ABSTRACT

Title of dissertation: MONGE–AMPÈRE ITERATION

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Dissertation directed by: Professor Yanir A. Rubinstein
Department of Mathematics

In this thesis we introduce a Monge–Ampère iteration to be a sequence of convex functions $\{\varphi_i\}_{i\in\mathbb{N}}$ which solve the sequence of Monge–Ampère equations $\det(\nabla^2\varphi_{i+1}) = h \circ \varphi_i$ with second boundary values $\nabla\varphi_{i+1}(\mathbb{R}^n) = A$ where h is a function of one variable and A is a bounded, convex set. Our main analytic theorem gives sufficient conditions on the function h and the set A so that a Monge–Ampère iteration $\{\varphi_i\}$, once correctly normalized, converges smoothly on compact sets to a convex solution of the Monge–Ampère equation $\det(\nabla^2\varphi) = h \circ \varphi$ with second boundary value $\nabla\varphi(\mathbb{R}^n) = A$. Monge–Ampère iterations $\{\varphi_i\}_{i\in\mathbb{N}}$ arise as a sequence of solutions to optimal transport problems, so our convergence result can be interpreted as breaking apart the Monge–Ampère equation $\det(\nabla^2\varphi) = h \circ \varphi$, $\nabla\varphi(\mathbb{R}^n) = A$ into a sequence of optimal transport problems.

We then turn to two geometric applications of our main theorem. The first application, when $h(t) = e^{-t}$, is to Ricci iteration, which was introduced by Rubinstein. We prove a sequence $\{\omega_i\}_{i\in\mathbb{N}}$ of toric Kähler metrics with fixed edge singularities solving the Kähler–Ricci iteration on a toric Fano manifold converges,

after being twisted by automorphisms, to a Kähler–Einstein metric with the same singularities. This extends the smooth Kähler–Ricci iteration convergence theorem of Darvas–Rubinstein [12] to edge metrics on toric Fano manifolds.

The second geometric application, when $h(t) = t^{-(n+2)}$, is to affine differential geometry. We introduce the affine iteration to be a sequence of graph immersions $f_i : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ such that the affine normal at $f_{i+1}(x)$ is a constant multiple of the position vector $f_i(x)$. Thus, the affine iteration is a sequence of prescribed affine normal problems. We prove for any affine iteration $\{f_i\}_{i\in\mathbb{N}}$ there exists a sequence of matrices $M_i \in Sl_{n+1}\mathbb{R}$ such that $\{M_i \cdot f_i(\mathbb{R}^n)\}$ converges smoothly to an affine sphere.

MONGE–AMPÈRE ITERATION

by

Ryan Hunter

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Advisory Committee:
Professor Yanir A. Rubinstein, Chair/Advisor
Professor Alexander Barg
Doctor Tamás Darvas
Professor Pierre-Emmanuel Jabin
Professor Richard A. Wentworth

Dedication

To my wife Quinn.

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Chapter 1: Introduction

1.1 Motivation and overview

The heart of this thesis is the approximation of solutions to complicated problems by a sequence of solutions to simpler problems. The simplest example of this idea, which is still relevant to our final goal, is the approximation of zeros of a continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$. One approach is to define the first-order ordinary differential equation

$$\dot{x}(t) = F(x(t)). \tag{1.1}$$

If a solution $x(t)$ to equation (1.1) exists for all time, and $\lim_{t\to\infty} x(t) = c$, then

$$\lim_{t\to\infty} \dot{x}(t) = \lim_{t\to\infty} F(x(t)) = F(c).$$

Thus, $F(c)$ must equal 0. Thus, if a solution $x(t)$ has a finite limit, then values of $x(t)$ approximate a zero of the function F.

We can simplify this approach by approximating the derivative $\dot{x}(t)$ in equation (1.1) by a finite difference $\dot{x}(t) \sim x(t+1) - x(t)$. The dynamical system

$$x_{i+1} = x_i + F(x_i) \tag{1.2}$$

is the forward Euler method for approximating solutions to equation (1.1). If $\{x_i\}$ is a sequence of points solving equation (1.2) and $\lim_{i\to\infty} x_i = c$, then $F(c) = 0$.

Thus, the points x_i are approximations for a zero of F. The main difficulty in utilizing these two approximation methods is that we do not know if a solution $x(t)$ or a sequence x_i will converge to a finite value. Thus, proving convergence is the essential first step to finding approximations.

Now we turn to the setting of Riemannian geometry. The main object of study is the pair (M, g) of a smooth manifold with a Riemannian metric. A fundamental problem in Riemannian geometry is finding metrics of constant curvature on a manifold M. In dimension 2 the curvature is entirely described by a single number, the Gaussian curvature, and all orientable, compact manifolds in dimension 2 admit a metric of constant Gaussian curvature.

For an n-dimensional manifold, the curvature is a tensor which is determined at each point by n choose 2 numbers called sectional curvatures. Asking for every sectional curvature to equal some constant is very restrictive, and there are many manifolds which do not admit metrics of constant sectional curvature. The Ricci curvature of g, denoted $\mathrm{Ric}(g)$, is a trace of the full curvature tensor and is the same tensor type as the metric. We define *Einstein metrics* to be those which satisfy

$$\mathrm{Ric}(g) = \mu\, g$$

for some constant μ. This generalization of constant Gaussian curvature is much less restrictive than metrics of constant sectional curvature.

The differential equation and dynamical system strategies described above have been employed in the study of Einstein metrics. In our motivating problem we approximated solutions to $F(x) = 0$ for x in $\mathbb{R}^n$. Now we want to approximate solutions to $-\operatorname{Ric}(g) + \mu\, g = 0$ for g in the infinite-dimensional space of metrics on M. The analogue of the first approximation method is the *Ricci flow*, which was introduced by Hamilton [22] and can be written as

$$\frac{\partial g(t)}{\partial t} = -\operatorname{Ric}(g(t)) + \mu\, g(t)$$

for metrics $g(t)$ on M. The fixed points of the Ricci flow are Einstein metrics, and if the Ricci flow converges, then the metrics $g(t)$ are approximations of an Einstein metric. As before, we can use a finite difference approximation for the time derivative to write down a dynamical system. The backwards Euler method is

$$g_{i+1} - g_i = -\operatorname{Ric}(g_{i+1}) + \mu\, g_{i+1}$$

for metrics g_i on M. This dynamical system is called the *Ricci iteration*, and it was first introduced by Rubinstein [41] [39]. Most of the results concerning the Ricci iteration have been made for Kähler manifolds [5] [13], with some exceptions like the work of Pulemotov–Rubinstein's on homogeneous spaces [36]. On compact Kähler manifolds Rubinstein [41] dealt with the case of nonpositive μ. The more difficult case is for positive μ, which simplifies to

$$\operatorname{Ric}(g_{i+1}) = g_i \tag{1.3}$$

when $\mu = 1$. Darvas–Rubinstein [12] proved when M is a compact Kähler manifold admitting a Kähler–Einstein metric, the Ricci iteration (1.3), after being twisted by an automorphism, converges smoothly to a Käher–Einstein metric.

Kähler manifolds are a subclass of complex manifolds, and the Kähler–Ricci iteration is a series of complex Monge–Ampère equations, meaning they depend on the determinant of the complex Hessian, denoted by $\det\left(\frac{\partial^2 f}{\partial z_i\,\partial \bar{z}_j}\right)$. In this thesis we study real Monge–Ampère analogues of the Kähler–Ricci iteration, meaning the equations depend on the determinant of the real Hessian, denoted by $\det(\nabla^2 f)$. Specifically, we will approximate convex solutions to the *second boundary value Monge–Ampère problem*

$$\begin{cases} \dfrac{\det(\nabla^2\varphi)}{\lambda(A)} = \dfrac{h\circ\varphi}{\|h\circ\varphi\|_1} \quad \text{on } \mathbb{R}^n \\ \nabla\varphi(\mathbb{R}^n) = A \end{cases} \tag{1.4}$$

by convex solutions to the sequence of simpler second boundary value Monge–Ampère equations

$$\begin{cases} \dfrac{\det(\nabla^2\varphi_{i+1})}{\lambda(A)} = \dfrac{h\circ\varphi_i}{\|h\circ\varphi_i\|_1} \quad \text{on } \mathbb{R}^n \\ \nabla\varphi_{i+1}(\mathbb{R}^n) = A \end{cases} \tag{1.5}$$

where λ is Lebesgue measure, $\|\cdot\|_1$ is the L^1 norm on $\mathbb{R}^n$, h is a function of one variable, and A is a bounded, convex set. We call a sequence of convex functions $\{\varphi_i\}$ which solve the sequence of Monge–Ampère equations (1.5) a *Monge–Ampère iteration*. This thesis is the first work about iterations of real Monge–Ampère equations.

1.2 Monge–Ampère iteration convergence

In Chapter 2 we provide sufficient hypotheses on the function h to guarantee the convergence of a Monge–Ampère iteration to a solution of the second boundary problem (1.4). Solutions to equation (1.5) are only unique up to an additive constant, so we must first normalize the solutions before we can obtain convergence. For this reason we define the *normalized Monge–Ampère second boundary value problem* to be

$$\begin{cases} \dfrac{\det(\nabla^2\varphi)}{\lambda(A)} = \dfrac{h\circ\varphi}{\|h\circ\varphi\|_1} \quad \text{on } \mathbb{R}^n \\ \nabla\varphi(\mathbb{R}^n) = A \\ \int_A \varphi^*\, d\lambda = -\tau \end{cases} \tag{1.6}$$

where φ^* is the Legendre transform of φ, and τ is a constant. We likewise define a *normalized Monge–Ampère iteration* to be a sequence of smooth convex functions $\{\varphi_i\}$ solving

$$\begin{cases} \dfrac{\det(\nabla^2\varphi_{i+1})}{\lambda(A)} = \dfrac{h\circ\varphi_i}{\|h\circ\varphi_i\|_1} \quad \text{on } \mathbb{R}^n \\ \nabla\varphi_{i+1}(\mathbb{R}^n) = A \\ \int_A \varphi^*_{i+1}\, d\lambda = -\tau. \end{cases} \tag{1.7}$$

In order to define the hypotheses which guarantee the convergence of a normalized Monge–Ampère iteration, we first define the space of continuous functions with at most linear growth

$$\mathcal{C} = \{\, f : \mathbb{R}^n \to (\tau,\infty) \mid f \text{ continuous, and } f(x)/(1+|x|) \text{ bounded} \,\}, \tag{1.8}$$

and the space of probability measures with finite first moments

$$\mathcal{P}_1 = \left\{ \mu \in \mathcal{P} \mid \int_{\mathbb{R}^n} |x|\, d\mu < \infty \right\}.$$

The natural pairing between $\mathcal{C}$ and $\mathcal{P}_1$ is

$$\langle f, \mu\rangle = \int_{\mathbb{R}^n} f\, d\mu.$$

The Monge–Ampère measure of a convex function φ is defined on Borel sets U by

$$\mathrm{MA}(\varphi)(U) = \lambda(\partial\varphi(U)),$$

where $\partial\varphi$ is the subgradient of φ. The Monge–Ampère measure is an extension of the measure $\det(\nabla^2\varphi)\,\lambda$ to any non-smooth convex function. Its definition is due to Alexandrov [1], and a thorough treatment can be found in Rauch–Taylor [37].

We define a functional $\mathcal{F} : \mathcal{C} \to \mathbb{R}$ by

$$\mathcal{F}(f) = H^{-1}(\|H \circ f\|_1) \quad \text{where} \quad H(t) = \int_t^\infty h\, d\lambda, \tag{1.9}$$

and a dual functional $\mathcal{G} : \mathcal{P}_1 \to \mathbb{R}$ by

$$\mathcal{G}(\mu) = \inf \{\, g(\langle f, \mu\rangle,\, \mathcal{F}(f)) \mid f \in \mathcal{C}\,\}, \tag{1.10}$$

for a function g of two variables, chosen so that the following hypotheses are satisfied:

Hypotheses 1.2.1.

(B1) h is smooth, positive, decreasing, and there exist $p > 0$ and $C > 0$ such that $h(t) \leq C\, t^{-(n+p+1)}$ for $t \gg 1$.

(B2) If solutions to equation (1.6) exist, then they are unique up to translations of $\mathbb{R}^n$.

(B3) $g(s,t)$ is differentiable, decreasing in t, and satisfies $g(s, g(s,t)) = t$.

(B4) If $\{\varphi_i\}$ is a sequence of smooth convex functions solving equation (1.7), then

$$g\Big(\Big\langle \varphi_{i+1}, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big)\Big) \leq \mathcal{F}(\varphi_i) \text{ for all } i.$$

We will always assume

$$A \subset \mathbb{R}^n \text{ is open, bounded, convex, and } \int_A y_i \, d\lambda = 0 \text{ for } i = 1, \ldots, n, \tag{1.11}$$

meaning the barycenter of A lies at the origin. Our main theorem is:

Theorem 1.2.2. *Assume A satisfies (1.11) and equation (1.7) satisfies Hypotheses 1.2.1. Let $\{\varphi_i\}$ be a normalized Monge–Ampère iteration solving equation (1.7), and let $\widetilde{\varphi}_i(x) = \varphi_i(x + a_i)$ for a_i such that $\varphi_i(a_i) = \inf \varphi_i$. Then, there exists a smooth, convex solution φ to equation (1.6) such that $\widetilde{\varphi}_i$ converges to φ on compact sets in every $C^{k,\alpha}$ norm.*

Theorem 1.2.2 proves that under Hypotheses 1.2.1, normalized Monge–Ampère iterations approximate solutions of the Monge–Ampère second boundary value problem.

For each step of the iteration, the function φ_{i+1} arises as the solution to an optimal transport problem. Specifically, if φ_{i+1} solves equation (1.7), then

$$(\nabla \varphi_{i+1})_\# \left(\frac{h \circ \varphi_i}{\|h \circ \varphi_i\|_1} \lambda \right) = \frac{\lambda\big|_A}{\lambda(A)},$$

and $\nabla \varphi_{i+1}$ minimizes the cost

$$\int_{\mathbb{R}^n} |x - T(x)|^2 \frac{h \circ \varphi_i}{\|h \circ \varphi_i\|_1} \, d\lambda$$

over all maps $T : \mathbb{R}^n \to A$ which also push forward the probability measure $\dfrac{h \circ \varphi_i}{\|h \circ \varphi_i\|_1} \lambda$ to the uniform measure on A. The numerical approximation of φ_{i+1} is facilitated by this optimal transportation interpretation, as is shown in the work of Lindsey–Rubinstein [26] and references therein. This demonstrates the potential of using a Monge–Ampère iteration to approximate a solution of equation (1.6).

The Monge–Ampère second boundary problem (1.4) has been studied by Berman and Berndtsson [4] for $h(t) = e^{-t}$, and by Klartag [24] for $h(t) = t^{-(n+p+1)}$ when $p > 0$. Using their work, we show Hypotheses 1.2.1 are satisfied for $h(t) = e^{-t}$ in Section 3.1 and for $h(t) = t^{-(n+p+1)}$ when $p > 0$ in Section 4.1. Theorem 1.2.2 could possibly be applied to other functions $h(t)$ once their respective Monge–Ampère second boundary values problems are studied.

1.3 Geometric applications

As we mentioned in Section 1.1, the Monge–Ampère iteration is designed to be a real Monge–Ampère analogue of the complex Monge–Ampère equation defining the Kähler–Ricci iteration. Our first geometric application is to show Theorem 1.2.2 recovers Darvas–Rubinstein's result [12] on the convergence of the Kähler–Ricci iteration in the special case of toric Kähler manifolds. We also extend their result to prove convergence of the Kähler–Ricci iteration for metrics with edge singularities on toric Kähler manifolds. This is the content of Chapter 3.

Our second geometric application is to affine differential geometry. We define an affine iteration to be a certain prescribed affine normal problem on certain affine immersions. We show that Theorem 1.2.2 implies the affine iteration converges to an affine sphere. This is the content of Chapter 4.

1.3.1 Kähler–Ricci iteration

We begin Chapter 3 by proving Hypotheses 1.2.1 are satisfied for $h(t) = e^{-t}$ when A satisfies (1.11). In particular, we use the main theorem of Berman and Berndtsson [4] to prove Hypothesis (B2). After showing the other hypotheses are satisfied, we apply Theorem 1.2.2 to prove the following:

Theorem 1.3.1. *Assume $A \subset \mathbb{R}^n$ satisfies (1.11), and fix $\tau \in \mathbb{R}$ and the function $h(t) = e^{-t}$. If $\{\phi_i\}$ is a sequence of smooth, strictly convex functions solving the Monge–Ampère iteration (1.7), then there exist constants $a_i \in \mathbb{R}^n$ such that $\widetilde{\phi}_i(x) = \phi_i(x + a_i)$ converges to ϕ, a smooth convex solution to equation (1.6), on compact sets in every $C^{k,\alpha}$ norm.*

The remainder of Chapter 3 is devoted to interpreting Theorem 1.3.1 in terms of the Kähler–Ricci iteration on toric Kähler manifolds.

Let X be a compact Kähler manifold with positive first Chern class. The Kähler–Ricci iteration is the sequence of prescribed Ricci curvature problems

$$\operatorname{Ric}(\omega_{i+1}) = \omega_i \tag{1.12}$$

for $\{\omega_{i+1}\}$ Kähler metrics in the $c_1(X)$, the first Chern class of X. The Kähler–Ricci iteration can be thought of as a discretization of the Kähler–Ricci flow, which is given by $\partial_t \omega = -\operatorname{Ric}(\omega) + \omega$ when $c_1(X)$ is positive. Each step of the Kähler–Ricci iteration admits a unique, smooth solution by the Calabi–Yau Theorem [47], and Theorem 1.2 of Darvas–Rubinstein [12] proves that when X admits a Kähler–Einstein metric, there exist automorphisms g_i such that $g_i^* \omega_i$ converges smoothly to a Kähler–Einstein metric.

Each step of the Kähler–Ricci iteration is a complex Monge–Ampère equation, but when X is a toric Kähler manifold the equation reduces to a real Monge–Ampère equation. Toric Kähler manifolds of complex dimension n are Kähler manifolds with an effective Hamiltonian holomorphic $T^n = (S^1)^n$ action, and they are characterized by certain compact, convex polytopes $P \subset \mathbb{R}^n$ called Delzant polytopes. The T^n action on X is free on an open dense subset of X biholomorphic to $\mathbb{C}^{*n}$. Kähler metrics ω on X are described by potentials in this open orbit:

$$\omega\Big|_{\mathbb{C}^{*n}} = \sqrt{-1}\partial\overline{\partial}\,\phi.$$

When ω is invariant under the T^n action, the potential ϕ only depends on $x_i = \log|z_i|^2$, so it can be thought of as a smooth, strictly convex function on $\mathbb{R}^n$. Work of Guillemin [21] shows the potential of ω satisfies $\nabla\phi(\mathbb{R}^n) = \operatorname{Int} P$ and certain asymptotics at infinity. The Kähler–Ricci iteration can be written in terms of these potentials as the Monge–Ampère iteration

$$\begin{cases} \dfrac{\det(\nabla^2\phi_{i+1})}{\lambda(P)} = \dfrac{e^{-\phi_i}}{\|e^{-\phi_i}\|_1} \\ \nabla\phi_{i+1}(\mathbb{R}^n) = \operatorname{Int} P. \end{cases} \tag{1.13}$$

A T^n invariant Kähler metric ω given by $\omega\Big|_{\mathbb{C}^{*n}} = \sqrt{-1}\partial\overline{\partial}\,\phi$ in the open orbit is Kähler–Einstein if it solves the second boundary problem

$$\begin{cases} \dfrac{\det(\nabla^2\phi)}{\lambda(P)} = \dfrac{e^{-\phi}}{\|e^{-\phi}\|_1} \\ \nabla\phi(\mathbb{R}^n) = \operatorname{Int} P. \end{cases} \tag{1.14}$$

Wang and Zhu [46] proved that Kähler–Einstein metrics exist on toric Kähler manifolds if and only if the barycenter of P lies at the origin, which corresponds to condition (1.11) in Theorem 1.2.2. Thus, if P is a Delzant polytope with barycenter at the origin, then after normalizing the additive constants of $\{\phi_i\}$ the Ricci iteration convergence result of Darvas–Rubinstein [12] implies the existence of constants $a_i \in \mathbb{R}^n$ such that $\phi_i(x + a_i)$ converges smoothly to ϕ, a solution of equation (1.14). Thus, Darvas–Rubinstein's theorem proves Theorem 1.3.1 in the special case when the closure of A is a Delzant polytope.

Theorem 1.3.1 holds more generally for any convex set A with barycenter at the origin. The theorem of Darvas–Rubinstein does not imply Theorem 1.7 in a majority of these cases.

Conversely, we can use Theorem 1.3.1 to recover Darvas–Rubinstein's Kähler–Ricci iteration convergence results on toric Kähler manifolds. Moreover, we can extend their results to prove the convergence of a singular Kähler–Ricci iteration. The extension of their result to the singular setting introduces new difficulties because we must work on the noncompact open orbit of the toric manifold.

We say a Kähler metric ω has an edge singularity of angle β along a divisor D given locally by $\{z_1 = 0\}$ if ω is asymptotic to the reference metric

$$\sqrt{-1}\left(|z_1|^{2(\beta-1)}\, dz_1 \wedge d\bar{z}_1 + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i \right).$$

The term $\sqrt{-1}\,|z|^{2(\beta-1)}\, dz \wedge d\bar{z}$ is the flat metric on the cone which is obtained from $\mathbb{C}$ by cutting out a sector of angle $2\pi(1-\beta)$ and gluing the exposed edges. The angle β lies in $(0, 1]$ with $\beta = 1$ corresponding to smooth metrics. The Ricci curvature of such a singular metric can be interpreted as a current, meaning it is a form with distributional coefficients. The singular part $\mathrm{Ric}(\omega)$ is $(1-\beta)\,[D]$, where $[D]$ is the current of integration along the divisor D. We can extend the definition of the Kähler–Ricci iteration to be

$$\mathrm{Ric}(\omega_{i+1}) = \mu\,\omega_i + \sum_{i=1}^{k}(1-\beta_i)\,[D_i] \tag{1.15}$$

for ω_{i+1} Kähler metrics in $c_1(X)$ with edge singularities of angle β_i along the divisors D_i. We will show when X is a toric Kähler manifold and D_i are certain divisors, Kähler–Ricci iteration (1.15) converges to a metric ω with the same edge singularities solving the Kähler–Einstein equation

$$\mathrm{Ric}(\omega) = \mu\,\omega_i + \sum_{i=1}^{k}(1-\beta_i)\,[D_i]. \tag{1.16}$$

Specifically, we restrict our attention to toric manifolds X which are Fano, meaning $c_1(X) > 0$, and choose the corresponding Delzant polytopes P which correspond to the anticanonical line bundle. These Fano polytopes are of the form

$$P = \textstyle\bigcap_{i=1}^{M}\{\, y \in \mathbb{R}^n \mid l_i(y) \geq 0 \,\} = \bigcap_{i=1}^{M}\{\, y \mid 1 + \langle n_i, y\rangle \geq 0 \,\}$$

where $n_i \in \mathbb{Z}^n$ are primitive over $\mathbb{Z}$. Each facet of the boundary of P equals $\{l_i(y) = 0\}$ for some i, and each facet corresponds to a divisor D_i which is invariant under the T^n action. In section 3.3.2 we prove the following generalization of Darvas–Rubinstein's theorem in the case of toric Kähler manifolds.

Corollary 1.3.2. *Let X_P be a toric Fano manifold associated to a Fano polytope $P = \bigcap_{i=1}^{M}\{\, l_i(y) \geq 0 \,\} = \bigcap_{i=1}^{M}\{\, 1 + \langle n_i, y\rangle \geq 0 \,\}$. Let $\mu \in (0, \min\{l_i(P_c)^{-1} \mid i = 1, \ldots, M\,\}]$ be a constant, and let $\beta_i = \mu\, l_i(P_c)$ for P_c the barycenter of P. Let $\{\omega_i\}$ be a sequence of Kähler metrics in $c_1(X_P)$, with edge singularities of angle β_i along* *the toric divisors D_i, which solve the Kähler–Ricci iteration (1.15). Then there exist automorphisms g_i such that $\{g_i^* \omega_i\}$ converges smoothly on compact subsets of the open orbit to a Kähler–Einstein metric with the same edge singularities.*

1.3.2 Affine iteration

We begin Chapter 4 by proving Hypotheses 1.2.1 are satisfied for $h(t) = t^{-(n+p+1)}$ such that $p > 0$ when A satisfies (1.11). In particular, we use the main theorem of Klartag [24] to prove Hypothesis (B2). After showing the hypotheses are satisfied, we apply Theorem 1.2.2 to prove the following:

Theorem 1.3.3. *Assume $A \subset \mathbb{R}^n$ satisfies (1.11), and fix $p > 0$, $\tau < 0$, and the function $h(t) = t^{-(n+p+1)}$. If $\{\phi_i\}$ is a sequence of smooth, strictly convex functions solving the Monge–Ampère iteration (1.7), then there exist constants $a_i \in \mathbb{R}^n$ such that $\widetilde{\phi}_i(x) = \phi_i(x + a_i)$ converges to ϕ, a smooth convex solution to equation (1.6), on compact sets in every $C^{k,\alpha}$ norm.*

The remainder of Chapter 4 is devoted to interpreting Theorem 1.3.3 with $p = 1$ in terms of affine differential geometry.

Affine differential geometry is concerned with immersions $f : M^n \hookrightarrow \mathbb{R}^{n+1}$ and their properties which are *equiaffine*, meaning they are invariant under volume preserving affine transformations of the form $x \mapsto A\,x + v$ for $A \in Sl_{n+1}\mathbb{R}$ and $v \in \mathbb{R}^{n+1}$. The affine normal $\xi : M \to T\,\mathbb{R}^{n+1}\big|_{f(M)}$ is a uniquely defined equiaffine transversal vector field which plays the role of the Euclidean unit normal from Riemannian geometry.

An important object of study in affine geometry are *affine spheres* which are immersions f satisfying

$$f(x) + c\,\xi = x_0$$

for a constant $c \in \mathbb{R}$ and a fixed $x_0 \in \mathbb{R}^n$. The affine sphere equation implies the affine normals, once scaled by a constant, all meet at a point x_0 called the *center* of the affine sphere. We will study the case $c > 0$ where the immersion is called an *elliptic affine sphere*. When M is compact, the only affine spheres are ellipsoids as proven by Blaschke [6] for $n = 2$ and Deicke [15] in higher dimensions. By further work of Calabi [10] and Cheng–Yau [11], completeness of an elliptic affine sphere implies compactness, and so the only complete examples of elliptic affine spheres are ellipsoids.

To allow for a larger class of elliptic affine spheres we consider incomplete immersions. Specifically, we study immersions which are graphs of the Legendre transforms of convex functions. For A a compact subset of $\mathbb{R}^n$ we define the *Legendre graph immersion over* A to be the immersion $f_\phi : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ given by

$$f_\phi(x) = (\nabla\phi(x), \langle x, \nabla\phi(x)\rangle - \phi(x)) = (\nabla\phi(x),\, \phi^*(\nabla\phi(x)))$$

for $\phi : \mathbb{R}^n \to \mathbb{R}$, a smooth, strictly convex function such that $\nabla\phi(\mathbb{R}^n) = A$. The affine normal of this immersion is given by

$$\xi_\phi(x) = -(\nabla\psi(x), \langle x, \nabla\psi(x)\rangle - \psi(x)),$$

where $\psi(x) = \det(\nabla^2\phi)^{-1/(n+2)}$. If ϕ solves the second boundary Monge–Ampère equation (1.6) with $h(t) = t^{-(n+2)}$, then $\psi = \|\phi\|_{-(n+2)}^{-1}\,\phi$, so the affine normal is

$$\xi_\phi(x) = -\|\phi\|_{-(n+2)}^{-1}\, f_\phi(x),$$

and it follows that f_ϕ is an elliptic affine sphere.

Given a Legendre graph immersion f_ψ we can ask whether there exists a Legendre graph immersion f_ϕ such that $\xi_\phi = -f_\psi$. This leads us to the *prescribed affine normal problem for Legendre graph immersions*: If $A \subset \mathbb{R}^n$ is an open, bounded convex set and $\psi : \mathbb{R}^n \to \mathbb{R}$ is a smooth, positive, strongly convex function such that $\nabla\psi(\mathbb{R}^n) = A$, does there exist a Legendre graph immersion f_ϕ over A such that

$$\xi_\phi(x) = -c\, f_\psi(x)$$

for some constant $c > 0$. In Proposition 4.3.5 we prove that the prescribed affine normal problem has a smooth solution ϕ which is unique up to an additive constant.

We define the *affine iteration over* A to be a sequence $\{f_i := f_{\phi_i}\}$ of Legendre graph immersions over A, such that f_{i+1} solves the prescribed affine normal problem for f_i, and the functions ϕ_i are normalized by $\int_A \phi_i^*\, d\lambda = -\tau < 0$.

The integral condition $\int_A \phi_i^*\, d\lambda = -\tau < 0$ fixes the nonuniqueness from solving the prescribed affine normal problem. A normalized Monge–Ampère iteration $\{\phi_i\}$ for $h(t) = t^{-(n+2)}$ corresponds to an affine iteration f_i over A. Theorem 1.3.3 implies the sequence $\{\phi_i\}$ converges, after translations, to a function ϕ which solves equation (1.6), so f_ϕ is an affine sphere. In Section 4.3 we make this argument rigorous in the proof of the following corollary:

Corollary 1.3.4. *Assume $A \subset \mathbb{R}^n$ satisfies (1.11). Let ϕ_0 be a smooth, strictly convex function such that $\nabla\phi(\mathbb{R}^n) = A$, and $\int_A \phi_0^*\, d\lambda = -\tau < 0$. Let f_0 be the Legendre graph immersion of ϕ_0. Then there exists an affine iteration $\{f_i\}_{i=0}^\infty$ over A and $M_i \in Sl_{n+1}\mathbb{R}$ such that $M_i \cdot f_i(\mathbb{R}^n)$ converge smoothly to an elliptic affine* *sphere with center at the origin.*

1.4 A Dirichlet problem associated to toric Kähler–Einstein metrics

In Chapter 5 we prove an extension of a result by Mabuchi [29] [30]. In these two expository papers Mabuchi demonstrated how smooth Kähler–Einstein metrics on certain toric Kähler manifolds can be transformed into solutions of an associated Monge–Ampère Dirichlet problem. The following is a summary of his result.

Let ϕ be an open orbit potential for a Kähler–Einstein metrics on the toric Kähler manifold $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $Bl_3\,\mathbb{P}^2$, the blowup of $\mathbb{P}^2$ at three points. Such a potential is a smooth convex function solving

$$\begin{cases} \det(\nabla^2\phi) = e^{-\phi} \\ \nabla\phi(\mathbb{R}^2) = \text{Int } P \end{cases}$$

for a polytope P associated to the toric Kähler manifold. Using the change of variables $y = \nabla\phi(x)$, the matrix-valued function

$$(\nabla^2\phi) + \frac{1}{3} y\, y^T$$

extends to a smooth function on a neighborhood of P. In fact, there is a strongly convex function H which is smooth up to the boundary of P and solves

$$\text{adj}\,(\nabla^2 H) = (\nabla^2\phi) + \frac{1}{3} y\, y^T,$$

where adj denotes the adjugate of a matrix. If we define

$$\Omega = \nabla H(\text{Int } P),$$

then the Legendre transform of H can be used to define $\chi : \Omega \to \mathbb{R}$ which solves the Dirichlet problem

$$\begin{cases} det(\nabla^2 \chi) = (-\chi)^{-5/2} & \text{on } \Omega \\ \chi = 0 & \text{on } \partial\Omega. \end{cases}$$

The domain Ω associated to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $Bl_3\,\mathbb{P}^2$ can be computed using an asymptotic expansion of H near the boundary of P.

Our first extension of Mabuchi's theorem is to consider Kähler–Einstein metric ω with edge singularities solving

$$\mathrm{Ric}(\omega) = \mu\,\omega_i + \sum_{i=1}^{k} (1 - \beta_i)\,[D_i]$$

on any toric Fano manifold, whereas Mabuchi only considered smooth Kähler–Einstein metrics. Let ϕ be the open orbit potential for ω. If P is the Fano polytope associated to the toric manifold and P_c is its barycenter, then the function $\varphi(x) = \phi(x) - \langle P_c, x\rangle$ solves

$$\begin{cases} \det(\nabla^2 \varphi) = e^{-\mu\,\varphi} \\ \nabla\varphi(\mathbb{R}^n) = \mathrm{Int}\ P - P_c. \end{cases}$$

Using the change of variables $y = \nabla\varphi(x)$ we prove the matrix valued function

$$(\nabla^2 \varphi) + \frac{\mu}{3} y\,y^T$$

extends to a smooth function on a neighborhood of P. Mabuchi proved this smooth extension in only three cases, whereas we use the Guillemin boundary conditions on ϕ to give a rigorous proof for any toric Fano manifold.

At this point we restrict ourselves to the five toric Fano manifolds in complex dimension 2: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $Bl_k\,\mathbb{P}^2$ for $k = 1, 2, 3$, the blowup of $\mathbb{P}^2$ at up to three points. These are the manifolds Mabuchi studied plus $BL_k\,\mathbb{P}^2$ for $k = 1, 2$. For each of these manifolds we prove there exists a strongly convex function H which is smooth up to the boundary of $P - P_c$ and solves

$$\operatorname{adj}(\nabla^2 H) = (\nabla^2 \varphi) + \frac{\mu}{3} y\, y^T.$$

If we define

$$\Omega_\mu = \nabla H(\text{Int } P - P_c),$$

then the Legendre transform of H can be used to define $\chi : \Omega_\mu \to \mathbb{R}$ which solves the Dirichlet problem

$$\begin{cases} det(\nabla^2 \chi) = (-\chi)^{-5/2} & \text{on } \Omega_\mu \\ \chi = 0 & \text{on } \partial\Omega_\mu. \end{cases} \tag{1.17}$$

This result shows the domains Ω found by Mabuchi each lie in a one parameter family of domains Ω_μ. We compute the domain Ω_μ associated to each of the five toric Fano surfaces using expansions of H near the boundary of $P - P_c$.

We summarize this result in the following theorem.

Theorem 1.4.1. *Let X_P be a toric Fano manifold in complex dimension 2, and let P be the associated Fano polytope. For each Kähler–Einstein metric ω with edge singularities solving equation (1.16) there is a domain Ω_μ and a corresponding function $\chi : \Omega_\mu \to \mathbb{R}$ solving equation (1.17).*

1.5 Organization

Chapter 2: Convergence of the Monge–Ampère iteration

In this Chapter we prove Theorem 1.2.2. We begin in Section 2.1 with the analysis background needed for the proof. We focus first on properties of the Legendre transform, and second on the Monge–Ampère measure and the associated regularity of Monge–Ampère equations. In Section 2.2 we give the proof of Theorem 1.2.2 with a motivation of Hypotheses 1.2.1 in Subection 2.2.1.

Chapter 3: Kähler–Ricci iteration on toric manifolds

In Section 3.1 we prove Theorem 1.3.1, which amounts to verifying Hypotheses 1.2.1 for $h(t) = e^{-t}$. In Section 3.2 we show how to construct toric Kähler manifolds from Delzant polytopes, and we prove the necessary facts about its line bundles, divisors, and Kähler metrics. In Section 3.3 we use Theorem 1.3.1 to prove Corollary 1.3.2 about the convergence of the Kähler–Ricci iteration of metrics with edge singularities on toric Kähler manifolds. After the proof we show the functionals $\mathcal{F}$ and $\mathcal{G}$ can be interpreted in terms of the Ding functional and Mabuchi K-energy from Kähler geometry.

Chapter 4: Affine iteration

In Section 4.1 we prove Theorem 1.3.3, which amounts to verifying Hypotheses 1.2.1 for $h(t) = t^{-(n+p+1)}$ when $p > 0$. In Section 4.2 we discuss the background for affine immersions. We define the affine normal and prove a number of equivalent conditions to an immersion being an affine sphere. In Section 4.3 we use Theorem 1.3.3 to prove Corollary 1.3.4 about the convergence of the affine iteration of Legendre graph immersions. After the proof we give affine geometric interpretations of $\mathcal{F}$ and $\mathcal{G}$.

Chapter 5: A Dirichlet problem associated to toric Kähler–Einstein metrics

We prove Theorem 1.4.1 in Sections 5.1 and 5.2. Section 5.1 is devoted to proving an analytic result about the smooth extension of $(\nabla^2\varphi)^{-1}$ to a neighborhood of its domain P. Section 5.2 follows the work of Mabuchi [29] [30] to prove that this smooth extension can be used to produce the domain Ω and the function χ in Theorem 1.4.1. In Section 5.3 we compute the domains Ω in the cases where Theorem 1.4.1 applies.

Chapter 2: Convergence of the Monge–Ampère iteration

The goal of this chapter is to prove Theorem 1.2.2 on the convergence of the normalized Monge–Ampère iteration. In Section 2.1 we provide the analysis background needed, and we prove or give references for the analytic lemmas which we use within the proof.

In Section 2.2 we prove Theorem 1.2.2. We begin in subsection 2.2.1 with a motivation of Hypotheses 1.2.1 and a brief explanation of their role in the proof. In Subsection 2.2.2 we outline the proof and break it down into five steps. In Subsections 3.3 – 3.7 we prove each step of the outline.

2.1 Analysis background

This section is organized as follows. In Subsection 2.1.1 we discuss convex functions and most importantly the concept of the subdifferential. In Subsection 2.1.2 we use the subdifferential to define the Monge–Ampère measure which is used to define weak solutions of Monge–Ampère equations. Then we describe Caffarelli's work on the regularity of weak solutions to Monge–Ampère equations and prove a corollary of Caffarelli's theorems which is applicable to our problem. Subsection 2.1.3 is devoted to the Legendre transform of convex functions, and Subsection 2.1.4 defines the Wasserstein distance on probability measures.

2.1.1 Convex functions

A set $\Omega \subset \mathbb{R}^n$ is *convex* if for every pair of points x_0, x_1 in Ω, the line segment $\{\, t\, x_1 + (1-t)\, x_0 \mid t \in [0,1] \,\}$ lies in Ω. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is *convex* if its epigraph

$$\mathrm{epi}(f) = \{\, (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \geq f(x) \,\}$$

is a convex set. In particular, this definition implies the domain of f

$$\mathrm{dom}(f) = \{\, x \in \mathbb{R}^n \mid f(x) < \infty\},$$

which is just the projection of $\mathrm{epi}(f)$ onto the first n coordinates, must be a convex set. Thus, we often write $f : \Omega \to \mathbb{R}$, where $\Omega = \mathrm{dom}(f)$. The epigraph definition of convex functions is equivalent to the definition that f is convex if

$$f(t\, x_1 + (1-t)\, x_0) \leq t\, f(x_1) + (1-t)\, f(x_0) \text{ for } x_0,\ x_1 \text{ in } \Omega \text{ and } t \text{ in } [0,1].$$

If f is C^2, there is a local classification of convex functions based on the Hessian matrix $\nabla^2 f$ of second partial derivatives.

Lemma 2.1.1. *If $f : \Omega \to \mathbb{R}$ is a C^2 function, then f is convex if and only if $\nabla^2 f(x) \geq 0$ for every $x \in \Omega$.*

Proof. Assume $\nabla^2 f(x) \geq 0$ for every $x \in \Omega$. To show f is convex, it is enough to show that for every pair of points $x_0 \neq x_1$ in Ω, the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = f(t\, x_1 + (1-t)\, x_0) := f(x_t)$$

is convex, where x_t is the line segment from x_0 to x_1. The second derivative of g satisfies

$$g''(t) = (x_1 - x_0)^T \left(\nabla^2 f(x_t)\right)(x_1 - x_0) \geq 0,$$

because $\nabla^2 f$ is nonnegative definite. Thus, it suffices to show that for a function h of one variable, $h'' \geq 0$ implies h is convex. We can assume $x_0 \leq x_t \leq x_1$ without loss of generality. $h'' \geq 0$ implies h' is an increasing function, so by the mean value theorem

$$\frac{h(x_t) - h(x_0)}{x_t - x_0} \leq h'(x_t) \leq \frac{h(x_1) - h(x_t)}{x_1 - x_t}.$$

If we ignore the middle term of the inequality, and multiply both sides by the positive term $t\,(1-t)\,(x_1 - x_0)$, we get

$$(1-t)\,(h(x_t) - h(x_0)) \leq t\,h(x_1) - t\,h(x_t),$$

which implies the convexity of h.

To prove the other direction, assume that f is convex. We wish to show for all unit vectors u and all points $x \in \Omega$, $u^T \left(\nabla^2 f(x)\right) u \geq 0$. The limit definition of the second derivative shows

$$u^T \left(\nabla^2 f(x)\right) u = \lim_{h \to 0} \frac{f(x - h\,u) - 2f(x) + f(x + h\,u)}{h^2}.$$

By the convexity of f, we know $f(x) \leq \dfrac{f(x - h\,u) + f(x + h\,u)}{2}$, so the limit is nonnegative for all values of h. Thus $u^T \left(\nabla^2 f(x)\right) u \geq 0$. □

When $f : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$, the convexity of f implies that the graph of f lies above the tangent plane to $(x, f(x))$. Specifically,

$$f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle \text{ for all } z \in \Omega.$$

This supporting hyperplane characterization of $\nabla f(x)$ is used for any convex function to define the *subgradient* of f at x by

$$\partial f(x) = \{\, y \in \mathbb{R}^n \mid f(z) \geq f(x) + \langle y, z - x \rangle \text{ for all } z \in \Omega \,\}.$$

The subdifferential can be equivalently defined by $\partial f(x) = \bigcap_{z \in \mathrm{dom}(f)} \{\, y \mid f(z) \geq f(x) + \langle y, z - x \rangle \,\}$, which shows $\partial f(x)$ is the intersection of halfspaces. Thus, $\partial f(x)$ is a convex set. A simple example of the subgradient is $f(x) = R\,|x|$ for which

$$\partial f(x) = \begin{cases} \overline{B_R} & x = 0 \\ \frac{R}{|x|} x & x \neq 0. \end{cases}$$

In this example, $\partial f(x)$ only contains more than one point at 0 where f is not differentiable. This fact is true for all convex functions, and we reference Theorem 25.1 of Rockafellar [38] without proof.

Lemma 2.1.2. *Let f be a convex function, and let x be any point in* $\mathrm{dom}(f)$*. If f is differentiable at x, then $\partial f(x) = \nabla f(x)$. Conversely, if the subgradient $\partial f(x)$ contains a single point, then f is differentiable at x.*

The subdifferential of a convex function may be empty for some x in $\mathrm{dom}(f)$. For example, $f(x) = -\sqrt{1 - |x|^2}$ has empty subdifferential when $|x| = 1$ because $|\nabla f(x)| \to \infty$ as $|x| \to 1$. The next lemma shows the points with empty subdifferential must always occur on the boundary.

Lemma 2.1.3. *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a convex function, then $\partial f(x)$ is nonempty for every x in* $\mathrm{Int}(\mathrm{dom}(f))$*.*

Proof. Let x be a point in $\mathrm{Int}(\mathrm{dom}(f))$. The point $(x, f(x)) \in \mathbb{R}^{n+1}$ lies in the boundary of $\mathrm{epi}(f)$, and since $\mathrm{epi}(f)$ is always a convex set, there is some supporting hyperplane to $\mathrm{epi}(f)$ at $(x, f(x))$. Specifically, there exists an outward pointing normal (y, y_{n+1}) such that for all points (z, z_{n+1}) in $\mathrm{epi}(f)$

$$\langle\, (y, y_{n+1}), (z, z_{n+1}) - (x, f(x))\, \rangle \leq 0.$$

If $y_{n+1} > 0$, then taking any point (z, z_{n+1}) in $\mathrm{epi}(f)$ and letting $z_{n+1} \to \infty$ contradicts the inequality, so $y_{n+1} \leq 0$. If $y_{n+1} = 0$, then $\langle y, z - x\rangle \leq 0$ for all z in $\mathrm{dom}(f)$, but this contradicts $x \in \mathrm{Int}(\mathrm{dom}(f))$ because some z in a small ball around x would force $\langle y, z - x\rangle > 0$. Thus $y_{n+1} < 0$, and we can normalize the inequality to have $y_{n+1} = -1$, so

$$\langle\, (y, -1), (z, z_{n+1}) - (x, f(x))\, \rangle \leq 0$$

which implies

$$z_{n+1} \geq f(x) + \langle y, z - x\rangle$$

for all (z, z_{n+1}) in $\mathrm{epi}(f)$. This implies $y \in \partial f(x)$. □

If $y \in \partial f(x)$, the hyperplane given by $z_{n+1} = f(x) + \langle y, z - x\rangle$ is called a *supporting hyperplane to f at x* because it touches the graph of f at $(x, f(x))$ and is less than or equal to the graph of f at all other points. A convex function $f : \Omega \to \mathbb{R}$ is said to be *strictly convex* if for every $x \in \Omega$ and every $y \in \partial f(x)$ the supporting hyperplane $z_{n+1} = f(x) + \langle z - x, y\rangle$ intersects the graph of f only at $(x, f(x))$. In other words, the graph of f does not contain any line segments. Another equivalent definition is that f is strictly convex if

$$f(t\, x_1 + (1-t)\, x_0) < t\, f(x_1) + (1-t)\, f(x_0) \quad \text{for } x_0 \neq x_1 \text{ in } \Omega \text{ and } t \text{ in } (0,1).$$

One can show that if f is a C^2 function then $\nabla^2 f > 0$ implies f is strictly convex, but in contrast to Lemma 2.1.1 the converse does not hold. A counterexample is the function $f(x) = x^4$ which is strictly convex, but $f''(0) = 0$.

The next lemma is a type of comparison principle for subgradients of convex functions. Roughly speaking, it compares a convex function f to a cone with its vertex on the graph of f and says if f becomes greater than the cone, then the subgradient of f must be larger than the subgradient of the cone.

Lemma 2.1.4. *Let f be a convex function on B_r, and let $a < b$ be constants such that $f(0) = a$ and $f(x) \geq b$ for $|x| = r$. Then*

$$B_{(b-a)/r} \subset \partial f(B_r).$$

Proof. It is enough to show that if $|y| = (b-a)/r$, then $y \in \partial f(B_r)$. $|y| = (b-a)/r$ implies

$$a + \langle y, x\rangle \leq b \leq f(x) \quad \text{for } |x| = r.$$

Let

$$c = \sup_{x \in B_r} \{a + \langle y, x\rangle - f(x)\}.$$

Considering 0 in the supremum shows $c \geq 0$. We claim

$$a + \langle y, x\rangle - c \leq f(x) \tag{2.1}$$

is a supporting hyperplane for f for some x_1 such that $|x_1| < r$. If $c = 0$, then $x_1 = 0$ gives equality in (2.1), proving the lemma. If $c > 0$ then let x_1 be any point in B_r attaining the supremum defining c. It remains to show $|x_1| < r$. If $|x_1| = r$

then

$$0 \geq a + \langle y, x_1 \rangle - f(x_1) = c > 0,$$

so $|x_1|$ must be less than r, and $y \in \partial f(x_1)$. □

Convex functions have good compactness properties, which we will utilize to extract convergent subsequences of the Monge–Ampère iteration. We recall Theorem 10.9 from Rockafellar [38]:

Lemma 2.1.5. *If $\{f_i\}_{i=1}^{\infty}$ are finite, convex functions on $\mathbb{R}^n$, and for each $x \in \mathbb{R}^n$ the sequence $\{f_i(x)\}_{i=1}^{\infty}$ bounded, then there exists a subsequence which converges to a finite, convex function uniformly on compact subsets of $\mathbb{R}^n$.*

The pointwise convergence of convex functions also implies one inclusion on the convergences of their subdifferentials, as shown in Theorem 24.5 from Rockafellar [38]:

Lemma 2.1.6. *If $\{f_i\}_{i=1}^{\infty}$ are finite, convex functions on $\mathbb{R}^n$ which converge pointwise to a finite, convex function f on $\mathbb{R}^n$, then for every $x \in \mathbb{R}^n$ and every $\epsilon > 0$ there exists i_0 such that for all $i \geq i_0$*

$$\partial f_i(x) \subset \partial f(x) + B_\epsilon.$$

In the special case when $\partial\varphi_i(\mathbb{R}^n) = A$, a bounded convex set, if φ_i converges pointwise to a finite, convex function φ, then Lemma 2.1.6 implies

$$\text{Int}\, A \subset \partial\varphi(\mathbb{R}^n).$$

This inclusion can be strict, as is shown by the following example. Let φ be any convex function with $\partial\varphi(\mathbb{R}^n) = A$, a convex set such that $A \subsetneq B_R$, and define

$$\varphi_i(x) = \max\{\varphi(x),\ -i + R\,|x|\,\}.$$

The sequence $\{\varphi_i\}$ converges to φ uniformly on compact sets, yet $\partial\varphi(\mathbb{R}^n) = A \subsetneq B_R = \partial\varphi_i(\mathbb{R}^n)$.

2.1.2 Monge–Ampère measure

The subgradient of a convex function generalized the gradient when $f : \Omega \to \mathbb{R}$ was not C^1. Although there is not a similar generalization for $\nabla^2 f$, there is a way to generalize $\det(\nabla^2 f)$ to any convex function. Consider the case when f is C^2 and strictly convex, so $y = \nabla f(x)$ is a bijective map. The change of variables formula implies, for any Borel set $U \subset \Omega$

$$\int_U \det(\nabla^2 f(x))\, dx_1 \wedge \cdots \wedge dx_n = \int_{\nabla f(U)} dy_1 \wedge \cdots dy_n = \lambda(\nabla f(U)),$$

where λ is Lebesgue measure. Thus, $\det(\nabla^2 f)\, dx_1 \wedge \cdots \wedge dx_n$ can be interpreted as a measure on Ω which equals $(\nabla f)^{-1}_{\#}(\lambda)$. We can use the subgradient in place of the gradient to generalize this push-forward definition to all convex functions. The *Monge–Ampère measure* of a convex function f is defined on all Borel sets $U \subset \Omega$ by

$$\mathrm{MA}(f)\,(U) = \lambda(\partial f(U)), \tag{2.2}$$

where $\partial f(U) = \bigcup_{x\in U} \partial f(x)$. When f is C^2, the mapping $x \mapsto \nabla\phi(x)$ is C^1, so by Sard's theorem

$$\mathrm{MA}(f)(\{\, x \mid \det(\nabla^2 f(x)) = 0\,\} = 0.$$

Thus, we can apply the change of variables $y = \nabla f(x)$ outside of this set, and it follows that

$$\mathrm{MA}(f) = \det(\nabla^2 f)\,\lambda.$$

The proof that $\mathrm{MA}(f)$ is in fact a Borel measure can be found in Rauch–Taylor [37].

The Monge–Ampère measure is used primarily as a way to define weak solutions of Monge–Ampère equations. A convex function f is said to be an *Alexandrov solution* of the Monge–Ampère equation

$$\det(\nabla^2 f(x)) = g(x) \tag{2.3}$$

if

$$\mathrm{MA}(f) = g\,\lambda$$

as Borel measures.

If f is a convex, C^2 solution to equation (2.3) for $g > 0$, then $\nabla^2 f > 0$ which implies f is strictly convex. This fact does not extend to Monge–Ampère measure. Specifically, even if f is an Alexandrov solution to $\det(\nabla^2 f) = g$ for $g > 0$, f may not be strictly convex. The first examples of this were shown by Pogorelov [35]. For $x = (x', x_n)$ we can define

$$f(x) = |x'|^{2-2/n}\,(1 + x_n^2) \tag{2.4}$$

which is not strictly convex because it is constant along the line $\{x' = 0\}$, but when $n \geq 3$, $\det(\nabla^2 f) = g(x)$ in the Alexandrov sense for a function $g > 0$. In fact, one can show that for a certain function h, $f(x) = |x'|^{2-2/n}\,h(x_n)$ solves $\det(\nabla^2 f) = 1$ in the Alexndrov sense. A more in depth discussion of Pogorelov's examples and estimates on the dimension of the set where f is not strictly convex can be found in Mooney [32] [33].

Pogorelov's example (2.4) is only in $C^{1,\alpha}$ for $\alpha = 1 - 2/n$, but one can show that $\det(\nabla^2 f) = g$ in the Alexandrov sense for g smooth and positive. This lack of regularity is a manifestation of the fact that regularity for Monge–Ampère equations only holds at points where f is strictly convex. Caffarelli [7] proved if f is a strictly convex Alexandrov solution of $\det(\nabla^2 f(x)) = g(x)$ and $g \in C^{0,\alpha}$, then $f \in C^{2,\alpha}$.

Although strict convexity is necessary to prove regularity, another result of Caffarelli provides a means of showing that Alexandrov solutions to certain Monge–Ampère equations are strictly convex. Let Ω be an open, convex, bounded set, and let $f : \Omega \to \mathbb{R}$ be a convex function. If f is not strictly convex, then there is some supporting hyperplane $z_{n+1} = f(x) + \langle z - x, y\rangle$ which intersects the graph of f at more than one point. Denote this set by

$$S_y = \{\, z \in \Omega \mid f(z) = f(x) + \langle z - x, y\rangle \,\}.$$

S_y is a convex set because it is the sublevel set of the convex function, so we can consider its *extreme points* which are the points in the boundary of S_y that are not convex combinations of other points in $\overline{S_y}$. Caffarelli [8] proved if f is an Alexandrov solution to $\det(\nabla^2 f) = g$ where $c^{-1} \leq g \leq c$ for some $c > 0$, then the extreme points of S_y must lie in the boundary of Ω. Caffarelli's result has a nice corollary when the domain of f is all of $\mathbb{R}^n$.

Lemma 2.1.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. If $\partial f(\mathbb{R}^n)$ has nonempty interior and* $\mathrm{MA}(f) = g$ *for g a positive, continuous function, then f is stricly* *convex.*

Proof. Assume f is not strictly convex. Then, for some point point $x \in \mathbb{R}^n$ and some point $y \in \partial f(x)$ the set

$$S_y = \{\, z \in \mathbb{R}^n \mid f(z) = f(x) + \langle z - x, y\rangle \,\}$$

contains more than one point. Since the graph of f lies above the supporting hyperplane $z_{n+1} = f(x) + \langle z - x, y\rangle$, it follows that S_y is the sublevel set of a convex function, so it is convex. Since $\mathrm{dom}(f) = \mathbb{R}^n$, it follows that f is continuous, so S_y is closed.

We claim the set S_y cannot contain any extreme points. Let p be any point in the boundary of S_y. Since g is positive and continuous, $0 < c < g(x) \leq C$ on $\overline{B_R(p)}$. Thus, f is an Alexandrov solution to $0 < c \leq \det(\nabla^2 f) \leq C$, so Caffarelli's theorem implies the extreme points of $S_y \cap \overline{B_R(p)}$ occur on the boundary. Thus, p is not an extreme point of S_y, so S_y has no extreme points.

Theorem 18.5.3 of Rockafellar [38] says any nonempty, closed convex set which contains no lines must contain at least one extreme point. The definition of S_y implies it must contain a line $z + t\,u$ for $t \in \mathbb{R}$. This implies $f(z+t\,u) = f(z) + t\,\langle u, y\rangle$, and we claim this contradicts $\partial f(\mathbb{R}^n)$ having nonempty interior. By adding a linear function to f, we can assume without loss of generality that $B_r \subset \partial f(\mathbb{R}^n)$ for some small $r > 0$, which implies $f(x) \geq -C + r\,|x|$ for some constant C. This lower bound contradicts $f(z + t\,u) = f(z) + t\,\langle u, y\rangle$ for either $t \to \infty$ or $t \to -\infty$, thus f is strictly convex. $\square$

2.1.3 Legendre transform

The *Legendre transform* of a function $f : \Omega \to \mathbb{R}$ is defined by

$$f^*(y) = \sup\{\, \langle x, y\rangle - f(x) \mid x \in \Omega \,\}.$$

Since f^* is a supremum of affine functions, f^* is convex and lower semicontinuous. The definition of the Legendre transform yields the same result if we define extend f to all of $\mathbb{R}^n$ by defining $f(x) = \infty$ for $x \notin \Omega$.

Lemma 2.1.8. *If f is a convex function, and $y \in \partial f(x)$, then $f^*(y) = \langle x, y\rangle - f(x)$.*

Proof. The definition of f^* trivially implies $f^*(y) \geq \langle z, y\rangle - f(z)$ for any z. Since $y \in \partial f(x)$, we have

$$f(z) \geq f(x) + \langle y, z - x\rangle$$

for all z, which trivially implies $\langle y, z\rangle - f(z) \leq \langle y, x\rangle - f(x)$. Thus,

$$f^*(z) = \sup\{\, \langle y, z\rangle - f(z) \mid z \in \mathbb{R}^n \,\} \leq \langle y, x\rangle - f(x).$$

□

By Lemma 2.1.8, the Legendre transform of f can be thought of as a function on the subgradients of f. The Legendre transform can be interpreted geometrically as follows: if $y \in \partial f(x)$, and $z_{n+1} = f(x) + \langle z - x, y\rangle$ is the supporting hyperplane at x, then $f^*(y) = \langle y, x\rangle - f(x)$ is the negative of the value of the supporting hyperplane at $z = 0$. When f is differentiable at x, $\partial f(x) = \nabla f(x)$, so Lemma 2.1.8 implies

$$f^*(\nabla f(x)) = \langle x, \nabla f(x)\rangle - f(x).$$

Lemma 2.1.9. *If f is a lower semicontinuous, convex function such that* $\mathrm{dom}(f)$ *has nonempty interior, then $f^{**} = f$.*

Proof. We first note the definition of the Legendre transform trivially implies $f^*(y) + f(x) \geq \langle y, x \rangle$ for all x and y. Thus,

$$f^{**}(x) = \sup \{ \langle y, x \rangle - f^*(y) \mid y \in \mathbb{R}^n \} \leq f(x).$$

Since the Legendre transform is always lower semicontinuous and we assumed f was lower semicontinuous, it suffices to prove $f(x) \leq f^{**}(x)$ on a dense subset of $\mathrm{dom}(f)$. Since $\mathrm{dom}(f)$ was assumed to have nonempty interior, Lemma 2.1.3 implies the set $\{x \in \mathrm{dom}(f) \mid \partial f(x) \text{ is nonempty} \}$ is dense in $\mathrm{dom}(f)$.

Thus, we can assume $x \in \mathrm{dom}(f)$ and $y \in \partial f(x)$. By Lemma 2.1.8 $f^*(y) = \langle y, x \rangle - f(x)$, so

$$f^{**}(x) = \sup \{ \langle x, y \rangle - f^*(y) \mid y \in \mathbb{R}^n \} \geq f(x)$$

by choosing $y \in \partial f(x)$ in the supremum. □

Lemma 2.1.9 is true in more generality for any lower semicontinuous, convex functions, but the proof involves a few more details, and we are only interested in the case when $\mathrm{dom}(f)$ has nonempty interior. The following properties of the Legendre transform follow from the definition.

- If $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$, then $g^*(y) \leq f^*(y)$ for all $y \in \mathbb{R}^n$.
- If $f_a(x) = f(x - a)$, then $f_a^*(y) = f^*(y) + \langle y, a \rangle$.
- If $f_a(x) = f(x) + \langle y, a \rangle$, then $f_a^*(y) = f^*(y - a)$.

- If $f_\lambda(x) = \lambda f(x/\lambda)$ for $\lambda > 0$, then $f_\lambda^*(y) = \lambda f^*(y)$.

In general, differentiability of a convex function does not imply any differentiability of its Legendre transform. For example,

$$f(x) = \begin{cases} 0 & |x| < 1 \\ \frac{1}{k}(|x|-1)^k & |x| \geq 1 \end{cases}$$

is a C^{k-1} function when $k \in \mathbb{N}$, but its Legendre transform

$$f^*(y) = |y| + \frac{k-1}{k}|y|^{k/(k-1)}$$

is not even differentiable at 0. This example shows how, roughly speaking, the Legendre transform maps portions of graphs which agree with affine functions to points with some singular behavior. But if f is strictly convex, and thus its graph has no affine portions, then f^* is differentiable. Specifically, we have the following lemma:

Lemma 2.1.10. *Let Ω be a convex set with nonempty interior, and let $f : \Omega \to \mathbb{R}$ be a strictly convex and C^1. Then f^* is differentiable at $\nabla f(x)$, and*

$$\nabla f^*(\nabla f(x)) = x.$$

Proof. By Lemma 2.1.2, it suffices to show that $\partial f^*(\nabla f(x)) = x$. We start by showing that $x \in \partial f^*(\nabla f(x))$, which is true even if f is not strictly convex. For any $z \in \mathbb{R}^n$

$$f^*(z) \geq \langle x, z\rangle - f(x) = \langle x, \nabla f(x)\rangle - f(x) + \langle x, z - \nabla f(x)\rangle.$$

By Lemma 2.1.8, $f^*(\nabla f(x)) = \langle x, \nabla f(x)\rangle - f(x)$, so the above inequality implies

$$f^*(z) \geq f^*(\nabla f(x)) + \langle x, z - \nabla f(x)\rangle,$$

for all $z \in \mathbb{R}^n$. Thus, $x \in \partial f^*(\nabla f(x))$ by definition.

Next we show the strict convexity of f implies $\partial f^*(\nabla f(x))$ contains only x. Assume by contradiction there exists a unit vector u and a constant ϵ such that $x+\epsilon\, u \in \partial f^*(\nabla f(x))$. Since the subdifferential is a convex set, $x+t\, u \in \partial f^*(\nabla f(x))$ for all t in the interval $[0, \epsilon]$. The definition of the subdifferential implies

$$\begin{aligned} f^*(z) &\geq f^*(\nabla f(x)) + \langle x + t\, u, z - \nabla f(x) \rangle \\ &= \langle x, \nabla f(x) \rangle - f(x) + \langle x + t\, u, z - \nabla f(x) \rangle \\ &= -f(x) + \langle x + t\, u, z \rangle - t\, \langle u, \nabla f(x) \rangle \end{aligned}$$

for all $z \in \mathbb{R}^n$. Since f is C^1, Lemma 2.1.9 implies $f^{**} = f$, so using the above inequality we find

$$\begin{aligned} f(x+t\, u) = f^{**}(x+t\, u) &= \sup \{ \langle x + t\, u, z \rangle - f^*(z) \mid z \in \mathbb{R}^n \} \\ &\leq f(x) + t\, \langle u, \nabla f(x) \rangle. \end{aligned}$$

But the supporting hyperplane for f at x implies $f(x+t\, u) \geq f(x) + \langle t\, u, \nabla f(x) \rangle$, so it follows that $f(x+t\, u) = f(x) + t\, \langle u, \nabla f(x) \rangle$, which contradicts the strict convexity of f. □

Lemma 2.1.10 gives another way of thinking about the Legendre transform when f is strictly convex. It is the function f^* whose gradient ∇f^* is the inverse map to ∇f. When f is strictly convex, Lemma 2.1.10 gives us the change of variables:

$$y = \nabla f(x) \text{ and } x = \nabla f^*(y),$$

which implies

$$f^*(\nabla f(x)) = \langle x, \nabla f(x) \rangle - f(x) \text{ and } f(\nabla f^*(y)) = \langle y, \nabla f^*(y) \rangle - f^*(y).$$

We can use this change of variables to define a formula for the integral of the Legendre transform.

Lemma 2.1.11. *Let Ω be an open convex set, and let $f : \Omega \to \mathbb{R}$ be convex and C^1. Then,*

$$\int_{\nabla f(\Omega)} f^* \, d\lambda = \int_\Omega \big(\langle x, \nabla f(x)\rangle - f(x)\big) \; \mathrm{MA}(f),$$

where $\mathrm{MA}(f)$ is the Monge–Ampère measure of f defined in equation (2.2).

Proof. The function $(\nabla f)^{-1}$ may not exist, but we can still define $(\nabla f)^{-1}(U) = \{\, x \mid \nabla f(x) \in U \,\}$. For any Borel $U \subset \Omega$ the Monge–Ampère measure satisfies

$$\begin{aligned} (\nabla f)_\#(\mathrm{MA}(f))\,(U) &= \mathrm{MA}(f)\,\big((\nabla f)^{-1}(U)\big) \\ &= \lambda\,\big(\,\nabla f((\nabla f)^{-1}(U))\,\big) = \lambda\,(U), \end{aligned}$$

so $(\nabla f)_\#(\mathrm{MA}(f)) = \lambda$. The definition for the pushforward of a measure implies for any measurable function g on $\nabla f(\Omega)$

$$\int_{\nabla f(\Omega)} g \, d\lambda = \int_\Omega g \circ \nabla f \;\, \mathrm{MA}(f).$$

The result follows by setting $g = f^*$ and recalling that for f which are C^1, $f^*(\nabla f(x)) = \langle x, \nabla f(x)\rangle - f(x)$. ◻

2.1.4 Wasserstein distance

The following background on probability measures can be found in Villani [45, Ch. 6]. Let $\mathcal{P}(X)$ denote the set of probability measure on normed linear space X. If we omit the reference to X, then it is assumed to be $\mathbb{R}^n$. A measure γ in $\mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$

is said to have first marginal $\mu \in \mathcal{P}$ and second marginal $\nu \in \mathcal{P}$ if

$$\gamma(A \times \mathbb{R}^n) = \mu(A) \text{ and } \gamma(\mathbb{R}^n \times B) = \nu(B),$$

for all Borels A, $B \subset \mathbb{R}^n$. The set of all γ with first marginal μ and second marginal ν is denoted $\pi(\mu, \nu)$. The *Wasserstein distance* between $\mu \in \mathcal{P}$ and $\nu \in \mathcal{P}$ is

$$\mathrm{Wass}(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y| \, d\gamma \mid \gamma \in \pi(\mu, \nu) \right\}, \tag{2.5}$$

where (x, y) are coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. $\mathrm{Wass}(\cdot, \cdot)$ satisfies the axioms of a distance on $\mathcal{P}$. The space of probability measures with *finite first moment* is given by

$$\mathcal{P}_1 = \left\{ \mu \in \mathcal{P} \mid \int_{\mathbb{R}^n} |x| \, d\mu < \infty \right\}.$$

Lemma 2.1.12. *If* μ, $\nu \in \mathcal{P}_1$ *then* $\mathrm{Wass}(\mu, \nu) < \infty$.

Proof. Let $\gamma = \mu \times \nu$ be the product measure of μ and ν. γ is a coupling of μ and ν so

$$\mathrm{Wass}(\mu, \, \nu) \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y| \, d\gamma \leq \int_{\mathbb{R}^n} |x| \, d\mu + \int_{\mathbb{R}^n} |y| \, d\nu < \infty.$$

□

Let $\{\mu_i\}_{i=1}^{\infty}$ and μ be probability measures in $\mathcal{P}_1$. Define *convergence in Wasserstein distance* by

$$\mu_i \to_1 \mu \text{ if and only if } \mathrm{Wass}(\mu_i, \, \mu) \to 0. \tag{2.6}$$

The following theorem [45, pg. 96] relates convergence in Wasserstein distance to weak convergence of measures. Recall, μ_i converges to μ *weakly*, denoted $\mu_i \Rightarrow \mu$, if

$$\int_{\mathbb{R}^n} f \, d\mu_i \to \int_{\mathbb{R}^n} f \, d\mu \text{ for all } f \in C_b,$$

the space of continuous, bounded functions.

Proposition 2.1.13. *Let $\{\mu_i\}_{i=1}^{\infty}$ and μ be probability measures in $\mathcal{P}_1$. The following are equivalent:*

(i) $\mu_i \to_1 \mu$.

(ii) $\mu_i \Rightarrow \mu$ *and*

$$\lim_{R\to\infty} \limsup_{i\to\infty} \left\{ \int_{\{\,|x|\geq R\,\}} |x| \, d\mu_i \right\} = 0.$$

(iii) For all continuous functions f with $|f(x)| \leq C\,(1+|x|)$,

$$\int_{\mathbb{R}^n} f \, d\mu_i \to \int_{\mathbb{R}^n} f \, d\mu.$$

2.2 Proof of Theorem 1.2.2

This section is organized as follows. In Subsection 2.2.1 we motivate Hypotheses 1.2.1 and briefly explain their main role in the proof. In Subsection 2.2.2 we outline the proof of Theorem 1.2.2 and break it down into five steps. In Sections 3.3 − 3.7 we prove each step of the outline.

2.2.1 Explanation of Hypotheses 1.2.1

Hypothesis (B1) :

If φ solves equation (1.6), then $h \circ \varphi$ is in $L^1(\mathbb{R}^n)$. Thus, it is natural to stipulate a decay condition on h to guarantee $\|h \circ \varphi\|_1 < \infty$. Assume h is a positive, decreasing function such that

$$h(t) \leq C\, t^{-(n+p+1)}, \text{ for } p > 1 \text{ when } t \gg 1.$$

If φ is convex and $\nabla\varphi(\mathbb{R}^n)$ contains the origin in its interior, then $\varphi(x) \geq r\,|x|$ for some r as $|x| \to \infty$. Since h is decreasing, $h(\varphi(x)) \leq h(r\,|x|)$ for large $|x|$, and the asymptotic bound implies $\|h \circ \varphi\|_1 < \infty$. The bounds on h also imply the bounds

$$0 < H(t) \leq c\,t^{-(n+p)} \text{ when } t \gg 1$$

for $H(t) = \int_t^\infty h\,d\lambda$. This bound implies $\mathcal{F}(f) = H^{-1}(\|H \circ f\|_1) < \infty$ for $f \in \mathcal{C}$. The smoothness and positivity of h are necessary to prove regularity of solutions to equation (1.7).

Hypothesis (B2) :

If the solutions to equation (1.6) were not unique up to translations, then the translated Monge–Ampère iteration $\widetilde{\varphi}_i(x) = \varphi_i(x + a_i)$, for a_i such that $\varphi_i(a_i) = \inf \varphi_i$, could have two subsequences converging to different solutions of equation (1.6).

Hypothesis (B3) :

In order for the function g to define a duality between $\mathcal{F}$ and $\mathcal{G}$ it must satisfy Hypothesis (B3). In the case of Legendre duality $g(s,t) = s - t$, which is easily seen to be decreasing in t and satisfy $g(s,\, g(s,t)) = t$. The definition of $\mathcal{G}$ implies

$$\mathcal{G}(\mu) \leq g(\langle f, \mu\rangle,\, \mathcal{F}(f)\,)$$

for all f in $\mathcal{C}$ and μ in $\mathcal{P}_1$. When g satisfies Hypothesis (B3), we can apply $g(\langle f, \mu\rangle, \cdot)$ to both sides of the equation to see

$$g(\langle f, \mu\rangle,\, \mathcal{G}(\mu)\,) \geq \mathcal{F}(f), \tag{2.7}$$

which characterizes the duality between $\mathcal{F}$ and $\mathcal{G}$.

Hypothesis (B4) :

If $\{\varphi_i\}$ is a sequence of convex functions solving the normalized Monge–Ampère iteration (1.7), then this hypothesis in conjunction with equation (2.7) implies

$$\mathcal{F}(\varphi_i) \geq g\Big(\Big\langle \varphi_{i+1}, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big)\Big) \geq \mathcal{F}(\varphi_{i+1}). \tag{2.8}$$

Thus $\{\mathcal{F}(\varphi_i)\}$ and $\left\{g\Big(\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big)\Big)\right\}$ are decreasing sequences along the normalized Monge–Ampère iteration. The idea of the proof of Theorem 1.2.2 is to show continuity for these two decreasing functionals, so any limit $\varphi_i \to \varphi$ will achieve equality between them. Then we show that equality is only achieved for solutions of equation (1.6).

Conditions (1.11) on A :

If φ is a solution of equation (1.6), then for $i = 1, \dots, n$

$$\begin{aligned}\int_A y_i \, d\lambda = \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_i} \det(\nabla^2 \varphi) \, d\lambda = \lambda(A) \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_i} \frac{h \circ \varphi}{\|h \circ \varphi\|_1} \, d\lambda \\ = \lambda(A) \, \|h \circ \varphi\|_1^{-1} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(H \circ \varphi) \, d\lambda,\end{aligned}$$

where H is an antiderivative of h. The decay condition on h implies, after an integration by parts, that the last integral is 0. Thus,

$$\int_A y_i \, d\lambda = 0 \text{ for } i = 1, \dots, n$$

which implies the barycenter of A must necessarily lie at the origin. This condition will be used in the proof of Lemma 2.2.4 which gives a lower bound for a convex function f in terms of $\lambda(A)$ and τ when f satisfies $\int_A f^* \, d\lambda = -\tau$.

2.2.2 Proof outline

We start by fixing notation for the proof. Assume $A \subset \mathbb{R}^n$ satisfies (1.11).

$\{\varphi_i\}$	smooth, convex solutions to the normalized Monge–Ampère iteration (1.7)
$\{a_i\}$	points such that $\varphi_i(a_i) = \inf \varphi_i$
$\{\widetilde{\varphi}_i\}$	the translated sequence $\widetilde{\varphi}_i(x) = \varphi_i(x + a_i)$

Step 1: Uniform growth estimate

We prove the translated sequence satisfies

$$\frac{\tau}{\lambda(A)} + r\,|x| \leq \widetilde{\varphi}_i(x) \leq C + R\,|x|,$$

where C, r, and R depend on A, τ, h, and the initial function φ_0 starting the iteration. The lower bound only depends on $\int_A \varphi_i^*\,d\lambda = -\tau$. The constant C in the upper bound depends on $\{\varphi_i\}$ solving the normalized Monge–Ampère iteration, but the $R\,|x|$ term only comes from $\nabla\varphi_i(\mathbb{R}^n) = A \subset B_R$ for some constant R depending only on A.

Step 2: Subsequence convergence and subgradient limits

The pointwise boundedness from step 1 implies any subsequence $\{\widetilde{\varphi}_{i'}\}$ will have a further subsequence

$$\widetilde{\varphi}_{i''} \to \varphi$$

converging uniformly on compact subsets of $\mathbb{R}^n$ to a convex function φ by compactness properties of convex functions. We use $\int_A \widetilde{\varphi}_i^*\,d\lambda = -\tau$ to show φ also has subgradient image $\partial\varphi(\mathbb{R}^n) = A$ up to a set of measure 0.

Step 3: Convergence of the Monge–Ampère measures

We prove if a subsequence $\varphi_{i'}$ converges to φ uniformly on compact sets, then

$$\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)} \to_1 \frac{\mathrm{MA}(\varphi)}{\lambda(A)},$$

where $\mu_i \to_1 \mu$ denotes convergence of probability measures in Wasserstein distance, which is defined in (2.6). Convergence in Wasserstein distance is equivalent to $\left\{\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)}\right\}$ converging to $\frac{\mathrm{MA}(\varphi)}{\lambda(A)}$ weakly and satisfying the tightness condition

$$\lim_{R\to\infty} \limsup_{i'\to\infty} \left\{ \int_{|x|\geq R} |x| \ \mathrm{MA}(\widetilde{\varphi}_{i'}) \right\} = 0.$$

The proof of weak convergence is a standard consequence of the uniform convergence on compact sets. The proof of the tightness condition relies on a uniform bound of the form $|a_{i+1} - a_i| \leq C$, which is essentially a bound on the rate the sequence $\{\varphi_i\}$ can drift horizontally.

Step 4: φ minimizes $g\Big(\Big\langle \cdot\,, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\cdot)\Big)$.

Recall the definition of $\mathcal{G}$:

$$\mathcal{G}(\mu) = \inf\{\, g(\,\langle f, \mu\rangle,\, \mathcal{F}(f)) \mid f \in \mathcal{C}\,\}.$$

Using the convergence $\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)} \to_1 \frac{\mathrm{MA}(\varphi)}{\lambda(A)}$, and continuity properties for $\mathcal{F}$ and $\mathcal{G}$ we prove that

$$\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) = g\Big(\Big\langle \varphi,\, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle,\, \mathcal{F}(\varphi)\Big).$$

This equation implies $f = \varphi$ is the minimizer in the definition of $\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big)$.

Step 5: $\{\widetilde{\varphi}_i\}$ converges smoothly to φ, which solves the Monge–Ampère equation (1.6).

We begin by showing φ, the uniform limit of a convergent subsequence $\{\widetilde{\varphi}_{i'}\}$, is a solution to equation (1.6). We take variations φ_ϵ and differentiate $g\Big(\Big\langle \varphi_\epsilon, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi_\epsilon)\Big)$ to show equation (1.6) is the Euler–Lagrange equation for this functional, and φ is a weak solution. The smoothness of φ results from the $C^{2,\alpha}$ estimates of Caffarelli and elliptic regularity. The subgradient image $\partial\varphi(\mathbb{R}^n) = A$ up to a set of measure 0 is then upgraded to $\nabla\varphi(\mathbb{R}^n) = A$. We also prove $\int_A \varphi^* \, d\lambda = -\tau$.

To prove convergence of the whole sequence $\{\widetilde{\varphi}_i\}$, we show the limit φ of every convergent subsequence $\widetilde{\varphi}_{i'}$ is unique. Since φ is a smooth solutions to equation (1.6), it is unique up to translations by Hypothesis (B2). The condition $\widetilde{\varphi}_i(0) = \inf \widetilde{\varphi}_i$ implies $\varphi(0) = \inf \varphi$, so it follows that φ is unique. Since every subsequence has a further subsequence which converges and the limits are the same, it follows that $\widetilde{\varphi}_i$ converges to φ. The smooth convergence is a consequence of Caffarelli's $C^{2,\alpha}$ estimates and elliptic regularity.

2.2.3 Uniform growth estimate

In this section we will prove step 1 of Section 2.2.2:

Proposition 2.2.1. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Then*

$$\tau + r\,|x| \le \widetilde{\varphi}_i(x) \le C + R\,|x|. \tag{2.9}$$

Before proving Proposition 2.2.1 we prove a lemma which shows the bounds (2.9) imply bounds on $\|h \circ \widetilde{\varphi}_i\|_1$ and $\|H \circ \widetilde{\varphi}_i\|_1$ which will be important for subsequent steps of the proof of Theorem 1.2.2.

Lemma 2.2.2. *Assume h satisfies Hypothesis (B1), and let H be defined by equation (1.9). Let $f : \mathbb{R}^n \to (a, \infty)$ be a convex function. If $f(x) \geq a + r\,|x|$ for $r > 0$, then there exists a constant c depending on a, h, and r such that*

$$\|h \circ f\|_1 \leq c \quad \text{and} \quad \|H \circ f\|_1 \leq c. \tag{2.10}$$

If $f(x) \leq C + R\,|x|$ for $R > 0$, then there exists a constant c depending on C, h, and R such that

$$\|h \circ f\|_1 \geq c > 0 \quad \text{and} \quad \|H \circ f\|_1 \geq c > 0. \tag{2.11}$$

Proof. By Hypothesis (B1), h is positive and decreasing, and there exists $\rho > 0$ such that $h(t) \leq C\,t^{-(n+p+1)}$ when $t \geq \rho$. This hypothesis implies $H(t) = \int_t^\infty h\,d\lambda$ is positive and decreasing, and $H(t) \leq C'\,t^{-(n+p)}$ when $t \geq \rho$.

Firstly, assume $f(x) \geq a + r\,|x|$. Since h and H are decreasing, it follows that

$$h \circ f(x) \leq h(a + r\,|x|) \quad \text{and} \quad H \circ f(x) \leq H(a + r\,|x|).$$

Thus we can estimate

$$\begin{aligned} \|h \circ f\|_1 &\leq n\,\omega_n \int_0^\infty h(a + r\,s)\,s^{n-1}\,ds \\ &\leq \omega_n \left(\frac{\rho - a}{r}\right)^n h(\tau) + n\,\omega_n \int_{(\rho-a)/r}^\infty C\,(a + r\,s)^{-(n+p+1)}\,s^{n-1}\,ds. \end{aligned}$$

The last integral converges because the integrand is less than $C'\,s^{-(2+p)}$ for large s and $p > 0$. Thus, $\|h \circ f\|_1$ is bounded above by a constant depending only on h, a, and r. The bound for $\|H \circ f\|_1$ is identical, except the last integrand will be less than $C'\,s^{(n+p)}$ for large s. Thus we can see that $p > 0$ is optimal for the convergence of $\|H \circ f\|_1$.

Secondly, assume $f(x) \le C + R\,|x|$. Since h and H are decreasing, it follows that

$$h \circ f(x) \ge h(C + R\,|x|) \quad \text{and} \quad H \circ f(x) \ge H(C + R\,|x|).$$

Thus, we can estimate

$$\begin{aligned} \|h \circ f\|_1 &\ge n\,\omega_n \int_0^\infty h(C + R\,s)\, s^{n+1}\, ds \\ &\ge \omega_n \left(\frac{\rho - C}{R}\right)^n h((\rho - C)/R) > 0. \end{aligned}$$

The lower bound for $\|H \circ f\|_1$ is completely analogous. □

Now we return to the proof of Proposition 2.2.1. We begin with two basic lemmas about the Legendre transform and translated sequences.

Lemma 2.2.3. *Assume A satisfies (1.11). Let f be a convex function such that $\int_A \varphi^*\, d\lambda = -\tau$. Assume $f(a_i) = \inf_{x\in\mathbb{R}^n}\{f(x)\}$, and let $\widetilde{f}(x) = f(x + a_i)$ be a translation of f. Then*

$$\inf_{x\in\mathbb{R}^n}\{\widetilde{f}(x)\} = \widetilde{f}(0)\,, \qquad \inf_{y\in A}\{\widetilde{f}^*(y)\} = \widetilde{f}^*(0) = -\widetilde{f}(0)\,, \textit{ and } \qquad \int_A \widetilde{f}^*\, d\lambda = -\tau.$$

Proof. By definition of $\widetilde{f}$,

$$\widetilde{f}(0) = f(a_i) = \inf_{x\in\mathbb{R}^n}\{f(x)\} = \inf_{x\in\mathbb{R}^n}\{\widetilde{f}(x)\}.$$

By the definition of the Legendre transform,

$$\widetilde{f}^*(0) = \sup_{x\in\mathbb{R}^n}\{-\widetilde{f}(x)\} = -\widetilde{f}(0).$$

In order to show $\widetilde{f}^*(0) = \inf\{\widetilde{f}^*\}$ we note

$$\widetilde{f}^*(y) = \sup_{x\in\mathbb{R}^n}\{\langle x, y\rangle - \widetilde{f}(x)\} \ge -\widetilde{f}(0) = \widetilde{f}^*(0).$$

The Legendre transforms of $\widetilde{f}$ and f are related by

$$\widetilde{f}^*(y) = \sup_{x\in\mathbb{R}^n}\{\langle x, y\rangle - f(x+a_i)\} = \sup_{x\in\mathbb{R}^n}\{\langle x-a_i, y\rangle - f(x)\} = f^*(y) - \langle a_i, y\rangle,$$

which implies

$$\int_A \widetilde{f}^*\, d\lambda = \int_A (f^*(y) - \langle a_i, y\rangle)\, d\lambda = -\tau,$$

because the barycenter of A is at the origin. □

Lemma 2.2.4. *Assume A satisfies (1.11), and let f be a convex function on $\mathbb{R}^n$. If $\int_A f^*\, d\lambda = -\tau < \infty$, then $f(x) \geq \tau/\lambda(A)$ for all x.*

Proof. If there were a point x_0 in $\mathbb{R}^n$ where $f(x_0) < \tau/\lambda(A)$, then $f^*(y) = \sup\{\,\langle x, y\rangle - f(x) \mid x \in \mathbb{R}^n\,\} > \langle x_0, y\rangle - \tau/\lambda(A)$. Since the barycenter of A lies at the origin,

$$\int_A f^*(y)\, d\lambda > \int_A \left(\langle x_0, y\rangle - \frac{\tau}{\lambda(A)}\right) d\lambda = -\tau, \tag{2.12}$$

which contradicts $\int_A f^*\, d\lambda = -\tau$. □

Substep 1: $\tau + r\,|x| \leq \widetilde{\varphi}_i(x)$

We recall the convex analysis Lemma 2.6 from Klartag [24].

Lemma 2.2.5. *Let $A \subset \mathbb{R}^n$ be convex with the origin in its interior. There exists $r > 0$, depending on A, with the following property: Let $\psi : A \to \mathbb{R}$ be convex and integrable. Assume $\psi(0) = \inf \psi$, and $\int_A \psi\, d\lambda \leq 0$. Then for $y \in A$,*

$$\psi(y) \leq \psi(0)/2 \quad \textit{when } |y| \leq r.$$

We illustrate Lemma 2.2.5 for $n = 1$ in Figure 2.1.

The r in Lemma 2.2.5 will become the r in $\tau + r\,|x| \leq \widetilde{\varphi}_i(x)$.

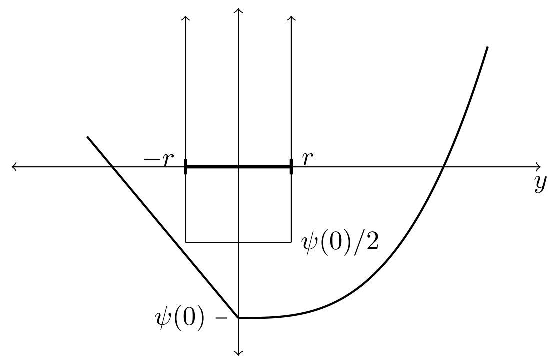

Figure 2.1: Diagram of Lemma 2.2.5

Lemma 2.2.6. *Assume A satisfies (1.11). Let f be a convex function such that* $\inf\{f\} = f(0)$ *and* $\int_A f^* \, d\lambda = -\tau$*. Then there exists an r depending on A such that*

$$f(x) \geq \Big(f(0) + \frac{\tau}{\lambda(A)}\Big)/2 + r\,|x| \geq \tau + r\,|x|. \tag{2.13}$$

Proof. Adding $\tau/\lambda(A)$ to f^* implies

$$\int_A \Big(f^* + \frac{\tau}{\lambda(A)}\Big)\, d\lambda = 0.$$

Thus, $f^* + \tau/\lambda(A)$ satisfies all the hypotheses of Lemma 2.2.5, so there exists an r depending only on A such that $f^*(y) + \tau/\lambda(A) \leq (f^*(0) + \tau/\lambda(A))/2$ when $|y| \leq r$. Equivalently,

$$f^*(y) \leq -\Big(f(0) + \frac{\tau}{\lambda(A)}\Big)/2 + \mathbb{1}_{B_r}, \tag{2.14}$$

where

$$\mathbb{1}_D(y) = \begin{cases} 0 & y \in D \\ \infty & y \notin D \end{cases}$$

is the convex indicator function of a convex set D. The order reversing property of the Legendre transform implies

$$f(x) = \sup_{y\in A}\{\langle x,y\rangle - f^*(y)\} \geq \sup_{y\in\mathbb{R}^n}\Big\{\langle x,y\rangle - \Big(-\Big(f(0)+\frac{\tau}{\lambda(A)}\Big)/2 + \mathbb{1}_{B_r}\Big)\Big\}$$
$$= \Big(f(0)+\frac{\tau}{\lambda(A)}\Big)/2 + \sup_{|y|\leq r}\{\langle x,y\rangle\} = \Big(f(0)+\frac{\tau}{\lambda(A)}\Big)/2 + r\,|x|,$$

which is the first inequality in equation (2.13). Lemma 2.2.4 implies $f(0) \geq \tau/\lambda(A)$, which yields the second inequality in equation (2.13). □

Lemma 2.2.3 implies $\inf\{\widetilde{\varphi}_i\} = \widetilde{\varphi}_i(0)$ and $\int_A \widetilde{\varphi}_i^*\,d\lambda = -\tau$, so we can apply Lemma 2.2.6 to $\widetilde{\varphi}_i$ to prove substep 1.

Substep 2: $C_1 \leq \mathcal{F}(\varphi_i) = \mathcal{F}(\widetilde{\varphi}_i) \leq C_2$

Before proving the upper bound in equation (2.9), we must prove $\mathcal{F}(\varphi_i)$ is bounded. $\mathcal{F}$ is translation invariant, so $\mathcal{F}(\widetilde{\varphi}_i) = \mathcal{F}(\varphi_i)$, and it is sufficient to show the boundedness of either. First we show $\mathcal{F}$ is bounded below.

Lemma 2.2.7. *Assume A satisfies (1.11) and h satisfies Hypothesis (B1). Let $\mathcal{F}$ be the functional defined in (1.9). There exists a constant C depending on A, τ, p, and h such that if f is convex and $\int_A f^*\,d\lambda = -\tau$, then $\mathcal{F}(f) \geq C$.*

Proof. Recall the definition (1.9) of $\mathcal{F}$:

$$\mathcal{F}(f) = H^{-1}(\|H\circ f\|_1) \quad \text{where} \quad H(t) = \int_t^\infty h\,d\lambda.$$

By Hypothesis (B1), h is positive, so H is a strictly decreasing function. Thus H^{-1} is strictly decreasing as well, so if we show $\|H\circ f\|_1 \leq C$, then the lower bound for $\mathcal{F}(f)$ will follow.

$\|H\circ f\|_1$ is invariant under translations of f so we can assume without loss of generality that $\inf\{f\} = f(0)$. Lemma 2.2.3 implies $\int_A f^*\,d\lambda = -\tau$ is unchanged by translations. Lemma 2.2.6 implies the lower bound

$$f(x) \geq \frac{\tau}{\lambda(A)} + r\,|x|.$$

Then, Lemma 2.2.2 implies $\|H \circ f\|_1 \leq C$ as desired. $\square$

The hypotheses of Lemma 2.2.7 are satisfied by φ_i, so $\mathcal{F}(\widetilde{\varphi}_i) = \mathcal{F}(\varphi_i) \geq C$. Now we prove the upper bound for $\mathcal{F}$ along the normalized Monge–Ampère iteration.

Lemma 2.2.8. *Assume the normalized Monge–Ampère iteration satisfies Hypotheses 1.2.1. Then*

$$\mathcal{F}(\varphi_{i-1}) \geq g\Big(\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big)\Big) \geq \mathcal{F}(\varphi_i), \tag{2.15}$$

and there exists β finite such that

$$\beta = \lim_{i\to\infty} \mathcal{F}(\varphi_i) = \lim_{i\to\infty} g\Big(\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big)\Big) \tag{2.16}$$

Proof. The first inequality in equation (2.15) is exactly Hypothesis (B4). Since φ_i is a convex function with $\nabla\varphi_i(\mathbb{R}^n) = A$ it follows that $\varphi_i \in \mathcal{C}$, defined in (1.8). Thus, by the definition of $\mathcal{G}$,

$$\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big) \leq g\Big(\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi_i)\Big).$$

After applying $g\Big(\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big\rangle, \cdot\Big)$ to both sides of the equation, Hypothesis (B3), which states $g(s,t)$ is a decreasing involution, implies

$$g\Big(\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big)\Big) \geq \mathcal{F}(\varphi_i),$$

proving the second inequality in equation (2.15). The two inequalities show that

$$\{\,\mathcal{F}(\varphi_i)\,\} \quad \text{and} \quad \Big\{\, g\Big(\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_i)}{\lambda(A)}\Big)\Big) \Big\}$$

are decreasing sequences. Lemma 2.2.7 implies $\mathcal{F}(\varphi_i)$ is bounded below, so both decreasing sequences converge to a common finite value β as in equation (2.16). $\square$

The two previous lemmas imply $C_1 \leq \mathcal{F}(\varphi_i) = \mathcal{F}(\widetilde{\varphi}_i) \leq C_2$.

Substep 3: $\widetilde{\varphi}_i(x) \leq C + R\,|x|$

Now we use the boundedness of $\mathcal{F}(\widetilde{\varphi}_i)$ to prove the upper bound for $\widetilde{\varphi}_i$.

Lemma 2.2.9. *Assume A satisfies conditions (1.11) and the normalized Monge–Ampère iteration satisfies Hypotheses 1.2.1. Then,*

$$\widetilde{\varphi}_i(x) \leq C + R\,|x|$$

for constants C and R depending on τ, A, and h.

Proof. $\mathcal{F}(\widetilde{\varphi}_i) = H^{-1}(\|H \circ \widetilde{\varphi}_i\|_1) \leq c$, and H is positive and decreasing, so

$$0 < H(c) \leq \|H \circ \widetilde{\varphi}_i\|_1.$$

We will first show there exists C, independent of i, such that $\widetilde{\varphi}_i(0) \leq C$. Lemma 2.2.6 implies

$$\widetilde{\varphi}_i(x) \geq (\widetilde{\varphi}_i(0) + \tau)/2 + r\,|x|.$$

As in the proof of Lemma 2.2.2, the bound for h from Hypothesis (B1) shows there exists C, R, and $p > 0$ such that

$$\|H\circ\widetilde{\varphi}_i\|_1 \leq \int_{B_R} H\Big(\Big(\widetilde{\varphi}_i(0)+\frac{\tau}{\lambda(A)}\Big)/2\Big)\,d\lambda + C\int_{\mathbb{R}^n\setminus B_R}\Big(\Big(\widetilde{\varphi}_i(0)+\frac{\tau}{\lambda(A)}\Big)/2+r\,|x|\Big)^{-(n+p)}\,d\lambda.$$

Since $H(t) = \int_t^\infty h\,d\lambda$, $\lim_{t\to\infty} H(t) = 0$ so

$$\lim_{\widetilde{\varphi}_i(0)\to\infty} \int_{B_R} H\Big(\Big(\widetilde{\varphi}_i(0) + \frac{\tau}{\lambda(A)}\Big)/2\Big)\,d\lambda = 0.$$

Likewise, $\left(\left(\widetilde{\varphi}_i(0)+\frac{\tau}{\lambda(A)}\right)/2+r\,|x|\right)^{-(n+p)}$ converges to the zero function pointwise as $\widetilde{\varphi}_i(0)\to\infty$, and they are all bounded above by $\left(\frac{\tau}{\lambda(A)}+r\,|x|\right)^{-(n+p)}$ so

$$\lim_{\widetilde{\varphi}_i(0)\to\infty} C\int_{\mathbb{R}^n\setminus B_R}\left(\left(\widetilde{\varphi}_i(0)+\frac{\tau}{\lambda(A)}\right)/2+r\,|x|\right)^{-(n+p)} d\lambda=0,$$

by the dominated convergence theorem. But we have the positive lower bound

$$0<H(c)\le\int_{B_R} H\left(\left(\widetilde{\varphi}_i(0)+\frac{\tau}{\lambda(A)}\right)/2\right)d\lambda+C\int_{\mathbb{R}^n\setminus B_R}\left(\left(\widetilde{\varphi}_i(0)+\frac{\tau}{\lambda(A)}\right)/2+r\,|x|\right)^{-(n+p)} d\lambda,$$

so the integrals cannot go to 0, and there must be some upper bound $C\ge\widetilde{\varphi}_i(0)$. If $R>0$ is a constant satisfying $A\subset B_R$, then $|\nabla\widetilde{\varphi}_i(x)|\le R$, and by integrating along lines from the origin it follows that

$$\widetilde{\varphi}_i(x)\le C+R\,|x|.$$

□

The three previous substeps complete the proof of Proposition 2.2.1.

2.2.4 Subsequence convergence and subgradient limits

In this section we will prove Step 2 of the outline:

Proposition 2.2.10. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Then every subsequence $\{\widetilde{\varphi}_{i'}\}$ of the translated sequence $\{\widetilde{\varphi}_i\}$ has a further subsequence $\{\widetilde{\varphi}_{i''}\}$ which converges uniformly on compact sets to some convex function φ for which* $\operatorname{Int}\partial\varphi(\mathbb{R}^n)=A$.

The subsequence convergence is a simple corollary of the compactness properties for locally, uniformly bounded convex functions.

Lemma 2.2.11. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration satisfies Hypotheses 1.2.1. Then every subsequence $\{\widetilde{\varphi}_{i'}\}$ has a further subsequence, uniformly convergent on compact sets.*

Proof. By Proposition 2.2.1,

$$\frac{\tau}{\lambda(A)} \leq \widetilde{\varphi}_{i'}(x) \leq C + R\,|x|.$$

Thus, for every fixed $x_0 \in \mathbb{R}^n$, the set $\{\widetilde{\varphi}_{i'}(x_0)\} \subset [\tau/\lambda(A),\, C + R\,|x_0|\,]$ is bounded. By Theorem 10.9 of Rockafellar [38] there exists a further subsequence $\{\widetilde{\varphi}_{i''}\}$ which converges to a convex function φ uniformly on compact subsets of $\mathbb{R}^n$. □

The convergence of the subgradients relies upon the fact that $\int_A \widetilde{\varphi}_i^*\, d\lambda = -\tau$ for each i. In general, if we only know convex functions f_i converge to f uniformly on compact sets, we may have $\partial f(\mathbb{R}^n) \subsetneq \partial f_i(\mathbb{R}^n)$.

The proof of the subgradient limit relies upon a convex analysis lemma which is a generalization of Lemma 2.2.5. Klartag proved that if ψ is a convex function such that $\int_A \psi \leq 0$, then there is an upper bound for ψ, depending on A and its minimum value, in a small ball around its minimum. This Lemma extends the upper bound to any open set away from the boundary of A.

Lemma 2.2.12. *Let $A \subset \mathbb{R}^n$ be convex. For each $\epsilon > 0$ define $A_\epsilon = \{\, y \mid B_\epsilon(y) \subset A\,\}$. There exists $C > 0$ depending on ϵ and $\lambda(A)$ with the following property: Let $\psi : A \to \mathbb{R}$ be convex, and assume $\int_A \psi\, d\lambda \leq 0$. Then*

$$\psi(y) \leq -C \inf_A\{\psi\} \quad \textit{when } y \in A_\epsilon.$$

Proof. We can make the simplifying assumption that $\lambda(A) = 1$. Define C to be any constant such that $C + 1 > 1/V$ where $V = \omega_n \epsilon^n/2$ is the volume of a half-ball of radius ϵ. Assume by contradiction to the conclusion of the lemma

$$\left\{\, y \in A \mid \psi(y) \le -C \ \inf\{\psi\} \,\right\} \text{ does not contain } A_\epsilon.$$

Since $\{\, y \mid \psi(y) \le -C \ \inf\psi \,\}$ is the sublevel set of a convex function, it is convex. Let y_0 be any point in the portion of the boundary of $\{\, y \mid \psi(y) \le -C \ \inf\psi \,\}$ which intersects A_ϵ. Then the supporting halfspace at y_0 lies outside $\{\, y \mid \psi(y) \le -C \ \inf\psi \,\}$, meaning there exists an outward normal v such that

$$\big(A \cap \{\, y \mid \langle y - y_0, v\rangle \ge 0 \,\}\big) \subset \{\, y \in A \mid \psi(y) \ge -C \ \inf\psi \,\}.$$

Since $y_0 \in A_\epsilon$ we can intersect the first set with $B_\epsilon(y_0)$ to get

$$B := \big(B_\epsilon(y_0) \cap \{\, y \mid \langle y - y_0, v\rangle \ge 0 \,\}\big) \subset \{\, y \in A \mid \psi(y) \ge -C \ \inf\psi \,\}.$$

This inclusion is shown in Figure 2.2.

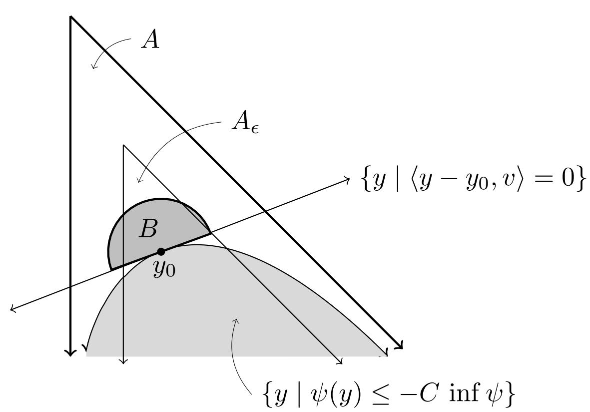

Figure 2.2: Diagram for the proof of Lemma 2.2.12

Then, since $\lambda(A) = 1$,

$$\int_A \psi(y)\, d\lambda \ge (1 - V)(\inf\psi) + V(-C \ \inf\psi) = (-\inf\psi)\,(-1 + (C + 1)\,V).$$

$\int_A \psi \, d\lambda \leq 0$ so $(-\inf \psi) \geq 0$. We chose C so that $C + 1 > 1/V$, and it follows that $\int_A \psi \, d\lambda > 0$, which is a contradiction since $\int_A \psi \, d\lambda \leq 0$. □

Lemma 2.2.13. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration satisfies Hypotheses 1.2.1. Let $\{\widetilde{\varphi}_{i'}\}$ be any subsequence which converges to φ uniformly on compact subsets of $\mathbb{R}^n$. Then* $\operatorname{Int} \partial\varphi(\mathbb{R}^n) = A$.

Proof. First we find a refined upper bound for $\widetilde{\varphi}_i$. Let

$$\mathbb{1}_D = \begin{cases} 0 & y \in D \\ \infty & y \notin D \end{cases}$$

be the convex indicator function of a convex set D. Then $\mathbb{1}_D^*(x) = \sup\{\, \langle x, y \rangle \mid y \in D \,\}$ is the cone emanating from the origin with subgradient image equal to $\overline{D}$.

By Lemma 2.2.3 $\inf \widetilde{\varphi}_i^* = \widetilde{\varphi}_i^*(0) = -\widetilde{\varphi}_i(0)$, so

$$\widetilde{\varphi}_i^* \geq \mathbb{1}_A + \widetilde{\varphi}_i^*(0) = \mathbb{1}_A - \widetilde{\varphi}_i(0).$$

The order reversing property of the Legendre transform implies

$$\widetilde{\varphi}_i(x) \leq \widetilde{\varphi}_i(0) + \mathbb{1}_A^*(x).$$

Proposition 2.2.1 implies $\{\widetilde{\varphi}_i(0)\}$ is bounded, so there is a constant C such that

$$\widetilde{\varphi}_i(x) \leq C + \mathbb{1}_A^*(x) \quad \text{for all } i. \tag{2.17}$$

Next we find a refined lower bound for $\widetilde{\varphi}_i$. The addition of a constant to every $\widetilde{\varphi}_i$ does not affect the convergence or the subgradients, so we can assume without loss of generality that $\int_A \widetilde{\varphi}_{i'}^* \, d\lambda = -\tau \leq 0$. Thus we can apply Lemma 2.2.12 to show that for every $\epsilon > 0$ there exists a constant C_ϵ such that

$$\widetilde{\varphi}_i^*(y) \leq \mathbb{1}_{A_\epsilon}(y) - C_\epsilon \left(\inf \widetilde{\varphi}_i^* \right) = \mathbb{1}_{A_\epsilon}(y) + C_\epsilon \, \widetilde{\varphi}_i(0).$$

Since $\{\widetilde{\varphi}_i(0)\}$ is bounded,

$$\widetilde{\varphi}_i^*(y) \leq \mathbb{1}_{A_\epsilon}(y) + C_\epsilon.$$

By the order reversing properties of the Legendre transform

$$\widetilde{\varphi}_i(x) \geq \mathbb{1}_{A_\epsilon}^*(x) - C_\epsilon. \tag{2.18}$$

Since the bounds (2.17) and (2.18) are independent of i, it follows that

$$\mathbb{1}_{A_\epsilon}^*(x) - C_\epsilon \leq \varphi(x) \leq C + \mathbb{1}_A^*(x) \quad \text{for all } i.$$

Thus, for every $\epsilon > 0$, $A_\epsilon \subset \operatorname{Int} \partial\varphi(\mathbb{R}^n) \subset A$. Letting $\epsilon \to 0$ finishes the proof. $\square$

2.2.5 Convergence of the Monge–Ampère measures

In this section we will prove step 3 of Section 2.2.2.

Proposition 2.2.14. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Assume a subsequence $\{\widetilde{\varphi}_{i'}\}$ converges to φ uniformly on compact sets. Then*

$$\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)} \to_1 \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \tag{2.19}$$

as defined in (2.6).

For this step, we will simplify notation by assuming $\lambda(A) = 1$. The same proofs hold in general by replacing $\mathrm{MA}(\varphi)$ by $\dfrac{\mathrm{MA}(\varphi)}{\lambda(A)}$. By Theorem 2.1.13, equation

(2.19) is equivalent to

$$\mathrm{MA}(\widetilde{\varphi}_{i'}) \Rightarrow \mu \quad \text{and} \quad \lim_{R\to\infty} \limsup_{i'\to\infty} \left\{ \int_{\{|x|\geq R\}} |x| \; \mathrm{MA}(\widetilde{\varphi}_{i'}) \right\} = 0.$$

The weak convergence $\mathrm{MA}(\widetilde{\varphi}_{i'}) \Rightarrow \mathrm{MA}(\varphi)$ is a consequence of $\varphi_{i'}$ converging to φ uniformly on compact sets by Lemma 2.2 from Trudinger and Wang [44].

The second condition, which is a tightness condition on the measures, relies upon a uniform bound $|a_{i+1} - a_i| \leq C$.

When $h(t) = e^{-z}$, the Monge–Ampère iteration corresponds to Ricci iteration on toric Kähler manifolds. In this case a_i is a toric automorphism acting on the potential φ_i, so this Lemma proves the automorphims which make the Ricci iteration have bounded differences. This fact was proven more generally for the Ricci iteration on compact Kähler manifolds admitting Kähler–Einstein metrics in Theorem 5.1 of Darvas–Rubinstein [12].

Lemma 2.2.15. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Then there exists $C > 0$ such that $|a_{i+1} - a_i| \leq C$ for all i.*

Proof. By translating both φ_i and φ_{i+1} we can assume $a_i = 0$, so we need to prove a uniform upper bound for $|a_{i+1}|$. In order to simplify notation, define

$$\alpha_{i+1} := \varphi_{i+1}(a_{i+1}) - \tau.$$

By Lemma 2.2.4 $\alpha_{i+1} \geq 0$. Moreover, equation (2.12) implies that if $\alpha_{i+1} = 0$, then $\varphi_{i+1} = \mathbb{1}_A^*$ which would contradict the smoothness of φ_{i+1}, so α_{i+1} is positive.

Lemma 2.2.6 implies $\varphi_{i+1}(x) \geq r\,|x - a_{i+1}| + \tau + \alpha_{i+1}/2$, and in particular

$$\varphi_{i+1}(x) \geq \tau + (3/2)\,\alpha_{i+1} \quad \text{for } x \text{ such that } |x - a_{i+1}| = \alpha_{i+1}/r.$$

This inequality is depicted in Figure 2.3.

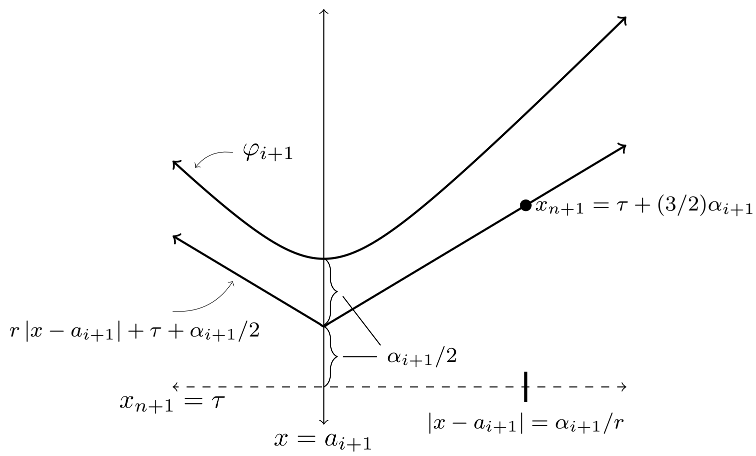

Figure 2.3: Diagram for the proof of Lemma 2.2.15

Lemma 2.1.4 implies

$$B_r \subset \nabla\varphi_{i+1}(B_{\alpha_{i+1}/r}(a_{i+1})),$$

and in particular

$$\omega_n r^n \leq \lambda\big(\nabla\varphi_{i+1}\big(B_{\alpha_{i+1}/r}(a_{i+1})\big)\big) = \int_{B_{\alpha_{i+1}/r}(a_{i+1})} \det(\nabla^2\varphi_{i+1})\, d\lambda.$$

Let

$$\epsilon = \sup \left\{\ \det(\nabla^2\varphi_{i+1}(x)) \mid x \in B_{\alpha_{i+1}/r}(a_{i+1}) \right\}.$$

By Proposition 2.2.1 there exists a constant $C > 0$ such that $\alpha_{i+1} \leq C$, so

$$\omega_n r^n \leq \epsilon\, \omega_n (C/r)^n, \tag{2.20}$$

and thus $(r^2/2C)^n \leq \epsilon$ is a uniform lower bound for ϵ. It remains to prove an upper bound for ϵ which goes to 0 as $|a_{i+1}| \to \infty$, independently of i. Since we assumed $\varphi_i(0) = \inf \varphi_i$, Proposition 2.2.1 implies

$$\tau + r\,|x| \leq \varphi_i(x) \leq C + R\,|x|.$$

By these bounds and Hypothesis (B1) on the iteration, there is some constant C' such that

$$\det(\nabla^2 \varphi_{i+1}(x)) = \frac{h \circ \varphi_i}{\|h \circ \varphi_i\|_1} \leq C'\,(\tau + r\,|x|)^{-(n+p+1)}$$

as $|x| \to \infty$. Thus

$$\epsilon \leq \sup\,\left\{\, C'\,(\tau + r\,|x|)^{-(n+p+1)} \mid x \in B_{C/r}(a_{i+1})\,\right\}$$

for $|a_{i+1}|$ large enough. Thus, $\epsilon \to 0$ as $|a_{i+1}| \to \infty$, and there is some constant C such that if $|a_{i+1}| \geq C$, then $\epsilon < (r^2/2C)^n$, contradicting equation (2.20). Thus $|a_{i+1}|$ has a uniform upper bound. □

Lemma 2.2.16. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Then $\{\mathrm{MA}(\widetilde{\varphi}_i)\}$ satisfies the tightness condition*

$$\lim_{R\to\infty} \limsup_{i\to\infty} \left\{ \int_{|x|\geq R} |x|\; \mathrm{MA}(\widetilde{\varphi}_i) \right\} = 0.$$

Proof. By Proposition 2.2.1,

$$\tau + r\,|x - a_i| \leq \varphi_i(x) \leq C + R\,|x - a_i|.$$

Thus, since h is decreasing and $\|h \circ \varphi_i\|_1$ has a positive lower bound by Proposition

2.2.2, there is some constant C' such that

$$\begin{aligned}\det(\nabla^2\widetilde{\varphi}_{i+1})(x) &= \det(\nabla^2\varphi_{i+1})(x+a_{i+1})\\ &= \frac{h\big(\varphi_i(x+a_{i+1})\big)}{\|h\circ\varphi_i\|_1}\\ &\le C'\,h\big(\tau + r|x-a_i+a_{i+1}|\big).\end{aligned}$$

And Hypothesis (B1) implies

$$h\big(\tau + r|x-a_i+a_{i+1}|\big) \le C'\left(\tau + r|x-a_i+a_{i+1}|\right)^{-(n+p+1)}$$

for $|x|\gg 1$. By Lemma 2.2.15, $|a_{i+1}-a_i|\le c$ for a constant c independent of i, so $|x-(a_i-a_{i+1})|\ge |x|-|a_i-a_{i+1}|\ge |x|-c$. Thus,

$$\det(\nabla^2\widetilde{\varphi}_{i+1})(x)\le C'\left(\tau - r\,c + r\,|x|\right)^{-(n+p+1)}$$

for $|x|\gg 1$. The tightness condition follows from

$$\begin{aligned}&\lim_{R\to\infty}\limsup_{i\to\infty}\left\{\int_{|x|\ge R}|x|\,\det(\nabla^2\widetilde{\varphi}_i)\,d\lambda)\right\}\\ &\qquad\le \lim_{R\to\infty}\int_{|x|\ge R}C'\,|x|\left(\tau - r\,c + r\,|x|\right)^{-(n+p+1)}d\lambda\\ &\qquad\le \lim_{R\to\infty}\int_{|x|\ge R}C''\,|x|^{-(n+p)}\,d\lambda.\end{aligned}$$

Since $|x|^{-(n+p)}$ is integrable for $p>0$, the limit is 0. □

Lemma 2.2.16 concludes the proof of Proposition 2.2.14.

2.2.6 φ minimizes $g\big(\big\langle\cdot,\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\big\rangle,\,\mathcal{F}(\cdot)\big)$

In this section we will prove step 4 of Section 2.2.2.

By the definition of $\mathcal{G}$,

$$\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) = \inf\Big\{ g\Big(\Big\langle f, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(f)\Big) \mid f \in \mathcal{C} \Big\} \leq g\Big(\Big\langle f, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(f)\Big)$$

for every $f \in \mathcal{C}$. If we can show that $\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) = g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi)\Big)$, then step 4 will be proven.

Proposition 2.2.17. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Assume a subsequence $\{\widetilde{\varphi}_{i'}\}$ converges to φ uniformly on compact sets. Then*

$$\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) = g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi)\Big). \tag{2.21}$$

To prove Proposition 2.2.17, we first note that Proposition 2.2.10 implies $\mathrm{Int}\,\partial\varphi(\mathbb{R}^n) = A$, so $\varphi(x) \leq C\,(1 + |x|)$. Thus $\varphi \in \mathcal{C}$, and

$$\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) \leq g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi)\Big).$$

The reverse inequality $\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) \geq g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi)\Big)$ relies upon the continuity of the pertinent functionals along the normalized Monge–Ampère iteration, which we prove in the next three lemmas.

Lemma 2.2.18. *Assume the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Assume a subsequence $\{\widetilde{\varphi}_{i'}\}$ converges to φ uniformly on compact sets. Then $\mathcal{F}(\widetilde{\varphi}_{i'}) \to \mathcal{F}(\varphi)$.*

Proof. By Hypothesis (B1), $h(t)$ is smooth and positive, so $H(t) = \int_t^\infty h\,d\lambda$ is smooth, positive, and strictly decreasing. In particular, H has a continuous inverse, H^{-1}. Since $\mathcal{F}(f) = H^{-1}\big(\|H \circ f\|_1\big)$, the lemma will follow from showing

$$\|H \circ \widetilde{\varphi}_{i'}\|_1 \to \|H \circ \varphi\|_1.$$

$\widetilde{\varphi}_{i'}$ converges to φ uniformly on compact sets, and since H is continuous, it follows that $H \circ \widetilde{\varphi}_{i'}$ converges to $H \circ \varphi$ pointwise. By Proposition 2.2.1 $\tau/\lambda(A) + r\,|x| \leq \widetilde{\varphi}_{i'}(x) \leq C + R\,|x|$, and Hypothesis (B1) also implies $H(t) \leq C\,t^{-(n+p)}$ as $t \to \infty$, so $H \circ \widetilde{\varphi}_{i'}$ are uniformly bounded by an L^1 function. Thus, $\|H \circ \widetilde{\varphi}_{i'}\|_1 \to \|H \circ \varphi\|_1$ by the dominated convergence theorem. □

Before the next lemma, we need to say a few words about double sequences $s_{i,j}$ indexed by $i, j \in \mathbb{N}$. We say that a double sequence $s_{i,j}$ converges to a if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|s_{i,j} - a| \leq \epsilon$ when $i, j \geq N$. In particular, convergence of the double sequence $s_{i,j}$ implies convergence of the diagonal sequence $s_{i,i}$.

We can also consider the iterated limits $\lim_{i\to\infty}(\lim_{j\to\infty} s_{i,j})$ and $\lim_{j\to\infty}(\lim_{i\to\infty} s_{i,j})$. But even if both of these limits exist, and equal the same value, the whole double sequence may not converge. For example, consider the double sequence $s_{i,j} = \dfrac{i\,j}{i^2 + j^2}$. Both of the iterated limits exist and equal 0, but $s_{i,j}$ does not converge as a double sequence, which is clear from $\lim_{i\to\infty} s_{i,i} = 1/2$ and $\lim_{i\to\infty} s_{i,2i} = 2/5$.

In order to deduce the convergence of the double sequence from the iterated limits, we need convergence of one iterated limit $\lim_{i\to\infty}(\lim_{j\to\infty} s_{i,j}) = a$ and uniform convergence of the inside limit $\lim_{j\to\infty} s_{i,j}$. We say $s_{i,j} \xrightarrow[j\to\infty]{} a_i$ uniformly if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$, independent of i, such that $j \geq N$ implies $|s_{i,j} - a_i| < \epsilon$ for all i.

In the next lemma we assume $\lambda(A) = 1$ for notational convenience. The result in general follows by scaling each equation by $\lambda(A)^{-1}$.

Lemma 2.2.19. *Assume the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Assume a subsequence* $\{\widetilde{\varphi}_{i'}\}$ *converges to* φ *uniformly on compact sets. Then* $\langle \widetilde{\varphi}_{i'}, \mathrm{MA}(\widetilde{\varphi}_{i'})\rangle \to \langle \varphi, \mathrm{MA}(\varphi)\rangle$.

Proof. We will prove, more generally, that the double sequence $\langle \widetilde{\varphi}_{i'}, \mathrm{MA}(\widetilde{\varphi}_{j'})\rangle \to \langle \varphi, \mathrm{MA}(\varphi)\rangle$.

Proposition 2.2.14 implies $\mathrm{MA}(\widetilde{\varphi}_{i'}) \to_1 \mathrm{MA}(\varphi)$, and by Lemma 2.1.13 this is equivalent to

$$\langle f, \mathrm{MA}(\widetilde{\varphi}_{i'})\rangle \to \langle f, \mathrm{MA}(\varphi)\rangle,$$

for all continuous f such that $f(x) \leq C\,(1+|x|)$ for some constant C. In particular, for every fixed i'

$$\lim_{j'\to\infty} \langle \widetilde{\varphi}_{i'}, \mathrm{MA}(\widetilde{\varphi}_{j'})\rangle = \langle \widetilde{\varphi}_{i'}, \mathrm{MA}(\varphi)\rangle.$$

Moreover, this converge is uniform because the uniform estimate $\tau + r\,|x| \leq \widetilde{\varphi}_{i'}(x) \leq C + R\,|x|$ implies

$$\big|\,\langle \widetilde{\varphi}_{i'}, \mathrm{MA}(\widetilde{\varphi}_{j'})\rangle - \langle \widetilde{\varphi}_{i'}, \mathrm{MA}(\varphi)\rangle\,\big| \leq R\,\big|\,\langle\,|x|, \mathrm{MA}(\widetilde{\varphi}_{j'})\rangle - \langle\,|x|, \mathrm{MA}(\varphi)\rangle\,\big| < \epsilon$$

for $j' \gg 1$, independent of i'.

Since $\widetilde{\varphi}_{i'}$ converges to φ pointwise and $\widetilde{\varphi}_{i'}$ are uniformly bounded,

$$\lim_{i'\to\infty} \langle \widetilde{\varphi}_{i'}, \mathrm{MA}(\varphi)\rangle = \langle \varphi, \mathrm{MA}(\varphi)\rangle$$

by the dominated convergence theorem. Thus, the iterated sequence converges

$$\lim_{i'\to\infty} \big(\, \lim_{j'\to\infty} \langle \widetilde{\varphi}_{i'},\, \mathrm{MA}(\widetilde{\varphi}_{j'})\rangle\,\big) = \langle \varphi, \mathrm{MA}(\varphi)\rangle,$$

and the inside limit converges uniformly. It follows that the double sequence converges. □

Lemma 2.2.20. *Assume the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Assume a subsequence $\{\widetilde{\varphi}_{i'}\}$ converges to φ uniformly on compact sets. Then* $\limsup_{i'\to\infty} \mathcal{G}\Big(\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)}\Big) \le \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big)$.

Proof. Recall

$$\mathcal{G}(\mu) = \inf\{\, g(\langle f, \mu\rangle,\, \mathcal{F}(f)) \mid f \in \mathcal{C}\,\}$$

where $\mathcal{C} = \{\, f : \mathbb{R}^n \to (\tau, \infty) \mid f \text{ continuous, and } f(x)/(1+|x|) \text{ bounded}\,\}$.

Let μ_i and μ be any measures in $\mathcal{P}_1$ such that $\mu_i \to_1 \mu$. Theorem 2.1.13 implies

$$\lim_{i\to\infty} \langle f, \mu_i\rangle = \langle f, \mu\rangle$$

for every fixed $f \in \mathcal{C}$. Since g is continuous, it follows that

$$\lim_{i\to\infty} g(\langle f, \mu_i\rangle,\, \mathcal{F}(f)) = g(\langle f, \mu\rangle,\, \mathcal{F}(f))$$

for every fixed $f \in \mathcal{C}$. Thus $\mathcal{G}$ is the infimum of continuous functionals, so $\mathcal{G}$ is upper semicontinuous, meaning

$$\limsup_{i\to\infty} \mathcal{G}(\mu_i) \le \mathcal{G}(\mu)$$

whenever $\mu_i \to_1 \mu$. The lemma then follows from Proposition 2.2.14 which says $\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)} \to_1 \frac{\mathrm{MA}(\varphi)}{\lambda(A)}$. □

Lemma 2.2.21. *Let $\{x_i\}$ and $\{y_i\}$ be sequences of real numbers with* $\lim_{i\to\infty} x_i = x$ *and* $\limsup_{i\to\infty} y_i \le y$ *for x and y finite. Let $g(s,t)$ be a continuous function such that $g(s,\cdot)$ is decreasing for all fixed s. Then*

$$\liminf_{i\to\infty} g(x_i, y_i) \ge g(x, y).$$

Proof. We need to show that $\liminf_{i\to\infty} g(x_i, y_i) \geq g(x, y) - \epsilon$ for all $\epsilon > 0$. For any $\epsilon > 0$ there exists a $\delta > 0$ such that $g(x, y + \delta) \geq g(x, y) - \epsilon/2$ because g is continuous at (x, y).

Since $\limsup y_i \leq y$, it follows that for i large enough, $y_i \leq y + \delta$. Since $g(s, \cdot)$ is decreasing for every fixed s, it follows that

$$g(x_i, y_i) \geq g(x_i, y + \delta) \text{ for } i \text{ large enough.}$$

Now using that $x_i \to x$ and the continuity of g we have that

$$g(x_i, y + \delta) \geq g(x, y + \delta) - \epsilon/2 \geq g(x, y) - \epsilon \text{ for } i \text{ large enough.}$$

Putting the previous two equations together, and taking the lim inf of both sides implies

$$\liminf g(x_i, y_i) \geq g(x, y) - \epsilon.$$

□

Lemma 2.2.22. *Assume the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Assume a subsequence $\{\widetilde{\varphi}_{i'}\}$ converges to φ uniformly on compact sets. Then*

$$\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) \geq g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi)\Big).$$

Proof. Recall equation (2.16) which we restate for the subsequence $\widetilde{\varphi}_{i'}$

$$\lim_{i\to\infty} \mathcal{F}(\widetilde{\varphi}_{i'}) = \lim_{i\to\infty} g\Big(\Big\langle \widetilde{\varphi}_{i'}, \frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)}\Big)\Big).$$

By Lemma 2.2.18, $\lim_{i'\to\infty} \mathcal{F}(\widetilde{\varphi}_{i'}) = \mathcal{F}(\varphi)$, so

$$\mathcal{F}(\varphi) = \lim_{i\to\infty} g\Big(\Big\langle \widetilde{\varphi}_{i'}, \frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)}\Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)}\Big)\Big).$$

Lemma 2.2.21 applies to the limit in the above equation because $\left\langle \widetilde{\varphi}_{i'}, \frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)} \right\rangle \to \left\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \right\rangle$ by Lemma 2.2.19 and $\limsup \mathcal{G}\Big(\frac{\mathrm{MA}(\widetilde{\varphi}_{i'})}{\lambda(A)}\Big) \leq \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big)$ by Lemma 2.2.20. Lemma 2.2.21 implies

$$\mathcal{F}(\varphi) \geq g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big)\Big).$$

Finally, by applying the decreasing function $g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \cdot\Big)$ to both sides of the previous equation, and using the fact that $g(s, g(s,t)) = t$, we get

$$g\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \mathcal{F}(\varphi)\Big) \leq \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big).$$

□

This concludes the proof of Proposition 2.2.17.

2.2.7 Convergence of the iteration

In this section we will prove step 5 of Section 2.2.2, thus completing the proof of Theorem 1.2.2.

Proposition 2.2.23. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration satisfies Hypotheses 1.2.1. Then, $\widetilde{\varphi}_i$ converges to φ, which is a smooth solution to equation (1.6), in every $C^{k,\alpha}$ norm on compact sets.*

We begin in the next two lemmas by proving the limit φ of any convergent subsequence $\widetilde{\varphi}_{i'}$ is a smooth solution to equation (1.6). In the first lemma we show that φ satisfies the differential equation and the second boundary value, and in the second lemma we show that φ satisfies the normalization $\int_A \varphi^* \, d\lambda = -\tau$.

Lemma 2.2.24. *Assume A satisfies (1.11) and the normalized Monge–Ampère iteration (1.7) satisfies Hypotheses 1.2.1. Let $\{\widetilde{\varphi}_{i'}\}$ be a subsequence which converges to φ uniformly on compact sets. Then φ is a smooth solution to*

$$\begin{cases} \dfrac{\det(\nabla^2\varphi)}{\lambda(A)} = \dfrac{h\circ\varphi}{\|h\circ\varphi\|_1} \\ \nabla\varphi(\mathbb{R}^n) = A. \end{cases}$$

Proof. By Proposition 2.2.14 and the definition of $\mathcal{G}$

$$g\Big(\Big\langle\varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(\varphi)\Big) = \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big) \le g\Big(\Big\langle f, \frac{\mathrm{MA}(\varphi)}{\lambda(A)}\Big\rangle, \mathcal{F}(f)\Big) \quad \text{for all } f\in\mathcal{C}, \tag{2.22}$$

where we recall

$$\mathcal{C} = \{\, f:\mathbb{R}^n\to(\tau,\infty) \mid f \text{ continuous, and } f(x)/(1+|x|) \text{ bounded}\,\}.$$

Consider a variation $\varphi_\epsilon = \varphi + \epsilon\, b$ for b a continuous, bounded function on $\mathbb{R}^n$. For ϵ small enough, $\varphi_\epsilon\in\mathcal{C}$. Hypothesis (B1) says h is positive and decreasing so $H(t) = \int_t^\infty h\,d\lambda$ is positive, strictly decreasing, and convex. Thus

$$-(t-s)\,h(s) \le H(t) - H(s) \le -(t-s)\,h(t),$$

for any values s, t. Letting $s=\varphi(x)$ and $t=\varphi_\epsilon(x)$ implies

$$\begin{aligned} -\int_{\mathbb{R}^n} b(x)\,h\big(\varphi(x)\big)\,d\lambda &\le \epsilon^{-1}\left(\int_{\mathbb{R}^n} H\big(\varphi_\epsilon(x)\big)\,d\lambda - \int_{\mathbb{R}^n} H\big(\varphi(x)\big)\,d\lambda\right) \\ &\le -\int_{\mathbb{R}^n} b(x)\,h\big(\varphi_\epsilon(x)\big)\,d\lambda, \end{aligned}$$

for $\epsilon>0$ with the reverse inequalities when $\epsilon<0$. The dominated convergence theorem implies

$$\int_{\mathbb{R}^n} b(x)\,h\big(\varphi_\epsilon(x)\big)\,d\lambda \to \int_{\mathbb{R}^n} b(x)\,h\big(\varphi(x)\big)\,d\lambda,$$

so

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \|H \circ \varphi_\epsilon\|_1 = \lim_{\epsilon \to 0} \epsilon^{-1} \left(\int_{\mathbb{R}^n} H\big(\varphi_\epsilon(x)\big)\, d\lambda - \int_{\mathbb{R}^n} H\big(\varphi(x)\big)\, d\lambda \right)$$
$$= - \int_{\mathbb{R}^n} b(x)\, h\big(\varphi(x)\big)\, d\lambda.$$

Since H is strictly decreasing and differentiable, it follows that H^{-1} is differentiable. Using the formula for the derivative of the inverse and the definition of $\mathcal{F}$, it follows that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{F}(\varphi_\epsilon) = \big(H^{-1}\big)'\big(\|H \circ \varphi\|_1\big) \cdot \left(- \int_{\mathbb{R}^n} h\big(\varphi(x)\big)\, b(x)\, d\lambda \right)$$
$$= \frac{1}{h\big(\mathcal{F}(\varphi)\big)} \left(\int_{\mathbb{R}^n} h\big(\varphi(x)\big)\, b(x)\, d\lambda \right).$$

The function $\epsilon \to \left\langle \varphi_\epsilon, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \right\rangle$ is linear in ϵ and it has derivative $\left\langle b, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \right\rangle$. The definition of $\mathcal{G}$ in Hypothesis (B3) assumes $g(s,t)$ is differentiable, so $g\Big(\left\langle \varphi_\epsilon, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \right\rangle, \mathcal{F}(\varphi_\epsilon)\Big)$ is differentiable at $\epsilon = 0$. Differentiating the right hand side of equation (2.22) for $f = \varphi_\epsilon$, and evaluating at its minimimum when $\epsilon = 0$ implies

$$0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g\Big(\Big\langle \varphi_\epsilon, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \mathcal{F}(\varphi_\epsilon)\Big) = \frac{\partial g}{\partial s}\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \mathcal{F}(\varphi)\Big) \Big\langle b, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle$$
$$+ \frac{\partial g}{\partial t}\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \mathcal{F}(\varphi)\Big) \frac{1}{h\big(\mathcal{F}(\varphi)\big)} \left(\int_{\mathbb{R}^n} h\big(\varphi(x)\big)\, b(x)\, d\lambda \right). \quad (2.23)$$

When $b = 1$ is the constant function, then using $\partial\varphi(\mathbb{R}^n) = A$ from Proposition 2.2.1 we find

$$\Big\langle b, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle = \lambda(A)^{-1} \int_{\mathbb{R}^n} \mathrm{MA}(\varphi) = \lambda(A)^{-1}\, \lambda\big(\partial\varphi(\mathbb{R}^n)\big) = 1.$$

So putting $b = 1$ into equation (2.23) shows

$$\frac{\partial g}{\partial s}\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \mathcal{F}(\varphi)\Big) = -\frac{\partial g}{\partial t}\Big(\Big\langle \varphi, \frac{\mathrm{MA}(\varphi)}{\lambda(A)} \Big\rangle, \mathcal{F}(\varphi)\Big) \frac{1}{h\big(\mathcal{F}(\varphi)\big)} \left(\int_{\mathbb{R}^n} h\big(\varphi(x)\big)\, d\lambda \right).$$

Plugging the previous equation back into equation (2.23) implies

$$\int_{\mathbb{R}^n} b(x)\,\frac{\mathrm{MA}(\varphi)}{\lambda(A)} = \frac{\int_{\mathbb{R}^n} h\big(\varphi(x)\big)\, b(x)\, d\lambda}{\int_{\mathbb{R}^n} h\big(\varphi(x)\big)\, d\lambda},$$

for any continuous, bounded b. It follows that

$$\frac{\mathrm{MA}(\varphi)}{\lambda(A)} = \frac{h\circ\varphi}{\|h\circ\varphi\|_1}\,\lambda. \tag{2.24}$$

so φ solves (1.6) in the Alexandrov sense. Since h is positive, Lemma 2.1.7 implies φ is strictly convex. Since $\partial\varphi(\mathbb{R}^n)$ is bounded, it follows that $h\circ\varphi \in C^{0,1}$. Since φ is strictly convex and $h\circ\varphi\in C^{0,1}$, Theorem 2 from Caffarelli [7] implies $\varphi\in C^{2,\alpha}$. Then φ is a classical solution of

$$\frac{\det(\nabla^2\varphi)}{\lambda(A)} = \frac{h\circ\varphi}{\|h\circ\varphi\|_1}.$$

Since h is smooth, elliptic regularity shows that φ is in fact smooth. Thus, φ is strictly convex, and $\partial\varphi(\mathbb{R}^n) = A$ is upgraded to $\nabla\varphi(\mathbb{R}^n) = A$. □

Next we show φ satisfies the normalization in equation (1.6). The proof will involve another double sequence argument as in Lemma 2.2.19.

Lemma 2.2.25. *Under the same assumptions as Lemma 2.2.24,*

$$\int_A \varphi^*\, d\lambda = -\tau.$$

Proof. Every $\widetilde{\varphi}_{i'}$ is a smooth, convex function with $\nabla\widetilde{\varphi}_{i'}(\mathbb{R}^n) = A$, so Lemma 2.1.11 implies

$$-\tau = \int_A \widetilde{\varphi}^*_{i'}\, d\lambda = \int_{\mathbb{R}^n} \big(\langle\nabla\widetilde{\varphi}_{i'}(x), x\rangle - \widetilde{\varphi}_{i'}(x)\big)\, \det(\nabla^2\widetilde{\varphi}_{i'})\, d\lambda.$$

By Lemma 2.2.19

$$\int_{\mathbb{R}^n} \widetilde{\varphi}_{i'}(x)\ \det(\nabla^2 \widetilde{\varphi}_{i'})\, d\lambda \to \int_{\mathbb{R}^n} \varphi\ \mathrm{MA}(\varphi), \tag{2.25}$$

and by Lemma 2.2.24 φ is smooth, so $\mathrm{MA}(\varphi) = \det(\nabla^2\varphi)\,\lambda$.

By Theorem 24.5 of Rockafellar [38], for every x in $\mathbb{R}^n$ and every $\epsilon > 0$ there exists i_0 such that $i' \geq i_0$ implies $\partial\widetilde{\varphi}_{i'}(x) \subset \partial\varphi(x) + B_\epsilon$. Since $\widetilde{\varphi}_{i'}$ and φ are smooth, this subgradient inclusion implies $\nabla\widetilde{\varphi}_{i'}$ converges pointwise to $\nabla\varphi(x)$. Also, $|\langle \nabla\widetilde{\varphi}_{i'}(x), x\rangle| \leq R\,|x|$, where R is a constant such that $A \subset B_R$. We consider the double sequence $\int_{\mathbb{R}^n} \langle \nabla\widetilde{\varphi}_{i'}(x), x\rangle\ \mathrm{MA}(\widetilde{\varphi}_{j'})$.

$$\lim_{j'\to\infty} \int_{\mathbb{R}^n} \langle \nabla\widetilde{\varphi}_{i'}(x), x\rangle\ \mathrm{MA}(\widetilde{\varphi}_{j'}) = \int_{\mathbb{R}^n} \langle \nabla\widetilde{\varphi}_{i'}(x), x\rangle\ \mathrm{MA}(\varphi),$$

because $\mathrm{MA}(\widetilde{\varphi}_{j'}) \to_1 \mathrm{MA}(\varphi)$. The convergence is uniform because of the bound $|\langle \nabla\widetilde{\varphi}_{i'}(x), x\rangle| \leq R\,|x|$. The iterated limit of the double sequence

$$\lim_{i'\to\infty} \left(\lim_{j'\to\infty} \int_{\mathbb{R}^n} \langle \nabla\widetilde{\varphi}_{i'}(x), x\rangle\ \mathrm{MA}(\widetilde{\varphi}_{j'}) \right) = \int_{\mathbb{R}^n} \langle \nabla\varphi(x), x\rangle\ \mathrm{MA}(\varphi),$$

converges by the dominated convergence theorem because the uniformly bounded $\nabla\widetilde{\varphi}_{i'}$ converge pointwise to $\nabla\varphi$. Thus the whole double sequence converges, and in particular the diagonal sequence converges:

$$\int_{\mathbb{R}^n} \langle \widetilde{\varphi}_{i'}(x), x\rangle\ \mathrm{MA}(\widetilde{\varphi}_{i'}) \to \int_{\mathbb{R}^n} \langle \nabla\varphi(x), x\rangle\ \mathrm{MA}(\varphi). \tag{2.26}$$

The limits (2.25) and (2.26) together imply

$$\begin{aligned} -\tau = \lim_{i'\to\infty} \int_{\mathbb{R}^n} \big(\langle \nabla\widetilde{\varphi}_{i'}(x), x\rangle - \widetilde{\varphi}_{i'}(x)\big)\ \mathrm{MA}(\widetilde{\varphi}_{i'}) \\ = \int_{\mathbb{R}^n} \big(\langle \nabla\varphi(x), x\rangle - \varphi(x)\big)\ \mathrm{MA}(\varphi) = \int_A \varphi^*\, d\lambda. \end{aligned}$$

□

Now we can finish the proof of Proposition 2.2.23.

Proof. Proposition 2.2.10 implies that every subsequence $\widetilde{\varphi}_{i'}$ has a convergent subsequence. If we can show that every convergent subsequence has the same unique limit, then this will imply the convergence of the whole sequence $\widetilde{\varphi}_i$.

Let $\widetilde{\varphi}_{i'}$ be a subsequence which converges to φ uniformly on compact sets. By Lemmas 2.2.24 and 2.2.25, φ is a smooth solution to equation (1.6). Also, since $\inf_{\mathbb{R}^n}\{\widetilde{\varphi}_i\} = \widetilde{\varphi}_i(0)$ and $\widetilde{\varphi}_{i'}$ converges to φ uniformly on compact sets, it follows that $\inf_{\mathbb{R}^n}\{\varphi\} = \varphi(0)$. By Hypothesis (B2), solutions to the second boundary value problem (1.6) are unique up to translation, so $\inf_{\mathbb{R}^n}\{\varphi\} = \varphi(0)$ implies φ is the same unique limit for any subsequence $\widetilde{\varphi}_{i'}$.

Now we must show that the convergence $\widetilde{\varphi}_i \to \varphi$ extends to $C^{k,\alpha}$. As in the proof of Lemma 2.2.25, $\nabla\widetilde{\varphi}_i$ converges to $\nabla\varphi$, so the smoothness of h implies $h \circ \widetilde{\varphi}_i$ converges to $h \circ \varphi$ in $C^{0,1}$ on compact sets. So in particular, the $C^{0,1}$ norm of $(h \circ \widetilde{\varphi}_i)/\|h \circ \widetilde{\varphi}_i\|$ has a uniform bound on each compact set.

Theorem 2 from Caffarelli [7] then implies $(h\circ\widetilde{\varphi}_i)/\|h\circ\widetilde{\varphi}_i\|$ has a uniform bound in $C^{2,\alpha}$ on a slightly smaller compact set. The compact embeddings of Hölder spaces implies there exists convergent subsequences in $C^{2,\beta}$ for some $\beta < \alpha$. But the limits of these subsequences are unique, so we have $C^{2,\beta}$ convergence. Bootstrapping this argument yields $C^{k,\alpha}$ convergence on compact sets. □

Chapter 3: Kähler–Ricci iteration on toric manifolds

The motivation for the definition of the Monge–Ampère iteration comes from the Ricci iteration. In this chapter we show the Monge–Ampère iteration (1.7) with $h(t) = e^{-t}$ is equivalent to the Ricci iteration on toric Kähler manifolds. We begin in Section 3.1 by proving Theorem 1.3.1 which shows the normalized Monge–Ampère iteration with $h(t) = e^{-t}$ converges.

In Section 3.2 we discuss the background on toric Kähler manifolds which is necessary for the geometric interpretation of Theorem 1.3.1. In Section 3.3 we define the Kähler–Ricci iteration on toric Kähler manifolds and prove Corollary 1.3.2 about the convergence of the Kähler–Ricci iteration.

3.1 Monge–Ampère iteration with $h(t) = e^{-t}$

The goal of this section is to prove Theorem 1.3.1 about the convergence of the Monge–Ampère iteration for $h(t) = e^{-t}$. The proof is purely analytic, and we defer the geometric interpretations to subsequent sections. In order to prove Theorem 1.3.1, we must verify Hypotheses 1.2.1 and apply Theorem 1.2.2.

Hypothesis (B1):

Clearly e^{-t} is smooth, positive, and decreasing. Also, it is bounded by $C\,t^{-(n+p+1)}$

when $t \gg 1$ and $p = 1$, for example.

Hypothesis (B2):

Theorem 1.1 on pages 651 of Berman and Berndtsson [4] says if the barycenter of A lies at the origin, then there exist smooth convex solutions φ to

$$\begin{cases} \det(\nabla^2 \varphi) = e^{-\varphi} \\ \nabla\varphi(\mathbb{R}^n) = A, \end{cases} \tag{3.1}$$

and they are unique up to translations by $\mathbb{R}^n$. A convex function φ solves equation (3.1) if and only if $\varphi + c$ solves

$$\begin{cases} \dfrac{\det(\nabla^2 \varphi)}{\lambda(A)} = \dfrac{e^{-\varphi}}{\|e^{-\varphi}\|_1} \\ \nabla\varphi(\mathbb{R}^n) = A, \end{cases} \tag{3.2}$$

so convex solutions to equation (3.2) are unique up to translation and an additive constant. Theorem 1 of [4] also proves that if φ is a convex solution to equation (3.1), then $\int_A \varphi^* \, d\lambda < \infty$. Thus, the normalization $\int_A \varphi^* \, d\lambda = -\tau$ is valid for any $\tau \in \mathbb{R}$, and convex solutions to the normalized Monge–Ampère second boundary problem (1.6) are unique up to translations.

Hypothesis (B3):

We define $g(s, t) = s - t$. To verify Hypothesis (B3) we note g is decreasing in t, and $g(s, g(s, t)) = s - (s - t) = t$.

Hypothesis (B4):

First, we compute $\mathcal{F}$. We integrate $H(t) = \int_t^\infty e^{-s} \, d\lambda = e^{-t}$, so $H^{-1}(t) = -\log(t)$, and

$$\mathcal{F}(f) = H^{-1}\big(\|H \circ f\|_1\big) = -\log\big(\|e^{-f}\|_1\big),$$

for any $f \in \mathcal{C} = \{f : \mathbb{R}^n \to (\tau, \infty) \mid f \text{ continuous, and } f(x)/(1+|x|) \text{ bounded}\}$.

Next we define $\mathcal{G}$ with $g(s,t) = s - t$.

$$\mathcal{G}(\mu) = \inf\{\, \langle f,\, \mu\rangle + \log\left(\|e^{-f}\|_1\right) \mid f \in \mathcal{C}\,\}.$$

For $g(s,t) = s - t$ Hypothesis (B4) says

$$\Big\langle \varphi_{i+1},\, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big\rangle - \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big) \le \mathcal{F}(\varphi_i).$$

To prove this inequality, we compute $\mathcal{G}(\mu)$ when μ has a continuous density. Lemma 3.1.1 can be interpreted as saying $\mathcal{G}(\mu) = -\operatorname{Ent}_\lambda(\mu)$, the relative entropy of μ with respect to Lebesgue measure λ.

Lemma 3.1.1. *Let h be a continuous, nonnegative function on $\mathbb{R}^n$ such that $h\,\lambda \in \mathcal{P}_1$. Then,*

$$\mathcal{G}(h\,\lambda) = -\int_{\mathbb{R}^n} h\log(h)\, d\lambda.$$

Proof. First, we show $-\int_{\mathbb{R}^n} h\log(h)\, d\lambda \le \mathcal{G}(h\lambda)$. It is sufficient to show

$$-\int_{\mathbb{R}^n} h\log(h)\, d\lambda \le \int_{\mathbb{R}^n} f\,h\, d\lambda + \log\left(\int_{\mathbb{R}^n} e^{-f}\, d\lambda\right) \tag{3.3}$$

for all $f \in \mathcal{C}$. The convexity of the function $x\log(x)$ implies the elementary inequality

$$t - s \le t\log(t) - t\log(s) = t\log(t/s) \tag{3.4}$$

for $t \ge 0$ and $s > 0$. Equation (3.4) is clearly true when $s = t$, and when $t = 0$ it is true by interpreting $t\log(t) = 0$ when $t = 0$. When $t > s$ the convexity of $x\log(x)$ implies

$$1 + \log(s) \le \frac{t\log(t) - s\log(s)}{t - s}.$$

Multiplying both sides by $t - s$ and simplifying implies equation (3.4). When $s > t$ the convexity of $x \log(x)$ implies

$$\frac{s\log(s) - t\log(s)}{s - t} \leq 1 + \log(s).$$

Multiplying both sides by $t - s$ switches the inequality and implies equation (3.4). Applying equation (3.4) with $t = h(x)$ and $s = e^{-f(x)}/\|e^{-f}\|_1$ yields

$$\begin{aligned}\frac{e^{-f(x)}}{\|e^{-f}\|_1} - h(x) &\leq h(x)\log(h(x)) - h(x)\log\left(\frac{e^{-f(x)}}{\|e^{-f}\|_1}\right)\\ &= h(x)\log(h(x)) + h(x)f(x) + \log(\|e^{-f}\|_1)\, h(x)\end{aligned}$$

Since $h\,\lambda \in \mathcal{P}_1$ it follows that $\|h\|_1 = 1$, so integrating over $\mathbb{R}^n$ implies

$$0 \leq \int_{\mathbb{R}^n} h\, \log(h)\, d\lambda + \int_{\mathbb{R}^n} f\, h\, d\lambda + \log\big(\|e^{-f}\|_1\big),$$

which is equivalent to equation (3.3).

Next, we show the reverse inequality: $-\int_{\mathbb{R}^n} h\log(h)\, d\lambda \geq \mathcal{G}(h\lambda)$. Consider

$$f_k(x) = \min\{-\log\big(h(x)\big),\, k\,(1 + |x|)\}. \tag{3.5}$$

By assumption h is continuous and integrable, so h is finite, which implies f_k is continuous and finite. Thus, $f_k \in \mathcal{C}$, and

$$\mathcal{G}(h\,\lambda) \leq \langle f_k, h\,\lambda\rangle + \log\big(\|e^{-f_k}\|_1\big). \tag{3.6}$$

$h\, f_k$ is an increasing sequence of functions which converge pointwise to $-h\log(h)$, so

$$\langle f_k, h\,\lambda\rangle = \int_{\mathbb{R}^n} f_k\, h\, d\lambda \to -\int_{\mathbb{R}^n} h\log(h)\, d\lambda$$

by the monotone convergence theorem.

e^{-f_k} converges to h pointwise, and $e^{-f_k(x)} \le e^{-f_1(x)}$ for all k. Since e^{-f_1} is integrable, the dominated convergence theorem implies

$$\|e^{-f_k}\|_1 \to \|h\|_1 = 1.$$

Thus $\log\left(\|e^{-f_k}\|_1\right) \to 0$, and taking the limit as $k \to \infty$ in equation (3.6) implies

$$\mathcal{G}(h\,\lambda) \le -\int_{\mathbb{R}^n} h \log(h)\, d\lambda.$$

□

We need one further lemma concerning the condition $\int_A \varphi_i^* = -\tau$ along the iteration. Lemma 3.1.2 can be thought of as an integral comparison principle for the Monge–Ampère measure in comparison to the traditional Monge–Ampère measure comparison principle of Rauch and Taylor [37].

Lemma 3.1.2. *If φ and ψ are two smooth, convex functions on $\mathbb{R}^n$ such that $\nabla\varphi(\mathbb{R}^n) = \nabla\psi(\mathbb{R}^n) = A$, and $\int_A \varphi^*\, d\lambda = \int_A \psi^*\, d\lambda$, then*

$$\langle \varphi, \mathrm{MA}(\varphi)\rangle \le \langle \psi, \mathrm{MA}(\varphi)\rangle.$$

Proof. The equality case of Legendre duality is $\varphi(x) + \varphi^*(\nabla\varphi(x)) = \langle x, \nabla\varphi(x)\rangle$. Integrating both sides of this equality against the Monge–Ampère measure of φ and using the change of variables $y = \nabla\varphi(x)$ and $x = \nabla\varphi^*(y)$ implies

$$\int_{\mathbb{R}^n} \varphi\ \mathrm{MA}(\varphi) + \int_A \varphi^*\, d\lambda = \int_A \langle y, \nabla\varphi^*(y)\rangle\, d\lambda.$$

Since ψ is convex, $\langle y, x\rangle \le \psi^*(y) + \psi(x)$ for all x and y, so

$$\int_A \langle y, \nabla\varphi^*(y)\rangle\, d\lambda \le \int_A \psi^*(y) + \psi(\nabla\varphi^*(y))\, d\lambda = \int_{\mathbb{R}^n} \psi\ \mathrm{MA}(\varphi) + \int_A \psi^*\, d\lambda.$$

Since $\int_A \varphi^* \, d\lambda = \int_A \psi^* \, d\lambda$, it follows that

$$\int_{\mathbb{R}^n} \varphi \ \mathrm{MA}(\varphi) \le \int_{\mathbb{R}^n} \psi \ \mathrm{MA}(\varphi).$$

□

Now we can use Lemma 3.1.1 and Lemma 3.1.2 to prove Hypothesis (B4).

Lemma 3.1.3. $\Big\langle \varphi_{i+1}, \dfrac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)} \Big\rangle - \mathcal{G}\Big(\dfrac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big) \le \mathcal{F}(\varphi_i).$

Proof. Since φ_i is smooth, $\mathrm{MA}(\varphi_i) = \det(\nabla^2 \varphi_i)\lambda$ for all i. Since $\int_A \varphi_i^* \, d\lambda = \int_A \varphi_{i+1}^* \, d\lambda$, Lemma 3.1.2 implies

$$\int_{\mathbb{R}^n} \varphi_{i+1} \frac{\det(\nabla^2 \varphi_{i+1})}{\lambda(A)} \, d\lambda \le \int_{\mathbb{R}^n} \varphi_i \frac{\det(\nabla^2 \varphi_{i+1})}{\lambda(A)} \, d\lambda.$$

Subtracting $\mathcal{G}\Big(\dfrac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big)$ from both sides and applying the equality from Lemma 3.1.1 yields

$$\begin{aligned}\Big\langle \varphi_{i+1}, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)} \Big\rangle - \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big) \le \int_{\mathbb{R}^n} \varphi_i \frac{\det(\nabla^2 \varphi_{i+1})}{\lambda(A)} \, d\lambda \\ + \int_{\mathbb{R}^n} \frac{\det(\nabla^2 \varphi_{i+1})}{\lambda(A)} \log\Big(\frac{\det(\nabla^2 \varphi_{i+1})}{\lambda(A)}\Big) \, d\lambda.\end{aligned}$$

Since $\{\varphi_i\}$ is a Monge–Ampère iteration solving equation (1.7), it follows that $\dfrac{\det(\nabla^2 \varphi_{i+1})}{\lambda(A)} = \dfrac{h \circ \varphi_i}{\|h \circ \varphi_i\|_1}$. Thus, the above inequality implies

$$\begin{aligned}\Big\langle \varphi_{i+1}, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)} \Big\rangle - \mathcal{G}\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big) &\le \int_{\mathbb{R}^n} \varphi_i \frac{e^{-\varphi_i}}{\|e^{-\varphi_i}\|_1} \, d\lambda + \int_{\mathbb{R}^n} \frac{e^{-\varphi_i}}{\|e^{-\varphi_i}\|_1} \log\Big(\frac{e^{-\varphi_i}}{\|e^{-\varphi_i}\|_1}\Big) \, d\lambda \\ &= \|e^{-\varphi_i}\|_1^{-1} \int_{\mathbb{R}^n} \varphi_i \, e^{-\varphi_i} \, d\lambda - \|e^{-\varphi_i}\|_1^{-1} \int_{\mathbb{R}^n} \varphi_i \, e^{-\varphi_i} \, d\lambda - \log\big(\|e^{-\varphi_i}\|_1\big) \int_{\mathbb{R}^n} \frac{e^{-\varphi_i}}{\|e^{-\varphi_i}\|_1} \, d\lambda \\ &= -\log\big(\|e^{-\varphi_i}\|_1\big) = \mathcal{F}(\varphi_i).\end{aligned}$$

□

Lemma 3.1.3 concludes the proof of Hypothesis (B4), so Theorem 1.2.2 implies Theorem 1.3.1.

3.2 Toric Kähler manifolds

In this section we give the necessary background on toric Kähler manifolds in order to define the Kähler–Ricci iteration on toric Kähler manifolds in Section 3.3.

A complex manifold X is called *Kähler* if it has a Hermitian metric h such that the associated two form $\omega(u,v) = \operatorname{Re} h(\sqrt{-1}u, v)$ is closed. A compact Kähler manifold X is *toric* if there is an effective Hamiltonian holomorphic action of the real torus $T^n = (S^1)^n$.

The Kähler form ω can be used to define the moment map of the Hamiltonian action of T^n. The Atiyah–Guillemin–Sternberg Theorem says the image of the moment map is a compact, convex polytope, which we denote by P.

Using the symplectic point of view, Delzant [16] proved P must be of a specific form. We say P is *Delzant* if for each vertex v there exists a transformation $A \in Sl_n(\mathbb{Z})$ such that

$$A\,(P - v) \cap B_\epsilon = \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1, \dots, n\} \cap B_\epsilon \text{ for some } \epsilon > 0, \tag{3.7}$$

where $A\,(P - v) = \{\, A\,(x - v) \mid x \in P\,\}$. In other words, a neighborhood of each vertex of P is $Sl_n(\mathbb{Z})$–equivalent to a neighborhood of the origin in the first orthant. In fact, Delzant proved a bijective correspondence between Delzant polytopes and symplectic toric manifolds. Later work by Guillemin [21] explained the connection to Kähler geometry and proved the bijective correspondence

$$\{\,\text{polarized, toric Kähler manifolds } (X_P, L_P)\,\} \longleftrightarrow \{\,\text{integral Delzant polytopes } P\,\}.$$

Polarized means the Kähler manifold X_P is paired with a certain line bundle L_P. Integral polytopes are polytopes which are the convex hull of points in $\mathbb{Z}^n$.

3.2.1 Construction of X_P

We will adapt the exposition of Donaldson [17] and Berman [3] to describe the construction of the toric Kähler manifold X_P from the Delzant polytope P. We choose the approach of defining X_P which elucidates the Kähler point of view and allows us to compute algebraic geometric properties of X_P most easily.

We begin by defining the important data associated to a Delzant polytope $P \subset \mathbb{R}^n$. A *face* of P is the intersection of P with a supporting hyperplane of P. A face of P has dimension k if it lies in an affine space of dimension k, but no smaller dimensional affine space. A dimension 0 face is called a *vertex*, and the codimension 1 faces of P are called *facets*. The Delzant condition implies P has dimension n, so the facets of P are dimension $(n-1)$.

We enumerate the facets of P by $\{F_i\}_{i=1}^{M}$. For every vertex v, the Delzant condition implies there are n facets adjacent to v. The mapping $P \mapsto A\,(P - v)$ from (3.7) maps each facet adjacent to v into some hyperplane perpendicular to a coordinate axis. For each vertex v, we define the map $\sigma : \{\, i \mid v \in F_i \,\} \to \{\, 1, \ldots, n \,\}$ by the condition

$$A\,(F_i - v) \subset {e_{\sigma(i)}}^{\perp}, \tag{3.8}$$

where $\{e_k\}$ are the standard unit normals in $\mathbb{R}^n$. If $\widetilde{v}$ is another vertex of P, we use $\widetilde{A}$ to denote the transformation in equation (3.7) and $\widetilde{\sigma}$ to denote the identification between facets F_i of P adjacent to $\widetilde{v}$ with the coordinate hyperplanes containing their image $\widetilde{A}(F_i - \widetilde{v})$.

Now we will describe the construction of X_P for any Delzant polytope P. Any toric Kähler manifold contains $\mathbb{C}^{*n}$ as an open, dense subset such that T^n acts on $\mathbb{C}^{*n}$ by

$$\theta \cdot z = \left(e^{\sqrt{-1}\theta_1}\, z_1, \, \ldots, \, e^{\sqrt{-1}\theta_n}\, z_n\right).$$

Thus, we can view X_P as a compactification of $\mathbb{C}^{*n}$ with an extension of the action of T^n. We will call the chart $\mathbb{C}^{*n}$ with coordinate z the *open orbit* of X_P because it is the orbit of any one point if we extend the action of T^n to an action of $\mathbb{C}^{*n}$ acting on itself. To define the compactification of the open orbit, we associate a coordinate chart $U \simeq \mathbb{C}^n$ with coordinate w to each vertex and define the transition functions.

Let v be any vertex of P, and let $U \simeq \mathbb{C}^n$ with coordinate w be a chart associated to v. The Delzant condition (3.7) implies there exists a matrix A in $Sl_n(\mathbb{Z})$ such that $A\,(P - v)$ agrees with the closed first orthant in some neighborhood of the origin. We define the transition functions between z and w by

$$w_i = \prod_j z_j^{A^{ji}} \qquad z_i = \prod_j w_j^{A_{ji}},$$

where A^{ij} are the components of A^{-1}. The transition functions are holomorphic for $z \in \mathbb{C}^{*n}$ and $w \in \mathbb{C}^{*n} \subset U$. If we define action-angle coordinates in $z \in \mathbb{C}^{*n}$ and $w \in \mathbb{C}^{*n} \subset U$ by

$$z_i = e^{\frac{x_i}{2} + \sqrt{-1}\alpha_i} \text{ and } w_i = e^{\frac{u_i}{2} + \sqrt{-1}\beta_i} \tag{3.9}$$

for x_i, $u_i \in \mathbb{R}$ and α_i, $\beta_i \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, then the transformations are defined by

$$x = A^T u \qquad \alpha = A^T \beta. \tag{3.10}$$

We also need to define the transition between the coordinate charts U and $\widetilde{U}$ associated to two vertices v and $\widetilde{v}$ with coordinates w and $\widetilde{w}$. On the open orbit $\mathbb{C}^{*n}$, we can compute

$$\begin{aligned} w_i &= \prod_j z_j^{A^{ji}} = \prod_j \left(\prod_k \widetilde{w}_k^{\widetilde{A}_{kj}} \right)^{A^{ji}} \\ &= \prod_k \widetilde{w}_k^{\sum_j \widetilde{A}_{kj} A^{ji}} = \prod_k \widetilde{w}_k^{(\widetilde{A}A^{-1})_{ki}}. \end{aligned}$$

Thus, the transition functions between w and $\widetilde{w}$ are

$$w_i = \prod_j \widetilde{w}_j^{(\widetilde{A}A^{-1})_{ji}} \qquad \widetilde{w}_i = \prod_j w_j^{(A\widetilde{A}^{-1})_{ji}}. \tag{3.11}$$

If there are k facets which contain both v and $\widetilde{v}$, we will show the transition function between w and $\widetilde{w}$, is defined on subsets of U and $\widetilde{U}$ which are both isomorphic to $\mathbb{C}^k \times \mathbb{C}^{*(n-k)}$.

Lemma 3.2.1. *For indices i such that $v \in F_i$ and $\widetilde{v} \in F_i$,*

$$A^T e_{\sigma(i)} = \widetilde{A}^T e_{\widetilde{\sigma}(i)},$$

for σ and $\widetilde{\sigma}$ defined in (3.8).

Proof. Since the facet F_i contains both vertices v and $\widetilde{v}$, there is some normal vector n_i such that

$$F_i \subset \{\, y \mid \langle y - v, n_i \rangle = 0 \,\} = \{\, y \mid \langle y - \widetilde{v}, n_i \rangle = 0 \,\}.$$

This inclusion implies $F_i - v \subset n_i^\perp$ and $F_i - \widetilde{v} \subset n_i^\perp$. If $y \in n_i^\perp$, then $A\,y \in ((A^T)^{-1} n_i)^\perp$, so it follows that

$$A\,(F_i - v) \subset \big((A^T)^{-1} n_i\big)^\perp \text{ and } \widetilde{A}\,(F_i - \widetilde{v}) \subset \big((\widetilde{A}^T)^{-1}\, n_i\big)^\perp.$$

By the definition of $e_{\sigma(i)}$, $A\,(F_i - v) \subset e_{\sigma(i)}{}^\perp$, so there are nonzero constants α and $\widetilde{\alpha}$ such that

$$(A^T)^{-1} n_i = \alpha\, e_{\sigma(i)} \text{ and } (\widetilde{A}^T)^{-1} n_i = \widetilde{\alpha}\, e_{\widetilde{\sigma}(i)}.$$

We can solve both equations for n_i and set them equal to each other to find

$$\frac{\alpha}{\widetilde{\alpha}} e_{\sigma(i)} = (A^T)^{-1}\, \widetilde{A}^T\, e_{\widetilde{\sigma}(i)}.$$

Since $(A^T)^{-1}\, \widetilde{A}^T$ is in $Sl_n(\mathbb{Z})$, it follows that $\alpha/\widetilde{\alpha}$ must lie in $\mathbb{Z}$. Solving instead for $e_{\widetilde{\sigma}(i)}$ shows that $\widetilde{\alpha}/\alpha$ must also lie in $\mathbb{Z}$, so either $\widetilde{\alpha}/\alpha = 1$ or $\widetilde{\alpha}/\alpha = -1$. But the sign of both α and $\widetilde{\alpha}$ are only determined by whether n_i points to the interior of P, or to the exterior of P, so both α and $\widetilde{\alpha}$ have the same sign. Thus $\alpha/\widetilde{\alpha} = 1$, so

$$A^T\, e_{\sigma(i)} = \widetilde{A}^T\, e_{\widetilde{\sigma}(i)}.$$

□

Lemma 3.2.2. *The transition functions (3.11) between w and $\widetilde{w}$ are holomorphic on the sets*

$$\big\{\, w \mid w_j \neq 0 \textit{ for } j \textit{ such that } \widetilde{v} \notin F_{\sigma^{-1}(j)} \,\big\}$$
$$= \big\{\, \widetilde{w} \mid \widetilde{w}_j \neq 0 \textit{ for } j \textit{ such that } v \notin F_{\widetilde{\sigma}^{-1}(j)} \,\big\} \simeq \mathbb{C}^k \times \mathbb{C}^{*(n-k)}.$$

Proof. The transition functions (3.11) between w and $\widetilde{w}$ are holomorphic on w, $\widetilde{w} \in \mathbb{C}^{*n}$ because $\widetilde{A}A^{-1}$ and $A\,\widetilde{A}^{-1}$ are in $Sl_n(\mathbb{Z})$.

If F_i is a facet of P such that $v \in F_i$ and $\widetilde{v} \in F_i$, then $w_{\sigma(i)}$ and $\widetilde{w}_{\widetilde{\sigma}(i)}$ can equal 0, so we must prove these terms always have nonnegative powers in the transition functions. By equation (3.11) the positivity of the powers of $w_{\sigma(i)}$ and $\widetilde{w}_{\widetilde{\sigma}(i)}$ in their transition functions is equivalent to

$$(\widetilde{A}A^{-1})_{\widetilde{\sigma}(i)k} \geq 0 \text{ and } (A\widetilde{A}^{-1})_{\sigma(i)k} \tag{3.12}$$

for $k = 1, \ldots, n$. Equation (3.12) follows from Lemma 3.2.1 because

$$\begin{aligned}(\widetilde{A}A^{-1})_{\widetilde{\sigma}(i)k} &= e_{\widetilde{\sigma}(i)}{}^T \widetilde{A}A^{-1}e_k = (\widetilde{A}^T e_{\widetilde{\sigma}(i)})^T A^{-1}e_k \\ &= (A^T e_{\sigma(i)})^T A^{-1}e_k = e^T_{\sigma(i)}e_k \geq 0.\end{aligned}$$

□

Lemma 3.2.3. *For indices i such that $v \in F_i$ and $\widetilde{v} \in F_i$, the coordinates w and $\widetilde{w}$ satisfy*

$$w_{\sigma(i)} = \widetilde{w}_{\widetilde{\sigma}(i)}\, f(\widetilde{w}_1, \ldots, \widehat{\widetilde{w}_{\widetilde{\sigma}(i)}}, \ldots, \widetilde{w}_n)$$

for f, a holomorphic function of the other $(n-1)$ variables.

Proof. The transition function between w and $\widetilde{w}$ given in equation (3.11) implies

$$w_{\sigma(i)} = \prod_j \widetilde{w}_j^{(A\widetilde{A}^{-1})_{\sigma(i)j}}.$$

By Lemma 3.2.2, the transitions are holomorphic, so we only need to prove the power of $\widetilde{w}_{\widetilde{\sigma}(i)}$ equals 1, which is equivalent to

$$(A\widetilde{A}^{-1})_{\sigma(i)\,\widetilde{\sigma}(i)} = 1. \tag{3.13}$$

Equation (3.13) follows from Lemma 3.2.1 because

$$(A\widetilde{A}^{-1})_{\sigma(i)\,\widetilde{\sigma}(i)} = {e_{\sigma(i)}}^T A\widetilde{A}^{-1} e_{\widetilde{\sigma}(i)} = (A^T e_{\sigma(i)})^T\, \widetilde{A}^{-1} e_{\widetilde{\sigma}(i)}$$
$$= (\widetilde{A}^T e_{\widetilde{\sigma}(i)})^T\, \widetilde{A}^{-1} e_{\widetilde{\sigma}(i)} = {e_{\widetilde{\sigma}(i)}}^T e_{\widetilde{\sigma}(i)} = 1.$$

□

We define

$$U \cap \widetilde{U} = \left\{\, w \mid w_j \neq 0 \text{ for } j \text{ such that } \widetilde{v} \notin F_{\sigma^{-1}(j)} \,\right\}$$
$$= \left\{\, \widetilde{w} \mid \widetilde{w}_j \neq 0 \text{ for } j \text{ such that } v \notin F_{\widetilde{\sigma}^{-1}(j)} \,\right\}$$

to be the overlap of the coordinate charts. The holomorphic transition functions between all the vertices satisfy the cocycle property, which can be seen easily in the action-angle coordinates because the transition functions sending $u \mapsto \widetilde{u}$ and $\beta \mapsto \widetilde{\beta}$ are given by $(\widetilde{A}A^{-1})^T$. On the points where action-angle coordinates are not defined, the transition functions send $\left\{\, w \mid w_j \neq 0 \text{ for } j \text{ such that } \widetilde{v} \notin F_{\sigma^{-1}(j)} \,\right\} \mapsto \left\{\, \widetilde{w} \mid \widetilde{w}_j \neq 0 \text{ for } j \text{ such that } v \notin F_{\widetilde{\sigma}^{-1}(j)} \,\right\}$ which trivially satisfies the cocycle property. Thus, X_P is a compact, complex manifold.

The T^n action on $\mathbb{C}^{*n}$ extends to each coordinate chart by

$$\theta \cdot w = \left(e^{\sqrt{-1}\, A^{j1}\theta_j}\, w_1,\, \ldots,\, e^{\sqrt{-1}\, A^{jn}\theta_j}\, w_n \right), \tag{3.14}$$

and one can check Lemma 3.2.1 implies the action is the same in the w and $\widetilde{w}$ coordinates. This agreement means T^n acts holomorphically on X_P. The action is clearly effective, so it remains to show X_P is Kähler and the T^n action is Hamiltonian with respect to any Kähler form ω. We also note that the open orbit $\mathbb{C}^{*n}$ is not necessary to define X_P, so the maps in $Sl_n(\mathbb{Z})$ sending $P - v$ to $P - \widetilde{v}$ completely determine the complex manifold X_P. For this reason, if P' is a polytope which is $Sl_n(\mathbb{Z})$ equivalent to P, then $X_{P'}$ is biholomorphic to X_P.

3.2.2 Construction of L_P

To show X_P is Kähler we will prove the stronger fact that X_P is projective by defining a very ample line bundle L_P on X_P. We will define X_P by describing its holomorphic sections.

We now assume that P is an integral Delzant polytope, so all of its vertices lie in $\mathbb{Z}^n$. For any point p in $\mathbb{Z}^n \cap P$, the integer lattice in P, we will define a section s_p of some line bundle. For every vertex v we define

$$q = A\,(p - v)$$

to be the image of p under the map $P \mapsto A\,(P - v)$ which defined the chart U and coordinate w associated to v. In this chart we define a local section by

$$s_p\big|_U = w^q,$$

where

$$w^q = \prod_i w_i^{q_i}.$$

In any other chart $\widetilde{U}$ with coordinate $\widetilde{w}$ the section s_p is locally given by

$$s_p\big|_{\widetilde{U}} = \widetilde{w}^{\widetilde{q}} \text{ for } \widetilde{q} = \widetilde{A}\,(p - \widetilde{v}).$$

The points q and $\widetilde{q}$ are related by $A^{-1}q + v = \widetilde{A}^{-1}\widetilde{q} + \widetilde{v}$.

The computation in Lemma 3.2.3 shows that the coordinates w and $\widetilde{w}$ are related by

$$w_i = \prod_j \widetilde{w}_j^{(\widetilde{A}A^{-1})_{ji}},$$

so in particular, for any $b \in \mathbb{R}$

$$w^{\,b} = \widetilde{w}^{\widetilde{A}A^{-1}b}.$$

The local representations of the section s_p are related by

$$\begin{aligned} s_p\big|_U &= w^{\,q} = \widetilde{w}^{\widetilde{A}A^{-1}q} = \widetilde{w}^{\widetilde{A}A^{-1}\,(A\widetilde{A}^{-1}\widetilde{q}+A(\widetilde{v}-v))} \\ &= \widetilde{w}^{\,\widetilde{q}}\,\widetilde{w}^{\,\widetilde{A}(\widetilde{v}-v)} = \widetilde{w}^{\,\widetilde{A}(\widetilde{v}-v)}\, s_p\big|_{\widetilde{U}}. \end{aligned}$$

Thus, if s_p is a section of a line bundle, then the transition functions are defined by

$$g_{U\widetilde{U}} = \widetilde{w}^{\,\widetilde{A}(\widetilde{v}-v)}.$$

To show the transition functions satisfy the cocycle condition, and thus define a line bundle, we will repeatedly apply the formula $w^{A\,b} = \widetilde{w}^{\widetilde{A}\,b}$.

$$\begin{aligned} g_{U\widetilde{U}}\, g_{\widetilde{U}U'}\, g_{U'U} &= \widetilde{w}^{\,\widetilde{A}(\widetilde{v}-v)}\, w'^{\,A'(v'-\widetilde{v})}\, w^{\,A(v-v')} \\ &= w^{A(\widetilde{v}-v)}\, w^{A(v'-\widetilde{v})}\, w^{A(v-v')} = 1. \end{aligned}$$

We define L_P to be the line bundle defined by the transition functions $g_{U\widetilde{U}}$. Since P is an integral polytope, the vertices lie in $\mathbb{Z}^n$ so the transition functions are holormorphic on their domain of definition. Since p was chosen to lie in $\mathbb{Z}^n$, the section s_p is also holomorphic. Since p was an arbitrary element of $P \cap \mathbb{Z}^n$ it follows that every integer lattice point in P defines a holomorphic section of L_P.

Lemma 3.2.4. *L_P is very ample.*

Proof. If $N = |P \cap \mathbb{Z}^n|$, then we can enumerate $\{p_i\}_{i=1}^N = P \cap \mathbb{Z}^n$. Let $\{s_i\}_{i=1}^N$ be the sections which are locally defined by $s_i\big|_{U_v} = w^{A\,(p_i - v)}$ in the coordinate chart associated to v. The sections of L_P define a map to complex projective space, $\iota : X_P \to \mathbb{P}^{N-1}$ by

$$\iota(z) = [\, s_1(z) : \cdots : s_N(z)\,].$$

If ι is an embedding, then by definition, L_P is very ample. In the coordinate chart U_v the map is given by

$$\iota\big|_U(w) = \left[\, w^{A\,(p_1 - v)} : \cdots : w^{A\,(p_N - v)} \,\right].$$

Since P is Delzant and integral, it follows that $\{0, e_1, \ldots, e_n\} \subset A\,(P - v)$. Thus, after possibly reindexing the points p_i, it follows that

$$\iota\big|_U(w) = \left[\, 1 : w_1 : \cdots : w_n : w^{A\,(p_{n+2} - v)} : \cdots : w^{A\,(p_N - v)} \,\right].$$

In this representation it is clear that ι is an embedding of U into $\mathbb{P}^{N-1}$. The map $\iota\big|_U$ is an embedding for every coordinate chart, so $\iota : X_P \to \mathbb{P}^{N-1}$ is an embedding of all of X_P. □

Lemma 3.2.4 shows that L_P is equivalently defined by

$$L_P = \iota^*(\mathcal{O}(1)).$$

The sections of $\mathcal{O}(1)$ are spanned by the monomials $\{z_i\}_{i=1}^N$. Each section s_i is the pullback of the monomial z_i under the Kodaira map of L_P. Thus, $\{s_i\}_{i=1}^N$ generates the space of holomorphic sections of L_P:

$$H^0(X_P, L_P) = \operatorname{span}_{\mathbb{C}}\{\, s_p \mid p \in \mathbb{Z}^n \cap P \,\}.$$

The embedding $\iota : X_P \to \mathbb{P}^{N-1}$ shows that X_P is Kähler because we can define a Kähler form on X_P as the pullback of the Fubini–Study form ω_{FS} on $\mathbb{P}^{N-1}$. This Kähler metric is a representative of 2π times the first Chern class,

$$\iota^*(\omega_{FS}) \in 2\pi\, c_1(L_P).$$

The definition of the Fubini–Study form implies the smooth Kähler potentials for $\iota^*(\omega_{FS})$ in each coordinate chart are given by

$$\iota^*(\omega_{FS})\big|_U := \sqrt{-1}\partial\overline{\partial}\,\psi = \sqrt{-1}\partial\overline{\partial}\,\log\left(\sum_{i=1}^{N} |w^{A\,(p_i - v)}|^2\right). \tag{3.15}$$

The T^n action leaves $\iota^*(\omega_{FS})$ invariant, so the potentials only depend on the action variable u defined in equation (3.9). Thus, the potential ψ on U defined in equation (3.15) is given by

$$\psi = \log\left(\sum_{i=1}^{N} e^{\langle A\,(p_i - v),\, u\rangle}\right).$$

The transition functions for the action-angle coordinates given in equation (3.10) imply

$$A^T\, u = \widetilde{A}^T\, \widetilde{u}.$$

The transition between u and $\widetilde{u}$ can be used to compute the difference between the potentials for $\iota^*(\omega_{FS})$ in coordinate patches U and $\widetilde{U}$:

$$\begin{aligned} \psi - \widetilde{\psi} &= \log\left(\sum_{i=1}^{N} e^{\langle A\,(p_i - v),\, u\rangle}\right) - \log\left(\sum_{i=1}^{N} e^{\langle \widetilde{A}\,(p_i - \widetilde{v}),\, \widetilde{u}\rangle}\right) \\ &= \log\left(\sum_{i=1}^{N} e^{\langle \widetilde{A}\,(p_i - v),\, \widetilde{u}\rangle}\right) - \log\left(\sum_{i=1}^{N} e^{\langle \widetilde{A}\,(p_i - v),\, \widetilde{u}\rangle + \langle \widetilde{A}\,(v - \widetilde{v}),\, \widetilde{u}\rangle}\right) \\ &= \langle \widetilde{A}\,(\widetilde{v} - v),\, \widetilde{u}\rangle = \langle A\,(\widetilde{v} - v),\, u\rangle. \end{aligned} \tag{3.16}$$

We will use equation (3.16) when we study arbitrary toric Kähler metrics in $2\pi\, c_1(L_P)$.

3.2.3 Examples

Projective space, $\mathbb{P}^n$

Consider the simplex $P_t = \mathrm{cvx}\{\,0,\, t\,e_1, \ldots, t\,e_n\,\} \subset \mathbb{R}^n$ for $t \in \mathbb{N}$ and $\{e_i\}$ the standard basis vectors in $\mathbb{R}^n$. This polytope in dimension 2 is shown in Figure 3.1.

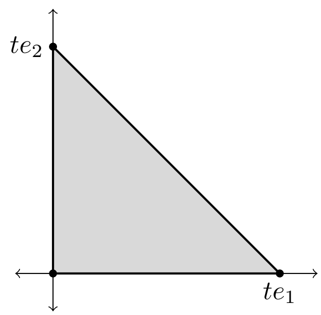

Figure 3.1: The polytope P_t associated to $\mathbb{P}^2$

The polytope P_t already aligns with the first orthant, so for the vertex $v = 0$ the associated matrix is $A = \mathrm{Id}$. Thus, if $z \in \mathbb{C}^{*n}$ is the coordinate in the open orbit, then the coordinate $w \in \mathbb{C}$ in the chart U associated to the vertex 0 satisfies $z = w$ when $w \in \mathbb{C}^{*n}$.

By the symmetry of P_t, it suffices to look at the chart associated to the vertex $v = t\,e_1$. If we translate P_t so that the vertex at v lies at the origin, we get

$$P_t - v = \mathrm{cvx}\{\,-te_1,\, 0,\, t(e_2 - e_1),\, \ldots,\, t(e_n - e_1)\,\}.$$

We choose A to be the map which sends $-e_1 \mapsto e_1$ and $e_i - e_1 \mapsto e_i$ for $i \geq 2$. Thus,

$$\widetilde{A} = \begin{pmatrix} -1 & -1 & -1 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

is a matrix in $Sl_n(\mathbb{Z})$ such that $A(P_t - v)$ agrees with the first orthant. One can show that $\widetilde{A}^{-1} = \widetilde{A}$, and applying the formula $w_i = \prod_j \widetilde{w}_j^{(\widetilde{A}A^{-1})_{ji}}$ yields

$$w_1 = \frac{1}{\widetilde{w}_1},\ w_i = \frac{\widetilde{w}_i}{\widetilde{w}_1} \text{ for } i \geq 2, \text{ and } \widetilde{w}_1 = \frac{1}{w_1},\ \widetilde{w}_i = \frac{w_i}{w_1} \text{ for } i \geq 2.$$

The transition functions are holomorphic on

$$\{\, w \mid w_1 \neq 0 \,\} = \{\, \widetilde{w} \mid \widetilde{w}_1 \neq 0 \,\} \simeq \mathbb{C}^* \times \mathbb{C}^{n-1}.$$

The rest of the charts are identical, up to a permutation of coordinates. The manifold X_{P_t} is biholomorphic to $\mathbb{P}^n$. We can exhibit the map $f : X_{P_t} \to \mathbb{P}^n$ explicitly by

$$f\big|_U(w_1, \dots, w_n) = [1 : w_1 : \dots : w_n], \text{ and } f\big|_{\widetilde{U}}(\widetilde{w}_1, \dots, \widetilde{w}_n) = [\widetilde{w}_1 : 1 : \widetilde{w}_2 : \dots : \widetilde{w}_n].$$

The change of coordinates shows that on $U \cap \widetilde{U}$

$$f(w_1, \dots, w_n) = [1 : w_1 : \dots : w_n] = \left[\frac{1}{w_1} : 1 : \frac{w_2}{w_n} : \dots : \frac{w_n}{w_1}\right] = f(\widetilde{w}_1, \dots, \widetilde{w}_n).$$

The line bundle L_{P_t} has transition functions $g_{U\widetilde{U}} = \widetilde{w}^{\widetilde{A}(\widetilde{v}-v)}$, which in the above charts are

$$g_{U\widetilde{U}} = \widetilde{w}_1{}^{-t}.$$

These are the transition functions for t times the hyperplane bundle, which is often denoted by $\mathcal{O}(t)$. The global holomorphic sections of $L_{P_t} = \mathcal{O}(t)$ are in one to one correspondence with the points of $P_t \cap \mathbb{Z}^n$. There are $(t^2+t)/2$ lattice points in P_t, so we know

$$\dim_{\mathbb{C}} H^0(X_{P_t},\, L_{P_t}) = \frac{t^2+t}{2}.$$

One point blow up of projective space $Bl_p\,\mathbb{P}^2$

Consider the polytope $P = \operatorname{cvx}\{\,0,\, e_1,\, 2\,e_2,\, e_1+e_2\,\}$ in $\mathbb{R}^2$ shown in Figure 3.2

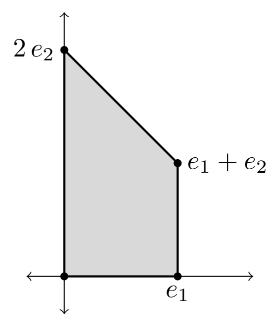

Figure 3.2: The polytope P associated to $Bl_p\,\mathbb{P}^2$

The polytope P aligns with the first orthant, so we have a coordinate patch U with coordinate w associated to the vertex 0. Rather than describe every coordinate chart on X_P and compute their transition functions, we will instead think of X_P as a projective variety under the Kodaira embedding $\iota : X_P \to \mathbb{P}^4$ associated to L_P. In the coordinate chart associated to the vertex 0, the embedding is given by

$$\iota\big|_U(w_1,\, w_2) = [\,1 : w_1 : w_2 : w_1\,w_2 : w_2^2\,].$$

Thus, $X_P \simeq \overline{\iota\big|_U(\mathbb{C}^2)}$, and we claim that X_P is biholomorphic to $Bl_p\,\mathbb{P}^2$. To prove this biholomorphism, we will show the functions which cut out the variety $\overline{\iota(\mathbb{C}^2)}$ in $\mathbb{P}^4$ realize X_P as $Bl_p\,\mathbb{P}^2$.

Let $[x_0 : \cdots : x_4]$ be homogeneous coordinates on $\mathbb{P}^4$. Then we can easily verify

$$\overline{\iota\big|_U(\mathbb{C}^2)} = \{\, [x_0 : \cdots : x_4] \in \mathbb{P}^4 \mid x_2\, x_3 - x_1\, x_4 = 0\,,\; x_0\, x_3 - x_1\, x_2 = 0\,,\; \text{and } x_2^2 - x_0\, x_4 = 0 \,\}.$$

Now we will show that this variety is biholomorphic to $Bl_p\,\mathbb{P}^2$. If we choose to blow up $\mathbb{P}^2$ at the point $p = [0 : 0 : 1]$, then $Bl_p\,\mathbb{P}^2$ is the subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ given by

$$\big\{ \big([\, z_0 : z_1 : z_2\,], [\, y_0 : y_1\,] \big) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z_0\, y_1 - z_1\, y_0 = 0 \big\}.$$

We can use the Segre embedding to map $\sigma : \mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ by

$$\sigma\,(\, [\, z_0 : z_1 : z_2\,], [\, y_0 : y_1\,]\,) = [\, z_0\, y_0 : z_0\, y_1 : z_1\, y_0 : z_1\, y_1 : z_2\, y_0 : z_2\, y_1\,].$$

But since we are embedding the variety $z_0\, y_1 - z_1\, y_0 = 0$, we can actually embed $Bl_p\,\mathbb{P}^2$ into the subvariety $x_1 - x_2 = 0$ which is a copy of $\mathbb{P}^4$ in $\mathbb{P}^5$. Or, after rearranging the coordinates so the embedding will agree with $\overline{\iota\big|_U(\mathbb{C}^2)}$, we can consider the embedding

$$\big\{ \big([\, z_0 : z_1 : z_2\,], [\, y_0 : y_1\,] \big) \mid z_0\, y_1 - z_1\, y_0 = 0 \big\} \mapsto [\, z_0\, y_0 : z_2\, y_0 : z_0\, y_1 : z_2\, y_1 : z_1\, y_1\,]. \tag{3.17}$$

Multiplying $z_0\, y_1 - z_1\, y_0 = 0$ by $z_2\, y_1$, $z_2\, y_0$, or $z_0\, y_1$ respectively implies

$$(z_0\, y_1)\,(z_2\, y_1) - (z_1\, y_1)\,(z_2\, y_0) = 0$$
$$(z_0\, y_0)\,(z_2\, y_1) - (z_0\, y_1)\,(z_2\, y_0) = 0$$
$$(z_1\, y_0)^2 - (z_0\, y_0)\,(z_1\, y_1) = 0.$$

Thus, the image of $Bl_p\,\mathbb{P}^2$ under the embedding (3.17) lies in a variety of $\mathbb{P}^4$ which is cut out by the equations

$$x_2\, x_3 - x_4\, x_1 = 0$$

$$x_0\, x_3 - x_2\, x_1 = 0$$

$$x_2^2 - x_0\, x_4 = 0.$$

One can show the embedding (3.17) surjects onto this variety, so

$$Bl_p\, \mathbb{P}^2 \simeq \{\, [x_0 : \cdots : x_4] \in \mathbb{P}^4 \mid x_2\, x_3 - x_1\, x_4 = 0\,,\ x_0\, x_3 - x_1\, x_2 = 0\,,\ \text{and } x_2^2 - x_0\, x_4 = 0\,\}.$$

This variety is equal to $\overline{\iota\big|_U(\mathbb{C}^2)}$, so $X_P \simeq Bl_p\, \mathbb{P}^2$. The line bundle L_P is equal to $\iota^*(\mathcal{O}(1))$.

3.2.4 Toric Divisors

The T^n action on a toric Kähler manifold X_P can be extended to an action by all of $\mathbb{C}^{*n}$. To define the extension, we just replace $e^{\sqrt{-1}\theta_i}$ by $z_i \in \mathbb{C}^*$ in equation (3.14). We say that a submanifold S of X_P is *invariant* if

$$\{\, z \cdot p \mid z \in \mathbb{C}^{*n} \text{ and } p \in S\,\} = S.$$

A *divisor* on a complex manifold is a formal linear combination

$$D = \sum a_i\, D_i$$

where $a_i \in \mathbb{R}$ and D_i are codimension 1 irreducible subvarieties. A divisor D on a toric Kähler manifold X_P is called *toric* if each submanifold D_i is invariant under the action of $\mathbb{C}^{*n}$.

Lemma 3.2.5. *The invariant codimension* 1 *submanifolds of* X_P *are in one to one correspondence with the facets of* P*.*

Proof. Let D be an invariant, codimension 1 submanifold. The definition of the $\mathbb{C}^{*n}$ action in equation (3.14) shows that locally, in the coordinate chart U associated to a vertex v, D must be given by

$$D\big|_U = \{\, w \mid w_{\sigma(i)} = 0 \,\},$$

for some i such that $v \in F_i$. If $\widetilde{v}$ is a vertex which also lies in F_i, then by Lemma 3.2.3

$$\widetilde{w}_{\widetilde{\sigma}(i)} = w_{\sigma(i)}\, f(w_1, \ldots, \widehat{w_{\sigma(i)}}, \ldots, w_n).$$

Thus, $\widetilde{w}_{\widetilde{\sigma}(i)}$ and $w_{\sigma(i)}$ vanish to the same order, and thus D is given by the vanishing of $\widetilde{w}_{\widetilde{\sigma}(i)}$ in the coordinate chart for each vertex v such that $v \in F_i$. Thus, there is a correspondence between invariant codimension 1 submanifolds of X_P and facets of P. □

An integral, Delzant polytope P has a unique description as the intersection of halfplanes which define its faces:

$$P = \textstyle\bigcap_{i=1}^{M} \{l_i(y) \geq 0\} \qquad \text{for} \qquad l_i(y) = \langle n_i, y \rangle + \lambda_i,$$

where $\lambda_i \in \mathbb{Z}$ and $n_i \in \mathbb{Z}^n$ is a primitive inward pointing normal, meaning its components are relatively prime over $\mathbb{Z}$. The enumeration of the halfspaces can be chosen to agree with the enumeration of the facets of P, so

$$F_i = \{\, y \in P \mid l_i(y) = 0 \,\}.$$

Using the above correspondence, we can enumerate the toric submanifolds $\{D_i\}_{i=1}^M$ of X_P. Since n_i is the primitive inward normal of F_i satisfying $\langle y - v, n_i \rangle = 0$ for all $y \in F_i$, and since $e_{\sigma(i)}$ is the primitive inward normal of $A\, F_i$ satisfying $\langle A\,(y - v), e_{\sigma(i)} \rangle = 0$ for all $y \in F_i$, it follows that

$$n_i = A^T e_{\sigma(i)} \tag{3.18}$$

for A and σ associated to each vertex $v \in F_i$. This identification is crucial for the proof of the following lemmas.

For a divisor D on X_P, we use $\mathcal{O}(D)$ to denote the line bundle associated to D.

Lemma 3.2.6. $\mathcal{O}(\sum_{i=1}^M \lambda_i D_i) = L_P$.

Proof. Consider the section s_0 of L_P associated to the lattice point 0. If $0 \in P$, then s_0 is a holomorphic section, but it is a well defined meromorphic section for any P. If we denote the divisor of a meromorphic section by (s), then $L_P = (s_0)$. The section (s_0) only has zeros and poles on the toric divisors D_i, so it suffices to prove that s_0 vanishes to order λ_i along the divisor D_i. In the chart U associated to a vertex v, s_0 is locally given by

$$s_0\big|_U = w^{-A\,v}.$$

Since $D_i \cap U = \{\, w \mid w_{\sigma(i)} = 0\,\}$, it follows that s_0 vanishes to order $(-A\,v)_{\sigma(i)}$ along D_i. Thus, it suffices to show $(-A\,v)_{\sigma(i)} = \lambda_i$.

Since v lies in F_i, it follows that

$$\langle n_i, v \rangle + \lambda_i = l_i(v) = 0.$$

By equation (3.18), $n_i = A^T e_{\sigma(i)}$, so

$$-\langle A^T e_{\sigma(i)}, v \rangle = \lambda_i.$$

Thus, $(-A\,v)_{\sigma(i)} = \lambda_i$, which concludes the proof of the lemma. □

Lemma 3.2.7. *$\mathcal{O}(\sum_{i=1}^M D_i) = -K_{X_P}$, where $-K_{X_P}$ is the anticanonical bundle of X_P.*

Proof. Let v and $\widetilde{v}$ be two adjacent vertices of P. The proof amounts to computing the transition functions for $\mathcal{O}(\sum D_i)$ and $-K_{X_P}$ on the intersection $U \cap \widetilde{U}$. The defining function for $(\sum D_i)\big|_U$ is $\prod_i w_i$, and likewise the defining function for $(\sum D_i)\big|_{\widetilde{U}}$ is $\prod_j \widetilde{w}_j$. Thus, the transition functions for $\mathcal{O}(\sum D_i)$ are given by

$$g_{U\widetilde{U}} = \frac{\prod_i w_i}{\prod_j \widetilde{w}_j}.$$

The transition function for $-K_{X_P}$ are, by definition, given by $g_{U\widetilde{U}} = \det\left(\dfrac{\partial w_i}{\partial \widetilde{w}_j}\right)$. Differentiating the transition function between w and $\widetilde{w}$ shows

$$\begin{aligned}\frac{\partial w_i}{\partial \widetilde{w}_j} &= \frac{\partial}{\partial \widetilde{w}_j} \prod_k \widetilde{w}_k^{(\widetilde{A}A^{-1})_{ki}} \\ &= (\widetilde{A}A^{-1})_{ji} \frac{w_i}{\widetilde{w}_j}.\end{aligned}$$

Thus, the transition functions for $-K_{X_P}$ are

$$g_{U\widetilde{U}} = \det\left(\frac{\partial w_i}{\partial \widetilde{w}_j}\right) = \det\left((\widetilde{A}A^{-1})^T\right) \frac{\prod_i w_i}{\prod_j \widetilde{w}_j} = \frac{\prod_i w_i}{\prod_j \widetilde{w}_j},$$

where $\det\left((\widetilde{A}A^{-1})^T\right) = 1$ because A and $\widetilde{A}$ are in $Sl_n(\mathbb{Z})$. The transition functions for $\mathcal{O}(\sum D_i)$ and $-K_{X_P}$ are equal, so $\mathcal{O}(\sum_{i=1}^M D_i) = -K_{X_P}$. □

A divisor D is said to be *linearly equivalent to* 0, which we denote by $D \sim 0$ if D is the divisor of some meromorphic function f, which we denote by $D = (f)$.

Lemma 3.2.8. *For any $\tau \in \mathbb{R}^n$, $D_\tau = \sum_{i=1}^M \langle n_i, \tau \rangle D_i \sim 0$.*

Proof. The lemma is clearly true for $\tau = 0$. Thus, it is sufficient to show that $\sum_{i=1}^M \langle n_i, e_j \rangle D_i \sim 0$ for $j = 1, \ldots n$. This linear equivalence to 0 will follow from the fact that

$$\sum_{i=1}^M \langle n_i, e_j \rangle D_i = (z_j),$$

where z_j is the meromorphic extension to X_P of the coordinate z_j on the open orbit $\mathbb{C}^{*n}$. The change of variables $z_j = \prod_k w_k^{A_{kj}}$ on the coordinate chart U implies

$$(z_j)\big|_U = \sum_{k=1}^n A_{kj} D_{\sigma^{-1}(k)}.$$

Equation (3.18) implies $n_{\sigma^{-1}(k)} = A^T e_k$, so

$$\langle n_{\sigma^{-1}(k)}, e_j \rangle = A_{kj}.$$

Thus,

$$(z_j)\big|_U = \sum_{k=1}^n \langle n_{\sigma^{-1}(k)}, e_j \rangle D_{\sigma^{-1}(k)},$$

and since this equation holds in every coordinate chart, it follows that $(z_j) = \sum_{i=1}^M \langle n_i, e_j \rangle D_i$. □

If $D \sim 0$, then the line bundle $\mathcal{O}(D)$ is trivial. Thus, the last three lemmas can be used to show that $L_P = -K_{X_P}$ if and only if there exists $\tau \in \mathbb{R}^n$ such that

$$\sum_{i=1}^M \lambda_i D_i + \sum_{i=1}^M \langle n_i, \tau \rangle D_i = \sum_{i=1}^M D_i. \tag{3.19}$$

In other words, $L_P = -K_{X_P}$ if and only if there exists $\tau \in \mathbb{R}^n$ such that $l_i(\tau) = 1$ for $i = 1, \ldots, M$.

A Kähler manifold X is called *Fano* if $-K_X$ is ample. When a toric variety X is Fano, there is a Delzant polytope P with $\lambda_i = 1$ for every facet such that $X_P = X$ and $L_P = -K_X$. For this reason, we call Delzant polytopes with $\lambda_i = 1$ *Fano* polytopes. In Figures 3.3 through 3.7 we show the Delzant Fano polytopes in dimension 2 with their integer lattice points.

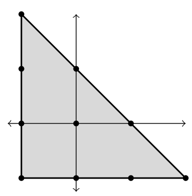

Figure 3.3: $\mathbb{P}^2$

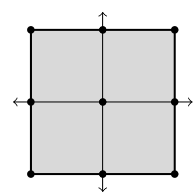

Figure 3.4: $\mathbb{P}^1 \times \mathbb{P}^1$

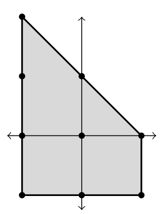

Figure 3.5: $Bl_1\,\mathbb{P}^2$

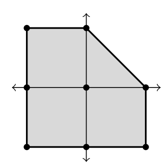

Figure 3.6: $Bl_2\,\mathbb{P}^2$

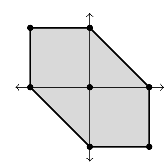

Figure 3.7: $Bl_3\,\mathbb{P}^2$

The last three figures are of $\mathbb{P}^2$ blown up at one, two, and three points respectively. In order for the blowup to still be toric, we can only blow up at the fixed points of the T^2 action, of which there are three on $\mathbb{P}^2$.

3.2.5 Smooth Kähler metrics on X_P

It is a natural assumption, coming from the symplectic point of view, to only consider Kähler metrics which are invariant under the T^n action. Let ω be such a toric Kähler metric. In each coordinate chart U associated to a vertex v there is a Kähler potential $\phi : U \to \mathbb{R}$ such that

$$\omega\big|_U = \sqrt{-1}\partial\overline{\partial}\,\phi.$$

The potential ϕ is a function of the coordinate $w \in U \simeq \mathbb{C}^n$, but on $\mathbb{C}^{*n} \subset U$ we can also consider ϕ as a function of the action-angle coordinates $u \in \mathbb{R}^n$ and $\gamma \in T^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ given by

$$w_i = e^{\frac{u_i}{2}+\sqrt{-1}\gamma_i}.$$

Then ϕ being invariant under the T^n action is equivalent to ϕ being a function of u only, so $\dfrac{\partial\phi}{\partial\gamma_i} = 0$. To compute $\sqrt{-1}\partial\overline{\partial}\,\phi$ in the action-angle coordinates we first compute

$$w_i\,\frac{\partial}{\partial w_i} = \frac{\partial}{\partial u_i} - \sqrt{-1}/2\,\frac{\partial}{\partial\gamma_i}, \tag{3.20}$$

and

$$\frac{dw_i}{w_i} = \frac{1}{2}\,du_i + \sqrt{-1}\,d\gamma_i,$$

and then compute

$$\begin{aligned}\omega\big|_{\mathbb{C}^{*n}\subset U} &= \sqrt{-1}\,\partial\overline{\partial}\,\phi\\ &= \sqrt{-1}\sum_{i,\,j=1}^{n}\frac{\partial^2\phi}{\partial u_i\,\partial u_j}\Big(\frac{1}{2}\,du_i + \sqrt{-1}\,d\gamma_i\Big)\wedge\Big(\frac{1}{2}\,du_j + \sqrt{-1}\,d\gamma_j\Big)\\ &= \sum_{i,\,j=1}^{n}\frac{\partial^2\phi}{\partial u_i\,\partial u_j}\,du_i\wedge d\gamma_j. \end{aligned}\tag{3.21}$$

Since ω is a positive 2-form, it follows that $\nabla^2\phi$ is positive-definite, and thus $\phi(u)$ is convex.

Now we assume that $\omega \in 2\pi\, c_1(L_P)$, where $c_1(L_P)$ is the first Chern class of L_P. Let $\widetilde{\phi}$ be the Kähler potential for ω in the chart $\widetilde{U}$ associated to the vertex $\widetilde{v}$, so

$$\omega\big|_{\widetilde{U}} = \sqrt{-1}\partial\overline{\partial}\,\widetilde{\phi} = \sum \frac{\partial^2\widetilde{\phi}}{\partial\widetilde{u}_i\,\partial\widetilde{u}_j}\, d\widetilde{u}_i \wedge d\widetilde{\gamma}_j,$$

where we are using the action-angle coordinates for $\widetilde{w}_i$ given by

$$\widetilde{w}_i = e^{\frac{\widetilde{u}_i}{2} + \sqrt{-1}\widetilde{\gamma}_i}.$$

Lemma 3.2.9. *Let (X_P, L_P) be a polarized toric Kähler manifold, and let ω be a toric Kähler metric in $2\pi\, c_1(L_P)$. Let U and $\widetilde{U}$ be charts associated to the vertices v and $\widetilde{v}$, and let*

$$\omega\big|_U = \sqrt{-1}\partial\overline{\partial}\,\phi, \text{ and } \omega\big|_{\widetilde{U}} = \sqrt{-1}\partial\overline{\partial}\,\widetilde{\phi}$$

for toric potentials satisfying $\phi(0) = \widetilde{\phi}(0)$. Then,

$$\phi - \widetilde{\phi} = \langle\, \widetilde{A}(\widetilde{v} - v),\, \widetilde{u}\,\rangle = \langle\, A\,(\widetilde{v} - v),\, u\,\rangle. \tag{3.22}$$

Proof. In Lemma 3.2.4 we showed the Kodaira map $\iota : X_P \hookrightarrow \mathbb{P}^{N-1}$ associated to L_P is an embedding. In particular, $\iota^*(\omega_{FS})$ is a toric Kähler metric in $2\pi\, c_1(L_P)$. Since ω and $\iota^*(\omega_{FS})$ are elements of the same cohomology class, there exists a smooth function g on X_P such that

$$\omega - \iota^*(\omega_{FS}) = \sqrt{-1}\partial\overline{\partial}\, g.$$

Both ω and $\iota^*(\omega_{FS})$ are toric, so g can be chosen to be toric as well. If ψ and $\widetilde{\psi}$ are the potentials for $\iota^*(\omega_{FS})$ as in equation (3.15), it follows that

$$\sqrt{-1}\partial\overline{\partial}\,(\psi - \phi - g) = 0, \text{ and } \sqrt{-1}\partial\overline{\partial}\,(\widetilde{\psi} - \widetilde{\phi} - g) = 0.$$

The computation in equation (3.21) shows that if a function f is toric and $\sqrt{-1}\partial\overline{\partial}\, f = 0$, then $f = a + \langle b,\, u\rangle$ in the action-angle coordinates. Thus,

$$\psi - \phi - g = a + \langle b,\, u\rangle, \text{ and } \widetilde{\psi} - \widetilde{\phi} - g = \widetilde{a} + \langle \widetilde{b},\, u\rangle.$$

Both ψ and ϕ are convex function defining smooth metrics. Since these metrics are smooth when $w_i = 0$ it follows that the limit as $u_i \to -\infty$ of both ψ and ϕ must be bounded. If either potential were asymptotic to a linear function with nonzero gradient as $u_i \to -\infty$ then the corresponding metric would be singular. Since g is bounded on X_P it follows that $\psi - \phi - g$ is bounded as $u_i \to -\infty$, so in particular b and $\widetilde{b}$ must equal 0. Thus,

$$\phi - \widetilde{\phi} = \psi - \widetilde{\psi} + a - \widetilde{a}.$$

The computation of $\psi - \widetilde{\psi}$ in equation (3.16) implies

$$\phi - \widetilde{\phi} = \langle\, \widetilde{A}(\widetilde{v} - v),\, \widetilde{u}\,\rangle + a - \widetilde{a},$$

and $\phi(0) = \widetilde{\phi}(0)$ implies $a - \widetilde{a} = 0$. □

Proposition 3.2.10. *Let $(X_P,\, L_P)$ be a polarized toric Kähler manifold associated to an integral Delzant polytope P. If ω is a toric Kähler metric in $2\pi\, c_1(L_P)$ and ϕ is a toric potential for ω in the coordinate chart U associated to the vertex v, then*

$$\overline{\nabla_u\, \phi(\mathbb{R}^n)} = A\,(P - v).$$

Proof. Since ϕ is a smooth Kähler potential for ω in U it follows that

$$\frac{\partial \phi}{\partial w_i} = \frac{1}{w_i}\frac{\partial \phi}{\partial u_i} = e^{-\frac{u_i}{2}-\sqrt{-1}\gamma_i}\frac{\partial \phi}{\partial u_i}$$

is smooth as $w_i \to 0$, or equivalently as $u_i \to -\infty$. Thus, for every fixed γ and fixed u_j for $j \neq i$,

$$\lim_{u_i \to -\infty} e^{-u_i/2}\,\frac{\partial \phi}{\partial u_i} = c \tag{3.23}$$

for some finite constant c which must be positive to guarantee that ϕ is convex. Thus, $\dfrac{\partial \phi}{\partial u_i}$ is asymptotic to $c\,e^{u_i/2}$ as $u_i \to -\infty$. This asymptotic relation is true for all i, so it follows that

$$\nabla_u\,\phi(\mathbb{R}^n) \subset \{\, y \in \mathbb{R}^n \mid y_i > 0 \text{ for all } i\,\}, \text{ and}$$

$$\lim_{u_i \to -\infty} \nabla\phi(u) \in \{\, y \in \mathbb{R}^n \mid y_i = 0 \text{ for some } i\,\}.$$

Thus, $\overline{\nabla_u\,\phi(\mathbb{R}^n)}$ lies in the first orthant and agrees with the first orthant in some neighborhood of the origin. If $\widetilde{\phi}$ is a toric potential for $\widetilde{\phi}$ in $\widetilde{U}$, then the same is true for $\overline{\nabla_{\widetilde{u}}\,\widetilde{\phi}(\mathbb{R}^n)}$. To compare these two sets, we can use equation (3.22) because $\omega \in 2\pi c_1(L_P)$. Since $(\widetilde{A}A^{-1})^T\,\widetilde{u} = u$ it follows that $\nabla_{\widetilde{u}} = \widetilde{A}A^{-1}\,\nabla_u$, and applying this gradient transformation to equation (3.22) shows

$$\widetilde{A}A^{-1}\nabla_u\,\phi = \nabla_{\widetilde{u}}\,\widetilde{\phi} + \widetilde{A}\,(\widetilde{v} - v).$$

Rearranging this formula, and evaluating the gradients on $\mathbb{R}^n$ implies

$$A^{-1}\,\nabla_u\,\phi(\mathbb{R}^n) + v = \widetilde{A}^{-1}\,\nabla_{\widetilde{u}}\,\widetilde{\phi} + \widetilde{v}.$$

Thus, $\overline{\nabla_u\,\phi(\mathbb{R}^n)}$ transitions in the same way as $A\,(P - v)$, and agrees with the boundary of the first orthant when $u_i \to -\infty$, so it follows that

$$\overline{\nabla_u\,\phi(\mathbb{R}^n)} = A\,(P - v),$$

for each vertex v with associated action coordinate u. □

If we use the corresponding Kähler potential ϕ on the open orbit $\mathbb{C}^{*n}$, then it follows that $\nabla\phi(\mathbb{R}^n) = \text{Int } P$. We will use the open orbit to show that the T^n action is Hamiltonian with respect to any Kähler metric. Since the T^n action is given by $\alpha_i \mapsto \alpha_i + \theta_i$ in the open orbit, the infinitesimal action is given by the vector fields $\left\{\frac{\partial}{\partial\alpha_i}\right\}$. In the open orbit $\mathbb{C}^{*n}$ we can compute the contraction of ω along these vector fields to see

$$\begin{aligned}\omega(\partial/\partial\alpha_k, \cdot) &= \sum_{ij} \phi_{ij}\, dx_i \wedge d\alpha_j\, (\partial/\partial\alpha_k, \cdot) \\ &= \sum_i \phi_{ik}\, dx_i = -d\,\phi_k.\end{aligned}$$

This computation shows that the T^n action is Hamiltonian, and the moment map equals $\nabla\phi$ in the open orbit $\mathbb{C}^{*n}$.

The Atiyah-Guillemin-Sternberg Theorem says the image of the moment map equals P, which is the symplectic analogue of Proposition 3.2.10 which says if $\omega \in 2\pi\, c_1(L_P)$, then $\nabla\phi(\mathbb{R}^n) = \text{Int } P$. The converse does not hold in general, meaning $\nabla\phi(\mathbb{R}^n) = \text{Int } P$ is not sufficient for $\omega\big|_{\mathbb{C}^{*n}}$ to extend smoothly to all of X_P. Guillemin [21] proved the following necessary and sufficient conditions for ϕ to be the potential of a smooth Kähler metric on X_P in terms of its Legendre transform ϕ^*.

Theorem 3.2.11. *Suppose $\phi \in C^\infty(\mathbb{R}^n)$ is a convex function. Then $\omega\big|_{\mathbb{C}^{*n}} = \sqrt{-1}\partial\overline{\partial}\,\phi$ defined on the open orbit of X_P extends to a smooth Kähler metric on all of X_P if and only if*

1. $\phi^*(y) - \sum_{i=1}^M l_i(y)\, \log\big(l_i(y)\big) \in C^\infty(P)$

2. *ϕ^* extends continuously to the boundary of P, and for every face F of every codimension $\phi^*\big|_F$ is smooth and strongly convex.*

$C^\infty(P)$ is the class of functions which have a smooth extension to an open neighborhood of P. We call the two conditions in Theorem 3.2.11 the Guillemin boundary conditions for ϕ.

3.2.6 Singular Kähler metrics on X_P

In this subsection we consider Kähler metrics with edge singularities. We refer the reader to the survey article by Rubinstein [42] and the work of Jeffres, Mazzeo, and Rubinstein [23] and the work of Donaldson [18]. Kähler metrics with edge singularities are, roughly speaking, metrics which are locally conical in one of the n complex dimensions. The model for a metric with edge singularities comes from the cone in one complex dimension.

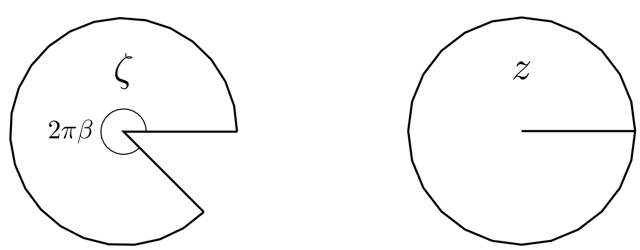

Figure 3.8: The construction of a cone by removing a sector of angle $2\pi(1-\beta)$

In Figure 3.8 we show the construction of a cone singularity. A sector of angle $2\pi(1-\beta)$ is removed from a disc and then the exposed sides are glued. In Figure 3.8 ζ is the Euclidean coordinate and ζ is the coordinate of the metric with a cone singularity. They are related by the map $\zeta = z^\beta$. The Euclidean metric on the cone is given by $\sqrt{-1}\, d\zeta \wedge d\overline{\zeta} = \sqrt{-1}\, \beta^2 |z|^{2(\beta-1)}\, dz \wedge d\overline{z}$. The parameter β is restricted to $0 < \beta \leq 1$, where $\beta = 1$ corresponds to smooth metrics.

More generally, if divisors $\{D_i\}_{i=1}^k$ are locally given by $D_i = \{z_i = 0\}_{i=1}^k$, then the model metric

$$\omega_\beta = \sqrt{-1} \left(\sum_{i=1}^{k} |z_i|^{2(\beta_i - 1)} dz_i \wedge d\overline{z}_i + \sum_{i=k+1}^{n} dz_i \wedge d\overline{z}_i \right) \tag{3.24}$$

has edge singularities of angle β_i along D_i.

A Kähler metrics ω is an edge metric with cone singularities $(\beta_1, \ldots, \beta_k)$ along divisors $(D_1, \ldots, D_k)$ if there is a chart U around every point such that $D_i \cap U = \{z_i = 0\}$, and when $\omega\big|_U = \sqrt{-1}\, g_{ij}\, dz_i \wedge d\overline{z}_j$ there exists a constant $C > 0$ such that

$$C^{-1}\, |z_i|^{2(\beta_i - 1)}\, \delta_{ij} \leq g_{ij} \leq C\, |z_i|^{2(\beta_i - 1)}\, \delta_{ij}$$

as positive semidefinite matrices. Conceptually, ω is an edge metric if it asymptotic to the reference metrics ω_β.

If ω is a metric on X_P with edge singularities along the toric divisors, and ω is in $2\pi\, c_1(L_P)$, then as before $\nabla\phi(\mathbb{R}^n) = \operatorname{Int} P$ for a Kähler potential ϕ in the open orbit. Guillemin's theorem can be adapted to find necessary and sufficient conditions for $\phi : \mathbb{R}^n \to \mathbb{R}$ to be the potential of a toric Kähler metric on X_P with edge singularities of angle β_i along the toric divisors D_i. We reference Datar [14, Proposition 4.2.1] for the proof of the following theorem.

Theorem 3.2.12. *Suppose $\phi \in C^\infty(\mathbb{R}^n)$ is a convex function. Then $\omega := \sqrt{-1}\partial\overline{\partial}\phi$ defined on the open orbit $\mathbb{C}^{*n}$ extends to a metric on X_P with edge singularities of angle β_i along the divisors D_i if and only if the Legendre transform ϕ^* of the potential satisfies*

1. $\phi^*(y) - \sum_{i=1}^{M} \beta_i^{-1} l_i(y) \log\big(l_i(y)\big) \in C^\infty(P)$

2. *ϕ^* extends continuously to the boundary of P, and for every face F of every codimension $\phi^*\big|_F$ is smooth and strongly convex.*

We end the discussion of Kähler metrics with computation on $\mathbb{P}^1$. Let $P = [0,1]$, and let U be the coordinate chart associated to the vertex 0 and $\widetilde{U}$ be the coordinate chart associated to the vertex 1. In the chart U consider a Kähler potential

$$\phi_\beta = \beta^{-1} \log\big(1 + |w|^{2\beta}\big) = \beta^{-1} \log\big(1 + e^{\beta u}\big).$$

Then we define a Kähler metric in the w coordinate by

$$\omega_\beta\big|_U = \sqrt{-1}\partial\overline{\partial}\,\phi_\beta = \sqrt{-1}\frac{\beta\,|w|^{2\beta-2}}{\big(1+|w|^{2\beta}\big)^2}\,dw \wedge d\overline{w},$$

and we can see that ω_β has an edge singularity of angle β at $w = 0$. In the action-angle coordinate ω is given by

$$\omega_\beta\big|_{\mathbb{C}^*\subset U} = \phi_\beta''\,du \wedge d\gamma = \frac{\beta\,e^{\beta u}}{\big(1+e^{\beta u}\big)^2}\,du \wedge d\gamma.$$

The integration

$$\int_{X_P} \omega = \int_{\mathbb{C}^*} \frac{\beta\,e^{\beta u}}{\big(1+e^{\beta u}\big)^2}\,du \wedge d\gamma = 2\pi \int_{\mathbb{R}} \frac{\beta\,e^{\beta u}}{\big(1+e^{\beta u}\big)^2}\,du = 2\pi$$

shows that ω_β lies in $2\pi\,c_1(H)$ where H is the hyperplane bundle on $\mathbb{P}^1$. In the coordinate patch $\widetilde{U}$ associated to the vertex 1 we have coordinate $\widetilde{w} \in \mathbb{C}$ and the action-angle coordinates $\widetilde{u}, \widetilde{\gamma}$ on $\mathbb{C}^*$. The transition functions are $\widetilde{w} = w^{-1}$ and $\widetilde{u} = -u$. Equation (3.22) shows that the Kähler potential $\widetilde{\phi}$ for ω_β in $\widetilde{U}$ is given by

$$\widetilde{\phi}_\beta = \phi_\beta + \langle \widetilde{A}(\widetilde{v} - v), \widetilde{u}\rangle = \beta^{-1}\log\big(1 + e^{-\beta\,\widetilde{u}}\big) - \widetilde{u} = \beta^{-1}\log\big(1 + e^{\beta\,\widetilde{u}}\big).$$

Thus, $\widetilde{\phi}_\beta$ has the same form as ϕ_β, so ω_β has a cone singularity of angle β at the points $\widetilde{w} = 0$ as well. We can show these cone angles agree with the Guillemin boundary conditions of ϕ^*_β by computing the Legendre transform.

The supremum in the definition of the Legendre transform is achieved when

$$y = \phi'_\beta(u) = \frac{e^{\beta\, u}}{1 + e^{\beta\, u}},$$

so we will use the relation $1 + e^{\beta\, u} = \dfrac{1}{1-y}$ to show

$$\begin{aligned} \phi^*_\beta(y) = y\, u - \phi_\beta(u) = y\, \beta^{-1} \log\left(\frac{y}{1-y}\right) - \beta^{-1} \log\left(\frac{1}{1-y}\right) \\ = \beta^{-1}\, y\, \log(y) + \beta^{-1}\,(1-y) \log(1-y). \end{aligned}$$

Theorem 3.2.12 implies ω_β has cone singularities of angle β on the divisors corresponding to the vertices 0 and 1 which are the points $w = 0$ and $\widetilde{w} = 0$, which agrees with our computation of ω_β. In particular, when $\beta = 1$, ω_1 is just the Fubini–Study form on $\mathbb{P}^1$.

3.3 Ricci iteration

The goal of this section is to prove Corollary 1.3.2 about the convergence of the Kähler–Ricci iteration on toric Fano manifolds. In Subsection 3.3.1 we will derive the Kähler–Einstein equation on toric Fano manifolds and recall the known results about existence and uniqueness. In Subsection 3.3.2 we will define the Kähler–Ricci iteration and prove Corollary 1.3.2. In Subsection 3.3.3 we will show that the functions $\mathcal{F}$ and $\mathcal{G}$ of the Monge–Ampère iteration for $h(t) = e^{-t}$ can be used to define the Ding functional and Mabuchi K-Energy on toric Kähler manifolds.

3.3.1 Kähler–Einstein metrics

Let X be a compact Kähler manifold, and let ω be a Kähler metric on X. If ω is given in local coordinates as

$$\omega = \sqrt{-1}\, g_{ij}\, dz_i \wedge d\overline{z}_j,$$

then we can define the Ricci form in local coordinates by

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\, \partial\overline{\partial} \log\big(\det(g_{ij})\big).$$

It is a standard result in Kähler geometry that for any Kähler metric ω,

$$\mathrm{Ric}(\omega) \in 2\pi\, c_1(X) := 2\pi\, c_1(-K_X),$$

and we reference Chapter 1 of Griffiths and Harris [20] for the proof. A metric is *Kähler–Einstein* if it satisfies

$$\mathrm{Ric}(\omega) = \mu\, \omega$$

for some constant $\mu \in \mathbb{R}$. After rescaling ω, we can assume $\mu \in \{-1, 0, 1\}$. These three cases respectively imply the anticanonical bundle $-K_X$ is negative, trivial, or positive, which provides a cohomological restriction to finding Kähler–Einstein metrics. For example, if Σ is a complex torus, then $X = \Sigma \times \mathbb{P}^1$ cannot admit a Kähler–Einstein metric because the anticanonical bundle of $\mathbb{P}^1$ is positive and the anticanonical bundle of Σ is trivial, so the anticanonical bundle of X does not have a sign.

When $-K_X$ is negative or trivial Kähler–Einstein metrics always exist as shown separately by Aubin [2] in the negative case and Yau [47] in the negative and trivial cases. In the positive case, when X if Fano, Kähler–Einstein metrics do not always exist. Partly for this reason, we generalize the definition of smooth Kähler–Einstein metrics to metrics with edge singularities.

Consider the reference metric ω_β defined in equation (3.24). This metric has edge singularities of angle β_i along the divisors $\{z_i = 0\}$. The Ricci form of ω_β is

$$\begin{aligned}\mathrm{Ric}(\omega_\beta) &= -\sqrt{-1}\,\partial\overline{\partial}\,\log\big(\det\big(\,|z_i|^{2(\beta_i-1)}\delta_{ij}\big)\big)\\ &= -\sqrt{-1}\,\partial\overline{\partial}\,\log\left(\prod_{i=1}^{n}|z_i|^{2(\beta_i-1)}\right)\\ &= \sum_{i=1}^{n}(1-\beta_i)\,\sqrt{-1}\,\partial\overline{\partial}\,\log\big(\,|z_i|^2\,\big).\end{aligned}$$

The above equation is 0 when $z_i \neq 0$ for all i, and is not defined when $z_i = 0$. But we can use currents, which are forms with distributional coefficients, to interpret $\sqrt{-1}\,\partial\overline{\partial}\,\log\big(|z_i|^2\big)$. The Poincaré-Lelong formula says

$$\sqrt{-1}\,\partial\overline{\partial}\,\log\big(|z_i|^2\big) = 2\pi\,T_{\{z_i=0\}},$$

where $T_{\{z_i=0\}}$ is the current of integration along the divisor $\{z_i = 0\}$. Since the $\sqrt{-1}\,\partial\overline{\partial}$ operator is essentially the Laplacian, this formula is a variant on the fact that $\pi^{-1}\log\big(|x|\big)$ is the fundamental solution for the Laplacian on $\mathbb{R}^2$. More details on currents of integration can be found in Chapter 3 of Griffiths and Harris [20].

For a codimension 1 submanifold D, we will use the shorthand notation

$$2\pi\,T_D = [D],$$

so that $[D]$ lies in $2\pi\,c_1(\mathcal{O}(D))$. Thus, if ω is a Kähler metric on a compact manifold X with edge singularities of angle β_i along divisors D_i, the singular part of $\mathrm{Ric}(\omega)$ is $\sum_{i=1}^{N}(1-\beta_i)\,[D_i]$. Thus, we define the Kähler–Einstein edge equation to be

$$\mathrm{Ric}(\omega) = \mu\,\omega + \sum_{i=1}^{N}(1-\beta_i)\,[D_i]. \tag{3.25}$$

Since $\mathrm{Ric}(\omega) \in 2\pi\, c_1(X)$, there are restrictions on the cohomology classes of ω and the divisors. We will specifically look at the case when $\omega \in 2\pi\, c_1(X)$ and $\sum_{i=1}^{N}(1-\beta_i)\,[D_i] \in (1-\mu)\,2\pi\, c_1(X)$, for $\mu \in (0,1]$ and $\beta_i \in (0,1]$. This equation reduces to the smooth case when $\alpha = 1$ and $\beta_i = 1$. The results of Aubin–Yau [2] [47] on the existence and uniqueness of Kähler–Einstein metrics were generalized to results on the existence and uniqueness of Kähler–Einstein edge metrics by Jeffres–Mazzeo–Rubinstein [23].

Now we return to toric Kähler manifolds. Let X_P be a toric Fano manifold, so that $L_P = -K_X$, and P is a Fano polytope, meaning

$$P = \textstyle\bigcap_{i=1}^{M}\{\, l_i(y) \geq 0\,\} = \bigcap_{i=1}^{M}\{\, 1 + \langle n_i, y\rangle \geq 0\,\}$$

for n_i, the primitive inward pointing normals to the facets F_i. We will only consider Kähler metrics $\omega \in 2\pi\, c_1(L_P) = 2\pi\, c_1(X)$, and by restricting to the case when ω is toric the edge singularities of ω can only occur on the toric divisors $\{D_i\}$. Lemmas 3.2.8 and 3.2.7 imply $\sum_{i=1}^{M}(1-\beta_i)\,[D_i] \in (1-\mu)\,2\pi\, c_1(X)$ if and only if

$$\sum_{i=1}^{M}(1-\beta_i)\,D_i + \sum_{i=1}^{M}\langle n_i,\, \tau\rangle D_i = \sum_{i=1}^{M}(1-\mu)D_i$$

for some $\tau \in \mathbb{R}^n$. For each i we can solve the above equation for β_i to find

$$\beta_i = \alpha\left(1 + \langle n_i,\, \tau/\mu\rangle\right) = \mu\, l_i(\tau/\mu).$$

Since τ was arbitrary, it follows that the angles $\{\beta_i\}$ must satisfy

$$\beta_i = \mu\, l_i(p) \text{ for some } p \in \mathrm{Int}\ P. \tag{3.26}$$

The restriction $p \in \operatorname{Int} P$ guarantees $\beta_i > 0$. In the next few lemmas we will prove that if equation (3.25) has a solution, then the point p in equation (3.26) must be the barycenter of P.

Lemma 3.3.1. *Let ω be a toric Kähler metric in $2\pi\, c_1(L_P)$ with edge singularities of angle β_i along the toric divisors D_i. In a chart U associated to any vertex v let $\omega|_U = \sqrt{-1}\,\partial\overline{\partial}\,\phi$ for $\phi : \mathbb{C}^n \to \mathbb{R}$ which is a function of only the angle coordinate $u \in \mathbb{R}^n$. Then for all sets $(-\infty, a]^n \subset \mathbb{R}^n$ there exists a constant $c > 0$ such that*

$$c^{-1} \leq \det(\nabla_u^2\,\phi)\, e^{-\sum_{i=1}^n \beta_{\sigma^{-1}(i)}\, u_i} \leq c$$

for $u \in (-\infty, a]^n$, where σ defined in equation (3.8).

Proof. Since ω has singularities of angle $\beta_{\sigma^{-1}(i)}$ along the divisor $w_i = 0$, it follows that ω is asymptotic to the reference metric

$$\omega_\beta = \sqrt{-1}\sum_{i=1}^n |w_i|^{2(\beta_{\sigma^{-1}(i)}-1)} dw_i \wedge d\overline{w}_j.$$

If $u \in (-\infty, a]^n$, then w lies in the compact set $\{\, w \mid |w_i|^2 \leq e^a \,\}$ so there is a constant $c > 0$ such that

$$c^{-1}\, e^{(\beta_{\sigma^{-1}(i)}-1)u_i}\,\delta_{ij} \leq \frac{\partial^2\phi}{\partial w_i\,\partial\overline{w}_j} \leq c\, e^{(\beta_{\sigma^{-1}(i)}-1)u_i}\,\delta_{ij} \tag{3.27}$$

as matrices. Equation (3.20) implies

$$\frac{\partial^2\phi}{\partial w_i\,\partial\overline{w}_j} = w_i^{-1}\overline{w}_j^{-1}\frac{\partial^2\phi}{\partial u_i\,\partial u_j},$$

so in particular

$$\det\left(\frac{\partial^2\phi}{\partial w_i\,\partial\overline{w}_j}\right) = \det(\nabla_u^2\phi)\prod_{i=1}^n |w_i|^2. \tag{3.28}$$

Taking determinants of the matrices in equation (3.27) implies

$$c^{-n} \prod e^{(\beta_{\sigma^{-1}(i)}-1)u_i} \leq \det(\nabla_u^2 \phi) \prod |w_i|^{-2} \det(\nabla_u^2 \phi) \leq c^n \prod e^{(\beta_{\sigma^{-1}(i)}-1)u_i}.$$

Since $|w_i|^2 = e^{u_i}$, it follows that

$$c^{-n} \leq e^{-\sum_{i=1}^n \beta_{\sigma^{-1}(i)}\, u_i} \det(\nabla_u^2\, \phi) \leq c^n.$$

□

Lemma 3.3.2. *Let X_P be a toric Fano manifold, and let ω and η be toric Kähler metrics in $2\pi\, c_1(L_P) = 2\pi\, c_1(X_P)$ such that ω has edge singularities of angle β_i along the toric divisors D_i. Assume ω and η satisfy*

$$\mathrm{Ric}(\omega) = \mu\, \eta + \sum_{i=1}^{M} (1-\beta_i)\, [D_i]$$

for some $\mu \in (0,1]$. If $\omega = \sqrt{-1}\, \partial\overline{\partial}\, \phi$ and $\eta = \sqrt{-1}\, \partial\overline{\partial}\, \psi$ in the chart U associated to the vertex v of P, then

$$\det(\nabla_u^2\, \phi) = e^{-\mu\, \psi + \sum_{i=1}^n \beta_{\sigma^{-1}(i)} u_i + c}$$

for some constant c.

Proof. Since ω and η are toric, their potentials are functions of the action variable u only. When $u \in \mathbb{R}^n$, then in the w coordinate $w_i \neq 0$ for all i. Thus, ω and η are smooth potentials in the set $\{\, u \in \mathbb{R}^n \,\}$, where they satisfy

$$\mathrm{Ric}(\omega)\big|_{\{\, u \in \mathbb{R}^n \,\}} = \mu\, \eta\big|_{\{\, u \in \mathbb{R}^n \,\}}.$$

In terms of potentials, the previous equation implies

$$\sqrt{-1}\, \partial\overline{\partial} \left(\log\left(\det\left(\frac{\partial^2 \phi}{\partial w_i\, \partial \overline{w}_j} \right)\right) + \mu\, \psi \right) = 0. \tag{3.29}$$

Equation (3.28) implies

$$\log\left(\det\left(\frac{\partial^2\phi}{\partial w_i\,\partial\overline{w}_j}\right)\right) = \log\big(\det(\nabla_u^2\phi)\big) + \sum_{i=1}^{n}\log\big(|w_i|^2\big),$$

and since $\partial\overline{\partial}\log\big(|w_i|^2\big) = 0$ when $w_i \neq 0$, equation (3.29) simplifies to

$$\sqrt{-1}\,\partial\overline{\partial}\,\big(\log\big(\det(\nabla_u^2\phi)\big) + \mu\,\psi\big) = 0.$$

We can add on any linear function of u, and still

$$\sqrt{-1}\,\partial\overline{\partial}\,\Big(\log\big(\det(\nabla_u^2\phi)\big) - \sum_{i=1}^{n}\beta_{\sigma^{-1}(i)}u_i + \mu\,\psi\Big) = 0.$$

Since the above function only depends on the variable u, it follows that its matrix of second partial derivatives is 0, so it must equal some affine function in u. If we can show that it is bounded on the set $(-\infty, a]^n$ for some a, then it must be constant. Equation 3.3.1 implies $\log\big(\det(\nabla_u^2\phi)\big) - \sum_{i=1}^{n}\beta_{\sigma^{-1}(i)}u_i$ is bounded on $(-\infty, a]$ for any a. Since η is a metric in $2\pi\,c_1(L_P)$, and ψ is a potential for η in the coordinate chart associated to v, it follows from equation (3.23) that ψ is asymptotic to $e^{-u_i/2}$ when $u_i \to -\infty$. In particular, ψ is bounded on $(-\infty, a]^n$. Thus,

$$\log\big(\det(\nabla_u^2\phi)\big) - \sum_{i=1}^{n}\beta_{\sigma^{-1}(i)}u_i + \mu\,\psi = c$$

for some constant c, which concludes the proof. □

Proposition 3.3.3. *Let X_P be a toric Fano manifold associated to a toric polytope $P = \bigcap_{i=1}^{M}\{\,l_i(y) \geq 0\,\} = \bigcap_{i=1}^{M}\{\,1 + \langle n_i, y\rangle \geq 0\,\}$, and let $\omega \in 2\pi\,c_1(L_P)$ be a metric with edge singularities of angle β_i along the toric divisors D_i. If ω satisfies*

$$\mathrm{Ric}(\omega) = \mu\,\omega + \sum_{i=1}^{M}(1-\beta_i)[D_i]$$

for $\mu \in (0, 1]$, then $\beta_i = \mu\,l_i(P_c)$ where P_c is the barycenter of P.

Proof. Equation (3.26) implies $\beta_i = \mu\, l_i(p)$ for some $p \in \operatorname{Int} P$. Thus, for ϕ a potential for ω in the chart U associated to v, Lemma 3.3.2 implies

$$\det(\nabla_u^2 \phi) = e^{-\mu\,\phi + \mu \sum_{i=1}^n l_{\sigma-1(i)}(p) u_i + c}$$

for some constant c. Since $\nabla\phi(\mathbb{R}^n) = A(P - v)$ the change of variables $y = \nabla\phi(u)$ implies

$$\begin{aligned}\int_{A(P-v)} y_k\, d\lambda &= \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial u_k} e^{-\mu\,\phi + \mu \sum_{i=1}^n l_{\sigma-1(i)}(p) u_i + c}\, d\lambda \\ &= -\mu^{-1} \int_{\mathbb{R}^n} \frac{\partial}{\partial u_k} \Big(e^{-\mu\,\phi + \mu \sum_{i=1}^n l_{\sigma-1(i)}(p) u_i + c} \Big)\, d\lambda \\ &\quad + l_{\sigma^{-1}(k)}(p) \int_{\mathbb{R}^n} e^{-\mu\,\phi + \mu \sum_{i=1}^n l_{\sigma-1(i)}(p) u_i + c}\, d\lambda.\end{aligned}$$

Integrating by parts shows that the first integral in the last line equals 0, and the second integral just equals $\lambda\big(A(P - v)\big)$. Thus,

$$\frac{\int_{A(P-v)} y_i\, d\lambda}{\int_{A(P-v)} 1\, d\lambda} = l_{\sigma^{-1}(i)}(p)$$

which shows that the ith component of the barycenter of $A(P - v)$ equals $l_{\sigma^{-1}(i)}(p)$. If we use P_c to denote the barycenter of P, then

$$e_i^T\, A(P_c - v) = l_{\sigma^{-1}(i)}(p). \tag{3.30}$$

Equation (3.18) implies $e_i^T\, A(p - v) = l_{\sigma^{-1}(i)}(p)$ for any point $p \in \mathbb{R}^n$, and since equation (3.30) holds for all u_i on every chart U, it follows that $p = P_C$. Thus, equation (3.26) says $\beta_i = \mu\, l_i(P_c)$. □

Proposition 3.3.3 and the restriction $\beta_i \in (0, 1]$ implies

$$0 < \mu \le \min_{i=1,\ldots,M} \left\{ l_i(P_c)^{-1} \right\} \le 1.$$

Li [25], relying on analytic results in Donaldson [18] and Jeffres–Mazzeo–Rubinstein [23], proved the quantity $\min_{i=1,\ldots,M}\left\{ l_i(P_c)^{-1} \right\}$ is equal to the greatest lower Ricci bound of X_P. This fact was used by Song and Wang [43] to prove the following theorem:

Theorem 3.3.4. *Let X_P be a toric Fano manifold associated to a toric polytope $P = \bigcap_{i=1}^M \{ l_i(y) \geq 0 \} = \bigcap_{i=1}^M \{ 1 + \langle n_i, y\rangle \geq 0 \}$. Then for*

$$\mu \in \big(0, \min_{i=1,\ldots,M} \{ l_i(P_c)^{-1} \} \big]$$

there is a toric Kähler metric $\omega \in 2\pi\, c_1(L_P)$ with edge singularities of angle $\beta_i = \mu\, l_i(P_c)$ solving

$$\mathrm{Ric}(\omega) = \mu\,\omega + \sum_{i=1}^M (1-\beta_i)[D_i].$$

3.3.2 Convergence of the Kähler–Ricci iteration

The Kähler–Ricci iteration on a Fano manifold X is a series of prescribed curvature problems

$$\mathrm{Ric}(\omega_{i+1}) = \omega_i, \text{ for } \omega_{i+1} \in 2\pi\, c_1(X).$$

The definition can be extended to metrics with edge singularities $\beta_i \in (0,1]$ along divisors D_i by

$$\mathrm{Ric}(\omega_{i+1}) = \mu\,\omega_i + \sum_{i=1}^M (1-\beta_i)[D_i] \text{ for } \omega_{i+1} \in 2\pi\, c_1(X),$$

and we recover the smooth definition when $\mu = 1$ and $\beta_i = 0$ for all i.

Let X_P be a toric Fano manifold with an associated Fano polytope

$$P = \textstyle\bigcap_{i=1}^M \{ l_i(y) \geq 0 \} = \bigcap_{i=1}^M \{ 1 + \langle n_i, y\rangle \geq 0 \}$$

where n_i are the inward pointing normals to the facets F_i which are primitive over $\mathbb{Z}$. Let μ be a constant in $\big(0, \min\{\, l_i(P_c)^{-1} \mid i = 1, \dots, n\,\}\big]$, and let $\beta_i = \mu\, l_i(P_c)$ where P_c is the barycenter of P. We will consider the Kähler–Ricci iteration

$$\mathrm{Ric}(\omega_{i+1}) = \mu\, \omega_i + \sum_{i=1}^{M} (1 - \beta_i)\, [D_i] = \mu\, \omega_i + \sum_{i=1}^{M} (1 - \mu\, l_i(P_c))\, [D_i] \tag{3.31}$$

where $\omega_i \in 2\pi\, c_1(L_P)$ have edge singularities of angle β_i along the toric divisors D_i.

Lemma 3.3.5. *Let $\{\omega_i\}$ be a sequence of metrics in $2\pi\, c_1(L_P)$ solving equation (3.31), and let $\omega_i = \sqrt{-1}\,\partial\bar{\partial}\,\phi_i$ in the open orbit of X_P for potentials ϕ_i which are functions of only the action variable x and satisfy $\nabla_x \phi_i(\mathbb{R}^n) = \mathrm{Int}\, P$. Then the functions ϕ_i satisfy*

$$\det(\nabla_x^2\, \phi_{i+1}) = e^{-\mu(\phi_i - \langle x, P_c\rangle) + c}$$

for some constant c.

Proof. Let $\{\psi_i\}$ be potentials for $\{\omega_i\}$ in any chart associated to a vertex v. The potentials ψ_i are a function of the action variable u, where $x = A^T\, u$ as in equation (3.10). Since $\omega_i \in 2\pi\, c_1(L_P)$, the potentials ψ_i are related to ϕ_i by

$$\psi_i = \phi_i + \langle x, v\rangle, \tag{3.32}$$

which is equivalent to equation (3.22) when we transition from the open orbit rather than another chart $\widetilde{U}$. Lemma 3.3.2 shows that the potentials ψ_i satisfy

$$\det(\nabla_u^2\, \psi_{i+1}) = e^{-\mu\, \psi_i + \sum_{k=1}^{n} \beta_{\sigma^{-1}(k)} u_k + c} \tag{3.33}$$

for some constant c. Since $\{\omega_i\}$ solves equation (3.31) we know that $\beta_k = \mu\, l_k(P_c)$. We can compute

$$\beta_{\sigma^{-1}(k)} = \mu\, l_{\sigma^{-1}(k)}(P_c) = \mu(\langle n_{\sigma^{-1}(k)},\, P_c\rangle + 1).$$

Since v lies in the facet $F_{\sigma^{-1}(k)}$ it follows that $l_{\sigma^{-1}(k)}(v) = 0$, or equivalently $1 = -\langle n_{\sigma^{-1}(k)},\, v\rangle$. Plugging $-\langle n_{\sigma^{-1}(k)},\, v\rangle$ in for 1 in the above equation implies

$$\beta_{\sigma^{-1}(k)} = \mu\,\langle\, n_{\sigma^{-1}(k)},\, P_c - v\,\rangle.$$

Equation (3.18) implies $n_{\sigma^{-1}(k)} = A^T\, e_k$, so

$$\beta_{\sigma^{-1}(k)} = \mu\,\langle\, e_k,\, A\,(P_c - v)\,\rangle.$$

Thus,

$$\sum_{k=1}^{n} \beta_{\sigma^{-1}(k)} u_k = \mu\,\langle u,\, A\,(P_c - v)\,\rangle.$$

Plugging the previous equation back into equation (3.33) implies

$$\det(\nabla_u^2\, \psi_{i+1}) = e^{-\mu\,(\psi_i - \langle u,\, A\,(P_c - v)\,\rangle) + c}.$$

The change of action coordinates $x = A^T\, u$ implies $\nabla_u^2 f = A\,\nabla_x^2 f\, A^T$, so in particular $\det(\nabla_u^2 f) = \det(\nabla_x^2 f)$ since $A \in Sl_n\,\mathbb{Z}$. Now we use relation (3.32) to show the above equation implies

$$\begin{aligned}\det(\nabla_x^2\, \phi_{i+1}) &= e^{-\mu\,(\psi_i - \langle u,\, A\,(P_c - v)\,\rangle) + c} \\ &= e^{-\mu\,(\phi_i + \langle x, v\rangle - \langle x,\, (P_c - v)\,\rangle) + c} \\ &= e^{-\mu(\,\phi_i - \langle x,\, P_c\rangle\,) + c},\end{aligned}$$

as desired. □

Now that we have a convenient description of the Kähler–Ricci iteration in the open orbit we can prove Corollary 1.3.2

Proof. In the open orbit ω_i is given in terms of the potentials

$$\omega_i = \sqrt{-1}\,\partial\overline{\partial}\,\phi_i$$

where ϕ_i are a function of only the action variable x, and $\nabla\phi_i(\mathbb{R}^n) = \text{Int } P$. Lemma 3.3.5 implies the ϕ_i satisfy the equations

$$\det(\nabla^2\,\phi_{i+1}) = e^{-\mu(\phi_i - \langle x, P_c\rangle)+c}$$

for some constant c. We introduce the auxiliary functions

$$\varphi_i(x) = \mu\,(\phi_i(x) - \langle x,\ P_c\rangle\,)$$

so $\nabla\varphi_i(\mathbb{R}^n) = \text{Int }\mu\,(P - P_c)$ has barycenter at the origin. The sequence $\{\varphi_i\}$ solves

$$\det(\nabla^2\varphi_{i+1}) = (\mu^n\,e^c)\,e^{-\varphi_i},$$

and we can remove reference to the constant c by normalizing

$$\frac{\det(\nabla^2\varphi_{i+1})}{\lambda\big(\mu\,(P - P_c)\big)} = \frac{e^{-\varphi_i}}{\|e^{-\varphi_i}\|_1}.$$

Adding a constant to φ_i keeps both sides of the above equation the same, so we can assume the φ_i satisfy $\int_{\mu(P-P_c)} \varphi_i^*\,d\lambda = 0$. Since the barycenter of $\mu(P - P_c)$ lies at the origin, Theorem 1.3.1 implies there exist constants $\{a_i\}$ such that the sequence $\{\widetilde{\varphi}_i(x) = \varphi_i(x + a_i)\}$ converges smoothly on compact subsets of $\mathbb{R}^n$ to a function φ which solves

$$\begin{cases} \dfrac{\det(\nabla^2\varphi)}{\lambda\big(\mu\,(P - P_c)\big)} = \dfrac{e^{-\varphi}}{\|e^{-\varphi}\|_1} \\ \nabla\varphi(\mathbb{R}^n) = \text{Int }\mu\,(P - P_c) \\ \int_A \varphi^*\,d\lambda = 0. \end{cases}$$

If we define $\phi(x) = \mu^{-1}\,\varphi(x) + \langle x,\, P_c\rangle$, then $\{\widetilde{\phi}_i(x) := \phi_i(x+a_i)\}$ converges smoothly on compact subsets of $\mathbb{R}^n$ to ϕ_i. Also $\nabla\phi(\mathbb{R}^n) = \mathrm{Int}\ P$, and ϕ solves

$$\det(\nabla^2\,\phi) = e^{-\mu(\phi - \langle x,\, P_c\rangle) + c}$$

for some constant c. If we define $\omega = \sqrt{-1}\,\partial\overline{\partial}\,\phi$ on the open orbit, then

$$\mathrm{Ric}(\omega)\big|_{\mathbb{C}^{*n}} = \mu\,\omega\big|_{\mathbb{C}^{*n}}.$$

If g_i is the unique automorphism of X_P which is given by $g(z_i) = e^{a_i/2}\,z_i$ on the open orbit, then the metrics

$$\widetilde{\omega}_i = \sqrt{-1}\,\partial\overline{\partial}\,\widetilde{\phi}_i = g_i^*(\omega_i)$$

converge smoothly to ω on compact subsets of the open orbit. It remains to show that ω extends to a metric with edge singularities of angle β_i along D_i. The potential ϕ for ω solves

$$\det(\nabla^2\phi) = e^{-\mu(\phi - \langle x,\, P_c\rangle) + c},$$

so, by the results of Jeffres–Mazzeo–Rubinstein [23] and Song and Wang [43], ω is a toric Kähler metric $\omega \in 2\pi\,c_1(L_P)$ with edge singularities of angle $\beta_i = \mu\,l_i(P_c)$ solving

$$\mathrm{Ric}(\omega) = \mu\,\omega + \sum_{i=1}^{M}(1-\beta_i)[D_i].$$

□

3.3.3 Ding Functional and Mabuchi K-Energy

In this section we will consider only smooth Kähler metrics, even though everything extends easily to the edge case as shown in Rubinstein [42]. For the definitions of all the functionals in this section we refer to equations (5) and (8) in [12]. Let X_P be a smooth toric Fano manifold, and let ω be a reference metric in $2\pi\, c_1(X_P)$. If $\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ is another Kähler metric, then the *Aubin-Mabuchi functional* $\mathrm{AM}(\varphi)$ is defined implicitly by

$$\frac{d}{dt}\Big|_{t=0} \mathrm{AM}(\varphi + t\, v) = \frac{1}{V}\int_{X_P} v\ \mathrm{MA}_{\mathbb{C}}(\varphi), \quad \mathrm{AM}(0) = 0,$$

where $\mathrm{MA}_{\mathbb{C}}(\varphi) = (\sqrt{-1}\partial\overline{\partial}\,\varphi)^n$ is the complex Monge-Ampere operator, and $V = \int_{X_P} \omega^n$ is the volume of X_P. Since X_P is Fano, we can choose ω so that $\mathrm{Ric}(\omega)$ is a positive form. If φ is a toric invariant function, then

$$\mathrm{MA}_{\mathbb{C}}(v) = (\sqrt{-1}\partial\overline{\partial}\, v)^n = (v_{ij}\, dx_i \wedge d\alpha_j)^n = \det(\nabla^2 v)\, dx_1 \wedge \cdots \wedge dx_n \wedge d\alpha_1 \wedge \cdots \wedge d\alpha_n.$$

Since X_P is toric, we can write any metric in terms of its potential in the open orbit $\mathbb{C}^{*n}$. We denote the reference metric ω and and any other metric ω_φ by

$$\omega = \sqrt{-1}\partial\overline{\partial}\,\psi \quad \omega_\varphi = \sqrt{-1}\partial\overline{\partial}\,(\psi + \varphi) = \sqrt{-1}\partial\overline{\partial}\,\phi.$$

It is convenient to define the Aubin-Mabuchi functional on the Kähler potential ϕ without regard to the reference metric ω. This simplification does not affect the definition because we assume $\mathrm{AM}(0) = 0$.

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathrm{AM}(\phi + t\, v) &= \frac{1}{V}\int_{X_P} v\ \det(\nabla^2 \phi) = dx_1 dx_n \wedge d\alpha_1 \wedge \cdots \wedge d\alpha_n \\ &= \frac{1}{(2\pi)^n\, \lambda(P)} (2\pi)^n \int_{\mathbb{R}^n} v\ \det(\nabla^2\phi)\, dx_1 \wedge \cdots \wedge dx_n \\ &= \frac{d}{dt}\Big|_{t=0} \Big(-\lambda(P)^{-1} \int_P \phi_t^*(y)\, dy_1 \wedge \cdots \wedge dy_n \Big), \end{aligned}$$

where we used the change of variables $y = \nabla\phi(x)$ and the first variation formula for the Legendre transform in the last equality. We refer the reader to [40, pg. 85] for a proof and exposition of the variations of the Legendre transform. Thus, for a convex function ϕ, the Aubin-Mabuchi functional of the metric $\sqrt{-1}\partial\overline{\partial}\,\phi$ is given by

$$\mathrm{AM}(\phi) = -\frac{1}{\lambda(P)}\int_P \phi^*\,d\lambda.$$

Now we return to the reference metric ω. Define $f_\omega \in C^\infty(X_P)$ to be the unique function satisfying

$$\sqrt{-1}\partial\overline{\partial}\, f_\omega = \mathrm{Ric}(\omega) - \omega, \qquad \int_{X_P} e^{f_\omega}\omega^n = V.$$

Let $\omega = \sqrt{-1}\partial\overline{\partial}\,\psi$ in the open orbit coordinates, and assume ω is toric invariant. We can add a constant to ψ so that

$$f_\omega = -\log\big(\det(\nabla^2\psi)\big) - \psi + \langle a, x\rangle$$

for some constant $a \in \mathbb{R}^n$, which we will show must equal 0. Since ω was chosen so that $\mathrm{Ric}(\omega)$ is a positive form in $2\pi\, c_1(X_P)$, it follows that both $-\log\big(\det(\nabla^2\psi)\big)$ and ψ are potentials for Kähler metrics in the same class. In particular, they both have gradient images equal to Int P. If $a \neq 0$, then f_ω is unbounded because the gradient of f_ω equals a as $|x| \to \infty$. The unboundedness of f_ω contradicts the fact that f_ω is a smooth function on X_P, so $a = 0$.

The *Ding functional* $\mathcal{D}(\omega_\varphi)$ is given by

$$\mathcal{D}(\omega_\varphi) = -\,\mathrm{AM}(\varphi) - \log\frac{1}{V}\int_{X_P} e^{f_\omega-\varphi}\,\omega^n.$$

Since $\omega_\varphi = \sqrt{-1}\partial\overline{\partial}\,(\psi{+}\varphi) = \sqrt{-1}\partial\overline{\partial}\,\phi$ we will write $\mathcal{D}$ as a function of the potential

ϕ,

$$\begin{aligned}\mathcal{D}(\phi) &= \lambda(P)^{-1}\int_P \phi^*\,d\lambda - \log\frac{1}{\lambda(P)}\int_{\mathbb{R}^n} e^{-\log\,\det(\nabla^2\psi)-\psi-(\phi-\psi)}\,\det(\nabla^2\psi)\,d\lambda\\ &= \lambda(P)^{-1}\int_P \phi^*\,d\lambda - \log\frac{1}{\lambda(P)}\int_{\mathbb{R}^n} e^{-\phi}\,d\lambda\\ &= \lambda(P)^{-1}\int_P \phi^*\,d\lambda - \log\Big(\int_{\mathbb{R}^n} e^{-\phi}\,d\lambda\Big) + \log\big(\lambda(P)\big).\end{aligned}$$

Thus, on the toric variety X_P the Ding functional is related to the functional $\mathcal{F}$ by

$$\mathcal{D}(\phi) = \mathcal{F}(\phi) + \lambda(P)^{-1}\int_P \phi^*\,d\lambda + \log\big(\lambda(P)\big).$$

The *Mabuchi K-energy* is given by

$$\mathcal{K}(\phi) = \lambda(P)^{-1}\Big(\int_{\mathbb{R}^n}\phi\,\det(\nabla^2\phi)\,d\lambda + \int_{\mathbb{R}^n}\det(\nabla^2\phi)\,\log\big(\det(\nabla^2\phi)\big)\,d\lambda + \int_P \phi^*\,d\lambda\Big),$$

as is shown in Section 4.2 of Berman and Berndttson [4] and Section 2.4 of Donaldson [17]. In Section 3.1 we showed

$$\mathcal{G}\Big(\frac{\mathrm{MA}(\varphi)}{\lambda(P)}\Big) = -\int_{\mathbb{R}^n}\frac{\det(\nabla^2\phi)}{\lambda(P)}\,\log\Big(\frac{\det(\nabla^2\phi)}{\lambda(P)}\Big)\,d\lambda.$$

Thus,

$$\mathcal{K}(\phi) = \Big\langle \phi,\,\frac{\mathrm{MA}(\phi)}{\lambda(P)}\Big\rangle - \mathcal{G}\Big(\frac{\mathrm{MA}(\phi)}{\lambda(P)}\Big) + \log\big(\lambda(P)\big) + \int_P \phi^*\,d\lambda.$$

If $\{\phi_i\}$ are potentials for $\{\omega_i\}$ solving the Ricci iteration (3.31), then Lemma 2.2.8 implies

$$\mathcal{F}(\phi_i) \geq \Big\langle \phi_{i+1},\,\frac{\mathrm{MA}(\phi_{i+1})}{\lambda(P)}\Big\rangle - \mathcal{G}\Big(\frac{\mathrm{MA}(\phi_{i+1})}{\lambda(P)}\Big) \geq \mathcal{F}(\phi_{i+1}).$$

Adding $\int_P \phi^*\,d\lambda + \log\big(\lambda(P)\big)$ to every term we get for X_P a toric Kähler manifold, along the Ricci iteration

$$\mathcal{D}(\phi_i) \geq \mathcal{K}(\phi_{i+1}) \geq \mathcal{D}(\phi_{i+1}).$$

This inequality is a special case of a more general fact from Kähler geometry, that along the Ricci iteration

$$\mathcal{D}(\omega_i) \geq \mathcal{K}(\omega_{i+1}) \geq \mathcal{D}(\omega_{i+1}),$$

which is shown in Rubinstein [41].

Chapter 4: Affine iteration

Monge–Ampère second boundary value problems appear in affine differential geometry as the equation of affine spheres. In this chapter we show the Monage-Ampère iteration (1.7) with $h(t) = t^{-(n+2)}$ can be interpreted as a sequence of prescribed affine normal problems for affine immersions, and that the iteration converges to an affine sphere. We begin in Section 4.1 by proving Theorem 1.3.3 which shows the normalized Monge–Ampère iteration with $h(t) = t^{-(n+p+1)}$ for $p > 0$ converges.

In Section 4.2 we discuss the background regarding affine immersions which is necessary for the geometric interpretation of the Monge–Ampère iteration with $h(t) = t^{-(n+2)}$. In Section 4.3 we define a subclass of affine immersions called Legendre graph immersions, and we show that we can solve a prescribed affine normal problem for these immersions. Then we define the affine iteration as a sequence of prescribed affine normal problems on Legendre graph immersions, and prove prove Corollary 1.3.4 establishing its convergence.

4.1 Monge–Ampère iteration for $h(t) = t^{-(n+p+1)}$

To prove Theorem 1.3.3 about the convergence of the Monge–Ampère iteration for $h(t) = t^{-(n+p+1)}$ when $p > 0$ we must verify Hypotheses 1.2.1 and apply Theorem

1.2.2.

Hypothesis (B1):

When $p > 0$ Hypothesis (B1) is satisfied trivially.

Hypothesis (B2):

Klartag proved [24, Theorem 3.10], that the second boundary problem

$$\begin{cases} \dfrac{\det(\nabla^2\varphi)}{\lambda(A)} = \dfrac{\varphi^{-(n+p+1)}}{\|\varphi^{-(n+p+1)}\|_1} \\ \nabla\varphi(\mathbb{R}^n) = A \end{cases} \tag{4.1}$$

has smooth, strictly convex solutions, unique up to translations $\varphi(x-a)$ and dilations $\varphi_a(x) = a\,\varphi(x/a)$ if and only if A has barycenter at the origin.

Lemma 4.1.1. *If φ is a smooth, convex solution to equation (4.1) for $p > 0$, then $\int_A \varphi^*\,d\lambda < 0$.*

Proof. The Legendre transform of φ is given by $\varphi^*(\nabla\varphi(x)) = \langle x, \nabla\varphi(x)\rangle - \varphi(x)$. Thus, the change of variables $y = \nabla\varphi(x)$ implies

$$\int_A \varphi^*\,d\lambda = \int_{\mathbb{R}^n} \big(\langle x, \nabla\varphi(x)\rangle - \varphi(x)\big)\,\det(\nabla^2\varphi)\,d\lambda.$$

Since φ satisfies equation (4.1),

$$\begin{aligned} &\int_{\mathbb{R}^n} \big(\langle x, \nabla\varphi(x)\rangle - \varphi(x)\big)\,\det(\nabla^2\varphi)\,d\lambda \\ &\qquad\qquad = C\int_{\mathbb{R}^n} \langle x,\, \nabla\varphi(x)\rangle\,\varphi(x)^{-(n+p+1)}\,d\lambda - C\int_{\mathbb{R}^n} \varphi^{-(n+p)}\,d\lambda \end{aligned}$$

for some constant $C > 0$. We can simplify the first integral on the second line by an integration by parts. We restrict to the ball of radius R to show

$$\begin{aligned}\int_{B_R} \langle x, \nabla\varphi(x)\rangle \, \varphi^{-(n+p)} \, d\lambda &= \frac{-1}{n+p} \int_{B_R} \langle x, \nabla(\varphi^{-(n+p)})\rangle \, d\lambda \\ &= \frac{n}{n+p} \int_{B_R} \varphi^{-(n+p)} \, d\lambda - \frac{1}{n+p} \int_{\partial B_R} \varphi^{-(n+p)} \left\langle x, \frac{x}{|x|} \right\rangle dS\end{aligned}$$

where dS is the surface measure on the sphere of radius R. Since φ is positive and convex there are positive constants c and r such that $\varphi(x) \geq c + r\,|x|$. Thus, we can bound the integral

$$\int_{\partial B_R} \varphi^{-(n+p)} \left\langle x, \frac{x}{|x|} \right\rangle dS \leq (c + r\,R)^{-(n+p)} \, R \, R^{n-1} \, \omega_n.$$

This bound goes to 0 as $R \to \infty$ because $p > 0$. Thus, we know

$$C \int_{\mathbb{R}^n} \langle x, \, \nabla\varphi(x)\rangle \, \varphi(x)^{-(n+p+1)} \, d\lambda = C \, \frac{n}{n+p} \int_{\mathbb{R}^n} \varphi^{-(n+p)} \, d\lambda.$$

Returning to the integral of the Legendre transform we see

$$\int_A \varphi^* \, d\lambda = -C \, \frac{p}{n+p} \int_{\mathbb{R}^n} \varphi^{-(n+p)} \, d\lambda < 0.$$

□

The Legendre transform of the dilation satisfies

$$\varphi_a^*(y) = \sup_{x \in \mathbb{R}^n} \{ \, \langle x, y\rangle - \varphi_a(x) \, \} = a \sup_{x \in \mathbb{R}^n} \{ \, \langle x/a, y\rangle - \varphi(x/a) \, \} = a \, \varphi^*(y).$$

If φ is a solution to equation (4.1) then $\int_A \varphi^* \, d\lambda < 0$ by Lemma 4.1.1, so specifying $\int_A \varphi^* \, d\lambda = -\tau < 0$ is equivalent to specifying a unique dilation. Thus the second boundary problem (1.6) has smooth, strictly convex solutions unique up to translations.

Hypothesis (B3):

We define $g(s,t) = s/t$. To verify Hypothesis (B3) we note g is decreasing in t when s is positive, and $g(s, g(s,t)) = s/(s/t) = t$.

Hypothesis (B4):

We let $s = n + p$ for notational convenience and define $\mathcal{F}_s$ to be the functional $\mathcal{F}$ associated to $h(t) = t^{-(s+1)}$. First, we compute $\mathcal{F}_s$. We integrate $H(t) = \int_t^\infty x^{-(s+1)}\, d\lambda = s^{-1}\, t^{-s}$, so $H^{-1}(t) = (s\, t)^{-1/s}$. Thus,

$$\mathcal{F}_s(f) = H^{-1}\big(\|H \circ f\|_1\big) = \left(s \int_{\mathbb{R}^n} \frac{1}{s}\, f^{-s}\, d\lambda\right)^{-1/s} = \|f\|_{-s}$$

for any $f \in \mathcal{C} = \{ f : \mathbb{R}^n \to (\tau, \infty) \mid f \text{ continuous, and } f(x)/(1+|x|) \text{ bounded} \}$.

Next, we compute the definition of $\mathcal{G}_s$ defined in equation (1.10) with $g(s,t) = s/t$:

$$\mathcal{G}_s(\mu) = \inf \big\{ \langle f,\, \mu \rangle\, \|f\|_{-s}^{-1} \mid f \in \mathcal{C} \big\}.$$

For $g(s,t) = s/t$ Hypothesis (B4) says

$$\Big\langle \varphi_{i+1},\, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)} \Big\rangle\, \mathcal{G}_s\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big)^{-1} \le \mathcal{F}_s(\varphi_i).$$

To prove this inequality, we compute $\mathcal{G}_s(\mu)$ when μ has a continuous density.

Proposition 4.1.2. *Let h be a continuous, nonnegative function on $\mathbb{R}^n$. If $h\,\lambda$ is in $\mathcal{P}_1$, then*

$$\mathcal{G}_s(h\,\lambda) = \|h\|_{\frac{s}{s+1}}.$$

Proof. This proof relies on the reverse Hölder inequality which states that if $p \in (0,1)$ and $q < 0$ are numbers such that $1/p + 1/q = 1$, and $f > 0$, $h \ge 0$ are functions on $\mathbb{R}^n$, then

$$\|h\|_p \, \|f\|_q \leq \int_{\mathbb{R}^n} f \, h \, d\lambda. \tag{4.2}$$

Applying (4.2) with $f \in \mathcal{C}$, h defined in the proposition, $p = s/(s+1)$, and $q = -s$ shows

$$\|h\|_{\frac{s}{s+1}} \, \|f\|_{-s} \leq \langle f, \, h\lambda \rangle.$$

Since $f \in \mathcal{C}$ was arbitrary, it follows that

$$\|h\|_{\frac{s}{s+1}} \leq \inf\{ \, \langle f, \, h\lambda \rangle \, \|f\|_{-s}^{-1} \mid f \in \mathcal{C} \, \} = \mathcal{G}_s(h \, \lambda). \tag{4.3}$$

To prove the reverse inequality consider

$$f_k(x) = \min\{h(x)^{-1/(s+1)}, \, k \, (1 + |x|)\}. \tag{4.4}$$

By assumption h is continuous and integrable, so h is finite, which implies f_k is continuous and positive. Thus, $f_k \in \mathcal{C}$, and

$$\mathcal{G}_s(\mu) \leq \langle f_k, \mu \rangle \, \mathcal{F}_s(f_k)^{-1}.$$

f_k is an increasing sequence of positive functions which converge pointwise to ${g_\mu}^{-1/(s+1)}$, so

$$\langle f_k, \mu \rangle = \int_{\mathbb{R}^n} f_k \, h \, d\lambda \to \int_{\mathbb{R}^n} h^{\frac{s}{s+1}} \, d\lambda = \|h\|_{\frac{s}{s+1}}^{\frac{s+1}{s}}$$

by the monotone convergence theorem.

f_k^{-s} converges pointwise to $h^{\frac{s}{s+1}}$, and f_k^{-s} is dominated by f_1^{-s} for all k. f_1^{-s} is integrable because it is the maximum of $h^{\frac{s}{s+1}}$ and $(1+|x|)^{-s}$ which are both positive and integrable. $h^{\frac{s}{s+1}}$ is integrable by (4.3) and the finiteness of $\mathcal{G}_s$, and $(1 + |x|)^{-s}$ is integrable because $s > n$. Thus,

$$\|f_k\|_{-s}^{-1} \to \left(\int_{\mathbb{R}^n} h^{\frac{s}{s+1}} \, d\lambda \right)^{1/s} \tag{4.5}$$

by the dominated convergence theorem. Together, (4.4) and (4.5) imply

$$\mathcal{G}_s(h\,\lambda) \leq \|h\|_{\frac{s}{s+1}}.$$

$\square$

Now we use Lemmas 3.1.2 and 4.1.2 to prove Hypothesis (B4).

Lemma 4.1.3. $\mathcal{F}_s(\varphi_i) \geq \Big\langle \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}, \varphi_{i+1} \Big\rangle\, \mathcal{G}_s\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big)^{-1}.$

Proof. φ_{i+1} is smooth, so $\mathrm{MA}(\varphi_{i+1}) = \det(\nabla^2\varphi_{i+1})\,\lambda$, and since $\{\varphi_i\}$ is a Monge–Ampère iteration satisfying equation (1.7)

$$\int_{\mathbb{R}^n} |x|\ \mathrm{MA}(\varphi_{i+1}) = \|\varphi_i\|_{-(s+1)}^{s+1} \int_{\mathbb{R}^n} |x|\,\varphi_i(x)^{-(s+1)}\,d\lambda < \infty$$

because φ_i has linear growth and $s > n$. Thus $\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)} \in \mathcal{P}_1$, and Proposition 4.1.2 implies

$$\mathcal{G}_s\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big) = \Big\|\frac{\det(\nabla^2\varphi_{i+1})}{\lambda(A)}\Big\|_{\frac{s}{s+1}}.$$

Since $\int_A \varphi_i^*\,d\lambda = \int_A \varphi_{i+1}^*\,d\lambda$, Lemma 3.1.2 implies

$$\langle \varphi_{i+1},\, \mathrm{MA}(\varphi_{i+1})\rangle \leq \langle \varphi_i, \mathrm{MA}(\varphi_{i+1})\rangle.$$

By equation (1.7) defining the Monge–Ampère iteration,

$$\Big\langle \varphi_i, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)} \Big\rangle = \Big(\int_{\mathbb{R}^n} \varphi_i^{-(s+1)}\,d\lambda\Big)^{-1} \int_{\mathbb{R}^n} \varphi_i^{-s}\,d\lambda.$$

Combining the previous equations shows

$$
\begin{aligned}
&\Big\langle \varphi_{i+1}, \frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)} \Big\rangle \mathcal{G}_s\Big(\frac{\mathrm{MA}(\varphi_{i+1})}{\lambda(A)}\Big)^{-1} \\
&\qquad \le \left(\int_{\mathbb{R}^n} \varphi_i^{-(s+1)}\, d\lambda\right)^{-1} \left(\int_{\mathbb{R}^n} \varphi_i^{-s}\, d\lambda\right) \left\|\frac{\det(\nabla^2\varphi_{i+1})}{\lambda(A)}\right\|_{\frac{s}{s+1}}^{-1} \\
&\qquad = \left(\int_{\mathbb{R}^n} \varphi_i^{-(s+1)}\, d\lambda\right)^{-1} \left(\int_{\mathbb{R}^n} \varphi_i^{-s}\, d\lambda\right) \\
&\qquad\qquad\qquad \left(\int_{\mathbb{R}^n} \varphi_i^{-s}\, d\lambda\right)^{-\frac{s+1}{s}} \left(\int_{\mathbb{R}^n} \varphi_i^{-(s+1)}\, d\lambda\right) \\
&\qquad = \mathcal{F}_s(\varphi_i).
\end{aligned}
$$

□

Lemma 4.1.3 concludes the proof of Hypothesis (B4), so Theorem 1.2.2 implies Theorem 1.3.3.

4.2 Affine immersions

Affine differential geometry is concerned with properties of submanifolds of $\mathbb{R}^{n+1}$ which are *equiaffine*, meaning they are invariant under volume preserving affine transformations, $x \mapsto A\,x + v$ for $A \in Sl_{n+1}\mathbb{R}$ and $v \in \mathbb{R}^{n+1}$. The study of affine immersions $f : M^n \hookrightarrow \mathbb{R}^{n+1}$ parallels that of Riemmannian immersions. For Riemannian immersions, the standard metric on $\mathbb{R}^{n+1}$ is used to define the unit normal N to $f(M)$, and the splitting $T_{f(x)}\mathbb{R}^{n+1} = f_*(T_xM) + \mathrm{span}\{N_x\}$ is used to define the shape operator which describes the curvature of the embedding.

Since the metric on $\mathbb{R}^{n+1}$ is not invariant under $Sl_{n+1}\,\mathbb{R}$, the study of affine immersions relies instead on the standard volume form on $\mathbb{R}^{n+1}$ to define a transversal vector field which is equiaffine. In Subsection 4.2.1 we define a unique equiaffine transversal vector field called the affine normal, and then we use the affine normal to define equiaffine analogues for the Riemannian definitions of the induced metric and shape operator. In Subsection 4.2.2 we define several $Sl_{n+1}\mathbb{R}$ invariant measures on affine immersions with a particular focus on the affine surface area measure. In Subsection 4.2.3 we define affine spheres, which historically are at the heart of early work on real Monge–Ampère equations.

4.2.1 Affine normal

Consider a smooth immersion f of a manifold M^n as a hypersurface in $\mathbb{R}^{n+1}$:

$$f : M^n \hookrightarrow \mathbb{R}^{n+1}.$$

In order to decompose vectors in $T\,\mathbb{R}^{n+1}$ into a component tangent to $f(M)$ and a component transversal to $f(M)$ we choose a transversal vector field

$$\xi : M \to T\big|_{f(M)}\mathbb{R}^{n+1}.$$

A rule which assigns a transversal vector field ξ_f to every immersion f is equiaffine if for every $A \in Sl_{n+1}\mathbb{R}$ and $v \in \mathbb{R}^n$ defining $g(x) = A\,x + v$ we have

$$g_*(\xi_f) = \xi_{g\circ f}.$$

We note that the Euclidean unit normal only satisfies the above property for orthogonal matrices A, so the Euclidean normal is not equiaffine. We will show equiaffine transversal vector fields are unique up to scaling, and moreover there is an equiaffine way to normalize their length. We do so by defining a metric, connection, and volume forms based on an arbitrary transversal vector field ξ and then showing under a unique choice of ξ these objects are equiaffine.

Let $\xi \in T\big|_{f(M)}\mathbb{R}^{n+1}$, be an arbitrary vector field transversal to $f(M)$, meaning ξ_x and $f_*(T_xM)$ span $T\big|_{f(x)}\mathbb{R}^{n+1}$, where ξ_x are the pointwise values of ξ. We decompose $T\big|_{f(M)}\mathbb{R}^{n+1}$ as

$$T_{f(x)}\mathbb{R}^{n+1} = f_*(T_xM) + \operatorname{span}\{\xi_x\}.$$

We can differentiate vector fields on $T\big|_{f(M)}\mathbb{R}^{n+1}$ using the flat connection D on $\mathbb{R}^{n+1}$ by extending them smoothly to a neighborhood of $f(M)$. The formula

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\,\xi \tag{4.6}$$

defines a torsion free connection ∇ on TM and a symmetric bilinear form h on TM. Differentiating the vector field ξ,

$$D_X\xi = -f_*(S\,X) + \tau(X)\,\xi, \tag{4.7}$$

defines an endomorphism S of TM and a one form τ. These definitions are equiaffine, meaning if $\widetilde{f}(x) = v + A\,f(x)$ and $\widetilde{\xi} = A\,\xi$ for $v \in \mathbb{R}^n$ and $A \in Sl_n\,\mathbb{R}$, then $\widetilde{h} = h$, $\widetilde{\nabla} = \nabla$, $\widetilde{S} = S$ and $\widetilde{\tau} = \tau$. The next proposition, whose proof we take from Nomizu and Sasaki [34, Proposition 1.3], shows the converse is also true.

Proposition 4.2.1. *Let $f, \widetilde{f} : M \to \mathbb{R}^{n+1}$ be two immersions with transversal vector fields ξ and $\widetilde{\xi}$ respectively. If $\nabla = \widetilde{\nabla}$, $h = \widetilde{h}$, $S = \widetilde{S}$, and $\tau = \widetilde{\tau}$ then there exist $A \in Sl_{n+1}\mathbb{R}$ and $v \in \mathbb{R}^n$ such that $\widetilde{f} = v + Af$.*

Proof. Define $L : T_{f(x)}\mathbb{R}^{n+1} \to T_{\widetilde{f}(x)}\mathbb{R}^{n+1}$ by

$$L(f_*X) = \widetilde{f}_*X \qquad\qquad L(\xi_x) = \widetilde{\xi}_x.$$

by identifying all the tangent spaces of $\mathbb{R}^{n+1}$ we can think of L as a map from M to $Gl_n \mathbb{R}$. To prove L is in fact constant on M we compute

$$\begin{aligned}(D_X L)(f_*Y) &= D_X\big(L(f_*Y)\big) - L\big(D_X(f_*Y)\big)\\ &= D_X\big(L(f_*Y)\big) - L\big(f_*\nabla_X Y + h(X,Y)\xi\big)\\ &= D_X(\widetilde{f}_*Y) - \widetilde{f}_*(\widetilde{\nabla}_X Y) - \widetilde{h}(X,Y)\widetilde{\xi} = 0,\end{aligned}$$

and likewise

$$\begin{aligned}(D_X L)(\xi) &= D_X\widetilde{\xi} - L(D_X\xi)\\ &= -\widetilde{f}_*(\widetilde{S}X) + \widetilde{\tau}(X)\widetilde{\xi} - (-\widetilde{f}_*(SX) + \tau(X)\widetilde{\xi}) = 0.\end{aligned}$$

The previous computations show that $D_X(\widetilde{f} - L\,f) = 0$ so it is equal to a constant vector v in $\mathbb{R}^n$ so $\widetilde{f}(x) = v + Lf(x)$. □

It is important to note that h, ∇, S, and τ depend on both the immersion f and the choice of transversal vector ξ. Before proceeding, we will compute a simple example

Example 4.2.1. Let $f(x) = \big(x, F(x)\big)$ for some smooth function F. Let $\{e_i\}_{i=1}^{n+1}$ be the standard basis for $T\,\mathbb{R}^{n+1}$. Let $\xi = e_{n+1}$ be the transversal vector field. Since ξ is constant, equation (4.7) implies $S = 0$ and $\tau = 0$. Let $\{\partial_i = \partial/\partial x_i\}_{i=1}^{n}$ be a basis of vector fields on $M = \mathbb{R}^n$. Thus

$$f_*(\partial_i) = e_i + F_i\, e_{n+1}$$

where $F_i = \partial F/\partial x_i$.

$$D_{\partial_i} f_*(\partial_j) = F_{ij}\, e_{n+1}$$

so equation (4.6) implies $h(\partial_i, \partial_j) = F_{ij}$, and $\nabla_{\partial_i}\partial_j = 0$.

In the previous example, h is a positive-definite bilinear form if and only if $\nabla^2 F > 0$. In particular, if F is a strongly convex function, then h is positive-definite. We will only consider the case when $f : M \hookrightarrow \mathbb{R}^{n+1}$ is a *nondegenerate convex immersion*, meaning $f(M)$ is locally the graph of a convex function, and h is positive-definite at every point.

For $f : M \to \mathbb{R}^{n+1}$ a nondegenerate convex immersion, h is called the *affine metric*, ∇ is called the *affine connection*, and S the *affine shape operator*.

In order to define a unique, equiaffine normal vector, we introduce two volume forms on M. The first is the intrinsic volume form determined by the affine metric h:

$$\omega_h(X_1, \ldots, X_n) = \det(h(X_i, X_j)_{ij})^{1/2}. \tag{4.8}$$

The second volume form is extrinsic and defined in terms of the transversal vector field ξ:

$$\theta(X_1, \ldots, X_n) = \det\big(f_*(X_1), \ldots, f_*(X_n), \xi\big). \tag{4.9}$$

The volume form ω_h is equiaffine because it is defined in terms of the affine metric h which is equiaffine. To see that the volume form θ is equiaffine, consider the immersion $\widetilde{f}(x) = v + A\,f(x)$ and the vector field $\widetilde{\xi} = A\,\xi$ for $v \in \mathbb{R}^{n+1}$ and

$L \in SL_{n+1}\,\mathbb{R}$. Then,

$$\begin{aligned}\widetilde{\theta}(X_1,\ldots,X_n) &= \det\big(\widetilde{f}_*(X_1),\ldots,\widetilde{f}_*(X_n),\widetilde{\xi}\big)\\ &= \det\big(A\,f_*(X_1),\ldots,A\,f_*(X_n),\,A\xi\big)\\ &= \det(A)\,\theta(X_1,\ldots,X_n),\end{aligned}$$

and $\det(A)=1$, so $\widetilde{\theta}=\theta$.

In the next proposition we will prove there is a unique transversal vector field ξ which satisfies conditions based on the equiaffine properties of the immersion f. The following proof is taken from Nomizu and Sasaki [34, Theorem 3.1].

Proposition 4.2.2. *Let $f : M \to \mathbb{R}^{n+1}$ be nondegenerate convex immersion. There exists a unique transersal vector field $\xi \in T\,\mathbb{R}^{n+1}\big|_{f(M)}$ such that ξ points to the concave side of $f(M)$, and the induced affine structure satisfies $\tau = 0$ and $\theta = \omega_h$.*

Proof. Let ξ be any transversal vector field pointing to the concave side of $f(M)$. Consider the related transversal vector field $\widetilde{\xi} = \phi\,\xi + f_*(Z)$ for ϕ a smooth, positive function, and Z a vector field on M. $\widetilde{\xi}$ also points to the concave side of $f(M)$ because ϕ is positive, and we will show there is a unique choice of ϕ and Z which satisfy $\tau = 0$ and $\omega_h = \theta$. Equation (4.6) for ξ and $\widetilde{\xi}$ is given by

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\,\xi$$

$$D_X f_*(Y) = f_*(\widetilde{\nabla}_X Y) + \widetilde{h}(X,Y)\,\widetilde{\xi} = f_*\big(\widetilde{\nabla}_X Y + \widetilde{h}(X,Y)\,Z\big) + \widetilde{h}(X,Y)\,\phi\,\xi.$$

Equating the transversal components implies $h = \phi\,\widetilde{h}$. Equation (4.7) for ξ and $\widetilde{\xi}$ is given by

$$D_X\widetilde{\xi} = -f_*\big(\widetilde{S}(X) - \widetilde{\tau}(X)\,Z\big) + \widetilde{\tau}(X)\phi\,\xi$$

$$\begin{aligned} D_X\widetilde{\xi} &= D_X\big(\phi\,\xi + f_*(Z)\big) \\ &= X(\phi)\,\xi - \phi\,f_*\big(S(X)\big) + \phi\,\tau(X)\,\xi + f_*(\nabla_X Z) + h(X,Z)\,\xi \\ &= -f_*\big(\phi\,S(X) - \nabla_X Z\big) + \big(X(\phi) + \phi\,\tau(X) + h(X,Z)\big)\,\xi. \end{aligned}$$

Equating the transversal components implies $\widetilde{\tau}(\cdot) = \tau(\cdot) + d\,\log(\phi)(\cdot) + \phi^{-1}h(Z,\cdot)$. Since h is nondegenerate, there exists a vector field Z such that $\widetilde{\tau} = 0$. It remains to show that ϕ can be chosen so $\widetilde{\theta} = \omega_{\widetilde{h}}$. The volume forms $\widetilde{\theta}$ and θ are related by

$$\widetilde{\theta}(X_1,\ldots,X_n) = \det\big(f_*(X_1),\ldots,f_*(X_n),\phi\,\xi + f_*(Z)\big) = \phi\,\theta(X_1,\ldots,X_n),$$

and ω_h and $\omega_{\widetilde{h}}$ are related by

$$\begin{aligned} \omega_{\widetilde{h}}(X_1,\ldots,X_n) &= \det\Big(\widetilde{h}(X_i,X_j)_{ij}\Big)^{1/2} \\ &= \det\big(\phi^{-1}h(X_i,X_j)_{ij}\big)^{1/2} = \phi^{-n/2}\,\omega_h(X_1,\ldots,X_n), \end{aligned}$$

where we used the fact that $h = \phi\widetilde{h}$. The two previous equations imply $\widetilde{\theta} = \phi\theta$ and $\omega_{\widetilde{h}} = \phi^{-n/2}\omega_h$. If we define ϕ to be

$$\phi = \left(\frac{\omega_h}{\theta}\right)^{2/n+2}, \tag{4.10}$$

then $\omega_{\widetilde{h}} = \widetilde{\theta}$.

To prove uniqueness, assume two such transversal vector fields ξ and $\widetilde{\xi}$ exist. They are related by $\widetilde{\xi} = \phi\,\xi + f_*(Z)$ for some positive function ϕ and some vector field Z. The conditions $\omega_h = \theta$ and $\omega_{\widetilde{h}} = \widetilde{\theta}$ imply $\phi = 1$. Conditions $\widetilde{\tau} = \tau = 0$ and $\phi = 0$ imply $h(Z,\cdot) = 0$, so $Z = 0$ as well. Thus $\widetilde{\xi} = \xi$. □

The unique transversal vector field ξ is defined in terms of equiaffine constraints, and thus the vector field ξ is equiaffine as well. We call ξ the *affine normal* of f. Nondegenerate convex immerions $f : M \to \mathbb{R}^{n+1}$ paired with their unique affine normals ξ are called *affine immersions*. Affine immersions induce an equiaffine structure which consists of an affine metric h, an affine connection ∇, and an affine shape operator S which satisfy the equations

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\,\xi \qquad \text{and} \qquad D_X \xi = -f_*(S(X)).$$

In this affine immersion we have equality between the volume forms θ and ω_h. We define the *affine surface area measure* Ω to be integration over this volume form. For any Borel $U \subset M$ lying in a coordinate patch of M with coordinate x, the affine surface area measure is given by

$$\Omega(U) = \int_U \omega_h = \int_U \det(h_{ij})^{1/2}\, dx_1 \wedge \cdots \wedge dx_n. \tag{4.11}$$

4.2.2 $SL_{n+1}\mathbb{R}$ invariant measures

In equation (4.11) we defined the affine surface area measure as integration against the Riemannian volume form of the affine metric. In this section we define the cone measure of an immersion, and give a dual formulation of affine surface area. We draw from the exposition by Klartag [24] and Nomizu and Sasaki [34].

Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate convex immersion. For any Borel $U \subset M$ we define the *cone measure* by

$$\mu(U) = (n+1)\,\lambda\big(\operatorname{cvx}(0, f(U))\big),$$

where λ is Lebesgue measure on $\mathbb{R}^{n+1}$ and $\operatorname{cvx}(0, f(U))$ is the convex hull of the origin and $f(U)$. The cone measure is $(n+1)$ times the volume of the cone over $f(U)$ with vertex at the origin. In Lemma 4.2.3 we will show when $f(x)$ is transversal to $f(M)$ the cone measure is absolutely continuous with respect to the surface measure induced from the immersion $f : M \hookrightarrow \mathbb{R}^{n+1}$.

Let $N(x)$ denote the Euclidean unit normal to $f(M)$ at $f(x)$, with the orientation chosen so that $\langle \xi_x, N(x)\rangle \geq 0$. Let $dV_{f(M)} = f^*(\iota_N \, dV_{\mathbb{R}^{n+1}})$ be the induced volume form of M, and let dV_{S^n} be the standard volume form on the sphere of radius 1. The Euclidean unit normal can be thought of as the Gauss map $N : M \to S^n$. We can define the Gaussian curvature, $\kappa : S^n \to \mathbb{R}$, of the immersion f as a function of the Euclidean unit normals to $f(M)$ by the formula

$$dV_{f(M)} = N^*\Big(\frac{1}{\kappa}\, dV_{S^n}\Big).$$

Lemma 4.2.3. *Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate convex immersion. If $f(x)$ is transversal to $f(M)$ for every x in M, then for every Borel $U \subset M$*

$$\mu(U) = \int_U |\,\langle f(x), N(x)\rangle\,|\, dV_{f(M)} = \int_{N(U)} |\,\langle f(N^{-1}(z)), z\rangle\,|\, \kappa^{-1}(z)\, dV_{S^n}. \quad (4.12)$$

Proof. If A is a Borel set in $\mathbb{R}^n \times \{0\}$, and $p = (p', p_{n+1})$ is a point in $\mathbb{R}^{n+1}$, then the formula for the volume of a Euclidean cone implies

$$(n+1)\, \lambda_{n+1}\big(\operatorname{cvx}(\,p, A)\,\big) = p_{n+1}\lambda_n(A) = |\langle p, e_{n+1}\rangle|\, \lambda_n(A),$$

where λ_n is n-dimensional Lebesgue measure, and e_{n+1} is the unit vector perpendicular to A. The vector $-p$ points from the vertex of the cone $\operatorname{cvx}(\,p, A)$ to a point in A. Thus, infinitesimally, the cone measure of $\operatorname{cvx}(\,0, f(U))$ is equal to the measure of $f(U)$ times $|\langle f(x), N(x)\rangle|$ for $x \in U$ because $f(x)$ points from the vertex of the cone to a point in $f(U)$. This implies

$$d\mu = |\,\langle f(x), N(x)\rangle\,|\, dV_{f(M)}.$$

The measure of an arbitrary Borel set U follows by integration over U. The second equality in equation (4.12) follows from the definition of the Gaussian curvature. □

The inner product $\langle f(x), N(x)\rangle$ is referred to as the *support function* of f. We will use the shorthand

$$\rho(x) = \langle f(x), N(x)\rangle, \tag{4.13}$$

so when $f(x)$ is transversal to $F(M)$, the cone measure can be written

$$\mu(U) = \int_U |\rho(x)|\, dV_{f(M)} = \int_{N(U)} |\rho(N^{-1}(z))|\, \kappa^{-1}(z)\, dV_{S^n}.$$

Since μ is defined as the volume of a cone in $\mathbb{R}^{n+1}$ with its vertex at the origin, it is SL_{n+1}-invariant, but in contrast to the affine surface area measure, the cone measure is not translation invariant.

Next we will show the affine surface area (4.11) can be expressed in terms of the Gaussian curvature. First we need the following lemma.

Lemma 4.2.4. *Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate convex affine immersion with affine normal ξ. Let N_x be the Euclidean unit normal pointing to the concave side of $f(M)$, and let $\kappa(N_x)$ be the Gaussian curvature of $f(M)$ at $f(x)$. Then*

$$\langle \xi_x, N_x\rangle = \kappa^{1/(n+2)}(N_x).$$

Proof. We follow the proof of Proposition 4.2.2 with N as the transversal vector field to $f(M)$.

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\, N.$$

Since the transversal vector field was chosen to be the Euclidean unit normal, h is the Riemmanian second fundamental form of the immersion. Also, the extrinsic volume form defined in equation (4.9) equals the induced volume form from $\mathbb{R}^{n+1}$.

$$\theta(X_1, \ldots, X_n) = \det\big(f_*(X_1), \ldots, f_*(X_n), N\big) = dV_{f(M)}(X_1, \ldots, X_n),$$

Recall that $\omega_h(X_1, \ldots, X_N) = \det\big(h(X_i, X_j)_{ij}\big)^{1/2}$. If $\{X_i\}$ is a unimodular basis for θ, then $\omega_h(X_1, \ldots, X_n) = \kappa^{1/2}$ by the definition of κ as the determinant of the Riemmanian shape operator S defined by $h(X,Y) = \langle f_*(S(X)), f_*(Y)\rangle$. In the notation of Proposition 4.2.2 we have

$$\frac{\omega_h}{\theta} = \kappa^{1/2}.$$

Equation (4.10) from the proof of Proposition 4.2.2 shows that the affine normal ξ is given by

$$\xi = \left(\frac{\omega_h}{\theta}\right)^{2/(n+2)} N + f_*(Z) = \kappa^{1/(n+2)}\, N + f_*(Z) \tag{4.14}$$

for some vector field Z on M. Taking the inner product with the unit normal shows

$$\langle \xi, N\rangle = \kappa^{1/(n+2)}.$$

□

Proposition 4.2.5. *Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate convex affine immersion, and let Ω be the affine surface area measure defined in equation (4.11). Then for* *every Borel* $U \subset M$,

$$\Omega(U) = \int_A \kappa(N(x))^{1/(n+2)}\, dV_{f(M)} = \int_{N(U))} \kappa^{-(n+1)/(n+2)}\, dV_{S^n}. \qquad (4.15)$$

Proof. Since $\Omega(U) = \int_U \omega_h$, the first inequality is equivalent to showing $\theta = \kappa(N(x))\, dV_{f(M)}$. Since ξ is the affine normal, the volume form ω_h is equal to the volume form θ defined in equation(4.9) as

$$\theta = f^*(\iota_\xi\, dV_{\mathbb{R}^{n+1}}),$$

where $\iota_\xi\, dV_{\mathbb{R}^{n+1}}$ is the contraction of the Euclidean volume form along the affine normal ξ. The contraction only depends on the component of ξ which is perpendicular to $f(M)$, so it follows that

$$\theta = \langle \xi, N\rangle\, f^*(\iota_N\, dV_{\mathbb{R}^n}) = \langle \xi, N\rangle\, dV_{f(M)}.$$

Lemma 4.2.4 shows that $\langle \xi, N\rangle = \kappa^{1/(n+2)}$, so

$$\theta = \kappa(N(x))^{1/(n+2)}\, dV_{f(M)}.$$

The last equality follows from the Riemannian geometry fact that $N^*(\kappa^{-1}\, dV_{S^n}) = dV_{f(M)}$. □

Lastly we give a dual definition of affine surface area which was introduced by Lutwak in [28]:

$$\Omega(U) = \inf\{ \left(\int_{N(U)} \alpha(z)\, \kappa(z)^{-1}\, dV_{S^n}\right)^{\frac{n+1}{n+2}} \left(\int_{N(U)} \alpha(z)^{-(n+1)}\, dV_{S^n}\right)^{\frac{1}{n+2}} \mid \alpha \in C_+(S^n)\, \}, \qquad (4.16)$$

where $C_+(S^n)$ are continuous, positive functions on S^n.

We refer to Lutwak [28] for the equivalence between definition (4.16) and (4.15), though we will prove the equivalence in the case of immersions which are graphs of Legendre transforms in Subsection 4.3.2. Lutwak also realized that the two integrals inside the infimum of defintion (4.16) can be interpreted in the nonsmooth case. When $f(M)$ is the boundary of any convex set with the origin in its interior, then Lutwak showed the first integral is a power of the mixed volume of $f(M)$ and the polar set Q°, where Q is the convex set whose radial definition function is α^{-1}. The second integral is a power of the volume of Q.

4.2.3 Affine spheres and dual affine immersions

An affine immersion is called an *affine sphere* if the shape operator $S = \gamma\, I$ for some constant γ. Affine spheres are grouped into three families based on the sign of γ. The case $\gamma < 0$ is called *hyperbolic*, $\gamma = 0$ is called parabolic, and $\gamma > 0$ is called elliptic. Affine spheres also have a geometric interpretation in terms of the affine normals.

Proposition 4.2.6. *Let $f : M \to \mathbb{R}^{n+1}$ be a convex immersion, ξ be the affine normal, and S be the induced shape operator. Then $S = 0$ if and only if the ξ are all parallel. And $S = \gamma I$ with $\gamma \neq 0$ if and only if $f(x) + \gamma^{-1}\xi_x = y_0$ a fixed point which is called the center of the affine sphere.*

Proof. If $S = 0$ then ξ is parallel because $D_X\xi = -f_*(S(X)) = 0$ for all vector fields X on M. Conversely, if all of the ξ are parallel, then there exists some function γ

such that $\gamma\,\xi$ is a parallel vector field. Then we compute

$$0 = D_X(\gamma\xi) = X(\gamma)\,\xi - \gamma f_*(S(X)).$$

Each term in the expression is 0, so $S = 0$.

If $S = \gamma\, I$ then

$$D_X\big(f(x) + \gamma^{-1}\xi\big) = f_*(X) - \gamma^{-1} f_*\big(S(X)\big) = 0$$

for all vector fields X. Thus, $f(x) + \gamma^{-1}\xi_x = y_0$ for some constant point y_0. Conversely, if $f(x) + \gamma\,\xi$ is constant for some function γ, then

$$0 = D_X\big(f(x) + \gamma\,\xi\big) = f_*(X) + X(\gamma)\xi - \gamma\, f_*\big(S(X)\big).$$

The tangent and transversal components are both equal to 0 so γ is a constant and $S(X) = \gamma^{-1}X$ for said constant. □

We illustrate the equivalent affine sphere definition from Proposition 4.2.6 in Figure 4.1.

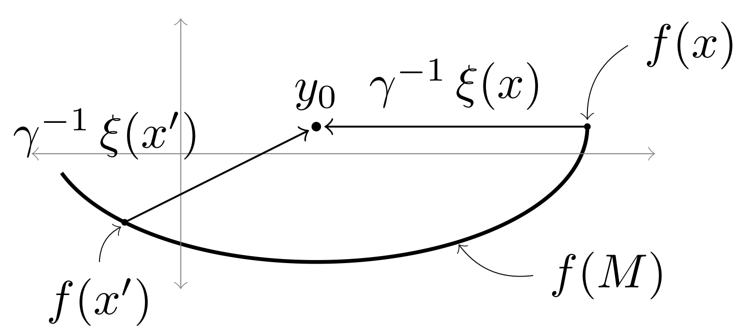

Figure 4.1: An Affine Sphere with center y_0

The affine normals all meeting at a single point is a natural affine geometric restriction because it is invariant under volume preserving affine transformations.

Example 4.2.2. Let $M = S^n$ be the sphere which we think of as $\{\, x \in \mathbb{R}^{n+1} \mid |x| = R\,\}$. We can define the immersion of S^n as a sphere of radius R by $f(x) = R\,x$. By equation (4.14) the affine normal is given by

$$\xi = \kappa^{1/(n+2)}\, N + f_*(Z)$$

for some vector field Z on S^n which satisfies $h(Z, X) = -X(\kappa^{1/(n+2)})$ for every vector field X on S^n. Since the immersion f has constant Gaussian curvature $\kappa = R^{-n}$ it follows that $Z = 0$. Thus,

$$\xi(x) = \kappa^{1/(n+2)}\, N(x) = R^{-n/(n+2)}\, N(x) = -R^{(2n+2)(n+2)} f(x)$$

which shows f is an affine sphere with center at the origin. Since affine spheres are invariant under volume preserving affine transformations, this shows all ellipses are affine spheres.

It turns out that these are the only complete examples, but there are many more incomplete elliptic affine spheres.

There are many conditions on affine immersions which are equivalent to being an affine sphere.

Lemma 4.2.7. *Let $f : M \hookrightarrow \mathbb{R}^{n+1}$ be a nondegenerate convex immersion. Then f is an affine sphere with center at the origin if and only if there exists a constant C such that*

$$\rho(x) = C\, \kappa(N(x))^{1/(n+2)}$$

for every point x in M.

Proof. Consider the function $g : M \to \mathbb{R}$ defined by

$$g(x) = \frac{\langle f(x),\, N(x)\rangle}{\langle \xi(x),\, N(x)\rangle}.$$

By Lemma 4.2.4, $\langle \xi, N\rangle = \kappa^{1/(n+2)}$, so $g(x) = \rho(x)\,\kappa(N(x))^{-1/(n+2)}$. Thus, it suffices to prove f is an affine sphere with center at the origin if and only if $g = C$ for some constant C.

If f is an affine sphere with its center at the origin, then $f(x) = C\,\xi(x)$ for some constant C. Thus,

$$g(x) = \frac{\langle C\,\xi,\, N(x)\rangle}{\langle \xi(x),\, N(x)\rangle} = C.$$

Now assume that $g(x) = C$ for all x. For all immersions, g satisfies

$$f(x) = f_*(Z) + g(x)\,\xi$$

for some vector field $Z \in TM$. Differentiating this with respect to an arbitrary vector field $X \in TM$ and applying equations (4.6) and (4.7) implies

$$\begin{aligned} f_*(X) &= D_X\, f_*(Z) + D_X(g\,\xi) \\ &= f_*(\nabla_X Z) + h(X,\, Z)\,\xi + X(g)\,\xi - g\, f_*(S\,X). \end{aligned}$$

Equating the transversal components implies $h(X, Z) = -X(g)$, and since g was assumed to be constant, it follows that $h(X, Z) = 0$ for all $X \in TM$. Since we assumed f was a nondegenerate affine immersion, h is a nondegenerate bilinear form, so $Z = 0$. Thus, $f(x) = g(x)\,\xi = C\,\xi$, so f is an affine sphere. □

The previous lemma implies the following proposition:

Proposition 4.2.8. *Let $f : M \hookrightarrow \mathbb{R}^{n+1}$ be a nondegenerate convex immersion such that $f(x)$ is always transversal to $f(M)$. Then f is an affine sphere with center at the origin if and only if there is a constant C such that*

$$\mu = C\,\Omega,$$

where μ is the cone measure on M, and Ω is the affine surface area measure on M.

Proof. Since f is a nondegenerate convex immersion with $f(x)$ transversal to $F(M)$, it follows that $d\mu = |\rho|\, dV_{f(M)}$. By Proposition 4.2.5, the affine surface area measure is given by $d\Omega = \kappa^{1/(n+2)}\, dV_{f(M)}$. Thus, $\mu = C\,\Omega$ if and only if $|\rho| = C\,\kappa^{1/(n+2)}$. By Lemma 4.2.7, $|\rho| = C\,\kappa^{1/(n+2)}$ is equivalent to being an affine sphere with center at the origin. □

Affine spheres naturally arise in dual pairs. We define the *dual affine immersion* $\nu : M \hookrightarrow \mathbb{R}^{n+1}$ by

$$\nu(x) = \rho(x)^{-1}\, N(x),$$

where ρ is defined in equation (4.13) and N is the Euclidean unit normal. Since the Euclidean normal to the immersion $\nu(x)$ is given by $f(x)/|f(x)|$, it follows that the dual of ν is f again. The following proposition was known to Calabi, and first written down by Gigena [19].

Proposition 4.2.9. *If $f(M)$ is a nondegenerate convex affine sphere with its center at the origin, then the dual immersion $\nu(M)$ is also an affine sphere with its center at the origin. The dual affine spheres are of the same type (elliptic, parabolic, or hyperbolic).*

Another useful relation between affine spheres and their duals is the following proposition.

Proposition 4.2.10. *If $f : M \hookrightarrow \mathbb{R}^{n+1}$ is a nondegenerate convex immersion such that $f(x)$ is transversal to $f(M)$, then f is an affine sphere if and only if there exists a constant C such that*

$$\mu_{f(M)} = C\, \mu_{\nu(M)},$$

where μ_f and μ_ν are the cone measure of f and its dual ν respectively.

The proof of this proposition can be found in Corollary 5.9 of Klartag [24]. The previous proposition and Proposition 4.2.8 imply that when f is an affine sphere, the cone measures and surface area measures associated to both f and ν are all proportional to one another.

4.3 Affine iteration of Legendre graph immersions

The goal of this section is to prove Corollary 1.3.4. In Subsection 4.3.1 we define the Legendre graph immersion, which is an affine immersion that is globally the graph of the Legendre transform of a convex function. Then in Subsection 4.3.3 we define the problem of finding a Legendre graph immersion with a prescribed affine normal. Solving this problem is equivalent to solving one step of the Monge–Ampère iteration with $h(t) = t^{-(n+2)}$. In Subsection 4.3.1 we define the affine iteration of Legendre graph immersions as an iterative sequence of prescribed affine normal problems. Then we prove Corollary 1.3.4 about the convergence of the affine iteration to an affine sphere.

4.3.1 Legendre graph immersions

A large class of examples of affine immersions are graphs of convex functions. Specifically, we will study the graphs of Legendre transforms of convex functions with bounded gradient image.

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, strictly convex function. We define the *Legendre graph immersion* $f_\phi : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ by

$$f_\phi(x) = \big(\nabla\phi(x), \langle x, \nabla\phi(x)\rangle - \phi(x)\big) = \big(\nabla\phi(x),\, \phi^*(\nabla\phi(x))\big). \tag{4.17}$$

When $\nabla\phi(\mathbb{R}^n) = A$ then we say f_ϕ is a *Legendre graph immersion over* A, to capture that $f_\phi(\mathbb{R}^n)$ is the graph of the Legendre transform ϕ^* over the domain A.

Lemma 4.3.1. *Let ϕ be a smooth, strictly convex function. If f_ϕ is the Legendre graph immersion defined in (4.17), then the affine normal of f_ϕ is given by*

$$\xi = -\big(\nabla\psi(x), \langle x, \nabla\psi(x)\rangle - \psi(x)\big) \quad \textit{where} \quad \psi(x) = \det\big(\nabla^2\phi(x)\big)^{-1/(n+2)}.$$

Proof. By Lemma 4.2.2 we must verify that ξ points to the concave side of $f(\mathbb{R}^n)$, and the induced affine structure satisfies $\tau = 0$ and $\theta = \omega_h$.

Since ϕ is strictly convex, it follows that $\psi > 0$. To verify that ξ points to the concave side of f, it is enough to show it at one point. Since ϕ is strictly convex, ψ is positive, so $\xi(0) = \big(-\nabla\psi(0), \psi(0)\big)$ has positive e_{n+1} component. Since $f(\mathbb{R}^n)$ is the graph of a convex function, it follows that ξ points to the concave side of $f(\mathbb{R}^n)$.

Now we compute the affine metric, shape operator, and the volume forms θ and ω_h. Let $\partial_i = \partial/\partial x_i$, and let $\{e_i\}_{i=1}^{n+1}$ be the standard basis for $T\mathbb{R}^{n+1}$. Then

$$f_*(\partial_j) = \phi_{jk}\, e_k + \phi_{jk}\, x_k\, e_{n+1}$$

where repeated indices are summed from 1 to n. Differentiating once more shows

$$\begin{aligned} D_{\partial_i} f_*(\partial_j) &= \phi_{ijk}\, e_k + (\phi_{ijk}\, x_k + \phi_{ij})\, e_{n+1} \\ &= \left(\phi_{ijl}\, \phi^{lk} - (n+2)^{-1}\, \phi_{ij}\, \phi^{lm}\, \phi_{lmp}\, \phi^{kp}\right)\, f_*(\partial_k) + \psi^{-1}\, \phi_{ij}\, \xi \end{aligned} \tag{4.18}$$

where ϕ^{ij} are the components of $(\nabla^2\phi)^{-1}$, which is well defined becuase ϕ is strictly convex. To verify the second inequality, we must expand $\psi^{-1}\, \phi_{ij}\, \xi$. Utilizing the identity $\psi_i = -(n+2)^{-1}\, \psi\, \phi^{kl}\, \phi_{kli}$ shows

$$\begin{aligned} \psi^{-1}\, \phi_{ij}\, \xi &= -\psi^{-1}\, \phi_{ij}\, \left(\psi_k\, e_k + \left(\langle x, \nabla\psi\rangle - \psi\right) e_{n+1}\right) \\ &= (n+2)^{-1}\, \phi_{ij}\, \phi^{pq}\, \phi_{pqk}\, e_k\, + \left((n+2)^{-1}\phi_{ij}\, \phi^{pq}\, \phi_{pqk}\, x_k + \phi_{ij}\right)\, e_{n+1}. \end{aligned}$$

Plugging this into equation (4.18) along with the definition of $f_*(\partial_k)$ verifies the equality. The definition of h given in (4.6) along with equation (4.18) implies

$$h_{ij} := h(\partial_i, \partial_j) = \psi^{-1}\, \phi_{ij}. \tag{4.19}$$

Now we differentiate ξ to verify that $\tau = 0$.

$$D_{\partial_i}\xi = -\psi_{ij}\, e_j - \psi_{ij}\, x_j\, e_{n+1} = -\psi_{ij}\, \phi^{jk}\, f_*(\partial_k).$$

The definition of S given in (4.7) along with the previous equation implies

$$S(\partial_i) = \psi_{ij}\, \phi^{jk}\, e_k \quad \text{and} \quad \tau = 0. \tag{4.20}$$

The last step to verifying ξ is the affine normal is to show the two volume forms ω_h and θ defined in equations (4.8) and (4.9) are equal. It is enough to verify their equality on the basis $\{\partial_i\}$.

$$\omega_h(\partial_1, \cdots, \partial_n) = \det(h_{ij})^{1/2} = \left(\psi^{-n}\, \det(\phi_{ij})\right)^{1/2} = \psi^{-(n+1)},$$

because $\det(\nabla^2\phi) = \psi^{-(n+2)}$.

$$\begin{aligned}\theta(\partial_1, \cdots, \partial_n) &= \det\begin{pmatrix} f_*(\partial_i) & \cdots & f_*(\partial_n) & \xi \end{pmatrix} \\ &= \det\begin{pmatrix} \nabla^2\phi & -\nabla\psi \\ x^T\,\nabla^2\phi & \psi - \langle x, \nabla\psi\rangle \end{pmatrix} \\ &= \det(\nabla^2\phi)\left(\psi - \langle x, \nabla\psi\rangle + x^T\,\nabla^2\phi\,(\nabla^2\phi)^{-1}\,\nabla\psi\right) = \psi^{-(n+1)},\end{aligned}$$

where we used the formula

$$\det\begin{pmatrix} A & u \\ v^T & a \end{pmatrix} = \det(A)\,(a - v^T\,A^{-1}\,u).$$

Thus, $\omega_h = \theta$, and ξ is the affine normal. □

Now we will derive a formula for the dual immersion $\nu : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ and find conditions for both $f(\mathbb{R}^n)$ and $\nu(\mathbb{R}^n)$ to be elliptic affine spheres. First we must derive a formula for the Euclidean normal to $f(\mathbb{R}^n)$, which we will denote N_f. If we let $y = \nabla\phi(x)$, then the image of the immersion f is a graph of the form $(y, \phi^*(y))$. The normal to this graph is

$$N_f(x) = \left(\frac{\nabla\phi^*(y)}{\sqrt{1+|\nabla\phi^*(y)|^2}}, \frac{-1}{\sqrt{1+|\nabla\phi^*(y)|^2}}\right) = \left(\frac{x}{\sqrt{1+|x|^2}}, \frac{-1}{\sqrt{1+|x|^2}}\right). \quad (4.21)$$

Then the support function $\rho_f(x) = \langle N_f(x), f(x)\rangle$ is given by

$$\rho_f(x) = \frac{\phi(x)}{\sqrt{1+|x|^2}}.$$

And finally, the dual affine immersion $\nu(x) = \rho_f(x)^{-1}\,N_f(x)$ is given by

$$\nu_\phi(x) = \left(\frac{x}{\phi(x)}, \frac{-1}{\phi(x)}\right). \quad (4.22)$$

Lemma 4.3.2. *Let ϕ be a smooth, positive, strictly convex satisfying $\nabla\phi(\mathbb{R}^n) = A$, a bounded convex set with the center in its interior. Then the boundary of the dual immersion $\nu_\phi(\mathbb{R}^n)$ equals*

$$\partial A^\circ \times \{0\}$$

where $A^\circ = \{\, y \in \mathbb{R}^n \mid \langle x, y\rangle \leq 1 \text{ for all } x \in A \,\}$ is the polar set of A, and ∂A° is its boundary.

Proof. Since ϕ is a strictly convex function, the boundary of $\nu_\phi(\mathbb{R}^n)$ equals

$$\{\, \lim_{r\to\infty} \nu_\phi(r\,x) \mid x \in S^{n-1} \,\}.$$

Since the origin lies in the interior of A, the convex function ϕ is proper, so

$$\lim_{r\to\infty} \frac{-1}{\phi(r\,x)} = 0.$$

Thus, it remains to show

$$\lim_{r\to\infty} \frac{r\,x}{\phi(r\,x)} \in \partial A^\circ.$$

For every $\epsilon > 0$ define

$$A_\epsilon = \{\, y \in A \mid B_\epsilon(y) \subset A \,\}.$$

Since $\nabla\phi(\mathbb{R}^n) = A$ it follows that for every $\epsilon > 0$ there is a constant C_ϵ such that

$$C_\epsilon + \sup_{y\in A_\epsilon}\{\, \langle x, y\rangle \,\} \leq \phi(x) \leq \phi(0) + \sup_{y\in A}\{\, \langle x, y\rangle \,\}.$$

It follows that for r large enough

$$\frac{r}{\phi(0) + \sup_{y\in A}\{\, \langle r\,x, y\rangle \,\}} \leq \frac{r}{\phi(r\,x)} \leq \frac{r}{C_\epsilon + \sup_{y\in A_\epsilon}\{\, \langle r\,x, y\rangle \,\}}.$$

Taking the limit as $r \to \infty$ implies

$$\frac{1}{\sup_{y \in A}\{\, \langle r\, x, y \rangle \,\}} \leq \lim_{r \to \infty} \frac{r}{\phi(r\, x)} \leq \frac{1}{\sup_{y \in A_\epsilon}\{\, \langle r\, x, y \rangle \,\}}.$$

Since ϵ was arbitrary, it follows that

$$\lim_{r \to \infty} \frac{r\, x}{\phi(r\, x)} = \frac{x}{\sup_{y \in A}\{\, \langle x, y \rangle \,\}}.$$

For all $y \in A$

$$\left\langle \frac{x}{\sup_{y \in A}\{\, \langle x, y \rangle \,\}}, \, y \right\rangle \leq 1,$$

so $\dfrac{x}{\sup_{y \in A}\{\, \langle x, y \rangle \,\}} \in A^\circ$. For arbitrarily small ϵ the point $(1+\epsilon)\dfrac{x}{\sup_{y \in A}\{\, \langle x, y \rangle \,\}}$ does not lie in A°, so $\dfrac{x}{\sup_{y \in A}\{\, \langle x, y \rangle \,\}}$ must lie in the boundary of A°. □

In the language of Klartag [24] we say the immersion ν_ϕ has *anchor* A°. We depict a Legendre graph immersion and its dual immersion in Figure 4.2. The dashed lines in the top figure form the boundary of the convex set A which the immersion is a graph over. The dashed lines in the second picture form $\partial A^\circ \times \{0\}$, which is the boundary of the dual immersion, as guaranteed by Lemma 4.3.2.

Proposition 4.3.3. *Let $A \subset \mathbb{R}^n$ satisfy conditions (1.11), and let ϕ be a strictly convex function solving the Monge–Ampère second boundary problem (1.6) with $h(t) = t^{-(n+2)}$ and $\tau > 0$. Then the immersion f defined in equation (4.17) and its dual affine immersion ν defined in equation (4.22) are elliptic affine spheres.*

Proof. $f(\mathbb{R}^n)$ is an elliptic affine sphere if and only if the affine shape operator S of f satisfies $S = \gamma I$ for $\gamma > 0$. In equation (4.20) we computed

$$S(\partial_i) = \psi_{ij}\, \phi^{jk}\, e_k$$

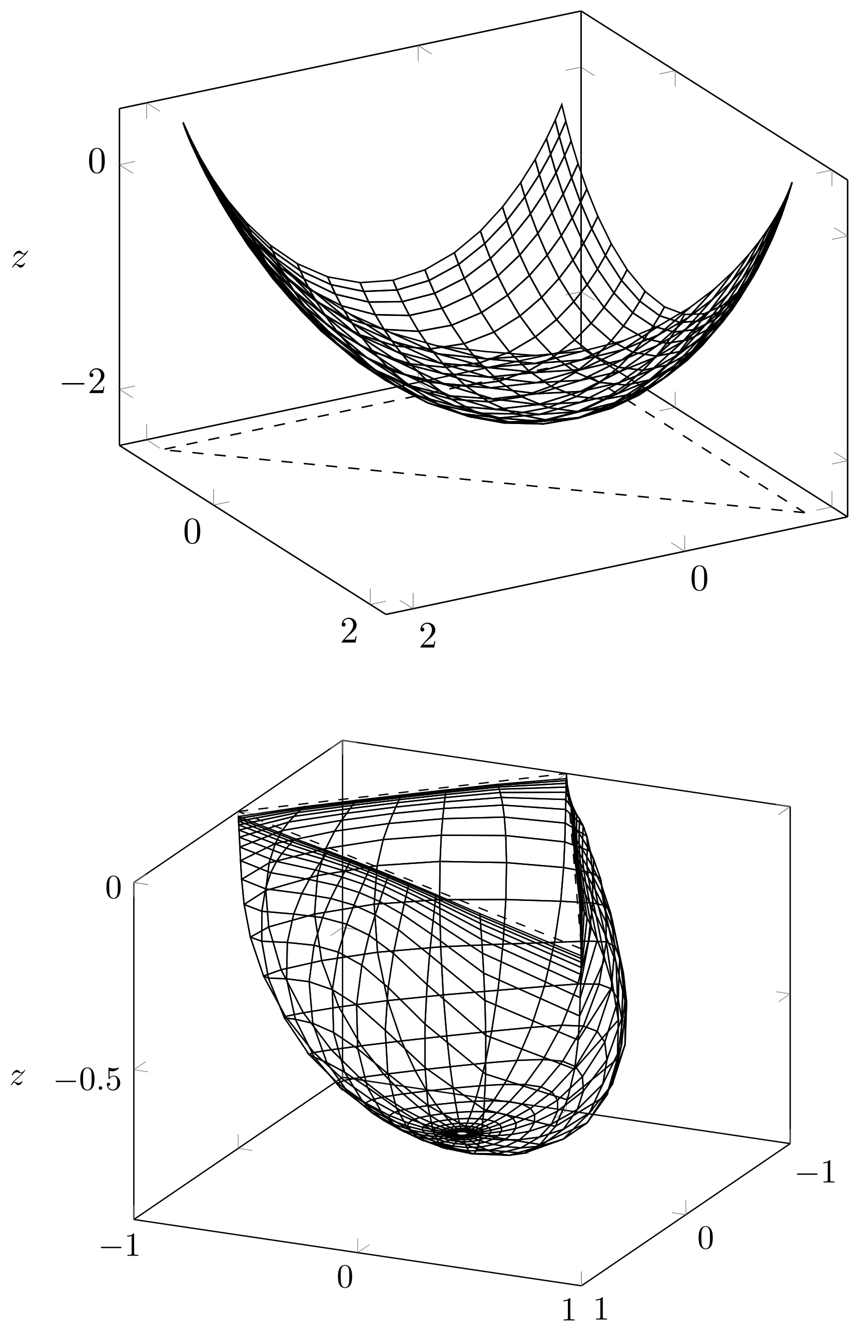

Figure 4.2: A Legendre graph immersion over A and its dual immersion

where $\psi = \det(\nabla^2\phi)^{-1/(n+2)}$. Since ϕ solves equation (1.6) it follows that

$$\psi = \lambda(A)^{-1/(n+2)} \, \|\phi\|_{-(n+2)}^{-1} \, \phi,$$

so the shape operator simplifies to

$$S(\partial_i) = \lambda(A)^{-1/(n+2)} \, \|\phi\|_{-(n+2)}^{-1} \, \phi_{ij} \, \phi^{jk} \, e_k = \lambda(A)^{-1/(n+2)} \, \|\phi\|_{-(n+2)}^{-1} \, \partial_i.$$

Thus, $f(\mathbb{R}^n)$ is an elliptic affine sphere with $\gamma = \lambda(A)^{-1/(n+2)} \, \|\phi\|_{-(n+2)}^{-1} > 0$. Proposition 4.2.9 implies that the dual immersion $\nu(\mathbb{R}^n)$ is also an elliptic affine sphere. □

4.3.2 Affine surface area of Legendre graph immersions

Next, we want to verify that the three definitions of the affine surface area are equivalent.

Affine surface area definition 1:

Equation (4.11) defines the affine surface area as

$$\Omega_f(U) = \int_U \omega_h = \int_U \det(h_{ij})^{1/2} \, dx_1 \wedge \cdots \wedge dx_n.$$

We derived the affine metric h of f in equation (4.19) to be $h_{ij} := h(\partial_i, \partial_j) = \psi^{-1} \, \phi_{ij}$, where $\psi = \det(\nabla^2\phi)^{-1/(n+2)}$. Thus, $\det(h_{ij}) = \psi^{-n} \, \det(\nabla^2\phi) = \psi^{-(2n+2)}$, and the affine surface are measure is given by

$$\Omega(U) = \int_U \psi^{-(n+1)} \, d\lambda = \int_U \det(\nabla^2\phi)^{\frac{n+1}{n+2}} \, d\lambda. \tag{4.23}$$

Affine surface area definition 2:

Equation (4.15) defines the affine surface area as

$$\Omega_f(U) = \int_U \kappa_f\big(N_f(x)\big)^{1/(n+2)} \, dV_{f(\mathbb{R}^n)}.$$

Since $f(\mathbb{R}^n)$ is the graph of ϕ^* we can use the coordinate $y = \nabla\phi$ and the formula for the volume form of a graph of a function to find

$$dV_{f(\mathbb{R}^n)} = \big(1 + |\nabla\phi^*(y)|^2\big)^{1/2}\, dy = \big(1 + |x|^2\big)^{1/2}\, \det(\nabla^2\phi(x))\, dx$$

where $dx = dx_1 \wedge \cdots \wedge dx_n$ is shorthand for the standard volume form on $M = \mathbb{R}^n$. We derived the formula for the Gauss map N_f in equation (4.21), and it can be used to compute

$$N_f^*(dV_{S^n}) = \big(1 + |x|^2\big)^{-(n+1)/2}\, dx.$$

Now we use the identity $dV_f = \kappa_f\big(N_f(x)\big)^{-1}\, N_f^*(dV_{S^n})$ to show

$$\kappa_f\big(N_f(x)\big) = \big(1 + |x|^2\big)^{-(n+2)/2}\, \det\big(\nabla^2\phi(x)\big)^{-1}.$$

Putting the formulas for $\kappa_f\big(N_f(x)\big)$ and dV_f into equation (4.15) yields

$$\begin{aligned}\Omega_f(U) &= \int_U \Big(\big(1 + |x|^2\big)^{-(n+2)/2}\, \det\big(\nabla^2\phi(x)\big)^{-1}\Big)^{1/(n+2)} \big(1 + |x|^2\big)^{1/2}\, \det\big(\nabla^2\phi(x)\big)\, dx \\ &= \int_U \det(\nabla^2\phi)^{\frac{n+1}{n+2}}\, d\lambda.\end{aligned}$$

Affine surface are definition 3:

Equation (4.16) gives the dual definition of affine surface area as

$$\Omega(U) = \inf\left\{ \left(\int_{N(U)} \alpha(z)\, \kappa(z)^{-1}\, dV_{S^n}\right)^{\frac{n+1}{n+2}} \left(\int_{N(U)} \alpha(z)^{-(n+1)}\, dV_{S^n}\right)^{\frac{1}{n+2}} \mid \alpha \in C_+(S^n) \right\},$$

where $C_+(S^n)$ are continuous, positive functions on S^n. Using the formulas for N_f, κ_f, and $N_f^*(dV_{S^n})$ derived above, it follows that

$$\begin{aligned}\Omega(U)^{\frac{n+2}{n+1}} &= \inf_{\alpha \in C_+(S^n)} \Bigg\{ \left(\int_U \alpha\big(N_f(x)\big)\, \kappa_f\big(N_f(x)\big)^{-1}\, N_f^*(dV_{S^n})\right) \\ &\qquad\qquad \left(\int_U \alpha\big(N_f(x)\big)^{-(n+1)}\, N_f^*(dV_{S^n})\right)^{\frac{1}{n+1}} \Bigg\} \\ &= \inf_{\alpha \in C_+(S^n)} \left\{ \left(\int_U \widehat{\alpha}(x)\, \det\big(\nabla^2\phi(x)\big)\, dx\right) \left(\int_U \widehat{\alpha}(x)^{-(n+1)}\, dx\right)^{\frac{1}{n+1}} \right\},\end{aligned}$$

where we define $\widehat{\alpha} \in C_+(\mathbb{R}^n)$ by

$$\widehat{\alpha}(x) = \sqrt{1+|x|^2}\,\alpha\Big(\frac{x}{\sqrt{1+|x|^2}}, \frac{-1}{\sqrt{1+|x|^2}}\Big).$$

Since $\alpha \in C_+(S^n)$, it follows that $0 < \alpha(z) \leq C$. Thus, $\widehat{\alpha}$ is continuous, and

$$\widehat{\alpha}(x) \leq C\,\sqrt{1+|x|^2} \leq C\,(1+|x|),$$

so $\widehat{\alpha} \in \mathcal{C} = \{\, f : \mathbb{R}^n \to (0,\infty) \mid f \text{ is continuous, and } f(x)/(1+|x|) \text{ is bounded}\,\}$. This characterization of $\widehat{\alpha}$ implies

$$\Omega_f(\mathbb{R}^n)^{\frac{n+2}{n+1}} = \lambda(A)\,\inf\Big\{\,\Big\langle \widehat{\alpha}, \frac{\mathrm{MA}(\phi)}{\lambda(A)}\Big\rangle\,\|\widehat{\alpha}\|^{-1}_{-(n+1)} \mid \widehat{\alpha} \in \mathcal{C}\,\Big\} = \lambda(A)\,\mathcal{G}\Big(\frac{\mathrm{MA}(\phi)}{\lambda(A)}\Big).$$

Proposition 4.1.2 implies $\mathcal{G}\Big(\frac{\mathrm{MA}(\phi)}{\lambda(A)}\Big) = \Big\|\frac{\det(\nabla^2\phi)}{\lambda(A)}\Big\|_{\frac{n+1}{n+2}}$, so the previous equation again verifies that

$$\Omega_f(\mathbb{R}^n) = \int_{\mathbb{R}^n} \det(\nabla^2\phi)^{\frac{n+1}{n+2}}\,d\lambda.$$

Thus, Proposition 4.1.2 can be viewed as a proof of Lutwak's theorem that the dual definition of the affine surface area is equal to the usual definition in the case when the immersion f is given by equation (4.17).

The last observation we want to make about the immersion f is the interpretation of the functional $\mathcal{F}(\phi) = \|\phi\|_{-(n+1)}$.

Lemma 4.3.4. *Let ϕ be a smooth, positive, strictly convex function. If f is the immersion defined in equation (4.17), and ν is its dual immersion defined in equation (4.22), then*

$$\mathcal{F}(\phi)^{-(n+1)} = \mu_\nu(\mathbb{R}^n)$$

where μ_ν is the cone measure of the dual immersion.

Proof. The cone measure of the dual immersion is given by

$$d\mu_\nu = \rho_\nu \, dV_\nu$$

where ρ_ν is the support function of ν, and dV_ν is the induced volume form of the immersion. Since ν is dual to f, it follows that its normal vector field $N_\nu(x) = f(x)/|f(x)|$. The definitions of the dual map and the support function imply

$$\rho_\nu = \langle N_\nu, \nu \rangle = \left\langle \frac{f}{|f|}, \frac{N_f}{\langle N_f, f\rangle} \right\rangle = |f|^{-1}.$$

A computation shows that

$$dV_\nu = \nu^*(\iota_{N_\nu} dV_{\mathbb{R}^{n+1}}) = \phi^{-(n+1)}|f| \, dx.$$

Thus,

$$\mathcal{F}(\phi)^{-(n+1)} = \int_{\mathbb{R}^n} \phi^{-(n+1)} \, d\lambda = \mu_\nu(\mathbb{R}^n).$$

□

4.3.3 Prescribed affine normals

We define the *prescribed affine normal problem for Legendre graph immersions* as follows. Fix an open, bounded, convex set $A \subset \mathbb{R}^n$. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, positive, strictly convex function such that $\nabla\psi(\mathbb{R}^n) = A$, and let f_ψ be the Legendre graph immersion over A, defined in (4.17). The problem is to find a Legendre graph immersion f_ϕ over A such that the affine normal ξ of f_ϕ satisfies

$$\xi(x) = -c\, f_\psi(x)$$

for some constant $c > 0$.

Proposition 4.3.5. *Let A be an open, convex set with the origin contained in its interior. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is a smooth, positive, strictly convex function such that $\nabla\psi(\mathbb{R}^n) = A$, and if f_ψ is its Legendre graph immersion over A, then there is a Legendre graph immersion f_ϕ over A, unique up to an additive constant, which solves the prescribed affine normal problem for Legendre graph immersions.*

Proof. Let B_r be some small ball around the origin which is contained in A. Since $\nabla\psi(\mathbb{R}^n) = A$, it follows that $\psi(x) \geq r\,|x| - c$ for some constant c. Since ψ is also positive, this lower bound implies $\psi^{-(n+2)}$ is integrable. It follows that

$$\frac{\psi(x)^{-(n+2)}}{\int_{\mathbb{R}^n} \psi^{-(n+2)}\, d\lambda}\,\lambda$$

is a smooth, positive probability measure on $\mathbb{R}^n$. McCann proved in [31, Main Theorem] that the equation

$$\begin{cases} \dfrac{\det(\nabla^2\phi)}{\lambda(A)} = \dfrac{\psi(x)^{-(n+2)}}{\int_{\mathbb{R}^n} \psi^{-(n+2)}\, d\lambda} \\ \nabla\phi(\mathbb{R}^n) = A \end{cases}$$

has convex solutions which are unique up to an additive constant. Since ψ is smooth and positive, it follows from the regularity theory of Caffarelli [9] that ϕ is smooth, and thus strictly convex. Thus, f_ϕ as defined in equation (4.17) is a Legendre graph immersion over A.

Since $\det(\nabla^2\phi) = c\,\psi$ for $c = \lambda(A)^{-1/(n+2)}\|\psi\|^{-1}_{-(n+2)}$, Proposition 4.3.1 implies the affine normal of f_ϕ is given by

$$\xi(x) = -c\left(\nabla\psi(x), \psi^*\big(\nabla\psi(x)\big)\right) = -c\, f_\psi(x).$$

Thus f_ϕ solves the prescribed affine normal problem, and ϕ is unique up to the addition of a constant. $\square$

If we choose the additive constant so that $\phi > 0$, then we can solve the prescribed affine normal problem for f_ϕ and create a sequence of Legendre graph immersions. If the barycenter of A is the origin, then the integral condition $\int_A \phi^* \, d\lambda = -\tau < 0$ forces $\phi(x) > 0$.

4.3.4 Convergence of the affine iteration

We define the *affine iteration of Legendre graph immersions over* A to be a sequence $\{f_i := f_{\phi_i}\}$ of Legendre graph immersions over A, an open convex set with barycenter at the origin, such that f_{i+1} solves the prescribed affine normal problem for f_i, and $\int_A \phi_i^* \, d\lambda = -\tau < 0$.

Now we prove Corollary 1.3.4 which states that there exist $M_i \in Sl_{n+1}\mathbb{R}$ such that $M_i \cdot f_i(\mathbb{R}^n)$ converges smoothly to an elliptic affine hemisphere.

Proof. Since A satisfies (1.11) and $\int_A \phi_0^* \, d\lambda = -\tau < 0$, Lemma 2.2.4 implies $\phi_0(x) \geq \tau > 0$ for all $x \in \mathbb{R}^n$. Then Proposition 4.3.5 implies there exists a smooth solution f_1 to the prescribed normal normal problem for f_0, unique up to an additive constant. But there is a unique solution if we stipulate the normalization $\int_A \phi_1^* \, d\lambda = -\tau$. We can apply apply this argument iteratively to find a sequence $\{f_i\}$ which is an affine iteration over A.

The functions $\{\phi_i\}$ which define the Legendre graph immersions $\{f_i\}$ are smooth solutions to the Monge–Ampère iteration (1.7) with $h(t) = t^{-(n+2)}$. Theorem 1.2.2 with $h(t) = t^{-(n+2)}$, whose hypotheses we verified in Section 4.1, implies there exists a sequence of constants $\{a_i\}$ such that $\widetilde{\phi}_i(x) = \phi_i(x + a_i)$ converges smoothly to ϕ, which is a smooth, convex solution of the Monge–Ampère equation (1.6). By Proposition 4.3.3, the Legendre graph immersion f_ϕ associated to ϕ is an elliptic affinesphere.

The Legendre transform of the translated sequence satisfy $\widetilde{\phi}^*_i(y) = \phi^*_i(y) - \langle a, y\rangle$. Since the image of Legendre graph immersions are simply the graph of the Legendre transform, it follows that

$$\widetilde{f}_i(\mathbb{R}^n) = \begin{pmatrix} I & 0 \\ a_i^T & 1 \end{pmatrix} f_i(\mathbb{R}^n).$$

Thus, if we define $M_i = \begin{pmatrix} I & 0 \\ a_i^T & 1 \end{pmatrix}$, then it follows that $M_i \cdot f_i(\mathbb{R}^n)$ converges smoothly to $f_\phi(\mathbb{R}^n)$ which is an elliptic affine hemisphere with its center at the origin. □

Finally, we will make a remark about the decreasing functionals along the affine iteration. As a consequence of Lemma 2.2.8 and the fact that Hypotheses 1.2.1 hold for $h(t) = t^{-(n+2)}$ as shown in Section 4.1, we have the inequalities

$$\mathcal{F}(\phi_i) \geq \left\langle \phi_{i+1}, \frac{\mathrm{MA}(\phi_{i+1})}{\lambda(A)} \right\rangle \mathcal{G}\Big(\frac{\mathrm{MA}(\phi_{i+1})}{\lambda(A)}\Big)^{-1} \geq \mathcal{F}(\phi_{i+1})$$

when ϕ_i solves the Monge–Ampère iteration with $h(t) = t^{-(n+2)}$. Due to the calculation of these functionals in Section 4.3.1, this is equivalent to

$$\mu_{\nu_i}(\mathbb{R}^n)^{-\frac{1}{n+1}} \geq \mu_{f_{i+1}}(\mathbb{R}^n)\, \Omega_{f_{i+1}}(\mathbb{R}^n)^{-\frac{n+2}{n+1}} \geq \mu_{\nu_{i+1}}(\mathbb{R}^n)^{-\frac{1}{n+1}}.$$

The right inequality, which comes from the definition of $\mathcal{G}$ and the equivalence of normal and dual definitions of affine surface area, is a special case of the inequality in Lemma AB of Lutwak [27]. The left inequality says that the reverse inequality holds along the affine iteration, and that equality is achieved when the immersion is an affine sphere.

Chapter 5: A Dirichlet problem associated to Kähler–Einstein metrics

The goal of this chapter is to prove Theorem 1.4.1, which is an extension of a result of Mabuchi [29] [30]. We recall Mabuchi's theorem which states if ϕ is an open orbit potential for a smooth Kähler–Einstein metric on $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $Bl_3\,\mathbb{P}^2$, then ϕ can be used to construct a domain Ω and a convex function $\chi : \Omega \to \mathbb{R}$ solving

$$\begin{cases} \det(\nabla^2 \chi) = (-\chi)^{-5/2} & \text{on } \Omega \\ \chi = 0 & \text{on } \partial\Omega. \end{cases}$$

Mabuchi's proof can be broken into two key steps. The first step is to show $(\nabla^2 \phi^*)^{-1}(y)$, the inverse of the Hessian of the Legendre transform, extends to a smooth matrix-valued function in a neighborhood of P – the polytope associated to the toric Kähler manifold. The second step is to show there exists a smooth convex function H in this neighborhood of P which solves

$$\operatorname{adj}(\nabla^2 H) = (\nabla^2 \phi^*)^{-1}(y) + \frac{1}{3} y\, y^T$$

where adj denotes the adjugate of a matrix. Furthermore,

$$\Omega = \nabla H(\text{Int } P)$$

is a bounded convex domain, and H^* can be used to define the function χ on Ω solving the above Dirichlet Monge–Ampère equation with 0 boundary values.

The proof of Theorem 1.4.1 extends each of these two steps. In the first step we prove $(\nabla^2\phi^*)^{-1}(y)$ extends to a smooth matrix-valued function in a neighborhood of P for potentials ϕ of any smooth or singular metric on any toric Kähler manifold. Mabuchi's proof used the fact that ω was a Kähler–Einstein metric on one of the three toric Kähler manifolds that he studied.

In the second step we show that Mabuchi's technique of defining H, Ω, and χ can be extended to Kähler–Einstein metrics with edge singularities solving

$$\mathrm{Ric}(\omega) = \mu\,\omega + \sum_{i=1}^{N}(1-\beta_i)\,[D_i]$$

on any of the five toric Fano surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $Bl_k\,\mathbb{P}^2$ for $k = 1, 2, 3$.

We prove Theorem 1.4.1 in Sections 5.1 and 5.2. In Section 5.3 we compute the domains Ω in the five cases where Theorem 1.4.1 applies.

5.1 A smooth extension of $(\nabla^2\phi^*)^{-1}$

Since ϕ is an open orbit potential for a toric metric on a toric Kähler manifold with edge singularities, we know ϕ satisfies the Guillemin conditions of Theorem 3.2.12 for some Delzant polytope P. Thus, it suffices to prove the following proposition:

Proposition 5.1.1. *If $\phi : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function which satisfies the Guillemin boundary conditions of Theorem 3.2.12 for a Delzant polytope P, then the matrix valued function $(\nabla\phi^*)^{-1}$ defined on* Int P *extends to a smooth function on a neighborhood of P.*

In order to prove Proposition 5.1.1 we first introduce some notation. For a fixed n consider the following set of multiindices:

$$\mathcal{J}_k = \{\, J = (J_1\ \ldots, J_n) \text{ a multiindex} \mid J_i \in \{0,1\} \text{ and } J_i = 0 \text{ for } i > k \,\}.$$

For example,

$$\mathcal{J}_2 = \{\, (0,\ldots,0),\ (1,0,\ldots,0),\ (0,1,0,\ldots,0),\ (1,1,0,\ldots,0) \,\}.$$

If $b = (b_1,\ldots,b_n)$ is a vector of real constants, and $J = (J_1,\ldots,J_n) \in \mathcal{J}_k$, then we denote a multiindex power of b by

$$b^J = \prod_{i=1}^{n} b_i^{J_i}.$$

For any two multiindices J and K and any n-by-n matrix A, $A_{J\oplus K}$ is the matrix A with the i^{th} row removed if $J_i = 1$ and the i^{th} column removed if $K_i = 1$. Thus, if $J \in \mathcal{J}_k$, $A_{J\oplus J}$ is dimension $n - |J|$ where $|J| = \sum_{i=1}^{n} J_i$.

Lemma 5.1.2. *If A is an n-by-n matrix A, $b = (b_1,\ldots,b_n)$ is a vector in $\mathbb{R}^n$, and $\{e_i\}$ is the standard basis for $\mathbb{R}^n$, then*

$$\det\left(A + \sum_{i=1}^{k} b_i\, e_i e_i^T\right) = \sum_{J\in\mathcal{J}_k} b^J \det(A_{J\oplus J})$$

where we use the convention that the determinant of the empty matrix is 1.

Proof. We will prove the lemma by induction on k. The case $k = 0$ trivially says $\det(A) = \det(A)$. Now assume that the lemma is true up to k. We can use the formula for the determinant of a rank-one perturbation to compute the $(k+1)^{st}$ formula:

$$\det\left(A+\sum_{i=1}^{k+1} b_i\, e_i e_i^T\right) = \det\left(\left(A+\sum_{i=1}^{k} b_i\, e_i e_i^T\right) + b_{k+1}\, e_{k+1} e_{k+1}^T\right)$$

$$= \det\left(A+\sum_{i=1}^{k} b_i\, e_i e_i^T\right) + b_{k+1}\, e_{k+1}^T \,\mathrm{adj}\left(A+\sum_{i=1}^{k} b_i\, e_i e_i^T\right) e_{k+1}$$

$$= \sum_{\mathcal{J}_k} b^J \det(A_{J\oplus J}) + b_{k+1}(-1)^{k+1+k+1} \det\left(\left(A+\sum_{i=1}^{k} b_i\, e_i e_i^T\right)_{k+1}\right)$$

$$= \sum_{\mathcal{J}_{k+1}} b^J \det(A_{J\oplus J})$$

□

For $n = 2$ and $k = 2$, Lemma 5.1.2 simply says

$$\det\begin{pmatrix} a_{11}+b_1 & a_{12} \\ a_{21} & a_{22}+b_2 \end{pmatrix} = (a_{11}\, a_{22} - a_{12}\, a_{21}) + b_1\, a_{22} + b_2\, a_{11} + b_1\, b_2.$$

Now we can apply Lemma 5.1.2 to $\det(\nabla^2\phi^*)$. Recall that ϕ^* is defined on the Delzant polytope

$$P = \bigcap_{i=1}^{M}\{\, l_i(y) \geq 0 \,\} = \bigcap_{i=1}^{M}\{\, \lambda_i + \langle n_i, y\rangle \geq 0 \,\}.$$

Lemma 5.1.3. *If $\phi : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function which satisfies the Guillemin boundary conditions of Theorem 3.2.12 for a Delzant polytope P, then*

$$\det(\nabla^2\phi^*) = F(y)\prod_{i=1}^{M} l_i(y)^{-1} \tag{5.1}$$

where $F \in C^\infty(P)$, meaning F is smooth up to the boundary of P, and $F(y) > 0$ for all y in the closed set P.

Proof. Define F on Int P to satisfy equation (5.1). The strong convexity of ϕ^* implies $F(y) > 0$ on Int P. Now consider $y' \in \partial P$, the boundary of P. After a suitable linear coordinate change and reindixing the boundary defining functions $\{l_i\}$ we can assume $l_i(y) = y_i + \lambda_i$ for $i = 1, \ldots, n$, $l_i(y') = 0$ for $i = 1, \ldots, k$, and $l_i(y') > 0$ for $i = k+1, \ldots, M$. By the Guillemin boundary conditions of Theorem 3.2.12 there is some neighborhood $U \ni y'$ such that $U \cap \partial P \subset \left(\partial P \cap \left(\bigcup_{i=1}^{k} \{l_i(y) = 0\}\right)\right)$ and

$$\phi^*\big|_{U\cap P}(y) = \sum_{i=1}^{k} \beta_i^{-1}(y_i + \lambda_i)\, \log(y_i + \lambda_i) + h(y),$$

where h is smooth up to the boundary of $U \cap P$. It follows that

$$\nabla^2\phi^*\big|_{U\cap P} = \nabla^2 h + \sum_{i=1}^{k} \beta_i^{-1}(y_i + \lambda_i)^{-1} e_i\, e_i^T,$$

and we can compute its determinant using Lemma 5.1.2.

$$\begin{aligned}
\det(\nabla^2\phi^*)\big|_{U\cap P} &= \det\left(\nabla^2 h + \sum_{i=1}^{k} \beta_i^{-1}(y_i + \lambda_i)^{-1} e_i\, e_i^T\right) \\
&= \sum_{\mathcal{J}_k} \left(\beta^J (y+\lambda)^J\right)^{-1} \det\left((\nabla^2 h)_{J\oplus J}\right) \\
&= \prod_{i=1}^{k} \beta_i^{-1} l_i(y)^{-1} \sum_{\mathcal{J}_k} \beta^{J'} (y+\lambda)^{J'} \det\left((\nabla^2 h)_{J\oplus J}\right),
\end{aligned}$$

where $J' = -J + e_1 + \ldots e_k$. The definition of F implies

$$F\big|_U = \prod_{i=1}^{k} \beta_i^{-1} \sum_{\mathcal{J}_k} \beta^{J'} (y+\lambda)^{J'} \det\left((\nabla^2 h)_{J\oplus J}\right).$$

Since $h \in C^\infty(U)$, the previous computation shows that F extends smoothly to the boundary of $\partial P \cap U$. In order to show that $F > 0$ on $\partial P \cap U$ it suffices to show $\sum_{\mathcal{J}_k} \beta^{J'} (y+\lambda)^{J'} \det\left((\nabla^2 h)_{J\oplus J}\right) > 0$ on $\partial P \cap U$.

Note that since $J_i = 0$ for $i \geq k+1$ it follows that $J'_i \geq 0$ for all i. For each multiindex such that J' is not identically 0 we have that $(y+\lambda)^{J'} = 0$ on $\partial P \cap U$. Thus,

$$\sum_{\mathcal{J}_k} \beta^{J'} (y+\lambda)^{J'} \det\left((\nabla^2 h)_{J\oplus J}\right) = \det\left((\nabla^2 h)_{K\oplus K}\right)$$

where $K = e_1 + \dots e_k$. But $(\nabla^2 h)_{K\oplus K} = (\nabla^2\varphi^*)_{K\oplus K}$, and $(\nabla^2\varphi^*)_{K\oplus K} > 0$ on $\partial P \cap U$ by the second hypothesis of the Guillemin boundary conditions. This shows $F(y) > 0$ on ∂P.

This shows the existence of a smooth extension in a neighborhood of every $y' \in \partial P$. To get an extension defined on a neighborhood of all of P we can apply Whitney's extension theorem. □

In the next lemma we show the same is true for the minors of $\nabla^2\phi^*$ except the functions F may be 0 on certain boundary components.

Lemma 5.1.4. *Under the same hypotheses as Lemma 5.1.3,*

$$\det\left((\nabla^2\phi^*)_{e_j\oplus e_k}\right) = F(y)\prod_{i=1}^{M} l_i(y)^{-1},$$

for $j, k = 1, \dots, n$, where $F \in C^\infty(P)$ and $F(y) \geq 0$ for all $y \in P$.

Proof. The proof is almost identical to the proof of the previous lemma. The only difference is that the linear change of coordinates in the beginning of the proof will change the minor whose determinant we are taking. Thus we will have

$$\det\left((\nabla^2\phi^*)_{e_j\oplus e_k}\right) = G(\nabla^2 h)\prod_{i=1}^{k} \beta_i^{-1} l_i(y)^{-1},$$

where G is a polynomial function of the entries of $\nabla^2 h$. This will still be smooth up to the boundary, so as before the function F defined by

$$F(y) = \det\left((\nabla^2\phi^*)_{e_j\oplus e_k}\right)\prod_{i=1}^{M} l_i(y)$$

will be smooth up to the boundary. The strict convexity of ϕ^* implies $\det\left((\nabla^2\phi^*)_{e_j\oplus e_k}\right) > 0$ on Int P, so $F(y) \geq 0$ on P. □

Now we can prove Proposition 5.1.1.

Proof. We compute each component of $(\nabla^2\phi^*)^{-1}$ using Cramer's rule:

$$(\phi^*)^{ij}(y) = \det\big(\nabla^2\phi^*(y)\big)^{-1}(-1)^{i+j}\det\big((\nabla^2\phi^*(y))_{e_j\oplus e_e}\big).$$

By Lemmas 5.1.3 and 5.1.4 there are functions $F, G \in C^\infty(P)$ with $F > 0$ and $G \geq 0$ such that

$$\det\big(\nabla^2\phi^*(y)\big)^{-1}(-1)^{i+j}\det\big((\nabla^2\phi^*(y))_{e_j\oplus e_i}\big) = \frac{G(y)\prod_{i=1}^M l_i(y)^{-1}}{F(y)\prod_{i=1}^M l_i(y)^{-1}}.$$

Since F, $G \in C^\infty(P)$, it follows that there is some open set containing P where F, G are defined and smooth. We can choose this open set small enough so that $F(y) > 0$ which implies that $G(y)/F(y) = (\phi^*)^{ij}$ is a smooth extension to a neighborhood of P. □

5.2 Proof of Theorem 1.4.1

First we recall the setup for Theorem 1.4.1. Let X_P be a toric Fano manifold with an associated Fano polytope

$$P = \textstyle\bigcap_{i=1}^M \{\, 1 + \langle n_i, y\rangle \geq 0 \,\}$$

as described in the end of Subsection 3.2.4. Let $\omega \in c_1(X_P)$ be a Kähler–Einstein metric with edge singularities of angle β_i along the toric divisors D_i solving

$$\mathrm{Ric}(\omega) = \mu\,\omega + \sum_{i=1}^N (1-\beta_i)\,[D_i].$$

If $\omega\big|_{\mathbb{C}^{*n}} = \sqrt{-1}\,\partial\overline{\partial}\,\phi$ is a potential in the open orbit, then ϕ is a smooth, strongly convex function on $\mathbb{R}^n$ which satisfies the Guillemin boundary conditions of Theorem

3.2.12. If we define

$$\varphi(x) = \phi(x) - \langle P_c,\, x\rangle,$$

where P_c is the barycenter of P, then φ satisfies the same Guillemin boundary conditions, but over the domain $P - P_c$.

Lemma 3.3.5 implies

$$\begin{cases} \det(\nabla^2\varphi) = e^{-\mu\,\varphi} \\ \nabla\varphi(\mathbb{R}^n) = \text{Int } P - P_c. \end{cases}$$

To prove Theorem 1.4.1 we closely follow the proof by Mabuchi [29] [30].

Lemma 5.2.1. *For each $i = 1, \ldots n$ and for $y \in P - P_c$*

$$\mu\, y_i + \sum_{j=1}^{n} \frac{\partial}{\partial y_j} (\varphi^*)^{ij}(y) = 0,$$

where $(\varphi^*)^{ij}$ is defined on a neighborhood of $P - P_c$ by Lemma 5.1.1.

Proof. We will show this by direct computation on Int $P - P_c$ and it will extend to all of P by continuity. Subscripts denote differentiation with respect to x when they are applied to φ and with respect to y when they are applied to φ^*. The change of variables $y = \nabla\varphi(x)$ and $x = \nabla\varphi^*(y)$ imply

$$\begin{aligned} \mu\, y_i + \sum_{j=1}^{n} \frac{\partial}{\partial y_j} (\varphi^*)^{ij} &= \mu\, \varphi_i + \sum_{j,k=1}^{n} \frac{\partial x_k}{\partial y_j} \frac{\partial}{\partial x_k} (\varphi_{ij}) \\ &= \mu\, \varphi_i + \sum_{j,k} \varphi^*_{jk}\, \varphi_{ijk}. \end{aligned}$$

Now we use the formula $\frac{d}{dt} \log \det(A(t)) = \mathrm{tr}(A^{-1} A'(t))$ to show

$$\mu \varphi_i + \sum_{j,k} \varphi^*_{jk} \varphi_{ijk} = \mu \varphi_i + \mathrm{tr}\left((\nabla^2\varphi)^{-1}(\nabla^2\varphi)_i\right)$$

$$= \mu \varphi_i + \frac{\partial}{\partial x_i} \log \det(\nabla^2\varphi)$$

$$= \mu \varphi_i - \mu\varphi_i = 0.$$

□

Now we consider the matrix valued function in $C^\infty(P - P_c)$ given by

$$(\nabla^2\varphi^*)^{-1} + \frac{\mu}{n+1} y\, y^T.$$

We wish to show that this matrix equals $\mathrm{adj}(\nabla^2 H)$ for some function $H \in C^\infty(P - P_c)$, where adj denotes the adjugate operation. The equation we are trying to solve is thus

$$\mathrm{adj}(\nabla^2 H) = (\nabla^2\varphi^*)^{-1} + \frac{\mu}{n+1} y\, y^T. \tag{5.2}$$

From now on we restrict ourselves to dimension $n = 2$ where the adjugate has a particularly simple form, and we have that $\mathrm{adj}(\,\mathrm{adj}(A)) = A$. Thus our problem reduces to finding an $H \in C^\infty(P)$ such that

$$\nabla^2 H = \begin{pmatrix} \varphi_{22} + \frac{\mu}{3} y_2^2 & -\varphi_{12} - \frac{\mu}{3} y_1 y_2 \\ -\varphi_{12} - \frac{\mu}{3} y_1 y_2 & \varphi_{11} + \frac{\mu}{3} y_1^2 \end{pmatrix}.$$

In order to find the primitive function H on P it is necessary that the matrix be defined on a neighborhood of P, which is guaranteed by Proposition 5.1.1. It is also necessary that the matrix on the right hand side is symmetric, which obviously holds. Finally, it is sufficient to check the compatibility of the third partial derivatives. Thus, we need to show

$$\frac{\partial}{\partial y_2}\left(\varphi_{22}+\frac{\mu}{3}\,y_2^2\right)=\frac{\partial}{\partial y_1}\left(-\varphi_{12}-\frac{\mu}{3}\,y_1\,y_2\right),\ \text{and}$$
$$\frac{\partial}{\partial y_2}\left(-\varphi_{12}-\frac{\mu}{3}\,y_1\,y_2\right)=\frac{\partial}{\partial y_1}\left(\varphi_{11}+\frac{\mu}{3}\,y_1^2\right).$$

These equations are exactly the content of Lemma 5.2.1, so there exists $H \in C^{\infty}(P)$ satisfying equation 5.2. H is only unique up the addition of an affine function, but we can choose certain normalizations.

Lemma 5.2.2. *The constant term of H can be chosen so that*

$$e^{-\mu\varphi}=\mu\,(H-\langle y,\,\nabla H\rangle).$$

Proof. We start by computing

$$\begin{aligned}\frac{\partial}{\partial y_1}e^{-\mu\varphi}&=-\mu e^{-\mu\varphi}\left(\varphi_1\frac{\partial x_1}{\partial y_1}+\varphi_2\frac{\partial x_2}{\partial y_1}\right)=-\mu e^{-\mu\varphi}\left(y_1\varphi^*_{11}+y_2\varphi^*_{12}\right)\\&=-\mu\left(y_1\varphi_{22}-y_2\varphi_{12}\right),\end{aligned}$$

using the fact that $\det(\nabla^2\varphi)=e^{-\mu\varphi}$ and the formula for the inverse of a 2-by-2 matrix. Now we use the fact that $\varphi_{22}=H_{11}-\frac{\mu}{3}y_2^2$ and $\varphi_{12}=-H_{12}-\frac{\mu}{3}y_1\,y_2$. This implies

$$\begin{aligned}-\mu\left(y_1\varphi_{22}-y_2\varphi_{12}\right)&=-\mu\left(y_1\left(H_{11}-\frac{\mu}{3}y_2^2\right)+y_2\left(H_{12}+\frac{\mu}{3}y_1\,y_2\right)\right)\\&=-\mu\left(y_1H_{11}+y_2H_{12}\right)\\&=\frac{\partial}{\partial y_1}\mu\left(H-y\cdot\nabla H\right).\end{aligned}$$

A completely analagous calculation shows that $\dfrac{\partial}{\partial y_2} e^{-\mu\varphi} = \dfrac{\partial}{\partial y_2}\mu\,(H - y \cdot \nabla H)$, so we can add a constant to H such that

$$e^{-\mu\varphi} = \mu(H - y \cdot \nabla H).$$

□

Now H is unique up to a choice of a linear function. This choice will not have any effect on the rest of the results other than to translate the domain $\nabla H(P - P_c)$.

Lemma 5.2.3. *H is strongly convex on* Int $P - P_c$.

Proof. We need to show that $u^T(\nabla^2 H)u > 0$ on Int $P - P_c$ for any $u \in \mathbb{R}^2$, $u \neq 0$.

$$u^T(\nabla^2 H)u = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \varphi_{22} + \dfrac{\mu}{3} y_2^2 & -\varphi_{12} - \dfrac{\mu}{3} y_1\, y_2 \\ -\varphi_{12} - \dfrac{\mu}{3} y_1\, y_2 & \varphi_{11} + \dfrac{\mu}{3} y_1^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$= \begin{pmatrix} u_1 & -u_2 \end{pmatrix} \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{12} & \varphi_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} + \frac{\mu}{3}\,(y_2 u_1 - y_1 u_2)^2 > 0$$

On the last line, the first term is positive on Int $P - P_c$ because φ is strongly convex on $\mathbb{R}^n$. Here we are evaluating $\nabla^2\varphi$ on $P - P_c$ using $(\nabla^2\varphi)(\nabla\varphi^*(y))$. □

Denote the Legendre transform by $(H)^*(z) = \psi(z)$ where

$$z = \nabla H(y) \qquad \text{and} \qquad \psi(z) = \langle y,\, z\rangle - H(y).$$

We can use this formula for the Legendre transform because H is strongly convex. We now define the domain

$$\Omega = \nabla H(\text{Int } P - P_c).$$

We know Ω is bounded because H extends smoothly to a neighborhood of P, which implies that $|\nabla H(y)| < \infty$ for $y \in P$. Now we show ψ satisfies a type of Monge-Ampere equation on Ω.

Lemma 5.2.4. *ψ satisfies the equation*

$$\begin{cases} 1 = -\mu \, \det(\nabla^2\psi)\, \psi + \dfrac{\mu}{3}(\nabla\psi)^T \left(\text{adj}(\nabla^2\psi)\right)(\nabla\psi) & \textit{on } \Omega \\ \psi = 0 & \textit{on } \partial\Omega. \end{cases}$$

Proof. From Lemma 5.2.2 and the definition of ψ we have

$$-\mu\psi = \mu \left(H - \langle y, \nabla H\rangle\right) = e^{-\mu\varphi} = \det(\nabla^2\varphi).$$

Then using the fact that $\text{adj}(\nabla^2 H) = (\nabla^2\varphi^*)^{-1} + \frac{\mu}{3} y\, y^T$ we have

$$\begin{aligned} -\mu\psi &= \det \begin{pmatrix} H_{22} - \frac{\mu}{3}y_1^2 & -H_{12} - \frac{\mu}{3}y_1\, y_2 \\ -H_{12} - \frac{\mu}{3}y_1\, y_2 & H_{11} - \frac{\mu}{3}y_2^2 \end{pmatrix} \\ &= \det(\nabla^2 H) - \frac{\mu}{3} y^T (\nabla^2 H)\, y \end{aligned}$$

Now using that $(\nabla^2 H)(\nabla^2\psi) = \text{Id}$ we have

$$\begin{aligned} 1 = \det(\nabla^2\psi)\det(\nabla^2 H) &= \mu \, \det(\nabla^2\psi)\left(-\psi + \frac{1}{3}y^T(\nabla^2 H)y\right) \\ &= -\mu \, \det(\nabla^2\psi)\,\psi + \frac{\mu}{3}(\nabla\psi)^T \left(\text{adj}(\nabla^2\psi)\right)(\nabla\psi). \end{aligned}$$

To prove the boundary values, we use the relation $-\mu\psi = \det(\nabla^2\varphi)$. Since $\nabla\varphi(\mathbb{R}^n) = \text{Int } P - P_c$ is bounded, $\lim_{|x|\to\infty} \det(\nabla^2\varphi) = 0$. And $x \to \infty$ corresponds to $y \to \partial P$ and $z \to \partial\Omega$. Thus $\lim_{z\to\partial\Omega} \psi(z) = 0$. □

The last function to introduce is $\chi \in C^\infty(\Omega)$ which is define by

$$\chi(z) = -\left(-\left(\frac{9}{4}\mu\right)^{1/3} \psi(z)\right)^{2/3}. \tag{5.3}$$

Lemma 5.2.5. *χ satisfies the equation*

$$\begin{cases} \det(\nabla^2\chi) = (-\chi)^{-5/2} & \text{on } \Omega \\ \chi = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. The boundary values follow immediately from the equation for ψ. Consider $\chi(z) = -(-c\,\psi(z))^{2/3}$ for $c > 0$. We compute

$$\begin{aligned} \chi_i &= \frac{2}{3}c\,(-c\psi)^{-1/3}\,\psi_i \\ \chi_{ij} &= \frac{2}{9}c^2\,(-c\psi)^{-4/3}\,\psi_i\,\psi_j + \frac{2}{3}c\,(-c\psi)^{-1/3}\,\psi_{ij}, \end{aligned}$$

and using the formula for the determinant of a rank-one perturbation and the PDE satisfied by ψ it follows that

$$\begin{aligned} \det(\nabla^2\chi) &= \frac{4}{9}c^2\,(-c\psi)^{-5/3}\left((-c\psi)\,\det(\nabla^2\psi) + \frac{1}{3}c\,(\nabla\psi)^T\,\text{adj}(\nabla^2\psi)(\nabla\psi)\right) \\ &= \frac{4}{9}c^2\,(-c\psi)^{-5/3}\left(\frac{c}{\mu}\right) = \frac{4c^3}{9\mu}\,(-\chi)^{-5/2}. \end{aligned}$$

Setting $c = \left(\frac{9}{4}\mu\right)^{1/3}$ finishes the proof. □

5.3 Examples

In this section we compute the domain Ω of the functions ψ and χ for the five toric Fano surface. In the cases where a Kähler–Einstein metric is known explicitly, this will be an explicit computation. In the other cases we will find an expansion for H near $\partial(P - P_c)$ in order to find $\Omega = \nabla H(\text{Int } P - P_c)$. We use the symmetries of φ to simplify the computations, so we discuss those first.

5.3.1 Symmetries of Kähler–Einstein metrics

The function φ solving

$$\begin{cases} \det(\nabla^2\varphi) = e^{-\mu\,\varphi} \\ \\ \nabla\varphi(\mathbb{R}^n) = \text{Int } P - P_c \end{cases}$$

is unique up to translations. If $A\,(P - P_c) = P - P_c$ for some $A \in Sl_n\mathbb{R}$, then $\widetilde{\varphi}(x) = \varphi(A^T x)$ satisfies the same differential equation by virtue of the change of variable formulas

$$\nabla\widetilde{\varphi}(x) = A\,\nabla\varphi(A^T x) \qquad \text{and} \qquad \nabla^2\widetilde{\varphi}(x) = A\,\nabla^2\varphi(A^T x)\,A^T.$$

Uniqueness up to translations implies $\varphi(A^T x) = \varphi(x - a)$, and a translation of φ can be chosen such that $\varphi(A^T x) = \varphi(x)$ for A such that $A\,(P - P_c) = P - P_c$. This implies a symmetry for the Legendre transform of φ as well.

$$\varphi^*(A^{-1}\,y) = \sup_x \{\, \langle x,\, A^{-1}y\rangle - \varphi(x)\} = \sup_x \{\, \langle x,\, y\rangle - \varphi(A^T x)\} = \varphi^*(y).$$

Since $A\,(P - P_c) = P - P_c$ if and only if $A^{-1}\,(P - P_c) = P - P_c$ it follows that $\varphi^*(A\,y) = \varphi^*(y)$.

Now we restrict ourselves to the case when $n = 2$ and examine the symmetries of H. To investigate the symmetries of H, we define $\widetilde{H}(y) = H(A\,y)$ for $A \in Sl_2\mathbb{R}$ such that $A\,(P - P_c) = P - P_c$.

$$\begin{aligned}
\nabla^2 \widetilde{H}(y) &= A^T\, \nabla^2 H(A\,y)\, A \\
&= A^T \left(e^{-\mu(A\,y\cdot\nabla\varphi^*(A\,y) - \varphi^*(A\,y))}\, \nabla^2\varphi^*(A\,y) + \frac{\mu}{3}\, \mathrm{adj}((A\,y\,y^T\,A^T) \right) A \\
&= e^{-\mu(y\cdot\nabla\phi^*(y) - \varphi^*(y))} \nabla^2\varphi^*(y) + \frac{\mu}{3} A^T\, A^{T-1} y\, y^T A^{-1}\, A \\
&= \nabla^2 H(y)
\end{aligned}$$

Thus $\nabla^2 H$ is invariant under linear transformations which preserve P. As stated before, H is only unique up to the addition of linear function, so this linear function can be chosen so that the invariance of $\nabla^2 H$ is inherited by H.

Summarizing, if $A\,(P - P_c) = P - P_c$ for $A \in Sl_2\,\mathbb{R}$ it follows that

$$\varphi(A^T\,x) = \varphi(x),\ \varphi^*(A\,y) = \varphi^*(y),\ \text{and}\ H(A\,y) = H(y).$$

The final observation is that the domain $\Omega = \nabla H(P - P_c)$ will have similar symmetries to P. If $A\,(P - P_c) = (P - P_c)$ for $A \in Sl_2\,\mathbb{R}$, then $A^T\,\Omega = \Omega$.

5.3.2 $\mathbb{P}^2$

The Fano polytope associated to $\mathbb{P}^2$ is

$$P = \cap_{i=1}^3 \{l_i(y) \geq 0\},$$

for $l_1(y) = y + 1$, $l_2(y) = y_2 + 1$, and $l_3(y) = -y_1 - y_2 + 1$. The center of mass of P is 0. The Kähler potential ϕ of the Kähler–Einstein metric $\omega = \sqrt{-1}\partial\overline{\partial}\,\phi$ which satisfies $\nabla\phi(\mathbb{R}^2) = \text{Int } P$ and $\det(\nabla^2\phi) = e^{-\mu\phi}$ is given by

$$\phi(x) = -\frac{2}{\mu}\log(3\,\mu) - x_1 - x_2 + \frac{3}{\mu}\log\left(1 + e^{\mu\, x_1} + e^{\mu\, x_2}\right).$$

The coordinates $y = \nabla\phi(x)$ satisfy the relations

$$1 - y_1 - y_2 = \frac{3}{1 + e^{\mu\, x_1} + e^{\mu\, x_2}}, \quad e^{\mu\, x_1} = \frac{y_1 + 1}{1 - y_1 - y_2}, \text{ and } \quad e^{\mu\, x_2} = \frac{y_2 + 1}{1 - y_1 - y_2}.$$

Together these show

$$\phi^*(y) = \frac{1}{\mu}\log\left(\frac{\mu^2}{3}\right) + \sum_{i=1}^{3}\frac{1}{\mu}\, l_i(y)\log(l_i(y)),$$

so the cone angle along each divisor is $\beta_i = \mu\, l_i(0) = \mu$.

$$H = \frac{\mu}{6}(y_1^2\, y_2 + y_1\, y_2^2) + \frac{\mu}{3}(y_1^2 + y_1\, y_2 + y_2^2) + \frac{\mu}{3}.$$

We have chosen the normalization for H to have the same symmetries as ϕ^*. Then the domain for ψ, $\Omega = \nabla H(P)$, can be expressed as

$$\Omega = \left\{\left(\frac{6}{\mu}z_1 + 4\right) \geq \left(\frac{3}{\mu}z_2 + \frac{1}{2}\right)^2\right\} \bigcap \left\{\left(\frac{6}{\mu}z_2 + 4\right) \geq \left(\frac{3}{\mu}z_1 + \frac{1}{2}\right)^2\right\}$$
$$\bigcap \left\{\frac{15}{2} - \frac{18}{\mu^2}(z_1 - z_2)^2 - \frac{6}{\mu}(z_1 + z_2) \geq 0\right\}.$$

A closed form solution can be found for ψ in the y coordinates:

$$\psi(\nabla H(y)) = y \cdot \nabla H(y) - H(y) = -\frac{\mu}{3}(1 + y_1)\,(1 + y_2)\,(1 - y_1 - y_2).$$

We show the domains P and Ω associated to $\mathbb{P}^2$ in Figures 5.1 and 5.2.

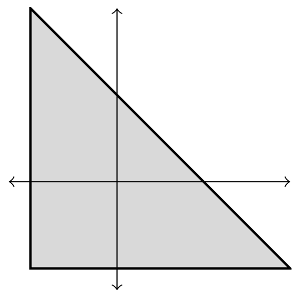

Figure 5.1: P for $\mathbb{P}^2$

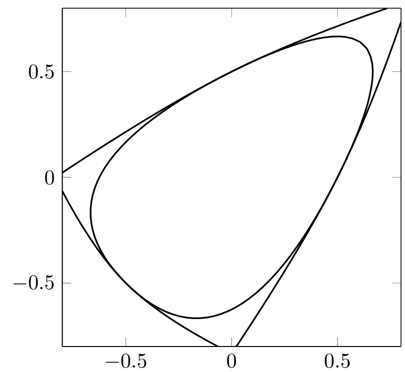

Figure 5.2: Ω for $\mathbb{P}^2$, $\mu = 1$

5.3.3 $\mathbb{P}^1 \times \mathbb{P}^1$

The Fano polytope associated to $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$P = \cap_{i=1}^4 \{l_i(y) \geq 0\},$$

for $l_1(y) = 1 + y_1$, $l_2(y) = 1 + y_2$, $l_3(y) = 1 - y_1$, and $l_4(y) = 1 - y_2$. The center of mass of P is 0. The Kähler potential ϕ of the Kähler–Einstein metric $\omega = \sqrt{-1}\partial\overline{\partial}\,\phi$ which satisfies $\nabla\phi(\mathbb{R}^2) = Int\,P$ and $\det(\nabla^2\phi) = e^{-\mu\phi}\ P$ is given by

$$\phi(x) = -\frac{2}{\mu}\log(2\mu) - x_1 - x_2 + \frac{2}{\mu}\,\log(1 + e^{\mu\,x_1}) + \frac{2}{\mu}\,\log(1 + e^{\mu\,x_2}).$$

The coordinates $y = \nabla\phi(x)$ satisfy the relations

$$e^{\mu\,x_i} = \frac{1 + y_i}{1 - y_i}.$$

Together these show

$$\phi^*(y) = \frac{2}{\mu}\log\left(\frac{\mu}{2}\right) + \sum_{i=1}^{4}\frac{1}{\mu}\,l_i(y)\log(l_i(y)),$$

so the cone angle along each divisor is $\beta_i = \mu\,l_i(0) = \mu$.

$$H = \frac{\mu}{4}(y_1^2 + y_2^2) - \frac{\mu}{12}y_1^2\,y_2^2 + \frac{\mu}{4}.$$

We have chosen the normalization for H to have the same symmetries as ϕ^*. Then the domain for ψ, $\Omega = \nabla H(P)$, can be expressed as

$$\Omega = \left\{3 - \left(\frac{3}{\mu}z_2\right)^2 - \frac{6}{\mu}z_1 \geq 0\right\} \bigcap \left\{3 - \left(\frac{3}{\mu}z_1\right)^2 - \frac{6}{\mu}z_2 \geq 0\right\}$$
$$\bigcap \left\{3 - \left(\frac{3}{\mu}z_2\right)^2 + \frac{6}{\mu}z_1 \geq 0\right\} \bigcap \left\{3 - \left(\frac{3}{\mu}z_1\right)^2 + \frac{6}{\mu}z_2 \geq 0\right\}$$

A closed form solution can be found for ψ in the y coordinates:

$$\psi(\nabla H(y)) = y \cdot \nabla H(y) - H(y) = -\frac{\mu}{4}(1+y_1)(1-y_1)(1+y_2)(1-y_2).$$

We show the domains P and Ω associated to $\mathbb{P}^1 \times \mathbb{P}^1$ in Figures 5.3 and 5.4.

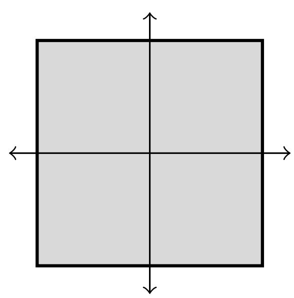

Figure 5.3: P for $\mathbb{P}^1 \times \mathbb{P}^1$

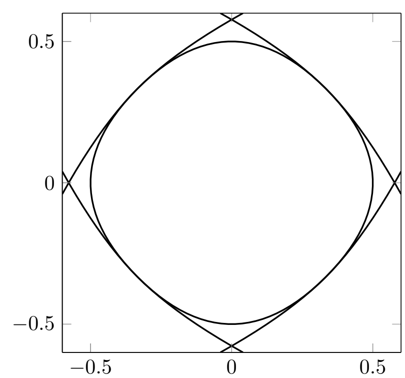

Figure 5.4: Ω for $\mathbb{P}^1 \times \mathbb{P}^1$, $\mu = 1$

5.3.4 $Bl_3\,\mathbb{P}^2$

The Fano polytope associated to $Bl_3\,\mathbb{P}^2$, the blowup of $\mathbb{P}^2$ at three points, is

$$P = \cap_{i=1}^{6}\{l_i(y) \geq 0\},$$

for $l_1(y) = 1+y_1$, $l_2(y) = 1+y_1+y_2$, $l_3(y) = 1+y_2$, $l_4(y) = 1-y_1$, $l_5(y) = 1-y_1-y_2$, and $l_6(y) = 1-y_2$. The center of mass of P is 0. There exists a Kähler Einstein metric ω solving

$$Ric(\omega) = \mu\,\omega + \sum_{i=1}^{6}(1-\mu)D_i,$$

with angles $\beta_i = \mu$ along each divisor. The potential ϕ for ω in the x coordinates associated to P does not have a closed form solution, but we can exploit the symmetries of the metric to find an expansion of the associated function H near ∂P which will enable us to compute $\Omega = \nabla H(P - P_c)$.

H is only unique up to the addition of a linear function, so we can choose H such that for every $A \in Sl_2\,\mathbb{R}$ such that $A\,P = P$, $H(A\,y) = H(y)$. The group of A with this property is the dihedral group of order 12 corresponding to the symmetries of a hexagon. In order to find $\Omega = \nabla H(P)$ we only need to compute ∇H on one edge of ∂P, and then we will use the symmetry to complete the description.

Consider the edge $y_2 - 1 = 0$, where $-1 \le y_1 \le 0$. Along the entire boundary $\nabla^2\phi = 0$, so

$$\nabla^2 H(y) = \begin{pmatrix} \frac{\mu}{3}y_2^2 & -\frac{\mu}{3}y_1\,y_2 \\ -\frac{\mu}{3}y_1\,y_2 & \frac{\mu}{3}y_1^2 \end{pmatrix}.$$

Thus along $y_2 = 1$ we have $H_{11} = \dfrac{\mu}{3}$. The symmetry of H implies $H(-1,1) = H(0,1)$ so we can integrate H_{11} along this edge to find

$$H\Big|_{\{y_2=1\}} = \frac{\mu}{6}\left(y_1 + \frac{1}{2}\right)^2 + c,$$

for some constant c which we do not need to determine. This immediately implies

$$H_1\Big|_{\{y_2=1\}} = \frac{\mu}{3}\left(y_1 + \frac{1}{2}\right).$$

In order to find H_2 along this edge, we just integrate its value from $(-1,1)$ which we know from the symmetry is $H_2(-1,1) = -H_1(-1,1) = \dfrac{\mu}{6}$.

$$H_2(y_1,1) = \frac{\mu}{6} + \int_{-1}^{y_1} H_{12}(t,1)\,dt = \frac{\mu}{6} - \int_{-1}^{y_1} \frac{\mu}{3} t\,dt = \frac{\mu}{6}\left(2 - y_1^2\right).$$

Thus along this edge

$$\frac{6}{\mu} z_1 = 2y_1 + 1 \qquad \text{and} \qquad \frac{6}{\mu} z_2 = 2 - y_1^2.$$

The corresponding edge of Ω is thus

$$\left\{ \left(\frac{3}{\mu} z_1 - \frac{1}{2} \right)^2 = 2 - \frac{6}{\mu} z_2 \right\}$$

from $(z_1, z_2) = (-\mu/6, \mu/6)$ to $(\mu/6, \mu/3)$.

Applying the dihedral symmetry of Ω, we can find all of Ω.

$$\Omega = \bigcap_{i=1}^{6} \{\gamma_i(z) \geq 0\},$$

$$\begin{cases}
\gamma_1 = 2 - 6/\mu\, z_2 - (3/\mu\,(z_1 - z_2) + 1/2)^2 \\
\gamma_2 = 2 - 6/\mu\, z_1 - (3/\mu\, z_2 - 1/2)^2 \\
\gamma_3 = 2 + 6/\mu\, z_2 - (3/\mu\, z_1 + 1/2)^2 \\
\gamma_4 = 2 + 6/\mu\, z_2 - (3/\mu\,(z_2 - z_1) + 1/2)^2 \\
\gamma_5 = 2 + 6/\mu\, z_1 - (3/\mu\, z_2 + 1/2)^2 \\
\gamma_6 = 2 - 6/\mu\, z_2 - (3/\mu\, z_1 - 1/2)^2
\end{cases}$$

We show the domains P and Ω associated to $BL_3\,\mathbb{P}^2$ in Figures 5.5 and 5.6.

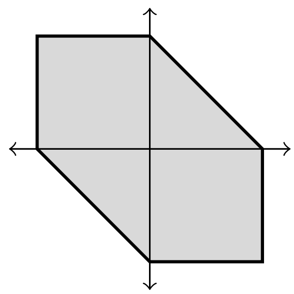

Figure 5.5: P for $BL_3\,\mathbb{P}^2$

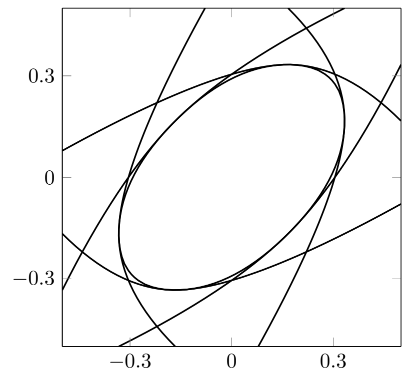

Figure 5.6: Ω for $BL_3\,\mathbb{P}^2$, $\mu = 1$

5.3.5 $Bl_1\,\mathbb{P}^2$

The Fano polytope associated to $Bl_1\,\mathbb{P}^2$, the blowup of $\mathbb{P}^2$ at one point is

$$P = \cap_{i=1}^{4}\{l_i(y) \geq 0\},$$

for $l_1(y) = 1 + y_1$, $l_2(y) = 1 + y_1 + y_2$, $l_3(y) = 1 + y_2$, and $l_4(y) = 1 - y_1 - y_2$. The center of mass of P is $(1/12, 1/12)$. There exists Kähler Einstein metric ω in L_P solving

$$Ric(\omega) = \mu\,\omega + (1 - \frac{13}{12}\mu)D_1 + (1 - \frac{7}{6}\mu)D_2 + (1 - \frac{13}{12}\mu)D_3 + (1 - \frac{5}{6}\mu)D_4,$$

with angles $\beta_1 = \beta_3 = 13/12\,\mu$, $\beta_2 = 7/6\,\mu$, and $\beta_4 = 5/6\,\mu$. The potential ϕ for ω inherits a $\mathbb{Z}_2$ symmetry from P, and likewise $\varphi(x) = \phi(x) - \langle x,\, P_c\rangle$ has the same symmetry. We will use the symmetry $\varphi(y_1,\, y_2) = \varphi(y_2, y_1)$ to find an expansion of the associated function H near $\partial P - P_c$ which will enable us to compute $\Omega = \nabla\varphi(P - P_c)$.

To facilitate the computation we will denote

$$P - P_c = \text{cvx}\,\{(-b, -a)\,(-a, -b),\ (c, -b),\ (-b, c)\,\}$$

where $a = 1/12$, $b = 13/12$ and $c = 23/12$. The polytope $P - P_c$ is symmetric across the line $y_1 = y_2$, so we can assume that $H(-b, -a) = H(-a, -b)$ and likewise $H(c, -b) = H(-b, c)$. Since H is only unique up to the addition of an affine function, we will further assume that both of these values are 0.

Consider the edge $y_2 = -b$ of $P - P_c$ where $-a \le y_1 \le c$. The function

$$g(t) = H(-a + t, -b)$$

for $0 \le t \le 2$ expresses the values of H along this edge. On the adjacent edge $y_1 + y_2 = -(a + b)$ we use the function

$$f(t) = H(-b + t, -a - t)$$

for $0 \le t \le 1$ to express the value of H. In particular, $f(1) = g(0) = H(-a, -b)$. Along these edges $\nabla^2 \varphi = 0$, so

$$\nabla^2 H(y) = \begin{pmatrix} \frac{\mu}{3} y_2^2 & -\frac{\mu}{3} y_1\, y_2 \\ -\frac{\mu}{3} y_1\, y_2 & \frac{\mu}{3} y_1^2 \end{pmatrix}. \tag{5.4}$$

It follows that

$$f''(t) = (H_{11} - H_{12} - H_{21} + H_{22})(-b + t, -a - t) = \frac{\mu}{3}(a + b)^2.$$

Since $f(0) = f(1) = 0$, it follows that

$$f(t) = -\frac{\mu}{6}(a + b)^2\, t + \frac{\mu}{6}(a + b)^2\, t^2.$$

This implies

$$H_1(-b+t,-a-t) - H_2(-b+t,-a-t) = f'(t) = -\frac{\mu}{6}(a+b)^2 + \frac{\mu}{3}(a+b)^2\, t. \quad (5.5)$$

We can do the same computation for g to show

$$g(t) = -\frac{\mu}{3}b^2\, t + \frac{\mu}{6}b^2\, t^2,$$

and

$$H_1(-a+t,-b) = g'(t) = -\frac{\mu}{3}b^2 + \frac{\mu}{3}b^2\, t \quad (5.6)$$

Since $y_1 = -a+t$ for the function g, it follows that along the edge $y_2 = -b$

$$z_1 = -\frac{\mu}{3}b^2 + \frac{\mu}{3}b^2\,(a+y_1).$$

Now we need to compute $z_2 = H_2$ along the edge $y_2 = -b$ in order to find the relationship between z_1 and z_2 along this boundary component. Evaluating equation 5.5 at $t=1$ implies

$$H_1(-a,-b) - H_2(-a,-b) = f'(1) = \frac{\mu}{6}(a+b)^2,$$

and evaluating equation 5.6 at $t=0$ implies

$$H_1(-a,-b) = g'(0) = -\frac{\mu}{3}b^2.$$

Together these two equations imply

$$H_2(-a,-b) = -\frac{\mu}{3}b^2 - \frac{\mu}{6}(a+b)^2.$$

We can use this value of H_2 to compute

$$H_2(-a+t,-b) = -\frac{\mu}{3}b^2 - \frac{\mu}{6}(a+b)^2 + \int_0^t H_{21}(-a+s,-b)\,ds.$$

Equation (5.4) implies $H_{21} = -\frac{\mu}{3} y_1\, y_2$ along the boundary, so

$$\begin{aligned} H_2(-a+t,-b) &= -\frac{\mu}{3}b^2 - \frac{\mu}{6}(a+b)^2 + \frac{\mu}{3}b\int_0^t (-a+s)\,ds \\ &= -\frac{\mu}{3}b^2 - \frac{\mu}{6}(a+b)^2 + \frac{\mu}{3}b\big(-a\,t + \frac{1}{2}t^2\big). \end{aligned}$$

Since $t = a + y_1$, we get

$$z_2 = -\frac{\mu}{3}b^2 - \frac{\mu}{6}(a+b)^2 - \frac{\mu}{3}ab(a+y_1) + \frac{\mu}{6}b\,(a+y_1)^2.$$

The equations for z_1 and z_2 imply ∇H sends the boundary component $y_2 = -b$ of $P - P_c$ to the boundary component

$$\frac{3}{\mu}z_2 = -b^2 - \frac{1}{2}(a+b)^2 - ab\big(1 + \frac{3}{\mu\, b^2}z_1\big) + \frac{b}{2}\big(1 + \frac{3}{\mu\, b^2}z_1\big)^2$$

of Ω. By the symmetry of $P - P_c$ the boundary component $y_1 = -b$ gets mapped to

$$\frac{3}{\mu}z_1 = -b^2 - \frac{1}{2}(a+b)^2 - ab\big(1 + \frac{3}{\mu\, b^2}z_2\big) + \frac{b}{2}\big(1 + \frac{3}{\mu\, b^2}z_2\big)^2.$$

The other two boundary components can be computed in a similar way. We show the domains P and Ω associated to $BL_1\,\mathbb{P}^2$ in Figures 5.7 and 5.8.

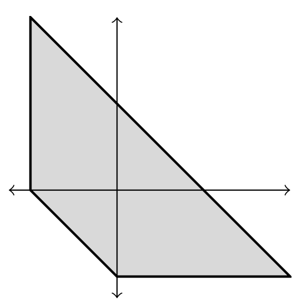

Figure 5.7: P for $BL_1\,\mathbb{P}^2$

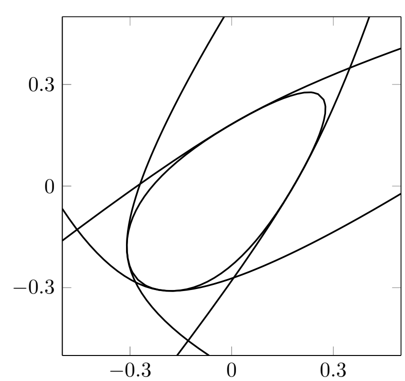

Figure 5.8: Ω for $BL_1\,\mathbb{P}^2$, $\mu = 1/2$

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