuza (2004) [71] analyzes a horizontally differentiated market in which the auctioneer has private information about product characteristics and bidders have horizontally

differentiated preferences over the product space. He shows that when releasing information is costly to the auctioneer, the auctioneer has incentives to release less than efficient level of information. Coleff and Garcia (2014) [72] study the optimal release of information in a procurement auction in which sellers can choose their horizontal product characteristics according to the auctioneer's reported preference. They show that it is not optimal for the auctioneer to send public information to all sellers under presence of entry cost. Closely related to this paper, Colucci et al. (2015) [73] compare the performance of different information provision schemes under first-score auctions and second-score auctions in a model with differentiated bidders whose qualities are private information to the auctioneer. However, they assume bidders' costs are heterogenous and commonly known in the model, while I adopt Dasgupta (1990) [74]'s production model and assume bidders' costs are determined by their own investment decisions and a random variable. In Ganuza (2004) [71]'s model, the auctioneer's information provision will alter the bidders' perception of their own values. In Coleff and Garcia (2014) [72], the auctioneer's information provision will alter the equilibrium profile of bidders' horizontal locations and the number of bidders. In Colucci et al (2015) [73], the auctioneer's information provision will change the bidders' bidding strategies in the first score auction. This study is different from the above studies in the sense that the auctioneer's information provision will alter the profile of bidder's values by changing their investment incentives.

This chapter is also closely related to the strand of literature on studying bidders' pre-auction investment incentives under different auction mechanisms. Most of

this literature focus on studying suppliers' investment incentives in sealed-bid auctions for a homogenous product. A common goal of these studies is to compare the equilibrium investment levels induced by the auction mechanism to the socially optimal investment level, and compare the performance of different mechanisms based on their efficiency in inducing pre-auction investments (Piccione and Tan 1996 [68]; Bag 1997 [69]). However, there exists no mechanism that can uniquely implements ex ante efficient investment when suppliers can only make investment decisions simultaneously prior to the auction (Arozamena and Cantillon 2004 [70]; Li et al. 2006 [67]; Hatfield et al. 2015 [75]; Tomoeda 2015 [76]). Different from these previous studies that focus on finding socially-optimal investment-inducing mechanism, the goal of this study is to find an information provision scheme that maximizes the auctioneer's ex ante expected revenue in a second score auction, given the presence of differentiated sellers and pre-auction investment opportunity.

4.3   Model

4.3.1   Environment

An auctioneer wishes to procure one unit of an indivisible product that may come in different varieties. There are N risk-neutral potential suppliers $i \in \{1, 2, \cdots, N\}$ providing imperfect substitutes that feature different varieties of this product[1]. The product characteristic of each supplier is exogenous and observable to the auctioneer. The auctioneer values the specific product of each supplier differently. There

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[1]In this paper, I use feminine pronoun for the auctioneer and masculine pronouns for bidders.

are two stages of the game: investment stage and auction stage. The time line of the game is presented as below:

1. **t=1:** At the beginning of the investment stage, the auctioneer announces the allocation and payment rules of a second score auction and the information disclosure policy. The auctioneer can choose to publicly announce the entire profile of her valuations to all suppliers, or to privately inform each supplier her value for that supplier, or to conceal this information.

2. **t=2:** N suppliers enter the game. The auctioneer observes the product characteristics of each supplier and privately learns her valuation over their products $\{q_i,\}_{i=1}^N$. Each q_i measures the match between the auctioneer's private taste over product design and supplier i's product characteristics, and q_i is called bidder i's quality in the remaining of this chapter. Assuming preference is quasilinear in price, then the auctioneer's utility from purchasing supplier i's product at price p_i is

$$U(q_i, p_i) = q_i - p_i \tag{4.1}$$

If the auctioneer does not disclose any information, then all suppliers have common belief that q_i is independently and identically distributed according to distribution $G(.)$ on $[\underline{q}, \bar{q}]$. Furthermore, assume $\underline{q} > g(0) + \bar{\eta}$ and there is no outside buying options so that it is always ex post efficient for the auctioneer to purchase the product from one of the potential suppliers.

3. **t=3:** The auctioneer sends a private signal $\hat{q}_i \in \{\{q_i\}_{i=1}^N, q_i, \emptyset\}$ to every bidder i according to the information policy chosen at $t = 1$.

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4. **t=4:** After observing the signal provided by the auctioneer, each supplier i makes an investment $k_i \in \mathbb{R}^+$ simultaneously to reduce the cost of his product given common cost-reducing technology $g(.)$. k_i is the sunk cost of investment.

5. **t=5:** At the beginning of the auction stage, each supplier i receives a random cost shock η_i that is independently and identically distributed according to a commonly known uniform distribution $H(.)$ on $[\underline{\eta}, \bar{\eta}]$.

   Following Dasgupta (1990) [74]'s production cost model, I assume the total production cost of supplier i is given by

   $$c_i = c(k_i, \eta_i) = g(k_i) + \eta_i \tag{4.2}$$

   in which $g' < 0$, $g'' > 0$, $\lim_{k \to 0} -g'(k) = \infty$, and $\lim_{k \to \infty} -g'(k) = 0$, so the cost reducing investment exhibits decreasing returns.

   Each supplier i's "value" v_i as the total trading surplus that he can provide by selling the product to the auctioneer is therefore given by

   $$v_i = v(q_i, k_i, \eta_i) = q_i - g(k_i) - \eta_i \tag{4.3}$$

6. **t=6:** Each supplier submits bid b_i that represents the minimum payment he is willing to accept to provide the product in a second score auction. The scoring rule used in the auction is

   $$\sigma_i = q_i - b_i \tag{4.4}$$

   The auctioneer announces scores of all the bidders at end of the auction. The highest-score bidder i wins the contract and receives a payment equals to the

bid of the supplier j with the second highest score, adjusted by their quality difference: $p_i = b_j + q_i - q_j$.

4.3.2 Equilibrium of Second Score Auction

I will first show that truth-telling is still a dominant strategy for suppliers in the second score procurement auction when each supplier's value depends on the auctioneer's information q_i.

Corollary 4.1. *In the second score procurement auction with differentiated suppliers selling imperfect substitutes, it is still a dominant strategy for each supplier to bid his true production cost c_i.*

Proof. See Appendix. □

Since the auctionner privately knows the qualities of all bidders, by submitting a bid b_i, the value profile of all bidders $\{v_i\}_{i=1}^N$ will be revealed. Therefore, the second score auction can be written as a direct revelation mechanism in which the arguments of the allocation rule and the payment rule is the profile of bidders' values $\{v_i\}_{i=1}^N$.

4.3.3 Equilibrium of Investment Stage

In this section, I will characterize each supplier's optimal investment strategy. At the investment stage, the suppliers choose investment levels to maximize their expected payoffs in the auction, given that all suppliers report truthfully in the second score auction.

Define $\hat{F}_{-i}(.|q_{-i}, k^*_{-i})$ as the distribution of the highest value among bidder i's $(N-1)$ opponents' values at the beginning of the auction stage, before the random cost shocks η are realized. Then $\hat{F}_{-i}(.|q_{-i}, k^*_{-i})$ depends on opposing bidders' qualities q_{-i} and equilibrium investment strategies k^*_{-i}.

The incentive compatibility of the second score auction implies that the expected payoff of bidder i with value v_i is given by

$$\Pi_i(v_i) = \Pi_i(\underline{v}) + \int_{\underline{v}}^{v_i} \hat{F}_{-i}(\tau|q_{-i}, k^*_{-i})d\tau \tag{4.5}$$

At the investment stage, each supplier chooses an investment level k^*_i that maximizes the expected payoff in the auction as a best response to opponents' investments k^*_{-i}, given the distribution of qualities and random cost shocks, and the information provided by the auctioneer $\hat{q}_i$.

Definition 4.1. *A profile of investments chosen at investment stage $\{k^*_i\}_{i=1}^N$ is an equilibrium under information provision scheme $\hat{q}_i$ if for all i,*

$$k^*_i \in \operatorname*{argmax}_{k_i} E_{q,\eta}\left[\int_{\underline{v}}^{v_i(q_i,k_i,\eta_i)} \hat{F}_{-i}(\tau|k^*_{-i}, q_{-i})d\tau\Big|\hat{q}_i\right] - k_i \tag{4.6}$$

Let $\hat{V}(v_1, v_2, \cdots, v_N)$ denote the second highest value given a profile of values $\{v_i\}_{i=1}^N$. Then $\hat{V}(v_1, v_2, \cdots, v_N)$ is the auctioneer's ex-post revenue given $\{v_i\}_{i=1}^N$.

The auctioneer's problem is to choose $\hat{q}_i \in \left\{\{q_j\}_{j=1}^N, q_i, \emptyset\right\}$ to induce a profile of values $(v_1, \cdots, v_N)$ that yields the highest ex ante expected revenue in the auction, given that suppliers will play equilibrium investment strategy in the investment stage given the information provided by the auctioneer.

Definition 4.2. *The auctioneer's problem of optimizing information provision is*

$$\max_{\hat{q}\in\{\{q_j\}_{j=1}^N,q_i,\emptyset\}} E_\eta\left[\hat{V}\left(v_1,v_2,\cdots,v_N\right)\right]$$

$$s.t. \quad v_i = q_i - g(k_i^*) - \eta_i \tag{4.7}$$

$$k_i^* \in \operatorname*{argmax}_{k_i} E_{q,\eta}\left[\int_{\underline{v}}^{q_i-g(k_i)-\eta_i} \hat{F}_{-i}\left(\tau\big|k_{-i}^*,q_{-i}\right)d\tau\Big|\hat{q}_i\right] - k_i \quad \forall i$$

To study the impact of auctioneer's information provision of private valuations q_i, I will compare the suppliers' equilibrium investment strategies and expected revenues in the auction under concealment, private disclosure, and public disclosure. The main analysis will focus on the case where there are only $N = 2$ bidders. A discussion of the general case with $N \geq 2$ bidders will be provided in the end.

4.4 Equilibrium Investment Strategies with Two Sellers

In this section, I will analyze the suppliers' investment strategy when there are $N = 2$ bidders. Let i and j denote the identity of the 2 bidders. For each bidder i, the distribution of the opposing bidder j's value given bidder j's quality q_j and investment k_j^* at the beginning of the auction is given by

$$\hat{F}_{-i}\left(\tau\big|q_j,k_j^*\right) = Prob\left(q_j - g(k_j^*) - \eta_j \leq \tau\right)$$

$$= Prob\left(\eta_j \geq q_j - g(k_j^*) - \tau\right) \tag{4.8}$$

$$= 1 - H\left(q_j - g(k_j^*) - \tau\right)$$

Given the distribution of quality $G(q)$, let Q_1 and Q_2 denote the random variables that represent the highest order statistic and the second highest order statistic among bidders' qualities, respectively. Let (q_1, q_2) be realizations of (Q_1, Q_2). Then $q_1 = \max\{q_i, q_j\}$ and $q_2 = \min\{q_i, q_j\}$ for any realization of qualities $\{q_i, q_j\}$.

Define $\Delta(G)$ as the ex ante expected difference between q_1 and q_2 given distribution G:

$$\Delta(G) = E\big(q_1 - q_2\big|G\big) \tag{4.9}$$

$\Delta(G)$ represents the expected dispersion of the auctioneer's valuation on the two bidders' products, which in turn measures how much the auctioneer cares about non-price characteristics relative to cost. Mathematically, $\Delta(G)$ represents the expected difference between the first order statistics and the second order statistics among 2 draws given distribution $G(.)$. Holding the expected quality constant, a greater $\Delta(G)$ implies that the expected difference between the higher quality and the lower quality is larger, i.e., the auctioneer is willing to pay more for contracting with the high-quality supplier instead of the low-quality supplier. When $\Delta(G) = 0$, $q_1 = q_2 = E(q|G)$, this model turns into the standard procurement auction model without product differentiation: the auctioneer's valuation for any supplier's product equals $E(q|G)$ and is common knowledge. There is no difference between the three information provision schemes when $\Delta(G) = 0$.

Since the three information disclosure policies yields the same expected revenue when $\Delta(G) = 0$, I will next explore how the expected revenue under the three disclosure policies change when holding the expected quality constant and increasing the dispersion of qualities $\Delta(G)$ in the following analysis.

4.4.1 Equilibrium under Concealment of Quality

Under concealment of qualities, each supplier chooses investment strategy knowing only the distribution of (q_i, q_j) and distribution of (η_i, η_j). Note that at the time of investment, suppliers are ex ante identical with symmetric distribution of q_i and η_i.

Given any level of opponent's investment k_j, each bidder i chooses investment k_i^* that solves

$$\max_{k_i} \int_{q_i} \int_{q_j} \int_{\underline{v}}^{q_i - g(k_i) - E\eta_i} \left\{ 1 - H\big(q_j - g(k_j) - \tau\big) \right\} d\tau dG(q_j) dG(q_i) - k_i \qquad (4.10)$$

Take the first order condition will give supplier i's best response investment function $k_i^*(k_j)$ to the opponent's investment k_j. A subgame perfect equilibrium (k_i^C, k_j^C) is given by $k_i^C = k_i^*(k_j^C)$ and $k_j^C = k_j^*(k_i^C)$. By examining the first order condition and the second order condition of equation (4.10), the next proposition shows that the two bidders will chose identical investment k^C in equilibrium, in which k^C depends only on the cost reducing technology $g(.)$.

Proposition 4.1. *Under concealment of quality with $N = 2$, both suppliers will select an identical investment $k_i^C = k_j^C = k^C$ in a subgame perfect equilibrium. k^C does not depend on $G(.)$.*

Proof. See Appendix. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4.1 comes from the ex ante symmetry across bidders at the time when they make investment decisions. At the optimal level of investment, the

marginal expected return from investment should equal the marginal cost of invest-ment, given that the opponent also invests optimally. Given the ex-ante symmetry of the bidders, the expected return of investment in auction is always equivalent for two bidders, and the marginal cost of investment depends only on technology $g(.)$. Therefore, the equilibrium investment k^C is identical across bidders and is independent of the quality distribution $G(q)$.

4.4.2 Equilibrium under Private Disclosure of Quality

Under private disclosure of quality, suppose a symmetric perfect Bayesian equilibrium investment strategy $k^D : [\underline{q}, \bar{q}] \to \mathbb{R}^+$ exists. Each supplier's optimal investment strategy k_i^D solves

$$\max_{k_i} \int_{q_j} \int_{\underline{v}}^{q_i - g(k_i) - E\eta_i} \left\{ 1 - H(q_j - g(k^D(q_j)) - \tau) \right\} d\tau dG(q_j) \qquad (4.11)$$

The equilibrium investment strategy of each bidder $k_i^D = k^D(q_i)$ is characterized by the first order condition of i's objective function given in equation (4.11).

The next proposition shows that privately disclosing quality q_i to each bidder will induce ex ante high quality suppliers to invest more aggressively compared to low quality suppliers. The symmetric equilibrium investment strategy $k^D(q_i)$ is increasing in q_i.

Proposition 4.2. *When there are only 2 bidders, under private disclosure of quality, the perfect Bayesian equilibrium investment $k^D(q_i)$ is increasing in q_i.*

Proof. See Appendix. □

Proposition 4.2 comes from the fact that the optimal investment decision of each bidder depends on the expected return of investment in the auction. Suppliers with higher quality products has higher expected probability of winning the auction than suppliers with lower quality products. The former has higher expected return for any given level of investment.

4.4.3 Equilibrium under Public Disclosure of Quality

Now suppose the auctioneer publicly announce the entire quality profile $\{q_i, q_j\}$ at the beginning of investment stage to all bidders. Under the public disclosure of quality, each bidder will hold different belief over the distribution of its opponent's value. Given $\{q_i, q_j\}$, and any level of opponent's invesment k_j, each bidder i will choose investment strategy k_i^* that solves

$$\max_{k_i} \int_{\underline{v}}^{q_i - g(k_i) - E\eta_i} \left(1 - H\big(q_j - g(k_j) - \tau\big)\right) d\tau \tag{4.12}$$

Under public disclosure of (q_i, q_j) to each bidder, the best response investment $k_i^*(k_j; q_i, q_j)$ to opponent's investment k_j is characterized by the first order condition of i's objective function given by equation (4.12) with k_j^A replaced by k_j. Let (k_i^A, k_j^A) denote the subgame perfect equilibrium investment profile under public disclosure (announcement) of qualities. For any quality profile (q_i, q_j), the subgame perfect equilibrium investment profile under public information disclosure (k_i^A, k_j^A) is defined as $k_i^A(q_i, q_j) = k_i^*(k_j^A; q_i, q_j)$ and $k_j^A(q_i, q_j) = k_j^*(k_i^A; q_i, q_j)$, in which $k_i^*(.; q_i, q_j)$ and $k_j^*(.; q_i, q_j)$ are each bidder's best response function.

The next proposition shows that given the same cost reducing technology

$g(.)$, publicly disclosing all bidders' qualities will further induce the high quality supplier to invest more aggressively, and the low quality supplier to invest less aggressively. Each bidder's equilibrium investment $k_i^A(q_i, q_j)$ under announcement of entire quality profile is increasing in $(q_i - q_j)$.

Proposition 4.3. *When there are only 2 bidders, under public disclosure of qualities* (q_i, q_j), *each bidder's best response investment* $k_i^*(k_j; q_i, q_j)$ *is increasing in* $(q_i - q_j)$ *and decreasing in* k_j. *The subgame perfect equilibrium investment* $k_i^A(q_i, q_j)$ *is increasing in* $(q_i - q_j)$.

Proof. See Appendix. $\square$

Proposition 4.3 comes from the fact that the higher quality bidder has higher expected return from investment, as the expected probability of winning the auction is higher. When the higher quality bidder knows exactly his ex ante advantage before the auction starts, his investment incentive will be stronger, while the lower quality bidder will be discouraged from investing given this information. This is because the pre-auction investments are strategic substitutes between bidders, and knowing that opponent has a low quality for certain will make the high quality bidder believe that the investment of the opponent is also low, which further increases the expected return from investment.

4.4.4 Revenue Comparison

At the beginning of auction, the expected value of bidder i with quality q_i and investment k_i before the realization of random cost component η_i is given by

$$V(k_i, q_i) = \int_{\underline{\eta}}^{\bar{\eta}} \left(q_i - g(k_i) - \eta_i \right) dH(\eta_i)$$

$$= q_i - g(k_i) - E\eta_i \tag{4.13}$$

Define $V(k_1^L, q_1)$ and $V(k_2^L, q_2)$ as the equilibrium expected value of the high quality supplier and the low quality supplier under information policy $L \in \{C, D, A\}$, in which C represents concealment, D represents private disclosure and A represents public disclosure (announcement), at the beginning of auction, given their equilibrium investments k_1^L, k_2^L under realizations $Q_1 = q_1, Q_2 = q_2$:

$$V(k_1^L, q_1) = q_1 - g(k_1^L) - E\eta$$

$$V(k_2^L, q_2) = q_2 - g(k_2^L) - E\eta \tag{4.14}$$

Under concealment of qualities, $k_1^C = k_2^C = k^C$. Under private disclosure of qualities, $k_1^D = k^D(q_1)$ and $k_2^D = k^D(q_2)$. Under public disclosure of qualities, $k_1^A = k_1^A(q_1, q_2)$ and $k_2^A = k_2^A(q_2, q_1)$.

The ex ante expected winner's payoff in the auction under policy $L \in \{C, D, A\}$ given distribution G is given by

$$E\Pi^L(G) = E[V(k_1^L, q_1) - V(k_2^L, q_2)|G] \tag{4.15}$$

The ex ante expected revenue to the auctioneer under policy $L \in \{C, D, A\}$ given distribution G is given by

$$ER^L(G) = E[V(k_2^L, q_2)|G] \tag{4.16}$$

As mentioned at the beginning of this chapter, when the auctioneer does not care about non-price characteristics and $\Delta(G) = 0$, the three information disclosure policy gives the same expected revenue:$ER^C(G) = ER^D(G) = ER^A(G)$. I will next analyze how the expected revenues change under the three different information provision policies as $\Delta(G)$ increases from 0 in order to compare the revenues of the three information provision policies when $\Delta(G) > 0$.

The next proposition shows that the expected revenues $ER^C(G)$, $ER^D(G)$ and $ER^A(G)$ are decreasing in $\Delta(G)$ under all three information provision schemes, when holding the expected quality constant. It can be shown that the negative impact of increasing $\Delta(G)$ on $ER^C(G)$ is weaker than that on $ER^D(G)$ and $ER^A(G)$, at any level of $\Delta(G) > 0$. This implies that when there are only 2 bidders, the ex ante expected revenue to the auctioneer is always highest under concealment of quality among the three information schemes.

Proposition 4.4. *When there are only 2 bidders, the expected revenue to the auctioneer $ER^L(G)$ is decreasing in $\Delta(G)$ for all $L \in \{C, D, A\}$. Moreover,*

$$\frac{dER^A(G)}{d\Delta(G)} < \frac{dER^C(G)}{d\Delta(G)} < 0$$
$$\frac{dER^D(G)}{d\Delta(G)} < \frac{dER^C(G)}{d\Delta(G)} < 0$$

(4.17)

When $\Delta(G) = 0$, $ER^C(G) = ER^D(G) = ER^A(G)$

When $\Delta(G) > 0$, $ER^C(G) > ER^D(G)$ and $ER^C(G) > ER^A(G)$. Both $ER^C(G) - ER^D(G)$ and $ER^C(G) - ER^A(G)$ are increasing in $\Delta(G)$.

Proof. See Appendix. □

Proposition 4.4 implies that when there are only 2 bidders, it is always optimal

for the auctioneer to conceal their qualities. When the auctioneer discloses her private values to the bidders, the lower quality bidder will be discouraged from making investments, which leads to lower expected value of the lower quality bidder and lower expected revenue in the auction.

The result of Proposition 4.4 comes from the fact that $\Delta(G)$ represents the dispersion of quality distribution G. Holding the expected quality constant and increasing $\Delta(G)$ will generate a mean preserving spread of the original distribution, under which it is more likely to observe a high value of q_1 and a low value of q_2. This is the only source that drives the fact that $E\Pi^C(G)$ being increasing in $\Delta(G)$ and $ER^C(G)$ being decreasing in $\Delta(G)$ under concealment of quality, as the equilibrium investment k^C is independent of G. This source also present under private disclosure of quality and public disclosure of quality. However, under private disclosure and public disclosure of quality, increasing $\Delta(G)$ will not only decrease the expected value of q_2, but also decrease the expected investment of the lower quality bidder, as the low quality bidder will be discouraged from investment by receiving a low quality signal. Therefore, the impact of $\Delta(G)$ on expected revenue is stronger when the auctioneer discloses her values than that when the auctioneer conceals her values. Moreover, the difference between expected revenues under any two information schemes is increasing in $\Delta(G)$, as the bidders' investment incentives will be affected by the information provided by the auctioneer more significantly when auctioneer cares more about non-price characteristics.

4.5 Conclusions

This chapter studies the information provision problem in a procurement auction where the auctioneer has private subjective valuations over the suppliers' products, and suppliers have opportunity to invest in cost reduction prior to entering the auction. In this paper, I analyze the equilibrium investment strategies of suppliers under concealment of auctioneer's private valuations, private disclosure of auctioneer's valuation, and public disclosure of auctioneer's valuations, and provide a revenue comparison among these three information provision schemes under the presence of 2 bidders. The main conclusions are summarized as below:

First, disclosing the auctioneer's private valuation over each supplier's quality will induce high quality suppliers to invest more aggressively and discourage low quality suppliers from making investments. This result comes from the fact that each bidder's expected return from investment is increasing in his quality. Therefore, providing more information will induce a more dispersed distribution of values in the auction through this differentiation effect at the investment stage. When there are only two bidders, providing more information will discourage the lower quality bidder from investment and reduce the expected revenue. This leads to the result that concealment gives the highest expected revenue among the three information provision schemes considered in this paper.

Second, when one information scheme dominates the other information scheme under given distribution G, the benefit of the better scheme over the worse scheme increases in the dispersion of quality $\Delta(G)$. This result comes from the fact that

$\Delta(G)$ measures how much the auctioneer cares about qualities relative to costs. When the auctioneer cares more about qualities, the impact of quality differentiation on bidders' investment incentives is stronger, and providing information on this quality differentiation has greater impact on the equilibrium distribution of values.

I will next provide a brief discussion on the more general case when there are $N \geq 2$ bidders. Define $\Delta(G, N) = E[q_1 - q_N | G, N]$. Then $\Delta(G, N)$ measures the dispersion of qualities among N bidders given distribution $G(.)$. It is natural to conjecture that given a fixed number of bidders N and distribution G, concealment is optimal only if N is small enough s.t. the expected value of second order statistics is decreasing in the dispersion of qualities. When N is large enough s.t. $E(q_2 | G, N)$ is increasing in $\Delta(G, N)$, then the rank order of expected revenues under three information provision policies will be reversed, and public disclosure will provide the highest expected revenue. When N approaches infinity, it is always optimal to publicly disclose all qualities. This result is consistent with Ganuza (2004) [71]'s finding that the optimal level of information provision is increasing in the number of bidders. When there are only 2 bidders, competition in auction is weak, and disclosing the auctioneer's private information will give more informational rent to the winner. In contrast, when the number of bidders is large, disclosing more information will promote competition among the high quality bidders and will increase the expected revenue. When the number of bidders approaches infinity so that the model approaches a perfectly competitive market where each seller captures zero informational rent, it is optimal for the auctioneer to disclose all information.

The findings in this chapter suggest a few directions for future research. First,

this chapter assumes that participation in the auction is costless and the number of bidders in the auction N is exogenous. Since the provision of the auctioneer's information also changes the ex ante expected payoff to the winner, it would be interesting to allow endogenous entry of bidders. If the quality information is disclosed before bidders make entry decisions, then low quality bidders will not enter the auction, which reduces the degree of competition and lowers the auctioneer's expected revenue. When disclosing more information is optimal under exogenous entry, the positive impact of information disclosure on expected revenue through inducing higher quality bidders investing more will be offset by the negative impact through preventing low quality bidders from participating. On the other hand, when there are very few bidders so that concealing information is optimal given this fixed number of bidders, disclosing information will yield higher expected payoff to the winner and therefore induce more bidders to enter, so the optimal information disclosure scheme again becomes ambiguous. The next step of this research may introduce entry cost to the model and study how the revenue ranking of three information provision schemes change when number of bidders is also endogenously determined by the information provision scheme.

Second, this chapter assumes that providing information to bidders is costless to the auctioneer, which is not a practical assumption, as communication between the auctioneer and bidders usually comes at a cost. When providing information is costly, the benefit of information disclosure to the auctioneer may be outweighed by the cost of communication. When the cost of information provision is independent of number of bidders, it would be optimal to disclose quality when N is large enough

since the benefit of information provision increases in N. However, when the cost of information provision also increases in N, the optimal level of information provision becomes ambiguous, and the next step of this study may include providing a characterization of the optimal level of information provision when providing information is costly.

Appendix A: Proofs for Chapter 2

Proof of Lemma 2.1:

Proof. I first show that if an equilibrium bidding strategy in a one-dimensional as-
sortative position auction is symmetric and strictly increasing, then the equilibrium
must be efficient. Let $\beta(x)$ denote the equilibrium bidding strategy. $\beta'(x) > 0$ im-
plies that $\beta(x_i) > \beta(x_j)$ for any $x_i > x_j$. Under the assortative ranking rule, bidder
i is placed above bidder j if $x_i > x_j$, so the equilibrium allocation must be efficient.

I next show that if an equilibrium of a one-dimensional assortative position
auction is efficient, then the equilibrium bidding strategy must be strictly increasing
and symmetric across bidders. Suppose an efficient equilibrium $\big(\beta_1(x_1), \beta_2(x_2), \cdots, \beta_N(x_N)\big)$
exists in a one-dimensional assortative position auction, then a bidder who receives
signal x_i must be placed above a bidder who receives a lower signal $x_j < x_i$ if both
win some position in equilibrium. For an arbitrary bidder i, take any value $x_i' > x_i$,
then there is positive probability that some of bidder i's opponents receive signals
between x_i and x_i', i.e., there exists $j \neq i$ with signal $x_j \in (x_i, x_i')$. Efficiency requires
that j is placed below i when bidder i receives x_i', and j is placed above i when i

receives x_i. Under the assortative ranking rule, this requires

$$\beta_j(x_j) < \beta_i(x_i')$$

$$\beta_j(x_j) > \beta_i(x_i)$$

(A.1)

Suppose $\beta_i(x_i') \leq \beta_i(x_i)$, then for any value of $\beta_j(x_j)$, it is impossible for condition (A.1) to hold, which yields a contradiction. Therefore, at any efficient equilibrium, bidder i must bid strictly higher when receiving signal x_i' than receiving signal x_i, i.e., for every bidder i, we must have

$$x_i' > x_i \quad \rightarrow \quad \beta_i(x_i') > \beta_i(x_i) \tag{A.2}$$

Therefore, every bidder must use a strictly increasing bidding strategy in an efficient equilibrium, so $\beta_i'(x_i) > 0$ for all i.

Next, suppose there exists an efficient equilibrium that is not symmetric, i.e., there exists $i \neq j$ s.t. $\beta_i(\hat{x}) \neq \beta_j(\hat{x})$ at some $\hat{x} \in [0, \bar{x}]$. Without loss of generality, assume $\beta_i(\hat{x}) < \beta_j(\hat{x})$ for some $\hat{x} \in [0, \bar{x}]$. Since $\beta_i(.)$ and $\beta_j(.)$ are continuous, there exists some $x_i, x_j \in [0, \bar{x}]$ s.t. $x_j < \hat{x} < x_i$, but $\beta_i(x_i) < \beta_j(x_j)$. Under the assortative ranking rule, this means that bidder j who receives the lower signal x_j will get a higher position than bidder i who receives the higher signal $x_i > x_j$, which contradicts the efficiency assumption. Therefore, if an efficient equilibrium exists in a one-dimensional assortative position auction, then the equilibrium bidding strategy must be symmetric across bidders, i.e., $\beta_i(.) = \beta(.)$ for all i.

□

Proof of Lemma 2.2:

Proof. Define $v^{1,2}(x_i, y_1, y_2)$ as bidder i's expected value per click conditional on her

own signal equals to x_i, the highest signal among her opponents Y_1 equals to y_1, the

second highest signal among her opponents Y_2 equals to y_2:

$$v^{1,2}(x_i, y_1, y_2) = E\big[v_i\big|X = x_i, Y_1 = y_1, Y_2 = y_2\big] \tag{A.3}$$

Suppose a monotonic Bayesian equilibrium bidding strategy $\beta(.)$ exists. For

any arbitrary bidder i, suppose all of i's opposing bidders follow the monotonic

Bayesian equilibrium bidding strategy $\beta(.)$. Let $\beta^{-1}(.)$ denote the inverse function

of $\beta(.)$. Then bidder i's best response bid b_i^* maximizes

$$
\begin{aligned}
\Pi(b_i|x_i) =& \int_0^{\beta^{-1}(b_i)} \int_0^{y_1} \alpha_1 \Big[v^{1,2}(x_i, y_1, y_2) - \beta(y_1)\Big] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1 \\
&+ \int_{\beta^{-1}(b_i)}^{\bar{x}} \int_0^{\beta^{-1}(b_i)} \alpha_2 \Big[v^{1,2}(x_i, y_1, y_2) - \beta(y_2)\Big] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1
\end{aligned}
\tag{A.4}
$$

in which $g_i^{2,1}(y_2, y_1|x_i)$ is the conditional joint density function of (Y_2, Y_1) given $X =$

x_i. Let $g_{1|2}(y_1|y_2, x_i)$ and $g_{2|1}(y_2|y_1, x_i)$ be conditional marginal densities of Y_1 given

(Y_2, X) and Y_2 given (Y_1, X) respectively. Let $g_1(y_1|x_i)$ and $g_2(y_2|x_i)$ be conditional

marginal densities of Y_1 and Y_2 given $X = x_i$ respectively, then $g_i^{2,1}(y_2, y_1|x_i) =$

$g_{1|2}(y_1|y_2, x_i)g_2(y_2|x_i) = g_{2|1}(y_2|y_1, x_i)g_1(y_1|x_i)$.

Take derivative of the objective function (A.4) with respect to b_i:

$$
\begin{aligned}
\frac{d\Pi(b_i|x_i)}{db_i} =& \frac{g_1(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))} \int_0^{\beta^{-1}(b_i)} \alpha_1 \Big[v^{1,2}(x_i, \beta^{-1}(b_i), y_2) - b_i\Big] g_{2|1}\big(y_2\big|\beta^{-1}(b_i), x_i\big) dy_2 \\
&- \frac{g_1(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))} \int_0^{\beta^{-1}(b_i)} \alpha_2 \Big[v^{1,2}(x_i, \beta^{-1}(b_i), y_2) - \beta(y_2)\Big] g_{2|1}\big(y_2\big|\beta^{-1}(b_i), x_i\big) dy_2 \\
&+ \frac{g_2(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))} \int_{\beta^{-1}(b_i)}^{\bar{x}} \alpha_2 \Big[v^{1,2}(x_i, y_1, \beta^{-1}(b_i)) - b_i\Big] g_{1|2}\big(y_1\big|\beta^{-1}(b_i), x_i\big) dy_1
\end{aligned}
\tag{A.5}
$$

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Since $\beta(x_i)$ is an equilibrium, it is optimal for bidder i to bid $b_i^* = \beta(x_i)$ when her opponents follow $\beta(.)$. Evaluate $\frac{d\Pi(b_i)}{db_i}$ at $b_i^* = \beta(x_i)$:

$$
\begin{aligned}
\frac{d\Pi(\beta(x_i)|x_i)}{db_i} =& \frac{g_1(x_i|x_i)}{\beta'(x_i)} \int_0^{x_i} \alpha_1 \Big[v^{1,2}(x_i, x_i, y_2) - \beta(x_i) \Big] g_{2|1}(y_2|x_i, x_i) dy_2 \\
& - \frac{g_1(x_i|x_i)}{\beta'(x_i)} \int_0^{x_i} \alpha_2 \Big[v^{1,2}(x_i, x_i, y_2) - \beta(y_2) \Big] g_{2|1}(y_2|x_i, x_i) dy_2 \quad \text{(A.6)} \\
& + \frac{g_2(x_i|x_i)}{\beta'(x_i)} \int_{x_i}^{\bar{x}} \alpha_2 \Big[v^{1,2}(x_i, y_1, x_i) - \beta(x_i) \Big] g_{1|2}(y_1|x_i, x_i) dy_1
\end{aligned}
$$

According to the definition of $v^1(x_i, x_i)$ and $v^2(x_i, x_i)$, equation (A.6) is equivalent to

$$
\begin{aligned}
\frac{d\Pi(\beta(x_i)|x_i)}{db_i} =& \frac{g_1(x_i|x_i)}{\beta'(x_i)} \alpha_1 \Big[v^1(x_i, x_i) - \beta(x_i) \Big] \\
& - \frac{g_1(x_i|x_i)}{\beta'(x_i)} \alpha_2 \Big[v^1(x_i, x_i) - \int_0^{x_i} \beta(y_2) g_{2|1}(y_2|x_i, x_i) dy_2 \Big] \quad \text{(A.7)} \\
& + \frac{g_2(x_i|x_i)}{\beta'(x_i)} \alpha_2 \Big[v^2(x_i, x_i) - \beta(x_i) \Big]
\end{aligned}
$$

Bidding $b_i^* = \beta(x_i)$ maximizes $\Pi(b_i|x_i)$ only if $\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = 0$. Setting equation (A.7) to zero and rearrange yields

$$
\beta(x_i) = \frac{g_1(x_i|x_i) \Big[(\alpha_1 - \alpha_2) v^1(x_i, x_i) + \alpha_2 \int_0^{x_i} \beta(y_2) g_{2|1}(y_2|x_i, x_i) dy_2 \Big] + g_2(x_i|x_i) \alpha_2 v^2(x_i, x_i)}{\alpha_1 g_1(x_i|x_i) + \alpha_2 g_2(x_i|x_i)}
$$

(A.8)

This is a Volterra equation of the second kind. In the one-dimensional GSP Auction with 2 positions, if a monotonic equilibrium bidding strategy $\beta^G(x_i)$ exists, then $\beta^G(x_i)$ must satisfy the Volterra equation (A.8) for all $x_i \in [0, \bar{x}]$.

$\square$

Proof of Lemma 2.3:

Proof. Suppose a monotonic symmetric Bayesian equilibrium bidding strategy $\beta(.)$ exists in the one-dimensional VCG auction. For an arbitrary bidder i, suppose all

of i's opposing bidders follow the monotonic Bayesian equilibrium bidding strategy $\beta(.)$. Let $\beta^{-1}(.)$ denote the inverse function of $\beta(.)$. Then bidder i's best response bid b_i^* maximizes

$$\Pi(b_i|x_i) = \int_0^{\beta^{-1}(b_i)} \int_0^{y_1} \left\{ \alpha_1 \left[v^{1,2}(x_i, y_1, y_2) - \beta(y_1) \right] + \alpha_2 \left[\beta(y_1) - \beta(y_2) \right] \right\} g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1$$
$$+ \int_{\beta^{-1}(b_i)}^{\bar{x}} \int_0^{\beta^{-1}(b_i)} \alpha_2 \left[v^{1,2}(x_i, y_1, y_2) - \beta(y_2) \right] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1$$
(A.9)

Take derivative with respect to b_i:

$$\frac{d\Pi(b_i|x_i)}{db_i} = \frac{g_1(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))} \int_0^{\beta^{-1}(b_i)} (\alpha_1 - \alpha_2) \left[v^{1,2}(x_i, \beta^{-1}(b_i), y_2) - b_i \right] g_{2|1}\left(y_2|\beta^{-1}(b_i), x_i\right) dy_2$$
$$+ \frac{g_2(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))} \int_{\beta^{-1}(b_i)}^{\bar{x}} \alpha_2 \left[v^{1,2}(x_i, y_1, \beta^{-1}(b_i)) - b_i \right] g_{1|2}\left(y_1|\beta^{-1}(b_i), x_i\right) dy_1$$
(A.10)

Since $\beta(x_i)$ is an equilibrium, $b_i^* = \beta(x_i)$ maximizes $\Pi(b_i|x_i)$ for any value of x_i. For all $x_i \in [0, \bar{x}]$, evaluate $\frac{d\Pi(b_i|x_i)}{db_i}$ at $b_i^* = \beta(x_i)$ gives

$$\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = \frac{g_1(x_i|x_i)}{\beta'(x_i)} \int_0^{x_i} (\alpha_1 - \alpha_2) \left[v^{1,2}(x_i, x_i, y_2) - \beta(x_i) \right] g_{2|1}\left(y_2|x_i, x_i\right) dy_2$$
$$+ \frac{g_2(x_i|x_i)}{\beta'(x_i)} \int_{x_i}^{\bar{x}} \alpha_2 \left[v^{1,2}(x_i, y_1, x_i) - \beta(x_i) \right] g_{1|2}\left(y_1|x_i, x_i\right) dy_1$$
(A.11)

According to the definition of $v^1(x_i, x_i)$ and $v^2(x_i, x_i)$, this is equivalent to

$$\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = \frac{g_1(x_i|x_i)}{\beta'(x_i)}(\alpha_1 - \alpha_2) \left[v^1(x_i, x_i) - \beta(x_i) \right] + \frac{g_2(x_i|x_i)}{\beta'(x_i)} \alpha_2 \left[v^2(x_i, x_i) - \beta(x_i) \right]$$
(A.12)

Bidding $\beta(x_i)$ maximizes $\Pi(b_i|x_i)$ only if $\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = 0$, which means that bidder i cannot increase $\Pi(b_i|x_i)$ by increasing or decreasing bid from $\beta(x_i)$ by any small amount. Set $\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = 0$ and rearrange the equation yields

$$\beta(x_i) = \frac{g_1(x_i|x_i)(\alpha_1 - \alpha_2)v^1(x_i, x_i) + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2}$$
(A.13)

which characterizes the unique equilibrium bidding strategy $\beta^V(x_i)$ in the one-dimensional VCG auction. $\square$

Proof of Proposition 2.1:

Proof. Suppose the unique equilibrium bidding strategy $\beta^G(x_i)$ characterized in Lemma 2.2 is continuous and strictly increasing in x_i given any CTR profile (α_1, α_2). First observe that since $\beta^G(.)$ is continuous, when x_i approaches $\bar{x}$, the equilibrium bid $\beta^G(x_i)$ characterized in Lemma 2.2 approaches $\beta^G(\bar{x})$:

$$\lim_{x_i \to \bar{x}} \beta^G(x_i) = \beta^G(\bar{x}) = \left(\frac{\alpha_1 - \alpha_2}{\alpha_1}\right) v^1(\bar{x}, \bar{x}) + \left(\frac{\alpha_2}{\alpha_1}\right) \int_0^{\bar{x}} \beta(y_2) g_{2|1}(y_2|\bar{x}, \bar{x}) dy_2 \quad (A.14)$$

Consider the case when α_2 approaches α_1, the equilibrium bid $\beta^G(\bar{x})$ approaches

$$\lim_{\alpha_2 \to \alpha_1} \beta^G(\bar{x}) = \int_0^{\bar{x}} \beta(y_2) g_{2|1}(y_2|\bar{x}, \bar{x}) dy_2 \quad (A.15)$$

which implies that $\beta^G(\bar{x})$ satisfies

$$\lim_{\alpha_2 \to \alpha_1} \int_0^{\bar{x}} \left(\beta^G(\bar{x}) - \beta^G(y_2)\right) g_{2|1}(y_2|\bar{x}, \bar{x}) dy_2 = 0 \quad (A.16)$$

However, equation (A.16) yields a contradiction to the assumption that $\beta^G(x_i)$ is strictly increasing in x_i, since for any strictly increasing function, $\beta^G(\bar{x}) > \beta^G(y_2)$ for any $0 \le y_2 < \bar{x}$ and $\beta^G(\bar{x}) = \beta^G(y_2)$ at $y_2 = \bar{x}$. Therefore, it is impossible for any strictly increasing $\beta^G(x_i)$ to satisfy equation (A.16) at $x_i = \bar{x}$. Since $\beta^G(x_i)$ approaches $\beta^G(\bar{x})$ when x_i approaches $\bar{x}$, this contradiction also applies to any x_i

sufficiently close to $\bar{x}$. Therefore, it is impossible for the equilibrium $\beta^G(x_i)$ characterized by Lemma 2.2 to be strictly increasing under every CTR profile. Since $\beta^G(x_i)$ is the unique equilibrium bidding strategy, there exists no monotonic equilibrium in the one-dimensional GSP auction with two positions under some CTR profile. Given the result of Lemma 2.1, this implies that there exists some number of positions K with some CTR profile such that no efficient equilibrium exists in the one-dimensional GSP auction. □

Proof of Proposition 2.2:

Proof. Define $\gamma(x_i; \alpha_1, \alpha_2)$ as the weighting factor in the equilibrium bidding function $\beta^V(x_i)$ characterized in Lemma 2.3:

$$\gamma(x_i; \alpha_1, \alpha_2) = \frac{g_1(x_i|x_i)(\alpha_1 - \alpha_2)}{g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2} \tag{A.17}$$

then the equilibrium bidding strategy characterized in Lemma 2.3 can be rewritten as

$$\beta^V(x_i) = \gamma(x_i; \alpha_1, \alpha_2)v^1(x_i, x_i) + \left(1 - \gamma(x_i; \alpha_1, \alpha_2)\right)v^2(x_i, x_i) \tag{A.18}$$

Take derivative of $\beta(x_i) = \gamma(x_i)v^1(x_i, x_i) + (1 - \gamma(x_i))v^2(x_i, x_i)$ with respect to x_i:

$$\frac{d\beta^V(x_i)}{dx_i} = \underbrace{\gamma(x_i)\left[\frac{\partial v^1(x_i, x_i)}{\partial x_i}\right] + (1 - \gamma(x_i))\left[\frac{\partial v^2(x_i, x_i)}{\partial x_i}\right]}_{\text{bid-increasing incentive from higher expected values}} + \underbrace{\frac{\partial \gamma(x_i)}{\partial x_i}\left[v^1(x_i, x_i) - v^2(x_i, x_i)\right]}_{\text{bid-shading incentive from the "winner's curse"}} \tag{A.19}$$

The first two terms in equation (A.19) capture the positive effect of greater expected values on $\beta^V(x_i)$ when x_i increases. As x_i increases, the expected values conditional on winning both position 1 and position 2.2 increase, which causes equilibrium bid

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$\beta^V(x_i)$ to increase. The last term captures the negative effect of the "winner's curse" on $\beta^V(x_i)$. As x_i increases, bidder i is more likely to win the first position at any monotonic equilibrium, which amplifies the "winner's curse." When the negative effect from the "winner's curse" dominates the positive effect from increased expected values, the sign of $\frac{d\beta^V(x_i)}{dx_i}$ can be negative.

Note that given any CTR profile (α_1, α_2), for any $x_i \in [0, \bar{x}]$, the magnitude of the "winner's curse," $v^1(x_i, x_i) - v^2(x_i, x_i)$, is multiplied by $\frac{\partial \gamma(x_i)}{\partial x_i}$. The later can be expressed as

$$\frac{\partial \gamma(x_i; \alpha_1, \alpha_2)}{\partial x_i} = \frac{(\alpha_1 - \alpha_2)\alpha_2 \left[\frac{\partial g_1(x_i|x_i)}{\partial x_i} g_2(x_i|x_i) - g_1(x_i|x_i) \frac{\partial g_2(x_i|x_i)}{\partial x_i} \right]}{\left[g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2 \right]^2} > 0 \quad (A.20)$$

For any CTR profile (α_1, α_2) satisfying $0 < \alpha_2 < \alpha_1$, take limit of $\frac{\partial \gamma(x_i; \alpha_1, \alpha_2)}{\partial x_i}$ when x_i approaches $\bar{x}$ yields

$$\lim_{x_i \to \bar{x}} \frac{\partial \gamma(x_i; \alpha_1, \alpha_2)}{\partial x_i} = -\left(\frac{\alpha_2}{\alpha_1 - \alpha_2} \right) \times \frac{1}{g_1(\bar{x}|\bar{x})} \times \frac{\partial g_2(\bar{x}|\bar{x})}{\partial x_i} \quad (A.21)$$

When α_2 is sufficiently close to α_1, the denominator becomes sufficiently close to 0 so that $\lim_{x_i \to \bar{x}} \frac{\partial \gamma(x_i; \alpha_1, \alpha_2)}{\partial x_i}$ approaches infinity. As long as $v^1(x_i, x_i) < v^2(x_i, x_i)$, the negative impact from "winner's curse" will be dominant when x_i is sufficiently close to $\bar{x}$ and α_2 is sufficiently close to α_1. Therefore, under any non-trivially interdependent values, when there are $K = 2$ positions, there always exists some CTR profile (α_1, α_2) in which α_2 is strictly lower than but sufficiently close to α_1 s.t. the equilibrium bid $\beta^V(x_i)$ is decreasing in x_i for values of x_i close to the upper boundary $\bar{x}$. This demonstrates that there always exists some number of

positions K with some CTR profile such that no efficient equilibrium exists in the one-dimensional VCG auction. $\square$

Proof of Lemma 2.4:

Proof. It is straightforward to see that if every bidder adopts a symmetric and strictly increasing bidding strategy for every position k in equilibrium of a K-dimensional assortative position auction, then the equilibrium allocation is always efficient. Let $\beta(x) = \big(\beta_1(x), \cdots, \beta_K(x)\big)$ be the symmetric equilibrium bidding strategy. Since $\beta_k(x)$ is strictly increasing for every k, the bidder with the highest signal will submit the highest bid for position 1 and win position 1. The bidder with the second highest signal will submit the highest bid among the rest of bidders and win position 2, etc. The equilibrium allocation will rank bidders according to their signals and therefore is efficient.

I will next show that an equilibrium of a K-dimensional assortative position auction is efficient only if every bidder uses a symmetric and strictly increasing bidding strategy $\beta_k(x)$ for any position k. Suppose an efficient equilibrium exists in a K-dimensional assortative auction, then a bidder who receives a signal x_i must be placed above a bidder who receives a lower signal $x_j < x_i$ if both bidders receive some position in equilibrium. Pick an arbitrary bidder i, for any position $k \in [1, K]$, take any value $x_i' > x_i$, then there is positive probability that there are exactly $(k-1)$ bidders who receive signals above x_i' and one bidder $j \neq i$ who receives signal $x_j \in (x_i, x_i')$. Efficiency requires that bidder i wins position k if bidder i receives signal x_i', and bidder j wins position k if bidder i receives signal x_i. With the K-

dimensional assortative ranking rule, bidder i's bid for position k must always be higher than bidder j's bid for position k when receiving x_i', and bidder i's bid for position k must always be lower than bidder j's bid for position k when receiving x_i:

$$\beta_{ik}(x_i') > \beta_{jk}(x_j)$$
$$\beta_{ik}(x_i) < \beta_{jk}(x_j)$$

(A.22)

This is only possible when $\beta_{ik}(x_i') > \beta_{ik}(x_i)$. Therefore, for every bidder i and every position k, we must have

$$x_i' > x_i \quad \rightarrow \quad \beta_{ik}(x_i') > \beta_{ik}(x_i)$$

(A.23)

which means $\beta_{ik}(x_i)$ is strictly increasing in x_i for every i and every k.

Next, I will show that any efficient equilibrium in a K-dimensional assortative position auction must be symmetric across bidders. Suppose the equilibrium is not symmetric, i.e., there exists some $k \in [1, K]$ and $i \neq j$ s.t. $\beta_{ik}(\hat{x}) \neq \beta_{jk}(\hat{x})$ for some $\hat{x} \in [0, \bar{x}]$. Without loss of generality, assume $\beta_{ik}(\hat{x}) > \beta_{jk}(\hat{x})$. Since $\beta_{ik}(.)$ and $\beta_{jk}(.)$ are continuous, there exists x_i, x_j s.t. $x_i < \hat{x} < x_j$, and $\beta_{ik}(x_i) > \beta_{jk}(x_j)$. There is positive probability that there are exactly $(k-1)$ bidders other than i and j receive signals above x_j. Since $x_i < \hat{x} < x_j$, efficiency requires that bidder j wins position k. However, with the K-dimensional assortative ranking rule, $\beta_{ik}(x_i) > \beta_{jk}(x_j)$ implies that bidder j cannot win position k, which yields a contradiction. Therefore, it is impossible to have $\beta_{ik}(\hat{x}) \neq \beta_{jk}(\hat{x})$ for any i, j, any $k \in [1, K]$, and any value of $\hat{x}$. In any efficient equilibrium, each bidder must use a symmetric bidding strategy $\beta_{ik}(.) = \beta_k(.)$ for every position k. $\qquad \square$

Proof of Proposition 2.3:

Proof. For any arbitrary bidder i, let $g_i^{\{k\}}(y_k, \cdots, y_1 | x_i)$ be the joint density of $(Y_k, Y_{k-1}, \cdots, Y_1)$ conditional on $X = x_i$, according to the joint distribution of signals $F(x_1, \cdots, x_N)$. Define $v^{\{k\}}(x_i; y_1, y_2, \cdots, y_k)$ as bidder i's expected value per click conditional on her own signal X equals to x_i, the highest signal Y_1, the second highest signal Y_2, ..., the k-th highest signal Y_k received by her opponents equals to $(y_1, y_2, \cdots, y_k)$:

$$v^{\{k\}}(x_i; y_1, y_2, \cdots, y_k) = E\left[v_i \middle| X = x_i, Y_1 = y_1, Y_2 = y_2, \cdots, Y_k = y_k\right] \qquad \text{(A.24)}$$

Suppose all of bidder i's opposing bidders follow the monotonic Bayesian equilibrium bidding strategy $\beta(.) = (\beta_1(.), \beta_2(.), \cdots, \beta_K(.))$ in the K-dimensional GSP auction. Let $\beta_k^{-1}(.)$ denote the inverse function of $\beta_k(.)$. Then bidder i's best response bid $(b_i^{1*}, b_i^{2*}, \cdots, b_i^{K*})$ maximizes

$$\int_0^{\beta_1^{-1}(b_i^1)} \alpha_1 \left[v^{\{1\}}(x_i, y_1) - \beta_1(y_1)\right] g_i^{\{1\}}(y_1 | x_i) dy_1$$
$$+ \int_{\beta_1^{-1}(b_i^1)}^1 \int_0^{\beta_2^{-1}(b_i^2)} \alpha_2 \left[v^{\{2\}}(x_i, y_1, y_2) - \beta_2(y_2)\right] g_i^{\{2\}}(y_2, y_1 | x_i) dy_2 dy_1$$
$$+ \int_{\beta_1^{-1}(b_i^1)}^1 \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_0^{\beta_3^{-1}(b_i^3)} \alpha_3 \left[v^{\{3\}}(x_i, y_1, y_2, y_3) - \beta_3(y_3)\right] g_i^{\{3\}}(y_3, y_2, y_1 | x_i) dy_3 dy_2 dy_1$$
$$+ \cdots$$
$$+ \int_{\beta_1^{-1}(b_i^1)}^1 \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_{\beta_3^{-1}(b_i^3)}^{y_2} \cdots \int_0^{\beta_K^{-1}(b_i^K)} \alpha_K \left[v^{\{K\}}(x_i, y_1, \cdots, y_K) - \beta_K(y_K)\right] g_i^{\{K\}}(y | x_i) dy$$
$$\text{(A.25)}$$

Define $\Pi_k^G \left(x_i, y_1, \cdots, y_k\right)$ as

$$\Pi_k^G \left(x_i, y_1, \cdots, y_k\right) = \alpha_k \left[v^{\{k\}}(x_i, y_1, \cdots, y_k) - \beta_k(y_k)\right] \times g_i^{\{k\}}(y_k, \cdots, y_1 | x_i) \quad \text{(A.26)}$$

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Then the objective function (A.25) can be rewritten as

$$\Pi(b_i|x_i) = \underbrace{\int_0^{\beta_1^{-1}(b_i^1)} \Pi_1^G\big(x_i, y_1\big) dy_1}_{A_1}$$

$$+ \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_0^{\beta_2^{-1}(b_i^2)} \Pi_2^G\big(x_i, y_1, y_2\big) dy_2 dy_1}_{A_2}$$

$$+ \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_0^{\beta_3^{-1}(b_i^3)} \Pi_3^G\big(x_i, y_1, \cdots, y_3\big) dy_3 dy_2 dy_1}_{A_3}$$

$$+ \cdots$$

$$+ \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_{\beta_3^{-1}(b_i^3)}^{y_2} \cdots \int_0^{\beta_K^{-1}(b_i^K)} \Pi_K^G\big(x_i, y_1, \cdots, y_K\big) dy_K \cdots dy_1}_{A_K}$$

$$(A.27)$$

Let A_k denote the k-th term in the objective function (A.27). The definitions of A_1, A_2 and A_3 are shown in the objective function (A.27). For any $k \geq 3$, A_k is given by

$$A_k = \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}}}_{(k-1)} \int_0^{\beta_k^{-1}(b_i^k)} \Pi_k^G\big(x_i, y_1, \cdots, y_k\big) dy_k \cdots dy_2 dy_1$$

$$(A.28)$$

The first order condition with respect to $b_i^1, b_i^2, \cdots, b_i^K$ is given by

$$\frac{\partial A_1}{\partial b_i^1} + \frac{\partial A_2}{\partial b_i^1} + \frac{\partial A_3}{\partial b_i^1} + \cdots + \frac{\partial A_K}{\partial b_i^1} = 0$$

$$\frac{\partial A_2}{\partial b_i^2} + \frac{\partial A_3}{\partial b_i^2} + \cdots + \frac{\partial A_K}{\partial b_i^2} = 0$$

$$\cdots \qquad (A.29)$$

$$\frac{\partial A_{K-1}}{\partial b_i^{K-1}} + \frac{\partial A_K}{\partial b_i^{K-1}} = 0$$

$$\frac{\partial A_K}{\partial b_i^K} = 0$$

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Since each b_i^k enters $A_k, A_{k+1}, \cdots, A_K$, but does not enter any $A_{k'}$ with $k' < k$.

For any $1 \leq k \leq K$, take derivative of A_k with respect to b_i^k and replacing b_i^{n*} by $\beta_n(x_i)$ for all $n \in \{1, 2, \cdots, K\}$ yields

$$\frac{\partial A_k}{\partial b_i^k} = \frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \Pi_k^G\Big(x_i, y_1, \cdots, y_{k-1}, \beta_k^{-1}(b_i^k)\Big) dy_{k-1} \cdots dy_1$$

$$= \frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \Pi_k^G\Big(x_i, y_1, \cdots, y_{k-1}, x_i\Big) dy_{k-1} \cdots dy_1$$

$$= \frac{g_k(x_i|x_i)}{\beta_k'(x_i)} \alpha_k \Big[v^k(x_i, x_i) - b_i^k\Big]$$

$$\text{(A.30)}$$

Take derivative of A_{k+1} with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all $n \in \{1, 2, \cdots, K\}$ yields

$$\frac{\partial A_{k+1}}{\partial b_i^k} = -\frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_0^{\beta_{k+1}^{-1}(b_i^{k+1})} \Pi_{k+1}^G\Big(x_i, y_1, \cdots, \beta_k^{-1}(b_i^k), y_{k+1}\Big)$$

$$dy_{k+1} dy_{k-1} \cdots dy_1$$

$$= -\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_0^{x_i} \Pi_{k+1}^G\Big(x_i, y_1, \cdots, x_i, y_{k+1}\Big) dy_{k+1} dy_{k-1} \cdots dy_1$$

$$= -\frac{g_k(x_i|x_i)}{\beta_k'(x_i)} \alpha_k \Big[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1}\big(y_{k+1}\big|x_i, x_i\big)\Big]$$

$$\text{(A.31)}$$

Take derivative of A_{k+2} with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all $n \in \{1, 2, \cdots, K\}$ yields

$$\frac{\partial A_{k+2}}{\partial b_i^k} = \frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \times$$

$$\int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_{\beta_{k+1}^{-1}(b_i^{k+1})}^{\beta_k^{-1}(b_i^k)} \int_0^{\beta_{k+2}^{-1}(b_i^{k+2})} \Pi_{k+2}^G\Big(x_i, y_1, \cdots, \beta_k^{-1}(b_i^k), y_{k+1}, y_{k+2}\Big) dy_{k+2} dy_{k+1} \cdots dy_1$$

$$= -\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_{x_i}^{x_i} \int_0^{x_i} \Pi_{k+2}^G\Big(x_i, y_1, \cdots, x_i, y_{k+1}, y_{k+2}\Big) dy_{k+2} dy_{k+1} dy_{k-1} \cdots dy_1$$

$$= 0$$

$$\text{(A.32)}$$

This is because the integral of any continuous function on $[x_i, x_i]$ is zero. For any

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A_n with $n \geq k+2$, $\frac{\partial A_n}{\partial b_i^k}$ also contains an integral on $[x_i, x_i]$. Therefore,

$$\frac{\partial A_n}{\partial b_i^k} = 0, \quad \forall n \neq k, k+1 \tag{A.33}$$

Therefore, the K first order conditions of the objective function characterized in (A.29) can be simplified to

$$\frac{\partial A_k}{\partial b_i^k} + \frac{\partial A_{k+1}}{\partial b_i^k} = 0, \quad \forall k \in \{1, 2, \cdots, K-1\}$$
$$\frac{\partial A_K}{\partial b_i^K} = 0 \tag{A.34}$$

For the last position K, the equilibrium bid $b_i^{K*} = \beta_K(x_i)$ is defined by $\frac{\partial A_K}{\partial b_i^K} = 0$, i.e.,

$$\frac{g_K(x_i|x_i)}{\beta_K'(x_i)} \alpha_K \left[v^K(x_i, x_i) - \beta_K(x_i) \right] = 0 \tag{A.35}$$

so the Bayesian equilibrium bidding strategy for the last position K in the K-dimensional GSP auction is

$$\beta_K(x_i) = v^K(x_i, x_i) \tag{A.36}$$

For any position $k \in \{1, 2, \cdots, K-1\}$, the equilibrium bid $b_i^{k*} = \beta_k(x_i)$ is characterized by $\frac{\partial A_k}{\partial b_i^k} + \frac{\partial A_{k+1}}{\partial b_i^k} = 0$, i.e.,

$$\frac{g_k(x_i|x_i)}{\beta_k'(x_i)} \left[\alpha_k \left[v^k(x_i, x_i) - \beta_k(x_i) \right] - \alpha_{k+1} \left[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1|k}(y_k|x_i, x_i) \right] \right] = 0 \tag{A.37}$$

Rearranging equation (A.37) gives the equilibrium bidding strategy $\beta_k(x_i)$ for any position above K in the K-dimensional GSP auction:

$$\beta_k(x_i) = v^k(x_i, x_i) - \frac{\alpha_{k+1}}{\alpha_k} \left[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1|k}(y_k|x_i, x_i) \right], \quad \forall k \in [1, K-1] \tag{A.38}$$

$\square$

Proof of Proposition 2.4:

Proof. Suppose all of bidder i's opposing bidders follow a monotonic Bayesian equilibrium bidding strategy $\beta(x) = \big(\beta_1(.), \beta_2(.), \cdots, \beta_K(.)\big)$ in the K-dimensional VCG auction. Let $\beta_k^{-1}(.)$ denote the inverse function of $\beta_k(.)$. Let $g_i^{\{K\}}(y_K, \cdots, y_1|x_i)$ be the joint density of $(Y_K, Y_{K-1}, \cdots, Y_1)$ conditional on $X = x_i$. Let $v^{\{K\}}(x_i; y_1, y_2, \cdots, y_K)$ be bidder i's expected value per click conditional on her own signal X equals to x_i, the highest signal Y_1, the second highest signal Y_2, ... the K-th highest signal Y_K received by her opponents equal to $(y_1, y_2, \cdots, y_K)$. Define $\Pi_k^V\big(x_i, y_1, \cdots, y_K\big)$ as

$$\Pi_k^V\big(x_i, y_1, \cdots, y_K\big) = \left[\alpha_k v^{\{K\}}(x_i, y_1, \cdots, y_K) - \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1})\beta_j(y_j)\right] \times g_i^{\{K\}}(y_K, \cdots, y_1|x_i)$$

(A.39)

Then bidder i's best response bid $(b_i^{1*}, b_i^{2*}, \cdots, b_i^{K*})$ maximizes

$$\Pi(b_i|x_i) = \underbrace{\int_0^{\beta_1^{-1}(b_i^1)} \int_0^{y_1} \int_0^{y_2} \cdots \int_0^{y_{K-1}} \Pi_1^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_1}$$

$$+ \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_0^{\beta_2^{-1}(b_i^2)} \int_0^{y_2} \cdots \int_0^{y_{K-1}} \Pi_2^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_2}$$

$$+ \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_0^{\beta_3^{-1}(b_i^3)} \cdots \int_0^{y_{K-1}} \Pi_3^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_3}$$

$$+ \cdots$$

$$+ \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_{\beta_3^{-1}(b_i^3)}^{y_2} \cdots \int_0^{\beta_K^{-1}(b_i^K)} \Pi_K^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_K}$$

(A.40)

Let B_k denote the k-th term in equation (A.40). B_1, B_2 and B_3 are given in

equation (A.40). For all $k \geq 3$, each B_k can be expressed as

$$B_k = \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}}}_{(k-1)} \int_{0}^{\beta_k^{-1}(b_i^k)} \underbrace{\int_{0}^{y_k} \cdots \int_{0}^{y_{K-1}}}_{(K-k)} \Pi_k^V\left(x_i, y_1, y_2, \cdots, y_K\right) dy_K \cdots dy_1$$

(A.41)

Since each b_i^k only enters $B_k, B_{k+1}, \cdots, B_K$, but not enter any $B_{k'}$ with $k' < k$, the first order condition of the objective function (A.40) with respect to $(b_i^1, b_i^2, \cdots, b_i^K)$ is given by

$$\frac{\partial B_1}{\partial b_i^1} + \frac{\partial B_2}{\partial b_i^1} + \frac{\partial B_3}{\partial b_i^1} + \cdots + \frac{\partial B_K}{\partial b_i^1} = 0$$

$$\frac{\partial B_2}{\partial b_i^2} + \frac{\partial B_3}{\partial b_i^2} + \cdots + \frac{\partial B_K}{\partial b_i^2} = 0$$

$$\cdots \tag{A.42}$$

$$\frac{\partial B_{K-1}}{\partial b_i^{K-1}} + \frac{\partial B_K}{\partial b_i^{K-1}} = 0$$

$$\frac{\partial B_K}{\partial b_i^K} = 0$$

Take derivative of B_k with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all $n \in \{1, 2, \cdots, K\}$ yields

$$\frac{dB_k}{db_i^k} = \frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \times$$

$$\int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_{0}^{\beta_k^{-1}(b_i^k)} \cdots \int_{0}^{y_{K-1}} \Pi_k^V\left(x_i, y_1, \cdots, y_{k-1}, \beta_k^{-1}(b_i^k), y_{k+1}, \cdots, y_K\right) dy_K \cdots dy_1$$

$$= \frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_{0}^{x_i} \cdots \int_{0}^{y_{K-1}} \Pi_k^V\left(x_i, y_1, \cdots, y_{k-1}, x_i, y_{k+1}, \cdots, y_K\right) dy_K \cdots dy_1$$

(A.43)

Take derivative of B_{k+1} with respect to b_i^k, and replacing b_i^n by $\beta_n(x_i)$ for all

$n \in \{1, 2, \cdots, K\}$ yields

$$\frac{dB_{k+1}}{db_i^k} = -\frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \times$$

$$\int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_0^{\beta_{k+1}^{-1}(b_i^{k+1})} \cdots \int_0^{y_{K-1}} \Pi_{k+1}^V \Big(x_i, y_1, \cdots, y_{k-1}, \beta_k^{-1}(b_i^k), y_{k+1}, \cdots, y_K \Big)$$

$$dy_K \cdots dy_1$$

$$= -\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_0^{x_i} \cdots \int_0^{y_{K-1}} \Pi_{k+1}^V \Big(x_i, y_1, \cdots, y_{k-1}, x_i, y_{k+1}, \cdots, y_K \Big) dy_K \cdots dy_1$$

$$\text{(A.44)}$$

Take derivative of B_{k+2} with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all

$n \in \{1, 2, \cdots, K\}$ yields

$$\frac{dB_{k+2}}{db_i^k} = -\frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \times$$

$$\int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_{\beta_{k+1}^{-1}(b_i^{k+1})}^{\beta_k^{-1}(b_i^k)} \int_0^{\beta_{k+2}^{-1}(b_i^{k+2})} \cdots \int_0^{y_{K-1}} \Pi_{k+2}^V \Big(x_i, y_1, \cdots, \beta_k^{-1}(b_i^k), \cdots, y_K \Big) dy$$

$$= -\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_{x_i}^{x_i} \int_0^{x_i} \cdots \int_0^{y_{K-1}} \Pi_{k+2}^V \Big(x_i, y_1, \cdots, y_{k-1}, x_i, y_{k+1}, \cdots, y_K \Big) dy_K \cdots dy_1$$

$$= 0$$

$$\text{(A.45)}$$

since the integral of any continuous function on $[x_i, x_i]$ is zero. At the equilibrium

where $b_i = \beta(x_i)$, $\frac{dB_n}{db_i^k}$ contains an integral on $[x_i, x_i]$ for any B_n with $n \geq k+2$, so

$\frac{dB_n}{db_i^k} = 0$ for all $n \neq k, k+1$. Therefore, the first order conditions characterized in

equation (A.42) becomes

$$\frac{dB_k}{db_i^k} + \frac{dB_{k+1}}{db_i^k} = 0, \quad \forall k \in [1, K-1]$$

$$\frac{dB_K}{db_i^K} = 0$$

$$\text{(A.46)}$$

For the last position K, the equilibrium bid $b_i^{K*} = \beta_K(x_i)$ is characterized by

$\frac{dB_K}{db_i^K} = 0$:

$$\frac{1}{\beta_K'(x_i)} \int_{x_i}^1 \cdots \int_{x_i}^{y_{K-2}} \int_0^{x_i} \Pi_K^V\Big(x_i, y_1, \cdots, y_{k-1}, x_i\Big) dy_{K-1} \cdots dy_1$$

$$= \frac{1}{\beta_K'(x_i)} \int_{x_i}^1 \cdots \int_{x_i}^{y_{K-2}} \int_0^{x_i} \alpha_K \Big[v^{\{K\}}(x_i, y_1, \cdots, y_{K-1}, x_i) - \beta_K(x_i)\Big] g_i^{\{K\}}(x_i, y_{K-1}, \cdots, y_1 | x_i) dy_K \cdots dy_1$$

$$= \frac{g_K(x_i | x_i)}{\beta_K'(x_i)} \alpha_K \Big[v^K(x_i, x_i) - \beta_K(x_i)\Big]$$

$$= 0$$

$$(A.47)$$

so the equilibrium bidding strategy for the last position K in the K-dimensional VCG auction is given by

$$\beta_K(x_i) = v^K(x_i, x_i) \tag{A.48}$$

For any position $1 \leq k \leq K - 1$, the equilibrium bid $b_i^{k*} = \beta_k(x_i)$ is characterized by $\frac{dB_k}{db_i^k} + \frac{dB_{k+1}}{db_i^k} = 0$:

$$\frac{1}{\beta_k'(x_i)} \int_{x_i}^1 \cdots \int_0^{x_i} \cdots \int_0^{y_{K-1}} \Big[\Pi_k^V(x_i, y_1, \cdots, x_i, \cdots, y_K) - \Pi_{k+1}^V(x_i, y_1, \cdots, x_i, \cdots, y_K)\Big] dy_K \cdots dy_1$$

$$= \frac{1}{\beta_k'(x_i)} \int_{x_i}^1 \cdots \int_0^{x_i} \cdots \int_0^{y_{K-1}} (\alpha_k - \alpha_{k+1}) \Big[v^{\{K\}}(x_i, y_1, \cdots, x_i, \cdots, y_K) - \beta_k(x_i)\Big] dy_K \cdots dy_1$$

$$= \frac{g_k(x_i | x_i)}{\beta_k'(x_i)} (\alpha_k - \alpha_{k+1}) \Big[v^k(x_i, x_i) - \beta_k(x_i)\Big]$$

$$= 0$$

$$(A.49)$$

Therefore, for any position above the last position K, the equilibrium bidding strategy $\beta_k(x_i)$ in the K-dimensional VCG auction is given by

$$\beta_k(x_i) = v^k(x_i, x_i), \quad \forall k \in \{1, 2, \cdots, K - 1\} \tag{A.50}$$

$\square$

Proof of Proposition 2.5:

Proof. First consider the case when no bidder has dropped out. When there are more than K bidders remaining in the auction, each bidder will not drop out until the expected payoff from the last position K falls below zero. Suppose all the opposing bidders adopt strategy b_N^* defined in proposition 2.5, $b_N^*(x_i) = v^{(K)}(x_i, x_i, \cdots, x_i)$. When all bidders are in the auction, at any price p, bidder i wins the last position K by dropping out right now only if there are $(N-K)$ bidders drop out simultaneously at this price, i.e., the lowest $(N-K)$ value bidders have the same signal $Y_K = Y_{K+1} = \cdots = Y_{N-1} = y_K$. Therefore, given that all opponents follow strategy $b_N^*(x)$, bidder i's expected value conditional on winning K is

$$\alpha_K v^{(K)}(x_i, y_K, \cdots, y_K) = \alpha_K E[v_i | X = x_i, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{N-1} = y_K]$$

$$(A.51)$$

Bidder i's expected payment conditional on winning K is

$$\alpha_K v^{(K)}(y_K, y_K, \cdots, y_K) = \alpha_K E[v_i | X = y_K, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{N-1} = y_K]$$

$$(A.52)$$

The expected payoff from the last position K is non-negative for bidder i if and only if $x_i \geq y_K$. By using strategy b_N^*, bidder i will win position K or some position above K if and only if $x_i \geq y_K$, so b_N^* is the best response bidding strategy for each bidder i when all bidders are sill in the auction, assuming all other bidders also adopt strategy b_N^*. This is an ex-post equilibrium, since b_N^* is bidder i's optimal strategy for any realization of opposing bidders' signals x_{-i}.

Next, consider the case when $(N - n)$ bidders have dropped out, but $n \geq K + 1$ bidders are still in the auction so that the allocation of no position has been

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determined. Similar to the case with N active bidders, each bidder will not drop out until the expected payoff from the last position K falls below zero. However, the expected payoff from the last position is now calculated conditional on the revealed signals of the $(N - n)$ drop-out bidders, $Y_n = y_n, \cdots, Y_{N-1} = y_{N-1}$, in which $y_n, y_{n+1}, \cdots, y_{N-2}, y_{N-1}$ are inferred from $b_N^*(y_{N-1}) = p_N$, $b_{N-1}^*(y_{N-2}|p_N) = p_{N-1}$, $b_{n+1}^*(y_n|p_N, \cdots, p_{n+2}) = p_{n+1}$. Assume all the remaining opposing bidders adopt strategy b_n^*. At any price p, bidder i will win the position K by dropping out at the current price only if the lowest-value $(n - K)$ bidders among the active bidders drop out simultaneously, i.e., they have the same signal $Y_K = \cdots = Y_{n-1} = y_K$. Bidder i's expected value upon winning K is

$$\alpha_K v^{(K)}\Big(x_i, \underbrace{y_K, \cdots, y_K}_{(n-K)}, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N - n) \text{ lowest signals}}\Big)$$

$$= \alpha_K E\big[v_i \big| X = x_i, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{n-1} = y_K, Y_n = y_n, \cdots, Y_{N-1} = y_{N-1}\big]$$

$$(A.53)$$

Her payment upon winning K is

$$\alpha_K v^{(K)}\Big(y_K, \underbrace{y_K, \cdots, y_K}_{(n-K)}, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N - n) \text{ lowest signals}}\Big)$$

$$= \alpha_K E\big[v_i \big| X = y_K, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{n-1} = y_K, Y_n = y_n, \cdots, Y_{N-1} = y_{N-1}\big]$$

$$(A.54)$$

Therefore, it is profitable to stay in the auction if and only if $x_i \geq y_K$. By using bidding strategy b_n^*, bidder i will win a position no lower than K if and only if $x_i \geq y_K$, so b_n^* is the best response bidding strategy for each bidder i when there are $K < n < N$ bidders in the auction. This is an ex-post equilibrium, since b_n^* is the best response given any realization of other bidders' signals.

Next, consider the case when $n \leq K$ bidders are left in the auction. When there are $n \leq K$ bidders left in the auction, all the remaining bidders will win some position, so the drop-out price of each bidder i only affect which position she gets. In equilibrium, a bidder with signal x_i should be indifferent between getting the current lowest position n at price p_{n+1} and the next best position $(n-1)$ at a higher price. Note that bidder i wins position $(n-1)$ at a higher price b only if the lowest-value remaining bidder drops out at b. Assuming that all remaining opposing bidders adopt strategy b_n^*, bidder i's expected payoff from winning the next best position $(n-1)$ given the revealed signals $(y_{n-1}, \cdots, y_N)$ is

$$
\begin{aligned}
E\Pi_{n-1} =& \alpha_{n-1} \left[v^{(n-1)}(x_i, y_{n-1}, y_n, \cdots, y_N) - b \right] \\
=& \alpha_{n-1} \left[v^{(n-1)}(x_i, y_{n-1}, y_n, \cdots, y_N) - v^{(n-1)}(y_{n-1}, y_{n-1}, y_n, \cdots, y_N) \right] \\
&+ \alpha_n \left[v^{(n-1)}(y_{n-1}, y_{n-1}, y_n, \cdots, y_N) - p_{n+1} \right]
\end{aligned}
$$

(A.55)

since

$$
b = v^{(n-1)}(y_{n-1}, y_{n-1}, y_n, \cdots, y_N) - \frac{\alpha_n}{\alpha_{n-1}} \left[v^{(n-1)}(y_{n-1}, y_{n-1}, y_n, \cdots, y_N) - p_{n+1} \right]
$$

(A.56)

Bidder i's expected payoff from winning the current lowest position n, given $(Y_{n-1}, \cdots, Y_N)$ is

$$
E\Pi_n = \alpha_n \left[v^{(n-1)}(x_i, y_{n-1}, y_n \cdots, y_N) - p_{n+1} \right]
$$

(A.57)

Subtracting equation (A.57) from equation (A.55), the expected payoff from staying in the auction and getting position $(n-1)$ is higher than the expected payoff from

dropping out right now and getting position n if and only if

$$E\Pi_{n-1} - E\Pi_n = (\alpha_{n-1} - \alpha_n)\Big[v^{(n-1)}(x_i, y_{n-1}, y_n \cdots, y_N) - v^{(n-1)}(y_{n-1}, y_{n-1}, y_n \cdots, y_N)\Big] \geq 0$$

(A.58)

Inequality (A.58) holds if and only if $x_i \geq y_{n-1}$. Therefore, by using bidding strategy b_n^*, bidder i wins a position no lower than $(n-1)$ if and only if $x_i \geq y_{n-1}$, so b_n^* is the best response bidding strategy for bidder i when there are $n < K$ bidders remain in the auction. This is an ex-post equilibrium at the time when n bidders are left in the auction, since b_n^* is bidder i's optimal strategy for any realization of the other bidders signals x_{-i}. Therefore, $(b^*, \cdots, b^*)$ characterized in Proposition 2.5 is an ex-post equilibrium in the Generalized English Auction with interdependent values. $\qquad\square$

Proof of Proposition 2.6:

Proof. I first compare expected revenues of the K-dimensional GSP auction and the K-dimensional VCG auction, and then compare expected revenues of the K-dimensional VCG auction and the GEA.

Let $\beta^V(x_i) = \big(\beta_1^V(x_i), \beta_2^V(x_i), \cdots, \beta_K^V(x_i)\big)$ and $\beta^G(x_i) = \big(\beta_1^G(x_i), \beta_2^G(x_i), \cdots, \beta_K^G(x_i)\big)$ denote the Bayesian equilibrium bidding strategies in the K-dimensional VCG auction and K-dimensional GSP auction, respectively. According to the characterization of $\beta^V(x_i)$ and $\beta^G(x_i)$ in Propositions 3 and 4, the expected prices for the last position K in the K-dimensional VCG auction and the K-dimensional GSP auction

are as follows:

$$E\left[p^{V,(K)}\right] = \alpha_K E\left[\beta_K^V(Y_K)\Big|\{Y_{K-1} > X > Y_K\}\right] = \alpha_K E\left[v^K(Y_K, Y_K)\Big|\{Y_{K-1} > X > Y_K\}\right]$$

$$E\left[p^{G,(K)}\right] = \alpha_K E\left[\beta_K^G(Y_K)\Big|\{Y_{K-1} > X > Y_K\}\right] = \alpha_K E\left[v^K(Y_K, Y_K)\Big|\{Y_{K-1} > X > Y_K\}\right]$$

$$(A.59)$$

For any position $1 \le k \le K - 1$, the expected price $E\left[p^{V,(k)}\right]$ in the K-dimensional VCG auction and the expected price $E\left[p^{G,(k)}\right]$ in the K-dimensional GSP auction are given below:

$$E\left[p^{V,(k)}\right] = (\alpha_k - \alpha_{k+1})E\left[\beta_k^V(Y_k)\Big|\{Y_{k-1} > X > Y_k\}\right] + E\left[p^{V,(k+1)}\right]$$

$$= (\alpha_k - \alpha_{k+1})E\left[v^k(Y_k, Y_k)\Big|\{Y_{k-1} > X > Y_k\}\right] + E\left[p^{V,(k+1)}\right]$$

$$E\left[p^{G,(k)}\right] = \alpha_k E\left[\beta_k^G(Y_k)\Big|\{Y_{k-1} > X > Y_k\}\right]$$

$$= \alpha_k E\left[v^k(Y_k, Y_k) - \left[\frac{\alpha_{k+1}}{\alpha_k}v^k(Y_k, Y_k) - E\left[\beta_{k+1}^G(Y_{k+1})\right]\right]\Big|\{Y_{k-1} > X > Y_k\}\right]$$

$$= (\alpha_k - \alpha_{k+1})E\left[v^k(Y_k, Y_k)\Big|\{Y_{k-1} > X > Y_k\}\right] + E\left[p^{G,(k+1)}\right]$$

$$(A.60)$$

Equation (A.59) and (A.60) imply that

$$E\left[p^{V,(k)}\right] - E\left[p^{V,(k+1)}\right] = E\left[p^{G,(k)}\right] - E\left[p^{G,(k+1)}\right], \quad \forall k \in \{1, 2, \cdots, K-1\}$$

$$E\left[p^{V,(K)}\right] = E\left[p^{G,(K)}\right]$$

$$(A.61)$$

which means the expected prices for the last position K are the same, and the expected difference in prices between any two adjacent positions are the same. Therefore,

$$E\left[p^{V,(k)}\right] = E\left[p^{G,(k)}\right], \quad \forall k \in \{1, 2, \cdots, K\} \tag{A.62}$$

which directly implies that the K-dimensional VCG auction and the K-dimensional GSP auction are revenue equivalent.

Alternatively, the revenue equivalence between the K-dimensional VCG auction and the K-dimensional GSP auction can be proved by showing that the expected payments of each bidder are the same in two auctions. First consider the case of $K = 2$ positions. The expected payments by a bidder with signal x_i in the K-dimensional VCG auction and the K-dimensional GSP auction are given by

$$m^V(x_i) = Pr(x_i \geq Y_1)E\left[(\alpha_1 - \alpha_2)\underbrace{v^1(Y_1, Y_1)}_{\beta_1^V(Y_1)} + \alpha_2\underbrace{v^2(Y_2, Y_2)}_{\beta_2^V(Y_2)}\,\middle|\,x_i \geq Y_1\right]$$

$$+ Pr(Y_2 \leq x_i < Y_1)E\left[\alpha_2\underbrace{v^2(Y_2, Y_2)}_{\beta_2^V(Y_2)}\,\middle|\,Y_2 \leq x_i < Y_1\right]$$

$$m^G(x_i) = Pr(x_i \geq Y_1)E\left[\alpha_1\underbrace{\left[v^1(Y_1, Y_1) - \frac{\alpha_2}{\alpha_1}v^1(Y_1, Y_1) + \frac{\alpha_2}{\alpha_1}E[v^2(Y_2, Y_2)|Y_1]\right]}_{\beta_1^G(Y_1)}\,\middle|\,x_i \geq Y_1\right]$$

$$+ Pr(Y_2 \leq x_i < Y_1)E\left[\alpha_2\underbrace{v^2(Y_2, Y_2)}_{\beta_2^G(Y_2)}\,\middle|\,Y_2 \leq x_i < Y_1\right]$$

$$(A.63)$$

The only difference between $m^V(x_i)$ and $m^G(x_i)$ comes from the term $E[v^2(Y_2, Y_2)|Y_1 \leq x_i]$ in $m^V(x_i)$ and $E\left[E[v^2(Y_2, Y_2)|Y_1]\,\middle|\,Y_1 \leq x_i\right]$ in $m^G(x_i)$. According to the Law of Iterated Expectation,

$$E\left[E[v^2(Y_2, Y_2)|Y_1]\,\middle|\,Y_1 \leq x_i\right] = E\left[v^2(Y_2, Y_2)\,\middle|\,Y_1 \leq x_i\right] \qquad (A.64)$$

which implies $m^V(x_i) = m^G(x_i)$. Similar argument applies for any $K \geq 2$ positions. Since the expected payments of a bidder with the same signal x_i are the same in two auctions, the K-dimensional GSP auction and the K-dimensional VCG auction are always revenue equivalent.

I next compare expected revenue of the GEA and the K-dimensional VCG auction. The expected prices for the last position K in GEA and K-dimensional

VCG auction are as follows:

$$E\left[p^{E,(K)}\right] = \alpha_K E\left[v^{(K)}(Y_K, Y_K; Y_{K+1}, Y_{K+2}, \cdots, Y_{N-1}) \Big| \{Y_{K-1} > X > Y_K\}\right]$$

$$E\left[p^{V,(K)}\right] = \alpha_K E\left[v^K(Y_K, Y_K) \Big| \{Y_{K-1} > X > Y_K\}\right]$$

(A.65)

According to Milgrom and Weber (1982) [2]'s Linkage Principle, $E[p^{E,(K)}] \geq E[p^{V,(K)}]$.

A formal proof is given below:

$$v^K(x_i, y_K) = E\left[v_i \Big| X = x_i, Y_K = y_K\right]$$

$$= E\left[E\left[v_i \Big| X, Y_K, Y_{K+1}, \cdots, Y_{N-1}\right] \Big| X = x_i, Y_K = y_K\right] \quad \text{(A.66)}$$

$$= E\left[v^{(K)}(X, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = x_i, Y_K = y_K\right]$$

For $x_i > y_K$, we have

$$v^K(y_K, y_K) = E\left[v^{(K)}(X, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = y_K, Y_K = y_K\right]$$

$$= E\left[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = y_K, Y_K = y_K\right] \quad \text{(A.67)}$$

$$\leq E\left[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = x_i, Y_K = y_K\right]$$

Therefore,

$$E\left[p^{V,(K)}\right] = \alpha_K E\left[v^K(Y_K, Y_K) \Big| \{Y_{K-1} > X > Y_K\}\right]$$

$$\leq \alpha_K E\left[E\left[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) | X, Y_K\right] \Big| \{Y_{K-1} > X > Y_K\}\right]$$

$$= \alpha_K E\left[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| \{Y_{K-1} > X > Y_K\}\right]$$

$$= E\left[p^{E,(K)}\right]$$

(A.68)

so the expected price for the last position K is weakly higher in the GEA than in the K-dimensional VCG auction.

For any position $k < K$, the increment in expected price between position k

and position $k+1$ in GEA and K-dimensional VCG auction are as follows:

$$E\left[p^{E,(k)} - p^{E,(k+1)}\right] = (\alpha_k - \alpha_{k+1})E\left[v^{(k)}(Y_k, Y_k; Y_{k+1}, \cdots, Y_{N-1})\Big|\{Y_{k-1} > X > Y_k\}\right]$$

$$E\left[p^{V,(k)} - p^{V,(k+1)}\right] = (\alpha_k - \alpha_{k+1})E\left[v^k(Y_k, Y_k)\Big|\{Y_{k-1} > X > Y_k\}\right]$$

$$\text{(A.69)}$$

Applying the Linkage Principle again, we have

$$E\left[v^k(Y_k, Y_k)\Big|\{Y_{k-1} > X > Y_k\}\right] \leq E\left[v^{(k)}(Y_k, Y_k; Y_{k+1}, \cdots, Y_{N-1})\Big|\{Y_{k-1} > X > Y_k\}\right]$$

$$\text{(A.70)}$$

so the increment in expected price between any two adjacent positions is weakly higher in the GEA than in the K-dimensional VCG:

$$E\left[p^{E,(k)}\right] - E\left[p^{E,(k+1)}\right] \geq E\left[p^{V,(k)}\right] - E\left[p^{V,(k+1)}\right], \quad \forall k \in \{1, 2, \cdots, K-1\} \quad \text{(A.71)}$$

Since the expected price for the last position is weakly higher in GEA, and the increment in expected price between any two positions above the last position is also weakly higher in GEA, the expected price for every position is weakly higher in the GEA than in the K-dimensional VCG auction. Therefore, expected revenue in the GEA is weakly higher than expected revenue in the K-dimensional VCG auction. □

Proof of Corollary 2.5[1]:

Proof. The proof of Corollary 2.5 is based on two lemmas. Lemma A.1 provides a characterization of ex-post IC and IR mechanism under affiliated signals. Lemma A.2 characterizes the ex-ante expected revenue in any ex-post IC and IR mechanism.

---

[1]The proof of Corollary 2.5 follows from Myerson (1981) [40], Ulku (2013) [45] and Li (2017) [44].

Lemma A.1. *For any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1-A3** and signal distribution $F(x)$ satisfying assumptions **A4-A5**, a mechanism (q, p) is ex-post IC and IR if and only if for all bidder i, for any signal profile (x_i, x_{-i}), $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and the ex-post utility $u_i(x_i, x_{-i})$ satisfies*

$$u_i(x_i, x_{-i}) = u_i(0, x_{-i}) + \int_0^{x_i} \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] q_i(s, x_{-i}) ds, \quad \forall \, x_{-i} \tag{A.72}$$

$$u_i(0, x_{-i}) \geq 0, \quad \forall \, x_{-i} \tag{A.73}$$

Proof. I first show that any ex-post IC and IR mechanism satisfies the characterization in Lemma A.1, then show that any mechanism satisfying the conditions in Lemma A.1 must be ex-post IC and IR.

Suppose (q, p) is an ex-post IC and IR mechanism. According to the definition of ex-post IC, for all bidder i, for any true signal profile (x_i, x_{-i}) and bidder i's reported signal x_i',

$$
\begin{aligned}
u_i(x_i, x_{-i}) &\geq q_i(x_i', x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i', x_{-i}) \\
&= u_i(x_i', x_{-i}) + q_i(x_i', x_{-i}) \Big[v_i(x_i, x_{-i}) - v_i(x_i', x_{-i}) \Big]
\end{aligned}
\tag{A.74}
$$

which implies

$$
\begin{aligned}
u_i(x_i, x_{-i}) &\geq u_i(x_i', x_{-i}) + q_i(x_i', x_{-i}) \Big[v_i(x_i, x_{-i}) - v_i(x_i', x_{-i}) \Big] \\
u_i(x_i', x_{-i}) &\geq u_i(x_i, x_{-i}) + q_i(x_i, x_{-i}) \Big[v_i(x_i', x_{-i}) - v_i(x_i, x_{-i}) \Big]
\end{aligned}
\tag{A.75}
$$

which can be rewritten as

$$
\begin{aligned}
q_i(x_i', x_{-i}) \Big[v_i(x_i, x_{-i}) - v_i(x_i', x_{-i}) \Big] &\leq u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) \\
q_i(x_i, x_{-i}) \Big[v_i(x_i, x_{-i}) - v_i(x_i', x_{-i}) \Big] &\geq u_i(x_i, x_{-i}) - u_i(x_i', x_{-i})
\end{aligned}
\tag{A.76}
$$

Inequality (A.76) implies that $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and $u_i(x_i, x_{-i})$ has partial derivative

$$\frac{\partial u_i(x_i, x_{-i})}{\partial x_i} = q_i(x_i, x_{-i}) \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \tag{A.77}$$

integrate both sides, get

$$u_i(x_i, x_{-i}) = \int_0^{x_i} \left[q_i(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds + u_i(0, x_{-i}) \tag{A.78}$$

Ex-post IR implies $u_i(x_i, x_{-i}) \geq 0$ for all i. Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i and $v_i(x_i, x_{-i})$ is strictly increasing in x_i, equation (A.78) implies that $u_i(0, x_{-i}) \leq u_i(x_i, x_{-i})$ for all x_i, given any x_{-i}, so $u_i(x_i, x_{-i}) \geq 0$ for all x_i, given any x_{-i}, only if $u_i(0, x_{-i}) \geq 0$ given any x_{-i}. Therefore, any ex-post IC and IR mechanism must satisfy equation (A.78), $q_i(x_i, x_{-i})$ increasing in x_i, and $u_i(0, x_{-i}) \geq 0$.

I next show that any mechanism (q, p) that satisfies equation (A.78), $q_i(x_i, x_{-i})$ increasing in x_i, and $u_i(0, x_{-i}) \geq 0$ for any x_{-i} must be ex-post IC and IR.

Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i, $\frac{\partial v_i(s, x_{-i})}{\partial s} > 0$, and $u_i(x_i, x_{-i}) = u_i(0, x_{-i}) + \int_0^{x_i} \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] q_i(s, x_{-i}) ds$, it is trivial that $u_i(x_i, x_{-i}) \geq u_i(0, x_{-i})$ for all $x_i \geq 0$, given any x_{-i}, so $u_i(0, x_{-i}) \geq 0$ for all x_{-i} implies ex-post IR.

Suppose $x_i < x_i'$, then

$$\begin{aligned}
u_i(x_i', x_{-i}) &= u_i(x_i, x_{-i}) + \int_{x_i}^{x_i'} \left[q_i(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds \\
&\geq u_i(x_i, x_{-i}) + \int_{x_i}^{x_i'} \left[q_i(x_i, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds \\
&= u_i(x_i, x_{-i}) + \left[q_i(x_i, x_{-i}) \left(v_i(x_i', x_{-i}) - v_i(x_i, x_{-i}) \right) \right]
\end{aligned} \tag{A.79}$$

This directly implies ex-post IC. $\qquad\square$

The next lemma provides a characterization of the seller's expected revenue in any ex-post IC and IR mechanism.

Lemma A.2. *In any ex-post IC and IR mechanism, the ex-ante expected revenue is given by*

$$ER = \int_x \sum_i \left\{ q_i(x_i, x_{-i}) \left\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})} \times \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right\} \right\} f(x) dx$$
$$- \int_{x_{-i}} \sum_i u_i(0, x_{-i}) f_{-i|0}(x_{-i}|0) dx_{-i}$$

(A.80)

Proof. Following equation (A.78) in Lemma A.1, the ex-ante expected payoff to bidder i in any ex-post IC and IR mechanism is given by

$$E_x\left[u_i(x_i, x_{-i})\right]$$
$$= \int_{x_{-i}} u_i(0, x_{-i}) dF(x_{-i}|0) + \int_x \int_0^{x_i} q_i(s, x_{-i})\left[\frac{\partial v_i(s, x_{-i})}{\partial s}\right] ds\, f(x) dx$$
$$= \int_{x_{-i}} u_i(0, x_{-i}) dF(x_{-i}|0) + \int_{x_{-i}} \int_0^{\bar{x}} \int_0^{x_i} q_i(s, x_{-i})\left[\frac{\partial v_i(s, x_{-i})}{\partial s}\right] ds\, f_i(x_i|x_{-i}) dx_i\, f_{-i}(x_{-i}) dx_{-i}$$
$$= \int_{x_{-i}} u_i(0, x_{-i}) dF(x_{-i}|0) + \int_{x_{-i}} \int_0^{\bar{x}} \int_s^{\bar{x}} q_i(s, x_{-i})\left[\frac{\partial v_i(s, x_{-i})}{\partial s}\right] f_i(x_i|x_{-i}) dx_i\, ds\, f_{-i}(x_{-i}) dx_{-i}$$
$$= \int_{x_{-i}} u_i(0, x_{-i}) dF(x_{-i}|0) + \int_{x_{-i}} \int_0^{\bar{x}} \left(1 - F_i(s|x_{-i})\right) q_i(s, x_{-i})\left[\frac{\partial v_i(s, x_{-i})}{\partial s}\right] ds\, f_{-i}(x_{-i}) dx_{-i}$$
$$= \int_{x_{-i}} u_i(0, x_{-i}) dF(x_{-i}|0) + \int_{x_{-i}} \int_0^{\bar{x}} \left(1 - F_i(x_i|x_{-i})\right) q_i(x_i, x_{-i})\left[\frac{\partial v_i(x_i, x_{-i})}{\partial x_i}\right] dx_i\, f_{-i}(x_{-i}) dx_{-i}$$
$$= \int_{x_{-i}} u_i(0, x_{-i}) dF(x_{-i}|0) + \int_x \left[\frac{1 - F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})} q_i(x_i, x_{-i}) \frac{\partial v_i(x_i, x_{-i})}{\partial x_i}\right] f(x) dx$$

(A.81)

The ex-ante expected total surplus of the auction is given by

$$TS = \sum_i \int_x v_i(x_i, x_{-i}) q_i(x_i, x_{-i}) f(x) dx$$

(A.82)

The ex-ante expected revenue equals to the expected total surplus subtracted by the expected total payoff to all bidders:

$$ER = \sum_i \int_x v_i(x_i, x_{-i}) q_i(x_i, x_{-i}) f(x) dx$$

$$- \sum_i \left\{ \int_{x_{-i}} u_i(0, x_{-i}) f(x_{-i}|0) dx_{-i} + \int_x \left\{ \frac{1 - F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})} q(x_i, x_{-i}) \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right\} f(x) dx \right\}$$

$$= \sum_i \int_x \left\{ q_i(x_i, x_{-i}) \times \left\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})} \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right\} \right\} f(x) dx$$

$$- \sum_i \int_{x_{-i}} u_i(0, x_{-i}) f(x_{-i}|0) dx_{-i}$$

$$(A.83)$$

$\square$

According to the definition of marginal revenue $MR_i(x_i, x_{-i})$, the seller's problem is to maximize

$$ER = \int_x \sum_i \left\{ q_i(x_i, x_{-i}) MR_i(x_i, x_{-i}) \right\} f(x) dx - \int_{x_{-i}} \sum_i u_i(0, x_{-i}) f_{-i|0}(x_{-i}|0) dx_{-i}$$

$$(A.84)$$

subject to no reserve price, $u_i(0, x_{-i}) \geq 0$ for any x_{-i}, $q_i(x_i, x_{-i})$ increasing in x_i, and the feasibility constraint. When MR_i is strictly increasing in x_i, the expected revenue can be maximized by setting $u_i(0, x_{-i}) = 0$ for all x_{-i}, and allocating higher CTR to bidders with higher MR_i. Therefore, under regularity condition **R2**, the optimal allocation rule q^* is given by

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{X}^k(x_{-i}) \leq x_i < \hat{X}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{X}^K(x_{-i}) \end{cases} \qquad (A.85)$$

in which $[\hat{X}^k(x_{-i}), \hat{X}^{k-1}(x_{-i})]$ is the interval of value that bidder i's signal can take

such that bidder i has the k-th highest $MR_i(x_i, x_{-i})$ given her opponents' report x_{-i}.

The ex-post IC and IR conditions given in Lemma A.1 can be jointly written as

$$q_i(x_i, x_{-i})v_i(x_i, x_{-i}) - \int_0^{x_i} q_i(s, x_{-i})\frac{\partial v_i(s, x_{-i})}{\partial s}ds - p_i(x_i, x_{-i}) = u_i(0, x_{-i}) \geq 0, \quad \forall x_{-i}$$

(A.86)

for all bidder i. Choose $p_i^*(x_i, x_{-i}) = q_i^*(x_i, x_{-i})v_i(x_i, x_{-i}) - \int_0^{x_i} q_i^*(s, x_{-i})\frac{\partial v_i(s, x_{-i})}{\partial s}ds$, then $p_i^*(x_i, x_{-i})$ satisfies both constraint. Therefore, (q^*, p^*) is an optimal auction subject to no reserve price among all ex-post IC and IR mechanisms. □

Proof of Proposition 2.7[2]:

Proof. To show that (q^*, p^*) characterized in Corollary 2.5 is optimal subject to no reserve price among all Bayesian IC and IR mechanisms when bidders have independent signals, I first characterize the optimal Bayesian mechanism subject to no reserve price with independent signals, then show it is equivalent to (q^*, p^*). The proof is based on two lemmas presented below:

Lemma A.3. *For any value function $v_i(x_i, x_{-i})$ satisfying assumptions $\boldsymbol{A1}$-$\boldsymbol{A3}$, when bidders' signals are independently and identically distributed, a mechanism (q, p) is Bayesian IC and IR if for every bidder i, for any report of signals $x = (x_i, x_{-i})$, the expected CTR $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and the interim*

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[2]The proof of Proposition 2.7 follows from Myerson (1981) [40].

expected utility $U_i(x_i)$ satisfies

$$U_i(x_i) = U_i(0) + \int_{x_{-i}} \int_0^{x_i} \left[\frac{\partial v_i(s, x_{-i})}{\partial s}\right] q_i(s, x_{-i}) ds f_{-i}(x_{-i}) dx_{-i} \qquad \text{(A.87)}$$

$$U_i(0) \geq 0 \qquad \text{(A.88)}$$

Proof. I first show that any Bayesian IC and IR mechanism can be characterized by the conditions in Lemma A.3, then finish the proof by showing that any mechanism satisfying the characterization in Lemma A.3 must be Bayesian IC and IR.

According to the definition of Bayesian IC mechanism, for all bidder i, for any signal profile (x_i, x_{-i}) and bidder i's reported signal x_i',

$$\begin{aligned}
U_i(x_i) &\geq \int_{x_{-i}} \left[q_i(x_i', x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i', x_{-i})\right] f_{-i}(x_{-i}) dx_{-i} \\
&= U_i(x_i') + \int_{x_{-i}} \left[q_i(x_i', x_{-i})\left(v_i(x_i, x_{-i}) - v_i(x_i', x_{-i})\right)\right] f_{-i}(x_{-i}) dx_{-i}
\end{aligned} \qquad \text{(A.89)}$$

which implies

$$\begin{aligned}
U_i(x_i) &\geq U_i(x_i') + \int_{x_{-i}} \left[q_i(x_i', x_{-i})\left(v_i(x_i, x_{-i}) - v_i(x_i', x_{-i})\right)\right] f_{-i}(x_{-i}) dx_{-i} \\
U_i(x_i') &\geq U_i(x_i) + \int_{x_{-i}} \left[q_i(x_i, x_{-i})\left(v_i(x_i', x_{-i}) - v_i(x_i, x_{-i})\right)\right] f_{-i}(x_{-i}) dx_{-i}
\end{aligned} \qquad \text{(A.90)}$$

i.e.,

$$\begin{aligned}
\int_{x_{-i}} \left[q_i(x_i', x_{-i})\left(v_i(x_i, x_{-i}) - v_i(x_i', x_{-i})\right)\right] f_{-i}(x_{-i}) dx_{-i} &\leq U_i(x_i) - U_i(x_i') \\
\int_{x_{-i}} \left[q_i(x_i, x_{-i})\left(v_i(x_i, x_{-i}) - v_i(x_i', x_{-i})\right)\right] f_{-i}(x_{-i}) dx_{-i} &\geq U_i(x_i) - U_i(x_i')
\end{aligned} \qquad \text{(A.91)}$$

Therefore, $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and $U_i(x_i)$ has derivative

$$\frac{dU_i(x_i)}{dx_i} = \int_{x_{-i}} q_i(x_i, x_{-i}) \left[\frac{\partial v_i(x_i, x_{-i})}{\partial x_i}\right] f_{-i}(x_{-i}) dx_{-i} \qquad \text{(A.92)}$$

integrate both sides yields

$$U_i(x_i) = \int_{x_{-i}} \int_0^{x_i} \left[\frac{\partial v_i(s, x_{-i})}{\partial s}\right] q_i(s, x_{-i}) ds f_{-i}(x_{-i}) dx_{-i} + U_i(0) \qquad \text{(A.93)}$$

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Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i and $v_i(x_i, x_{-i})$ is strictly increasing in x_i, equation (A.93) implies that $U_i(0) \le U_i(x_i)$ for all x_i. Therefore, $U_i(x_i) \ge 0$ for all $x_i \in [0, \bar{x}]$ only if $U_i(0) \ge 0$.

I next show that any mechanism (q, p) that satisfies the characterization in Lemma A.3 must be Bayesian IC and IR. Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i, $\frac{\partial v_i(s, x_{-i})}{\partial s} > 0$, and $U_i(x_i) = U_i(0) + \int_{x_{-i}} \int_0^{x_i} \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] q_i(s, x_{-i}) ds f_{-i}(x_{-i}) dx_{-i}$, it is trivial that $U_i(x_i) \ge U_i(0)$ for all x_i, so $U_i(0) \ge 0$ implies Bayesian IR.

Suppose $x_i < x_i'$, then

$$
\begin{aligned}
U_i(x_i') &= U_i(x_i) + \int_{x_{-i}} \int_{x_i}^{x_i'} q_i(s, x_{-i}) \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds f_{-i}(x_{-i}) dx_{-i} \\
&\ge U_i(x_i) + \int_{x_{-i}} \int_{x_i}^{x_i'} q_i(x_i, x_{-i}) \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds f_{-i}(x_{-i}) dx_{-i} \qquad \text{(A.94)} \\
&= U_i(x_i) + \int_{x_{-i}} \left[q_i(x_i, x_{-i}) \Big(v_i(x_i', x_{-i}) - v_i(x_i, x_{-i}) \Big) \right] f_{-i}(x_{-i}) dx_{-i}
\end{aligned}
$$

This directly implies Bayesian IC. $\qquad \square$

The result of Lemma A.3 leads to the following lemma that gives an expression of the seller's expected revenue in any Bayesian IC and IR mechanism.

Lemma A.4. *For any Bayesian IC and IR mechanism that satisfy the conditions in Lemma A.3, the ex-ante expected revenue is given by*

$$
ER = \int_x \sum_i \left\{ q_i(x_i, x_{-i}) \left\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i)}{f_i(x_i)} \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right\} \right\} f(x) dx - \sum_i U_i(0)
$$
(A.95)

Proof. The ex-ante expected payoff to an bidder i in any Bayesian IC and IR auction

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is

$$E_{x_i}[U_i(x_i)] = U_i(0) + \int_x \int_0^{x_i} q_i(s, x_{-i}) \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds \, f(x) dx$$

$$= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \int_0^{x_i} q_i(s, x_{-i}) \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds \, f_i(x_i) dx_i \, f_{-i}(x_{-i}) dx_{-i}$$

$$= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \int_s^{\bar{x}} q_i(s, x_{-i}) \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] f_i(x_i) dx_i \, ds \, f_{-i}(x_{-i}) dx_{-i}$$

$$= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \left(1 - F_i(s) \right) q_i(s, x_{-i}) \left[\frac{\partial v_i(s, x_{-i})}{\partial s} \right] ds \, f_{-i}(x_{-i}) dx_{-i}$$

$$= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \left(1 - F_i(x_i) \right) q_i(x_i, x_{-i}) \left[\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right] dx_i \, f_{-i}(x_{-i}) dx_{-i}$$

$$= U_i(0) + \int_x \left[\frac{1 - F_i(x_i)}{f_i(x_i)} q_i(x_i, x_{-i}) \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right] f(x) dx \tag{A.96}$$

The ex-ante expected total surplus of the auction is given by

$$TS = \sum_i \int_x v_i(x_i, x_{-i}) q_i(x_i, x_{-i}) f(x) dx \tag{A.97}$$

The ex-ante expected revenue equals to the expected total surplus subtracted by the expected total payoff to all bidders:

$$ER = \sum_i \int_x v_i(x_i, x_{-i}) q_i(x_i, x_{-i}) f(x) dx$$

$$- \sum_i \left\{ U_i(0) + \int_x \left\{ \frac{1 - F_i(x_i)}{f_i(x_i)} q(x_i, x_{-i}) \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right\} f(x) dx \right\}$$

$$= \sum_i \int_x \left\{ q_i(x_i, x_{-i}) \times \left\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i)}{f_i(x_i)} \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right\} \right\} f(x) dx - \sum_i U_i(0) \tag{A.98}$$

$\square$

According to the definition of $MR_i(x_i, x_{-i})$ with independent signals, the seller's problem is to maximize

$$ER = \int_x \sum_i \left\{ q_i(x_i, x_{-i}) \times MR_i(x_i, x_{-i}) \right\} f(x) dx - \sum_i U_i(0) \tag{A.99}$$

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subject to no reserve price, $U_i(0) \geq 0$, $q_i(x_i, x_{-i})$ being weakly increasing in x_i, and the feasibility constraint. Since $U_i(0)$ is a constant, it is optimal to set $U_i(0) = 0$. The expected revenue is maximized by assigning higher q_i to bidders with higher $MR_i(x_i, x_{-i})$. Under this allocation rule, the constraint that $q_i(x_i, x_{-i})$ being weakly increasing in x_i is satisfied if $MR_i(x_i, x_{-i})$ is strictly increasing in x_i. Therefore, given that $MR_i(x_i, x_{-i})$ is strictly increasing in x_i, the optimal allocation rule $q(x_i, x_{-i})$ is given by

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{X}^k(x_{-i}) \leq x_i < \hat{X}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{X}^K(x_{-i}) \end{cases} \tag{A.100}$$

in which $[\hat{X}^k(x_{-i}), \hat{X}^{k-1}(x_{-i})]$ is the interval of value that bidder i's signal x_i can take such that bidder i has the k-th highest $MR_i(x_i, x_{-i})$ given opponents' report x_{-i}.

The Bayesian IC and IR conditions given in Lemma A.3 can be jointly written as

$$\int_{x_{-i}} \left\{ q_i(x)v_i(x) - \int_0^{x_i} q_i(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds - p_i(x) \right\} f_{-i}(x_{-i}) dx_{-i} = U_i(0) \geq 0 \tag{A.101}$$

for all bidder i. Choose $p_i^*(x) = q_i^*(x)v_i(x) - \int_0^{x_i} q_i^*(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds$, then $p_i^*(x)$ satisfies the joint constraint. Therefore, (q^*, p^*) is the optimal position auction subject to no reserve price among all Bayesian IC and IR mechanisms when bidders have independent signals. □

Proof of Proposition 2.8:

Proof. Under regularity conditions **R1**-**R3**, it is trivial that given any profile of signals, the rank ordering of signals is equivalent to the rank ordering of values $v_i(x_i, x_{-i})$ as well as the rank ordering of marginal revenues $MR_i(x_i, x_{-i})$, so for any bidder i, given any opponents' report x_{-i}, we must have

$$\hat{x}^k(x_{-i}) = \hat{X}^k(x_{-i}), \quad \forall k \tag{A.102}$$

in which $\hat{x}^k(x_{-i})$ is the minimum value that bidder i's signal can take such that i has the k-th highest value $v_i(x_i, x_{-i})$ given x_{-i}, and $\hat{X}^k(x_{-i})$ is the minimum value that bidder i's signal can take such that i has the k-th highest marginal revenue $MR_i(x_i, x_{-i})$ given x_{-i}. Therefore, the allocation rule of the optimal auction (q^*, p^*) defined in Corollary 2.5 is the same as the allocation rule of the Generalized-VCG mechanism (q^V, p^V). Replacing $\hat{X}^k(x_{-i})$ by $\hat{x}^k(x_{-i})$ in the optimal auction (q^*, p^*) defined in Corollary 2.5 yields

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{x}^k(x_{-i}) \le x_i < \hat{x}^{k-1}(x_{-i}) \\ \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases} \tag{A.103}$$

$$p_i^*(x_i, x_{-i}) = q_i^*(x_i, x_{-i}) v_i(x_i, x_{-i}) - \int_0^{x_i} q_i^*(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds \tag{A.104}$$

I next substitute equation (A.103) into equation (A.104) to characterize the optimal payment rule p^*. Note that the term $q_i^*(s, x_{-i}) \times \frac{\partial v_i(s, x_{-i})}{\partial s}$ inside the integral in $p^*(x_i, x_{-i})$ is given by

$$q_i^*(s, x_{-i}) \times \frac{\partial v_i(s, x_{-i})}{\partial s} = \begin{cases} \alpha_k \frac{\partial v_i(s, x_{-i})}{\partial s} & \text{if} \quad \hat{x}^k(x_{-i}) \le s < \hat{x}^{k-1}(x_{-i}) \\ \\ 0 & \text{if} \quad s < \hat{x}^K(x_{-i}) \end{cases} \tag{A.105}$$

so the integral of $q_i^*(s, x_{-i})\frac{\partial v_i(s,x_{-i})}{\partial s}$ on $[0, x_i]$ is given by

$$\int_0^{x_i} q_i^*(s, x_{-i})\frac{\partial v_i(s, x_{-i})}{\partial s}ds$$

$$= \begin{cases} \alpha_k \int_{\hat{x}^k(x_{-i})}^{x_i} \left[\frac{\partial v_i(s,x_{-i})}{\partial s}\right]ds + \sum_{j=k+1}^K \left\{\alpha_j \int_{\hat{x}^j(x_{-i})}^{\hat{x}^{j-1}(x_{-i})} \left[\frac{\partial v_i(s,x_{-i})}{\partial s}\right]ds\right\} & \text{if} \quad \hat{x}^k(x_{-i}) \le x_i < \hat{x}^{k-1}(x_{-i}) \\ \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases}$$

$$= \begin{cases} \alpha_k \left[v_i(x_i, x_{-i}) - v_i(\hat{x}^k, x_{-i})\right] + \sum_{j=k+1}^K \alpha_j \left[v_i(\hat{x}^{j-1}, x_{-i}) - v_i(\hat{x}^j, x_{-i})\right] & \text{if} \quad \hat{x}^k(x_{-i}) \le x_i < \hat{x}^{k-1}(x_{-i} \\ \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases}$$

(A.106)

Substitute the optimal allocation rule q^* given in equation (A.103) and the integral given in equation (A.106) into the optimal payment rule p^* yields

$$p_i^*(x) = \begin{cases} \sum_{j=k}^K (\alpha_j - \alpha_{j+1})v_i(\hat{x}^j, x_{-i}) & \text{if} \quad x_i \in [\hat{x}^k(x_{-i}), \hat{x}^{k-1}(x_{-i})] \\ \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases} \quad \text{(A.107)}$$

which is equivalent to the Generalized-VCG payment rule. Therefore, under regularity conditions **R1**-**R3**, the Generalized-VCG mechanism is the optimal position auction subject to no reserve price among all ex-post IC and IR mechanisms.

I next compare the expected revenue of the Generalized-VCG mechanism to expected revenues of the GEA, the K-dimensional GSP auction and the K-dimensional VCG auction. Since expected revenue of the GEA is higher than the other two static auctions, showing that the Generalized-VCG mechanism yields higher expected revenue than the GEA is sufficient for proving the revenue ranking provided in Proposition 2.8.

The ex-ante expected price for the last position K in the Generalized-VCG

mechanism and GEA are given by

$$E\left[p^{G-VCG,(K)}\right] = \alpha_K E\left[v_i(Y_K, Y_1, \cdots, Y_{N-1})\Big|\{Y_{K-1} > X > Y_K\}\right]$$

$$E\left[p^{E,(K)}\right] = \alpha_K E\left[v^{(K)}(Y_K, Y_K, \cdots, Y_{N-1})\Big|\{Y_{K-1} > X > Y_K\}\right]$$

(A.108)

According to the Linkage Principle,

$$E\left[p^{G-VCG,(K)}\right] \geq E\left[p^{E,(K)}\right] \tag{A.109}$$

For any position $1 \leq k \leq K - 1$, the expected price in the Generalized-VCG

mechanism and GEA are given by

$$E\left[p^{G-VCG,(k)} - p^{G-VCG,(k+1)}\right] = (\alpha_k - \alpha_{k+1})E\left[v_i(Y_k, Y_1, \cdots, Y_k, Y_{k+1}, \cdots, Y_{N-1})\Big|\{Y_{k-1} > X > Y_k\}\right]$$

$$E\left[p^{E,(k)} - p^{E,(k+1)}\right] = (\alpha_k - \alpha_{k+1})E\left[v^{(k)}(Y_k, Y_k, Y_{k+1}, \cdots, Y_{N-1})\Big|\{Y_{k-1} > X > Y_k\}\right]$$

(A.110)

Applying the Linkage Principle again yields

$$E\left[p^{G-VCG,(k)}\right] - E\left[p^{G-VCG,(k+1)}\right] \geq E\left[p^{E,(k)}\right] - E\left[p^{E,(k+1)}\right] \tag{A.111}$$

which implies that $E\left[p^{G-VCG,(k)}\right] \geq E\left[p^{E,(k)}\right]$ for all position k, so the expected

revenue of the Generalized-VCG mechanism is higher than the expected revenue of

the GEA, which is in turn higher than the expected revenue of K-dimensional VCG

auction and K-dimensional GSP auction under affiliated signals.

In the special case of independent signals, it is trivial that

$$E\left[p^{G-VCG,(K)}\right] = E\left[p^{E,(K)}\right]$$

$$E\left[p^{G-VCG,(k)} - p^{G-VCG,(k+1)}\right] = E\left[p^{E,(k)} - p^{E,(k+1)}\right]$$

(A.112)

which means $E\left[p^{G-VCG,(k)}\right] = E\left[p^{E,(k)}\right]$ for all position k, so the expected revenue

of the Generalized-VCG mechanism is equivalent to the expected revenue of the

GEA, which is in turn equivalent to the expected revenue of the K-dimensional VCG auction and the K-dimensional GSP auction under independent signals. It follows that the optimal revenue subject to no reserve price among all Bayesian IC and IR mechanisms is practically implementable by the GEA, the K-dimensional GSP auction, and the K-dimensional VCG auction under independent signals. $\quad\square$

Appendix B: Proofs for Chapter 3

Proof of Corollary 3.2:

Proof. For any bidder i, for any $n \in \{1, 2, \cdots, K\}$, given a profile of bidder i's opponents' reported costs $\hat{c}_{-i}$, let $nmin\{\hat{c}_{-i}\}$ be the n-th lowest cost among bidder i's opponents' reported costs $\hat{c}_{-i}$. Define

$$\hat{C}_n(\hat{c}_{-i}) = \min \left\{ nmin\{\hat{c}_{-i}\}, P(n) - (n-1)\delta \right\} \tag{B.1}$$

Suppose $P(n+1) - \hat{c}_n < n\delta$, so that it is not efficient to sell more than n licenses, bidder i will win one out of n licenses in the VCG mechanism if and only if

$$\hat{c}_i \leq nmin\{\hat{c}_{-i}\}, \quad \text{and} \quad P(n) - \hat{c}_i \geq (n-1)\delta \tag{B.2}$$

i.e., $\hat{c}_i \leq \hat{C}_n(\hat{c}_{-i})$.

The payoff of winning one out of n licenses to bidder i with true cost c_i is

$$u_n(c_i, \hat{c}_{-i}) = P(n) - c_i - \max \left\{ P(n) - nmin\{\hat{c}_{-i}\}, (n-1)\delta \right\} \tag{B.3}$$

Since $u_n(c_i, \hat{c}_{-i}) \geq 0$ if and only if $c_i \leq \hat{C}_n(\hat{c}_{-i})$ for any n, it is a dominant strategy to report $\hat{c}_i = c_i$ in the VCG mechanism. $\square$

Proof of Lemma 3.2:

Proof. I will first prove that given any reserve prices $0 = r_1 < r_2$ and $r_2 > P(2) - \bar{c}$, then there must exists some types of bidders bidding below r_2 in equilibrium, i.e., r_2 is binding for some bidders.

Suppose all bidders bid above r_2 in equilibrium, then total supply will be 2 for certain and bidders have a dominant strategy of bidding their true value of winning one out of 2 licenses, $\beta(c_i) = P(2) - c_i$. However, for any bidders with costs $c_i > P(2) - r_2$, $P(2) - c_i < r_2$, which contradicts the assumption that all bidders bid above r_2. Therefore, if $r_2 > P(2) - \bar{c}$, then there must exist some types of bidders who bid below r_2 in equilibrium.

Next, I will prove that no symmetric monotonic equilibrium bidding strategy exists in a uniform-price auction with reserve prices (r_1, r_2) s.t. $r_1 = 0, r_2 > P(2) - \bar{c}$.

Suppose a symmetric strictly decreasing Bayesian equilibrium bidding strategy $\beta(.)$ exists, given that some bidders must bid below r_2 in equilibrium, there must exist some threshold type c^* s.t. $\beta(c_i) < r_2$ if $c_i > c^*$, and $\beta(c_i) > r_2$ if $c_i < c^*$.

For any bidder i with $b_i < r_2$, the only possible winning outcome is to win one exclusive license if the lowest-cost opponent bids $\beta(y_1) < b_i < r_2$. Given all opponents adopt bidding strategy $\beta(.)$, bidder i's optimal bid b_i solves

$$\max_{b_i} \int_{\beta^{-1}(b_i)}^{\bar{c}} [P(1) - c_i - \beta(y_1)] dG_1(y_1) \tag{B.4}$$

where $G_1(y_1)$ is the distribution of the lowest cost among bidder i's opponents. The first order condition shows that

$$\beta(c_i) = P(1) - c_i \quad \text{if} \quad c_i \in (c^*, \bar{c}] \tag{B.5}$$

$P(1) - c_i < r_2$ for all $c_i > c^*$ implies $c^* = P(1) - r_2$. Note that as long as $b_i < r_2$,

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increasing b_i only increases the probability of winning when total supply is 1 and has no impact on probability of winning when total supply is 2. Therefore, the equilibrium bidding strategy is the same as the equilibrium strategy in a second-price auction with a fixed supply of 1 license when a bidder's cost is high enough to satisfy $P(1) - c_i < r_2$.

When $P(1) - \bar{c} > r_2 > P(2) - \bar{c}$, then $P(1) - r_2 > \bar{c}$, which means $c^* = \bar{c}$ and all bidders must bid above r_2 in equilibrium, contradicting the assumption that r_2 is binding for some bidders. Therefore, no monotonic separating equilibrium exists when $P(1) - \bar{c} > r_2 > P(2) - \bar{c}$.

When $r_2 > P(1) - \bar{c}$, then $P(1) - r_2 < \bar{c}$, which means $c^* = P(1) - r_2$ and $\beta\Big(P(1) - r_2\Big) = r_2$ in equilibrium. Next, consider any bidder i with $b_i > r_2$, bidder i will win one exclusive license only if its lowest-cost opponent submits a bid below r_2, i.e., $y_1 > c^*$. Bidder i will win one out of two licenses when its lowest-cost opponent bids above r_2 and the second-lowest cost opponent bids below b_i. Given all opponents adopt bidding strategy $\beta(.)$, bidder i's optimal bid b_i solves

$$
\max_{b_i} \int_{c^*}^{\bar{c}} \Big[P(1) - c_i - \beta(y_1) \Big] dG_1(y_1)
$$
$$
+ \int_{\underline{c}}^{c^*} \int_{\beta^{-1}(b_i)}^{c^*} \Big[P(2) - c_i - \beta(y_2) \Big] dG_2(y_2|y_1) G_1(y_1) \qquad \text{(B.6)}
$$
$$
+ \int_{\underline{c}}^{c^*} \int_{c^*}^{\bar{c}} \Big[P(2) - c_i - r_2 \Big] dG_2(y_2|y_1) G_1(y_1)
$$

where $G_n(y_n)$ is the distribution of the n-th lowest cost. Note that b_i only enters the second term. The first order condition implies

$$
\beta(c_i) = P(2) - c_i \quad \text{if} \quad c_i \in [\underline{c}, c^*] \qquad \text{(B.7)}
$$

This is because as long as $b_i \geq r_2$, increasing bid does not affect the probability

of winning when the total supply is 1 but only affect the probability of winning when the total supply is 2, since the total supply equals to 1 only if the lowest-cost opponent bids below r_2, which is not affected by bidder i's own bid. Therefore, the equilibrium bidding strategy is the same as the equilibrium bidding strategy in a uniform-price auction with a fixed supply of 2 when a bidder's cost is low enough to satisfy $P(2) - c_i \geq r_2$. Set $P(2) - c_i = r_2$ yields $c^* = P(2) - r_2 < P(1) - r_2$, which yields a contradiction to the assumption that there exists a single type c^* s.t. $\beta(c_i) > r_2$ for all $c_i < c^*$ and $\beta(c_i) < r_2$ for all $c_i > c^*$.

Given that

$$
\beta(c_i) = \begin{cases} P(1) - c_i & \text{if} \quad c_i \in (P(1) - r_2, \bar{c}] \\ \\ P(2) - c_i & \text{if} \quad c_i \in [\underline{c}, P(2) - r_2] \end{cases} \tag{B.8}
$$

There exists no bidding strategy for $c_i \in (P(2) - r_2, P(1) - r_2]$ such that $\beta(c_i)$ can be strictly monotonic over $[\underline{c}, \bar{c}]$. $\qquad \square$

Proof of Lemma 3.3:

Proof. For bidders with costs $c_i \in \left[P(2) - r_2, \min\{P(1) - r_2, \bar{c}\} \right]$, winning a license when total supply is 2 yields a non-positive payoff. Since increasing bid when $b_i \geq r_2$ only increases probability of winning when total supply is 2 and does not affect probability of winning when total supply is 1, any bid $b_i > r_2$ is dominated by bidding $b_i = r_2$.

On the other hand, winning a license when total supply is 1 yields a positive payoff for these bidders. Since total supply equals to 1 only if $\beta(y_1) < r_2$, $P(1) - c_i - r_2 > P(1) - c_i - \beta(y_1) > 0$ for all $c_i < \min\{P(1) - r_2, \bar{c}\}$. As long as $b_i < r_2$,

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decreasing bid does not affect probability of winning when total supply is 2 but only decreases probability of winning when total supply is 1, so any bid $b_i < r_2 - \epsilon$ for some arbitrarily small ϵ is dominated by bidding $b_i = r_2 - \epsilon$.

Therefore, bidders with costs $c_i \in \left[P(2) - r_2, \min\{P(1) - r_2, \bar{c}\} \right]$ will only bid r_2 or $r_2 - \epsilon$ in equilibrium. Pooling must occur for these bidders. $\qquad\square$

Proof of Lemma 3.4:

Proof. For bidder with costs $c_i \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\} \right]$, either r_2 or $r_2 - \epsilon$ is an optimal bidding strategy according to Lemma 3.3. Moreover, there is a trade-off between bidding r_2 and $r_2 - \epsilon$. Compared to bidding $r_2 - \epsilon$, bidding r_2 improves the probability of winning $\Pi_1(c_i) = P(1) - c_i - r_2 + \epsilon > 0$ when the total supply is 1 and the highest bid among opponents equals to $r_2 - \epsilon$, while also improves the probability of winning $\Pi_2(c_i) = P(2) - c_i - \beta(y_1) < P(2) - c_i - r_2 < 0$ when the total supply is 2 and the highest bid among opponents is no lower than r_2.

Note that both $\Pi_1(c_i)$ and $\Pi_2(c_i)$ are decreasing in c_i, so the gain from winning when total supply is 1 is diminishing while the loss from winning when total supply is 2 is increasing when c_i is greater.

For bidders with costs c_i sufficiently close to $P(2) - r_2$,

$$
\begin{aligned}
\lim_{c_i \to P(2) - r_2} \Pi_1(c_i) &= \lim_{c_i \to P(2) - r_2} P(1) - c_i - r_2 = P(1) - P(2) = \delta \\
\lim_{c_i \to P(2) - r_2} \Pi_2(c_i) &= \lim_{c_i \to P(2) - r_2} P(2) - c_i - r_2 = 0
\end{aligned}
\tag{B.9}
$$

so the gain from winning when supply equals to 1 is strictly positive while the loss from winning when supply equals to 2 is zero. The expected payoff from bidding r_2 is strictly higher than expected payoff from bidding $r_2 - \epsilon$.

If $P(1) - r_2 \leq \bar{c}$, then for bidders with costs c_i sufficiently close to $P(1) - r_2$,

$$\lim_{c_i \to P(1) - r_2} \Pi_1(c_i) = \lim_{c_i \to P(1) - r_2} P(1) - c_i - r_2 = 0$$

$$\lim_{c_i \to P(1) - r_2} \Pi_2(c_i) = \lim_{c_i \to P(1) - r_2} P(2) - c_i - r_2 = P(2) - P(1) = -\delta < 0$$

(B.10)

so the gain from winning when supply equals to 1 is zero while the loss from winning when supply equals to 2 is strictly negative. The expected payoff from bidding r_2 is strictly lower than expected payoff from bidding $r_2 - \epsilon$.

If $P(1) - r_2 > \bar{c}$, note that all bidders bidding at least r_2 is never an equilibrium, since the total supply will be certain to be 2, and all bidders with $c_i \in (P(2) - r_2, \bar{c}]$ will suffer a loss $\Pi_2(c_i) < 0$ and will have incentive to deviate to bidding $r_2 - \epsilon$. Since $\Pi_2(c_i)$ is decreasing in c_i, those bidders with costs c_i sufficiently close to $\bar{c}$ will bid $r_2 - \epsilon$ in equilibrium.

Similarly, all bidders with $c_i \in (P(2) - r_2, \bar{c}]$ bidding $r_2 - \epsilon$ is not an equilibrium, since bidding r_2 will increase the probability of winning $\Pi_1(c_i)$ when total supply equals to 1. Since $\Pi_1(c_i)$ is decreasing in c_i, those bidders with costs c_i sufficiently close to $P(2) - r_2$ will bid r_2 in equilibrium.

Therefore, there must exists $\hat{c} \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\} \right]$ such that all bidders with costs $c_i \in \left(P(2) - r_2, \hat{c} \right]$ bid r_2 and all bidders with costs $c_i \in \left[\hat{c}, \min\{P(1) - r_2, \bar{c}\} \right]$ bid $r_2 - \epsilon$ in equilibrium. $\quad\square$

Proof of Proposition 3.2:

Proof. Consider the equilibrium drop-out price of some arbitrary bidder i when there are $k \geq 2$ bidders left in the auction. When the current clock price $p < r_2$, bidder i will win at the current clock price p only if all the rest of $(k-1)$ bidders

drop out simultaneously at p, which gives him a payoff of $P(1) - c_i - p$, since all $(k-1)$ opponents dropping out at a price $p < r_2$ implies the total supply is 1.

Therefore, when $p < r_2$, a bidder will stay in the auction as long as the value of winning one exclusive license is above the current price. If $P(1) - r_2 \geq \bar{c}$, then no bidder will drop out before $r_2 - \epsilon$. If $P(1) - r_2 < \bar{c}$, bidders with costs $c_i \in (P(1) - r_2, \bar{c}]$ will drop out before price reaches r_2 and bidders with costs $c_i \in [\underline{c}, P(2) - r_2]$ will stay in the auction while $p < r_2$. The optimal drop-out strategy for all these bidders is $P(1) - c_i$.

However, for bidders with costs $c_i \in \left(P(2) - r_2, \min\{P(1) - r_2, \bar{c}\} \right]$, winning when total supply equals to 2 yields a negative payoff, and winning when total supply equals to 1 yields a positive payoff. To maximize probability of winning when $p < r_2$ and minimize probability of winning when $p \geq r_2$, these bidders will either drop out at $p = r_2 - \epsilon$ or at $p = r_2$. Since the gain from winning when total supply is 1 is decreasing in bidder's cost, and the loss from winning when total supply is 2 is increasing in bidder's cost, there always exists some $\hat{c}_2^k$ such that a bidder i with cost $c_i = \hat{c}_2^k$ is indifferent between dropping out at $r_2 - \epsilon$ and r_2 given that there are $k \geq 2$ bidders left in the auction at price of $p < r_2$.

A similar proof can be applied to any $n \in \{2, 3, \cdots, K\}$. When the clock price $p \in [r_n, r_{n+1})$, all bidders with costs $c_i \geq P(n) - r_n$ have dropped out. Only the equilibrium dropping out strategy of bidders with costs $c_i < P(n) - r_n$ needs to be discussed.

At any $p \in [r_n, r_{n+1})$, a bidder will win by dropping out at p only if all the other $(k-1)$ bidders drop out simultaneously at p, which gives him a payoff of

$P(n) - c_i$. Therefore, bidders with costs $c_i \in \left(P(n) - r_{n+1}, P(n) - r_n \right]$ will drop out

at $P(n) - c_i$ when $p \in [r_n, r_{n+1})$. Bidders with costs $c_i \in \left[\underline{c}, P(n+1) - r_{n+1} \right]$ will

stay in the auction while $p < r_{n+1}$. Their optimal drop out strategy is also $P(n) - c_i$.

For bidders with costs $c_i \in \left(P(n+1) - r_{n+1}, P(n) - r_{n+1} \right]$, winning when total

supply equals to $(n+1)$ yields a negative payoff, while winning when total supply

equals to n yields a positive payoff. There exists $\hat{c}_{n+1}^k$ such that a bidder i with cost

$c_i = \hat{c}_{n+1}^k$ is indifferent between dropping out at $r_{n+1} - \epsilon$ and r_{n+1} given that there

are $k \geq 2$ bidders left in the auction at a price $p \in [r_n, r_{n+1})$.

When the clock price $p \geq r_K$, only bidders with $c_i \in \left[\underline{c}, P(K) - r_K \right]$ are still

active. Since the total supply will be K for certain, it is a dominant strategy for

the remaining bidders to bid $P(K) - c_i$.

$\square$

Proof of Proposition 3.3:

Proof. For all $n \in \{1, 2, \cdots, K\}$, let $\Pi_i^n(b_i, b_{-i})$ denote bidder i's payoff from auction

given bidding profile (b_i, b_{-i}) conditional on total supply $S = n$. Let $P^n(b_i, b_{-i})$

denote the probability that total supply $S = n$ given bidding profile (b_i, b_{-i}). Let

$P^n\left(b_i, b_{-i} \middle| b_i^n \geq r_n\right)$ denote the probability that total supply $S = n$ given bidding

profile (b_i, b_{-i}) conditional on $b_i^n \geq r_n$. Then

$$P^n(b_i, b_{-i}) = P^n\left(b_i, b_{-i} \middle| b_i^n \geq r_n\right) \times \mathbb{1}\{b_i^n \geq r_n\} \tag{B.11}$$

Each bidder i's objective function is to maximize

$$\sum_{n=1}^{K} \Pi_i^n(b_i, b_{-i}) \times P^n\left(b_i, b_{-i} \middle| b_i^n \geq r_n\right) \times \mathbb{1}\{b_i^n \geq r_n\} \tag{B.12}$$

For any possible level of supply n at which $r_n > P(n) - c_i$, $\Pi_i^n(b_i, b_{-i}) \leq 0$ for all (b_i, b_{-i}). It is a dominant strategy to bid $b_i^n = 0$ when $c_i \in (P(n) - r_n, \bar{c}]$.

For all possible level of supply n at which $r_n \leq P(n) - c_i$, each bidder i needs to maximize $\Pi_i^n(b_i, b_{-i}) \times P^n(b_i, b_{-i} | b_i^n \geq r_n) \times \mathbb{1}\{b_i^n \geq r_n\}$.

Let N denote the set of all bidders. Let S_n denote the set of bidders who win a license between rounds $(R1)$ to (Rn). Then $N \setminus S_{n-1}$ is the set of bidders whose bids b_i^n will be considered in round (Rn). Consider the unconditional probabilities $P^n(b_i, b_{-i})$ first.

For $n = 1$,

$$P^1(b_i, b_{-i}) = \begin{cases} 1 & \text{if} \quad \max\{b_j^1\}_{j \in N} \geq r_1, \quad \text{and} \\ & \max\{b_j^2\}_{j \in N \setminus S_1} < r_2 \\ 0 & \text{else} \end{cases} \tag{B.13}$$

For all $n \in \{2, 3, \cdots, K - 1\}$,

$$P^n(b_i, b_{-i}) = \begin{cases} 1 & \text{if} \quad \max\{b_j^k\}_{j \in N \setminus S_{k-1}} \geq r_k \quad \forall k \in \{1, 2, \cdots, n\}, \quad \text{and} \\ & \max\{b_j^{n+1}\}_{j \in N \setminus S_n} < r_{n+1} \\ 0 & \text{else} \end{cases} \tag{B.14}$$

For $n = K$,

$$P^K(b_i, b_{-i}) = \begin{cases} 1 & \text{if} \quad \max\{b_j^k\}_{j \in N \setminus S_{k-1}} \geq r_k \quad \forall k \in \{1, 2, \cdots, K\} \\ 0 & \text{else} \end{cases} \tag{B.15}$$

Note that conditional on $b_i^n \geq r_n$, changing b_i^n does not affect $P^{\tilde{n}}(b_i, b_{-i})$ for all possible supply level $\tilde{n} \in \{1, 2, \cdots, K\}$. Therefore, the conditional probability

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$P^n\left(b_i, b_{-i} \mid b_i^n \geq r_n\right)$ is not affected by b_i^n. Moreover, conditional on winning a license before round (Rn), strategically bidding $b_i^n < r_n$ also does not affect $P^n(b_i, b_{-i})$, as b_i^n is not counted in round (Rn) if bidder i wins a license before round (Rn), so the bidders cannot strategically shade their bids for higher supply levels to lower the final supply level conditional on winning.

Each bidder's problem becomes choosing b_i^n to maximize $\Pi_i^n(b_i, b_{-i})$ if $r_n \leq P(n) - c_i$, and choosing $b_i^n = 0$ otherwise. Since $\Pi_i^n(b_i, b_{-i})$ denote the payoff from winning a license conditional on $S = n$, each bidder's dominant strategy is to bid his or her true value $P(n) - c_i$ conditional on $S = n$. □

Proof of Proposition 3.4:

Proof. For all $n \in \{K, K-1, \cdots, 1\}$, first consider each active bidder's bidding strategy if $k(r_n) \geq n$ s.t. the auction ends in round (Rn). When $k(r_n) = n$, the market clearing price is r_n and the auction ends immediately, so there is no need to discuss bidders' strategies. When $k(r_n) > n$, the auction is equivalent to an ascending clock auction with fixed supply of n identical items and $k(r_n)$ bidders with single-unit demands. It is well known that dropping out at true value $P(n) - c_i$ is a dominant strategy for each bidder.

I will next analyze each bidder's strategy at the beginning for each round n, at the time of which the clock price is set to be $p = r_n$. Each bidder needs to decide whether to state "in" or "out" in the auction. For each bidder i, participating in round (Rn) implies that bidder i will pay at least a price of r_n conditional on winning in round (Rn) when the total supply is n. First, consider the strategy for any bidder

i with cost $c_i > P(n) - r_n$:

- If there are less than $(n-1)$ opponents stating "in" at clock price $p = r_n$, then stating "in" and "out" yields the same payoff of zero in (Rn) since no allocation occurs in (Rn) in both cases and the auction will proceed to (Rn-1) with the same starting price.

- If there are exactly $(n-1)$ opponents stating "in" at clock price $p = r_n$, then bidder i wins a license with a negative payoff of $P(n) - c_i - r_n < 0$ by stating "in" and gets payoff of zero by stating "out" in (Rn).

- If there are more than $(n-1)$ opponents stating "in" at clock price $p = r_n$, then bidder i gets a non-positive payoff by stating "in" and gets a zero payoff by stating "out" in (Rn), since the auction will end in (Rn) with a price strictly higher than r_n.

Therefore, for bidders with costs $c_i > P(n) - r_n$, stating "out" at clock price $p = r_n$ in a weakly dominant strategy. Next, consider the strategies for any bidder with cost $c_i \leq P(n) - r_n$:

- If there are less than $(n-1)$ opponents stating "in" at the price of r_n, then stating "in" and "out" in round (Rn) yields the same payoff since no allocation occurs at round (Rn) in both cases.

- If there are exactly $(n-1)$ opponents stating "in" at the price of r_n, that implies there are $(n-1)$ opponents with costs $c_j \leq P(n) - r_n$, since no bidder with costs $c_j > P(n) - r_n$ will state "in" as proved above. If bidder i state

"in" at $p = r_n$, the auction ends immediately and bidder i gets a payoff of $P(n) - c_i - r_n \geq 0$. If bidder i state "out" at $p = r_n$, the auction will proceed to round (Rn-1). Bidder i will get a payoff of zero if not participating in (Rn-1). Suppose bidder i participate. Since all bidders who state "in" at $p = r_n$ are required to remain in the auction when $p = r_{n-1}$, all of these $(n-1)$ active bidders will be active in round (Rn-1) and the auction will end in (Rn-1) with probability of one. Let y_{n-1} denote the cost of the bidder who has the highest cost among those $(n-1)$ active bidders in (Rn), then $y_{n-1} \leq P(n) - r_n$. This bidder's drop out price is $P(n-1) - y_{n-1}$ in round (Rn-1). If bidder i drops out before $P(n-1) - y_{n-1}$, then bidder i gets zero payoff. If bidder i drops out after $P(n-1) - y_{n-1}$ and wins one out of $(n-1)$ licenses, then bidder i gets payoff of $y_{n-1} - c_i$. However, since $y_{n-1} \leq P(n) - r_n$, $y_{n-1} - c_i \leq P(n) - c_i - r_n$. Therefore, bidder i is weakly worse off by stating "out" than by stating "in" under every possible situation when there are exactly $(n-1)$ opponents stating "in" at $p = r_n$.

- If there are more than $(n-1)$ opponents stating "in" at clock price $p = r_n$, then bidder i will gets a non-negative payoff by stating "in" at $p = r_n$ and gets a zero payoff by stating "out" at $p = r_n$, since the auction will end in round (Rn). Bidder i has a positive probability of winning and gets a positive payoff in (Rn) by stating "in" and will be eliminated from auction by stating "out".

Therefore, for bidders with costs $c_i \leq P(n) - r_n$, stating "in" at clock price $p = r_n$ in a weakly dominant strategy. When $r_n = (n-1)\delta$ for all n, the Walrasian clock

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auction implements the efficient allocation. □

Proof of Lemma 3.5[1]:

Proof. I will first show that any Bayesian IC and IR mechanism must satisfy the characterizations in Lemma 3.5, then I will show that any mechanism satisfying the characterizations in Lemma 3.6 must be Bayesian IC and IR.

Suppose (μ, t) is a Bayeisan IC and IR mechanism. According to the Bayesian IC and IR condition, for any bidder i, for any true cost profile (c_i, c_{-i}) and bidder i's reported cost c_i',

$$
\begin{aligned}
U_i(c_i) &\geq \int_{c_{-i}} \left\{ \sum_{n=1}^{K} \pi(c_i, n) \mu_i^{(n)}(c_i', c_{-i}) - t_i(c_i', c_{-i}) \right\} f_{-i}(c_{-i}) dc_{-i} \\
&\geq \int_{c_{-i}} \left\{ \sum_{n=1}^{K} \left[\pi(c_i', n) + (c_i' - c_i) \right] \times \mu_i^{(n)}(c_i', c_{-i}) - t_i(c_i', c_{-i}) \right\} f_{-i}(c_{-i}) dc_{-i} \\
&\geq \int_{c_{-i}} \left\{ \sum_{n=1}^{K} \pi(c_i', n) \mu_i^{(n)}(c_i', c_{-i}) - t_i(c_i', c_{-i}) \right\} f_{-i}(c_{-i}) dc_{-i} \\
&\quad + \int_{c_{-i}} \left\{ \sum_{n=1}^{K} (c_i' - c_i) \times \mu_i^{(n)}(c_i', c_{-i}) \right\} f_{-i}(c_{-i}) dc_{-i} \\
&\geq U_i(c_i') + (c_i' - c_i) \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i', c_{-i}) f_{-i}(c_{-i}) dc_{-i}
\end{aligned}
\tag{B.16}
$$

Therefore,

$$
U_i(c_i) \geq U_i(c_i') + (c_i' - c_i) \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i', c_{-i}) f_{-i}(c_{-i}) dc_{-i}
$$

$$
U_i(c_i') \geq U_i(c_i) + (c_i - c_i') \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}
\tag{B.17}
$$

---

[1]The proof of Lemma 3.5 follows from Myerson (1981) [40].

which can be rewritten as

$$U_i(c_i) - U_i(c_i') \geq (c_i' - c_i) \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i', c_{-i}) f_{-i}(c_{-i}) dc_{-i}$$

$$U_i(c_i) - U_i(c_i') \leq (c_i' - c_i) \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} \tag{B.18}$$

The inequalities imply that if $c_i' > c_i$, then $U_i(c_i) - U_i(c_i') > 0$.

Suppose $c_i' > c_i$. Divide both sides by $(c_i' - c_i)$ and take limit:

$$U_i'(c_i) = -\lim_{c_i' \to c_i} \frac{U_i(c_i) - U_i(c_i')}{c_i' - c_i} = -\int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} < 0 \quad \text{(B.19)}$$

Therefore, the Bayesian IC condition implies

$$U_i(c_i) = U_i(\bar{c}) + \int_{c_i}^{\bar{c}} \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(s, c_{-i}) f_{-i}(c_{-i}) dc_{-i} ds \tag{B.20}$$

For all $c_i' \geq c_i$,

$$\int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i', c_{-i}) f_{-i}(c_{-i}) dc_{-i} \leq \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} \tag{B.21}$$

Since $U_i'(c_i) < 0$, the Bayesian IR condition $U_i(c_i) \geq 0$ for all $c_i \in [\underline{c}, \bar{c}]$ implies

$$U_i(\bar{c}) \geq 0 \tag{B.22}$$

Therefore, any Bayesian IC and IR mechanism must satisfy the characterization in Lemma 3.5.

I will next show that any mechanism (μ, t) that satisfies the characterization in Lemma 3.5 must be Bayesian IC and IR.

Equation (B.20) and inequality (B.21) implies that $U_i(\bar{c}) \leq U_i(c_i)$ for all $c_i \leq \bar{c}$. $U_i(\bar{c}) \geq 0$ implies Bayesian IR.

Suppose $c'_i > c_i$, then

$$
\begin{aligned}
U_i(c_i) =& U_i(c'_i) + \int_{c_i}^{c'_i} \left[\int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(s, c_{-i}) f_{-i}(c_{-i}) dc_{-i} \right] ds \\
\geq& U_i(c'_i) + \int_{c_i}^{c'_i} \left[\int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c'_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} \right] ds \qquad \text{(B.23)} \\
=& U_i(c'_i) + (c'_i - c_i) \times \int_{c_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(c'_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}
\end{aligned}
$$

According to inequality (B.17), this condition implies Bayesian IC. $\qquad \square$

Proof of Lemma 3.6:

Proof. For each bidder i, the ex-ante expected payoff is given below:

$$
\begin{aligned}
E_{c_i}\left[U_i(c_i) \right] =& U_i(\bar{c}) + \int_c \int_{c_i}^{\bar{c}} \sum_{n=1}^{K} \mu_i^{(n)}(s, c_{-i}) ds f(c) dc \\
=& U_i(\bar{c}) + \int_{c_{-i}} \int_{\underline{c}}^{\bar{c}} \int_{c_i}^{\bar{c}} \sum_{n=1}^{K} \mu_i^{(n)}(s, c_{-i}) ds f_i(c_i) dc_i f_{-i}(c_{-i}) dc_{-i} \\
=& U_i(\bar{c}) + \int_{c_{-i}} \int_{\underline{c}}^{\bar{c}} \left[\int_{\underline{c}}^{s} \sum_{n=1}^{K} \mu_i^{(n)}(s, c_{-i}) f_i(c_i) dc_i \right] ds f_{-i}(c_{-i}) dc_{-i} \\
=& U_i(\bar{c}) + \int_{c_{-i}} \int_{\underline{c}}^{\bar{c}} F_i(s) \sum_{n=1}^{K} \mu_i^{(n)}(s, c_{-i}) ds f_{-i}(c_{-i}) dc_{-i} \qquad \text{(B.24)} \\
=& U_i(\bar{c}) + \int_{c_{-i}} \int_{\underline{c}}^{\bar{c}} F_i(c_i) \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) dc_i f_{-i}(c_{-i}) dc_{-i} \\
=& U_i(\bar{c}) + \int_c \left[\frac{F_i(c_i)}{f_i(c_i)} \times \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \right] f(c) dc
\end{aligned}
$$

The total surplus generated in the auction is given by

$$
TS = \sum_i \int_c \left[\sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \pi(c_i, n) \right] f(c) dc \qquad \text{(B.25)}
$$

The seller's revenue can be derived by subtracting the total payoff of bidders

from the total surplus:

$$
\begin{aligned}
ER =& TS - \sum_i E_{c_i}\Big[U_i(c_i)\Big] \\
=& \sum_i \int_c \Big\{ \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \pi(c_i, n) \Big\} f(c) dc \\
& - \sum_i \Big\{ U_i(\bar{c}) + \int_c \Big\{ \frac{F_i(c_i)}{f_i(c_i)} \times \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \Big\} f(c) dc \Big\} \\
=& \sum_i \int_c \Big\{ \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \times \Big\{ \pi(c_i, n) - \frac{F_i(c_i)}{f_i(c_i)} \Big\} \Big\} f(c) dc - \sum_i U_i(\bar{c}) \\
=& \sum_i \int_c \Big\{ \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \times \Big\{ P(n) - c_i - \frac{F_i(c_i)}{f_i(c_i)} \Big\} \Big\} f(c) dc - \sum_i U_i(\bar{c})
\end{aligned}
$$
(B.26)

$\square$

Proof of Proposition 3.5:

Proof. The seller's problem is to maximize

$$
\begin{aligned}
ER =& \sum_i \int_c \Big\{ \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \times \Big\{ P(n) - c_i - \frac{F_i(c_i)}{f_i(c_i)} \Big\} \Big\} f(c) dc - \sum_i U_i(\bar{c}) \\
=& \sum_i \int_c \Big\{ \sum_{n=1}^{K} \mu_i^{(n)}(c_i, c_{-i}) \times MR(c_i, n) \Big\} f(c) dc - \sum_i U_i(\bar{c})
\end{aligned}
$$
(B.27)

subject to $U_i(\bar{c}) \geq 0$ and feasibility constraint. It is optimal to set $U_i(\bar{c}) = 0$. For every possible supply level n, it is optimal to allocate one license to each of the n highest marginal revenue bidders. Since $MR(c_i, n)$ is decreasing in c_i, it is equivalent to say that conditional on supply level being equal to n, it is optimal to allocate one license to each of the n lowest cost bidders.

Since the marginal impact on total revenue by selling the n-th license is $MR(c_{(n)}, n) - (n-1)\delta$, to determine the optimal level of supply n^*, we need to

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find n^* that satisfy

$$MR\big(c_{(n)}, n\big) - (n-1)\delta \geq 0$$

<div align="right">(B.28)</div>

$$MR\big(c_{(n+1)}, n+1\big) - n\delta < 0$$

Therefore, the optimal allocation rule is

$$\mu_i^{*(n)}(c_i, c_{-i}) = \begin{cases} 1 & \text{if } c_i \leq C^n(c_{-i}), \\[2mm] & MR\big(C^n(c_i, c_{-i}), n\big) - (n-1)\delta \geq 0, \\[2mm] & \text{and } MR\big(C^{n+1}(c_i, c_{-i}), n+1\big) - n\delta < 0 \\[2mm] 0 & \text{else} \end{cases}$$

<div align="right">(B.29)</div>

for all $n \in \{1, 2, \cdots, K-1\}$, and

$$\mu_i^{*(K)}(c_i, c_{-i}) = \begin{cases} 1 & \text{if } c_i \leq C'^K(c_{-i}), \\[2mm] & MR\big(C^K(c_i, c_{-i}), K\big) - (K-1)\delta \geq 0, \\[2mm] 0 & \text{else} \end{cases}$$

<div align="right">(B.30)</div>

where $C^n(c_i, c_{-i})$ is the n-th lowest cost given a profile of reported costs (c_i, c_{-i}).

Any tie is broken randomly.

The IC and IR conditions imply

$$U_i(\bar{c}) = U_i(c_i) - \int_{c_i}^{\bar{c}} \int_{c_{-i}} \sum_{n=1}^{K} \mu^{(n)}(s, c_{-i}) ds f_{-i}(c_{-i}) dc_{-i}$$

$$= \int_{c_{-i}} \left\{ \sum_{n=1}^{K} \mu^{(n)}(c_i, c_{-i})\big[P(n) - c_i\big] - \int_{c_i}^{\bar{c}} \sum_{n=1}^{K} \mu^{(n)}(s, c_{-i}) ds - t_i(c_i, c_{-i}) \right\} f_{-i}(c_{-i}) dc_{-i}$$

$$= 0$$

<div align="right">(B.31)</div>

Set $t_i^*(c_i, c_{-i})$ to be

$$t_i^*(c_i, c_{-i}) = \sum_{n=1}^{K} \mu^{(n)}(c_i, c_{-i})\big[P(n) - c_i\big] - \int_{c_i}^{\bar{c}} \sum_{n=1}^{K} \mu^{(n)}(s, c_{-i}) ds$$

<div align="right">(B.32)</div>

Then (μ^*, t^*) is an optimal mechanism among all Bayesian IC and IR mechanisms.

$\square$

Appendix C: Proofs for Chapter 4

Proof of Corollary 4.1:

Proof. Let b_i denote the bid submitted by bidder i. Since q_i is known to the auctioneer, define the adjusted bid of i as $\hat{v}_i = q_i - b_i$. The true value of bidder i is given by $v_i = q_i - c_i$. Reporting $b_i > c_i$ will lead to $\hat{v}_i < v_i$ and losing the auction when the supplier could have profitably won the auction with $\hat{v}_i = v_i$. Reporting $b_i < c_i$ will lead to $\hat{v}_i > v_i$ and winning the auction with negative payoff when $v_i - \hat{v}_j < 0$. Therefore, as in standard second-price auctions, it is a dominant strategy for each supplier to report true value v_i by submitting bid equals to marginal cost c_i truthfully. $\square$

Proof of Proposition 4.1:

Proof. Under the concealment of quality q_i, each supplier's objective function at the investment stage is

$$\max_{k_i} \int_{q_i} \int_{q_j} \int_{\underline{v}}^{q_i - g(k_i) - E\eta_i} \left\{ 1 - H(q_j - g(k_j^*) - \tau) \right\} d\tau dG_j(q_j) dG_i(q_i) - k_i \quad \text{(C.1)}$$

The first order condition is given by

$$-g'(k_i^*) \times \underbrace{\left\{ \int_{q_i} \int_{q_j} \left\{ 1 - H(q_j - g(k_j^*) - q_i + g(k_i^*) + E\eta_i) \right\} dG(q_j) dG(q_i) \right\}}_{\text{expected probability of winning}} - 1 = 0$$

$$\text{(C.2)}$$

Given the symmetry of the two bidders, the first order condition for the bidders is symmetric, which means we must have $k_i^* = k_j^*$ in equilibrium.

Since bidders are ex ante identical, in any symmetric equilibrium, the ex-ante expected probability of winning the auction is always $\frac{1}{2}$, i.e.,

$$\int_{q_i} \int_{q_j} \left\{ 1 - H(q_j - g(k_j^*) - q_i + g(k_i^*) + E\eta_i) \right\} dG(q_j) dG(q_i) = \frac{1}{2} \qquad \text{(C.3)}$$

The first order condition can be therefore written as

$$-g'(k_i^*) \times \frac{1}{2} - 1 = 0$$
$$-g'(k_i^*) = 2 \qquad \text{(C.4)}$$

The symmetric equilibrium investment under concealment of quality $k^C = k_i^* = k_j^*$ is therefore independent of the distribution $G(.)$ and $H(.)$. For any given cost reducing technology $g(.)$, the equilibrium investment k^C under quality concealment is identical across bidders and identical under any distribution of quality G. $\qquad \square$

Proof of Proposition 4.2:

Proof. When the auctioneer privately discloses q_i, each supplier i's objective function is

$$\max_{k_i} \int_{q_j} \int_{\underline{v}}^{q_i - g(k_i) - E\eta_i} \left\{ 1 - H\big(q_j - g(k^D(q_j)) - \tau\big) \right\} d\tau dG(q_j) - k_i \qquad \text{(C.5)}$$

The first order condition of each supplier's objective function is

$$-g'(k_i^D) \times \int_{q_j} \left\{ 1 - H\big(q_j - g\big(k^D(q_j)\big) - q_i + g(k_i^D) + E\eta_i\big) \right\} dG(q_j) - 1 = 0 \text{ (C.6)}$$

Suppose SOC < 0 s.t. an equilibrium exists. $k_i^D = k^D(q_i)$ characterized by FOC is the equilibrium investment strategy of supplier i with quality q_i. Take total

differentiation of FOC with respect to k_i^D and q_i:

$$\frac{dk_i^D}{dq_i} = -\frac{\int_{q_j} H'\big(q_j - g\big(k^D(q_j)\big) - q_i + g(k_i^D) + E\eta_i\big)(-g'(k_i^D))dG(q_j)}{SOC} > 0 \quad \text{(C.7)}$$

since $H'(.) > 0$, $-g'(.) > 0$, and the denominator < 0 by second order condition.

Therefore, the equilibrium investment k_i^D is increasing in each supplier's quality q_i

when the auctioneer discloses q_i at the investment stage.

$\square$

Proof of Proposition 4.3:

Proof. Under public disclosure of qualities, the objective function for bidder i given

information (q_i, q_j) and the opponent's investment k_j is

$$\max_{k_i} \int_{\underline{v}}^{q_i - g(k_i) - E\eta_i} \left\{1 - H\big(q_j - g(k_j) - \tau\big)\right\}d\tau - k_i \quad \text{(C.8)}$$

Each bidder i's best response investment $k_i^*(k_j; q_i, q_j)$ to any level of opponent's

investment k_j is characterized by

$$-g'(k_i^*) \times \left\{1 - H\big(q_j - g(k_j) - q_i + g(k_i^*) + E\eta_i\big)\right\} - 1 = 0 \quad \text{(C.9)}$$

Take total differentiation of the best response function with respect to k_i^* and $\big(q_i - q_j\big)$:

$$\frac{\partial k_i^*}{\partial(q_i - q_j)} = -\frac{H'\big(-(q_i - q_j) + g(k_i^*) - g(k_j) + E\eta_i\big)\big(-g'(k_i^*)\big)}{SOC} > 0 \quad \text{(C.10)}$$

since $H'(.) > 0$, $-g'(k_i^*) > 0$, and $SOC < 0$. Therefore, the best response investment

of i to any investment level of j will shift to the right when quality difference $(q_i - q_j)$

increases.

Take total differentiation of FOC with respect to k_i^* and k_j:

$$\frac{\partial k_i^*}{\partial k_j} = -\frac{H'\big(q_j - g(k_j) - q_i + g(k_i^*) + E\eta_i\big)g'(k_j)\big(-g'(k_i^*)\big)}{SOC} < 0 \qquad \text{(C.11)}$$

since $H'(.) > 0$, $-g'(k_i^*) > 0$, $g'(.) < 0$, and SOC < 0. So the best response investment of i is decreasing in the opponent's investment k_j under any announced quality (q_i, q_j).

The intersection of $k_i^*(k_j; q_i, q_j)$ and $k_j^*(k_i; q_i, q_j)$ gives the equilibrium investments (k_i^A, k_j^A). For each bidder i, assuming the opponent is playing the equilibrium k_j^A, then k_i^A is characterized by the first order condition given by

$$-g'(k_i^A) \times \Big\{1 - H\big(q_j - g(k_j^A) - q_i + g(k_i^A) + E\eta_i\big)\Big\} - 1 = 0 \qquad \text{(C.12)}$$

Suppose SOC < 0 s.t. an equilibrium exists. Take total differentiation of FOC with respect to k_i^A and $(q_i - q_j)$:

$$\frac{dk_i^A}{d(q_i - q_j)} = -\frac{H'\big(-(q_i - q_j) + g(k_i^A) - g(k_j^A) + E\eta_i\big)\big(-g'(k_i^A)\big)}{SOC} > 0 \qquad \text{(C.13)}$$

since $H'(.) > 0$, $-g'(k_i^A) > 0$, and $SOC < 0$. Therefore, the equilibrium investment of supplier i is increasing in the announced quality difference $(q_i - q_j)$. $\qquad\square$

Proof of Proposition 4.4:

Proof. The expected revenue under concealment of quality is given by

$$ER^C(G) = E\Big[V(k^C, q_2)\Big|G\Big] = E\Big[q_2 - g(k^C) - E\eta\Big|G\Big] \qquad \text{(C.14)}$$

in which k^C is independent of G and q. The total effect of $\Delta(G)$ on $ER^C(G)$ is

given by

$$\frac{dER^C(G)}{d\Delta(G)} = \frac{dE\left[q_2 - g(k^C) - E\eta \middle| G\right]}{d\Delta(G)} \tag{C.15}$$
$$= \frac{dE(q_2|G)}{d\Delta(G)} < 0$$

The expected revenue to the auctioneer under private disclosure of quality is given by

$$ER^D(G) = E\left[V(k_2^D, q_2)|G\right] = E\left[q_2 - g(k^D(q_2)) - E\eta \middle| G\right] \tag{C.16}$$

Holding the expected quality constant and increasing the dispersion $\Delta(G)$ will decrease the expected value of the low quality and decrease the expected investment of the low quality supplier. The total impact of $\Delta(G)$ on $ER^D(G)$ is

$$\frac{dER^D(G)}{d\Delta(G)} = \frac{dE\left[q_2 - g(k^D(q_2)) - E\eta \middle| G\right]}{d\Delta(G)}$$
$$= \frac{dE(q_2|G)}{d\Delta(G)} - g'(k_2^D)k^{D'}(q_2)\frac{dE(q_2|G)}{d\Delta(G)} \tag{C.17}$$
$$= \frac{dE(q_2|G)}{d\Delta(G)} \times \left\{1 - g'(k_2^D)k^{D'}(q_2)\right\} < 0$$

Since $1 - g'(k_2^D)k_2^{D'}(q_2) > 1$,

$$\frac{dER^D(G)}{d\Delta(G)} < \frac{dER^C(G)}{d\Delta(G)} \tag{C.18}$$

i.e., the negative impact of increased dispersion in G on ER^D is greater than on ER^C. Subtracting $ER^D(G)$ from $ER^C(G)$ gives

$$\frac{d(ER^C(G) - ER^D(G))}{d\Delta(G)} = g'(k_2^D)k^{D'}(q_2)\frac{dE(q_2|G)}{d\Delta(G)} > 0 \tag{C.19}$$

which also implies that the difference in expected qualities under concealment and under private disclosure is increasing in $\Delta(G)$.

The expected revenue to the auctioneer under public disclosure of quality is

$$ER^A(G) = E\left[V(k_2^A, q_2)\Big|G\right] = E\left[q_2 - g(k_2^A(q_2, q_1)) - E\eta\Big|G\right] \tag{C.20}$$

Holding the expected quality constant and increasing the dispersion $\Delta(G)$ will increase the expected difference $(q_1 - q_2)$ and decrease the expected investment of the low quality supplier. The total impact of $\Delta(G)$ on $ER^A(G)$ is

$$\begin{aligned}
\frac{dER^A(G)}{d\Delta(G)} &= \frac{dE\left[q_2 - g(k_2^A(q_2, q_1)) - E\eta\Big|G\right]}{d\Delta(G)} \\
&= \frac{dE(q_2|G)}{d\Delta(G)} - g'(k_2^A)\frac{dk_2^A(q_2, q_1)}{d(q_2 - q_1)}\frac{dE(q_2 - q_1|G)}{d\Delta(G)} \tag{C.21} \\
&= \frac{dE(q_2|G)}{d\Delta(G)} - g'(k_2^A)\frac{dk_2^A(q_2, q_1)}{d(q_2 - q_1)}(-1) < 0
\end{aligned}$$

Subtracting $ER^A(G)$ from $ER^C(G)$ gives

$$\frac{d(ER^C(G) - ER^A(G))}{d\Delta(G)} = g'(k_2^A)\frac{dk_2^A(q_2, q_1)}{d(q_2 - q_1)}(-1) > 0 \tag{C.22}$$

Therefore, $\frac{dER^A(G)}{d\Delta(G)} < \frac{dER^C(G)}{d\Delta(G)}$, and the difference in expected revenues is increasing in $\Delta(G)$.

Since $ER^C(G) = ER^D(G) = ER^A(G)$ when $\Delta(G) = 0$ and

$$\begin{aligned}
\frac{dER^A(G)}{d\Delta(G)} &< \frac{dER^C(G)}{d\Delta(G)} < 0 \\
\frac{dER^D(G)}{d\Delta(G)} &< \frac{dER^C(G)}{d\Delta(G)} < 0
\end{aligned} \tag{C.23}$$

We have

$$ER^C(G) > ER^D(G), \quad \text{and} \quad ER^C(G) > ER^A(G) \tag{C.24}$$

for any distribution $G(.)$ that satisfies $\Delta(G) > 0$ when there are 2 bidders, and the difference in expected revenues is increasing in $\Delta(G)$. $\qquad\square$

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