THE METAPLECTIC CASE OF THE WEIL-SIEGEL FORMULA

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ABSTRACT

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The Weil-Siegel formula, in the form developed by Weil, asserts the equality of a special value of an Eisenstein series with the integral of a related theta series. Recently, Kudla and Rallis have extended the formula into the range in which the Eisenstein series fails to converge at the required special value, so that Langlands' meromorphic analytic continuation must be used. In the case addressed by Kudla and Rallis, both the Eisenstein series and the integral of the theta series are automorphic forms on the adelic symplectic group. This thesis concentrates on extending the Weil-Siegel formula in the case in which both functions are automorphic forms on the two-fold metaplectic cover of the adelic symplectic group. First of all, a concrete model of the global metaplectic cover mentioned above is constructed by modifying the local formulas of Rao. Next, the meromorphic analytic continuation of the Eisenstein series is shown to be holomorphic at the special value in question. In the course of this work, we develop the functional equation and find all poles of an interesting family of local zeta-integrals similar to those studied in a paper of Igusa. Finally, the Weil-Siegel formula is proven in many cases by the methods of Kudla and Rallis.

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0. Introduction

The Weil-Siegel formula, in the form developed by Weil in [W1] and [W2], asserts the equality of a special value of an Eisenstein series with the integral of a related theta series. Recently, Kudla and Rallis [K-R1], [K-R2] have extended the formula into the range in which the Eisenstein series fails to converge at the required special value, so that Langlands' meromorphic analytic continuation must be used. In the case addressed by Kudla and Rallis, both the Eisenstein series and the integral of the theta series are automorphic forms on the adelic symplectic group.

The goal of this thesis is to extend the Weil-Siegel formula in the case in which both functions are automorphic forms on the two-fold metaplectic cover of the adelic symplectic group. In the course of this work, the following problems arise:

(1) A concrete model of the metaplectic cover mentioned above must be constructed. Although Weil develops the abstract existence and uniqueness of such a cover in [W1], there seem to be no concrete formulas available in the literature for the global 2-cocycle which defines the cover, with the exception of the $Sp(1) = SL(2)$ case (see [G]).

(2) The meromorphic analytic continuation of the Eisenstein series must be shown to be holomorphic at the special value in question.

(3) The Weil-Siegel formula itself must be shown to hold.

This last goal has been only partially fulfilled, in that there are some remaining cases for which the formula should hold, but for which the proof is incomplete.

Now we describe the setting of the Weil-Siegel formula in more detail. To begin with, fix a number field k, and write $\mathbb{A}$ for the ring of adeles of k. We then consider a symplectic vector space $W = k^{2n}$ with a standard symplectic form $<,>$, and an m-dimensional k vector space V with a non-degenerate symmetric form $(,)$. Let the automorphism groups of these two spaces be denoted by

$$G = Sp(W) \qquad \text{and} \qquad H = O(V).$$

Tensoring the spaces V and W yields a new symplectic space $\mathbf{W}$ with form $\ll,\gg = (,) \otimes <,>$. We then have an embedding $G \times H \hookrightarrow Sp(\mathbf{W}) \cong Sp(mn, k)$ so that the images of G and H form a dual reductive pair in $Sp(\mathbf{W})$. Considering the adelic points of these groups, we obtain dual pairs

$$G(\mathbb{A}) \times H(\mathbb{A}) \hookrightarrow Sp(\mathbf{W})_{\mathbb{A}}$$

Given a fixed character ψ of $k\backslash\mathbb{A}$, the oscillator or Weil representation $\omega = \omega_\psi$ of $Sp(\mathbf{W})_{\mathbb{A}}$ is a certain projective representation acting on the space $\mathcal{S}(V(\mathbb{A})^n)$ of Schwartz-Bruhat functions on $V(\mathbb{A})^n$. This defines a two-fold metaplectic cover

$$\widetilde{Sp}(\mathbf{W})_{\mathbb{A}} \xrightarrow{\pi} Sp(\mathbf{W})_{\mathbb{A}},$$

with ω lifting to an honest group representation of $\widetilde{Sp}(\mathbf{W})_{\mathbb{A}}$. There is also a unique splitting $Sp(\mathbf{W})_k \to \widetilde{Sp}(\mathbf{W})_{\mathbb{A}}$ of the k-rational points, and so we identify $Sp(\mathbf{W})_k$ with its image. This is developed in a paper of Weil [W1]. Then $\pi^{-1}(G(\mathbb{A}))$ forms a two-fold cover of $G(\mathbb{A})$ which is non-trivial if and only if $m = \dim(V)$ is odd. We will write ω for the oscillator representation restricted to $\pi^{-1}(G(\mathbb{A}))$. For any subgroup $L \subseteq G(\mathbb{A})$, we let $\tilde{L}$ denote $\pi^{-1}(L)$ in the case where m is odd.

Now given a function $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$, we may consider the theta function

$$\theta(g,h,\varphi) = \sum_{x \in V(k)^n} \omega(g)\varphi(h^{-1}x) \quad \text{for} \quad g \in \pi^{-1}(G(\mathbb{A})),\ h \in H(\mathbb{A}).$$

By [W1], this function is left $G(k)$-invariant. If we integrate out the orthogonal variable, we obtain a function

$$(\,*\,) \qquad I(g,\varphi) = \int_{H(k)\backslash H(\mathbb{A})} \theta(g,h,\varphi)\,dh$$

which converges if either

(1) $\{V, (\ ,\ )\}$ is anisotropic, so that $H(k)\backslash H(\mathbb{A})$ is compact, or

(2) $m - \alpha > n + 1$, where $\alpha \geq 1$ is the dimension of a maximal isotropic subspace of V.

Given these restrictions, equation $(*)$ above defines an automorphic form on

$$G(k)\backslash G(\mathbb{A}), \qquad \text{if } m \text{ is even, or}$$

$$G(k)\backslash \widetilde{G}(\mathbb{A}), \qquad \text{if } m \text{ is odd.}$$

In preparation for defining the Eisenstein series, we take a maximal parabolic subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix} \in G \right\}$$

and write $P = M \cdot N$, where

$$M = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \in G \mid a \in GL(n) \right\} \quad \text{and}$$
$$N = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \in G \mid b = {}^t b \right\}.$$

If K is a standard maximal compact subgroup of $G(\mathbb{A})$, then we have the Iwasawa decompositions $G(\mathbb{A}) = N(\mathbb{A}) \cdot M(\mathbb{A}) \cdot K$ and $\widetilde{G}(\mathbb{A}) = N(\mathbb{A}) \cdot \widetilde{M}(\mathbb{A}) \cdot \widetilde{K}$ (there is a natural splitting $N(\mathbb{A}) \hookrightarrow \widetilde{G}(\mathbb{A})$, so this last is well-defined). Finally, we need the functions on $G(\mathbb{A})$ and $\widetilde{G}(\mathbb{A})$ defined by

$$g = \left\{ \begin{matrix} n\, m(a)\, k \\ \\ n\, \widetilde{m(a)}\, \tilde{k} \end{matrix} \right\} \longmapsto |a(g)| \stackrel{d}{=} |\det(a)| \quad \text{(adelic abs. value)}.$$

We may then define an Eisenstein series for a $\widetilde{K}$ -finite function $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$ by

$$E(g, s, \varphi) = \sum_{\gamma \in P(k) \backslash G(k)} \Phi(\gamma g, s, \varphi) \ ,$$

where $g \in \pi^{-1}(G(\mathbb{A}))$, $s \in \mathbb{C}$, and

$$\Phi(g, s, \varphi) = |a(g)|^{s - s_o} \omega(g)\, \varphi(0)$$

with

$$s_o = \frac{m}{2} - \frac{n+1}{2}.$$

This series converges absolutely for $Re(s) > \frac{n+1}{2}$, and has a meromorphic continuation to the s -plane, and a functional equation relating values at s to those at $-s$ (see Arthur [A] for the $G(\mathbb{A})$ case, and Morris [M] for the $\widetilde{G}(\mathbb{A})$ case).

The original Weil-Siegel formula [W2] asserts that as long as $m > 2n+2$ (so that $s_o > \frac{n+1}{2}$), we have

$$E(g, s_o, \varphi) = I(g, \varphi) .$$

In other words, this identity holds in all cases in which the Eisenstein series is absolutely convergent at $s = s_o$. In two recent papers, [K-R1] and [K-R2], Kudla and Rallis extend the identity in the case where both sides are automorphic forms on $G(\mathbb{A})$ (i.e., when $m = \dim(V)$ is even):

THEOREM [K-R2]. *Let m be even, and let α be the dimension of a maximal isotropic subspace of $V(k)$. Assume that $\alpha = 0$ or that $m - \alpha > n+1$. Then for all K -finite $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$*

(i) $E(g, s, \varphi)$ is holomorphic at $s = s_o$, and

(ii) $E(g, s_o, \varphi) = \kappa \cdot I(g, \varphi)$ for all $g \in G(k)\backslash G(\mathbb{A})$, where

$$\kappa = \begin{cases} 1, & \textit{if} \quad m > n+1 \\ 2, & \textit{if} \quad m \leq n+1 \end{cases} .$$

This thesis concerns the case in which m is odd and $\alpha = 0$. The main result follows:

THEOREM. *Let $\{V, (\,,)\}$ be an anisotropic symmetric k -vector space of odd dimension m , and let $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$ be a $\widetilde{K}$ -finite function. Define the constant κ by*

$$\kappa = \begin{cases} 1, & \textit{if} \quad m > n+1 \text{ or } m = 1 \\ 2, & \textit{if} \quad 1 < m \leq n+1. \end{cases}$$

Then the Eisenstein series $E(g,s,\varphi)$ *is holomorphic at* $s = s_o$ *for all pairs* (m,n) *, and the equality*

$$E(g,s_o,\varphi) = \kappa \cdot I(g,\varphi), \qquad g \in G(k)\backslash\widetilde{G}(\mathbb{A})$$

holds in the following cases:

(i) $m = 1$,

(ii) $m = 3,\ n = 1$ *or* 2 ,

(iii) $3 < m \leq n+1$,

(iv) $m > n+3$.

If we accept Conjecture 10.2.3 (see chapter 10), then all cases with $m = n+2$ *and* $m = n+3$ *also hold, with the exception of* $(m,n) = (7,4)$.

The proof begins with a quick reduction to the problem of proving the equality of the constant terms:

$$E_{\widetilde{P}}(g,s_o,\varphi) = \kappa \cdot I_{\widetilde{P}}(g,\varphi) = \kappa \cdot \Phi(g,s_o,\varphi)$$

(the last equality being immediate by interchanging the order of integration). Next, the constant term of the Eisenstein series is easily written as the sum of $n+1$ terms:

$$E_{\widetilde{P}}(g,s,\varphi) = \Phi(g,s,\varphi) + \sum_{r=1}^{n-1} E^r_{\widetilde{P}}(g,s,\varphi) + M(s)\Phi(g,s) \ ,$$

where the first term will match $I_{\widetilde{P}}(g,\varphi)$ at $s = s_o$, the middle $n-1$ terms restrict to degenerate Eisenstein series on $\widetilde{M}(\mathbb{A})$ ($\cong$ a two-fold cover of

$GL(n, \mathbb{A})$), and where

$$(**) \qquad M(s)\Phi(g,s) = \int_{N(\mathbb{A})} \Phi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} xg, s)\, dx$$

is a $\widetilde{G}(\mathbb{A})$ -intertwining operator from a certain induced representation space to another. One then shows that the meromorphic continuations of all of these terms are finite at s_o , and that all but the first term either vanish or cancel each other at s_o , or in the cases $1 < m \leq n+1$, that exactly one of them survives to match the first term $\Phi(g, s_o, \varphi)$, giving the constant $\kappa = 2$ of the theorem.

The first difficulty arising in the proof was to come up with a workable model for the global two-fold cover of $G(\mathbb{A})$ (i.e., a global Weil representation and cocycle). This was done by suitably modifying the local Weil representation and cocycle developed in Rao [R]. At that point, most of the new work arises in studying the local analogue of the intertwining operator in $(**)$ above. A generalization of the method of Gindikin-Karpelevich is developed to simplify the image of the spherical vectors (defined by $\Phi_v^o(k,s) = 1$ at almost all places v satisfying $K_v \hookrightarrow \widetilde{K_v}$) under the local $M_v(s)$.

For the remaining finite places v , the possible poles of $M_v(s)$ when applied to arbitrary sections $\Phi_v(g,s)$ are determined by generalizing the work of Piatetski-Shapiro and Rallis in the appendix to section 4 of [PS-R2]. In the process, we derive the functional equations for a different type of zeta integral than that considered in [PS-R2] and [I1], and give a proof of some difficult details omitted in the appendix cited above. The zeta integrals in question are of the form

$$Z^F(s,\varphi) = \int_{Sym_n(k_v)} |\det X|^{s-\frac{n+1}{2}} F(X)\varphi(X)\, dX \ ,$$

where $Sym_n(k_v) = \{X \in M_{n,n}(k_v) \mid X = {}^tX\}$ and φ is a Schwartz function on this last space. In [I1], Igusa considers the case in which $F(X) = F(\det X)$ is a character of $k_v^\times$, and shows the existence of a functional equation of the form

$$Z^F(s,\hat{\varphi}) = \sum_{\substack{d \in Sym_n(k_v)/Gl(n) \\ \det(d) \neq 0}} c_d^F(s) \cdot Z_d^F(\tfrac{n+1}{2} - s, \varphi)$$

for some meromorphic functions $c_d^F(s)$, where Z_d^F indicates integration over the $Gl(n)$ -orbit of the representative $d \in Sym_n(k_v)$.

In computing the poles of $M_v(s)$, we find the factors $c_d^F(s)$ for functions of the form

$$F_\delta(X) = \frac{(\det X, \delta)_v}{\gamma_v(\det X, \frac{1}{2}\psi_v)} h_v(X) \ ,$$

where $\delta \in k_v^\times$ is fixed. Here $(\,,)_v$ is the local Hilbert symbol, $h_v(X)$ is the Hasse invariant of X , and $\gamma_v(a, \frac{1}{2}\psi_v) = \frac{\gamma_v(\psi_v(\frac{1}{2}a\cdot))}{\gamma_v(\psi_v(\frac{1}{2}\cdot))}$ is a quotient of Weil indices (in the notation of Rao [R]). While these F are not equivalent to characters of $k_v^\times$, they do have the following nice properties:

(1) they are defined on $Gl(n)$ -orbits,

(2) $F\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = F(A)F(B)$ for symmetric non-degenerate matrices A and B , and

(3) in dimension 1 , they are characters of second-degree on $k_v^\times/(k_v^\times)^2$.

The first two chapters will be concerned with developing a concrete model of the metaplectic group $\widetilde{Sp}(n, \mathbb{A})$ and with deriving formulas for the restriction of the Weil representation to the subgroups comprising a dual reductive pair. In chapter 3, the Weil-Siegel formula is described more fully using the notation and results of the first two chapters. Chapter 4 describes some relevant elements of the theory of automorphic forms as modified to apply to metaplectic covers, and gives a reduction argument essentially allowing us to prove Weil-Siegel for the constants terms. In chapters 5 through 7, the intertwining operator which appears in the expression for the constant term of E is studied thoroughly, and the locations of its poles are determined (at least when applied to "Weil-Siegel" sections). It is in these three chapters, and especially in chapter 6, that most of the new work of the thesis occurs. Chapter 8 concerns the middle terms appearing in the constant term of E, these terms being Eisenstein series on a two-fold cover of $GL(n, \mathbb{A})$. It is shown that the properties of these series can be obtained from those of the Eisenstein series on $GL(n, \mathbb{A})$ studied in [K-R1]. In the final two chapters, the poles and zeros of the various terms in the constant term are tabulated, proving the holomorphy of the constant term of E at s_o, and deriving the Weil-Siegel identity in many cases by the method of [K-R1].

§1.1 The symplectic group. We begin the construction by fixing a number field k. For any place v of k and corresponding absolute value $||_v$, let

$$
\begin{aligned}
k_v &= \text{the completion of } k \text{ with respect to } ||_v, \\
&\quad \text{and for } v < \infty, \\
\mathcal{O}_v &= \text{the local ring of integers of } k_v \\
&= \{z \in k_v : |z|_v \leq 1\}, \\
\mathcal{P}_v &= \text{the maximal prime ideal of } \mathcal{O}_v \\
&= \{z \in k_v : |z|_v < 1\} = \pi_v \mathcal{O}_v \\
\mathcal{U}_v &= \mathcal{O}_v \sim \mathcal{P}_v = \text{ the units of } \mathcal{O}_v \\
&= \{z \in k_v : |z|_v = 1\}.
\end{aligned}
$$

The ring of adeles of k will always be denoted by $\mathbb{A}$. Fix an integer $n \geq 1$, and suppose that $(W, <,>)$ is a symplectic vector space of dimension $2n$ over k, so that $<,>$ is a non-degenerate, skew-symmetric, k-bilinear form on W. The **symplectic group** $Sp(W)$ is the group of linear automorphisms of W which preserve the form $<,>$.

We will frequently identify $Sp(W)$ with the subgroup $Sp(n,k)$ of $GL(2n,k)$ defined as follows: setting $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in GL(2n,k)$, we get a symplectic form on k^{2n} (viewed as a space of row vectors) via

$$(w_1, w_2) \mapsto w_1 J\, {}^t w_2,$$

writing ${}^t v$ for the transpose of a vector or matrix. We then define $Sp(n,k)$ to be $\{g \in GL(2n,k) \mid gJ\,{}^t g = J\}$. Letting matrices act on row vectors in k^{2n} by right multiplication, it is clear that $Sp(n,k)$ is the symplectic group of linear transformations of k^{2n} preserving the form defined by J above. The isomorphism between $Sp(W)$ and $Sp(n,k)$ comes by the choice of a **symplectic basis** for $(W, <,>)$: that is, a basis $\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$ chosen such that $< e_i, e_j^* >= \delta_{ij}$, and $< e_i, e_j >=< e_i^*, e_j^* >= 0$ for all i and j with $1 \leq i, j \leq n$.

Given our choice of basis above, we will write $X = \mathrm{span}_k\{e_1, \ldots, e_n\}$ and $Y = \mathrm{span}_k\{e_1^*, \ldots, e_n^*\}$, so that X and Y are maximal isotropic subspaces of W, placed in duality via $<,>$. We then have $W = X \oplus Y$.

It is necessary to say something here about the notational convention used to represent elements of the group $Sp(W)$. An element $g \in Sp(W)$ may be naturally written in block form as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where we are thinking of these entries as being on the one hand linear transformations (acting on the right) $a : X \to X$, $b : X \to Y$, $c : Y \to X$, and $d : Y \to Y$. In this guise, we write $(x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (xa + yc, xb + yd) \in X \oplus Y$ for $x \in X$ and $y \in Y$. This is thoroughly explained in the beginning of Weil [W1]. On the other hand, in certain contexts it will be very convenient to recall that we have fixed a symplectic basis, and so we may consider a, b, c, and d as n by n matrices. Hopefully this will not cause much confusion; the sliding back and forth between viewpoints should make the statements and proofs to come more concise and easier to follow.

Now, tensoring all of the vector spaces above with k_v over k, we write $W_v = W(k_v) = W(k) \otimes_k k_v$, etc., for the local objects attached to a place $v \in \Sigma_k$, where

$$\Sigma_k = \{\text{all places of } k\}.$$

Consider a finite place v, and let $W_v^o \subset W_v$ be the $\mathcal{O}_v$-lattice

$$W_v^o = \operatorname{span}_{\mathcal{O}_v}\{e_1, \ldots, e_n^*\}.$$

Then letting $K_v \subset Sp(W_v)$ be the subgroup of transformations preserving W_v^o, we see that K_v is a maximal compact (open) subgroup of $Sp(W_v)$. As in Tate's thesis [T], the adelized symplectic group $Sp(W)_{\mathbb{A}} \stackrel{d}{=} Sp(W(\mathbb{A}))$ is the restricted direct product of the groups $\{Sp(W_v)\}_{v\in\Sigma_k}$ with respect to the collection $\{K_v\}_{v<\infty}$.

In this chapter, we intend to construct a non-trivial two-fold topological covering group $Mp(W)_{\mathbb{A}}$ of $Sp(W)_{\mathbb{A}}$, and related local two-fold covers for each $v \in \Sigma_k$. The foundation for the construction is Weil's paper [W1], in which he first contructs a cover of $Sp(n, \mathbb{A})$ with kernel isomorphic to the circle, and then goes on to prove (abstractly) that this cover contains a two-fold cover of $Sp(n, \mathbb{A})$, which is then unique by the work of Moore [Me]. Although this was all done some years ago, there appears to be no concrete construction available in the literature for the *global* metaplectic group, aside from the $n = 1$ case. This case was done by Kubota, whose work is summarized nicely in [G].

We begin by reviewing the construction of the local metaplectic group and Weil representation developed by Rao in [R], and then renormalize his representation in §1.6, using it to construct a global metaplectic group and Weil representation. First we must consider the Heisenberg group.

§1.2 The Heisenberg group. We work over a local field k_v for the time being, and fix a continuous character $\psi_v : k_v^+ \to \mathbf{T}$, where k_v^+ denotes the additive group of k_v, and $\mathbf{T} =$ the unit circle in $\mathbf{C}$. Choose a Haar measure $d_v x$ on k_v which is self-dual with respect to the identification $k_v \to \hat{k}_v$ given by $x \longmapsto \{z \mapsto \psi_v(xz)\}$. In other words, defining the **Schwartz-Bruhat functions** on k_v by

$$\mathcal{S}(k_v) \stackrel{def}{=} \begin{cases} \text{the usual Schwartz functions on } k_v, & \text{if } k_v = \mathbf{R}, \text{ or } \mathbf{C} \\ \text{the locally constant, compactly sup-} & \text{otherwise,} \\ \text{ported functions on } k_v, & \end{cases}$$

and defining the Fourier transform of $f \in \mathcal{S}(k_v)$ by

$$\hat{f}(x) = \int_{k_v} f(z)\psi_v(xz)\, d_v z,$$

the Fourier inversion formula holds: $(\hat{f})\hat{\ }(-x) = f(x)$. For all of this, see Tate [T]. Define the **Heisenberg group** $\mathcal{H}(W_v) = \mathcal{H}_v$ to be the set $W_v \times k_v$ with group law

$$(w,t)\cdot(w',t') = (w + w', t + t' + \tfrac{1}{2} < w, w' >).$$

It is easily seen that the center of $\mathcal{H}_v$ is the set

$$\{(0,t) \in \mathcal{H}_v \mid t \in k_v\} \simeq k_v^+,$$

so that we may consider ψ_v to be a character of the center in a natural way. Abusing notation a bit, we write $\mathcal{H}(Y_v) = \{(y,t) \in \mathcal{H}_v \mid y \in Y_v\}$, and note that since Y_v is isotropic, $\mathcal{H}(Y_v) \simeq Y_v \times k_v$ as a group, and ψ_v extends to a character of $\mathcal{H}(Y_v)$ via $\psi_v(y,t) \overset{d}{=} \psi_v(t)$.

Next, we induce ψ_v from $\mathcal{H}(Y_v)$ to $\mathcal{H}_v$: define $\mathrm{Ind}_{\mathcal{H}(Y_v)}^{\mathcal{H}(W_v)}(\psi_v)$ to be the space of measurable functions $f : \mathcal{H}_v \to \mathbf{C}$ such that

(1) $f(hg) = \psi_v(h)f(g)$ for all $h \in \mathcal{H}(Y_v)$, $g \in \mathcal{H}_v$, and

(2) $\int_{X_v} |f(x,0)|^2\,dx < \infty$.

We let $\rho = \rho_{\psi_v}$ denote the representation of $\mathcal{H}_v$ acting on the space above by

$$[\rho(h_1)f](h_2) = f(h_2h_1),$$

and one checks that the center k_v of $\mathcal{H}_v$ acts by the character ψ_v . In fact, ρ is an irreducible unitary representation of $\mathcal{H}_v$, and since it has central character ψ_v , the Stone-Von Neumann theorem guarantees us that it is unique up to unitary equivalence.

While the above is a natural development of this representation, a more commonly used model is the Schrödinger model. Since functions in $\mathrm{Ind}_{\mathcal{H}(Y_v)}^{\mathcal{H}(W_v)}(\psi_v)$ are determined by their values on $\mathcal{H}(Y_v)\backslash\mathcal{H}_v \simeq \{[x,0] \in \mathcal{H}_v \mid x \in X_v\} \simeq X_v$, and noting (2) above in the definition of $\mathrm{Ind}_{\mathcal{H}(Y_v)}^{\mathcal{H}(W_v)}(\psi_v)$, it is easily seen that the map

$$\mathrm{Ind}_{\mathcal{H}(Y_v)}^{\mathcal{H}(W_v)}(\psi_v) \longrightarrow L^2(X_v)$$

$$f \longmapsto \{x \mapsto f([x,0])\}$$

is an isomorphism of Hilbert spaces. Carrying over the $\mathcal{H}_v$ -action, and denoting the new representation by $\overline{U} = \overline{U}_{\psi_v}$, we obtain the formulas

$$\overline{U}([0,t])\varphi(x) = \psi_v(t)\varphi(x)$$

$$\overline{U}([y_o,0])\varphi(x) = \psi_v(< x, y_o >)\varphi(x)$$

$$\overline{U}([x_o,0])\varphi(x) = \varphi(x + x_o) \tag{1.2.1}$$

for $\varphi \in L^2(X_v)$, and for all $x, x_o \in X_v, y \in Y_v$, and $t \in k_v$. This representation of $\mathcal{H}_v$ in $L^2(X_v)$ is called the **Schrödinger model.**

We now notice that there is a right action of $g \in Sp(W_v)$ on $\mathcal{H}_v$ via $(w,t) \mapsto (w,t)^g = (wg,t)$, and that g fixes the center of $\mathcal{H}_v$. Hence, for any $g \in Sp(W_v)$, we may define a new representation $\overline{U}^g$ of $\mathcal{H}_v$ via $\overline{U}^g(h) = \overline{U}(h^g)$. Since the set of operators in $L^2(X_v)$ defined in this way is the same as that determined by $\overline{U}$, it is easily seen that $\overline{U}^g$ is another irreducible unitary representation of $\mathcal{H}_v$ with central character ψ_v . But the Stone-Von Neumann theorem tells us that $\overline{U}^g$ must then be equivalent to $\overline{U}$, so that there is a unitary operator ξ_g on $L^2(X_v)$ (unique up to a scalar multiple from $\mathbf{T}$ by Schur's lemma) satisfying

$$\xi_g^{-1} \circ \overline{U}(h) \circ \xi_g = \overline{U}(h^g) \qquad \text{for all } h \in \mathcal{H}_v. \tag{1.2.2}$$

We will call any system of operators $\{\xi_g\}_{g \in G_v}$ satisfying (1.2.2) a **projective Weil representation.**

§1.3 Rao's construction of the local Weil representation, I. Our two-fold cover of $Sp(W_v)$ will come from making a nice choice for each $g \in Sp(W_v)$ of an operator ξ_g on $L^2(X_v)$ satisfying (1.2.2), and such that $\xi_g \circ \xi_{g'} = \pm \xi_{gg'}$. Such a projective representation is developed by Rao [R], and we will use his choice as a start.

For the time being, we will fix our local field k_v and write G_v for $Sp(W_v)$. We occasionally omit the subscript v when the notation would be cumbersome. Following Rao, we begin by defining, for any j with $0 \leq j \leq n$,

$$\Omega_j = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_v \mid \operatorname{rank}(c) = j \right\}.$$

We let

$$P_v = \{g \in G_v \mid Y_v g = Y_v\} = \left\{ \begin{pmatrix} a & b \\ 0 & \check{a} \end{pmatrix} \in G_v \right\},$$

where $\check{a} = {}^t a^{-1}$. For any subset $S \subset \{1, 2, \ldots, n\}$, define an element τ_S in G_v by

$$e_i \cdot \tau_S = \begin{cases} e_i, & \text{if } i \notin S \\ -e_i^*, & \text{if } i \in S \end{cases}, \qquad e_i^* \cdot \tau_S = \begin{cases} e_i^*, & \text{if } i \notin S \\ e_i, & \text{if } i \in S. \end{cases}$$

From lemmas 2.14 and 2.15 of [R], we have:

Lemma 1.3.1 (Bruhat decomposition). *G_v can be written as the following disjoint union:*

$$G_v = \coprod_{j=0}^{n} \Omega_j,$$

where for any subset $S \subset \{1,2,\ldots,n\}$ with $j = |S|$, we have $\Omega_j = P_v \tau_S P_v$.

Rao first constructs a (local) projective representation $r : G_v \to \mathrm{Un}(L^2(X_v))$ using the Bruhat decomposition. Here, we use the notation $\mathrm{Un}(H)$ to stand for the group of unitary operators on a arbitrary Hilbert space H. For any $S \subset \{1,2,\ldots,n\}$, we let S' be the complement of S and write $X_S = \mathrm{span}_{k_v}\{e_i \mid i \in S\}$, so that $X_v = X_S \oplus X_{S'}$. Given any $x = \sum_{i \in S} x_i e_i$, we let $d_S x = \prod_{i \in S} dx_i$ be the Haar measure on X_S. Note that this is the dual measure to the measure $d_S y$ on Y_S (in the obvious notation) with respect to the dual pairing $(x,y) \mapsto \psi(< x,y >)$ on $X_S \times Y_S$, and using the analogous notion of Fourier transform. The projective representation r is defined as follows:

DEFINITION 1.3.2.

(1) *For* $p = \begin{pmatrix} a & b \\ 0 & \check{a} \end{pmatrix} \in P_v$, *and* $\varphi \in L^2(X_v)$, *we define*

$$r(p)\varphi(x) = |\det a|^{\frac{1}{2}} \psi_v(\tfrac{1}{2} < xa, xb >)\varphi(xa).$$

(2) *For any set* S *as above, and* $\varphi \in \mathcal{S}(X)$ *(the Schwartz-Bruhat functions on* X *), define*

$$r(\tau_S)\varphi(x) = \int_{X_S} \psi_v(< z, x_S{}^T >)\varphi(x_{S'} + z)\, d_S z,$$

where we write $x \in X_v = X_S \oplus X_{S'}$ *as* $x = x_S + x_{S'}$, *and let* $\tau = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. *This is just a partial Fourier transform of* f *with* *respect to the vector subspace* X_S *of* X_v *. Since* $\mathcal{S}(X_v)$ *is dense in* $L^2(X_v)$ *, this defines* $r(\tau_S)$ *on all of* $L^2(X_v)$ *, as is usual in Fourier analysis.*

(3) *Given an arbitrary* $g \in G_v$ *, choose a decomposition* $g = p_1 \tau_S p_2$ *as in Lemma 1.3.1, and define*

$$r(g) = r(p_1)r(\tau_S)r(p_2).$$

Rao proves that this last definition gives a well-defined operator for every element $g \in G_v$, and that these operators $r(g)$ satisfy (1.2.2). Actually, he uses a slightly different Heisenberg group and a different representation $\overline{U}$, but his conclusions translate to give the statement above. Rao also gives uniqueness conditions satisfied by r (see Theorem 3.6 [R]) and proves the following useful fact:

PROPOSITION 1.3.3. *For any arbitrary subset* S *of* $\{1, 2, \ldots, n\}$ *, let* $r_S(\cdot)$ *denote the standard (projective) Weil representation of* $Sp(W_S)$ *corresponding to the data* $W_S = X_S \oplus Y_S$ *, the symplectic basis being* $\{e_j, e_j^* \mid j \in S\}$ *. Let* $S_1, \ldots S_m$ *be a partition of* $\{1, 2, \ldots, n\}$ *and let* $g_j \in Sp(W_{S_j})$ *be given. If*

$$g = \operatorname{diag}(g_1, \ldots, g_m) \in Sp(W_v)$$

then on the space $\bigotimes_{i=1}^{m} \mathcal{S}(X_{S_i})$ *, we have*

$$r(g) = r_{S_1}(g_1) \otimes \ldots \otimes r_{S_m}(g_m).$$

Note that for the finite places, $\mathcal{S}(X_v) = \bigotimes_{i=1}^{m} \mathcal{S}(X_{S_i})$.

§1.4 Characters of second degree and the cocycle. The system of operators defined above by r gives only a **projective** representation because it fails to respect the group law: for any $g_1, g_2 \in G_v$, we know only that

$$r(g_1) \circ r(g_2) = c(g_1, g_2) r(g_1 g_2)$$

for some constant $c(g_1, g_2)$ depending on the g_i . This comes from the fact that the operators on both sides of the equation above satisfy (1.2.2) for $g = g_1 g_2$, which shows that this constant $c(g_1, g_2)$, called the **cocycle** defined by r , must in fact take values in the unit circle $\mathbf{T}$. In order to investigate the properties of this cocycle, we must first discuss characters of second degree.

This material comes mainly from Weil [W1].

DEFINITION 1.4.1. *Let A be any locally compact abelian group. A continuous function $f : A \to \mathbf{T}$ is called* **a character of second degree** *on A if the mapping*

$$A \times A \longrightarrow \mathbf{T}$$

$$(x, y) \longmapsto f(x+y) f(x)^{-1} f(y)^{-1}$$

is a bicharacter (i.e. a group homomorphism in each variable). Then writing $\hat{A}$ for the analytic dual group of A and denoting the pairing of $x \in A$ with $x^* \in \hat{A}$ by $[[x, x^*]] = [[x^*, x]] \in \mathbf{T}$, we see that we have

$$f(x+y) f(x)^{-1} f(y)^{-1} = [[x, y\rho]] = [[y, x\rho]]$$

for some continuous homomorphism $\rho = \rho(f) : A \to \hat{A}$. *Call* $\rho(f)$ *the* **symmetric morphism associated to** f . f *is said to be* **non-degenerate** *if* $\rho(f)$ *is an isomorphism.*

Given two locally compact abelian groups A and B with fixed Haar measures da and db , respectively, we define the modulus $|\alpha|$ of an isomorphism $\alpha : A \to B$ by

$$\int_B F(b)\, db = |\alpha| \int_A F(a\alpha)\, da$$

for $F \in C_c(B)$. Now fix a locally compact abelian group A, and choose Haar measures da and da^* on A and $\hat{A}$, respectively, such that the measures are dual with respect to the Fourier transforms defined by

$$\mathcal{F}(F)(a^*) = \int F(a)[[a, a^*]]\, da \quad \text{and}$$
$$\mathcal{F}(F^*)(a) = \int F(a^*)[[a, a^*]]\, da^*$$

for $F \in C_c(A)$ and $F^* \in C_c(\hat{A})$. Then Weil proves that for a non-degenerate character of second degree f on A , there exists a constant $\gamma(f)$ of absolute value 1, which we will call the **Weil index** of f , such that

$$\mathcal{F}(\Phi * f) = \gamma(f)|\rho(f)|^{-\frac{1}{2}}\mathcal{F}(\Phi) \cdot g$$

for all $\Phi \in C_c(A)$, where g is the character of second degree on $\hat{A}$ defined by $g(a^*) = f(a^*\rho^{-1})^{-1}$. Here $\Phi * f$ stands for the usual convolution of Φ and f . See section 14 of [W1] for details.

The appendix of Rao's paper [R] introduces a notation for the above which is extremely useful. Let k_v be a fixed local field, and let η_v be any non-trivial additive character of k_v. Then for any $a \in k_v^\times$, we will let $a\eta_v$ denote the character $x \mapsto \eta_v(ax)$. Now define $\gamma_v(\eta_v)$ to be the Weil index of the character of second degree given by $x \mapsto \eta_v(x^2)$, and let

$$\gamma_v(a, \eta_v) \stackrel{def}{=} \frac{\gamma_v(a\eta_v)}{\gamma_v(\eta_v)}$$

for any $a \in k_v^\times$. The Weil index of such characters is closely related to the **Hilbert symbol** of k_v, which is defined to be

$$(a, b)_v \stackrel{def}{=} \begin{cases} +1 & \text{if } ax^2 + by^2 = z^2 \text{ has a solution } (x, y, z) \neq (0,0,0) \text{ in } k_v^3, \\ -1 & \text{otherwise.} \end{cases}$$

Its most important properties are given in

PROPOSITION 1.4.2. *The Hilbert symbol* $(\,,)_v$ *of* k_v *is a non-degenerate bicharacter of the group* $k_v^\times/(k_v^\times)^2$ *if* $k_v \not\cong \mathbf{C}$. *In other words, for any* $a,\ b,\ c \in k_v^\times$,

(1) $(ab, c)_v = (a, c)_v (b, c)_v$,

(2) $(ac^2, b) = (a, b)$, *and*

(3) *if* a *is not a square, then there exists* $d \in k_v^\times$ *such that* $(a, d)_v = -1$.

Furthermore, if k *is a number field with completion* k_v *at* $v \in \Sigma_k$, *then we have the product formula*

$$\prod_{v \in \Sigma_k} (a, b)_v = 1$$

for all $a, b \in k$.

See Serre [Se] for details. Now, from the appendix of [R], we have the following properties of the Weil index of a non-trivial character η_v .

PROPOSITION 1.4.3 [R]. *For all* $a, b, c \in k_v$, $\gamma_v(ac^2, \eta_v) = \gamma_v(a, \eta_v)$ *and the function* $a \mapsto \gamma_v(a, \eta_v)$ *is a character of second degree on* $k_v^\times/(k_v^\times)^2$. *Moreover,*

$$\gamma_v(ab, \eta_v)\gamma_v(a\eta_v)^{-1}\gamma_v(b, \eta_v)^{-1} = (a, b)_v.$$

The following properties follow immediately:

(1) $\gamma_v(a, c\eta_v) = (a, c)_v\gamma_v(a, \eta_v)$.

(2) $\gamma_v(-1, \eta_v) = \gamma_v(\eta_v)^{-2}$.

(3) $\gamma_v(a, \eta_v)^2 = (-1, a)_v = (a, a)_v$.

(4) $\gamma_v(a, \eta_v)^4 = 1$ *and* $\gamma_v(\eta_v)^8 = 1$.

We will have frequent need for the following facts describing the behavior of both the Hilbert symbol and the Weil index when the local field is non-archimedean and has odd residual characteristic:

LEMMA 1.4.4. *Let* k_v *be a non-archimedean local field whose residue field has odd characteristic. Then*

(1) *the Hilbert symbol* $(\,,)_v$ *of* k_v *is trivial on the set* $\mathcal{U}_v \times \mathcal{U}_v$.

(2) *If the character* η_v *of* k_v^+ *is trivial on the set* $\mathcal{P}_v^{2a}$ *for some* $a \in \mathbf{Z}$, *but non-trivial on* $\mathcal{P}_v^{2a-1}$, *then the Weil index* $\gamma_v(x, \eta_v)$ *will equal* 1

for all $x \in \mathcal{U}_v$.

The first fact above is standard: see [Se], for example. The second is contained in the appendix of [R].

Given this background, we have the following theorem, which tells us how to compute $c(\cdot,\cdot)$.

THEOREM 1.4.5 (RAO).

(1) $c(p_1 g_1 p, p^{-1} g_2 p_2) = c(g_1, g_2)$ *for all* $p, p_1, p_2 \in P_v$ *and* $g_1, g_2 \in G_v$ *arbitrary.*

(2) $c(\tau_{S_1}, \tau_{S_2}) = 1$ *for any* $S_1, S_2 \subset \{1, 2, \ldots, n\}$.

(3) *If* $S_1, S_2, \ldots, S_m$ *is a partition of* $\{1, 2, \ldots, n\}$ *and* $c_S(\cdot,\cdot)$ *denotes the cocycle of* r_S *for any* S *(see Proposition 1.3.3), then*

$$c(g, g') = \prod_{j=1}^{m} c_{S_j}(g_j, g'_j)$$

where $g = \mathrm{diag}(g_1, \ldots, g_m)$ *and* $g' = \mathrm{diag}(g'_1, \ldots, g'_m)$.

(4) $c(\tau \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}, \tau) =$ *the Weil index of*

$$x \mapsto \psi(\tfrac{1}{2} < x, x \cdot \rho >) \qquad \textit{where } x \in X .$$

§1.5 Rao's construction of the local Weil representation, II. The purpose behind this work with the cocycle is to eventually construct a new group on which the projective representation will be a true group representation. Ignoring for the moment the fact that our cocycle c takes values in $\mu_8 = \{\text{all } 8^{\text{th}} \text{ roots of unity in } \mathbf{C}\}$, and supposing it were μ_2 -valued, we could define a group with underlying set $Sp(W_v) \times \mu_2$ and with group law

$$[g,\epsilon] \cdot [g',\epsilon'] = [gg', \epsilon\epsilon' c(g,g')].$$

Then a new representation given by $r(g,\epsilon)\varphi = \epsilon \cdot r(g)\varphi$ would satisfy

$$\begin{aligned} r([g,\epsilon][g',\epsilon'])\varphi &= r(gg', \epsilon\epsilon' c(g,g'))\varphi = \epsilon\epsilon' c(g,g') r(gg')\varphi \\ &= \epsilon\epsilon' r(g) r(g')\varphi = r(g,\epsilon) r(g',\epsilon')\varphi. \end{aligned}$$

Our new group would then be a two fold covering group of $Sp(W_v)$, which may be either a trivial or a non-trivial covering (may split or not).

Now we will write $r = r_v$ and $c = c_v$ to indicate the place v . As it turns out, we may renormalize r_v so that the cocycle does have values in μ_2 . Weil proved this indirectly in [W1]. It was done explicitly by Kubota (see [G]) for $Sp(1) = SL(2)$, and by Rao for the general case. We state Rao's results.

LEMMA 1.5.1 (RAO). *There exists a unique map* $g \mapsto x(g)$ *of* $Sp(W_v)$ *into*

$k_v^\times/(k_v^\times)^2$ *such that the following properties hold:*

(1) $x(p_1 g p_2) = x(p_1)x(g)x(p_2)$ *for all* $p_1, p_2 \in P_v$.

(2) $x(\tau_S) = 1$ *for all subsets* $S \subset \{1, 2, \ldots, n\}$.

(3) $x(p) = \det(p|_Y) \cdot (k_v^\times)^2$ *for all* $p \in P_v$.

Moreover, such a function is uniquely defined by

$$x(p_1 \tau_S p_2) = \det((p_1 p_2)|_Y) \cdot (k_v^\times)^2.$$

We then define the normalizing constants

$$m_v(g) = \gamma_v(x(g), \tfrac{1}{2}\psi_v)^{-1}\gamma_v(\tfrac{1}{2}\psi_v)^{-j}$$

for $g \in \Omega_j = P_v \tau_S P_v$, with $j = |S|$. This gives us a new system of operators

$$\tilde{r}_v(g) \stackrel{def}{=} m_v(g) r_v(g), \qquad \text{for } g \in G_v$$

with cocycle $\tilde{c}_v(\cdot, \cdot)$ defined by

$$\tilde{r}_v(g_1)\tilde{r}_v(g_2) = \tilde{c}_v(g_1, g_2)\tilde{r}_v(g_1 g_2). \tag{1.5.1}$$

It is elementary that c_v and $\tilde{c}_v$ have the relationship

$$\tilde{c}_v(g_1, g_2) = \Delta m_v(g_1, g_2) \cdot c_v(g_1, g_2) \tag{1.5.2}$$

where, for any function $d : G_v \to \mathbf{C}$, we will write

$$\Delta d(g_1, g_2) = \frac{d(g_1)d(g_2)}{d(g_1 g_2)}.$$

Equation (1.5.2) above states that the 2-cocycles $\tilde{c}_v$ and c_v are cohomologous (they differ by a coboundary Δm_v). For more on the relationships between group extensions and cohomology, see Jacobson ([J], pp. 363–369) and Moore [Me].

Rao derives the following explicit formula for $\tilde{c}_v$:

THEOREM 1.5.2 (RAO). *For any* $g_1, g_2 \in G_v$,

$$\begin{aligned}\tilde{c}_v(g_1, g_2) =& (x(g_1), x(g_2))_v \cdot (-x(g_1)x(g_2), x(g_1 g_2))_v \cdot \\ & ((-1)^l, \det \rho)_v \cdot [(-1, -1)_v]^{l(l+1)/2} \cdot h_v(\rho)\end{aligned}$$

where

ρ is the Leray invariant $-q(Y, Yg_1, Yg_2^{-1})$

$h_v(\rho)$ is the Hasse invariant of ρ,

$(\,,\,)_v$ is the Hilbert symbol, and

$2l = j_1 + j_2 - j - \dim\rho$ *for $g_1 \in \Omega_{j_1}$, $g_2 \in \Omega_{j_2}$,* and $g_1g_2 \in \Omega_j$.

Note: the Leray invariant is the isometry class of a certain inner product space associated to a triple of isotropic subspaces of the space W_v. This is explained in full in [R].

As an immediate corollary of this formula, we see that $\tilde{c}_v$ has values in μ_2. Notice also that if $k_v \cong \mathbf{C}$, then $\tilde{c}_v \equiv 1$, and so $\tilde{r}_v = r_v$ is a representation. We also have the following computational properties:

COROLLARY 1.5.3 (RAO). *Let $p, p_1, p_2 \in P_v$ and $g, g_1, g_2 \in G_v$ and $S_1, S_2 \subset \{1, 2, \ldots, n\}$ be arbitrary. Then*

(1) $\tilde{c}_v(\tau_{S_1}, \tau_{S_2}) = [(-1,-1)_v]^{l(l+1)/2}$ where $l = |S_1 \cap S_2|$.

(2) $\tilde{c}_v(p, g) = \tilde{c}_v(g, p) = (x(p), x(g))_v$.

(3) $\tilde{c}_v(p_1g_1, g_2p_2) = \tilde{c}_v(g_1, g_2) \cdot \frac{(x(p_1),x(g_1))_v\ (x(g_2),x(p_2))_v}{(x(p_1),x(p_2))_v\ (x(p_1p_2),x(g_1g_2))_v}$.

(4) $\tilde{c}_v(g_1p^{-1}, pg_2) = \tilde{c}_v(g_1, g_2)\ (x(p), -x(g_1)x(g_2))_v$.

§1.6 A renormalization. The procedure above describes a local projective Weil representation which may now be used to construct an honest group representation of a two-fold cover of G_v . We would ultimately like, however, to construct a nice projective representation of $Sp(W)_{\mathbb{A}} = \prod_v' Sp(W_v)$ (the restricted direct product with respect to $\{K_v\}_{v<\infty}$, as in §1.0) from these local representations using the method presented in Flath [F]. This method involves taking a "restricted tensor product" of the local representation spaces over all the places of k , as will be explained below. Unfortunately, we need to accomplish another renormalization of $\tilde{r}_v$ before this is possible. At this point, we depart from Rao's construction.

First of all, rather than considering the local spaces $L^2(X_v)$, we will work with the subspace $\mathcal{S}(X_v)$ of Schwartz-Bruhat functions on X_v . As this space is dense in L^2 , there is really no loss in doing this. In each space $\mathcal{S}(X_v)$, with finitely many exceptions, we pick out a certain distinguished vector which is almost a K_v -fixed vector for $\tilde{r}_v$.

Given our global number field k , fix a finite set of "bad" places

$$S_k = \{v \in \Sigma_k \mid \text{(i) } k_v = \mathbb{R},\ \mathbb{C}\text{ , or (ii) } v|2 \text{ for } v < \infty\text{ , or (iii) } \delta_v \neq \mathcal{O}_v\ \},$$

where δ_v is the local different of k . So $v \notin S_k$ implies that v is a finite, odd place of k which has $\mathcal{O}_v$ for a different.

We choose once and for all a global character

$$\psi : \mathbb{A} \longrightarrow \mathbb{T}$$

$$\psi = \prod_v \psi_v$$

which is trivial on k (embedded on the diagonal as usual). For convenience, we will also assume that the characters ψ_v are chosen as in Tate [T], at least for those places $v \notin S_k$. Since any other choice of global character will differ only by an element of k, this is not much of a restriction. One implication of this restriction is that for $v \notin S_k$, $\{x \in k_v \mid \psi_v(xy) = 1 \ \ \forall y \in \mathcal{O}_v\}$ is just the set $\mathcal{O}_v$ itself. This will imply that $\gamma_v(u, \psi_v) = 1$ for all $u \in \mathcal{U}_v$ if $v \notin s_k$ (see Proposition A.13 of [R]).

Fix a choice of symplectic basis of W as before, and for $v \notin S_k$, define the subgroup of $\mathcal{H}_v$

$$\mathcal{H}_v^o = \{(w,t) \in \mathcal{H}_v \mid w \in W_v^o = \mathrm{span}_{\mathcal{O}_v}\{e_1, \ldots, e_n^*\}, \text{ and } t \in \mathcal{O}_v\}.$$

LEMMA 1.6.1. *Given a place $v \notin S_k$, any vector $\varphi \in \mathcal{S}(X_v)$ fixed by $\mathcal{H}_v^o$ in the Schrödinger model must be a multiple of the characteristic function φ_v^o of $X_v^o = \mathrm{span}_{\mathcal{O}_v}\{e_1, \ldots, e_n\}$.*

Proof. Referring to equation (1.2.1), we see that φ is fixed by $\mathcal{H}_v^o$ if and only if, for all $x \in X_v$,

$$\begin{aligned} \psi_v(t)\varphi(x) &= \varphi(x) \qquad \text{for all } t \in \mathcal{O}_v, \\ \psi_v(< x, y_o >)\varphi(x) &= \varphi(x) \qquad \text{for all } y_o \in Y_v^o, \text{ and} \\ \varphi(x + x_o) &= \varphi(x) \qquad \text{for all } x_o \in X_v^o. \end{aligned}$$

From the definition of ψ_v in Tate, we see that the first equation holds automatically. Similarly, the second equation tells us that whenever $x \notin X_v^o$, we must have $\varphi(x) = 0$. The last equation then shows that $\varphi(x) = \varphi(0)$ for all $x \in X_v^o$. The conclusion follows. □

Now, we see from equation (1.2.2) that taking any fixed $g \in K_v = \mathrm{stab}(W_v^o) \subset G_v$, we must have

$$\overline{U}(h) \circ \tilde{r}_v(g)\varphi_v^o = \tilde{r}_v(g) \circ \overline{U}(h^g)\varphi_v^o = \tilde{r}_v(g)\varphi_v^o \qquad \text{for all } h \in \mathcal{H}_v^o .$$

In other words, $\tilde{r}_v(g)\varphi_v^o$ is an $\mathcal{H}_v^o$ -fixed vector, and so by the lemma must be some multiple of φ_v^o . Hence, for $v \notin S_k$, and any $g \in K_v$, we define the constant $\lambda_v(g)$ by

$$\tilde{r}_v(g)\varphi_v^o = \lambda_v(g)\varphi_v^o. \tag{1.6.1}$$

Since $\tilde{r}_v(g)$ is a unitary operator, it is immediate that $\lambda_v(g) \in \mathsf{T}$. We shall prove that, in fact, $\lambda_v : K_v \to \mu_2 = \{\pm 1\}$. To do this, we need the following decomposition of K_v .

LEMMA 1.6.2. *Identifying* K_v *with* $Sp(n, \mathcal{O}_v)$, *we write for each* $j = 0, 1, 2, \ldots, n$

$$\tau_j \overset{def}{=} \begin{pmatrix} 1_{n-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_j \\ 0 & 0 & 1_{n-j} & 0 \\ 0 & 1_j & 0 & 0 \end{pmatrix},$$

and define

$$P^o = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathcal{O}_v) \mid c \equiv 0 \pmod{\mathcal{P}_v} \right\}.$$

Then K_v *can be written as the disjoint union*

$$K_v = \coprod_{j=0}^{n} P^o \tau_j P^o.$$

Proof. There is a natural map $K_v = Sp(n, \mathcal{O}_v) \xrightarrow{\overline{(\ )}} Sp(n, \mathsf{F}_q)$ where $q = |\mathcal{O}_v/\mathcal{P}_v|$, given by reduction mod $\mathcal{P}_v$. Since F_q is a field (albeit finite), we have the usual Bruhat decomposition

$$Sp(n, \mathsf{F}_q) = \coprod_{j=0}^{n} \overline{P}^o \overline{\tau}_j \overline{P}^o,$$

where $\overline{P}^o$ is the image of P^o, and so $\overline{P}^o = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in Sp(n, \mathsf{F}_q) \mid c = 0 \right\}$. The proof follows easily by pulling back this decomposition. □

LEMMA 1.6.3. *If* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in P^o$, *then we may write*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \check{a} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$

where we have

(1) $a \in GL(n, \mathcal{O}_v) = \{x \in M(n, \mathcal{O}_v) \mid x^{-1}$ *exists, and* $x^{-1} \in M(n, \mathcal{O}_v)\}$,

(2) $ca^{-1} \equiv 0 \pmod{\mathcal{P}_v}$, *and*

(3) *each of the three matrices on the right lie in* $P^o \subset K_v$.

Proof. (1) follows easily by considering reduction modulo $\mathcal{P}_v$. The decomposition, as well as (2) and (3), are easy consequences of the definition of $Sp(n, k_v)$: for matrices $a, b, c, d \in M(n, k_v)$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in Sp(n, k_v) \Leftrightarrow a\,{}^t d - b\,{}^t c = 1_n$, $a\,{}^t b = b\,{}^t a$, and $c\,{}^t d = d\,{}^t c$. □

Using the information above, we begin our computation of $\lambda_v(\cdot)$.

LEMMA 1.6.4. *For* $v \notin S_k$, *any* $p \in P_v \cap K_v$, *and any set* $S \subset \{1, 2, \ldots, n\}$, *the following hold:*

(1) $\tilde{r}_v(p)\varphi_v^o = \varphi_v^o$,

(2) $\tilde{r}_v(\tau_S)\varphi_v^o = \varphi_v^o$,

(3) $\tilde{r}_v(\tau_S^{-1})\varphi_v^o = \varphi_v^o$, *and hence*

(4) $\lambda_v(p) = \lambda_v(\tau_S) = \lambda_v(\tau_S^{-1}) = 1$.

Proof. (1) Let $p = \begin{pmatrix} a & b \\ & \check{a} \end{pmatrix} \in P_v \cap K_v$. Then by definition,

$$\lambda_v(p) = \lambda_v(p)\varphi_v^o(0) = \tilde{r}_v(p)\varphi_v^o(0) = m_v(p)|a|_v^{\frac{1}{2}}\psi_v(0)\varphi_v^o(0) = m_v(p),$$

since $a \in GL(n, \mathcal{O}_v)$. Now

$$m_v(p) = \gamma_v(\det \check{a}, \tfrac{1}{2}\psi_v)^{-1} = \gamma_v(\frac{\det a}{2}\psi_v)/\gamma_v(\tfrac{1}{2}\psi_v) = 1,$$

as $\det(a)$ and 2 are in $\mathcal{U}_v$, and $v \notin S_k$.

(2) Referring to Definition 1.3.2 and Lemma 1.6.1, we write

$$X^o = X_S^o \oplus X_{S'}^o \subset X = X_S \oplus X_{S'}$$

for the $\mathcal{O}_v$ -lattices defined by the various subbases of $\{e_1, \ldots, e_n\}$, and let $\varphi_v^o, \varphi_{v,S}^o$, and $\varphi_{v,S'}^o$ be the characteristic functions of X^o, X_S^o , and $X_{S'}^o$ in the appropriate spaces of functions. Then it is easily seen that

$\varphi_v^o(x) = \varphi_{v,S}^o(x_S)\cdot\varphi_{v,S'}^o(x_{S'})$ for all $x = x_S + x_{S'}$ (pointwise multiplication).

So we then have

$$\begin{aligned} \tilde{r}_v(\tau_S)\varphi_v^o(x) &= m_v(\tau_S)\int_{X_S} \psi_v(< z, x_S\tau >)\varphi_v^o(x_{S'} + z)\, d_S z \\ &= m_v(\tau_S)\varphi_{v,S'}^o(x_{S'})\int_{X_S^o} \psi_v(< z, x_S\tau >)\, d_S z \end{aligned}$$

which, for $z = \sum_{i\in S} z_i e_i$, and $x_S = \sum_{i\in S} x_i e_i$

$$= m_v(\tau_S)\varphi^o_{v,S'}(x_{S'}) \int\limits_{\mathcal{O}_v^n} \psi_v(\sum_{i\in S} < z_i e_i, x_i e_i^* >) \prod_{i\in S} d_v z_i$$

$$= m_v(\tau_S)\varphi^o_{v,S'}(x_{S'}) \prod_{i\in S} \int\limits_{\mathcal{O}_v} \psi_v(z_i x_i)\, d_v z_i.$$

But now ψ_v is a character and $\mathcal{O}_v$ is a compact group, so we may use the basic identity

$$\int\limits_{\mathcal{O}_v} \psi_v(ab)\, d_v b = \begin{cases} 0, & a \notin \mathcal{O}_v \\ \text{meas}(\mathcal{O}_v) & a \in \mathcal{O}_v. \end{cases}$$

Since we have chosen our character ψ_v nicely (as in Tate) for $v \notin S_k$, we have measure $(\mathcal{O}_v) = 1$, and so we conclude that

$$\tilde{r}_v(\tau_S)\varphi^o_v(x) = m_v(\tau_S)\varphi^o_{v,S'}(x_{S'})\varphi^o_{v,S}(x_S) = m_v(\tau_S)\varphi^o_v(x).$$

(A much quicker proof would be to note that

$$\lambda_v(\tau_S) = \tilde{r}_v(\tau_S)\varphi^o_v(0) = m_v(\tau_S) \int\limits_{X_S} \varphi^o_v(z)\, d_S z = m_v(\tau_S),$$

but this is not as interesting.) Finally, since 2 is a unit in $\mathcal{O}_v$ and $x(\tau_S) = 1$, we have $m_v(\tau_S) = \gamma_v(x(\tau_S), \frac{1}{2}\psi_v)^{-1}\gamma_v(\frac{1}{2}\psi_v)^{-j} = 1$.

(3) By equation (1.5.1), $\tilde{r}_v(\tau_S^{-1})\tilde{r}_v(\tau_S)\varphi^o_v = \tilde{c}_v(\tau_S^{-1}, \tau_S)\varphi^o_v$ and so by (2) above, we see that $\tilde{r}_v(\tau_S^{-1})\varphi^o_v = \tilde{c}_v(\tau_S^{-1}, \tau_S)\varphi^o_v$. The conclusion follows easily from Corollary 1.5.3. □

Proposition 1.6.5. *Let $v \notin S_k$ be a fixed place of k. Given an element $g \in K_v = Sp(n, \mathcal{O}_v)$, suppose we have written $g = g_1\tau_j g_2 \in P^o\tau_j P^o$ as in Lemma 1.6.2. Then the following formula holds:*

$$\lambda_v(g) = \tilde{c}_v(g_1, \tau_j g_2)\ \tilde{c}_v(\tau_j, g_2)\ \tilde{c}_v(\tau, n(-c_1 a_1^{-1})\tau^{-1})\ \tilde{c}_v(\tau, n(-c_2 a_2^{-1})\tau^{-1})\cdot$$

$$(\det(a_1), x(n'(c_1 a_1^{-1})))_v \cdot (\det(a_2), x(n'(c_2 a_2^{-1})))_v$$

where we write $\tau = \tau_n$, $g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ for $i = 1, 2$, and

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad , \quad n'(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

for any $b = {}^t b \in M(n, k_v)$. In particular, notice that this proves that

$$\lambda_v(g) \in \mu_2 = \{\pm 1\} \qquad \text{for any } g \in K_v .$$

Proof. Let $g = g_1 \tau_j g_2$ as above. Then dropping the v subscripts,

$$\tilde{r}(g)\varphi^o = \tilde{c}(g_1, \tau_j g_2)\ \tilde{r}(g_1)\ \tilde{r}(\tau_j g_2)\varphi^o = \tilde{c}(g_1, \tau_j g_2)\ \tilde{c}(\tau_j, g_2)\ \tilde{r}(g_1)\ \tilde{r}(\tau_j)\ \tilde{r}(g_2)\varphi^o.$$

Since we know that each of these last three operators acts by a constant ($g_1, \tau_j, g_2 \in K_v$), we need only compute the contribution of each to the total. By the preceding lemma, $\tilde{r}(\tau_j)$ fixes φ^o, so it contributes nothing. Now, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P^o$, then

$$\begin{aligned} \tilde{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varphi^o &= \tilde{c}(n'(ca^{-1}), \begin{pmatrix} a & b \\ & \check{a} \end{pmatrix})\ \tilde{r}(n'(ca^{-1}))\ \tilde{r}\begin{pmatrix} a & b \\ & \check{a} \end{pmatrix}\varphi^o \\ &= (\det(a), x(n'(ca^{-1})))_v\ \tilde{r}(n'(ca^{-1}))\varphi^o \end{aligned}$$

by Lemmas 1.6.3, 1.6.4 and Corollary 1.5.3. But writing $n'(ca^{-1}) = \tau n(-ca^{-1})\tau^{-1}$, we see that

$$\begin{aligned} \tilde{r}(n'(ca^{-1}))\varphi^o =& \tilde{c}(\tau, n(-ca^{-1})\tau^{-1})\ \tilde{c}(n(-ca^{-1}), \tau^{-1}) \cdot \\ & \tilde{r}(\tau)\ \tilde{r}(n(-ca^{-1}))\ \tilde{r}(\tau^{-1})\varphi^o \\ =& \tilde{c}(\tau, n(-ca^{-1})\tau^{-1})\,\varphi^o \end{aligned}$$

by reference to Lemma 1.6.4. Combining all the constants gives us the needed formula. □

In order to renormalize $\tilde{r}_v(\cdot)$ so that φ_v^o is a K_v -fixed vector for almost all places, we must first extend the definition of λ_v from K_v to all of G_v in a manner suited to computations. We will do this by first fixing a generator π_v of the maximal ideal $\mathcal{P}_v$ of $\mathcal{O}_v$, and defining a homomorphism

$$u : P_v \to \mathcal{U}_v \quad \text{via} \quad u(p) = x(p)\pi_v^{-\mathrm{ord}_{\mathcal{P}}(x(p))}.$$

In other words, if $x(p) = u\pi_v^r$ for $u \in \mathcal{U}_v$, then $u(p) \stackrel{def}{=} u$. Finally, recall the **Iwasawa decomposition** of G_v , which states that $G_v = P_v \cdot K_v$.

PROPOSITION 1.6.6. *For any place* $v \notin S_k$ *, the formula*

$$\lambda_v(pk) \stackrel{def}{=} \lambda_v(k)\ (u(p), x(k))_v \quad \text{for all } p \in P_v \text{ , and } k \in K_v$$

gives a well-defined extension of λ_v *to all of* $G_v = Sp(n, k_v)$ *. This extension satisfies the following two properties.*

(1) $\lambda_v(p) = 1$ *for all* $p \in P_v$ *.*

(2) *For any* $p \in P_v$ *and* $g \in G_v$ *,*

$$\lambda_v(pg) = \lambda_v(g)\ (u(p), x(k))_v \ ,$$

where k *is any element of* K_v *such that* $g = p'k$ *for some* $p' \in P_v$ *.*

Proof. First, we must show this is well-defined: if $g \in G_v$ is written as $g = p_1k_1 = p_2k_2$ for $p_1, p_2 \in P_v$ and $k_1, k_2 \in K_v$, then let $p = k_1k_2^{-1} =$

$p_1^{-1}p_2 \in P_v \cap K_v$, so that $k_1 = pk_2$ and $p_1 = p_2p^{-1}$. Using p_1 and k_1 in the definition of $\lambda_v(g)$ yields

$$\lambda_v(k_1)\ (u(p_1), x(k_1))_v = \lambda_v(pk_2)\ (u(p_2p^{-1}), x(pk_2))_v$$

$$= [\lambda_v(k_2)\ (x(p), x(k_2))_v]\,[(u(p_2), x(k_2))_v\,(u(p), x(k_2))_v]$$

$$= \lambda_v(k_2)\ (u(p_2), x(k_2))_v$$

which is the definition using p_2 and k_2 . The first set of brackets above comes from

$$\lambda_v(pk_2) = \tilde{r}_v(pk_2)\,\varphi_v^o(0) = \tilde{c}_v(p, k_2)\ \tilde{r}_v(p)\ \tilde{r}_v(k_2)\,\varphi_v^o$$

$$= (x(p), x(k_2))_v\ \lambda_v(k_2)\ \tilde{r}_v(p)\,\varphi_v^o(0) = \lambda_v(k_2)\ (x(p), x(k_2))_v.$$

The second uses the fact that $(\,,)_v \equiv 1$ on $\mathcal{U}_v \times \mathcal{U}_v$ for $v \notin S_k$ (see Lemma 1.4.4), and that $x(p) \in \mathcal{U}_v$. This proves that the definition is independent of the decomposition used for an arbitrary $g \in G_v$. Both (1) and (2) are easy consequences of the definition. □

This allows us to renormalize $\tilde{r}_v$ as follows:

DEFINITION 1.6.7. *Let v be any place of k . The local projective Weil (or oscillator) representation which we will use is defined by the formula*

$$\omega_v(g) = \lambda_v(g)\tilde{r}_v(g) \qquad \textit{for } g \in G_v$$

acting on the space $L^2(X_v)$, or on the subspace $S(X_v)$ of smooth vectors when specified. Here we take $\lambda_v \equiv 1$ if $v \in S_k$. The cocycle of ω_v will be *denoted by* $\beta_v(\cdot,\cdot)$, *and satisfies*

$$\beta_v = [\Delta\lambda_v] \cdot \tilde{c}_v.$$

Among other things, one should note that for $v \notin S_k$, β_v satisfies

$$\beta_v(k_1, k_2) = 1 \qquad \text{for all } k_1, k_2 \in K_v .$$

This is easily seen by using φ_v^o in the definition of the cocycle defined by an operator.

§1.7 **The global Weil representation.** We may now construct the **restricted tensor product** $\mathcal{S}(X(\mathbb{A}))$ of the local spaces $\mathcal{S}(X_v)$ with respect to the collection of vectors $\{\varphi_v^o\}_{v \notin S_k}$. As in Flath [F], for any finite set Σ with $S_k \subset \Sigma \subset \Sigma_k$, we let $\mathcal{S}(X)_\Sigma = \bigotimes_{v\in\Sigma} \mathcal{S}(X_v)$. (All tensor products are over $\mathbf{C}$.) For $\Sigma \subset \Sigma'$ finite, define a map $\mathcal{S}(X)_\Sigma \to \mathcal{S}(X)_{\Sigma'}$ by $\otimes_{v\in\Sigma}\varphi_v \mapsto (\otimes_{v\in\Sigma}\varphi_v) \otimes (\otimes_{v\in\Sigma'\sim\Sigma}\varphi_v^o)$. This gives a directed system, and hence we define the restricted tensor product

$$\mathcal{S}(X(\mathbb{A})) = \bigotimes_v{}' \mathcal{S}(X_v) \quad \text{to be the direct limit} \quad \varinjlim_{\substack{\Sigma \text{ finite} \\ S_k \subset \Sigma \subset \Sigma_k}} \mathcal{S}(X)_\Sigma.$$

As a practical matter, this is spanned by the images of elements $\otimes_{v\in\Sigma}\varphi_v$ which we shall write as $\otimes_v'\varphi_v$, understanding that $\varphi_v = \varphi_v^o$ for $v \notin \Sigma$. Such vectors will be identified tacitly in the remainder of the paper with functions on $X(\mathbb{A})$: if $\varphi = \otimes_v'\varphi_v$, and $x \in X(\mathbb{A}) \overset{def}{=} X \otimes_k \mathbb{A} \cong \prod_v' X_v$ (restricted direct product with respect to X_v^o) then write $x = (x_v)_{v\in\Sigma_k} = (x_v)$ where $x_v \in X_v^o$

for almost all $v \in \Sigma_k$, and define $\varphi(x) = \prod_v \varphi_v(x_v)$. Since $\varphi_v^o(x_v) = 1$ for $x_v \in X_v^o$, this infinite product is well-defined. Hence we have a well-defined complex vector space of functions on $X(\mathbb{A})$, which will be called **Schwartz-Bruhat** functions (see [W1]). One can show that the space $\mathcal{S}(X(\mathbb{A}))$, when endowed with the product measure $dx = \prod_v d_v x_v$ (see Tate), is dense in $L^2(X(\mathbb{A}))$. Since we have renormalized our local representation so that φ_v^o is a K_v -fixed vector for ω_v at almost all places v, each element g of $G(\mathbb{A}) \stackrel{def}{=} Sp(W)_{\mathbb{A}} = \prod_v' Sp(W_v)$ (restricted w.r.t. $\{K_v\}_{v<\infty}$) defines an operator on $\mathcal{S}(X(\mathbb{A}))$ via

$$\omega(g)\varphi = \bigotimes_v{}' \omega_v(g_v)\varphi_v, \qquad \text{where}$$
$$g = (g_v) \in G(\mathbb{A}) \quad \text{and} \quad \varphi = \otimes_v' \varphi_v \in \mathcal{S}(X(\mathbb{A})).$$

As above, this extends to $L^2(X(\mathbb{A}))$ by a density argument. It is clear that this new projective representation has cocycle given by

$$\beta(g, g') = \prod_v \beta_v(g_v, g_v')$$

where the product will only involve a finite number of -1 terms. Note also that $\mathcal{H}(\mathbb{A}) = \prod_v' \mathcal{H}_v$ (restricted with respect to $\{\mathcal{H}_v^o\}$) acts on $\mathcal{S}(X(\mathbb{A}))$ via $\overline{U} = \bigotimes_v{}' \overline{U}_v$, and that $\overline{U}$ and ω satisfy the adelic analogue of equation (1.2.2). See Howe [H] for more information on the local to global transition.

§1.8 The local and global metaplectic groups. Following the general strategy in Weil [W1], we finally define the **local metaplectic group**, as a topological group, to be

$$Mp(W_v) = \left\{(g, \xi) \in Sp(W_v) \times \mathrm{Un}(L^2(X_v)) \mid \xi = \pm\omega_v(g)\right\}.$$

Here we are considering this to be a topological subgroup of the product group $Sp(W_v) \times \mathrm{Un}(L^2(X_v))$, where the first group has the topology inherited from $M(2n, k_v)$ (k_v is a topological field), and the second is given the strong operator topology. This last topology is defined by taking a fundamental system of neighborhoods of the identity consisting of sets of the form

$$\left\{\xi \in \mathrm{Un}(L^2(X_v)) : \|\xi\Phi_i - \Phi_i\|_2 \leq 1 \text{ for all } i,\ 1 \leq i \leq m\right\}$$

where we let $\{\Phi_1, \ldots, \Phi_m\}$ vary over all finite subsets of elements in $L^2(X_v)$.

There are two natural maps to be considered here: projection on the first factor gives a two-fold covering homomorphism that we will usually denote by

$$\pi : Mp(W_v) \longrightarrow Sp(W_v),$$

and projection on the second factor gives a unitary group representation of $Mp(W_v)$. As in [W1], one may show that π is a continuous, open surjection. The map π also gives a **non-trivial** cover if and only if $k_v \neq \mathbf{C}$. Non-trivial here means that there is no continuous group homomorphism $s : Sp(W_v) \to Mp(W_v)$ satisfying $\pi \circ s = \mathrm{id}$ (such a map would be called a **splitting** of the covering), or equivalently, that $Mp(W_v)$ is not isomorphic as a topological group to $Sp(W_v) \times \mathbf{Z}/2\mathbf{Z}$ in a manner compatible with π . This is proven abstractly in Weil, and again in Rao using the concrete realization developed there.

It is important to note, however, that we do have splittings over certain subgroups of $Sp(W_v)$. Specifically, if $v \notin S_k$, then the map

$$K_v \longrightarrow Mp(W_v)$$

$$g \longmapsto (g, \omega_v(g))$$

is an isomorphism onto its image, and gives a compact open neighborhood of the identity in $Mp(W_v)$. We may use this to realize the topology on $Mp(W_v)$ if we like. Similarly, for the remaining finite places $v < \infty,\ v \in S_k$, there is a compact open subgroup $K_v^o \subset K_v$ on which the cocycle β_v is trivial, and for which one may make the analogous statements. Another important splitting, which holds for any $v \in \Sigma_k$ (or later globally), concerns the subgroups defined by

$$N_v = \{n(b) \mid b = b^t \in M(n, k_v)\}$$

and is again given by

$$n(b) \mapsto (n(b), \omega_v(n(b))).$$

The **global metaplectic group** $Mp(W)_{\mathbb{A}}$ is defined in virtually the same way as the local group:

$$Mp(W)_{\mathbb{A}} = \left\{(g, \xi) \in Sp(W)_{\mathbb{A}} \times \mathrm{Un}(L^2(X(\mathbb{A}))) \mid \xi = \pm\omega(g)\right\},$$

with the topology also defined similarly. Projection on the first and second factors defines, respectively, a covering map, which we shall still call π : $Mp(W)_{\mathbf{A}} \to Sp(W)_{\mathbf{A}}$, and a unitary representation of $Mp(W)_{\mathbf{A}}$.

Rather than using these rather cumbersome definitions for routine calculations, we will usually identify $Mp(W_v)$ and $Mp(W)_{\mathbf{A}}$ with the groups formed from the sets $Sp(W_v) \times \mu_2$ and $Sp(W)_{\mathbf{A}} \times \mu_2$, respectively, with multiplication defined as in §1.5 via the cocycles β_v and β . The isomorphisms are given by

$$Sp(W_v) \times \mu_2 \xrightarrow{\sim} Mp(W_v)$$

$$[g, \epsilon] \longmapsto (g, \epsilon \cdot \omega_v(g))$$

and similarly for the adelic case. These "new" groups we will denote by $\widetilde{Sp}(W_v)$ and $\widetilde{Sp}(W)_{\mathbf{A}}$, and they will take the topologies pulled back from $Mp(W_v)$ and $Mp(W)_{\mathbf{A}}$. While the $\widetilde{Sp}$ definitions are more convenient for algebraic computations, we really need to refer back to the "true" Mp definitions for considerations of topology. The covering maps $\pi : \widetilde{Sp}(\cdot) \to Sp(\cdot)$ are now given by $\pi([g,\epsilon]) = g$, and the Weil representations by

$$\omega_v([g,\epsilon]) = \epsilon \cdot \omega_v(g), \text{ and } \omega([g,\epsilon]) = \epsilon \cdot \omega(g).$$

If $k_v \cong \mathbf{C}$, then we still define $\widetilde{Sp}(W_v)$ as above for pedagogical reasons, but note that the cocycle is trivial, and so $\widetilde{Sp}(W_v) \cong Sp(W_v) \times \mu_2$ in this case.

Although the global metaplectic group $\widetilde{Sp}(W)_{\mathbf{A}}$ is *not* a restricted direct product of local groups, there is a local-global relationship which will be important later. Regarding the maximal compact subgroups K_v, $v \notin S_k$, as actual subgroups of the $\widetilde{Sp}(W_v)$ as above (via $g \mapsto [g,1]$), we may form the restricted direct product $\prod'_v \widetilde{Sp}(W_v)$ with respect to the K_v. This then gives us the following commutative diagram:

$$
(1.8.1) \qquad \begin{array}{ccc}
\prod'_v \widetilde{Sp}(W_v) & \xrightarrow{p} & \widetilde{Sp}(W)_{\mathbb{A}} \\
 & \searrow & \downarrow \pi \\
 & & \prod'_v Sp(W_v) = Sp(W)_{\mathbb{A}}
\end{array}
$$

The vertical and diagonal maps are the obvious ones, while the horizontal map is given by $\prod'_v [g_v, \epsilon_v] \stackrel{p}{\longmapsto} [\prod_v g_v, \prod_v \epsilon_v]$. This may be used to relate the representations ω_v on $\widetilde{Sp}(W_v)$ to the global representation ω on $\widetilde{Sp}(W)_{\mathbb{A}}$. The group $\prod'_v \widetilde{Sp}(W_v)$ acts on $\mathcal{S}(X(\mathbb{A}))$ by the representation $\otimes'_v \omega_v$. But as we see in the following diagram,

$$
\begin{array}{ccc}
\prod'_v \widetilde{Sp}(W_v) & \stackrel{\otimes'_v \omega_v}{\longrightarrow} & \mathrm{Un}(\mathcal{S}(X(\mathbb{A}))) \\
p \downarrow & \nearrow \omega & \\
\widetilde{Sp}(W)_{\mathbb{A}} & &
\end{array}
$$

this map factors through to give ω. This allows us, for example, to represent a function such as

$$
\widetilde{Sp}(W)_{\mathbb{A}} \longrightarrow \mathbf{C}
$$

$$
g \longmapsto \omega(g)\varphi(0)
$$

for $\varphi = \otimes'_v \varphi_v \in \mathcal{S}(X(\mathbb{A}))$, as a tensor product of other functions:

$$
\omega(g)\varphi(0) = \otimes'_v \omega_v(g_v)\varphi_v(0) \quad \text{for} \quad g = p(\prod_v g_v).
$$

Next, we must consider the embedding of the k-rational points in the global metaplectic group. The usual embedding of $k \hookrightarrow \mathbb{A}$ as a discrete subgroup given by $x \mapsto (x, x, \dots)$ induces an embedding

$$Sp(W) = Sp(W)_k \hookrightarrow Sp(W)_{\mathbb{A}}$$

(the subscript k is added for emphasis). One of the most important facts proven in [W1] (abstractly) is that there is an analogous splitting of $\widetilde{Sp}(W)_{\mathbb{A}}$ over $Sp(W)_k$. Given the concrete formulas developed above, it is easy to see what this should be.

PROPOSITION 1.8.1. *There is a unique homomorphism* $l : Sp(W)_k \to \widetilde{Sp}(W)_{\mathbb{A}}$ *such that the diagram*

$$\begin{array}{ccc} & & \widetilde{Sp}(W)_{\mathbb{A}} \\ & l \nearrow & \downarrow \pi \\ Sp(W)_k & \longrightarrow & Sp(W)_{\mathbb{A}} \end{array}$$

commutes. It is defined by $l(g) = [g, \lambda(g)]$, *where* $\lambda(g) = \prod_v \lambda_v(g)$.

Proof. Writing $G(k), G(\mathbb{A})$, and $\widetilde{G}(\mathbb{A})$ for the three groups, suppose there are two such homomorphisms: $l_1, l_2 : G(k) \to \widetilde{G}(\mathbb{A})$. Then since they are lifts of the bottom mapping in the diagram, they must be of the form $l_i(g) = [g, a_i(g)]$ for some functions $a_i : G(k) \to \mu_2$, $i = 1, 2$. The fact that the l_i are homomorphisms amounts to saying that

$$\beta(g, g') = \frac{a_1(gg')}{a_1(g)a_1(g')} = \frac{a_2(gg')}{a_2(g)a_2(g')} \quad \text{for all } g, g' \in G(k) .$$

This means that the map $G(k) \to \mu_2$ given by $g \mapsto a_1(g)a_2(g)$ is a homomorphism. But $G(k)$ equals its own commutator subgroup, so this last map must be trivial, showing that $a_1 = a_2$, and hence that $l_1 = l_2$. This proves uniqueness.

To prove that the map l given above is in fact a homomorphism, we first need to check that $\lambda(g) = \prod_v \lambda_v(g)$ is well-defined. Writing $g = p_1\tau_S p_2 \in P(k)\tau_S P(k)$, choose a finite set Σ (with $S_k \subset \Sigma \subset \Sigma_k$) large enough so that if $v \notin \Sigma$ then $p_1, p_2 \in P(k) \cap K_v$. By Corollary 1.5.3, we see that $v \notin \Sigma$ then implies that $\tilde{c}_v(p_1, \tau_S p_2) = 1 = \tilde{c}_v(\tau_S, p_2)$, so that $\tilde{r}_v(g) = \tilde{r}_v(p_1)\tilde{r}_v(\tau_S)\tilde{r}_v(p_2)$. By the definition of Σ , then, each of these three operators fixes φ_v^o . Hence $\lambda_v(g) = 1$ for $v \notin \Sigma$, and so $\lambda(g)$ is well-defined.

We must now note that the cocycles $\tilde{c}_v$ are "well-behaved" globally with respect to the k -rational points $G(k)$ of G . The claim is that $\prod_v \tilde{c}_v(g_1, g_2) = 1$ for $g_1, g_2 \in G(k)$. To prove this, we use Rao's explicit formula for $\tilde{c}_v(\cdot,\cdot)$ given in Theorem 1.5.2. First of all, all we need to know about the Leray invariant mentioned in the theorem is that it is the isometry class of a certain inner product space, or quadratic module, attached to g_1 and g_2 . We can realize this concretely as a non-degenerate quadratic form ρ *with coefficients in* k (in this case) which is then thought of as acting on various finite-dimensional vector spaces $V_v = V \otimes_k k_v$ where v varies over Σ_k , and V is a fixed k -vector space. Now the product formula for the Hilbert symbol tells us that for any $a, b \in k$, we have $\prod_v (a,b)_v = 1$. Since the Hasse invariant $h_v(\rho)$ of ρ may be defined by means of the Hilbert symbol (as in Serre [Se]), it is also a basic fact that $\prod_v h_v(\rho) = 1$ (ρ is defined over k). This proves our claim about the $\tilde{c}_v(\cdot,\cdot)$.

Finally, we show that $l(g) = [g, \lambda(g)]$ is a homomorphism. For $g_1, g_2 \in G(k)$ arbitrary,

$$\beta(g_1, g_2) = \prod_v \left[\tilde{c}_v(g_1, g_2) \frac{\lambda_v(g_1)\lambda_v(g_2)}{\lambda_v(g_1 g_2)} \right] = \prod_v \left[\frac{\lambda_v(g_1)\lambda_v(g_2)}{\lambda_v(g_1 g_2)} \right] = \frac{\lambda(g_1)\lambda(g_2)}{\lambda(g_1 g_2)},$$

recalling that $\lambda_v \overset{def}{=} 1$ if $v \in S_k$. But this is exactly what we needed to show (perhaps it would look more natural if we had written $l(g) = [g, \lambda(g)^{-1}]$). □

Since this mapping $Sp(W)_k \to \widetilde{Sp}(W)_{\mathbb{A}}$ is unique, and an injection, we will often ignore it and identify $Sp(W)_k$ with its image in $\widetilde{Sp}(W)_{\mathbb{A}}$. One of the principal applications of this splitting is in the following theorem, proven by Weil in [W1], and stated in this form in [H].

THEOREM 1.8.2. *Let* $G = Sp(W)$ *as before. There exists a unique (up to a scalar multiple) linear functional* Θ *on* $\mathcal{S}(X(\mathbb{A}))$ *satisfying* $\Theta(\overline{U}(h)\varphi) = \Theta(\varphi)$ *for all* $h \in \mathcal{H}(k)$. *This functional is also* $G(k)$ *-invariant under the action of the Weil representation of* $\widetilde{G}(\mathbb{A})$. *It is given by the following formula:*

$$\Theta(\varphi) = \sum_{x \in X(k)} \varphi(x) \qquad \text{for } \varphi \in \mathcal{S}(X(\mathbb{A})).$$

This functional is called the **Theta distribution**. It will be used to construct one of the automorphic forms occurring in the Weil-Siegel formula.

§2.1 Construction of an (O,Sp) dual pair. It turns out, as discussed in Howe's article [H] in the Corvallis proceedings, that we wish to consider not the Weil representation of the metaplectic group $\widetilde{Sp}$, but rather the restriction of the Weil representation to subgroups of this last, and especially to the inverse image under $\pi : \widetilde{Sp} \to Sp$ of "dual reductive pairs" G and H in Sp.

At this point, the level of exposition will go up somewhat, and we will assume, for example, such things as the definition of a linear algebraic group.

DEFINITION 2.1.1 [H]. *Let L be a linear algebraic group defined over k. A pair (G, H) of algebraic subgroups of L is called a* **dual reductive pair** *if*

(1) *G and H are reductive groups (they are algebraically connected and have trivial unipotent radical), and*

(2) *G is the centralizer of H in L, and vice versa.*

We will focus on constructing a particular dual reductive pair of type (O, Sp) in a larger symplectic group. Begin by fixing a symplectic vector space $(W, <,>)$ over k as before, with a fixed symplectic basis

$$\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$$

(i.e. W has dimension $2n$). Since we have fixed this basis, we will write $G = Sp(n)$ for the symplectic group, so that $G(k) = Sp(n, k)$ is the group of transformations of W which preserve the form $<,>$. Now consider a k-vector space V of finite dimension m, furnished with a non-degenerate symmetric bilinear form $(\,,)$. Let $H = O(V)$ denote the orthogonal group of transformations of the space $(V, (\,,))$, also writing $H(k)$ to emphasize that we are looking at the k-points of the group. While it is not completely necessary, it will ease the exposition to come if we choose a basis $\{v_1, \ldots, v_m\}$ for V. Identifying V with k^m (viewed as a space of column vectors), the form $(\,,)$ gives rise to a matrix $Q = ((v_i, v_j))$, so that

$$(x, y) = {}^t x Q y \quad \text{and} \quad H = O(Q) = \{h \in GL(m, k) \mid {}^t h Q h = Q\}.$$

It is worth mentioning that we are following the convention of letting H act on V on the left, while G acts on W on the right. This seems to be standard practice in the literature, and it would probably be too confusing to attempt to change. As before, vector spaces and groups over the various completions of k and over the adeles will be denoted by $W_v, G_v = G(k_v), H(\mathbb{A})$, etc.

Taking the tensor product of V and W gives a new symplectic space $\mathbf{W}$ via

$$\mathbf{W} = V \otimes_k W \quad \text{with form} \quad \ll, \gg = (\,,) \otimes <,>,$$

by which we mean that $\ll v \otimes w, v' \otimes w' \gg = (v, v') < w, w' >$. It is easily checked that this defines a non-degenerate symplectic form on $\mathbf{W}$. We choose a symplectic basis for $\mathbf{W}$ by setting

$$e_{ij} = v_i \otimes e_j \quad \text{and} \quad e_{ij}^* = \sum_{f=1}^{m} (Q^{-1})_{fi} (v_f \otimes e_j^*)$$

for any i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$. The complete polarization $W = X \oplus Y$ of W introduced in chapter 1 defines another complete polarization on the new space $\mathbf{W}$ via $\mathbf{X} = V \otimes X$ and $\mathbf{Y} = V \otimes Y$. If it were necessary, we could choose an ordering here—say the dictionary order on (i, j) . But one can actually show that Rao's formulas for the Weil representation of $\widetilde{Sp}(\mathbf{W}_v)$ on $\mathcal{S}(\mathbf{X}_v)$ depend only on the choice of a complete polarization $\mathbf{W} = \mathbf{X} \oplus \mathbf{Y}$ of $\mathbf{W}$, and not on the choice of bases for $\mathbf{X}$ and $\mathbf{Y}$. So aside from taking the e_{ij} 's before the e^*_{ij} 's, we leave the ordering unspecified.

Remembering that Sp acts on the right, we then have a monomorphism

$$A : H(k) \times G(k) \longrightarrow Sp(\mathbf{W}) \qquad \text{defined by}$$

$$(h, g) \longmapsto h \otimes g,$$

where the action of $h \otimes g$ on $\mathbf{W}$ is given by $(v \otimes w) \cdot (h \otimes g) = (h^{-1}v) \otimes (wg)$. Note then that $[(v \otimes w) \cdot (h_1 \otimes g_1)] \cdot (h_2 \otimes g_2) = (v \otimes w) \cdot (h_1 h_2 \otimes g_1 g_2)$. One may check that leaving the ordering of the (i, j) unspecified, the matrix of $A(1_V, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ with respect to the $\{e_{ij}, e^*_{ij}\}$ basis is given in block form by

$$\begin{pmatrix} 1 \otimes a & Q \otimes b \\ Q^{-1} \otimes c & 1 \otimes d \end{pmatrix} \in Sp(\mathbf{W}).$$

This means, for example, that

$$\begin{aligned} e_{ij}(1_V \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}) &= \sum_{(k,l)} \delta_{ik} a_{jl} e_{kl} + Q_{ik} b_{jl} e^*_{kl} \\ &= \sum_{(k,l)} (1 \otimes a)_{(i,j)(k,l)} e_{kl} + (Q \otimes b)_{(i,j)(k,l)} e^*_{kl} \end{aligned}$$

where (k,l) runs over $1 \le k \le m$, $1 \le l \le n$; and $(1 \otimes a)_{(i,j)(k,l)}$ is meant to indicate the $(i,j)^{\text{th}}$ row, $(k,l)^{\text{th}}$ column entry of $1 \otimes a$. Similarly, $A(h, 1_W)$ for $h \in O(V)$ will yield a matrix

$$\begin{pmatrix} {}^t h^{-1} \otimes 1 & 0 \\ 0 & h \otimes 1 \end{pmatrix} \in Sp(\mathbf{W}).$$

The corresponding local maps on $H_v \times G_v$ will be denoted by A_v . In any case, this embedding defines a dual reductive pair (H, G) in $Sp(\mathbf{W})$.

§**2.2 The covering of Sp(W).** For the rest of this section, fix a place $v \in \Sigma_k$, and suppose that our local character ψ_v has been chosen as a component of a global character ψ of $\mathbb{A}/k$ as in §1.6. We will also suppose that ψ_v has been used to construct projective Weil representations on G_v and $Sp(\mathbf{W}_v)$, and hence metaplectic covers $\widetilde{G}_v$ and $\widetilde{Sp}(\mathbf{W}_v)$, respectively.

First, we focus on $\Pi^{-1}(A_v(G_v)) \subset \widetilde{Sp}(\mathbf{W}_v)$ (see the diagram below). This is a two-fold cover of (a copy of) G_v , and so by a result of Moore [Me], if $k_v \not\cong \mathbb{C}$ then it must be either trivial or isomorphic to $\widetilde{G}_v$, the local metaplectic group. In a temporary notation, let $\overline{G}_v$ be the set $G_v \times \mu_2$, and consider the following diagram:

$$\begin{array}{ccc} \overline{G}_v & \xrightarrow{A_v \times 1} & \widetilde{Sp}(\mathbf{W}_v) \\ \big\downarrow & & \big\downarrow \Pi \\ G_v & \xrightarrow[A_v]{} & Sp(\mathbf{W}_v) \end{array}$$

where the vertical maps are the canonical projections, and the map $A_v \times 1$ is given by $(g, \epsilon) \mapsto (A_v g, \epsilon)$. To distinguish between the various cocyles and other objects attached to G_v and $Sp(\mathbf{W}_v)$ in chapter 1, we will use (temporarily)

$$\begin{array}{ll} \beta_v, \tilde{c}_v, \lambda_v, & \text{for objects belonging to } G_v, \text{ and} \\ B_v, \widetilde{C}_v, \Lambda_v, & \text{for those belonging to } Sp(\mathbf{W}_v). \end{array}$$

If we pullback the cocycle B_v to define a group structure on $\overline{G}_v$ via the cocycle $\overline{\beta}_v(g_1, g_2) \stackrel{def}{=} B_v(A_v g_1, A_v g_2)$ for $g_i \in G_v$, then $A_v \times 1$ clearly defines an injective homomorphism giving $\overline{G}_v \cong \Pi^{-1}(A_v(G_v))$. The question of whether $\Pi^{-1}(A_v(G_v))$ is a trivial cover may then be answered by examining whether $\overline{\beta}_v$ is cohomologically trivial or homologous to β_v. Using Rao's explicit formula for $\tilde{c}_v$ (Theorem 1.5.2) Kudla has calculated the following:

LEMMA 2.2.1[K2]. *With notation as above,*

$$\widetilde{C}_v(A_v g_1, A_v g_2) = \tilde{c}_v(g_1, g_2)^m \cdot \Delta d_v(g_1, g_2) \quad \textit{for all } g_1, g_2 \in G_v$$

where $d_v(g)$ denotes the function on G_v given by

$$d_v(g) = (x(g), \det V)_v^{mj+1} (x(g), -1)_v^{\frac{m(m-1)}{2}} (\det V, -1)_v^{\frac{j(j-1)}{2}} h_v(V)^j$$

for $g \in \Omega_j \subset G_v$.

Since the proof involves a rather long and technical computation, it will be omitted. From this lemma, we immediately obtain

LEMMA 2.2.2. *For any* $g_1, g_2 \in G_v$,

$$\overline{\beta}_v(g_1, g_2) = \beta_v(g_1, g_2)^m \, \Delta[(\Lambda_v \circ A_v) \cdot \lambda_v^m \cdot d_v](g_1, g_2).$$

Hence if $k_v \ncong \mathbf{C}$ *, then the cover* $\overline{G}_v \to G_v$ *defined above is trivial if and only if* m *is even. If* $k_v \cong \mathbf{C}$ *, then the cover is trivial in any case.*

Relating this to the two standard covers of G_v gives the following:

PROPOSITION 2.2.3. *Suppose that* $k_v \ncong \mathbf{C}$ *. Then there exists a unique lifting* $\widetilde{A}_v$ *of* A_v *to an embedding of either* $G_v \times \mu_2$ *(if* m *is even) or* $\widetilde{G}_v$ *(if* m *is odd) in* $\widetilde{Sp}(\mathbf{W}_v)$ *such that the following diagrams commute:*

$$\begin{array}{ccc} G_v \times \mu_2 & \xrightarrow{\widetilde{A}_v} & \widetilde{Sp}(\mathbf{W}_v) \\ \downarrow & & \downarrow \\ G_v & \xrightarrow[A_v]{} & Sp(\mathbf{W}_v) \end{array} \quad , \quad \begin{array}{ccc} \widetilde{G}_v & \xrightarrow{\widetilde{A}_v} & \widetilde{Sp}(\mathbf{W}_v) \\ \downarrow & & \downarrow \\ G_v & \xrightarrow[A_v]{} & Sp(\mathbf{W}_v) \end{array} .$$

This is defined in either case by $\widetilde{A}_v(g, \epsilon) = (A_v(g), \epsilon \cdot \delta_v(g))$ *where* $\delta_v(g) = \Lambda_v(A_v(g)) \cdot \lambda_v(g)^m \cdot d_v(g)$ *. If* $k_v \cong \mathbf{C}$ *, then the same definitions work, but* $\delta_v = 1$ *, and we have* $A_v : G_v \times \mu_2 \xrightarrow{\sim} Sp(\mathbf{W}_v) \times \mu_2$ *.*

Proof. First we show uniqueness, treating the two cases together. Suppose $\widetilde{A}_v^1$ and $\widetilde{A}_v^2$ are two liftings as above. Then taking $\left(\widetilde{A}_v^1\right)^{-1} \circ \left(\widetilde{A}_v^2\right)$, we obtain an automorphism of the group $G_v \times \mu_2$, where multiplication is defined either using β_v or not depending on the parity of m . In either case, one shows that such an automorphism must be of the form $(g, \epsilon) \mapsto (g, \epsilon \cdot f(g))$ for some group homomorphism $f : G_v \to \mu_2$. But G_v is simple, and so f must be identically 1.

The proof that the $\widetilde{A}_v$ defined above is in fact a homomorphism is an easy consequence of the preceding lemma. $\square$

The final task to be accomplished in this section is to write down formulas for the pullback from $\widetilde{Sp}(\mathbf{W}_v)$ to $\widetilde{G}_v$ of the Weil representation of the first group. We will only do this for the odd m case, as this is the situation which will concern us for the rest of the paper. The formulas for even m involve only a slight modification, and may be easily derived from the work in this section.

Using $\overline{\omega}_v$ to denote the Weil representation of the group $\widetilde{Sp}(\mathbf{W}_v)$, we make a small change in the way the vector space $\mathbf{W}_v$, and hence the space $L^2(\mathbf{X}_v)$, are realized, as is customary in the literature of dual reductive pairs. By definition, $\mathbf{X}_v = V_v \otimes X_v$ and $\mathbf{Y}_v = V_v \otimes Y_v$. Since we have fixed bases $\{e_1, \ldots, e_n\}$ and $\{e_1^*, \ldots, e_n^*\}$ of X and Y respectively, we consider the isomorphisms

$$\begin{array}{ccc} V_v^n \xrightarrow{\sim} \mathbf{X}_v & & V_v^n \xrightarrow{\sim} \mathbf{Y}_v \\ x = (x_1, \ldots, x_n) \longmapsto \sum_{i=1}^{n} x_i \otimes e_i & , & y = (y_1, \ldots, y_n) \longmapsto \sum_{i=1}^{n} y_i \otimes e_i^* . \end{array}$$

This realization gives $\mathbf{W}_v \cong V_v^n \oplus V_v^n$. One may then check that the symplectic form $\ll , \gg$ is given now by

$$\ll [x, y], [x', y'] \gg = \operatorname{tr}((x, y') - (x', y))$$

where

$$[x, y], [x', y'] \in V_v^n \oplus V_v^n, \ x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n), \text{ etc.}$$

and (x, y') stands for the $n \times n$ matrix of inner products (x_i, y'_j). When dealing with dual pairs, we will consider the Weil representation $\overline{\omega}_v$ of $\widetilde{Sp}(\mathbf{W}_v)$ to act on the space $L^2(V_v^n)$.

A word on Haar measures: given any symplectic space $(U, <,>)$ over k_v and additive character ψ_v of k_v, there is a natural isomorphism

$$U \xrightarrow{\sim} \hat{U} \quad \text{defined by} \quad u \mapsto \psi_v(< \cdot, u >).$$

As in §1.2, this isomorphism determines a unique self-dual Haar measure du on U. If we choose a symplectic basis $\{\varepsilon_1, \ldots, \varepsilon_r, \varepsilon_1^*, \ldots, \varepsilon_r^*\}$ for U, then for $u = \sum_{i=1}^r a_i\varepsilon_i + b_i\varepsilon_i^*$, it is easy to see that this measure is given by $du = da_1 \ldots da_r\, db_1 \ldots db_r$, where da_i and db_i are the Haar measures on k_v chosen in §1.2. On the other hand, our particular symplectic space $\mathbf{W}_v \cong V_v^n \oplus V_v^n$ inherits another measure from V_v: dz is uniquely determined by $V_v \xrightarrow{\sim} \hat{V}_v$, $z \mapsto \psi_v((\cdot, z))$, and this yields a measure $d[x, y]$ on $\mathbf{W}_v$ via

$$d[x, y] = dx\, dy = dx_1 \ldots dx_n\, dy_1 \ldots dy_n,$$

$$\text{where } x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in V_v^n.$$

Since we have changed our representation space to $L^2(V_v^n)$, it is certainly more natural to use the measure $dx = dx_1 \ldots dx_n$ in our formulas.

Changing notation from that defined in Chapter 1, let

$$\omega_v(g) \stackrel{def}{=} \overline{\omega}_v(\widetilde{A}_v(g)) \quad \text{for} \quad g \in \widetilde{G}_v.$$

Also write

$$
\begin{aligned}
P_v &= \left\{ \begin{pmatrix} a & b \\ & \check{a} \end{pmatrix} \in G_v \right\} = M_v N_v, \text{ where} \\
M_v &= \left\{ m(a) = \begin{pmatrix} a & \\ & \check{a} \end{pmatrix} \mid a \in \mathrm{GL}(n, k_v) \right\}, \\
N_v &= \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b = {}^t b \in M(n, k_v) \right\}.
\end{aligned}
$$

In the following lemma, and throughout, we will frequently change a function $c : k_v^\times \to \mathbf{C}$ into a function on $\mathrm{GL}(n, k_v)$ (without mentioning this explicitly) by composition with determinant, writing $c(a) \overset{def}{=} c(\det a)$ for $a \in \mathrm{GL}(n, k_v)$.

LEMMA 2.2.4. *Let* $\pi : \widetilde{G}_v \to G_v$ *be the usual covering map, and for any subgroup* $L \subset G_v$, *write* $\widetilde{L} = \pi^{-1}(L)$. *Then*

(1) *multiplication in* $\widetilde{M}_v$ *is given by*

$$
[m(a_1), \epsilon_1][m(a_2), \epsilon_2] = [m(a_1 a_2), \epsilon_1 \epsilon_2 (a_1, a_2)_v], \text{ and}
$$

(2) *the formula*

$$
\chi_v([m(a), \epsilon]) = \frac{\epsilon\,(a, (-1)^{\frac{m-1}{2}} \det(V))_v}{\gamma_v(a, \frac{1}{2}\psi_v)}
$$

defines a character $\chi_v : \widetilde{M}_v \to \mu_4$.

Proof. These facts are immediate from Corollary 1.5.3, and §1.4 . □

PROPOSITION 2.2.5. *Let* $m = \dim_k(V)$ *be odd, and fix notation as above. Then the pullback* ω_v *to* $\widetilde{G}_v$ *of the Weil representation associated to* $\widetilde{Sp}(\mathbf{W}_v)$, *acting in the space* $L^2(V_v^n)$ *, is determined by the following formulas.*

(1) *If* $g \in \widetilde{M}_v$ *and* $\pi(g) = m(a)$ *, then for* $\varphi \in L^2(V_v^n)$ *,*

$$\omega_v(g)\varphi(x) = \chi_v(g)\,|a|_v^{\frac{m}{2}}\,\varphi(xa),$$

where $xa = (x_1, \ldots, x_n)a \overset{def}{=} (\sum_{i=1}^n x_i a_{i1}, \ldots, \sum_{i=1}^n x_i a_{in})$.

(2) *If* $n(b) \in N_v \hookrightarrow \widetilde{G}_v$ *(see §1.8 regarding the splitting over* N_v *), then*

$$\omega_v(n(b))\varphi(x) = \psi_v(\tfrac{1}{2}tr(b(x,x))) \cdot \varphi(x),$$

where (x,x) stands for the $n \times n$ matrix of V -inner products (x_i, x_j).

(3) *For* $S \subset \{1,2,\ldots,n\}$ *with* $|S| = j$ *, write* $V_v^n = V_v^S \oplus V_v^{S'}$ *, so that* $x \in V_v^n$ *decomposes as* $x = x_S + x_{S'}$ *. Then for* $\varphi \in \mathcal{S}(V_v^n)$ *,*

$$\omega_v([\tau_S, 1])\varphi(x) = \gamma_v(V)^{-j} \int\limits_{V_v^S} \psi_v(-tr(x_S, z))\varphi(x_{S'} + z)\, d_S z.$$

Here, $tr(x_S, z)$ *stands for the trace of the* $j \times j$ *matrix with entries* (x_i, z_l) *for* $i, l \in S$ *;* $d_S z = \prod_{i \in S} dz_i$ *is a product of self-dual measures on* V_v *as above; and* $\gamma_v(V)$ *is the Weil index of the map*

$$\begin{aligned} V_v &\longrightarrow \mathbf{T} \\ z &\longmapsto \psi_v(\tfrac{1}{2}(z,z)). \end{aligned}$$

As before, this definition of $\omega_v([\tau_S, 1])$ *extends to define an operator on* $L^2(V_v^n)$ *.*

§2.3 The covering of O(V). Although we will be less concerned with the action of $\Pi^{-1}(A_v(H_v))$ on $L^2(V_v^n)$, it is still necessary to compute it explicitly once and for all. As in the previous section, let $\overline{H}_v = H_v \times \mu_2$, and consider the diagram

$$\begin{array}{ccc} \overline{H}_v & \xrightarrow{A_v \times 1} & \widetilde{Sp}(\mathbf{W}_v) \\ \downarrow & & \downarrow \\ H_v & \xrightarrow{A_v} & Sp(\mathbf{W}_v). \end{array}$$

From §2.1, recall that $A_v(H_v) \subset \mathbf{P}_v$, the maximal parabolic of $Sp(\mathbf{W}_v)$ stabilizing $\mathbf{Y}_v$. This makes the computations much easier. The cocycle B_v of $\widetilde{Sp}(\mathbf{W}_v)$, when pulled back to $\overline{H}_v$, gives just

$$\begin{aligned} B_v(A_v(h_1), A_v(h_2)) &= (\det(h_1 \otimes 1_n), \det(h_2 \otimes 1_n))_v \\ &= (\det(h_1), \det(h_2))_v^n \qquad \text{for all } h_1, h_2 \in H_v. \end{aligned}$$

Hence $\Pi^{-1}(A_v(H_v))$ is always a trivial cover when n is even, and is trivial in any case for finite places v not dividing 2 (recall that $h \in H_v \Rightarrow \det h = \pm 1$, and $v \nmid 2 \Rightarrow (\,,)_v \equiv 1$ on $\mathcal{U}_v \times \mathcal{U}_v$), or for real places. The action of $(h, \epsilon) \in \overline{H}_v$ on a function $\varphi \in L^2(V_v^n)$ is computed to be

$$(\overline{\omega}_v \circ A_v)(h, \epsilon)\varphi(x) = \epsilon \cdot \gamma_v((\det h)^n, \frac{1}{2}\psi_v)^{-1}\varphi(h^{-1}x)$$

where $h^{-1}x = h^{-1}(x_1, \ldots, x_n) = (h^{-1}x_1, h^{-1}x_2, \ldots, h^{-1}x_n)$. As explained in Kudla's paper [K1], rather than dealing with this non-standard 2-fold covering of H_v , it is traditional to twist the representation above by the character $\chi_v'(h, \epsilon) = \epsilon\gamma_v(\det h, \frac{1}{2}\psi_v)$ in the case where n is odd. So we will take the action of H_v on $L^2(V_v^n)$ to be just the left action given by:

$$\omega_v(h)\varphi(x) \stackrel{def}{=} \varphi(h^{-1}x).$$

Note that this action still commutes with the action of $\widetilde{G}_v$ on $L^2(V_v^n)$.

§2.4 Global coverings. Let $m = \dim V$ remain odd throughout this section. We wish to reproduce the local results of the last two sections for the global groups $G(\mathbb{A})$ and $H(\mathbb{A})$. In other words, lift

$$A : G(\mathbb{A}) \longrightarrow Sp(\mathbf{W})_{\mathbb{A}} \qquad \text{to a map}$$

$$\widetilde{A} : \widetilde{G}(\mathbb{A}) \longrightarrow \widetilde{Sp}(\mathbf{W})_{\mathbb{A}},$$

realize similarly a model for $\Pi^{-1}(A_v(H(\mathbb{A})))$, and write down formulas for the pullback of $\overline{\omega}$ (defined on $\widetilde{Sp}(\mathbf{W})_{\mathbb{A}}$) to $\widetilde{G}(\mathbb{A})$ and $H(\mathbb{A})$.

For the first of these tasks, consider the diagram

$$\text{(2.4.1)} \qquad \begin{array}{ccc} \prod'_v \widetilde{G}_v & \xrightarrow{\alpha} & \prod'_v \widetilde{Sp}(\mathbf{W}_v) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \widetilde{G}(\mathbb{A}) & \xrightarrow{\widetilde{A}} & \widetilde{Sp}(\mathbf{W})_{\mathbb{A}} \end{array}$$

where α is the product of the maps $\widetilde{A}_v$, and the vertical maps are given by diagram (1.8.1). If we show that the map α is in fact well-defined (maps into the *restricted* product), then it is trivial to verify that there exists a unique monomorphism $\widetilde{A}$ causing the diagram to commute. As the entire diagram lies over

$$G(\mathbb{A}) \xrightarrow{A=\prod A_v} Sp(\mathbf{W})_{\mathbb{A}},$$

this will yield the desired lifting of A . The condition on α amounts to asking that $\widetilde{A}_v : \widetilde{G}_v \to \widetilde{Sp}(\mathbf{W}_v)$ map one standard maximal compact $K_v \subset \widetilde{G}_v$ into the other $Sp(mn, \mathcal{O}_v) \subset \widetilde{Sp}(\mathbf{W}_v)$ for almost all $v \in \Sigma_k$. The following lemma shows that this is in fact the case.

LEMMA 2.4.1. *Let V_v^o be the $\mathcal{O}_v$ -lattice given by $\mathrm{span}_{\mathcal{O}_v}\{v_1, \ldots, v_n\}$. Writing $(V_v^o)^* = \{v \in V_v \mid (v, w) \in \mathcal{O}_v$ for all $w \in V_v^o\}$, define $S_{k,V} = S_k \cup \{v \in \Sigma_k \mid (V_v^o)^* \neq V_v^o\}$. Then for all places $v \notin S_{k,V}$,*

(1) $A_v(K_v) \subset Sp(mn, \mathcal{O}_v)$, *and*

(2) *the function $\delta_v : G_v \to \mu_2$ is identically 1 on K_v .*

With the identifications $K_v \subset \widetilde{G}_v$ and $Sp(mn, \mathcal{O}_v) \subset \widetilde{Sp}(\mathbf{W}_v)$ developed in §1.8, this proves that $\widetilde{A}_v(K_v) \subset Sp(mn, \mathcal{O}_v)$ for $v \notin S_{k,V}$, since $\widetilde{A}_v([g, \epsilon]) = [A_v(g), \epsilon\, \delta_v(g)]$. Thus, we have proven most of

PROPOSITION 2.4.2. *Let $m = \dim V$ be odd. Then there exists a unique monomorphism $\widetilde{A}$ which makes the following diagram commute:*

$$\begin{array}{ccc} \widetilde{G}(\mathbb{A}) & \xrightarrow{\widetilde{A}} & \widetilde{Sp}(\mathbf{W})_{\mathbb{A}} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi} \\ G(\mathbb{A}) & \xrightarrow{A} & Sp(\mathbf{W})_{\mathbb{A}}. \end{array}$$

It is defined by $\widetilde{A}([g, \epsilon]) = [A(g), \epsilon\, \delta(g)]$, where $\delta(g) = \prod_v \delta_v(g)$. $\widetilde{A}$ also preserves the k -rational points: in other words $\widetilde{A}(G(k)) \subset Sp(\mathbf{W})_k$.

Proof. Existence and uniqueness have already been proven. That the formula for $\widetilde{A}$ is as given is an easy consequence of the commutativity of diagram

(2.4.1). To show that $\widetilde{A}(G(k)) \subset Sp(\mathbf{W})_k$, note that $g \in G(k)$ embeds as $[g, \lambda(g)]$, and we have

$$\widetilde{A}([g, \lambda(g)]) = [A(g), \lambda(g)\delta(g)] = [A(g), \Lambda(A(g))].$$

This follows from the fact that $\prod_v d_v(g) = 1$ (see Lemma 2.2.1). □

As preparation for giving the formulas for the pullback of the Weil representation, we need to describe the two-fold cover of $M(\mathbb{A}) \cong GL(n, \mathbb{A})$ and a certain character of the cover.

LEMMA 2.4.3. *As in Lemma 2.2.4, write* $\pi : \widetilde{G}(\mathbb{A}) \to G(\mathbb{A})$ *for the covering homomorphism, and* $\widetilde{L} = \pi^{-1}(L)$ *for a subgroup* $L \subset G(\mathbb{A})$.

(1) *Multiplication in* $\widetilde{M}(\mathbb{A})$ *is given by*

$$[m(a_1), \epsilon_1][m(a_2), \epsilon_2] = [m(a_1 a_2), \epsilon_1 \epsilon_2 (a_1, a_2)_k],$$

where we write $(\alpha, \beta)_k = \prod_v (\alpha_v, \beta_v)_v$ *for the product of all the local Hilbert symbols, given* $\alpha = (\alpha_v), \beta = (\beta_v) \in \mathbb{A}$.

(2) *The unique embedding* $G(k) \hookrightarrow \widetilde{G}(\mathbb{A})$ *restricts to*

$$\begin{aligned} M(k) &\longrightarrow \widetilde{M}(\mathbb{A}) \\ m(a) &\longmapsto [m(a), 1]. \end{aligned}$$

(3) *For all finite places* v *not dividing 2, the subgroup* $M_v^o = \{m(a) \in M_v \mid a \in GL(n, \mathcal{O}_v)\}$ *of* M_v *also embeds in* $\widetilde{M}_v$ *via* $g \mapsto [g, 1]$.

(4) *The tensor product of all the local characters* $\otimes_v \chi_v$ *defines a character of the restricted direct product* $\prod'_v \widetilde{M}_v$ *with respect to the* M_v° *. This character factors through the product map* $\prod'_v \widetilde{M}_v \to \widetilde{M}(\mathbb{A})$ *to give a character*

$$\chi : \widetilde{M}(\mathbb{A}) \longrightarrow \mathbf{T} \qquad \text{satisfying} \qquad \chi([m(a), \epsilon]) = \frac{\epsilon(a, (-1)^{\frac{m-1}{2}} \det(V))_k}{\gamma(a, \frac{1}{2}\psi)}.$$

Here $\gamma(b\psi)$ *stands for the Weil index of the map* $\mathbb{A} \to \mathbf{T}$ *given by* $x \mapsto \psi(bx^2)$ *: it satisfies* $\gamma(b\psi) = \prod_v \gamma_v(b_v\psi_v)$ *. Global properties of* γ *are explained thoroughly in section 30 of [W1]. A consequence of the discussion there is that* $\chi(g) = 1$ *for all* $g \in M(k)$ *, so that* χ *is actually a character of* $\widetilde{M}(\mathbb{A})/M(k)$ *.*

Now, since the representation $\otimes'_v \overline{\omega}_v : \prod'_v \widetilde{Sp}(\mathbf{W}_v) \to \mathrm{Un}(L^2(V(\mathbb{A})^n))$ factors through to $\overline{\omega} : \widetilde{Sp}(\mathbf{W})_{\mathbb{A}} \to \mathrm{Un}(\mathcal{S}(V(\mathbb{A})^n))$ (see §1.8), we see from diagram (2.4.1) above that the formulas for

$$\omega \stackrel{def}{=} \overline{\omega} \circ \widetilde{A} : \widetilde{G}(\mathbb{A}) \longrightarrow \mathrm{Un}(\mathcal{S}(V(\mathbb{A})^n))$$

result immediately from the local formulas given in Proposition 2.2.5.

Proposition 2.4.4. *Let* m *be odd, and consider a function* $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$ *. Then the representation* ω *of* $\widetilde{G}(\mathbb{A})$ *defined above is determined by the following formulas. Notation follows closely that of Proposition 2.2.5.*

(1) *If* $g \in \widetilde{M}(\mathbb{A})$ *and* $\pi(g) = m(a)$*,* $a \in GL(n, \mathbb{A})$ *,*

$$\omega(g)\varphi(x) = \chi(g)|a|^{\frac{m}{2}}\varphi(xa).$$

(2) *If* $n(b) \in N(\mathbb{A}) \hookrightarrow \widetilde{G}(\mathbb{A})$, *then*

$$\omega(n(b))\varphi(x) = \psi(\tfrac{1}{2}tr(b(x,x))) \cdot \varphi(x).$$

(3) *For* $S \subset \{1,2,\ldots,n\}$ *with* $j = |S|$, $\tau_S \in G(k) \subset \widetilde{G}(\mathbb{A})$ *acts via*

$$\omega(\tau_S)\varphi(x) = \gamma(V)^{-j} \int\limits_{V(\mathbb{A})^S} \psi(-tr(x_S, z))\varphi(x_{S'} + z)\, d_S z$$

Here, we note that $\tau_S \in G(k)$ *embeds as* $[\tau_S, 1] \in \widetilde{G}(\mathbb{A})$, *and the measure* dz_i *on* $V(\mathbb{A})$ *is the product (as defined in Tate) of the local measures on the* V_v . *In addition, it is the unique self-dual measure on* $V(\mathbb{A})$ *with respect to the identification*

$$V(\mathbb{A}) \xrightarrow{\sim} V(\mathbb{A})^{\wedge} \text{ given by } \quad z \mapsto \psi((\cdot, z)).$$

As in §2.3, one checks that the pullback of the cocycle $B = \prod_v B_v$ on $\widetilde{Sp}(\mathbf{W})_{\mathbb{A}}$ to the group $\overline{H}(\mathbb{A})$ is given by

$$B(A(h_1), A(h_2)) = (\det h_1, \det h_2)_k^n,$$

and $\overline{H}(\mathbb{A})$ acts on $L^2(V(\mathbb{A})^n)$ via

$$(\overline{\omega} \circ A)(h, \epsilon)\varphi(x) = \epsilon\,\gamma((\det h)^n, \tfrac{1}{2}\psi)^{-1}\varphi(h^{-1}x).$$

Note that $H(k) \hookrightarrow \overline{H}(\mathbb{A})$ via $h \mapsto [h, 1]$, and if n is odd, we may twist the representation above by the character $\chi' : \overline{H}(\mathbb{A})/H(k) \to \mathbf{T}$ defined via

$$\chi'([h, \epsilon]) = \epsilon\,\gamma(\det h, \tfrac{1}{2}\psi).$$

Hence, as in the local case, we will use the $H(\mathbb{A})$ action defined simply by

$$\omega(h)\varphi(x) \stackrel{def}{=} \varphi(h^{-1}x).$$

3. The Weil-Siegel formula

§3.1 Introduction. The Weil-Siegel formula asserts that a special value of an Eisenstein series equals the integral of a related theta series, where both of these are automorphic forms on a two-fold cover of $Sp(n, \mathbb{A})$. In this chapter, we describe the general problem, state the cases in which the formula is known, and give the main result of this thesis.

Let all notation be as in the preceding chapter. Then suppressing the map A , we have a dual pair

$$\begin{array}{ccc} & & \widetilde{Sp}(\mathbf{W})_{\mathbb{A}} \\ & & \big\downarrow \Pi \\ H(\mathbb{A}) \times G(\mathbb{A}) & \longrightarrow & Sp(\mathbf{W})_{\mathbb{A}} \end{array}$$

which induces a representation of $H(\mathbb{A}) \times \Pi^{-1}(G(\mathbb{A}))$ in the space $\mathcal{S}(V(\mathbb{A})^n)$. For the purposes of this introduction, we make no assumptions about the parity of $m = \dim(V)$. If m is odd, then $\Pi^{-1}(G(\mathbb{A})) \cong \widetilde{G}(\mathbb{A})$, and as before, we write $\widetilde{L}$ for the inverse image under $\pi : \widetilde{G}(\mathbb{A}) \to G(\mathbb{A})$ of any subgroup $L \subset G(\mathbb{A})$. In the even case, $\Pi^{-1}(G(\mathbb{A}))$ is a trivial extension of $G(\mathbb{A})$, and all of our formulas reduce to formulas on $G(\mathbb{A})$.

First, recall the theta distribution $\Theta \in \mathrm{Hom}_{Sp(\mathbf{W})_k}(\mathcal{S}(V(\mathbb{A})^n), 1)$ from Theorem 1.8.2. Given a function $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$, we may consider the theta function

$$\theta(g,h,\varphi) \stackrel{def}{=} \Theta(\omega(g,h)\varphi) = \sum_{x\in V(k)^n} \omega(g)\varphi(h^{-1}x),$$

$$\text{for } h \in H(\mathbb{A}), \quad g \in \begin{cases} G(\mathbb{A}), & \text{if } 2 \mid m, \\ \widetilde{G}(\mathbb{A}), & \text{if } 2 \nmid m. \end{cases}$$

Since the k-points of $\widetilde{G}(\mathbb{A})$ and $H(\mathbb{A})$ are mapped into $Sp(\mathbf{W})_k$ (see Proposition 2.4.2), this function is left $G(k)$ and $H(k)$ invariant. If we integrate out the orthogonal variable, we obtain a function

$$I(g,\varphi) = \int_{H(k)\backslash H(\mathbb{A})} \theta(g,h,\varphi)\, dh \tag{3.1.1}$$

which converges if either

(1) $\{V(k),(\,,\,)\}$ is anisotropic, or

(2) $m-\alpha > n+1$, where $\alpha \geq 1$ is the dimension of a maximal isotropic subspace of V.

The first of these is a consequence of reduction theory, which implies that if $V(k)$ is anisotropic, then $H(k)\backslash H(\mathbb{A})$ is compact. The second is proven in Weil's paper [W2]. In either case, the Haar measure dh is normalized so that $H(k)\backslash H(\mathbb{A})$ has volume one. Given these restrictions, equation (3.1.1) above defines an automorphic form on

$$G(k)\backslash G(\mathbb{A}), \qquad \text{if } m \text{ is even, or}$$

$$G(k)\backslash \widetilde{G(\mathbb{A})}, \qquad \text{if } m \text{ is odd.}$$

Next, in order to define the Eisenstein series, we consider the maximal parabolic $P = M \cdot N$ as before. Choose standard maximal compact subgroups of the various G_v by taking K_v to be $Sp(n, \mathcal{O}_v)$ for $v < \infty$, $Sp(n, \mathbf{R}) \cap O(2n) \cong U(n)$ for $k_v \cong \mathbf{R}$, and $Sp(n, \mathbf{C}) \cap U(2n)$ for $k_v \cong \mathbf{C}$. Writing $K = \prod_v K_v \subset G(\mathbb{A})$, the local Iwasawa decompositions give rise to global decompositions

$$G(\mathbb{A}) = N(\mathbb{A}) \cdot M(\mathbb{A}) \cdot K, \qquad \text{and} \qquad \widetilde{G}(\mathbb{A}) = N(\mathbb{A}) \cdot \widetilde{M}(\mathbb{A}) \cdot \widetilde{K}.$$

Here, as always, we identify $N(\mathbb{A}) \hookrightarrow \widetilde{G}(\mathbb{A})$ as a subgroup. Now define functions on $G(\mathbb{A})$ and $\widetilde{G}(\mathbb{A})$ (respectively) by

$$g = \left\{ \begin{matrix} n\, m(a)\, k \\ \\ n\, \widetilde{m(a)}\, \tilde{k} \end{matrix} \right\} \longmapsto |a(g)| \overset{d}{=} |\det(a)| \quad \text{(adelic abs. value)}.$$

Note that while the element $a(g) \in GL(n, \mathbb{A})$ is defined only as a coset of a maximal compact of $GL(n, \mathbb{A})$, the adelic absolute value of this *is* well-defined. Finally, for $s \in \mathbf{C}$, $s_o = s_o(m, n) = \frac{m}{2} - \frac{n+1}{2}$, and a K-finite (respectively, $\widetilde{K}$-finite) function $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$, we define the section

$$\Phi(g, s, \varphi) = |a(g)|^{s - s_o} \omega(g)\, \varphi(0)$$

and hence an Eisenstein series

$$E(g, s, \varphi) = \sum_{\gamma \in P(k) \backslash G(k)} \Phi(\gamma g, s, \varphi) \quad \text{for} \quad g \in \left\{ \begin{matrix} G(\mathbb{A}) \\ \widetilde{G}(\mathbb{A}) \end{matrix} \right\}.$$

By either the standard theory of Eisenstein series (see Arthur [A]) or a metaplectic extension of this theory (see Morris [M]), this series converges absolutely for $\mathrm{Re}(s) > \frac{n+1}{2}$, and has a meromorphic continuation to the complex s-plane and a functional equation relating values at s to those at $-s$.

§3.2 Past work. The original Weil-Siegel formula [W2] asserts that as long as $m > 2n+2$ (so that $s_o = \frac{m}{2} - \frac{n+1}{2} > \frac{n+1}{2}$), we have

$$E(g, s_o, \varphi) = I(g, \varphi) \ .$$

In other words, this identity holds in all cases in which the Eisenstein series is absolutely convergent at $s = s_o$. In two recent papers, [K-R1] and [K-R2], Kudla and Rallis extend the identity in the case where both sides are automorphic forms on $G(\mathbb{A})$ (i.e., when $m = \dim(V)$ is even):

THEOREM 3.2.1. *[K-R2] Let m be even, and let α be the dimension of a maximal isotropic subspace of V . Assume that $\alpha = 0$ or that $m - \alpha > n+1$. Then for all K -finite $\varphi \in \mathbf{S}(V(\mathbb{A})^n)$*

(1) *$E(g, s, \varphi)$ is holomorphic at $s = s_o(m,n)$, and*

(2) *$E(g, s_o, \varphi) = \kappa \cdot I(g, \varphi)$ for all $g \in G(k) \backslash G(\mathbb{A})$, where*

$$\kappa = \begin{cases} 1, & \text{if } \ m > n+1, \\ 2, & \text{if } \ m \leq n+1. \end{cases}$$

§3.3 Statement of thesis results. This paper will be concerned with the case where m is odd and $\alpha = 0$. The main result is as follows:

THEOREM 3.3.1. *Let $\{V, (\ ,\ )\}$ be an anisotropic symmetric k -vector space of odd dimension m , and let $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$ be a $\widetilde{K}$ -finite function. Define the constant κ by*

$$\kappa = \begin{cases} 1, & \text{if } \ m > n+1 \text{ or } m = 1 \\ 2, & \text{if } \ 1 < m \leq n+1. \end{cases}$$

Then the Eisenstein series $E(g,s,\varphi)$ *is holomorphic at* $s = s_o$ *for all pairs* (m,n) *, and the equality*

$$E(g,s_o,\varphi) = \kappa \cdot I(g,\varphi), \qquad g \in G(k)\backslash\widetilde{G}(\mathbb{A})$$

holds in the following cases:

(i) $m = 1$,

(ii) $m = 3,\ n = 1$ or 2 ,

(iii) $3 < m \leq n+1$,

(iv) $m > n+3$.

If we accept Conjecture 10.2.3 (see chapter 10), then all cases with $m = n+2$ *and* $m = n+3$ *also hold, with the exception of* $(m,n) = (7,4)$.

We give a brief sketch of the proof, which follows along the lines of that in [K-R1]. For the remainder of the paper, let the hypotheses be as given in Theorem 3.3.1 above.

Noting that $(N(k)\backslash N(\mathbb{A}))^{\wedge} \cong N(k)$ via

$$\psi_T(n(b)) = \psi(\mathrm{tr}(bT)) \quad \text{for} \quad n(b) \in N(\mathbb{A}),\ n(T) \in N(k),$$

an automorphic form f on $G(k)\backslash\widetilde{G}(\mathbb{A})$ has Fourier coefficients (with respect to $\widetilde{P} = N\widetilde{M}$) given by

$$f_T(g) = \int_{N(k)\backslash N(\mathbb{A})} f(ng)\psi_{-T}(n)\,dn.$$

Here dn is normalized to give $\mathrm{vol}(N(k)\backslash N(\mathbb{A})) = 1$. With this definition in mind, define the **constant term of** f with respect to $\widetilde{P}$ to be the 0^{th} Fourier coefficient of f, and denote this by $f_{\widetilde{P}}$. The proof begins with a quick reduction to the problem of proving

(1) that $E_{\widetilde{P}}(g, s, \varphi)$ is holomorphic at $s = s_o$, and

(2) that the constant terms satisfy

$$E_{\widetilde{P}}(g, s_o, \varphi) = \kappa \cdot I_{\widetilde{P}}(g, \varphi).$$

But the constant term of I is easily computed: writing $\varphi' = \omega(g)\varphi$,

$$\begin{aligned}
I_{\widetilde{P}}(g, \varphi) &= \int\limits_{N(k)\backslash N(\mathbb{A})} \int\limits_{H(k)\backslash H(\mathbb{A})} \sum_{x \in V(k)^n} \omega(ng)\varphi(h^{-1}x)\, dh\, dn \\
&= \int\limits_{N(k)\backslash N(\mathbb{A})} \int\limits_{H(k)\backslash H(\mathbb{A})} \sum_{x \in V(k)^n} \psi(\tfrac{1}{2}\mathrm{tr}(b(h^{-1}x, h^{-1}x)))\varphi'(h^{-1}x)\, dh\, dn(b) \\
&= \int\limits_{H(k)\backslash H(\mathbb{A})} \sum_{x \in V(k)^n} \left(\int\limits_{N(k)\backslash N(\mathbb{A})} \psi(\tfrac{1}{2}\mathrm{tr}(b((x,x)))dn(b) \right) \varphi'(h^{-1}x)\, dh \\
&= \varphi'(0) = \omega(g)\varphi(0) = \Phi(g, s_o, \varphi)
\end{aligned}$$

since $n(b) \mapsto \psi(\frac{1}{2}\mathrm{tr}(bT))$, $T \in N(k)$, is a non-trivial character of the compact group $N(k)\backslash N(\mathbb{A})$ if and only if $T = 0$.

Next, the constant term of E is easily written as the sum of $n+1$ terms:

$$E_{\widetilde{P}}(g, s, \varphi) = \Phi(g, s, \varphi) + \sum_{r=1}^{n-1} E^r_{\widetilde{P}}(g, s, \varphi) + M(s)\Phi(g, s),$$

where the first term will match $I_{\widetilde{P}}(g, \varphi)$ at $s = s_o$, the middle $n-1$ terms restrict to degenerate Eisenstein series on $\widetilde{M(\mathbb{A})} \cong$ a two-fold cover of $GL(n, \mathbb{A})$, and where

$$M(s)\Phi(g,s) = \int\limits_{N(\mathbb{A})} \Phi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} xg, s)\, dx$$

is a $\widetilde{G(\mathbb{A})}$ -intertwining operator from a certain induced representation space to another. One then shows that the meromorphic continuations of all of these terms are finite at s_o , and that all but the first term either vanish or cancel each other at s_o , or in the cases $1 < m \leq n+1$, that exactly one of them survives to match the first term $\Phi(g, s_o, \varphi)$, giving the constant $\kappa = 2$ of the theorem.

§4.1 Notation. For the remainder of this paper, let the hypotheses be as given in Theorem 3.3.1. As the title indicates, this chapter will be concerned with the reduction of our problem to proving the holomorphicity at s_0 of $E_{\widetilde{P}}(s)$, and the equality of the constant terms of $E(s_o)$ and I. Although the proof basically parallels that given in sections 2 and 3 of [K-R1], there are differences which arise due to the fact that we are working with automorphic forms on a metaplectic cover of $G(\mathbb{A}) = Sp(n,\mathbb{A})$. First of all, we establish notation, repeating some definitions from [K-R1].

Fix the Borel subgroup B of G given by

$$B = \left\{ \begin{pmatrix} a & b \\ & \check{a} \end{pmatrix} \in G \mid a \text{ is upper triangular in } GL(n) \right\}.$$

Then a Levi decomposition of B is given by $B = T \ltimes N_B$, where

$$T = \{g = \mathrm{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) \in G\}$$

is a maximal torus of G, and

$$N_B = \left\{ \begin{pmatrix} 1 & & * & & & \\ & \ddots & & & * & \\ 0 & & 1 & & & \\ & & & 1 & & 0 \\ & & & & \ddots & \\ & & & * & & 1 \end{pmatrix} \in G \right\}$$

is the unipotent radical of B. The **standard parabolics** of G are defined to be those algebraic subgroups of G containing B. They are parameterized by sequences $\mathbf{s} = (s_1, \dots, s_r)$ of positive integers with $|\mathbf{s}| = \sum s_i \leq n$ (see [K1]). For $1 \leq s \leq n$, define $Y_s = \operatorname{span}\{e_1^*, \dots, e_s^*\}$. Then the parabolic $P_{\mathbf{s}}$ associated to $\mathbf{s}$ is defined to be the stabilizer of the flag of isotropic subspaces of W given by

$$Y_{s_1} \subset Y_{s_1+s_2} \subset \cdots \subset Y_{|\mathbf{s}|} .$$

The maximal (proper) parabolics among these are determined by a sequence of length 1: for any integer r, $1 \leq r \leq n$, let P_r be the maximal parabolic stabilizing Y_r. P_r then has unipotent radical given by

$$N_r = \left\{ \begin{pmatrix} 1_r & x & y & z \\ & 1_{n-r} & {}^t z & \\ & & 1_r & \\ & & -{}^t x & 1_{n-r} \end{pmatrix} \in G \right\},$$

and we may take as Levi factor

$$M_r = \left\{ \begin{pmatrix} h & & & \\ & a & & b \\ & & \check{h} & \\ & c & & d \end{pmatrix} \in G \right\} \cong GL(r) \times Sp(n-r).$$

Note that the parabolic used to define the Eisenstein series E, previously denoted by P, is now P_n. Defining $w_j = \tau_j^{-1}$ (see Lemma 1.6.2) and setting $N''_{n,r} = N_n \cap N_r$ and

$$Q_{n-r} = \left\{ m\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_n \mid a \in GL(r),\ d \in GL(n-r) \right\},$$

we also observe that

$$w_{n-r}^{-1} P_n w_{n-r} \cap N_n = N''_{n,r}$$

$$w_{n-r}^{-1} P_n w_{n-r} \cap M_n = Q_{n-r}.$$

Finally, denote by

$$N'_r = \left\{ n\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mid b \in M(n-r) \right\}$$

a subgroup such that $N_n = N'_r \cdot N''_{n,r}$.

§4.2 **Automorphic forms on metaplectic covers.** In this section we briefly describe the elements of the theory of automorphic forms which will be needed in this chapter, and the manner in which the theory for ordinary adele groups generalizes to the metaplectic situation. This last follows Morris [M].

The essential features possessed by the finite central covering $\pi : \widetilde{G}(\mathbb{A}) \to G(\mathbb{A})$ which allow the theory to carry over are the following:

(1) The extension must split over the k -rational points $G(k)$ of $G(\mathbb{A})$.

For any standard parabolic subgroup $P = M \cdot N \subset G$,

(2) there exists a splitting over $N(\mathbb{A})$ which is *natural* in the sense that if $P \supset Q$ are parabolics with $N_Q \supset N_P$, then the splittings are compatible.

(3) $\widetilde{P}(\mathbb{A}) = \mathrm{Norm}_{\widetilde{G}(\mathbb{A})}(N(\mathbb{A}))$.

(4) There is a positive integer l such that $(Z(M(\mathbb{A}))^l)^{\sim} \subset Z(\widetilde{M}(\mathbb{A}))$

Property (1) was developed fully in Proposition 1.8.1. The next two properties hinge on the following: any standard parabolic $P = M \cdot N \supset B = T \cdot N_B$

satisfies $N_B \supset N$. But the map

$$N_B(\mathbb{A}) \to \widetilde{N}_B(\mathbb{A})$$

$$u \mapsto [u, 1]$$

defines a splitting, since $N_B \subset P_n$ and we have $\beta(u_1, u_2) = (x(u_1), x(u_2))_k = (1,1)_k = 1$ for all $u_1, u_2 \in N_B$. Hence the splitting over $N_B(\mathbb{A})$ is compatible with (and defines by restriction) splittings over N_P for any P , showing (2). Property (3) follows from an easy cocycle computation, given that $P(\mathbb{A}) = \mathrm{Norm}_{G(\mathbb{A})}(N(\mathbb{A}))$, and that $x(g) = 1$ for any $g \in N(\mathbb{A})$. Finally, (4) is a result of the fact that $z \in Z(M(\mathbb{A})) \Rightarrow \beta(z, g) = (x(z), x(g))_k = \beta(g, z)$ for all $g \in G(\mathbb{A})$ (hence $l = 1$ suffices).

From the discussion above, we see that the definition of the **constant term** of an automorphic form made in §3.3 for the parabolic $\widetilde{P}_n$ remains perfectly valid for an arbitrary standard parabolic $\widetilde{P} = \widetilde{M} \cdot N$, although the interpretation as a Fourier coefficient only holds for $\widetilde{P}_n$ (since N_n is abelian). Note that $\widetilde{G}$ itself may be considered a parabolic with trivial unipotent radical, so that $f_{\widetilde{G}} = f$.

DEFINITION 4.2.1. *Let $f : \widetilde{G}(\mathbb{A}) \to \mathbf{C}$ be an automorphic form as described in Borel and Jacquet [B-J]. Then f is called a* **cusp form** *if $f_{\widetilde{P}} = 0$ for all proper parabolics $\widetilde{P}(\mathbb{A}) \subset \widetilde{G}(\mathbb{A})$.*

Now, if f is an automorphic form on $\widetilde{G}(\mathbb{A}) \supset \widetilde{P}(\mathbb{A}) = N(\mathbb{A})\widetilde{M}(\mathbb{A})$, then $m \mapsto f_{\widetilde{P}}(mg)$ defines an automorphic form on $\widetilde{M}(\mathbb{A})$. Note that if f is right $\widetilde{K}$ -finite, then for finitely many $g \in \widetilde{G}(\mathbb{A})$, the functions $m \mapsto f_{\widetilde{P}}(mg)$ determine $f_{\widetilde{P}}$ (by the Iwasawa decomposition). Since any automorphic form on $\widetilde{M}(\mathbb{A})$ is $Z(\widetilde{M}(\mathbb{A}))$ -finite, we may write

$$f_{\widetilde{P}}(mg) = \sum_{\zeta} f_{\widetilde{P},\zeta}(mg)$$

where ζ ranges over a finite set of characters of $Z(M(k))\backslash Z(\widetilde{M}(\mathbb{A}))$, and $f_{\widetilde{P},\zeta}$ is the component of $f_{\widetilde{P}}$ transforming with central character ζ .

DEFINITION 4.2.2. *We say that* f *is* **negligible along** *the parabolic* $\widetilde{P} = N \cdot \widetilde{M}$ *if, for all* $g \in \widetilde{G}(\mathbb{A})$, $f_{\widetilde{P},\zeta}(\cdot g)$ *is perpendicular to all cusp forms* α *on* $\widetilde{M}(\mathbb{A})$ *with central character* ζ . *For a given* α , *this means that*

$$\int_{Z(\widetilde{M}(\mathbb{A}))\cdot M(k)\backslash \widetilde{M}(\mathbb{A})} f_{\widetilde{P},\zeta}(mg)\overline{\alpha(m)}\, dm = 0.$$

LEMMA 4.2.3. *If an automorphic form* f *on* $\widetilde{G}(\mathbb{A})$ *is negligible along* $\widetilde{P}$ *for all parabolic subgroups* $\widetilde{P}$ *of* $\widetilde{G}(\mathbb{A})$, *including* $\widetilde{G}(\mathbb{A})$ *itself, then* f *is identically zero.*

Note. This lemma is mentioned (in the real group situation) in Harish-Chandra [HC]. It derives from the decomposition of $L^2(G(k)\backslash\widetilde{G}(\mathbb{A}))$ given in section 2 of Morris [M], which parallels that in Langlands [L2].

Similarly to the non-metaplectic case, we say that parabolics $\widetilde{P}$ and $\widetilde{Q}$ are **associate** if their Levi factors $\widetilde{M_P}(\mathbb{A})$ and $\widetilde{M_Q}(\mathbb{A})$ are conjugate by an element of $G(k)$. Let $\{\widetilde{P}\}$ denote the equivalence class of associates of $\widetilde{P}$.

DEFINITION 4.2.4. *A form* f *is said to be* **concentrated on** $\{\widetilde{P}\}$ *(or more briefly, on* $\widetilde{P}$ *) if* f *is negligible along all parabolics* $\widetilde{Q}$ *such that* $\widetilde{Q} \notin \{\widetilde{P}\}$.

Now, given two automorphic forms f and f' on $\widetilde{G}(\mathbb{A})$, suppose we know that they are concentrated on the Borel subgroup $\widetilde{B}(\mathbb{A})$. Then Lemma 4.2.3 above says that $f = f'$ if and only if $f_{\widetilde{B}} = f'_{\widetilde{B}}$. We may extend this as follows:

LEMMA 4.2.5. *Suppose that* f *and* f' *as above are concentrated on* $\widetilde{B}(\mathbb{A})$ *, and let* $P \supset B$ *be any parabolic. Then the following are equivalent.*

(1) $f = f'$

(2) $f_{\widetilde{B}} = f'_{\widetilde{B}}$

(3) $f_{\widetilde{P}} = f'_{\widetilde{P}}$

Proof. We have already noted (1) $\Leftrightarrow$ (2), and obviously (1) $\Rightarrow$ (3), so it suffices to show that (3) $\Rightarrow$ (2). But if $P = MN \supset B = TN_B$, then $N_B \supset N$, and we may write $N_B(\mathbb{A}) = N(\mathbb{A}) \oplus N'(\mathbb{A})$ for some subgroup $N' \subset N_B$. Supposing that $F_{\widetilde{P}} = 0$, we see that

$$\begin{aligned} F_{\widetilde{B}}(g) &= \int\limits_{N_B(k)\backslash N_B(\mathbb{A})} F(ug)\, du \\ &= \int\limits_{N'(k)\backslash N'(\mathbb{A})} \int\limits_{N(k)\backslash N(\mathbb{A})} F(nn'g)\, dn\, dn' = 0 \end{aligned}$$

and applying this to $F = f - f'$, we are done. $\square$

§4.3 **The reduction.** First of all, we must show that both $E(g,s,\varphi)$ and $I(g,\varphi)$ are concentrated on $\widetilde{B}$. We begin with the Eisenstein series, which makes it necessary to discuss the roots of G. Following [PS-R1], let $t = \mathrm{diag}(t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1})$ be a generic element of our fixed maximal torus T. Defining $x_i : T \to k^\times$ by $t^{x_i} = t_i$, the set of roots of T in G is given by

$$\Phi_G = \{\pm(x_i \pm x_j), \pm 2x_i \mid i < j\}$$

and a positive set of roots by

$$\Phi_G^+ = \{(x_i \pm x_j), 2x_i \mid i < j\}.$$

Considering $P_n = M_n N_n$ in G, the set of roots of T in $M_n \cong GL(n)$ is then

$$\Phi_{M_n} = \{\pm(x_i - x_j) \mid i < j\} \supset \Phi_{M_n}^+ = \Phi_{M_n} \cap \Phi_G^+.$$

Write $W_G = N_G(T)/C_G(T)$ for the Weyl group of T in G, and similarly W_{M_n} for the group associated to M_n. In studying the Eisenstein series E, we will need to use the relative Bruhat decomposition, which states that, for any parabolics Q and P containing B,

$$G(k) = \coprod_{w \in W_{M_Q} \backslash W_G / W_{M_P}} QwP.$$

For our purposes, we also will need the following combinatorial lemma from [PS-R1].

LEMMA 4.3.1. *There exists a unique set of coset representatives* Ω *for* $W_{M_n}\backslash W_G$ *satisfying*

$$wN_Bw^{-1} \cap P_n \subset N_B \quad \text{for all } w \in \Omega. \tag{4.3.1}$$

Noting that $|W_G| = n!\,2^n$ *and* $|W_{M_n}| = n!$, *the* 2^n *elements of* Ω *are described as follows: for each subset* $S \subset \{1,\ldots,n\}$, *write* $S = \{i_1,\ldots,i_k\}$ *with* $i_1 < i_2 < \cdots < i_k$, *and also* $\sim S = \{i_{k+1},\ldots,i_n\}$ *with* $i_{k+1} > i_{k+2} > \cdots > i_n$. *Then the element* $w \in \Omega$ *corresponding to* S *is given by*

$$w^{-1}: \begin{array}{c} x_1 \mapsto x_{i_1} \\ \vdots \\ x_k \mapsto x_{i_k} \\ x_{k+1} \mapsto -x_{i_{k+1}} \\ \vdots \\ x_n \mapsto -x_{i_n} \end{array},$$

which may be realized as a matrix in $G(k)$ *with entries equal to* ± 1 *or* 0.

Proof. To see that the w above satisfy (4.3.1), consider the following. An element $w \in W_G$ acts on a root $\alpha \in \Phi_G$ to give $w \cdot \alpha \in \Phi_G$ via $t^{w\cdot\alpha} \stackrel{def}{=} (w^{-1}tw)^{\alpha}$ for $t \in T$. N_B is generated by the root groups N_α, where α ranges over Φ_G^+. But $wN_\alpha w^{-1} = N_{w\cdot\alpha}$, and so it is an easy check that each of the $w \in \Omega$ above does satisfy $N_{w\cdot\alpha} \cap P_n \subset N_B$ for all $\alpha \in \Phi_G^+$. On the other hand, multiplying one of the $w \in \Omega$ on the left by a non-trivial element of $W_{M_n} \cong S_n$ amounts to introducing a permutation of the x_i's. The check mentioned above will show that any such permutation would destroy property (4.3.1). $\square$

We now have enough information to prove the following basic fact.

PROPOSITION 4.3.2. *$E(g,s,\Phi)$ is concentrated on the Borel subgroup $\widetilde{B}$.*

Proof. The proof is expanded from that in [K-R1], and modified to our situation.

First of all, we show that E is negligible along $\widetilde{G}$. E transforms according to the central character ξ given by $\xi([\pm 1_{2n},\epsilon]) = \epsilon$ for $[\pm 1_{2n},\epsilon] \in Z \overset{def}{=} Z(\widetilde{G}(\mathbb{A}))$. Suppose then that f is a cusp form on $\widetilde{G}(\mathbb{A})$ with central character ξ. By a standard unfolding argument,

$$\begin{aligned}
&< E, f >_{\widetilde{G}(\mathbb{A})} \\
&= \int_{Z\cdot G(k)\backslash \widetilde{G}(\mathbb{A})} E(g)\bar{f}(g)\,dg \\
&= \int_{Z\cdot G(k)\backslash \widetilde{G}(\mathbb{A})} \sum_{\gamma\in P_n(k)\backslash G(k)} \Phi(\gamma g, s)\bar{f}(\gamma g)\,dg \\
&= \int_{Z\cdot P_n(k)\backslash \widetilde{G}(\mathbb{A})} \Phi(g)\bar{f}(g)\,dg \\
&= \int_{ZN_n(\mathbb{A})M_n(k)\backslash \widetilde{G}(\mathbb{A})} \int_{ZN_n(k)M_n(k)\backslash ZN_n(\mathbb{A})M_n(k)} \Phi(g_1 g)\bar{f}(g_1 g)\,dg_1\,dg \\
&= \int_{ZN_n(\mathbb{A})M_n(k)\backslash \widetilde{G}(\mathbb{A})} \Phi(g) \int_{N_n(k)\backslash N_n(\mathbb{A})} \bar{f}(ng)\,dn\,dg \\
&= \int_{ZN_n(\mathbb{A})M_n(k)\backslash \widetilde{G}(\mathbb{A})} \Phi(g)\bar{f}_{\widetilde{P}_n}(g)\,dg = 0.
\end{aligned}$$

Next, suppose that $\widetilde{P} = \widetilde{M}N \supsetneq \widetilde{B}$ is a proper standard parabolic of $\widetilde{G}$. As above, we have the decomposition $G(k) = \coprod_w P_n w P$, where w runs over a set of coset representatives for $W_{M_n}\backslash W/W_M$ chosen from among those for

$W_{M_n} \backslash W$ given above. Hence

$$E_{\widetilde{P}}(g, s, \Phi) = \int_{N(k)\backslash N(\mathbb{A})} \sum_{\gamma \in P_n(k)\backslash G(k)} \Phi(\gamma n g, s)\, dn$$

$$= \sum_w \int_{N(k)\backslash N(\mathbb{A})} \sum_{\gamma \in P \cap (w^{-1} P_n w)\backslash P} \Phi(w\gamma n g, s)\, dn.$$

Now defining $M''_w = M \cap w^{-1} P_n w$ and $N''_w = N \cap w^{-1} P_n w$, we see that the sum splits up into a sum over the k -points of $M''_w \backslash M$ and $N''_w \backslash N$. So we have

$$E_{\widetilde{P}}(g, s, \Phi) = \sum_w \sum_{\gamma_1 \in M''_w \backslash M} \int_{N''_w(k)\backslash N(\mathbb{A})} \Phi(w n \gamma_1 g, s)\, dn.$$

Thus far, we have only used the fact that $\widetilde{P} = \widetilde{M} \ltimes N$, and that the change of variable $n' = \gamma n \gamma^{-1}$ on $N(\mathbb{A})$ has modulus one for $\gamma \in M(k)$. Next, note that the lemma above gives

$$w N''_w(\mathbb{A}) w^{-1} = w N(\mathbb{A}) w^{-1} \cap P_n(\mathbb{A}) \subset N_B(\mathbb{A}),$$

all taking place in $G(\mathbb{A})$. But we must check that this also works in $\widetilde{G}(\mathbb{A})$, given the embeddings $N''_w(\mathbb{A}) \subset N(\mathbb{A}) \subset N_B(\mathbb{A}) \hookrightarrow \widetilde{G}(\mathbb{A})$, and $w \in G(k) \hookrightarrow \widetilde{G}(\mathbb{A})$. In other words, writing

$$[w, \lambda(w)][n'', 1][w^{-1}, \lambda(w^{-1})] = [w n'' w^{-1}, \epsilon]$$

for $n'' \in N''_w(\mathbb{A})$, we must check that $\epsilon = 1$. Letting $p = w n'' w^{-1} \in N_B(\mathbb{A})$

(by the lemma),

$$\begin{aligned}\epsilon &= \lambda(w)\lambda(w^{-1})\beta(w,n'')\beta(wn'',w^{-1})\\ &= \lambda(w)\lambda(w^{-1})\frac{\lambda(w)\lambda(n'')}{\lambda(wn'')}\frac{\lambda(wn'')\lambda(w^{-1})}{\lambda(p)}\prod_v \tilde{c}_v(w,n'')\tilde{c}_v(wn'',w^{-1})\end{aligned}$$

(4.3.2)

$$= \prod_v (x(w),x(n''))_v \tilde{c}_v(pw,w^{-1}) = \prod_v \tilde{c}_v(w,w^{-1}) = 1$$

using Corollary 1.5.3, Definition 1.6.7, and the proof of Proposition 1.8.1, together with the fact that $x(p)=1$ for all $p \in N_B(\mathbb{A}) \subset P_n(\mathbb{A})$.

Having proven that $wN''_w(\mathbb{A})w^{-1} \subset N_B(\mathbb{A})$ in $\widetilde{G}(\mathbb{A})$, and given that $\Phi(g,s)$ is left $N_B(\mathbb{A})$-invariant, we may write

$$E_{\widetilde{P}}(g,s,\Phi) = \sum_w \sum_{\gamma_1 \in M''_w \backslash M} \Phi_w(\gamma_1 g, s) \tag{4.3.3}$$

$$\text{where} \qquad \Phi_w(g,s) = \int\limits_{N''_w(\mathbb{A})\backslash N(\mathbb{A})} \Phi(wng,s)\,dn.$$

Now, we note that for $m \in \widetilde{M}''_w(\mathbb{A}),\ g \in \widetilde{G}(\mathbb{A})$,

$$\Phi_w(mg,s) = \mu_w(m)\Phi_w(g,s)$$

for $\mu_w(m) = \chi(wmw^{-1})|a(wmw^{-1})|^{s+\rho_n}\mathrm{mod}_P(m)^{-1}$ a character of $M''_w\backslash\widetilde{M}''_w(\mathbb{A})$. Setting $U(\mathbb{A}) \overset{\text{def}}{=} \widetilde{M}(\mathbb{A}) \cap w^{-1}N_B(\mathbb{A})w$, $\widetilde{T}(\mathbb{A})\cdot U(\mathbb{A})$ is a Borel in $\widetilde{M}(\mathbb{A})$, and $\widetilde{T}(\mathbb{A})\cdot U(\mathbb{A}) \subset \widetilde{M}''_w(\mathbb{A}) \subset \widetilde{M}(\mathbb{A})$. Hence $\widetilde{M}''_w(\mathbb{A})$ is a parabolic of $\widetilde{M}(\mathbb{A})$. Note also that $Z(\widetilde{M}(\mathbb{A})) \subset \widetilde{T}(\mathbb{A}) \subset \widetilde{M}''_w(\mathbb{A})$, so the function $m \mapsto \Phi_w(mg)$ has central character $\mu_w|_{Z(\widetilde{M}(\mathbb{A}))}$. If $M''_w = M$, then the term in $E_{\widetilde{P}}(mg,s,\Phi)$ corresponding to w is a constant times a character of

$\widetilde{M}(\mathbb{A})$, and hence is perpendicular to cusp forms. If $M_w'' \subsetneq M$, then the term corresponding to w is an Eisenstein series on $\widetilde{M}(\mathbb{A})$, and so by an argument analogous to that given at the beginning of the proof, it is perpendicular to all cusp forms. This proves that E is negligible along $\widetilde{P}$ for $\widetilde{B} \subsetneq \widetilde{P} \subsetneq \widetilde{G}$, and completes the proof that E is concentrated on the Borel. $\square$

The same short proof used for Corollary 3.4 of [K-R1] goes over verbatim to give us:

COROLLARY 4.3.3. *$E(g, s, \Phi)$ and $E_{\widetilde{P}_n}(g, s, \Phi)$ have the same set of poles with the same orders.*

The proof that $I(g, \varphi)$ is also concentrated on $\widetilde{B}$ parallels that given in section 2 of [K-R1] (for m even) so closely that there is little point in repeating it here. The only computation which needs some comment is the derivation (in Lemma 2.4) of the expression for the Fourier coefficient $E_\beta(g, s, \Phi)$, where $\beta \in \mathrm{Sym}_n(k)$. A cocycle calculation is required at one point in the proof of that lemma, but this amounts to virtually the same check as that performed in equation (4.3.2) in Proposition 4.3.2 above. We shall therefore take it for granted that I is concentrated on $\widetilde{B}$.

Summarizing the results of the last two sections, we have:

THEOREM 4.3.4. *Subject to the conditions stated in Theorem 3.3.1, suppose that*

(1) *$E_{\widetilde{P}_n}(g, s, \varphi)$ is holomorphic at $s = s_o(m, n)$, and*

(2) $E_{\widetilde{P}_n}(g, s_o, \varphi) = \kappa I_{\widetilde{P}_n}(g, \varphi) = \kappa\Phi(g, s_o, \varphi)$.

Then Theorem 3.3.1 is proven. In other words, (1) *and* (2) *hold without the* $\widetilde{P}_n$ *subscript.*

§4.4 The constant term of E and the intertwining operators. For the next few chapters, we will study the analytic properties of the constant term of $E(g, s, \Phi)$ with respect to $\widetilde{P}_n$, and an intertwining operator which appears in it as a summand.

Setting $\widetilde{P} = \widetilde{P}_n$ in the proof of Proposition 4.3.2, note that $\{w_j\}_{j=0}^{n}$ gives a different set of coset representatives for $W_{M_n}\backslash W/W_{M_n}$ than used there, but one which still satisfies $w_{n-r}N''_{n,r}(\mathbb{A})w_{n-r}^{-1} \subset N_B(\mathbb{A})$ in $\widetilde{G}(\mathbb{A})$. Hence equation (4.3.3) remains valid and results in

$$(4.4.1) \qquad E_{\widetilde{P}_n}(g, s, \Phi) = \sum_{r=0}^{n} E^r_{\widetilde{P}_n}(g, s, \Phi)$$

where

$$E^r_{\widetilde{P}_n}(g, s, \Phi) = \sum_{\gamma \in Q_r\backslash M_n} \Phi_r(\gamma g, s) \quad \text{and}$$

$$\Phi_r(g, s) = \int_{N'_{n-r}(\mathbb{A})} \Phi(w_r n g, s)\, dn.$$

We note that since $N'_n = 1$, $N'_0 = N_n$, and $Q_0 = Q_n = M_n$, we have

$$E^0_{\widetilde{P}_n} = \Phi \quad \text{and} \quad E^n_{\widetilde{P}_n}(g, s, \Phi) = \Phi_n(g, s, \Phi) = \int_{N_n(\mathbb{A})} \Phi(w_n n g, s)\, dn.$$

For the middle $n-1$ terms, however, the summation over $Q_r \backslash M_n$ does not disappear, and these terms may be regarded as Eisenstein series on $\widetilde{M}_n(\mathbb{A}) \cong \widetilde{GL}(n, \mathbb{A})$. They will be studied in chapter 8.

Now we focus on the term Φ_n . For convenience, until further notice we write $P = MN$ in place of $P_n = M_n N_n$, and let $w = w_n$. Notice that for $n \in N(\mathbb{A}),\ m \in \widetilde{M}(\mathbb{A})$, and $g \in \widetilde{G}(\mathbb{A})$, Φ satisfies

$$\Phi(nmg, s, \varphi) = \chi(m)|a(m)|^{s+\rho_n}\Phi(g, s, \varphi), \tag{4.4.2}$$

where we let $\rho_n = \frac{n+1}{2}$. In addition, if $\varphi = \otimes'_v \varphi_v \in \mathcal{S}(V(\mathbb{A})^n)$ is factorizable, then we may write

$$\Phi(g, s, \varphi) = \bigotimes_v {}'\Phi_v(g_v, s, \varphi_v), \tag{4.4.3}$$

choosing $g_v \in \widetilde{G}_v$ such that $p(\prod_v g_v) = g$ and setting

$$\Phi_v(g_v, s, \varphi_v) = |a(g_v)|^{s-s_o(m,n)}\omega_v(g_v)\varphi_v(0).$$

Recall the discussion of $p : \prod'_v \widetilde{G}_v \to \widetilde{G}(\mathbb{A})$ in §1.8.

We develop this more generally. Let $\mu : \widetilde{M}(\mathbb{A}) \to \mathbb{C}^\times$ be a continuous quasi-character which is genuine (i.e. is not the pullback via π of a character of $M(\mathbb{A})$). Pull back μ to $\bar{\mu} : \prod'_v \widetilde{M}_v \to \mathbb{C}$ and write $\bar{\mu} = \otimes'_v \mu_v$ for quasi-characters μ_v of $\widetilde{M}_v$, as in Tate [T]. These μ_v are then genuine, since μ was. So it is apparent that $\mu(m) = \prod_v \mu_v(m_v)$ for any choice of $m_v \in \widetilde{M}_v$ with $p(\prod_v m_v) = m$. Rather than keeping track, we will write $\mu = \otimes'_v \mu_v$, regarding both sides as functions on $\widetilde{M}(\mathbb{A})$.

Now, for each place $v \in \Sigma_k$, let

$$I(\mu_v) = \{\Phi_v : \widetilde{G}_v \to \mathbf{C} \text{ smooth } \mid$$
$$\Phi_v(nmg) = \mu_v(m)|a(m)|_v^{\rho_n}\Phi_v(g) \quad \forall n \in N_v,\ m \in \widetilde{M}_v\},$$

where "smooth" means locally constant at the finite places, and the usual notion at the archimedean places. This defines a group representation of $\widetilde{G}_v$, where the action is by right translation on functions.

DEFINITION 4.4.1. *We will call a character* $\mu_v : \widetilde{M}_v \to \mathbf{T}$ **unramified** *if*

(1) v *is a place of* k *at which* $K_v \hookrightarrow \widetilde{G}_v$, *and*

(2) $\mu_v \equiv 1$ *on* $K_v \cap \widetilde{M}_v$.

If v is a place at which μ_v is unramified, then we may define a **spherical vector** $\Phi_v^o \in I(\mu_v)$ by $\Phi_v^o(k) = 1$ for all $k \in K_v \hookrightarrow \widetilde{G}_v$. This allows us to define the global induced $\widetilde{G}(\mathbb{A})$-space

$$I(\mu) = \bigotimes_v{}' I(\mu_v),$$

by taking the restricted tensor product with respect to the vectors $\{\Phi_v^o\}$. Since the characters μ and μ_v were genuine, we may identify the resulting space of functions, nominally defined on $\prod_v' \widetilde{G}_v$, with a space of smooth functions $\Phi : \widetilde{G}(\mathbb{A}) \to \mathbf{C}$ satisfying

$$\Phi(nmg) = \mu(m)|a(m)|^{\rho_n}\,\Phi(g) \quad \text{for } n \in N(\mathbb{A}),\ m \in \widetilde{M}(\mathbb{A}).$$

Returning now to our situation, (4.4.2) and (4.4.3) above, together with the fact that $\Phi_v(s,\varphi_v^o) = \Phi_v^o(s) \in I(\chi_v||_v^s)$ for any place $v \notin S_{k,V}$ (see Lemma 2.4.1), lead us to conclude that

$$\Phi(s,\varphi) \in I(\chi||^s)$$

for any $s \in \mathbb{C}$ and $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$. Abbreviating $I(s) \stackrel{def}{=} I(\chi||^s)$ (and similarly $I_v(s) \stackrel{def}{=} I(\chi_v||_v^s)$), we consider the whole collection $I = \coprod_{s\in\mathbb{C}} I(s)$ to be a fiber bundle of $\widetilde{K}$ -modules ($\widetilde{K}$ acts by right translation) lying over $\mathbb{C}$, with fiber

$$\mathcal{B} = \{f : \widetilde{K} \to \mathbb{C} \text{ smooth } \mid f(pk) = \chi(p)f(k)\ \forall p \in \widetilde{P}(\mathbb{A}) \cap \widetilde{K}\}.$$

Here, of course, we have extended χ to $\widetilde{P}(\mathbb{A}) = N(\mathbb{A})\widetilde{M}(\mathbb{A})$ by making it trivial on $N(\mathbb{A})$. For each $s \in \mathbb{C}$, we have a $\widetilde{K}$ -isomorphism $I(s) \xrightarrow{\sim} \mathcal{B}$ given by restriction to $\widetilde{K}$. Given a function $f \in I(s)$, let $f' \in \mathcal{B}$ denote its image under this map. This then gives a global trivialization of the bundle:

$$\begin{array}{ccc} I & \xrightarrow{\sim} & \mathcal{B} \times \mathbb{C} \\ \downarrow & \swarrow & \\ \mathbb{C} & & \end{array}$$

DEFINITION 4.4.2. *A map* $s \mapsto \Phi(s) \in I(s)$ *will be called a* **section** *if the map*

$$\mathbb{C} \to \mathcal{B}$$

$$s \mapsto \Phi'(s)$$

is continuous, where $\mathcal{B}$ *has the natural inner product*

$$< f_1, f_2 >= \int\limits_{\widetilde{P}(\mathbb{A})\cap\widetilde{K}\backslash\widetilde{K}} f_1(k)\bar{f}_2(k)\,dk.$$

A section $\Phi(s)$ *is a* **standard section** *if it comes from a fixed* $f \in \mathcal{B}$ *: in other words, if* $\Phi'(s)$ *is independent of* s *. We will say that* $\Phi(s)$ *is a* $\widetilde{K}$ **-finite section** *if* $\mathcal{B}_\Phi \stackrel{def}{=} \mathrm{span}_{\mathbf{C}}\{k \cdot \Phi'(s) \mid k \in \widetilde{K}, s \in \mathbf{C}\}$ *is a finite-dimensional (* $\widetilde{K}$ *-invariant) subspace of* $\mathcal{B}$ *. A* $\widetilde{K}$ *-finite section will be called* **holomorphic** *(or* **meromorphic***) if, choosing a fixed basis* $\{\Phi^1, \ldots, \Phi^r\}$ *for* $\mathcal{B}_\Phi$ *and writing*

$$\Phi'(s) = \alpha_1(s)\Phi^1 + \cdots + \alpha_r(s)\Phi^r,$$

the functions $\alpha_i : \mathbf{C} \to \mathbf{C}$ *are holomorphic (respectively meromorphic).*

Note that all of the above is also valid in the local situation, with $I_v(s)$ replacing $I(s)$, $\widetilde{K}_v$ replacing $\widetilde{K}$, and so forth. We will denote the set of $\widetilde{K}$ -finite sections of I by $I_{\widetilde{K}}$, and the subspace of $\widetilde{K}$ -finite vectors of $I(s)$ by $I(s)_{\widetilde{K}}$. It is clear then that $\Phi(\varphi) \in I_{\widetilde{K}}$ for $\varphi \in \mathcal{S}(V(\mathbb{A})^n)_{\widetilde{K}}$.

Now, for $\mathrm{Re}(s) > \rho_n$, the formula

$$M(s)\Phi(g,s) = \int\limits_{N(\mathbb{A})} \Phi(wng,s)\,dn$$

defines a $\widetilde{G}(\mathbb{A})$ -intertwining operator

$$M(s) : I(s)_{\widetilde{K}} \to I(-s)_{\widetilde{K}}.$$

The usual theory of Eisenstein series, extended in Morris [M], tells us that the intertwining operator above has a meromorphic continuation. In other words, given any holomorphic section $\Phi(s) \in I(s)_{\widetilde{K}}$, $M(s)\Phi(s)$ will be a meromorphic section taking values in $I(-s)$ for any $s \in \mathbb{C} \sim \{\text{the pole set}\}$.

Now, the measure dn on $N(\mathbb{A})$ which gives $\mathrm{vol}(N(k)\backslash N(\mathbb{A})) = 1$ equals $\prod_{i \leq j} db_{ij}$ for $n = n(b)$, where db_{ij} is the measure on $\mathbb{A}$ constructed in Tate's thesis. As explained there, we may also write $dn = \prod_v dn_v$, so that for $\mathrm{Re}(s) > \rho_n$ and factorizable $\Phi \in I_{\widetilde{K}}$, we have

$$M(s)\Phi(g,s) = \bigotimes_v{}' \int_{N_v} \Phi_v(w_v n_v g_v, s)\, dn_v.$$

The choice of inverse image of $n \in N(\mathbb{A})$ under $p : \prod_v' \widetilde{G}_v \to \widetilde{G}(\mathbb{A})$ being clear, we fix elements $w_v \in \widetilde{G}_v$ for each $v \in \Sigma_k$ by defining $w_v = [w, 1] \in \widetilde{G}_v$. It is then clear that $p(\prod_v w_v) = [w, 1] = [w, \lambda(w)] \in G(k) \subset \widetilde{G}(\mathbb{A})$. This then leads us to define the operator

$$M_v(s) : I_v(s) \longrightarrow I_v(-s)$$

via

$$M_v(s)\Phi_v(g_v, s) = \int_{N_v} \Phi_v(w_v n_v g_v, s)\, dn_v$$

for $\mathrm{Re}(s) > \rho_n$, so that

$$M(s) = \bigotimes_v{}' M_v(s)$$

with the identification explained above. To begin to see where the poles of the operator $M(s)$ lie, we must first determine what happens to the standard spherical sections $\Phi_v^o(s) \in I_v(s)$.

§5.1 The method of Gindiken and Karpelevich. For the remainder of the chapter, we fix a place $v \notin S_k$ for which χ_v is unramified, and frequently omit the subscript v. Since it is easily seen that $M(s)\Phi^o(gk,s) = M(s)\Phi^o(g,s)$ for all $g \in \widetilde{G}_v$, $k \in K_v$, it follows that

$$M(s)\Phi^o(s) = Z(s)\Phi^o(-s)$$

for some meromorphic function $Z(s)$. In this chapter, we will compute the function

$$Z(s) = M(s)\Phi^o(w^{-1},s) = \int_{N_n(k_v)} \Phi^o(wnw^{-1},s)\,dn$$

for $\mathrm{Re}(s) \gg 0$ and assign it the obvious meromorphic continuation. This will be accomplished by adapting the method of Gindiken-Karpelevich, as described by Langlands in [L1], to the metaplectic situation. The result is as follows:

THEOREM 5.1.1. *For a place $v \notin S_k$ at which χ_v is unramified,*

$$M_v(s)\Phi_v^o(s) = \frac{a_v(n,s)}{b_v(n,s)}\Phi_v^o(-s),$$

where

$$a_v(n,s) = \prod_{k=1}^{[\frac{n+1}{2}]} \zeta_v(2s-(n+1)+2k)$$

$$b_v(n,s) = \prod_{k=1}^{[\frac{n+1}{2}]} \zeta_v(2s+n+2-2k).$$

Here [ ] *stands for the greatest integer function.*

Using Langlands' notation for the most part, we write $\mathfrak{h}$ for the Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$ corresponding to the torus T, and denote the positive infinitesimal roots (acting on $\mathfrak{h}$) by

$$\Sigma^+ = \{x_i \pm x_j, 2x_i \mid i < j\}.$$

Let $\mathfrak{h}^*_{\mathbf{R}}$ be the real vector space spanned by these roots. For simple roots

$$\alpha_1 = x_1 - x_2,\ \ldots,\ \alpha_{n-1} = x_{n-1} - x_n,\ \alpha_n = 2x_n,$$

let $\alpha_i^\vee$ (H_{α_i} in Langlands) be the co-root in $\mathfrak{h}$ of α_i, so that if $\{x_i^*\}$ is a dual basis to $\{x_i\}$, we have $\alpha_i^\vee = x_i^* - x_{i+1}^*$ for $1 \le i \le n-1$, and $\alpha_n^\vee = x_n^*$. Then $\mathfrak{h}_{\mathbf{R}} \stackrel{def}{=} \mathrm{span}_{\mathbf{R}}\{\alpha_i^\vee \mid 1 \le i \le n\}$ is the dual space to the vector space $\mathfrak{h}^*_{\mathbf{R}}$. As on p. 23 of [L1], we choose an open half-space $R_* \subset \mathfrak{h}^*_{\mathbf{R}}$ via

$$R_* = \{v \in \mathfrak{h}^*_{\mathbf{R}} \mid \lambda_*(v) > 0\}$$

where we set $\lambda_* = -\sum_{i=1}^n x_i^*$. For *any* similarly defined half-space R, let Σ_R^- be the set of negative roots lying in R, and Σ_R^+ be the set of positive roots in $\overline{R}$ (the closure of R). Let $N(R)$ equal the subgroup of $N_B \hookrightarrow \widetilde{G}_v$ generated by the root groups N_α, where $-\alpha$ ranges over Σ_R^-. We then set

$$\overline{N}(R) = [w,1]N(R)[w,1]^{-1} \subset \widetilde{G}_v.$$

This is different from the situation in [L1], in that Langlands has no covering, and so he lets $\overline{N}(R)$ be the group generated by the roots in Σ_R^-. Note that conjugation by w changes the sign of all roots, so that $\pi(\overline{N}(R))$ is exactly Langlands' group.

With our initial choice R_*, it is easily seen that

$$\Sigma^-_{R_*} = \{-(x_i + x_j), -2x_i \mid i < j\}$$

$$\Sigma^+_{R_*} = \{x_i - x_j \mid i < j\}$$

and

$$N(R_*) = N_n,$$

so that

$$Z(s) = \int_{\overline{N}(R_*)} \Phi^o(x, s)\, dx. \tag{5.1.1}$$

The idea of the Gindiken-Karpelevich method is to "remove" one root at a time from $\Sigma^-_{R_*}$ by changing R_*, and in so doing, to write $Z(s)$ as a product of one-dimensional integrals which are easily evaluated.

As developed in Langlands, the method proceeds as follows. Given an open half-space R (at some intermediate step) determined by $\lambda_0 \in \mathfrak{h}_{\mathbf{R}}$,

$$R = \{v \in \mathfrak{h}^*_{\mathbf{R}} \mid \lambda_0(v) > 0\},$$

another element $\lambda_1 \in \mathfrak{h}_{\mathbf{R}}$ is found which satisfies the following properties:

(1)

$$\Sigma^-_R = \{\alpha < 0 \mid \lambda_1(\alpha) \geq 0\}$$

(α will always stand for a root).

(2) There exists a unique negative root $-\alpha_0$ such that $\lambda_1(-\alpha_0) = 0$, and so

$$\Sigma_R^- = \{-\alpha_0\} \coprod \{\alpha < 0 \mid \lambda_1(\alpha) > 0\}.$$

Now, defining $S = \{v \in \mathfrak{h}_{\mathbf{R}}^* \mid \lambda_1(v) > 0\}$, and taking Σ_S^- and Σ_S^+ as before, note that

$$\Sigma_R^- = \{-\alpha_0\} \coprod \Sigma_S^-.$$

Although it isn't explicitly noted in Langlands, one may also show easily that

$$\alpha \in \Sigma_R^+ \implies \lambda_1(\alpha) > 0 \tag{5.1.2}$$

and

$$\Sigma_S^+ = \Sigma_R^+ \coprod \{\alpha > 0 \mid \lambda_1(\alpha) = 0\} = \Sigma_R^+ \coprod \{\alpha_0\}.$$

So beginning with R_* , $\Sigma_{R_*}^-$, and $\Sigma_{R_*}^+$, we obtain a succession of half-spaces

$$R_*, \ldots, R, S, \ldots ,$$

each time removing a negative root $-\alpha_0$ from Σ_R^- to get Σ_S^- , and adding α_0 to Σ_R^+ to get Σ_S^+ .

Specializing this to our situation, we find that there is some order to the removal of the roots (which will be needed later).

Lemma 5.1.2. *At some stage in the above process, suppose we are removing* $-\alpha_0$ *from* Σ_R^- *with* λ_1 *.*

(1) *If* $-\alpha_0 = -2x_r$ *, then we have already removed all roots* $-(x_i + x_r)$ *with* $1 \leq i < r$ *.*

(2) *If* $-\alpha_0 = -(x_r + x_s)$ *with* $r < s$ *, then we have already removed all roots of the form*

$$-(x_i + x_s) \quad \text{with } 1 \leq i < r, \text{ and}$$

$$-(x_i + x_r) \quad \text{with } 1 \leq i < s.$$

Proof. First of all, note that for all i with $1 \leq i \leq n-1$, $x_i - x_{i+1} \in \Sigma_{R_*}^+ \subset \Sigma_R^+$, as Σ_R^+ only grows at each step. So writing $\lambda_1 = \sum_{i=1}^n a_i x_i^*$ uniquely for $a_i \in \mathbf{R}$, we have $\lambda_1(x_i - x_{i+1}) > 0$ for all $i \leq n-1$, which says that

$$a_1 > a_2 > \cdots > a_n \tag{5.1.3}$$

Now suppose that $-\alpha_0 = -2x_r$ is being removed by λ_1, so that $\lambda_1(\alpha_0) = -4a_r = 0$. If $1 \leq i < r$, then

$$\tfrac{1}{2}\lambda_1(-(x_i + x_r)) = -a_i - a_r = -a_i < 0$$

by equation (5.1.3). This says that $-(x_i + x_r)$, which was in $\Sigma_{R_*}^-$, is no longer in $\Sigma_R^- = \{\alpha < 0 \mid \lambda_1(\alpha) \geq 0\}$, and hence must have been removed at an earlier step.

Suppose that $-\alpha_0 = -(x_r + x_s)$, $(r < s)$, is being removed by λ_1 . Then $\lambda_1(-(x_r + x_s)) = 0 \Longrightarrow a_r = -a_s$. First, let $1 \leq i < r$. Then

$$\tfrac{1}{2}\lambda_1(-(x_i + x_s)) = -a_i - a_s = -a_i + a_r < 0,$$

since $a_i > a_r$ by (5.1.3), and so $-(x_i + x_s)$ has been removed. Finally, we consider $-(x_i + x_r)$ with $1 \leq i < s$. Since $\frac{1}{2}\lambda_1(-(x_i + x_r)) = -a_i - a_r$, we have $i < s \Longrightarrow a_i > a_s = -a_r \Longrightarrow 0 > -a_i - a_r$, and we are done. □

This will be useful in the following sense. If $[\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \epsilon]$ represents a variable in $\overline{N}(S)$, then we have the following diagrams for $B = {}^tB$ from the lemma:

$$\begin{array}{cc} & \begin{array}{ccc} & & r \end{array} \\ \begin{array}{c} \\ \\ r \\ \\ \\ \end{array} & \left(\begin{array}{ccc} & & 0 \\ & & \vdots \\ 0 & \cdots & 0 \\ & & \\ & & \end{array}\right) \end{array} \qquad \begin{array}{cc} & \begin{array}{ccccc} & & r & & s \end{array} \\ \begin{array}{c} \\ \\ r \\ \\ s \\ \\ \end{array} & \left(\begin{array}{ccccc} & & 0 & & 0 \\ & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & & \\ & & & & \end{array}\right) \end{array}$$

$$(5.1.4) \qquad \alpha_0 = 2x_r \qquad\qquad \alpha_0 = x_r + x_s$$

In other words, if $\alpha \notin \Sigma_S^-$ (if it has been removed), then the entry $b_{ij} = b_{ji}$ of B corresponding to α must be zero. By induction, it is also easy to see that any entry lying above or to the left of a zero entry is also zero. So at any stage in the process, B representing $\overline{N}(S)$ has zeros above a stepped line proceeding from lower left to upper right.

One further bit of notation. Set $\mathfrak{h}^*_{\mathbf{Z}} = \mathrm{span}_{\mathbf{Z}}\{\alpha_i \mid 1 \le i \le n\}$ where the $\{\alpha_i\}$ are the simple roots as before, and define $\Lambda : \mathfrak{h}^*_{\mathbf{Z}} \to \mathrm{Hom}(T, k_v^\times)$ in the obvious way: for $t = \mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \in T$, write

$$t^{\Lambda(x_i)} = t_i \qquad \text{for all } i,$$

and extend via $\Lambda(\sum a_i x_i) = \sum a_i \Lambda(x_i)$, using additive exponential notation. Λ defines a map into $\mathrm{Hom}(T, \mathbf{R}^\times)$ by further applying $||_v$. We then tensor $\mathfrak{h}^*_{\mathbf{Z}}$ and $\mathrm{Hom}(T, \mathbf{R}^\times)$ with $\mathbf{C}$ over the integers to obtain a map which associates to every $\mu \in \mathfrak{h}^*_{\mathbf{C}}$ a mapping

$$t \mapsto |t^{\Lambda(\mu)}|_v$$

in $\mathrm{Hom}(T, \mathbf{C}^\times)$. So $\mu = \sum c_i x_i$ gives $|t^{\Lambda(\mu)}|_v \overset{def}{=} \prod_{i=1}^n |t_i|_v^{c_i}$ for $c_i \in \mathbf{C}$. This also extends trivially to $\widetilde{T}_v$ by first applying π to $(t, \epsilon) \in \widetilde{T}_v$.

In order to apply the method, it will be convenient to realize $\Phi^o \in I(\chi_v |\det|^s)$ as lying in $\mathrm{Ind}_{\widetilde{B}_v}^{\widetilde{G}_v}(\chi_v \Lambda(\mu))$. In other words, find $\mu \in \mathfrak{h}^*_{\mathbf{C}}$ such that $t \in \widetilde{T}_v$, $n \in N_B$, and $g \in \widetilde{G}_v$ will give

$$\Phi^o(ntg) = \chi(t)|t^{\Lambda(\mu+\rho_B)}|\Phi^o(g),$$

where $\rho_B = \frac{1}{2}\sum_{\alpha>0}\alpha$. This requires that $|\det(t)|^{s+\rho_n} = |t^{\Lambda(\mu+\rho_B)}|$. An easy check shows that $\rho_B = \sum_{i=1}^n (n-i+1)x_i$, and so we need $(s+\frac{n+1}{2})\sum_{i=1}^n x_i = \mu + \sum_{i=1}^n (n-i+1)x_i$, which yields

$$\mu = \sum_{i=1}^n (s - \rho_n + i)x_i.$$

Now, we write $N^o \subset N_n$ for the one-parameter subgroup generated by the root vector $X_{\alpha_0} \in \mathfrak{g}$, and define

$$\overline{N}^o \stackrel{def}{=} [w,1]N^o[w,1]^{-1} \subset \widetilde{G}_v.$$

Then at any stage with $\Sigma_R^- = \{-\alpha_0\} \coprod \Sigma_S^-$ as above, we clearly have $\overline{N}(R) = \overline{N}^o \cdot \overline{N}(S)$. Define an embedding

$$\varphi_{\alpha_0} : SL(2)_v \longrightarrow G_v \quad \text{by}$$

$$\varphi_{\alpha_0}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \exp(xX_{\alpha_0}) \qquad \varphi_{\alpha_0}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \exp(xX_{-\alpha_0})$$

$$\text{and} \quad \varphi_{\alpha_0}\begin{pmatrix} \exp(a) & \\ & \exp(-a) \end{pmatrix} = \exp(a\alpha_0^\vee)$$

for $x, a \in k_v$, and a small. Now

$$\int_{\overline{N}(R)} \Phi^o(\bar{n}, s)\, d\bar{n} = \int_{\overline{N}^o}\int_{\overline{N}(S)} \Phi^o(\bar{n}_2\bar{n}_1, s)\, d\bar{n}_2\, d\bar{n}_1,$$

where we take $\bar{n}_1(x) \stackrel{def}{=} [w,1][\varphi_{\alpha_0}\left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix}\right), 1][w,1]^{-1}$, $x \in k_v$, as a variable of integration for $\overline{N}^o$. Note that if $x \in \mathcal{O}_v$, then

$$\bar{n}_1(x) = [\varphi_{\alpha_0}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, 1] \in K_v \hookrightarrow \widetilde{G}_v,$$

and Φ^o is right K_v -invariant. On the other hand, since

$$\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

we may write

(5.1.5) $$\bar{n}_1(x) = n_1(x)a_1(x)k_1(x)$$

with

$$n_1(x) = \begin{cases} [\varphi\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}, 1] & \text{for } x \notin \mathcal{O}_v, \\ 1 & \text{for } x \in \mathcal{O}_v, \end{cases}$$

$$a_1(x) = \begin{cases} [\varphi\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}, \epsilon(x)] & \text{for } x \notin \mathcal{O}_v, \\ 1 & \text{for } x \in \mathcal{O}_v, \end{cases}$$

and

$$k_1(x) = \begin{cases} [\varphi\begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix}, 1] & \text{for } x \notin \mathcal{O}_v, \\ [\varphi\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, 1] & \text{for } x \in \mathcal{O}_v. \end{cases}$$

The factor $\epsilon(x)$ above is determined by equation (5.1.5), and it is in fact equal to 1, as will be shown later. Given this decomposition,

$$\begin{aligned} \Phi^o(\bar{n}_2\bar{n}_1, s) &= \Phi^o(\bar{n}_2 n_1 a_1, s) = \Phi^o(n_1 a_1, s)\Phi^o(a_1^{-1} n_1^{-1} \bar{n}_2 n_1 a_1, s) \\ &= \Phi^o(a_1, s)\Phi^o(a_1^{-1} n_1^{-1} \bar{n}_2 n_1 a_1, s). \end{aligned}$$

Defining $N^+(S) \subset N_B \hookrightarrow \widetilde{G}_v$ by

$$N^+(S) = \prod_{\substack{\lambda_1(\alpha)>0 \\ \alpha>0}} N_\alpha,$$

our goal here is to show that

$$a_1^{-1} n_1^{-1} \bar{n}_2 n_1 a_1 \in N^+(S) \cdot \overline{N}(S). \tag{5.1.6}$$

This is done by Langlands on the G_v -level, but in our situation we must check that the cocycles work out correctly. This is essential, because although Φ^o is left N_B -invariant, it is certainly not invariant by $\widetilde{N}_B$ on the left. If we show that we can write

$$a_1^{-1} n_1^{-1} \bar{n}_2 n_1 a_1 = p \cdot \bar{n}_2' \in N^+(S) \cdot \overline{N}(S),$$

then it is easily seen that $\bar{n}_2 \mapsto \bar{n}_2'$ is a bijection of the set $\overline{N}(S)$, with a change of variables given by

$$d\bar{n}_2 = \prod_{\substack{\lambda_1(\alpha)<0 \\ \alpha>0}} |a_1^{\Lambda(-\alpha)}| \, d\bar{n}_2'$$

as in Langlands. Hence, we will have

$$(5.1.7) \quad \int_{\overline{N}(R)} \Phi^o(\bar{n}, s) \, d\bar{n} = \int_{k_v} \int_{\overline{N}(S)} \Phi^o(a_1, s) \Phi^o(a_1^{-1} n_1^{-1} \bar{n}_2 n_1 a_1, s) \, d\bar{n}_2 \, dx$$
$$= \left(\int_{k_v} \Phi^o(a_1(x), s) \prod_{\substack{\lambda_1(\alpha)<0 \\ \alpha>0}} |a_1^{\Lambda(-\alpha)}| \, dx \right) \left(\int_{\overline{N}(S)} \Phi^o(\bar{n}_2', s) \, d\bar{n}_2' \right)$$

This reduces the original integral over $\overline{N}(R_*)$ to a product of one-dimensional integrals which may then be further simplified.

Therefore, we must address the question of the cocycles.

LEMMA 5.1.3. *The factor $\epsilon(x)$ determined by equation (5.1.5) is in fact always equal to 1.*

Sketch of proof. $\epsilon(x)$ is determined by the necessity that

$$\bar{n}_1(x) = [w, 1][\varphi \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}, 1][w^{-1}, 1] = [\varphi \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \epsilon'(x)]$$

equal

$$[\varphi\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix},1][\varphi\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix},\epsilon(x)][\varphi\begin{pmatrix}0 & -1\\ 1 & x^{-1}\end{pmatrix},1].$$

This amounts to checking that

$$\beta(\varphi\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix},\varphi\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix})\cdot\beta(\varphi\begin{pmatrix}x^{-1} & 1\\ 0 & x\end{pmatrix},\varphi\begin{pmatrix}0 & -1\\ 1 & x^{-1}\end{pmatrix})\cdot\epsilon(x)$$
$$=\epsilon'(x)=\beta(w,\varphi\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix})\cdot\beta(w\varphi\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix},w^{-1}).$$

A tedious, but straightforward calculation leads to the result that $\epsilon(x)=\epsilon'(x)=1$. It will be omitted to save space. □

The hard part of the check of (5.1.6) comes in showing that $n_1^{-1}\bar{n}_2n_1\in N^+(S)\cdot\overline{N}(S)$, so we take care of the easier computation first.

Lemma 5.1.4. *Suppose that* $p\in N^+(S)$ *and* $\bar{n}\in\overline{N}(S)$ *. Then* $a_1^{-1}p\bar{n}a_1\in N^+(S)\cdot\overline{N}(S)$.

Proof. For either type of root α_0 , we have $a_1\in\widetilde{T}\subset\widetilde{B}=\widetilde{T}\cdot N_B$. We have already claimed (and it is easy to show) that $\widetilde{T}$ normalizes N_B . So it suffices to prove that $a_1^{-1}\bar{n}a_1\in\overline{N}(S)$. Writing $a_1=[m(t),1]$ for $t\in GL(n,k_v)$ diagonal, we have $a_1^{-1}=[m(t^{-1}),(t,t)_v]$. Now $a_1^{-1}\bar{n}a_1=a_1^{-1}wnw^{-1}a_1$ with $n\in N(S)\subset N_B$, and

$$a_1^{-1}w=[m(t^{-1}),(t,t)_v][w,1]=[wm(t),\beta(m(t^{-1}),w)(t,t)_v]$$
$$=[w,1][m(t),(t,t)_v].$$

Since $[1,(t,t)_v]\in Z_{\widetilde{G}_v}$, we see that

$$a_1^{-1}\bar{n}a_1=wa_1na_1^{-1}w^{-1}=wn'w^{-1}$$

for $n' \in N(S)$, so that $a_1^{-1}\bar{n}a_1 \in \overline{N}(S)$, as desired. $\square$

In the following exposition, if $\alpha_0 = 2x_r$, we will call this case I, and if $\alpha_0 = x_r + x_s$ with $r < s$, then case II. For ease of notation, take $x^{-1} \notin \mathcal{O}_v$ and consider that

$$n_1(x^{-1}) = [\varphi_{\alpha_0}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1] = [\begin{pmatrix} 1 & xE \\ 0 & 1 \end{pmatrix}, 1],$$

where we let $E = E_{\alpha_0}$ be either E_{rr} in case I or $E_{rs} + E_{sr}$ in case II. E_{ij} will always stand for the $n \times n$ matrix with 1 in the $(i,j)^{\text{th}}$ position, and 0's elsewhere.

For the variable of integration $\bar{n}_2 \in \overline{N}(S)$, write

$$\bar{n}_2 = [\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \epsilon_2(B)]$$

where $B = {}^tB \in M(n, k_v)$ is non-zero (generically) only at the places represented by roots $\alpha \in \Sigma_S^-$ (see Lemma 5.1.2 and diagram (5.1.4)) and where

$$\epsilon_2(B) = \beta(w, n(-B))\beta(wn(-B), w^{-1}) = \lambda(n'(B))\tilde{c}(wn(-B), w^{-1}).$$

As before, we have taken

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad n'(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \quad \text{for } b = {}^tb \in M(n)\,.$$

On the G_v-level,

$$\pi(n_1^{-1}\bar{n}_2 n_1) = \begin{pmatrix} 1 & -xE \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} 1 & xE \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - xEB & -x^2EBE \\ B & 1 + xBE \end{pmatrix}.$$

We wish to write this as $p \cdot \pi(\bar{n}_2')$ for $p \in N^+(S)$ and $\bar{n}_2' \in \overline{N}(S)$, which we know is possible by Langlands' work. To see what is needed, let $\pi(\bar{n}_2') = n'(\overline{B})$ and consider that

$$p = \begin{pmatrix} 1 - xEB & -x^2EBE \\ B & 1 + xBE \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\overline{B} & 1 \end{pmatrix} \in N_B$$

if and only if $B = (1 + xBE)\overline{B}$. Taking the transpose and rearranging, this amounts to requiring that $\overline{B} = B(1 + xEB)^{-1}$. We need the following lemma:

LEMMA 5.1.5. *With notation as above, EB has zeros on and below the diagonal. It satisfies*

$$(EB)^2 = 0 \quad \textit{in case I, and}$$

$$(EB)^3 = 0 \quad \textit{in case II.}$$

Proof. In case I, $E = E_{rr}$, and so $(EB)_{ij} = 0$ unless $i = r$. But also $(EB)_{rj} = b_{rj} = b_{jr} = 0$ if $r \geq j$ by Lemma 5.1.2. This shows the first claim. Next,

$$(EBE)_{ij} = \sum_{k,l} e_{ik} b_{kl} e_{lj}$$

is clearly zero for $(i,j) \neq (r,r)$. But $(EBE)_{rr} = b_{rr} = 0$ by reference to diagram (5.1.4). Hence $(EB)^2 = 0$.

In case II, $E = E_{rs} + E_{sr}$, and we see easily that $(EB)_{rj} = b_{sj} = 0$ for $r \geq j$ and $(EB)_{sj} = b_{rj} = 0$ for $s \geq j$ by Lemma 5.1.2 again. The other entries are zero also, showing that $(EB)_{ij} = 0$ for $i \geq j$. Now $(EBE)_{ij} = 0$

unless $i, j \in \{r, s\}$. We then check using diagram (5.1.4) that

$$(EBE)_{rs} = \sum_{k,l} e_{rk} b_{kl} e_{ls} = b_{sr} = 0,$$
$$(EBE)_{ss} = b_{rr} = 0, \text{ but}$$
$$(EBE)_{rr} = b_{ss} \neq 0 \text{ in general.}$$

So $EBE = b_{ss} E_{rr}$ in this case. But also

$$(BEB)_{rj} = \sum_{k,l} b_{rk} e_{kl} b_{lj} = b_{rs} b_{rj} + b_{rr} b_{sj} = 0$$

for all j , and so we see that $(EBE)(BEB) = (EB)^3 = 0$. □

Now, since $(1+N)^{-1} = 1 - N + N^2 - N^3 + - \ldots$ for N nilpotent, the lemma and the discussion preceding it imply that if we set

$$\overline{B} = \begin{cases} B - xBEB & \text{in case I} \\ B - xBEB + x^2 BEBEB & \text{in case II} \end{cases}$$

then

$$p = \begin{pmatrix} 1 - xEB + x^2(EB)^2 & -x^2 EBE \\ 0 & 1 + xBE \end{pmatrix}$$

lies in $N^+(S) \subset N_B$, and satisfies $\pi(n_1^{-1} \bar{n}_2 n_1) = p \cdot \pi(\bar{n}_2') = p \cdot n'(\overline{B})$. It is an easy check (and will be seen in the work to come) that $\overline{B}$ has zero entries wherever B does, or in other words, that $\pi(\bar{n}_2') \in \pi(\overline{N}(S))$.

Finally, we must show that the cocycle computation works out, which amounts to checking that

(5.1.8) $$[n_1^{-1}, 1][n'(B), \epsilon_2(B)][n_1, 1] = [p, 1][n'(\overline{B}), \epsilon_2(\overline{B})] \in \widetilde{G}_v$$

with $\epsilon_2(B) = \lambda(n'(B))\tilde{c}(wn(-B), w^{-1})$, and similarly for $\epsilon_2(\overline{B})$. Equation (5.1.8) holds if and only if

$$\begin{aligned}
&\beta(n_1^{-1}, n'(B))\beta(n_1^{-1}n'(B), n_1)\epsilon_2(B) = \beta(p, n'(\overline{B}))\epsilon_2(\overline{B}) \\
&\Longleftrightarrow \frac{\lambda(n_1^{-1})\lambda(n'(B))}{\lambda(n_1^{-1}n'(B))}\frac{\lambda(n_1^{-1}n'(B))\lambda(n_1)}{\lambda(n_1^{-1}n'(B)n_1)}\lambda(n'(B))\tilde{c}(wn(-B), w^{-1}) \\
&\quad = \frac{\lambda(p)\lambda(n'(\overline{B}))}{\lambda(pn'(\overline{B}))}\lambda(n'(\overline{B}))\tilde{c}(wn(-\overline{B}), w^{-1}) \\
&\Longleftrightarrow \tilde{c}(wn(-B), w^{-1}) = \tilde{c}(wn(-\overline{B}), w^{-1}) \\
&\Longleftrightarrow c(wn(-B), w^{-1}) = c(wn(-\overline{B}), w^{-1}).
\end{aligned}$$

Since B is a variable of integration representing $\bar{n}_2 \in \overline{N}(S)$, we may change B for $-B$. The work we have done so far gives us:

LEMMA 5.1.6. *With notation as above, we may write*

$$a_1^{-1}n_1^{-1}\bar{n}_2 n_1 a_1 = p \cdot \bar{n}_2'$$

with $p \in N^+(S)$ *and* $\bar{n}_2' \in \overline{N}(S)$ *if and only if, for all* B *representing* $\bar{n}_2 \in \overline{N}(S)$ *and all* $x^{-1} \notin \mathcal{O}_v$ *, we have*

$$c(wn(B), w^{-1}) = c(wn(\overline{B}), w^{-1})$$

where

$$\overline{B} = \begin{cases} B + xBEB & \text{in case I} \\ B + xBEB + x^2BEBEB & \text{in case II.} \end{cases}$$

Notice that if B is non-degenerate, then $c(wn(B), w^{-1})$ is the Weil index of $x \mapsto \psi_v(\frac{1}{2} < x, xB >)$, $x \in X(k_v)$. So the Lemma essentially amounts to proving that B and $\overline{B}$ have the same Weil indices. We shall prove this by showing that B and $\overline{B}$ are equivalent, under the $GL(n)$ action $B \mapsto AB\,{}^tA$, to matrices having the same Weil indices.

First of all, if $n = 1$, then we already have, a one-dimensional integral, and the Lemma is not needed. Next, suppose that $n > 1$, and that we have already removed all the roots $-2x_1$, $-(x_1+x_2), \ldots, -(x_1+x_n)$ in previous steps. Then B has

$$b_{11} = b_{12} = b_{21} = \cdots = b_{1n} = b_{n1} = 0,$$

and the non-zero portion of B sits in a smaller dimensional block. As we must be removing $\alpha_0 = -(x_i + x_j)$ with $1 < i \leq j$, we are completely reduced to a lower dimensional situation. So by induction, we have only two remaining cases to consider: first of all, we may suppose that $-\alpha_0 = -(x_1 + x_n)$ is the root currently being removed. Then as above, we know that $b_{11} = \cdots = b_{1n} = 0$. Hence

$$(BEB)_{ij} = \sum_{k,l} b_{ik} e_{kl} b_{lj} = b_{i1} b_{nj} + b_{in} b_{1j} = 0,$$

since $b_{i1} = b_{1i} = 0 = b_{1j}$, and so $B = \overline{B}$ in this case.

Otherwise, we may assume that $-(x_1+x_n) \in \Sigma_S^-$, so that except for a set of measure zero (in the space of integration of B), $b_{1n} \neq 0 \neq b_{nn}$. Of course, since we are removing *some* root $-\alpha_0$, we know that $b_{11} = 0$, as $-2x_1$ is the first root to go. We will describe a transformation taking $B \mapsto \mathrm{diag}(x, y, z)$, where $x, z \in k_v^\times$ and y is an $(n-2) \times (n-2)$ matrix (absent for $n = 2$).

Let A be the unipotent $n \times n$ matrix with j^{th} row

$$\begin{array}{ccccccc} & & & j & & & \\ \Big(0 & \dots & 0 & 1 & 0 & \dots & -b_{jn}b_{nn}^{-1} \Big) \end{array}$$

for $1 \leq j \leq n-1$, and with n^{th} row $(0, \dots, 0, 1)$. Let $C = AB\,{}^tA$. A simple computation leads to:

LEMMA 5.1.7.

(1) $c_{ij} = b_{ij} - \frac{b_{in}b_{jn}}{b_{nn}}$ *for* $1 \leq i, j \leq n-1$, *and specifically* $c_{11} = \frac{-(b_{1n})^2}{b_{nn}}$.

(2) $c_{nj} = 0$ *for* $1 \leq j \leq n-1$.

(3) $c_{nn} = b_{nn}$.

Next, let A' have 1^{st} row $(1, 0, \dots, 0)$ and j^{th} row

$$\begin{array}{cccccccc} & & & & j & & & \\ \Big(-c_{1j}c_{11}^{-1} & 0 & \dots & 0 & 1 & 0 & \dots & 0 \Big) \end{array}$$

for $2 \leq j \leq n-1$, and n^{th} row $(0, \dots, 0, 1)$ (here we use $b_{1n} \neq 0$). Set $D = A'C\,{}^tA'$.

LEMMA 5.1.8.

(1) $d_{11} = \frac{-(b_{1n})^2}{b_{nn}}$.

(2) $d_{1j} = 0$ *for* $2 \leq j \leq n-1$.

(3) $d_{in} = 0$ *for* $1 \leq i \leq n-1$.

(4) $d_{nn} = b_{nn}$.

(5) $d_{ij} = b_{ij} + b_{nn}\left(\frac{b_{1i}b_{1j}}{b_{1n}^2}\right) - b_{jn}\left(\frac{b_{1i}}{b_{1n}}\right) - b_{in}\left(\frac{b_{1j}}{b_{1n}}\right)$ *for* $2 \leq i, j \leq n-1$.

Proof. $d_{ij} = \sum_{kl} a_{ik}c_{kl}a_{jl}$ for all i, j (omitting the $'$ on a). So $d_{11} = \sum a_{1k}c_{kl}a_{1l} = c_{11} = \frac{-(b_{1n})^2}{b_{nn}} \Longrightarrow (1)$. Next, $2 \leq j \leq n-1 \Longrightarrow d_{1j} = \sum a_{1k}c_{kl}a_{jl} = \sum c_{1l}a_{jl} = c_{11}(-c_{1j}c_{11}^{-1}) + c_{1j}(1) = 0 \Longrightarrow (2)$. If $1 \leq i \leq n-1$, then $d_{in} = \sum a_{ik}c_{kl}a_{nl} = \sum a_{ik}c_{kn} = a_{in}c_{nn}$ (as $c_{kn} = 0$ for $1 \leq i \leq n-1$) and so this is $0 \Longrightarrow (3)$. $d_{nn} = \sum a_{nk}c_{kl}a_{nl} = a_{nn}c_{nn}a_{nn} = c_{nn} = b_{nn} \Longrightarrow (4)$. Finally, $2 \leq i,j \leq n-1 \Longrightarrow d_{ij} = \sum_{kl} a_{ik}c_{kl}a_{jl} = \sum_l(-c_{1i}c_{11}^{-1})c_{1l}a_{jl} + \sum_l(1)c_{il}a_{jl} = -c_{1i}c_{11}^{-1}(c_{11}a_{j1} + c_{1j}a_{jj}) + (c_{i1}a_{j1} + c_{ij}a_{jj}) = -c_{1i}c_{11}^{-1}(c_{11}(-c_{1j}c_{11}^{-1}) + c_{1j}) + (c_{i1}(-c_{1j}c_{11}^{-1}) + c_{ij}) = c_{ij} - \frac{c_{1i}c_{1j}}{c_{11}} = b_{ij} - \left(\frac{b_{in}b_{jn}}{b_{nn}}\right) - \left(\frac{-b_{nn}}{b_{1n}^2}\right)\left(b_{1i} - \frac{b_{1n}b_{in}}{b_{nn}}\right)\left(b_{1j} - \frac{b_{1n}b_{jn}}{b_{nn}}\right) = b_{ij} - \left(\frac{b_{in}b_{jn}}{b_{nn}}\right) + b_{nn}\left(\frac{b_{1i}b_{1j}}{b_{1n}^2}\right) - b_{jn}\left(\frac{b_{1i}}{b_{1n}}\right) - b_{in}\left(\frac{b_{1j}}{b_{1n}}\right) + \left(\frac{b_{in}b_{jn}}{b_{nn}}\right) \Longrightarrow (5)$. $\square$

Now we may relabel this last matrix, letting D denote the $(n-2)\times(n-2)$ matrix with entries labelled d_{ij}, $2 \leq i,j \leq n-1$. Then we see that

$$(A'A)B^{t}(A'A) = \operatorname{diag}\left(\frac{-(b_{1n}^2)}{b_{nn}}, D, b_{nn}\right).$$

Our goal is to show that when we similarly transform $\overline{B}$ to obtain $\overline{D}$, we have $D = \overline{D}$ (the other two entries will not matter). Unfortunately, this only holds for $\alpha_0 = x_1 + x_j$, $1 \leq j \leq n-1$.

If $\alpha_0 = x_r + x_s$, $1 < r \leq s$, then we must first apply a permutation matrix to B to switch the 1^{st} and r^{th} row and column. Let P be the $n \times n$ permutation matrix representing the transposition $(1,r) \in S_n$, so that

$$P = E_{1r} + E_{r1} + \sum_{\substack{i=2 \\ i \neq r}}^{n} E_{ii}.$$

Note that diagram (5.1.4) shows that the upper left-hand $r \times r$ block of B consists of zeros. Setting $B' = PB\,{}^tP$, we then see that

(1) $b'_{1j} = b_{rj}$ for $1 \leq j \leq n$,

(2) $b'_{rj} = b_{1j}$ for $1 \leq j \leq n$,

(3) $b'_{ij} = b_{ij}$ in the case that either $1 < i < r$ or $r < i, j \leq n$.

Since B' is symmetric, this completely describes it. So applying P first to B and then transforming as above (with A, A' depending on B' now), we see that

$$B \mapsto \operatorname{diag}(\frac{-(b'_{1n})^2}{b'_{nn}}, D', b'_{nn})$$

where D' has entries

$$d'_{ij} = b'_{ij} + b'_{nn}\left(\frac{b'_{1i}b'_{1j}}{(b'_{1n})^2}\right) - b'_{jn}\left(\frac{b'_{1i}}{b'_{1n}}\right) - b'_{in}\left(\frac{b'_{1j}}{b'_{1n}}\right)$$

for $2 \leq i, j \leq n-1$. To simplify this we consider three ranges.

LEMMA 5.1.9.

(1) *If* $2 \leq i < r$, *then* $d'_{ij} = b_{ij} - b_{in}\left(\frac{b_{rj}}{b_{rn}}\right)$.

(2) *If* $i = r$, *then* $d'_{rj} = b_{1j} - b_{1n}\left(\frac{b_{rj}}{b_{rn}}\right)$.

(3) *If* $r < i, j \leq n-1$, *then*

$$d'_{ij} = b_{ij} + b_{nn}\left(\frac{b_{ri}b_{rj}}{b_{rn}^2}\right) - b_{jn}\left(\frac{b_{ri}}{b_{rn}}\right) - b_{in}\left(\frac{b_{rj}}{b_{rn}}\right).$$

Proof. Let $2 \le i < r$. Then $b'_{1i} = b_{ri} = 0$. Hence

$$d'_{ij} = b'_{ij} - b'_{in}\left(\frac{b'_{1j}}{b'_{1n}}\right) = b_{ij} - b_{in}\left(\frac{b_{rj}}{b_{rn}}\right) \Longrightarrow (1).$$

Next, $d'_{rj} = b'_{rj} - b'_{rn}\left(\frac{b'_{1j}}{b'_{1n}}\right)$ since $b'_{1r} = b_{rr} = 0$. So $d'_{rj} = b_{1j} - b_{1n}\left(\frac{b_{rj}}{b_{rn}}\right) \Longrightarrow$ (2). Finally, if $r < i, j \le n-1$, then b'_{ij}, b'_{nn}, b'_{jn} , and b'_{in} are all equal to the unprimed quantities, and the remaining change follow easily. □

PROPOSITION 5.1.10. *Transforming* B *to* D' , *and* $\overline{B}$ *(as given in Lemma 5.2.5) to* $\overline{D}'$ *as above, we see that* $D' = \overline{D}'$.

COROLLARY 5.1.11. $c(wn(B), w^{-1}) = c(wn(\overline{B}), w^{-1})$.

Proof of Prop. 5.1.10. As above, let D' correspond to B , and $\overline{D}'$ to $\overline{B}$. We separate the proof for the two different types of root α_0 .

Case I: $\alpha_0 = 2x_r$. Here we have $\overline{B} = B + xBEB$, where B and x are thought of as variables of integration, and we assume that $b_{rn} \neq 0 \neq b_{nn}$ without loss of generality (this is true on a set of measure zero in B space). Since $E = E_{rr}$ here, it is easily computed that

$$\bar{b}_{ij} = b_{ij} + x b_{ri} b_{rj}.$$

So among other things, note that

$$1 \le i \le r \text{ or } 1 \le j \le r \Longrightarrow \bar{b}_{ij} = b_{ij}$$

$$(*) \qquad 1 \le i, j \le r \Longrightarrow \bar{b}_{ij} = b_{ij} = 0.$$

Also $\bar{b}_{rn} = b_{rn}$ and $\bar{b}_{nn} = b_{nn} + xb_{rn}^2$ are non-zero for almost all B, given any fixed x. So $\overline{B}$ meets all the criteria used in transforming B as above. Also note that all the quantities in parentheses in Lemma 5.1.9 are the same for either B or $\overline{B}$ by $(*)$. We show that $\bar{d}'_{ij} = d'_{ij}$.

Let $2 \leq i < r$. Then

$$\bar{d}'_{ij} = \bar{b}_{ij} - \bar{b}_{in}\left(\frac{b_{rj}}{b_{rn}}\right) = b_{ij} - b_{in}\left(\frac{b_{rj}}{b_{rn}}\right) = d'_{ij}$$

by $(*)$ and Lemma 5.1.9. If $i = r$, then $\bar{d}'_{rj} = \bar{b}_{1j} - \bar{b}_{1n}\left(\frac{b_{rj}}{b_{rn}}\right) = d'_{rj}$ again. Now supposing that $r < i, j \leq n-1$, we have a more difficult check. In this case

$$\begin{aligned}
\bar{d}'_{ij} =& \bar{b}_{ij} + \bar{b}_{nn}\left(\frac{b_{ri}b_{rj}}{b_{rn}^2}\right) - \bar{b}_{jn}\left(\frac{b_{ri}}{b_{rn}}\right) - \bar{b}_{in}\left(\frac{b_{rj}}{b_{rn}}\right) \\
=& (b_{ij} + xb_{ri}b_{rj}) + (b_{nn} + xb_{rn}^2)\left(\frac{b_{ri}b_{rj}}{b_{rn}^2}\right) \\
& -(b_{jn} + xb_{rj}b_{rn})\left(\frac{b_{ri}}{b_{rn}}\right) - (b_{in} + xb_{ri}b_{rn})\left(\frac{b_{rj}}{b_{rn}}\right) \\
=& d'_{ij} + x(b_{ri}b_{rj} + b_{ri}b_{rj} - b_{rj}b_{ri} - b_{ri}b_{rj}) = d'_{ij}.
\end{aligned}$$

Case II: $\alpha_0 = x_r + x_s$ $(r < s)$. In this case, we take $\overline{B} = B + xBEB + x^2BEBEB$. Now since $EBE = b_{ss}E_{rr}$ (see Lemma 5.1.5), we have $BEBEB = b_{ss}BE_{rr}B$, and the $(i,j)^{\text{th}}$ entry is then equal to $b_{ss}b_{ir}b_{rj}$. An easy computation then gives

$$\bar{b}_{ij} = b_{ij} + x(b_{ir}b_{sj} + b_{is}b_{rj}) + x^2 b_{ss}b_{ir}b_{rj}.$$

The statements in $(*)$ above still hold in this case, so that the quantities in parentheses in Lemma 5.1.9 are again the same for either B or $\overline{B}$.

If $2 \leq i \leq r$, then $\bar{d}'_{ij} = d'_{ij}$ is immediate by $(*)$, as all quantities from Lemma 5.1.9 involve only b_{kl} with $1 \leq k \leq r$, so that $\bar{b}_{kl} = b_{kl}$. If $r < i,j \leq n-1$, then

$$\begin{aligned}
\bar{d}'_{ij} =& \bar{b}_{ij} + \bar{b}_{nn}\left(\frac{b_{ri}b_{rj}}{b_{rn}^2}\right) - \bar{b}_{jn}\left(\frac{b_{ri}}{b_{rn}}\right) - \bar{b}_{in}\left(\frac{b_{rj}}{b_{rn}}\right) \\
=& b_{ij} + x(b_{ir}b_{sj} + b_{is}b_{rj}) + x^2 b_{ss}b_{ir}b_{rj} \\
& + (b_{nn} + x(2b_{rn}b_{sn}) + x^2 b_{ss}b_{rn}^2)\left(\frac{b_{ri}b_{rj}}{b_{rn}^2}\right) \\
& - (b_{jn} + x(b_{jr}b_{sn} + b_{js}b_{rn}) + x^2 b_{ss}b_{jr}b_{rn})\left(\frac{b_{ri}}{b_{rn}}\right) \\
& - (b_{in} + x(b_{ir}b_{sn} + b_{is}b_{rn}) + x^2 b_{ss}b_{ir}b_{rn})\left(\frac{b_{rj}}{b_{rn}}\right) \\
=& d'_{ij} \\
& + x\left[b_{ir}b_{sj} + b_{is}b_{rj} + 2\left(\frac{b_{sn}b_{ri}b_{rj}}{b_{rn}}\right) - \left(\frac{b_{jr}b_{sn}b_{ri}}{b_{rn}}\right)\right. \\
& \qquad \left. - b_{js}b_{ri} - \left(\frac{b_{ir}b_{sn}b_{rj}}{b_{rn}}\right) - b_{is}b_{rj}\right] \\
& + x^2 b_{ss}(b_{ir}b_{rj} + b_{ri}b_{rj} - b_{jr}b_{ri} - b_{ir}b_{rj}) \\
=& d'_{ij}.
\end{aligned}$$

□

Proof of Cor. 5.1.11. For almost all B, we may write $AB\,{}^tA = T$, where $T = \mathrm{diag}(\frac{-a^2}{b}, D', b)$ for some $a, b \in k_v^\times$. But then

$$c(wn(B), w^{-1}) = c(wm(A^{-1})n(T)m(A), w^{-1})$$

$$= c(m({}^tA)wn(T)m(A), m(A^{-1})m(A)w^{-1}) = c(wn(T), w^{-1})$$

using Theorem 1.4.5. Writing $w = w_j \in Sp(j)$ for any rank j, and using the nice properties of c with respect to direct sums from Theorem 1.4.5 again, the above equals

$$
\begin{aligned}
& c(wn(\tfrac{-a^2}{b}), w^{-1})\, c(wn(D'), w^{-1})\, c(wn(b), w^{-1}) \\
&= \gamma_v(\tfrac{-a^2}{2b}\psi_v)\, \gamma_v(\tfrac{b}{2}\psi_v)\, c(wn(D'), w^{-1}) \\
&= \gamma_v(-b, \tfrac{1}{2}\psi_v)\, \gamma_v(b, \tfrac{1}{2}\psi_v)\, c(wn(D'), w^{-1}) \\
&= (-b, b)_v\, \gamma_v(-1, \tfrac{1}{2}\psi_v)\, c(wn(D'), w^{-1}) \\
&= c(wn(D'), w^{-1}) = c(wn(\overline{D}'), w^{-1}) = c(wn(\overline{B}), w^{-1}).
\end{aligned}
$$

□

§5.2 Simplification. Given the above, by Lemma 5.1.6 we see that equation (5.1.7) is valid, and we set about the task of simplifying

$$Z_{\alpha_0}(s) \stackrel{def}{=} \int_{k_v} \Phi^o(a_1(x), s) \prod_{\substack{\lambda_1(\alpha)<0 \\ \alpha>0}} |a_1^{\Lambda(-\alpha)}|\, dx.$$

As in Langlands, this reduces to

$$Z_{\alpha_0}(s) = \int_{k_v} \chi_v(a_1(x))|a_1^{\Lambda(\mu+\frac{\alpha_0}{2})}|\, dx$$

with $\mu = \sum_{i=1}^n (s - \rho_n + i)x_i$. Simplifying the expression above yields:

LEMMA 5.2.1. *Let* $q_v = |\mathcal{O}_v/\mathcal{P}_v|$ *and define* $\zeta_v(s) \stackrel{def}{=} (1 - q_v^{-s})^{-1}$ *for* $s \in \mathbf{C}$ *and* $v \notin S_k$ *. Then*

$$Z_{\alpha_0}(s) = \frac{\zeta_v(2(s-\rho_n)+i+j)}{\zeta_v(2(s-\rho_n)+i+j+1)}$$

where $\alpha_0 = x_i + x_j$ *for* $1 \le i \le j \le n$.

Proof. Although the answer looks uniform with respect to the two types of root α_0 , the computations are different. In case I, with $\alpha_0 = 2x_i$, we have

$$a_1(x) = \begin{cases} [\varphi_{\alpha_0}\begin{pmatrix} x^{-1} & \\ & x \end{pmatrix}, 1] = [m(t), 1], & \text{for } x \notin \mathcal{O}_v \\ 1 & \text{for } x \in \mathcal{O}_v \end{cases}$$

where $t = t(x) = \mathrm{diag}(1, \ldots, x^{-1}, \ldots, 1) \in GL(n, k_v)$ (the x^{-1} occurs in the i^{th} place). So

$$Z_{\alpha_0}(s) = \int_{\mathcal{O}_v} dx + \int_{k_v \sim \mathcal{O}_v} \chi_v([m(t), 1])|m(t)^{\Lambda(\mu + x_i)}|\, dx.$$

Since we are using the measure dx on k_v from Tate, and $v \notin S_k$, we have $\int_{\mathcal{O}_v} dx = 1$. Note also that $\chi_v : \widetilde{M}_n(k_v) \to \mathsf{T}$ can be viewed as a character of a two-fold covering $\tilde{k}_v^\times$ of $k_v^\times$ defined via $\tilde{k}_v^\times = k_v^\times \times \mu_2$ with multiplication given by $[a, \epsilon][b, \tau] = [ab, \epsilon\tau(a, b)_v]$. For any finite place $v \nmid 2$, $\mathcal{U}_v \times 1$ is a subgroup by Lemma 1.4.4, and χ_v is unramified if trivial on $\mathcal{U}_v \times 1$. Note also that $\chi_v(ac^2, 1) = \chi_v(a, 1)$ for all $a, c \in k_v^\times$. For $x \notin \mathcal{O}_v$ then,

$$\chi_v([m(t), 1])\, |m(t)^{\Lambda(\mu + x_i)}| = \chi_v(x^{-1}, 1)\, |m(t)^{x_i}| \prod_{j=1}^{n} |m(t)^{x_j}|^{s - \rho_n + j}$$
$$= \chi_v(x, 1)\, |x^{-1}|\, |x^{-1}|^{s - \rho_n + i} = \chi_v(x, 1)\, |x|^{-(s - \rho_n + i + 1)}$$

So

$$\int_{k_v \sim \mathcal{O}_v} \chi_v(x, 1)|x|^{-(s - \rho_n + i + 1)}\, dx = \sum_{r=1}^{\infty} \int_{\pi_v^{-r}\mathcal{U}_v} \chi_v(x, 1)|x|^{-(s - \rho_n + i)} \frac{dx}{|x|}$$
$$= \sum_{r=1}^{\infty} |\pi_v|^{r(s - \rho_n + i)} \int_{\mathcal{U}_v} \chi_v(\pi_v^{-r} u, 1)\, du.$$

Now, if $r = 2j + 1$ then

$$\chi_v(\pi_v^{-r}u, 1) = \chi_v(\pi_v u, 1) = \chi_v(\pi_v, 1)\chi_v(u, (\pi_v, u)_v) = \chi_v(\pi_v, 1)(\pi_v, u)_v,$$

since we are at a place at which χ_v is unramified. Hence the summand above becomes

$$\chi_v(\pi_v, 1)|\pi_v|^{r(s-\rho_n+i)} \int_{\mathcal{U}_v} (\pi_v, u)_v \, du = 0.$$

because $v \notin S_k \Longrightarrow (\,,) \equiv 1$ on $\mathcal{U}_v \times \mathcal{U}_v$, and so $u \mapsto (\pi_v, u)_v$ is a non-trivial character of $\mathcal{U}_v$ by the non-degeneracy of the Hilbert symbol. If, on the other hand, $r = 2j$, then $\chi_v(\pi_v^{-r}u, 1) = \chi_v(u, 1) = 1$. So we are left with

$$\begin{aligned} Z_{\alpha_0}(s) =& 1 + \left(\int_{\mathcal{U}_v} du\right) \sum_{j=1}^{\infty} q_v^{-2j(s-\rho_n+i)} \\ =& 1 + (1 - q_v^{-1}) \frac{q_v^{-2(s-\rho_n+i)}}{1 - q_v^{-2(s-\rho_n+i)}} \end{aligned}$$

(du here is just Tate's additive Haar measure restriced to $\mathcal{U}_v$: see [T])

$$\begin{aligned} =& \frac{1 - q_v^{-2(s-\rho_n+i)} + q_v^{-2(s-\rho_n+i)} - q_v^{-2(s-\rho_n+i)-1}}{1 - q_v^{-2(s-\rho_n+i)}} \\ =& \frac{\zeta_v(2(s-\rho_n) + 2i)}{\zeta_v(2(s-\rho_n) + 2i + 1)} \end{aligned}$$

as desired.

In case II, let $\alpha_0 = x_i + x_j$ with $i < j$. The situation is similar, except that we must take

$$t(x) = \mathrm{diag}(1, \ldots, x^{-1}, \ldots, x^{-1}, \ldots, 1) \in GL(n, k_v)$$

(1's in all places except the i^{th} and j^{th}). Now we note that $\chi_v([m(t),1]) = \chi_v(x^{-2},1) = 1$ for all $x \in k_v \sim \mathcal{O}_v$. Hence, for such x , we have

$$\begin{aligned}
|m(t)^{\Lambda(\mu+\frac{\alpha_0}{2})}| &= |m(t)^{x_i}|^{\frac{1}{2}}|m(t)^{x_j}|^{\frac{1}{2}}\prod_{r=1}^{n}|m(t)^{x_r}|^{s-\rho_n+r} \\
&= |x^{-1}||x^{-1}|^{s-\rho_n+i}|x^{-1}|^{s-\rho_n+j} = |x^{-1}|^{2(s-\rho_n)+i+j+1},
\end{aligned}$$

and so

$$\begin{aligned}
Z_{\alpha_0}(s) =& 1 + \int_{k_v\sim\mathcal{O}_v} |x^{-1}|^{2(s-\rho_n)+i+j+1}\,dx \\
=& 1 + \sum_{l=1}^{\infty}\int_{\pi_v^{-l}\mathcal{U}_v} |x|^{-(2(s-\rho_n)+i+j)}\,\frac{dx}{|x|} \\
=& 1 + \left(\int_{\mathcal{U}_v} du\right)\sum_{l=1}^{\infty} q_v^{-l(2(s-\rho_n)+i+j)} \\
=& \frac{\zeta_v(2(s-\rho_n)+i+j)}{\zeta_v(2(s-\rho_n)+i+j+1)}
\end{aligned}$$

carrying out the computation as before. □

Finally, we finish the proof of the main result of this chapter:

Proof of Theorem 5.1.1. Given the discussion throughout this section, all that remains to be proven is that

$$Z(s) \stackrel{def}{=} Z_v(n,s) = \prod_{-\alpha_0\in\Sigma^-_{R_*}} Z_{\alpha_0}(s) = \frac{a_v(n,s)}{b_v(n,s)}.$$

Grouping together terms from the two types of roots, we have

$$Z_v(n,s) = \prod_{1\le i\le j\le n} \frac{\zeta_v(2(s-\rho_n)+i+j)}{\zeta_v(2(s-\rho_n)+i+j+1)}.$$

For $n = 1$, this yields $Z_v(1,s) = \frac{\zeta_v(2s)}{\zeta_v(2s+1)} = \frac{a_v(1,s)}{b_v(1,s)}$. Now, since $\rho_n = \rho_{n-1} + \frac{1}{2}$ for $n \geq 2$, it is easily seen that

$$\begin{aligned} Z_v(n,s) =& Z_v(n-1, s-\tfrac{1}{2}) \prod_{i=1}^{n} \frac{\zeta_v(2(s-\rho_n)+i+n)}{\zeta_v(2(s-\rho_n)+i+n+1)} \\ =& \frac{\zeta_v(2s)}{\zeta_v(2s+n)} Z_v(n-1, s-\tfrac{1}{2}). \end{aligned}$$

So it suffices to show that $\frac{a_v(n,s)}{b_v(n,s)}$ satisfies the same recursion relation. Now

$$\frac{a_v(n-1, s-\frac{1}{2})}{b_v(n-1, s-\frac{1}{2})} = \prod_{k=1}^{[\frac{n}{2}]} \frac{\zeta_v(2s-(n+1)+2k)}{\zeta_v(2s+n-2k)}.$$

If n is odd, then $\left[\frac{n}{2}\right] = \frac{n-1}{2}$ and $\left[\frac{n+1}{2}\right] = \frac{n+1}{2}$, so that

$$\begin{aligned} \frac{\zeta_v(2s)}{\zeta_v(2s+n)} \frac{a_v(n-1, s-\frac{1}{2})}{b_v(n-1, s-\frac{1}{2})} &= \frac{\zeta_v(2s) \prod_{k=1}^{\frac{n-1}{2}} \zeta_v(2s-(n+1)+2k)}{\zeta_v(2s+n) \prod_{k=2}^{\frac{n+1}{2}} \zeta_v(2s+n+2-2k)} \\ = \prod_{k=1}^{[\frac{n+1}{2}]} \frac{\zeta_v(2s-(n+1)+2k)}{\zeta_v(2s+n+2-2k)} &= \frac{a_v(n,s)}{b_v(n,s)}. \end{aligned}$$

If n is even, then $\left[\frac{n}{2}\right] = \frac{n}{2}$ and $\left[\frac{n+1}{2}\right] = \frac{n}{2}$, so

$$\frac{\zeta_v(2s)}{\zeta_v(2s+n)} \frac{a_v(n-1, s-\frac{1}{2})}{b_v(n-1, s-\frac{1}{2})} = \frac{\prod_{k=1}^{\frac{n}{2}} \zeta_v(2s-(n+1)+2k)}{\frac{\zeta_v(2s+n)}{\zeta_v(2s)} \prod_{k=2}^{\frac{n+2}{2}} \zeta_v(2s+n+2-2k)} = \frac{a_v(n,s)}{b_v(n,s)}.$$

This completes the proof. □

§6.1 Global setup. Returning to the global situation, recall that for a factorizable section $\Phi = \otimes_v' \Phi_v \in I_{\widetilde{K}}$, we have

$$M(s)\Phi(g,s) = \bigotimes_v{}' M_v(s)\Phi_v(g_v, s)$$

for $p(\prod_v g_v) = g \in \widetilde{G}(\mathbb{A})$. Let $S \supset S_k$ be a finite set of places of k such that $\Phi_v = \Phi_v^o$ for $v \notin S$. With the results of the last chapter, we may then write

$$M(s)\Phi(s) = \left(\prod_{v \notin S} \frac{a_v(n,s)}{b_v(n,s)}\right) \cdot \left(\bigotimes_{v\in S} M_v(s)\Phi_v(s)\right) \otimes \Phi_S^o(-s)$$

where

$$\Phi_S^o = \bigotimes_{v \notin S} \Phi_v^o$$

as in [K-R1]. We use the following notation for the zeta function of the field k .

DEFINITION 6.1.1. *For any place* $v \in \Sigma_k$ *, define the local zeta function* $\zeta_v(s)$ *of the field* k *via*

$$\zeta_v(s) = \begin{cases} \frac{N(\delta_v)^{s/2}}{1-N(\mathcal{P}_v)^{-s}} & \text{if } v < \infty \\ \pi^{s/2}\Gamma(s/2) & \text{if } v = \text{real} \\ (2\pi)^{-s}\Gamma(s) & \text{if } v = \text{complex.} \end{cases}$$

Here we let $N(\mathfrak{a})$ *denote the number of elements in* $\mathcal{O}_v/\mathfrak{a}$ *for an ideal* $\mathfrak{a} \subset \mathcal{O}_v$ *, extended multiplicatively to fractional ideals. As before,* $\delta = \prod_v \delta_v$ *stands for* *the global different of* k *, formally the product of local differents. The global zeta function* $\zeta(s)$ *of* k *is defined via* $\zeta(s) \stackrel{\text{def}}{=} \prod_v \zeta_v(s)$ *for* $Re(s) > 1$ *.*

Thus normalized, the global zeta function $\zeta(s)$ has the "nice" functional equation $\zeta(s) = \zeta(1-s)$. Extend the local functions a_v and b_v defined as in Theorem 5.1.1 to global functions via $a(n,s) = \prod_v a_v(n,s)$, and similarly for $b(n,s)$. These functions then have well known poles and at least partially known zeros. This suggests that we write

$$M(s)\Phi(s) = \frac{a(n,s)}{b(n,s)}\left(\bigotimes_{v\in S}\frac{b_v(n,s)}{a_v(n,s)}M_v(s)\Phi_v(s)\right)\otimes\Phi_S^o(-s). \tag{6.1.1}$$

In this chapter, we will find the poles of the operator $\frac{b_v(n,s)}{a_v(n,s)}M_v(s)$ for $v < \infty$. In fact, the following theorem will be proven.

THEOREM 6.1.2. *Let* v *be a finite place of* k *. For any holomorphic section* $\Phi_v \in I_{\widetilde{K}_v}$ *,*

$$\frac{1}{a_v(n,s)}M_v(s)\Phi_v(s)$$

is also holomorphic. Moreover, for any given value s_o *, there exists a section* Φ_v *such that the expression above does not vanish at* s_o *.*

An analogue of this theorem is proven in [PS-R2], but in that situation the function $a_v(n,s)$ (and hence the operator $M_v(s)$) has poles which are more or less shifted by $\frac{1}{2}$ unit on the real axis, reflecting some inherent difference between the "half-integral weight" and "integral weight" situations (in reference to classical Siegel modular forms).

§**6.2 Reduction.** For the rest of the chapter, fix a place $v < \infty$. The proof begins with a reduction of the analytic properties of the intertwining operator to those of a certain zeta function similar to those studied by Igusa in [I1]. We begin with a result from [PS-R2]:

LEMMA 6.2.1 [PS-R2]. *Let* A *be the set of holomorphic sections* $\Phi \in I_{\widetilde{K}_v}$ *which have support*$(\Phi(s)) \subset \widetilde{P}_v w \widetilde{P}_v$ *for all* $s \in \mathbf{C}$. *Then the analytic properties of the family of functions*

$$\{M_v(s)\Phi(w,s) \mid \Phi \in A\}$$

coincide with the analytic properties of the family of sections

$$\left\{M_v(s)\Phi_v(s) \mid \Phi_v \in I_{\widetilde{K}_v},\ \textit{holomorphic}\right\}.$$

In other words, if the operator $M_v(s)$ *has a pole of a certain order, then this pole will occur in a function* $s \mapsto M_v(s)\Phi(w,s)$ *for some section* $\Phi \in A$.

The proof of this lemma given in [PS-R2] goes over almost unchanged, noting that $\widetilde{P}_v w \widetilde{P}_v = \widetilde{P}_v w N_v$ in $\widetilde{G}_v$, and that in the statement above as well as in the proof, $w = [w, 1] \in \widetilde{G}_v$.

This now allows the problem to be reduced to studying a concrete class of zeta functions, rather than the more difficult operator $M_v(s)$.

LEMMA 6.2.2. *All sections in* A *may be written as finite linear combinations with holomorphic coefficients of sections of the form*

$$\Phi(n_1 m w n(b), s) = |a(m)|^{s+\rho_n} \chi_v(m)\phi(b)$$

for $n_1, n(b) \in N_v$ *and* $m \in \widetilde{M}_v$ *, where* $\phi \in \mathcal{S}(Sym_n(k_v))$ *is a Schwartz-Bruhat function on* $Sym_n(k_v)$ *which is independent of* s *.*

Sketch of proof. First note that $\Phi \in A$ has support which may be written as

$$\text{supp}(\Phi(s)) = \widetilde{P} w C_s$$

where C_s is a compact open subset of N which may possibly depend on s . Next, by $\widetilde{K}$ -finiteness, it is possible to prove that there exist compact open subgroups $C \supset C'$ (independent of s) such that

(1) $\Phi(s)$ is C' -invariant on the right for all s , and

(2) $\text{supp}(\Phi(s)) \subset \widetilde{P} w C$ for all s .

Letting $A^C_{C'}$ be the set of all such sections, we see that $A = \cup_{C,C'} A^C_{C'}$. For each s , we also have a map

$$A \to \mathcal{S}(\text{Sym}_n(k_v))$$

$$\Phi \mapsto \phi(s)$$

where $\phi(b,s) \overset{def}{=} \Phi(wn(b), s)$, which clearly takes $A^C_{C'}$ to a similarly defined space $\mathcal{S}(\text{Sym}_n(k_v))^C_{C'}$ which is finite dimensional. It is then easy to show the claim. □

Now fix a function $\phi \in \mathcal{S}(\text{Sym}_n(k_v))$ and let Φ be the section in A associated to $x \mapsto \phi(-x)$ as above. Let $\text{Sym}_n(k_v)^\times$ be the set of symmetric matrices with non-zero determinant.

LEMMA 6.2.3. *For* $Re(s) > \rho_n$,

$$M_v(s)\Phi(w,s) = \frac{((-1)^n, (-1)^{(m-1)/2}\det(V))_v}{\gamma_v((-1)^n, \eta)} \cdot Z^F(s,\phi),$$

where

$$Z^F(s,\phi) \stackrel{def}{=} \int\limits_{\mathrm{Sym}_n(k_v)} |\det(x)|^{s-\rho_n} F(x)\phi(x)\,dx,$$

and where $F : Sym_n(k_v)^\times \to \mathbf{T}$ *is a function given by*

$$F(x) = \frac{(\det(x),\delta)_v}{\gamma_v(\det(x), \frac{1}{2}\psi_v)} h_v(x).$$

Here $\delta = (-1)^{(m+1)/2}\det(V)$, *and* $h_v(x)$ *is the Hasse invariant of* x *viewed as a quadratic form on* k_v^n .

Proof. For $\mathrm{Re}(s) > \rho_n$,

$$M_v(s)\Phi(w,s) = \int\limits_{\mathrm{Sym}_n(k_v)} \Phi(wn(x)w,s)\,dx.$$

Note that the support condition on Φ means that we are actually integrating over $\mathrm{Sym}_n(k_v)^\times$, since

$$\pi(wn(x)w) = \begin{pmatrix} -1 & 0 \\ x & -1 \end{pmatrix} \in \Omega_n = P_v w N_v \iff \det(x) \neq 0.$$

Now by writing

$$\begin{pmatrix} -1 & 0 \\ x & -1 \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix},$$

we see that $wn(x)w = p(x)wn(-x^{-1})$, where $p(x) = \left(\begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix}, \epsilon(x)\right)$ and $\epsilon(x)$ is defined by

$$\beta_v(w,n(x))\;\beta_v(wn(x),w) = \epsilon(x)\;\beta_v(p,w)\;\beta_v(pw,n(-x^{-1})).$$

(writing p for $\pi(p(x))$). We must simplify this last equality. It reduces quickly to

$$\epsilon(x) = \tilde{c}_v(wn(x), w)\ \tilde{c}_v(p, w).$$

Now for ease of notation, write $\eta = \frac{1}{2}\psi_v$ (in Rao's notation, this takes $y \mapsto \psi_v(\frac{1}{2}y)$) and let $\eta \circ x$ stand for the character of second degree $z \mapsto \eta(< z, z \cdot x >)$ defined on $X_v = \mathrm{span}_{k_v}(e_1, \ldots, e_n)$. Then we see that

$$\begin{aligned}
\tilde{c}_v(wn(x), w) &= \frac{m_v(wn(x))\ m_v(w)}{m_v(wn(x)w)}\, c_v(\tau n(x), \tau) \\
&= \frac{[\gamma_v((-1)^n, \eta)^{-1}\,\gamma_v(\eta)^{-n}]^2}{\gamma_v(\det(x), \eta)^{-1}\,\gamma_v(\eta)^{-n}}\,\gamma_v(\eta \circ x) \\
&= \gamma_v(\eta)^{-n}\,\gamma_v((-1)^n, \eta)^{-2}\,\gamma_v(\det(x), \eta)\,\gamma_v(\eta \circ x) \\
&= (-1, -1)_v^n\,\gamma_v(\eta)^{-n}\,\gamma_v(\det(x), \eta)\,\gamma_v(\eta \circ x)
\end{aligned}$$

and also that $\tilde{c}_v(p, w) = (\det(-x), (-1)^n)_v$, giving us

$$\begin{aligned}
\epsilon(x) &= (\det(x), (-1)^n)_v\,\gamma_v(\det(x), \eta)^2\ \left[\gamma_v(\eta)^{-n}\,\gamma_v(\det(x), \eta)^{-1}\gamma_v(\eta \circ x)\right] \\
&= (\det(x), (-1)^{n+1})_v\,h_v(x).
\end{aligned}$$

Here, we use an identity for $h_v(x)$ from the appendix of [R]. Next, we note that the measure $d^\times x \stackrel{def}{=} \frac{dx}{|\det(x)|^{\rho n}}$ is invariant under the action $x \mapsto ax\,{}^t a$ of $a \in GL(n, k_v)$, and furthermore, that it remains unchanged under the change of variable $x \mapsto x^{-1}$. We then have

$$\begin{aligned}
M_v(s)\Phi(w,s) &= \int\limits_{\mathrm{Sym}_n(k_v)} |-x^{-1}|^{s+\rho_n} \chi_v(m(-x^{-1}), \epsilon(x))\, \phi(x^{-1})\, dx \\
&= \int\limits_{\mathrm{Sym}_n(k_v)} |x|^{-s} \chi_v(m(-x^{-1}), \epsilon(x))\, \phi(x^{-1})\, d^{\times}x \\
&= \int\limits_{\mathrm{Sym}_n(k_v)} |x|^{s}\, \chi_v(m(-x), \epsilon(x^{-1}))\, \phi(x)\, d^{\times}x.
\end{aligned}$$

Since $x = x(x^{-1})\,{}^t x$, we see that x and x^{-1} are in the same $GL(n)$ orbit, and so $h_v(x) = h_v(x^{-1})$, giving $\epsilon(x^{-1}) = \epsilon(x)$. Also note that

$$\gamma_v(\det(-x), \eta) = ((-1)^n, \det(x))_v\ \gamma_v((-1)^n, \eta)\ \gamma_v(\det(x), \eta).$$

Hence

$$\begin{aligned}
&M_v(s)\Phi(w,s) \\
&= \int\limits_{\mathrm{Sym}_n(k_v)} |x|^{s-\rho_n} \frac{(\det(x),(-1)^{n+1})_v\ h_v(x)\ (\det(-x),(-1)^{(m-1)/2}\det(V))_v}{(\det(x),(-1)^n)_v\ \gamma_v((-1)^n,\eta)\ \gamma_v(\det(x),\eta)} \phi(x)\, dx \\
&\qquad = \frac{((-1)^n, (-1)^{(m-1)/2}\det(V))_v}{\gamma_v((-1)^n, \eta)} \int\limits_{\mathrm{Sym}_n(k_v)} |x|^{s-\rho_n} F(x)\phi(x)\, dx
\end{aligned}$$

as claimed. □

§**6.3 Zeta integrals.** So the problem of determining the poles of the operator $M_v(s)$ for $v < \infty$ is reduced to finding the poles of the zeta integral

$$Z^F(s,\phi) = \int\limits_{\mathrm{Sym}_n(k_v)} |x|^{s-\rho_n} F(x)\phi(x)\, dx$$

with F as in Lemma 6.2.3. This is closely related to a family of zeta integrals studied in detail by Igusa in [I1].

Letting $Y = \mathrm{Sym}_n(k_v)^\times$ and $G = GL(n, k_v)$ for the remainder of this section, we will write $g[d] = gd\,{}^t g$ for $g \in G,\ d \in Y$. This gives a group action of G on Y which splits up Y into a finite number of disjoint open orbits:

$$Y = \coprod_{i=1}^{h} Y_i$$

where, for each i, we may choose $d_i \in Y$ diagonal such that $G[d_i] = Y_i$. Suppose, as in Igusa, that $c : k_v^\times \to \mathbf{T}$ is a continuous character, and set

$$Z_i^c(s, \phi) \stackrel{def}{=} \int_{Y_i} |x|^{s-\rho_n} c(x)\phi(x)\, dx$$

for each i, where $\det(x)$ is understood at the appropriate places. Igusa proves that the integral converges for $\mathrm{Re}(s) > \rho_n$ to a holomorphic function of s (actually to a rational function in q^{-s}) which has a meromorphic continuation to the s-plane, and satisfies a system of functional equations:

$$Z_i^c(s, \hat{\phi}) = \sum_{j=1}^{h} \gamma_{ij}^c(s) Z_j^{c^{-1}}(\rho_n - s, \phi)$$

for meromorphic functions $\gamma_{ij}^c(s)$ independent of ϕ. Here the Fourier transform $\hat{\phi}$ of ϕ is defined using the character $T \mapsto \psi_v(tr(T))$ of $\mathrm{Sym}_n(k_v)$.

In our situation, $F : Y \to \mathbf{T}$ is *not* induced from a character of $k_v^\times$, but does have the following nice properties:

(1) $F(gd\,{}^t g) = F(d)$ for any $g \in G,\ d \in Y$,

(2) $F\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = F(A)F(B)$ for symmetric, non-degenerate matrices A and B, and

(3) in dimension 1, F is a character of second-degree on $k_v^\times/(k_v^\times)^2$.

Nevertheless, it is easy to see that Z^F still satisfies a functional equation: we have

$$Z_i^F(s,\phi) = F(d_i) \int_{Y_i} |x|^{s-\rho_n} \phi(x)\,dx$$

by (1) above, and so

$$\begin{aligned} Z^F(s,\hat{\phi}) &= \sum_i F(d_i)\ Z_i^1(s,\hat{\phi}) \\ &= \sum_{ij} F(d_i)\ \gamma_{ij}(s)\ Z_j^1(\rho_n - s, \phi) \\ &= \sum_{ij} F(d_i)\ F(d_j)^{-1}\,\gamma_{ij}(s)\ Z_j^F(\rho_n - s,\phi). \end{aligned}$$

We then define meromorphic functions

$$c_{d_j}(s) = \sum_i F(d_i)\ F(d_j)^{-1}\,\gamma_{ij}(s)$$

so that Z^F has the functional equation

$$Z^F(s,\hat{\phi}) = \sum_{j=1}^{h} c_{d_j}(s)\ Z_j^F(\rho_n - s,\phi). \tag{6.3.1}$$

In order to determine the poles of $Z^F(s)$ (applied to arbitrary ϕ), it will be necessary first to explicitly solve for the factors $c_d(s)$. In [PS-R2], the analogous factors appearing in the functional equation are $c_j(s) = \sum_i \gamma_{ij}(s)$. Although a formula for these functions is stated in the appendix to section 4 of that paper, no derivation is given there. It should be noted that the method presented in the next section also allows the functional equation in [PS-R2] to be computed with very little additional effort.

§6.4 **Computation of the functional equation.** Since we will compute $c_d(s)$ by induction on n, it will often be denoted by $c(n,s,d)$ to emphasize the dimension. We will actually be interested only in the poles and zeros of $c_d(s)$, so we will feel free to discard any multiples of holomorphic non-vanishing functions, denoting this by $c_d(s) \equiv (\dots)$. We begin with the following computation:

PROPOSITION 6.4.1. *Taking all notation as in the previous section, for* $Re(s) > \rho_n$ *we have*

$$c(n,s,d) = |d|^s F(d)^{-1} \int_C |x|^{s-\rho_n} F(x)\psi(tr(xd))\,dx,$$

where $C =$ *any open compact subset of* $Sym_n(k_v)$ *such that* $supp(\hat{\phi}_d) \subset C$, *and where* ϕ_d *is the characteristic function of the* $GL(n,\mathcal{O}_v)$ *-orbit of* d *in* $Sym_n(k_v)$.

Proof. Fix $d \in Y = \mathrm{Sym}_n(k_v)^\times$, so that writing $L = GL(n,\mathcal{O}_v)$, $L[d] = \{gd\,{}^tg : g \in L\}$ is an open compact subset of Y with characteristic function ϕ_d. Applying both sides of equation (6.3.1) to ϕ_d,

$$Z^F(s,\hat{\phi}_d) = c_d(s) Z_d^F(\rho_n - s, \phi_d).$$

Now the right-hand side (RHS) of the above is

$$(\mathrm{RHS}) = c_d(s)F(d) \int_{L[d]} |x|^{-s}\,dx = c_d(s)\,|d|^{-s}\,m^+(L[d])\,F(d)$$

where m^+ is the measure defined by dx. With $d^\times x$ as before, the left-hand side is

$$(\text{LHS}) = \int\limits_{\mathrm{Sym}_n(k_v)} |x|^s F(x)\hat{\phi}_d(x)\, d^\times x$$
$$= \int\limits_{C} |x|^s F(x) \left(\int\limits_{\mathrm{Sym}_n(k_v)} \phi_d(y)\psi_v(\mathrm{tr}(xy))\, dy \right) d^\times x$$

for any compact open set $C \subset \mathrm{Sym}_n(k_v)$ containing $\mathrm{supp}(\hat{\phi}_d)$

$$= \int\limits_{\mathrm{Sym}_n(k_v)} \phi_d(y) \int\limits_{C} |x|^s F(x)\psi_v(\mathrm{tr}(xy))\, d^\times x\, dy$$
$$(6.4.1) \qquad = \int\limits_{L[d]} \int\limits_{C} |x|^s F(x)\psi_v(\mathrm{tr}(xy))\, d^\times x\, dy.$$

The map $L \to L[d]$ given by $g \mapsto gd\,{}^t g$ is surjective, and so $L/O(d)$ is homeomorphic to $L[d]$, where $O(d)$ is the stabilizer of d in L. Both spaces $L/O(d) \simeq L[d]$ have a unique left L-invariant measure given by $d^\times x$ on $L[d]$. Fixing a Haar measure dg on L so that $\int_L dg = 1$, we then have

$$\int\limits_{L} f(g)\, dg = \int\limits_{L/O(d)} \int\limits_{O(d)} f(\dot{a}b)\, db\, d\dot{a}$$

for any $f \in C^\infty(L)$, where $d\dot{a} \rightleftharpoons d^\times x$, and db is Haar measure on $O(d)$ normalized to make the equality hold. Now since the inner integral in (6.4.1) is may be viewed as a function on L via $y = gd\,{}^tg$, we may write

$$\begin{aligned}(\mathrm{LHS}) &= m^+(L[d]) \int\limits_L \int\limits_C |x|^s F(x)\psi_v(\mathrm{tr}(xgd\,{}^tg))\, d^\times x\, dg \\ &= m^+(L[d]) \int\limits_L \int\limits_C |\,{}^tgxg|^s F(\,{}^tgxg)\psi_v(\mathrm{tr}(\,{}^tgxgd))\, d^\times x\, dg \\ &= m^+(L[d]) \int\limits_C |x|^{s-\rho_n} F(x)\psi_v(\mathrm{tr}(xd))\, dx\end{aligned}$$

using $|\det g| = 1$ for $g \in L$, and the L -invariance of $d^\times x$. The result follows by equating the two sides. □

The next proposition forms the basis for the induction.

PROPOSITION 6.4.2. *Fix a $GL(n, k_v)$ orbit of $Sym_n(k_v)^\times$, and choose a representative d of that orbit of the form $d = \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$, where $d_1 \in \mathrm{diag}(n-1, k_v^\times)$ and $d_2 \in k_v^\times$. Then for any $n \geq 2$, we have*

$$c(n, s, d) = c(n-1, s-\tfrac{1}{2}, d_1)\; I(n-1, s, d_1, d_2),$$

where

$$I(n-1, s, d_1, d_2) = \int |c|^{s+\frac{n-1}{2}} F(c)\; \psi(c\cdot \mathrm{tr}(B\,{}^tBd_1) + cd_2)\; d^\times c\, dB, \tag{6.4.2}$$

the integration taking place over

$$\mathcal{A} = \left\{(B, c) \in (k_v)^{n-1} \times k_v \mid Bc\,{}^tB \in \mathcal{C}_1,\ Bc \in \mathcal{C}_2,\ \text{and } c \in \mathcal{C}_3\right\}$$

for sufficiently large additively-closed compact open sets $\mathcal{C}_i$ in the appropriate spaces. Note that the variable B here is a column matrix.

Proof. We proceed by applying a change of variables to the integral representation for $c_d(s)$ from the preceding proposition. As in [PS-R2], we note that defining

$$\mathrm{Sym}_n^{(1)}(k_v) = \left\{ \begin{pmatrix} U & V \\ {}^tV & c \end{pmatrix} \in \mathrm{Sym}_n(k_v) \mid U \in \mathrm{Sym}_{n-1}(k_v),\ V \in M(n-1,1),\ \text{and } c \in k_v^\times \right\},$$

there is a homeomorphism

$$\begin{gathered} \mathrm{Sym}_{n-1}(k_v) \times M(n-1,1) \times k_v^\times \xrightarrow{\sim} \mathrm{Sym}_n^{(1)}(k_v) \qquad \text{defined by} \\ (A,B,c) \mapsto \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ {}^tB & 1 \end{pmatrix} = \begin{pmatrix} A+Bc\,{}^tB & Bc \\ c\,{}^tB & c \end{pmatrix}. \end{gathered} \tag{6.4.3}$$

We begin by writing

$$c(n,s,d) = \int_{\mathcal{C}} \left| \begin{matrix} U & V \\ {}^tV & c \end{matrix} \right|^{s-\rho_n} F\begin{pmatrix} U & V \\ {}^tV & c \end{pmatrix} \psi\left(\mathrm{tr}\left(\begin{pmatrix} U & V \\ {}^tV & c \end{pmatrix} d\right)\right) dU\, dV\, dc$$

and making the change of variables $U = A + Bc\,{}^tB$, $V = Bc$ as above. Note once again that $F(gx\,{}^tg) = F(x)$ for $g \in GL(n)$ and $x \in \mathrm{Sym}_n(k_v)^\times$, and also that F behaves well with respect to direct sums. This yields

(6.4.4)

$$c(n,s,d) = \int |A|^{s-\rho_n} |c|^{s+\frac{(n-1)}{2}} F(A)F(c)\psi(\mathrm{tr}(Ad_1 + Bc\,{}^tBd_1) + cd_2)\, dA\, dB\, d^\times c$$

where we choose sets $\mathcal{C}_i$ so that (schematically) we may write

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ {}^t\mathcal{C}_2 & \mathcal{C}_3 \end{pmatrix},$$

and the integral is over the set

$$\{(A,B,c) \mid A + Bc^tB \in \mathcal{C}_1,\ Bc \in \mathcal{C}_2, \text{ and } c \in \mathcal{C}_3\}.$$

Next, we note that the original integral has the same value for *any* open compact set $\mathcal{C}$ such that $\operatorname{supp}(\hat{\phi}_d) \subset \mathcal{C} \subset \operatorname{Sym}_n(k_v)$, and so we choose $\mathcal{C}_1$ to be $\pi^N \cdot \operatorname{Sym}_{n-1}(\mathcal{O}_v)$ for sufficiently negative N, and claim that

$$\mathcal{I}(B,c) \stackrel{def}{=} \int_{-Bc^tB+\mathcal{C}_1} |A|^{s-\rho_n} F(A)\psi(\operatorname{tr}(Ad_1))\,dA \equiv \begin{cases} c(n-1, s-\frac{1}{2}, d_1) & \text{if } Bc^tB \in \mathcal{C}_1 \\ 0 & \text{otherwise.} \end{cases}$$

This follows from considering that in the first case $-Bc^tB + \mathcal{C}_1 = \mathcal{C}_1$, while in the second, $(-Bc^tB + \mathcal{C}_1) \cap \mathcal{C}_1 = \emptyset$, and so $\int_{\mathcal{C}_1} = \int_{(-Bc^tB+\mathcal{C}_1)\cup\mathcal{C}_1} = \int_{\mathcal{C}_1} + \int_{-Bc^tB+\mathcal{C}_1}$ by the remark above.

So we see that by performing the integration with respect to A first in equation (6.4.4), we obtain

$$\begin{aligned} c(n,s,d) &= \int_{\substack{Bc\in\mathcal{C}_2 \\ c\in\mathcal{C}_3}} |c|^{s+\frac{(n-1)}{2}} F(c)\ \psi(\operatorname{tr}(Bc^tBd_1) + cd_2)\ \mathcal{I}(B,c)\,dB\,d^\times c \\ &= c(n-1, s-\tfrac{1}{2}, d_1)\ I(n-1, s, d_1, d_2), \end{aligned}$$

as claimed. □

To simplify equation (6.4.2), we need the following lemma:

Lemma 6.4.3. *If we take the sets $\mathcal{C}_i$ to be $\mathcal{C}_1 = \pi^{N_1}\operatorname{Sym}_{n-1}(\mathcal{O}_v)$, $\mathcal{C}_2 = \pi^N(\mathcal{O}_v)^{n-1}$, and $\mathcal{C}_3 = \pi^N\mathcal{O}_v$, then the set of integration $\mathcal{A}$ in equation*

(6.4.2) equals

$$\{(B,c)\in(k_v)^{n-1}\times k_v \mid c\in\mathcal{P}_v^N,\ B_1^2c\in\mathcal{P}_v^{N_1},\dots,B_{n-1}^2c\in\mathcal{P}_v^{N_1}\},$$

where $B=\begin{pmatrix}B_1\\ \vdots\\ B_{n-1}\end{pmatrix}$ *, and we take* $N_1\geq N$ *.*

Proof. We let the C_i be as above, and take $N_1\geq N$. If we then consider B to be fixed, the condition that c must satisfy in order for (B,c) to be included in the original set of integration $\mathcal{A}$ amounts to:

$$\mathrm{ord}_v(c)\geq M\stackrel{def}{=}\max\{N,N-b_i,N_1-b_i-b_j\mid 1\leq i,j\leq n-1\},$$

where $b_i=\mathrm{ord}_v(B_i)$. But now we check that in fact $M=\max\{N,N_1-2b_i\mid 1\leq i\leq n-1\}$.

First, note that for fixed i we have $\max\{N,N-b_i,N_1-b_i-b_j\mid 1\leq j\leq n-1\}=\max\{N,N_1-b_i-b_j\mid 1\leq j\leq n-1\}$: if $b_i\geq 0$, then $N\geq N-b_i$, and we may omit $N-b_i$ from the list, while $b_i<0$ implies that $N_1-2b_i>N-b_i$ (assuming that $N_1\geq N$), with the same conclusion. Next, by similar reasoning, we see that for any fixed (i,j) with $1\leq i<j\leq n-1$, we have $\max\{N,N_1-b_i-b_j,N_1-2b_i,N_1-2b_j\}=\max\{N,N_1-2b_i,N_1-2b_j\}$.

So this proves that

$$(B,c)\in\mathcal{A}\Longleftrightarrow \mathrm{ord}_v(c)\geq\max\{N,N_1-2\mathrm{ord}_v(B_i)\mid 1\leq i\leq n-1\},$$

which proves the lemma. □

Using the lemma then, we see that

(6.4.5)

$$I(n-1,s,d_1,d_2) = \int\limits_{\mathcal{P}_v^N} |c|^{s+\frac{(n-1)}{2}} F(c)\,\psi(cd_2) \prod_{j=1}^{n-1} \left[\int\limits_{cB_j^2 \in \mathcal{P}_v^{N_1}} \psi(cd_1^{(j)} B_j^2)\, dB_j \right] d^\times c,$$

writing $d_1 = \mathrm{diag}(d_1^{(1)}, \ldots, d_1^{(n-1)})$. For our next step, we must recognize the integral in brackets as a Weil index.

Lemma 6.4.4. *For any non-archimedean local field* k_v , *let* ψ_v *be our fixed additive character, and let* $D \in \mathbf{Z}$ *be maximal such that* $\psi_v = 1$ *on* $\mathcal{P}_v^{-D}$. *Write* $2\mathcal{O}_v = \mathcal{P}_v^e$. *Then for any* $a, c \in k_v^\times$, *we have*

$$\int\limits_{cb^2 \in \mathcal{P}_v^M} \psi_v(cab^2)\, db = |2ca|^{-\frac{1}{2}} \gamma_v(ca\psi_v)$$

for any integer M *such that* $M \le -(\mathrm{ord}_v(a) + D + 2e + 1)$.

Proof. We compute this similarly to a calculation in Perrin [P]. First, we treat the case where $a = 1$ and $c \in \mathcal{U}_v$, writing $(*)$ for the integral above.

Then taking $a = 1$ and $c = u$, we have $(*) = \int_{b^2 \in \mathcal{P}^M} \psi(ub^2)\, db$. Noting that $\mathrm{ord}(b^2) \ge M \iff \mathrm{ord}(b) \ge [\frac{M+1}{2}] \overset{def}{=} L$, we have $(*) = \int \chi_{\mathcal{P}^L}(b) \psi(ub^2)\, db$, where $[\ ]$ is the greatest integer function, and χ_A is the characteristic function of A for any set A . Next, it is easy to see that

$$\int\limits_{\mathcal{P}^{-(D+e+L)}} \psi(2uby)\, dy = \begin{cases} m^+(\mathcal{P}^{-(D+e+L)}) & \text{if } b \in \mathcal{P}^L \\ 0 & \text{if } b \notin \mathcal{P}^L, \end{cases}$$

and so setting $A = \mathcal{P}^{-(D+e+L)}$, we have

$$\chi_{\mathcal{P}^L}(b) = m^+(A)^{-1} \int\limits_A \psi(2uby)\, dy.$$

Substituting this into the expression for $(*)$, we obtain

$$(*) = m^+(A)^{-1} \int \int_A \psi(ub^2 + 2uby)\, dy\, db.$$

Note that the order of integration is fixed here. We next complete the square, observing that

$$\begin{aligned}
&M \leq -(D + 2e + 1) \\
&\Longleftrightarrow 2L = 2\left[\frac{M+1}{2}\right] \leq -D - 2e \\
&\Longleftrightarrow -2(D + e + L) \geq -D \\
&\Longleftrightarrow uy^2 \in \mathcal{P}^{-D} \;\text{ for all } y \in A = \mathcal{P}^{-(D+e+L)}.
\end{aligned}$$

Thus

$$\begin{aligned}
(*) &= m^+(A)^{-1} \iint \chi_A(-y)\ \psi(u(b+y)^2)\, dy\, db \\
&= m^+(A)^{-1} \iint \chi_A(x - y)\ \psi(uy^2)\, dy\, dx.
\end{aligned}$$

But now this is in the form needed to apply Weil's Corollary 2 to Theorem 2 [W1]. Following Weil's notation, we let $f(y) = \psi(uy^2)$, so that given the standard dual pairing on k_v, $< x, y >= \psi(xy)$, this gives us $f(x + y)f(x)^{-1}f(y)^{-1} =< x, 2uy >$, and so the symmetric mapping ρ associated to f is $y \overset{\rho}{\mapsto} 2uy$. Weil's Corollary 2 then gives us

$$(*) = m^+(A)^{-1}\gamma_v(f)|2u|^{-\frac{1}{2}} \int \chi_A(x)\, dx,$$

and so using Rao's notation now, we reach the conlusion $(*) = |2u|^{-\frac{1}{2}}\gamma_v(u\psi)$, as desired.

Next, we consider the case where $a = 1$ and $c = u\pi$ for $u \in \mathcal{U}_v$. A calculation entirely similar to the one above yields $(*) = \int \chi_{\mathcal{P}^L}(b)\psi(u\pi b^2)\,db$, where $L = [\frac{M}{2}]$. Setting $B = \mathcal{P}^{-(D+e+L+1)}$, we have

$$(*) = m(B)^{-1} \int \int_B \psi(u\pi(b^2 + 2by))\,dy\,db.$$

As before, $M \leq -D - 2e - 1 \iff 1 - 2(D + e + L + 1) \geq -D \implies$ we may complete the square. This then gives us

$$\begin{aligned}(*) &= m(B)^{-1} \iint \chi_B(-y)\psi(u\pi(b+y)^2)\,dy\,db \\ &= m(B)^{-1} \iint \chi_B(x-y)f(y)\,dy\,db,\end{aligned}$$

where $f(y) = \psi(u\pi y^2)$, and hence $\rho = 2u\pi = 2c$. Another application of Weil's corollary then yields $(*) = |2c|^{-\frac{1}{2}}\gamma_v(c\psi)$ in Rao's notation.

So far, we have proven the lemma for $a = 1$ and $c \in \mathcal{U}_v \cup \pi\mathcal{U}_v$. Now let c and $a \in k_v^\times$ be arbitrary, and suppose that $M \leq -(\mathrm{ord}(a) + D + 2e + 1)$. Write $ca = \pi^{2i}x$ for $x \in \mathcal{U}_v \cup \pi\mathcal{U}_v$. Then we finally have

$$\begin{aligned}\int_{cb^2 \in \mathcal{P}_v^M} \psi_v(cab^2)\,db &= \int_{\pi^{2i}xb^2 \in a\mathcal{P}^M} \psi(\pi^{2i}xb^2)\,db \\ &= |\pi|^{-i} \int_{xy^2 \in a\mathcal{P}^M} \psi(xy^2)\,dy = |2\pi^{2i}x|^{-\frac{1}{2}}\gamma_v(x\psi)\end{aligned}$$

by our previous results. But now this last expression equals $|2ca|^{-\frac{1}{2}}\gamma_v(ca\psi_v)$, since the Weil index is defined on square classes of $k_v^\times$. $\square$

In order to finish the simplification of equation (6.4.5), we need the following computations.

PROPOSITION 6.4.5. *Let k_v be any non-archimedean local field, and for $a \in k_v^\times$, let $H(a)$ stand for the Hilbert symbol character $x \mapsto (x,a)_v$. For a character c of $k_v^\times$, let $\zeta_v(s,c)$ denote the function*

$$\zeta_v(s,c) = \begin{cases} (1-c(\pi)q_v^{-s})^{-1} & \text{if } c \text{ is ramified,} \\ 1 & \text{if } c \text{ is unramified.} \end{cases}$$

Fix $d \in k_v^\times$. Then for $M \ll 0$ depending only on $\mathrm{ord}_v(d)$, the following equations hold.

$$\int_{\mathcal{P}_v^M} |x|^s\, \psi(x)\, (x,d)_v\, d^\times x \equiv \frac{\zeta_v(s,H(d))}{\zeta_v(1-s,H(d))} \tag{6.4.6}$$

$$\int_{\mathcal{P}_v^M} |x|^s\, \psi(x)\, \gamma_v(-xd,\psi)\, d^\times x \equiv \frac{\zeta_v(2s)}{\zeta_v(1-2s)} \frac{\zeta_v(\frac{1}{2}-s,H(d))}{\zeta_v(\frac{1}{2}+s,H(d))} \tag{6.4.7}$$

Note that $\equiv$ means "equal up to a multiple of a holomorphic non-vanishing function" as before, and that we only require that M be sufficiently negative, where this is determined by d.

Since the proof of this proposition is technical and quite long, it will be omitted. We now have enough information to solve for $I(n-1,s)$ in terms of zeta functions.

LEMMA 6.4.6. *Let the notation be as in Proposition 6.4.2. Then we have*

$$I(2k+1,s,d_1,d_2) \equiv \frac{\zeta_v(s,H((-1)^k 2\delta d_1))}{\zeta_v(1-s,H((-1)^k 2\delta d_1))} \qquad \text{for } k \geq 0, \text{ and}$$

$$I(2k,s,d_1,d_2) \equiv \frac{\zeta_v(2s)}{\zeta_v(1-2s)} \frac{\zeta_v(\frac{1}{2}-s,H((-1)^k 2\delta d))}{\zeta_v(\frac{1}{2}+s,H((-1)^k 2\delta d))} \qquad \text{for } k \geq 1,$$

where d and d_1 stand for their respective determinants where appropriate, and δ is the constant appearing in the definition of F .

Proof. Using the results of Lemma 6.4.4, equation (6.4.5) simplifies to

$$I(n-1,s,d_1,d_2)=\int\limits_{\mathcal{P}_v^N}|c|^{s+\frac{(n-1)}{2}}F(c)\psi(cd_2)\prod_{j=1}^{n-1}\left[|2cd_1^{(j)}|^{-\frac{1}{2}}\gamma_v(cd_1^{(j)}\psi)\right]\,d^\times c.$$

Now, writing $\eta=\frac{1}{2}\psi$, from the basic facts about the Weil index we compute that

$$\begin{aligned}F(c)=\frac{(c,\delta)_v}{\gamma_v(c,\eta)}&=\gamma_v(c,\eta)^3(c,\delta)_v=\gamma_v(c,\eta)(c,-\delta)_v\\&=\gamma_v(-\delta,\eta)^{-1}\gamma_v(-\delta c,\eta)=\gamma_v(-2\delta,\psi)^{-1}\gamma_v(-2\delta c,\psi),\end{aligned}$$

and so continuing:

$$I(n-1)\equiv\int\limits_{\mathcal{P}_v^N}|c|^s\,\psi(cd_2)\,\gamma_v(-2\delta c,\psi)\prod_{j=1}^{n-1}\left[\gamma_v(cd_1^{(j)},\psi)\right]\,d^\times c.$$

We now need to specialize to $n-1$ even or odd.

Suppose that $n-1=2k+1\geq 1$. Notice that we have an even number of Weil indices, and that for constants $a,b\in k_v^\times$,

$$\begin{aligned}&\gamma_v(ac,\psi)\,\gamma_v(bc,\psi)=\gamma_v(abc^2,\psi)\,(ac,bc)_v\\&=\gamma_v(ab,\psi)\,(a,b)_v(c,abc)_v=\gamma_v(ab,\psi)\,(a,b)_v(c,-ab)_v.\end{aligned}$$

Discarding all the constant terms and temporarily writing $a_j = d_1^{(j)}$,

$$I(2k+1) \equiv \int_{\mathcal{P}_v^N} |c|^s \psi(cd_2)(c, 2\delta a_{2k+1})_v \prod_{i=1}^{k} (c, -a_i a_{i+k})_v \, d^\times c$$

$$\equiv \int_{\mathcal{P}_v^N} |c|^s \, \psi(cd_2) \, (c, (-1)^k 2\delta d_1)_v \, d^\times c$$

$$\equiv \int_{\mathcal{P}_v^{N'}} |x|^s \, \psi(x) \, (x, (-1)^k 2\delta d_1)_v \, d^\times x.$$

This case is then done by equation (6.4.6) above.

Next, suppose that $n - 1 = 2k \geq 2$. In this case, we see that

$$I(2k) \equiv \int_{\mathcal{P}_v^N} |c|^s \, \psi(cd_2) \, \gamma_v(-2\delta c, \psi) \prod_{i=1}^{k} (c, -a_i a_{i+k})_v \, d^\times c$$

$$\equiv \int_{\mathcal{P}_v^{N'}} |x|^s \, \psi(x) \, \gamma_v(-2\delta d_2 x, \psi) \, (x, (-1)^k d_1)_v \, d^\times x,$$

making the change of variable $x = cd_2$, and letting $d_2 \mathcal{P}^N = \mathcal{P}^{N'}$. Noting that

$$\gamma_v(-x(-1)^k 2\delta d_1 d_2, \psi) = \gamma_v(-2\delta d_2 x, \psi) \, \gamma_v((-1)^k d_1, \psi) \, (-2\delta d_2 x, (-1)_k d_1)_v,$$

we see that up to a constant,

$$\gamma_v(-2\delta d_2 x, \psi) \, (x, (-1)^k d_1)_v \equiv \gamma_v(-x(-1)^k 2\delta d, \psi),$$

where d stands for $\det(d) = \det \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$. Substituting in the expression above, we obtain

$$I(2k) \equiv \int_{\mathcal{P}_v^{N'}} |x|^s \psi(x) \gamma_v(-x(-1)^k 2\delta d, \psi) \, d^\times x,$$

which finishes the proof by equation (6.4.7) of the previous proposition. □

Having (more or less) explicitly solved for $I(n-1,s)$, we may now finish the computation of $c_d(s)$.

THEOREM 6.4.7. *For any integer $n \geq 1$, and any $GL(n)$ representative $d \in Sym_n(k_v)^\times$, we have the following formula:*

$$c(n,s,d) \equiv \begin{cases} \frac{a_v(n,s)}{b_v(n,-s)} \frac{\zeta_v(\frac{1}{2}-s,H(\triangle))}{\zeta_v(\frac{1}{2}+s,H(\triangle))} & \text{if } n \text{ is odd} \\ \frac{a_v(n,s)}{b_v(n,-s)} & \text{if } n \text{ is even,} \end{cases}$$

where $\triangle = (-1)^{\frac{n-1}{2}} \cdot 2\delta \det(d)$, and a_v and b_v are as in Theorem 5.1.1.

Before starting the proof, we state the following inductive formulas for a_v and b_v, which are easily checked.

LEMMA 6.4.8. *For $n \geq 2$,*

$$a_v(n,s) = \begin{cases} \zeta_v(2s)a_v(n-1,s-\frac{1}{2}) & \text{if } n \text{ is odd} \\ a_v(n-1,s-\frac{1}{2}) & \text{if } n \text{ is even, and} \end{cases}$$

$$b_v(n,-s) = \begin{cases} \zeta_v(1-2s)b_v(n-1,-(s-\frac{1}{2})) & \text{if } n \text{ is odd} \\ b_v(n-1,-(s-\frac{1}{2})) & \text{if } n \text{ is even.} \end{cases}$$

Proof of Theorem 6.4.7. Beginning with $n=1$, we compute from Proposition 6.4.1 that

$$c(1,s,d) \equiv \int_{\mathcal{P}^N} |c|^{s-1} F(c)\psi(cd)\, dc$$

for N sufficiently negative. But exactly as in the proof of Lemma 6.4.6, this reduces to

$$\begin{aligned} c(1,s,d) &\equiv \int_{\mathcal{P}^{N'}} |x|^s \psi(x)\gamma_v(-2\delta dx,\psi)\, d^\times x \\ &\equiv \frac{\zeta_v(2s)}{\zeta_v(1-2s)} \frac{\zeta_v(\frac{1}{2}-s,H(2\delta d))}{\zeta_v(\frac{1}{2}+s,H(2\delta d))}, \end{aligned}$$

this last by use of equation (6.4.7). This proves the theorem for $n = 1$.

Now suppose the theorem is true for all $n < n_o$, where $n_o \geq 2$. If $n_o = 2k+1$, then with $d = \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$, we have

$$\begin{aligned} c(n_o, s, d) &= c(2k, s-\tfrac{1}{2}, d_1) I(2k, s, d_1, d_2) \\ &\equiv \left(\frac{a_v(2k, s-\frac{1}{2})}{b_v(2k, -(s-\frac{1}{2}))}\right) \left(\frac{\zeta_v(2s)}{\zeta_v(1-2s)} \frac{\zeta_v(\frac{1}{2}-s, H((-1)^k 2\delta d))}{\zeta_v(\frac{1}{2}+s, H((-1)^k 2\delta d))}\right) \\ &\equiv \frac{a_v(n_o, s)}{b_v(n_o, -s)} \frac{\zeta_v(\frac{1}{2}-s, H((-1)^k 2\delta d))}{\zeta_v(\frac{1}{2}+s, H((-1)^k 2\delta d))} \end{aligned}$$

using Proposition 6.4.2 and Lemmas 6.4.6 and 6.4.8.

If, on the other hand, $n_o = 2k+2$, then

$$\begin{aligned} &c(n_o, s, d) \\ &= c(2k+1, s-\tfrac{1}{2}, d_1) I(2k+1, s, d_1, d_2) \\ &\equiv \left(\frac{a_v(2k+1, s-\frac{1}{2})}{b_v(2k+1, -(s-\frac{1}{2}))} \frac{\zeta_v(1-s, H((-1)^k 2\delta d_1))}{\zeta_v(s, H((-1)^k 2\delta d_1))}\right) \frac{\zeta_v(s, H((-1)^k 2\delta d_1))}{\zeta_v(1-s, H((-1)^k 2\delta d_1))} \\ &\equiv \frac{a_v(n_o, s)}{b_v(n_o, -s)}. \end{aligned}$$

This finishes the proof. □

§**6.5 Poles of the zeta integral.** With the functional equation of $Z^F(s)$ at our disposal, we may now prove the analogue of Piatetski-Shapiro and Rallis' theorem (p.206 [PS-R2]):

THEOREM 6.5.1. *For any $\phi \in \mathcal{S}(Sym_n(k_v))$, the function*

$$s \mapsto \frac{1}{a_v(n, s)} Z^F(s, \phi)$$

is entire, and for a given $s_o \in \mathbb{C}$ *we can find a function* ϕ_o *such that*

$$\left(\frac{1}{a_v(n,s)} Z^F(s,\phi_o)\right)\Bigg|_{s=s_o} \neq 0.$$

While [PS-R2] states this theorem for $Z(s,\phi)$ (i.e. $F = 1$) and with a suitably modified function a_v , the proof which we will give is essentially the same. It is included here in expanded form for completeness. Note, however, that the zeta functions $Z^F(s,\phi)$ and $Z(s,\phi)$ are *not* the same, and have different pole behavior. Since Theorem 6.5.1 above directly implies Theorem 6.1.2, we will finish the section with a brief proof of the latter.

Proof of Theorem 6.5.1. We proceed by induction on n , and follow the notation and outline of [PS-R2]. Consider, in the case $n = 1$, that any pole of $Z^F(s,\phi)$ results from a ϕ with $0 \in \operatorname{supp}(\phi)$. Hence, we may write

$$Z^F(s,\phi) = \int_{k_v} |x|^s F(x)\phi(x)\, d^\times x = f(s) + c \cdot \int_{\mathcal{P}^N} |x|^s F(x)\, d^\times x$$

where $f(s)$ is entire, and $\phi(x) = c$ on $\mathcal{P}^N$ (i.e. choose N large enough). But then, the last integral equals

$$c' \sum_{i=N'}^{\infty} \int_{2\delta\pi^i \mathcal{U}} |x|^s \gamma_v(-\frac{x}{2\delta}\psi)\, d^\times x$$

and letting $x = -2\delta\pi^i u$, we have

$$\begin{aligned}
&c' \sum_{i=N'}^{\infty} |2\delta\pi^i|^s \int_{\mathcal{U}} \gamma_v(\pi^i u\psi)\, du \\
&= c'|2\delta|^s \sum_{j\geq N'/2} \left[q^{-2js}\left(\int_{\mathcal{U}} \gamma_v(u\psi)\, du\right) + q^{-(2j+1)s}\left(\int_{\mathcal{U}} \gamma_v(u\pi\psi)\, du\right)\right] \\
&= c'|2\delta|^s \left(\frac{M_1 q^{-N's}}{1-q^{-2s}} + \frac{M_2 q^{-(N'+1)s}}{1-q^{-2s}}\right),
\end{aligned}$$

taking N' even without loss, and for appropriate constants c', M_1, M_2 . So we see that $\frac{1}{\zeta_v(2s)} Z^F(s,\phi)$ is entire for $n=1$ and for any $\phi \in \mathcal{S}(k_v)$.

To show that we can choose ϕ to make this non-vanishing, consider the following. Writing $2\mathcal{O}_v = \mathcal{P}^e$, note that $\mathcal{U}_{2e+1} \overset{def}{=} 1+\mathcal{P}^{2e+1}$ is an open subset of k_v contained in $\mathcal{U}^2 = \{u^2 \mid u \in \mathcal{U}\}$ (see Serre [Se]). If s_o is not a pole of $\zeta_v(2s)$, then let $\phi_o = \mathrm{Char}(\mathcal{U}_{2e+1})$, so that

$$Z^F(s,\phi_o)\big|_{s=s_o} = \int_{\mathcal{U}_{2e+1}} |x|^{s_o} F(x)\,dx = \int_{\mathcal{U}_{2e+1}} dx \neq 0.$$

since $F(x,1)=1$ for $x \in \mathcal{U}^2$. If, on the other hand, $\zeta_v(2s_o)^{-1} = 0$, then $s_o = \frac{\pi\sqrt{-1}l}{\log(q)}$ for some $l \in \mathbf{Z}$. We may then apply the functional equation as in the beginning of the proof of Proposition 6.4.1: let ϕ_1 be the characteristic function of $GL(n,\mathcal{O}_v)[1_n]$, so that

$$\frac{1}{\zeta_v(2s)} Z^F(s,\hat{\phi}_1) = g(s)\frac{\zeta_v(\frac{1}{2}-s,H(2\delta))}{\zeta_v(1-2s)\zeta_v(\frac{1}{2}+s,H(2\delta))}$$

where $g(s)$ is holomorphic and non-vanishing. It is clear then that this function cannot vanish at $s_o \in i\mathbf{R}$.

Now assume the theorem is true for $\mathrm{Sym}_l(k_v)$ for all $l<n$. Let

$$\mathrm{Sym}_n^{(t)} = \{\begin{pmatrix} X & {}^tY \\ Y & W \end{pmatrix} \in \mathrm{Sym}_n \mid W \in GL(t)\},$$

which is an open subset of Sym_n . Hence

$$\mathcal{S}_t = \{\phi \in \mathcal{S}(\mathrm{Sym}_n) \mid \mathrm{supp}(\phi) \subset \mathrm{Sym}_n^{(t)}\}$$

is a subspace of $\mathcal{S}(\mathrm{Sym}_n)$. Fix, for the sake of argument, a point $s_o \in \mathbf{C}$ at which $Z^F(s)$ has a pole. Write the Laurent series

$$Z^F(s,\phi) = \sum_{k \geq A} \ell_k(\phi)(s-s_o)^k,$$

so that for any $k \geq A$, $\ell_k : \mathcal{S}(\mathrm{Sym}_n) \to \mathbf{C}$ is a distribution, and we have $\ell_A \neq 0$ and $A < 0$. For any t with $1 \leq t \leq n$, let k_t be the smallest $k \in \mathbf{Z}$ such that $\ell_k|_{\mathcal{S}_t} \neq 0$. Since we could conceivably have $\ell_A|_{\mathcal{S}_t} = 0$, it is evident that $k_t \geq A$.

Now the fact that $\mathrm{Sym}_n^{(t)} \subset \mathrm{Sym}_n$ is open implies that we have exact sequences

$$0 \to \mathcal{S}_t \xrightarrow{i} \mathcal{S}(\mathrm{Sym}_n) \to \mathcal{S}(\mathrm{Sym}_n \sim \mathrm{Sym}_n^{(t)}) \to 0$$

and

$$0 \to \mathcal{S}(\mathrm{Sym}_n \sim \mathrm{Sym}_n^{(t)})^* \to \mathcal{S}(\mathrm{Sym}_n)^* \xrightarrow{i^*} \mathcal{S}_t^* \to 0$$

where we write V^* for the complex dual (taken without topology) of a $\mathbf{C}$ - vector space V . We now suppose that $k_t > A$. If this is true, then $\ell_A|_{\mathcal{S}_t} = 0$, which says that $i^*(\ell_A) = 0$, and so $\mathrm{supp}(\ell_A) \subset \mathrm{Sym}_n \sim \mathrm{Sym}_n^{(t)}$ by the exact sequence above. But since $A < 0$, it is also easy to see (as in the $n = 1$ case) that $\mathrm{supp}(\ell_A) \subset \{x \in \mathrm{Sym}_n \mid \det(x) = 0\}$: in other words, if $\mathrm{supp}(\phi) \subset \mathrm{Sym}_n^{(n)}$, then $Z^F(s,\phi)$ is entire. Next, consider that if we define $\phi^g(x) = \phi(gx\,{}^tg)$ for $g \in GL(n)$ and $x \in \mathrm{Sym}_n$, then $Z^F(s,\phi^g) = |\det(g)|^{-2s} Z^F(s,\phi)$. Hence $\ell_A(\phi^g) = |\det(g)|^{-2s_o}\ell_A(\phi)$ for all $g \in GL(n)$

and $\phi \in \mathcal{S}(\mathrm{Sym}_n)$: ℓ_A is called a **homogeneous** distribution. So it is clear that $\mathrm{supp}(\ell_A)$ is a $GL(n)$ -invariant subset of

$$\{x \mid \det(x) = 0\} \cap (\mathrm{Sym}_n \sim \mathrm{Sym}_n^{(t)})$$

and consists of a finite union of $GL(n)$ orbits, since this is true of $\{x \mid \det(x) = 0\}$.

Specializing to the case $t = 1$ now, it is an easy computation to show that the set

$$\mathrm{Sym}_n \sim \mathrm{Sym}_n^{(1)} = \left\{ \begin{pmatrix} U & V \\ {}^tV & 0 \end{pmatrix} \in \mathrm{Sym}_n \mid U \in \mathrm{Sym}_{n-1},\ V \in M(n-1,1) \right\}$$

contains no non-zero $GL(n)$ orbits. Thus ℓ_A is a homogeneous distribution with $\mathrm{supp}(\ell_A) = \{0\}$. Since we are at a finite place v , and $\mathcal{S}(\mathrm{Sym}_n)$ consists of finite linear combinations of characteristic functions of compact open sets, we see that

$$\lim_{\substack{E \searrow 0 \\ E \text{ compact open}}} \ell_A(\mathrm{Char}(E))$$

exists, and is non-zero. Hence $\ell_A(\phi) = c \cdot \phi(0)$ for some constant $c \in \mathbf{C}^\times$. But now by the homogeneity of ℓ_A , we see that $s_o \in \left(\frac{\pi\sqrt{-1}}{\log(q)}\right) \cdot \mathbf{Z}$.

Thus we have proven: if s_o is a pole of $Z^F(s)$ of order $-A$, and if $k_1 > A$, where k_1 is the smallest integer k such that $\ell_k(\mathcal{S}_1) \neq 0$, then $s_o \in \left(\frac{\pi\sqrt{-1}}{\log(q)}\right) \cdot \mathbf{Z}$. Now consider the functional equation

$$Z^F(s, \hat{\phi}) = \sum_d c_d(s) Z_d^F(\rho_n - s, \phi).$$

Since $\mathrm{Re}(\rho_n - s_o) = \rho_n$, we see that $Z_d^F(\rho_n - s, \phi)$ has no pole at s_o for any d . Hence it must be that some $c_d(s)$ has a pole at s_o of order at least $-A$. But

$$\frac{c(n,s,d)}{a_v(n,s)} = \begin{cases} \frac{1}{b_v(n,-s)} \frac{\zeta_v(\frac{1}{2}-s, H(\triangle))}{\zeta_v(\frac{1}{2}+s, H(\triangle))} & \text{if } n \text{ is odd} \\ \frac{1}{b_v(n,-s)} & \text{if } n \text{ is even} \end{cases}$$

by Theorem 6.4.7, and this can have no poles at s_o . This takes care of all poles s_o for which $k_1 > A$.

Next, suppose that $Z^F(s)$ has a pole at s_o , but that $\ell_A \neq 0$ on $\mathcal{S}_1$ (i.e. $k_1 = A$). Then $Z^F(s, \phi)$ has a pole of order $-A$ at s_o for some $\phi \in \mathcal{S}_1$, and it suffices to show that

$$\frac{1}{a_v(n,s)} Z^F(s, \phi)$$

is holomorphic at s_o for all $\phi \in \mathcal{S}_1$. But now referring to the homeomorphism given in equation (6.4.3) in the proof of Proposition 6.4.2, we see that

$$\mathcal{S}_1 \cong \mathcal{S}(M(n-1,1)) \otimes \mathcal{S}(\mathrm{Sym}_{n-1}) \otimes \mathcal{S}(k_v^\times)$$

and so without loss of generality, we may consider $\phi \in \mathcal{S}_1$ of the form $\phi =$

$\phi_1 \otimes \phi_2 \otimes \phi_3$ according to the isomorphism above. Then for $\mathrm{Re}(s) > \rho_n$,

$$
\begin{aligned}
Z^F(s,\phi) &= \int\limits_{\mathrm{Sym}_n^{(1)}} |x|^{s-\rho_n} F(x)\phi(x)\,dx \\
&= \iiint |A|^{s-\rho_n} |c|^{s-\rho_n+n-1} F(A)\,F(c)\,\phi_1(B)\,\phi_2(A)\,\phi_3(c)\,dA\,dB\,dc \\
&= \left(\int\limits_{\mathrm{Sym}_{n-1}} |A|^{(s-\frac{1}{2})-\rho_{n-1}} F(A)\phi_2(A)\,dA \right) \left(\int\limits_{M(n-1,1)} \phi_1(B)\,dB \right) \\
&\quad \times \left(\int\limits_{k_v^\times} |c|^{s+\frac{n-3}{2}} F(c)\phi_3(c)\,dc \right).
\end{aligned}
$$

As noted before, $\mathrm{supp}(\phi_3) \subset k_v^\times$ means that the last integral is entire, and so

$$Z^F(s,\phi) = f(s) Z^F(s - \tfrac{1}{2}, \phi_2)$$

where $f(s)$ is entire. Now we use the induction hypothesis, so that

$$\frac{1}{a_v(n-1, s-\frac{1}{2})} Z^F(s,\phi)$$

is entire. Referring back to Lemma 6.4.8, this last implies that $\frac{1}{a_v(n,s)} Z^F(s,\phi)$ is also entire, as desired.

Having shown that we can force $\frac{1}{a_v(n,s)} Z^F(s,\phi)$ to be non-zero at a general point s_o in the case $n = 1$, we may then use the inductive argument above to construct a function ϕ_o for general n , such that $Z^F(s,\phi_o) \neq 0$ in case $a_v(n,s_o)^{-1} \neq 0$. If s_o is a pole of $a_v(n,s)$, then this is even easier, and we may use the functional equation applied to the characteristic function of $GL(n,\mathcal{O}_v)[1_n]$ as in the $n = 1$ case. This finishes the proof. □

Proof of Theorem 6.1.2. By Lemmas 6.2.1 and 6.2.2, we must only prove that

$$\frac{1}{a_v(n,s)} M_v(s)\Phi_v(w,s)$$

is holomorphic when applied to sections of the form

$$n_1 m w n(b) \mapsto |a(m)|^{s+\rho_n} \chi_v(m)\phi(b)$$

for $\phi \in \mathcal{S}(\mathrm{Sym}_n(k_v))$. But by Lemma 6.2.3,

$$\frac{1}{a_v(n,s)} M_v(s)\Phi_v(w,s) = c \cdot \frac{1}{a_v(n,s)} Z^F(s,\phi),$$

and we have just proven that this last is entire. □

§6.6 Final results. Resuming now the discussion of §6.1, we were interested in the poles of the operator $\frac{b_v(n,s)}{a_v(n,s)} M_v(s)$ for $v < \infty$. The added zeta functions in b_v may now introduce some poles. We keep track of these only at the special value $s_o(m,n) = \frac{m}{2} - \frac{n+1}{2}$.

LEMMA 6.6.1.

$$\operatorname*{ord}_{s_o(m,n)} b_v(n,s) = \begin{cases} -1 & \text{if } 1 \le m \le n \\ 0 & \text{if } n+1 \le m \end{cases}$$

Proof. Recall that

$$b_v(n,s) = \prod_{k=1}^{[\frac{n+1}{2}]} \zeta_v(2s+n+2-2k)$$

and that $v < \infty$ implies that $\zeta_v(z)$ has poles on the real axis only at $z = 0$.

If n is even, then a pole occurs at $s_o(m,n)$

$$\iff 2s_o + n + 2 - 2k = 0 \quad \text{for } k = 1, 2, \dots, \frac{n}{2}$$

$$\iff m + 1 = 2k \quad \text{for } k = 1, 2, \dots, \frac{n}{2}$$

$$\iff 1 \leq \frac{m+1}{2} \leq \frac{n}{2}$$

$$\iff 1 \leq m \leq n - 1.$$

If n is odd, then b_v has a pole at s_o

$$\iff 1 \leq \frac{m+1}{2} \leq \frac{n+1}{2}$$

$$\iff 1 \leq m \leq n.$$

□

In order to control the simple pole which may possibly occur at s_o in the term $\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s)$, we need to take advantage of the special properties of **Weil-Siegel sections.** Following [K-R1], we will use this term to denote $\widetilde{K}_v$ -finite sections $\Phi_v(s)$ whose values at s_o lie in the image of $\mathcal{S}(V_v^n)$ under the $\widetilde{G}_v$ -intertwining map

$$\mathcal{S}(V_v^n) \longrightarrow I(s_o)_{\widetilde{K}_v}$$

$$\varphi_v \longmapsto \Phi_v(s_o, \varphi_v),$$

recalling that $\Phi_v(g, s_o, \varphi_v) = \omega_v(g)\varphi_v(0)$. Let $T_v : \mathcal{S}(V_v^n) \to I(-s_o)_{\widetilde{K}_v}$ denote the mapping above composed with

$$\frac{1}{a_v(n,s)} M_v(s)|_{s=s_o} : I(s_o)_{\widetilde{K}_v} \longrightarrow I(-s_o)_{\widetilde{K}_v}.$$

Note that T_v is $\widetilde{G}_v$ -intertwining, and also that with the action of $H_v = O(V)_v$ on $\mathcal{S}(V_v^n)$ given in §2.3, we have $T_v(\omega_v(h)\varphi_v) = T_v(\varphi_v)$ for any $\varphi_v \in \mathcal{S}(V_v^n)$. Hence

$$T_v \in \mathrm{Hom}_{\widetilde{G}_v \times H_v}(\mathcal{S}(V_v^n), I_v(-s_o) \otimes 1),$$

where 1 is the trivial representation of H_v . But now as in [K-R1], the following proposition applies:

PROPOSITION 6.6.2. *Let l be the dimension of a maximal isotropic subspace of V_v , and let*

$$h_v(m,n,l) = \dim_{\mathbf{C}} Hom_{\widetilde{G}_v \times H_v}(\mathcal{S}(V_v^n), I_v(-s_o) \otimes 1).$$

Then

(1) *if* $1 \leq m \leq n$, *then* $h_v(m,n,l) = 0$.

(2) *If* $n+1 \leq m \leq 2n-1$, *then*

$$h_v(m,n,l) \leq \begin{cases} 1, & \text{if } m-l \leq n+1 \\ 0 & \text{otherwise.} \end{cases}$$

(3) *If* $2n+1 \leq m$, *then*

$$h_v(m,n,l) \leq \begin{cases} 1, & \text{if } n \leq l \\ 0 & \text{otherwise.} \end{cases}$$

The proof is essentially the same as that in [K-R1].

As a result, we have

COROLLARY 6.6.3. *Let* $v \in \Sigma_k$ *be a finite place. If* $\Phi_v \in I_{\widetilde{K}_v}$ *is a Weil-Seigel section, then*

$$\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s)$$

is holomorphic at $s = s_o(m,n)$ *. Moreover, if* $n+1 \leq m$ *and* $h_v(m,n,l) = 0$ *in Proposition 6.6.2, then the expression above has a zero at* s_o *.*

§7.1 Coverings of the unitary group. In order to finish our analysis of the global intertwining operator, we still must analyse the analytic properties of

$$\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s)$$

when k_v is archimedean. For the time being, fix a place v for which $k_v \cong \mathbf{R}$. The properties of the expression above when applied to Weil-Siegel sections will turn out to be governed by those of the operator $\frac{b_v(s)}{a_v(s)} M_v(s)$ when applied to a certain section given by a character of $\widetilde{K}_v$. Hence we must study this last group.

Igusa's book [I2] contains many details of the real metaplectic group. First of all, he shows that

$$K_v = Sp(n, \mathbf{R}) \cap O(2n, \mathbf{R})$$

is a maximal compact subgroup of $Sp(n, \mathbf{R})$, and that any maximal compact subgroup is a conjugate of K_v. The group K_v is also naturally isomorphic to the $n \times n$ unitary group $U(n) = \{g \in GL(n, \mathbf{C}) \mid g\,{}^t\bar{g} = 1_n\}$ via

$$K_v \xrightarrow{\sim} U(n)$$

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mapsto \alpha + i\beta.$$

$U(n)$ is a compact connected Lie group with fundamental group $\pi_1(U(n)) \cong \mathbf{Z}$. This may be shown as follows: $U(n)$ acts transitively on $S^{2n-1} = \{x \in$

$\mathbf{C}^n$ (column vectors) $\mid {}^t x\bar{x} = 1\}$, and so defining the map

$$U(n) \xrightarrow{\varphi} S^{2n-1} \quad \text{by} \quad g \mapsto g \cdot e_n, \quad \text{where } e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbf{C}^n ,$$

we see that the stabilizer of e_n in $U(n)$ is a copy of $U(n-1)$ (for $n \geq 2$) and that $U(n)/U(n-1)$ is homeomorphic to S^{2n-1} . Now $U(1) \cong S^1$ is connected, and by induction, we may assume that $U(n-1)$ is also. Since φ is an open mapping onto a connected space and has connected fibers ($\simeq U(n-1)$), it is easy to show that $U(n)$ is connected also.

The mapping φ above also defines a principal fiber bundle with fiber $U(n-1)$. Attached to any such bundle is a long exact sequence of groups

$$\cdots \longrightarrow \pi_j(U(n-1)) \longrightarrow \pi_j(U(n)) \longrightarrow \pi_j(S^{2n-1}) \xrightarrow{\Delta} \pi_{j-1}(U(n-1)) \longrightarrow \cdots$$

called the homotopy sequence of the bundle (we omit base points). See Steenrod [St]. In our situation, this yeilds

$$0 = \pi_2(S^{2n-1}) \longrightarrow \pi_1(U(n-1)) \xrightarrow{\sim} \pi_1(U(n)) \longrightarrow \pi_1(S^{2n-1}) = 0$$

for $n \geq 2$. Hence $\pi_1(U(n)) \cong \pi_1(U(n-1)) \cong \cdots \cong \pi_1(U(1)) \cong \mathbf{Z}$.

Now by the usual theory of covering spaces (see [G-H]), for every subgroup H of $\pi_1(U(n), 1)$, there exists a (connected) covering space

$$(X_H, x_o) \xrightarrow{p} (U(n), 1)$$

which is unique up to equivalence, and such that $H = p_*(\pi_1(X_H, x_0))$. There is also a unique structure of a topological group which may be placed on X_H

so that x_o is the identity element and p is a group homomorphism. The covering will be $[\pi_1(U(n)) : H]$ to one.

We are of course interested in two-fold coverings, and so the preceding guarantees us that $U(n)$ has a unique connected two-fold covering group (up to equivalence of coverings), since $\mathbb{Z}$ has a unique subgroup of index two. While $\widetilde{K}_v$ will turn out to be a connected covering, it will also be helpful to consider another model. Let

$$\widetilde{U}(n) \stackrel{def}{=} \{(x,z) \in U(n) \times \mathbf{T} \mid \det(x) = z^2\}.$$

This is a topological group as a subgroup of $U(n) \times \mathbf{T}$, and it is easy to prove that it is path connected. Hence

$$\widetilde{U}(n) \xrightarrow{p} U(n), \qquad p(x,z) = x$$

is a model of the unique two-fold cover of $U(n)$. This model is useful to us because projection on the *second* factor naturally gives a character c of $\widetilde{U}(n)$ with the property that $c(g)^2 = \det(p(g))$. For this reason, we will call this character $\det^{\frac{1}{2}}$.

PROPOSITION 7.1.1. *All characters of the group $\widetilde{U}(n)$ defined above are of the form c_m for $m \in \mathbb{Z}$, where*

$$c_m(g) = \left[\det^{\frac{1}{2}}(g)\right]^m.$$

In other words, $\det^{\frac{1}{2}}$ generates the group of characters of $\widetilde{U}(n)$.

This is easily proven by the Weyl character formula or otherwise.

Next, we state the facts concerning $\widetilde{K}_v$, with a sketch of their proof.

PROPOSITION 7.1.2. *Given* $\pi : Mp(n,\mathbf{R}) \to Sp(n,\mathbf{R})$ *as defined in §1.8,* $\widetilde{K}_v = \pi^{-1}(Sp(n,\mathbf{R}) \cap O(2n,\mathbf{R}))$ *is a connected group, as is* $Mp(n,\mathbf{R})$ *itself. Hence, making the identification* $K_v \cong U(n)$ *, there is a covering isomorphism of* $\widetilde{K}_v$ *with* $\widetilde{U}(n)$ *.*

Sketch of Proof. Weil defines a cover of $Sp(n,\mathbf{R})$ by the circle (in [W1]) via

$$\mathcal{M} = \{(g,\xi) \in Sp(n,\mathbf{R}) \times \mathrm{Un}(L^2(\mathbf{R}^n)) \mid \xi^{-1} \circ \overline{U}(h) \circ \xi = \overline{U}(h^g) \text{ for all } h \in \mathcal{H}_{\mathbf{R}}\}$$

analogously to the definition of $Mp(n,\mathbf{R})$. He proves that $\mathcal{M} \xrightarrow{Pr_1} Sp(n,\mathbf{R})$ is a continuous open surjection, and so it is easy to see that $\mathcal{M}$ is connected, and hence path connected. Wallach, in [Wa], proves that $Mp(n,\mathbf{R}) = \overline{[\mathcal{M},\mathcal{M}]}$. This can be used to show that $Mp(n,\mathbf{R})$ is connected. Letting L be the connected component of the identity in $\widetilde{K}_v$, one can then show that $Mp(n,\mathbf{R})/L \to Mp(n,\mathbf{R})/\widetilde{K}_v$ is a $[\widetilde{K}_v : L]$ -to-one covering map. But $\pi_1(Mp(n,\mathbf{R})/\widetilde{K}_v) = \pi_1(Sp(n,\mathbf{R})/K_v) = \{1\}$, and so $[\widetilde{K}_v, L] = 1$, proving that $\widetilde{K}_v$ is connected. □

§**7.2 Gaussians.** In this section, we will develop certain special vectors in the Weil representation called Gaussians. These are eigenvectors for the action of the maximal compact subgroups of the metaplectic group. We do this first for the general metaplectic group, and then make modifications necessary for the dual pair situation.

To begin with, we drop most subscripts v and continue to work over the real numbers. In order to use results from other sources, it will be convenient to assume that our fixed character of $(\mathbf{R},+)$ is given by $\psi(x) = \exp(2\pi i x)$. In this case, Rao states that the Weil index of $x \mapsto \psi(ax^2)$ is given by

$$\gamma(a\psi) = \exp\left[\frac{\pi i}{4}\operatorname{sign}(a)\right], \tag{7.2.1}$$

so that

$$\gamma(a, \tfrac{1}{2}\psi) = \gamma(a,\psi) \overset{def}{=} \frac{\gamma(a\psi)}{\gamma(\psi)} = \begin{cases} 1, & \text{if } a > 0 \\ -i, & \text{if } a < 0. \end{cases}$$

Taking advantage of our fixed basis for W, and identifying $W = X \oplus Y$ with $\mathbf{R}^n \oplus \mathbf{R}^n$, it is proven in [I2] that if $F_0 \in \mathcal{S}(\mathbf{R}^n)$ is the function given by $F_0(x) = \exp(-\pi x\, {}^t x)$, then taking $g = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right) \in K = Sp(n,\mathbf{R}) \cap O(2n)$ with $\det(\beta) \neq 0$, we have

$$r(g)F_0 = |\det(\beta)|^{-\frac{1}{2}} \det(1_n + i\beta^{-1}\alpha)^{-\frac{1}{2}} F_0 .$$

See Lemma 11 on p.37 of [I2]. Here r is the same as Rao's unnormalized projective representation (or Weil's, for that matter), and we must be careful in choosing the second square root. This is done by declaring that

$$\det(i^{-1}(\tau - \beta^{-1}\alpha))^{\frac{1}{2}} \to \det(\operatorname{Im}(\tau))^{\frac{1}{2}} > 0$$

as $\operatorname{Re}(\tau) \to \beta^{-1}\alpha$, where $\tau \in \{z \in M(n,\mathbf{C}) \mid z = {}^t z \text{ and } \operatorname{Im}(z) > 0\}$. In any case, the important point is that we can use this as a starting point to prove:

PROPOSITION 7.2.1. *Let* $\widetilde{K} \subset Mp(n,\mathbf{R})$ *be the usual maximal compact subgroup, and let* $\rho : Mp(n,\mathbf{R}) \to Un(L^2(\mathbf{R}^n))$ *denote the Weil representation associated to* ψ *, given here by* $\rho(g,\xi) = \xi$ *. Then there is a continuous group homomorphism (character)*

$$D : \widetilde{K} \to \mathbf{T}$$

such that

$$\rho(g)F_0 = D(g) \cdot F_0 \quad \text{for all } g \in \widetilde{K}, \text{ and}$$

$$D(g)^2 = \det(\pi(g)),$$

where we identify $K = \pi(\widetilde{K})$ *with* $U(n)$ *via*

$$\left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right) \longleftrightarrow \alpha + i\beta.$$

We will denote this character by $\det^{\frac{1}{2}}$.

(1) Note that we are using the Weil representation used in the definition of $Mp(n,\mathbf{R})$, and *not* the representation coming from a larger metaplectic group in a dual-pair situation.

(2) Also note that this is the "same" character as the $\det^{\frac{1}{2}}$ defined on $\widetilde{U}(n)$, and that it in fact gives us a covering isomorphism $\widetilde{K} \to \widetilde{U}(n)$ over $U(n)$ via $g \mapsto (\pi(g), D(g))$.

Sketch of proof. By Igusa's result above, for $g = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right) \in \Omega_n \cap K$, we have

$$\tilde{r}(g)F_0 = A(g)F_0,$$

where $A(g) \in \mathbf{T}$ is a scalar given by

$$A(g) = m(g) \,|\det(\beta)|^{-\frac{1}{2}} \det(1_n + i\beta^{-1}\alpha)^{-\frac{1}{2}}.$$

Here, $m(g) = \gamma(x(g), \frac{1}{2}\psi)^{-1}\gamma(\frac{1}{2}\psi)^{-n}$, and it is easily shown that $x(g) = \det(-\beta)$. Hence with no ambiguity about sign

$$\begin{aligned} A(g)^2 &= \gamma(\det(-\beta), \tfrac{1}{2}\psi)^{-2}\,\gamma(\tfrac{1}{2}\psi)^{-2n}\,|\det(-\beta)|^{-1}\det(1 + i\beta^{-1}\alpha)^{-1} \\ &= (-1, \det(-\beta))_{\mathbf{R}} \left(e^{\frac{\pi i}{4}}\right)^{-2n} \operatorname{sign}(\det(-\beta))\det(-\beta - i\alpha)^{-1} \\ &= (i)^{-n}\det(-i1_n)^{-1}\det(\alpha - i\beta)^{-1} \\ &= \det(\alpha + i\beta), \end{aligned}$$

which is at least a good start.

Next, consider $\widetilde{K} = \{(g, \xi) \in Sp(n, \mathbf{R}) \times \mathrm{Un}(L^2(\mathbf{R}^n)) \mid g \in O(2n) \text{ and } \xi = \pm\tilde{r}(g)\}$ to be an actual subset of $Mp(n, \mathbf{R})$. Define a map $\mathcal{D} : \widetilde{K} \to L^2(\mathbf{R}^n)$ by $\mathcal{D}(g, \xi) = \xi \cdot F_0 \ (= \pm\tilde{r}(g)F_0)$. This is certainly a continuous mapping by definition of the topology on $Mp(n, \mathbf{R})$, and we know that $\mathcal{D}(g, \xi) = D(g, \xi)F_0$ for all $(g, \xi) \in \pi^{-1}(\Omega_n) \cap \widetilde{K}$, where $D(g, \xi)$ is a scalar in $\mathbf{T}$ satisfying $D(g, \xi)^2 = \det(g)$ by the above. It is not hard then to show that in fact D has an extension to all of $\widetilde{K}$ such that $\mathcal{D}(x) = D(x)F_0$ for all $x \in \widetilde{K}$. That D is a continuous homomorphism satisfying the required properties follows easily from the analogous properties of the Weil representation. □

Igusa also proves (in a slightly different form) that for each maximal compact subgroup L of $Mp(n, \mathbf{R})$, there is a unique one-dimensional subspace of $L^2(\mathbf{R}^n)$ which is L-invariant. Since all such L are conjugates of $\widetilde{K}$, these subspaces will be generated by translates of the vector F_0 above, which is called the **Gaussian** associated to $\widetilde{K}$. It is uniquely determined by the condition that $F_0(0) = 1$.

Returning to the dual pair $H(\mathbf{R}) \times G(\mathbf{R})$ in $Sp(mn, \mathbf{R})$, let $\overline{\omega}_v$ denote the Weil representation of $Mp(mn, \mathbf{R})$, and let $\omega_v = \overline{\omega}_v \circ \widetilde{A}_v$ be its pull back to the group $Mp(n, \mathbf{R}) \cong \widetilde{G}(\mathbf{R})$ as in chapter 2. Write K and $\underline{K}$ for the standard maximal compact subgroups of $G(\mathbf{R})$ and $Sp(mn, \mathbf{R})$, respectively. We wish to find an eigenvector for the representation $\omega_v|_{\widetilde{K}}$. The problem here is that $\widetilde{A}_v(\widetilde{K}) \not\subset \widetilde{\underline{K}}$ in general, so that we cannot use the Gaussian associated to $\overline{\omega}_v$ and $\widetilde{\underline{K}}$. To see what is happening we need a little theory, which will be stated without proof.

First of all, note that the group K is actually the centralizer of $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ in $G(\mathbf{R})$. The element J also satisfies $J^2 = -1$ and has the property that

$$[w_1, w_2]_J \stackrel{def}{=} < w_1, w_2 J >$$

is a symmetric positive-definite bilinear form on $W_{\mathbf{R}}$. Let

$$\mathcal{X}_W = \{ j \in \mathrm{Aut}_{\mathbf{R}}(W_{\mathbf{R}}) \mid j^2 = -1 \text{ and } [,]_j \text{ is positive-definite and symmetric} \}.$$

Then in fact, any element $j \in \mathcal{X}_W$ defines a subgroup $K_j \stackrel{def}{=} C_{G(\mathbf{R})}(j)$, which is also the set of elements in $G(\mathbf{R})$ which leave the form $[,]_j$ invariant. Hence each such K_j is a maximal compact subgroup of $G(\mathbf{R})$, and is conjugate to

$K_J = K$. There are bijections among the sets

$$G(\mathbf{R})/K, \qquad \mathcal{X}_W, \quad \text{and} \quad \{\text{all maximal compacts}\} \qquad \text{given by}$$

$$gK \rightleftharpoons gJg^{-1} \rightleftharpoons gKg^{-1}.$$

$\mathcal{X}_W$ is called the **symmetric space** of $(W_{\mathbf{R}}, <,>)$. There is a similar situation for the orthogonal group $H(\mathbf{R})$ of $(V_{\mathbf{R}}, (\,,))$, except that here we must define

$$\mathcal{X}_V = \{i \in \mathrm{Aut}_{\mathbf{R}}(V_{\mathbf{R}}) \mid i^2 = 1 \text{ and } (\cdot, \cdot i) \text{ is positive-definite and symmetric}\}.$$

It is easy to check that for each choice of $(i, j) \in \mathcal{X}_V \times \mathcal{X}_W$, we get an element of $\mathcal{X}_{V \otimes W}$ by taking $i \otimes j = A_v(i, j)$. In our case, we wish to find an eigenvector for K_v , so we will fix $j = J$ as above. Note that whatever i we choose, we will have

$$A_v(1_V, k)A_v(i, J)A_v(1_V, k^{-1}) = A_v(i, J) = i \otimes J$$

for all $k \in K$, and so $A_v(K) \subset \underline{K}_{i \otimes J}$, which must then be a conjugate of $\underline{K} = \underline{K}_{\underline{J}}$ (with $\underline{J} = \begin{pmatrix} 0 & 1_{mn} \\ -1_{mn} & 0 \end{pmatrix}$). The choice of the element i determines which maximal compact of $H(\mathbf{R})$ will have nice properties with respect to the Gaussian we choose below.

Although it is not necessary, to be concrete, suppose that we have fixed a basis $\{v_1, \ldots, v_m\}$ for V as in §2.1, with respect to which the inner product $(\,,)$ has matrix $Q = ((v_r, v_s))$. Fix some $i \in \mathcal{X}_V$, and let it have matrix I .

Then with respect to the $\{e_{rs}, e^*_{rs}\}$ basis of $\mathbf{W}_v$ described in §2.1, $A_v(I,J)$ has matrix

$$\begin{pmatrix} 0 & QI \otimes 1 \\ (QI)^{-1} \otimes (-1) & 0 \end{pmatrix}.$$

The matrix QI is positive definite, and so we may find a matrix $d \in GL(n, \mathbf{R})$ with $QI = d^t d$, so that

$$A_v(I,J) = m(d)\,\underline{J}\,m(d^{-1}).$$

Now we see that if the Gaussian associated to $\underline{K}$ is given by

$$F_{\underline{J}}(x) = \exp(-\pi < x, x\underline{J} >),$$

then the Gaussian associated to $m(d)\,\underline{K}\,m(d^{-1})$ is just a multiple of $\tilde{r}(m(d),1)F_{\underline{J}}$, which yields

$$F_{(I,J)}(x) \stackrel{def}{=} \exp(-\pi < x\,m(d), x\,m(d)\underline{J} >)$$

in the $L^2(\mathbf{X}_v)$ model. Adapting this to the $L^2(V_v^n)$ model, we write

$$V_v^n \xrightarrow{\sim} \mathbf{X}_v$$

$$x = (x_1, \ldots, x_n) \mapsto \sum_{r=1}^{n} x_r \otimes e_r$$

and see that this reduces to

$$F_{(I,J)}(x) = \exp(-\pi \operatorname{tr}(x, xI)),$$

where (x, xI) represents the $m \times m$ matrix with entries $(x_r, x_s I)$. Changing notation, let φ_v^o denote the Gaussian associated to I and J, which we assume to be fixed. We then know that $\omega_v(k)\varphi_v^o$ is a multiple of φ_v^o for all $k \in \widetilde{K}$. This multiple may be determined by writing $\widetilde{A}_v(k) = [m(d),1]\ \underline{k}\ [m(d),1]^{-1}$ for $\underline{k} \in \underline{\widetilde{K}}$, and noting that if $\omega_v(k)\varphi_v^o = c(k)\cdot\varphi_v^o$, then on the one hand,

$$\begin{aligned}
&\overline{\omega}_v([m(d),1]\ \underline{k}\ [m(d),1]^{-1})\ \overline{\omega}_v([m(d),1])F_0 \\
&= \det{}^{\frac{1}{2}}(\underline{k})\ \overline{\omega}_v(m(d))F_0 \\
&= c(k)\ \overline{\omega}_v(m(d))F_0.
\end{aligned}$$

On the other hand, $c(k)$ is a character of $\widetilde{K}_v$, and hence $c(k) = [\det^{\frac{1}{2}}(k)]^r$ for some $r \in \mathbf{Z}$. By diagonalizing Q and I and equating $\det(\underline{k})$ and $\det(k)^r$, it is easy to show that $r = p - q$, where $(V_v, (\,,\,))$ has signature (p,q). Thus we have proven:

Proposition 7.2.2. *For any element $I \in \mathcal{X}_W$, the Gaussian $\varphi_v^o \in L^2(V_v^n)$ associated to $I \otimes J$ is given by $\varphi_v^o(x) = \exp(-\pi\ tr(x, xI))$, and satisfies*

$$\omega_v(k)\varphi_v^o = [\det{}^{\frac{1}{2}}(k)]^{p-q}\varphi_v^o,$$

where (p,q) is the signature of $(V_v, (\,,\,))$. It is also clear from the formula for φ_v^o that if L is the maximal compact subgroup of H_v preserving the form $(\cdot, \cdot I)$, then φ_v^o is fixed by L.

Whenever $t \in \frac{1}{2} + \mathbf{Z}$, we will write $\det^t(k)$ to mean $[\det^{\frac{1}{2}}(k)]^{2t}$ for $k \in \widetilde{K}_v$.

§7.3 Bounding the order of poles. Note now that $\widetilde{P}_v \cap \widetilde{K}_v = \{[m(\alpha), \epsilon] \mid \alpha \in O(n, \mathbf{R})\}$ is isomorphic to a subgroup $\widetilde{O}(n)$ of $\widetilde{U}(n)$ lying over $O(n,\mathbf{R}) \subset$

$U(n)$. It may be seen either directly, or by our work with Gaussians, that the character χ_v of $\widetilde{M_v}$ restricts to $\det^{\frac{p-q}{2}}$ on $\widetilde{P}_v \cap \widetilde{K}_v = \widetilde{M_v} \cap \widetilde{K}_v$. For now, fix $l = \frac{p-q}{2} \in \frac{1}{2} + \mathbf{Z}$. Then the function $k \mapsto \det^l(k)$ on $\widetilde{K}_v$ lies in the space $\mathrm{Ind}_{\widetilde{P}_v \cap \widetilde{K}_v}^{\widetilde{K}_v}(\chi_v|_{\widetilde{P}_v \cap \widetilde{K}_v}) \cong \mathrm{Ind}_{\widetilde{O}(n)}^{\widetilde{U}(n)}(\det^l|_{\widetilde{O}(n)})$. Hence, we may define a standard section $\Phi_v^l(s) \in I_v(s)$ via

$$\Phi_v^l(k) = \det^l(k),$$

identifying $K_v = U(n)$. This is a Weil-Siegel section, since

$$\Phi_v^l(g,s) = \Phi_v(g,s,\varphi_v^o),$$

and as in [K-R1], we obtain:

PROPOSITION 7.3.1. *For any $\widetilde{K}_v$ -finite Weil-Siegel section Φ_v , we have*

$$\min_{k \in \widetilde{K}_v} \operatorname*{ord}_{s_o(m,n)} M_v(s)\Phi_v(k,s) \geq \operatorname*{ord}_{s_o(m,n)} M_v(s)\Phi_v^l(1,s).$$

The proof is the same as that in [K-R1], with obvious modifications. One point worth mentioning is that

$$\mathrm{Ind}_{\widetilde{O}(n)}^{\widetilde{U}(n)}(\det^l|_{\widetilde{O}(n)}) \cong \det^l \otimes \mathrm{Ind}_{\widetilde{O}(n)}^{\widetilde{U}(n)}(1)$$

as $\widetilde{U}(n)$ -modules, and thus the theorem in [B-G-G] which asserts multiplicity-one for $\mathrm{Ind}_{O(n)}^{U(n)}(1)$ still holds.

In any case, we are reduced to looking at $M_v(s)\Phi_v^l(1,s)$. Since

$$M_v(s)\Phi_v^l(gk,s) = \det^l(k)M_v(s)\Phi_v^l(g,s)$$

for all $g \in \widetilde{G}_v$ and $k \in \widetilde{K}_v$, we see that

$$M_v(s)\Phi_v^l(s) = d_{n,v}(s,l)\Phi_v^l(-s)$$

for some meromorphic function $d_{n,v}(s,l)$, as in [K-R1]. The following lemma gives the value of this function.

LEMMA 7.3.2. *For $l = \frac{p-q}{2}$, we have*

$$d_{n,v}(s,l) = \left(\sqrt{(-1)^n}e^{-\frac{n\pi i}{4}}\right)^{q-p} 2^{n(1-s)}\pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n(s)}{\Gamma_n\left(\frac{s+\rho_n+l}{2}\right)\Gamma_n\left(\frac{s+\rho_n-l}{2}\right)}$$

where

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}}\Gamma(s)\Gamma(s-\tfrac{1}{2})\ldots\Gamma(s-\tfrac{n-1}{2}).$$

Proof. We will solve for $M_v(s)\Phi_v^l([w,1]^{-1},s) = d_{n,v}(s,l)\det^{-l}([w,1])$. During the proof, let

$$S = \mathrm{Sym}_n(\mathbf{R}) \supset S^+ = \{x \in S \mid x > 0 \text{ (} x \text{ is positive definite)}\}.$$

We must simplify

$$\int_S \Phi_v^l([w,1]\,[n(-x),1]\,[w,1]^{-1},s)\,dx.$$

First of all, note that $x \in S \Longrightarrow 1+x^2 \in S^+ \Longrightarrow (1+x^2)^{-1} \in S^+$, and so we may choose a matrix $a = a(x) \in S^+$ such that $a\,{}^ta = (1+x^2)^{-1}$. This allows us to write

$$\begin{aligned}\begin{pmatrix} a^{-1} & -{}^ta\cdot x \\ 0 & {}^ta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} &= \begin{pmatrix} a^{-1} - {}^ta\cdot x^2 & -{}^ta\cdot x \\ {}^ta\cdot x & {}^ta \end{pmatrix} \\ &= \begin{pmatrix} {}^ta & -{}^ta\cdot x \\ {}^ta\cdot x & {}^ta \end{pmatrix} \in K_v\end{aligned}$$

so that

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} a(x) & * \\ 0 & \check{a}(x) \end{pmatrix} \begin{pmatrix} {}^t a & -{}^t a \cdot x \\ {}^t a \cdot x & {}^t a \end{pmatrix} \stackrel{def}{=} p(x)k(x) \in P_v \cdot K_v.$$

Hence, writing

$$\bar{n}(x) \stackrel{def}{=} [w,1]\,[n(-x),1]\,[w,1]^{-1} = [p(x),1]\,[k(x),\epsilon(x)], \tag{7.3.1}$$

we have

$$\Phi_v^l(\bar{n}(x),s) = |a(x)|^{s+\rho_n} \chi_v(a(x),1)\ \det{}^l(k(x),\epsilon(x)).$$

Now we compute that

$$|a(x)|^{s+\rho_n} = (\det a(x))^{s+\rho_n} = \left[\det(1+x^2)^{-\frac{1}{2}}\right]^{s+\rho_n} = \det(1+x^2)^{-(\frac{s+\rho_n}{2})},$$

where $a \in S^+$ implies that all determinants are positive real numbers, and there is no ambiguity about signs of the square roots. Also, we recall that

$$\chi_v(a(x),1) = \frac{(a, (-1)^{\frac{m-1}{2}} \det(V))_v}{\gamma_v(a, \frac{1}{2}\psi)} = 1,$$

again since $\det(a) > 0$. This leaves us with

$$\Phi_v^l(\bar{n}(x),s) = \det(1+x^2)^{-(\frac{s+\rho_n}{2})} \det{}^l(k(x),\epsilon(x)),$$

where the factor $\epsilon(x)$ is defined by equation (7.3.1).

Next, we extend the function $\mathrm{Log} \circ \det : S^+ \to \mathbf{R}$ to an open convex neighborhood of S^+ in $S(\mathbf{C}) \stackrel{def}{=} \mathrm{Sym}_n(\mathbf{C})$ as follows. The set $-i\mathsf{H}_n =$

$\{y + ix \in S(\mathbf{C}) \mid y > 0\}$ is open, convex, and contains S^+, so by analytic continuation we can define a function

$$\operatorname{logdet} : -i\mathsf{H}_n \to \mathbf{C}$$

so that it equals $\operatorname{Log}(\det(y))$ for $y \in S^+$. Now use this function to define a branch of $z \mapsto \det(z)^c$ on $-i\mathsf{H}_n$ for any $c \in \mathbf{C}$, so that $\det(1_n)^c = 1$. Define

$$f(x) = \det(1 + x^2)^{-(\frac{s+\rho_n}{2})} \det(a(x))^l \det(1 - ix)^l$$

using the branch constructed above. Considering that $x \mapsto \Phi_v^l(\bar{n}(x), s)$ is a smooth function of $x \in S$ for fixed s, and setting $F(x) = \Phi_v^l(\bar{n}(x), s) \cdot f(x)^{-1}$, it is easy to see that $F(x)^2 = 1$ by Proposition 7.2.1. Thus $F : S \to \{\pm 1\}$ is a continuous function on a connected set, forcing $F(x) = 1$. This proves that

$$\begin{aligned}\Phi_v^l(\bar{n}(x), s) &= \det(1 + x^2)^{-(\frac{s+\rho_n}{2})} \det(a(x))^l \det(1 - ix)^l \\ &= \det(1 + x^2)^{-(\frac{s+\rho_n+l}{2})} \det(1 - ix)^l,\end{aligned}$$

where the only term requiring care in its definition is $\det(1 - ix)^l = \exp(l \cdot \operatorname{logdet}(1 - ix))$.

It is then easy to show that

$$\operatorname{Log}(\det(1 + x^2)) = \operatorname{logdet}(1 + ix) + \operatorname{logdet}(1 - ix),$$

which then reduces our problem to one of solving for

$$(7.3.2) \qquad \mathcal{I}(\alpha, \beta) \stackrel{def}{=} \int_S \det(1 - ix)^{-\alpha} \det(1 + ix)^{-\beta}\, dx,$$

where $\alpha = \frac{s+\rho_n-l}{2}$, $\beta = \frac{s+\rho_n+l}{2}$, and $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta) \gg 0$. Some tools for simplifying this type of integral are given by Shimura in [Sh], and the proof from this point on is due to Kudla [K2].

We begin with the classical formula

$$\int_{S^+} e^{-\mathrm{tr}(zx)} \det(z)^{s-\rho_n}\, dz = \Gamma_n(s) \det(x)^{-s}$$

valid initially for $\mathrm{Re}(s) > \rho_n - 1$ and $x \in S^+$, and then for $x \in -i\mathsf{H}_n$ by continuation. For this, see (1.16), p.273 [Sh]. This yields

$$\det(1-ix)^{-\alpha} = \Gamma_n(\alpha)^{-1}(2\pi)^{n\alpha} \int_{S^+} e^{-2\pi\,\mathrm{tr}(t(1-ix))} \det(t)^{\alpha-\rho_n}\, dt,$$

which we may substitute into (7.3.2) to obtain

$$\begin{aligned}\mathcal{I}(\alpha,\beta) = \Gamma_n(\alpha)^{-1}(2\pi)^{n\alpha}\cdot \\ \int_{S^+} e^{-2\pi\mathrm{tr}(t)} \det(t)^{\alpha-\rho_n} \left[\int_S e(\mathrm{tr}(tx)) \det(1+ix)^{-\beta}\, dx\right] dt,\end{aligned}$$

writing $e(d) \overset{def}{=} \exp(2\pi i d)$. Shimura also proves ((1.23) p.274) that for any $b \in S^+$,

$$\Gamma_n(s)\, 2^{\frac{n(n-1)}{2}} \int_S e(\mathrm{tr}(tx)) \det(b+2\pi i x)^{-s}\, dx = \begin{cases} e^{-\mathrm{tr}(tb)} \det(t)^{s-\rho_n} & \text{if } t \in S^+ \\ 0 & \text{if } t \notin S^+. \end{cases}$$

Using this, for $t \in S^+$

$$\int_S e(\mathrm{tr}(tx)) \det(1+ix)^{-\beta}\, dx = \Gamma_n(\beta)^{-1}\, 2^{-\frac{n(n-1)}{2}}\, (2\pi)^{n\beta}\, e^{-\mathrm{tr}(2\pi t)} \det(t)^{\beta-\rho_n}$$

and so, setting $G_n(s) = \Gamma_n(s)\,(2\pi)^{-ns}$,

$$\begin{aligned}
\mathcal{I}(\alpha,\beta) &= G_n(\alpha)^{-1}G_n(\beta)^{-1}\,2^{-\frac{n(n-1)}{2}} \int\limits_{S+} e^{-\mathrm{tr}(4\pi t)} \det{}^{(\alpha+\beta-\rho_n)-\rho_n}(t)\,dt \\
&= G_n(\alpha)^{-1}G_n(\beta)^{-1}\,2^{-\frac{n(n-1)}{2}}\,\Gamma_n(\alpha+\beta-\rho_n)\,(4\pi)^{-n(\alpha+\beta-\rho_n)} \\
&= \frac{G_n(\alpha+\beta-\rho_n)}{G_n(\alpha)G_n(\beta)}\,2^{-n(\alpha+\beta-1)} \\
&= 2^{n(1-s)}\,\pi^{\frac{n(n+1)}{2}}\,\frac{\Gamma_n(s)}{\Gamma_n\left(\frac{s+\rho_n+l}{2}\right)\Gamma_n\left(\frac{s+\rho_n-l}{2}\right)} \\
&= d_{n,v}(s,l)\det{}^{-l}([w,1]).
\end{aligned}$$

The result is completed by noting that if $F_0(x) = \exp(-\pi x\,{}^t x)$, $x \in \mathbf{R}$, then $r(\tau)F_0(x) = \int_{\mathbf{R}^n} F_0(y)e(y\,{}^t x)\,dy = F_0(x)$, and $r(-1_{2n})F_0(x) = F_0(-x) = F_0(x)$, so that

$$\begin{aligned}
\det{}^{\frac{1}{2}}([w,1]) &= m_v(-1_{2n})\,m_v(\tau) = \gamma_v((-1)^n,\tfrac{1}{2}\psi)^{-1}\cdot\gamma_v(\tfrac{1}{2}\psi)^{-n} \\
&= e^{-\frac{n\pi i}{4}}\begin{cases} 1 & \text{if } n \text{ is even} \\ i & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

by equation (7.2.1). □

We now must compute $\operatorname{ord}_{s_o(m,n)} \frac{b_v(n,s)}{a_v(n,s)} d_{n,v}(s,l)$, for which we need the following information.

Lemma 7.3.3.

(1)

$$\operatorname*{ord}_{s_o(m,n)} a_v(n,s)^{-1} = 0$$

(2)

$$\operatorname*{ord}_{s_o(m,n)} b_v(n,s) = \begin{cases} -\left[\frac{n-m}{2}\right]-1 & \textit{if } 1 \le m \le n+1, \\ 0 & \textit{if } n+1 < m. \end{cases}$$

(3)

$$\operatorname*{ord}_{s_o(m,n)} \Gamma_n(s) = \begin{cases} -\left[\frac{n}{2}\right] & \text{if } 1 \le m \le n+1, \\ \frac{m-1}{2} - n & \text{if } n+1 < m < 2n, \\ 0 & \text{if } 2n < m. \end{cases}$$

(4) *Let* $a = \frac{m}{2} \pm l$, *so that* $a = p$ *or* q . *Then*

$$\operatorname*{ord}_{s_o(m,n)} \Gamma_n\left(\frac{s+\rho_n \pm l}{2}\right)^{-1} = \begin{cases} \left[\frac{n+1-a}{2}\right] & \text{if } 0 \le a \le n+1, \\ 0 & \text{if } n+1 < a. \end{cases}$$

Proof. Recall that $\Gamma(z)$ has simple poles at $\{0,-1,-2,\dots\}$ and no zeros. Again using the notation $f(s) \equiv g(s)$ if the quotient $f(s)/g(s)$ is holomorphic non-vanishing, we have

$$a_v(n,s) \equiv \prod_{k=1}^{\left[\frac{n+1}{2}\right]} \Gamma(s-\rho_n+k) \quad \text{and} \quad b_v(n,s) \equiv \prod_{k=1}^{\left[\frac{n+1}{2}\right]} \Gamma(s+\tfrac{n+2}{2}-k).$$

(1) At $s_o = \frac{m}{2} - \frac{n+1}{2}$, $s - \rho_n + k = \frac{m}{2} - (n+1) + k \in \frac{1}{2} + \mathbf{Z}$, which shows that a_v has no poles.

(2) Here $s_o + \frac{n+2}{2} - k = \frac{m+1}{2} - k$. Since m is odd, k contributes a pole $\Longleftrightarrow$ $(\frac{m+1}{2} \le k$ and $1 \le k \le [\frac{n+1}{2}]) \Longleftrightarrow \frac{m+1}{2} \le k \le [\frac{n+1}{2}]$. It is easily checked that

$$\begin{aligned} &\#\{k \in \mathbf{Z} \mid \tfrac{m+1}{2} \le k \le [\tfrac{n+1}{2}]\} \\ &= \#\{r \in \mathbf{Z} \mid 0 \le r \le [\tfrac{n+1}{2} - \tfrac{m+1}{2}]\} = \begin{cases} 0 & \text{if } n+1 < m \\ [\frac{n-m}{2}]+1 & \text{if } 1 \le m \le n+1. \end{cases} \end{aligned}$$

(3) $\Gamma_n(s) \equiv \prod_{k=0}^{n-1} \Gamma(s - \frac{k}{2})$ and $s_o - \frac{k}{2} = \frac{m-(n+1+k)}{2}$. The number of k which contribute poles is

$$N = \#\{k \in \mathbf{Z} \mid 0 \le k \le n-1 \text{ and } m-(n+1) \le k \text{ and } n \equiv k \pmod 2\}.$$

If $2n < m$, then $n-1 < m-(n+1)$, and so $N = 0$. If $n+1 < m < 2n$, then $0 < m-(n+1) < n-1$, and so the number of k with $m-(n+1) \leq k \leq n-1$ is $2n-m+1$, which is even. Hence $N = \frac{1}{2}(2n-m+1) = n - \frac{m-1}{2}$. If $1 \leq m \leq n+1$, then $m-(n+1) \leq 0$, so $N = \#\{k \mid 0 \leq k \leq n-1 \text{ and } n \equiv k \pmod 2\}$, which is easily seen to be $[n/2]$.

(4) Setting $a = \frac{m}{2} \pm l$, we see that $s_o + \rho_n \pm l = \frac{m}{2} \pm l = a$, so we wish to find $\mathrm{ord}_{s=a}\, \Gamma_n(s/2)^{-1}$, where $a \geq 0$. As before, the number of terms contributing a zero is

$$
\begin{aligned}
N &= \#\{k \in \mathbf{Z} \mid \frac{a-k}{2} \leq 0 \text{ and } 0 \leq k \leq n-1 \text{ and } k \equiv a \pmod 2\} \\
&= \#\{k \in \mathbf{Z} \mid a \leq k \leq n-1 \text{ and } k \equiv a \pmod 2\} \\
&= \begin{cases} [\frac{n+1-a}{2}] & \text{if } 0 \leq a \leq n+1, \\ 0 & \text{if } n+1 < a. \end{cases}
\end{aligned}
$$

□

Now we may collect all these facts together with Proposition 7.3.1 and Lemma 7.3.2 to give us:

PROPOSITION 7.3.4. *Let v be a real place of k , and suppose that $p > q$, where (p,q) is the signature of $(V_v, (\,,))$. Then for any $\widetilde{K}_v$ -finite Weil-Siegel section Φ_v ,*

$$
\operatorname*{ord}_{s_o(m,n)} \frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s) \geq \begin{cases} 0 & \text{if } p \leq n+1, \\ [\frac{p-n}{2}] & \text{if } q \leq n+1 < p \text{ and } m < 2n, \\ [\frac{n+1-q}{2}] & \text{if } q \leq n+1 < p \text{ and } 2n < m, \\ 0 & \text{if } n+1 < q. \end{cases}
$$

If $q < p$ *(since* $m = p + q$ *is odd,* $p \neq q$ *), then the result above holds with* p *and* q *interchanged. Notice that the lower bound on the order is actually attained by* $\Phi_v = \Phi_v^l$ *, which is Weil-Siegel, and that* $\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s)$ *is always holomorphic at* $s_o(m,n)$ *.*

Proof. Consider the following ranges:

$$\text{(i)}: \quad 1 \leq m \leq n+1$$

$$\text{(ii)}: \quad p \leq n+1 < p+q = m$$

$$\text{(iii)}: \quad q \leq n+1 < p \text{ and } m < 2n$$

$$\text{(iv)}: \quad q \leq n+1 < p \text{ and } 2n < m$$

$$\text{(v)}: \quad n+1 < q.$$

In range (i), since each pair of numbers $\{n-m, n\}$ and $\{n+1-p, n+1-q\}$ contains and odd and an even number, we see that

$$\begin{aligned} &-\left[\frac{n-m}{2}\right] - 1 - \left[\frac{n}{2}\right] + \left[\frac{n+1-p}{2}\right] + \left[\frac{n+1-q}{2}\right] \\ &= -\left(\frac{2n-m-1}{2}\right) - 1 + \left(\frac{2n+1-p-q}{2}\right) = 0. \end{aligned}$$

By Lemma 7.3.3, this proves that $\frac{b_v(s)}{a_v(s)} d_v(s)$ has $\text{ord} = 0$ at s_o , and we are done by Proposition 7.3.1.

In range (ii), $q < p \leq n+1 \Longrightarrow n+1 < m \leq 2n+1$. Then

$$\begin{aligned} &0 + \frac{m-1}{2} - n + \left[\frac{n+1-p}{2}\right] + \left[\frac{n+1-q}{2}\right] \\ &= \frac{m-1}{2} - n + \left(\frac{2n+1-m}{2}\right) = 0 \end{aligned}$$

again. This covers all possibilities if $p \le n+1$.

In range (iii), note that $n+1 < m < 2n$. Then

$$0 + \frac{m-1}{2} - n + 0 + \left[\frac{n+1-q}{2}\right] = \left[\frac{p-n}{2}\right].$$

In range (iv), the order is $0+0+0+\left[\frac{n+1-q}{2}\right]$, and in range (v) all terms in Lemma 7.3.3 have order 0. $\square$

§7.4 The complex places. If $v \in \Sigma_k$ is a complex place, then the cocycle $\tilde{c}_v$ on $G_v \cong Sp(n,\mathbb{C})$ is trivial, so that Weil's r_v is an honest group representation of G_v. Hence at these places, we do not need to deal with a covering group. As in [K-R1], there is again a special vector $\varphi_v^o \in S(V_v^n)$ such that

$$\omega_v(k)\varphi_v^o = \varphi_v^o$$

for all $k \in K_v \cong Sp(n) = Sp(n,\mathbb{C}) \cap U(2n)$. Letting $\Phi_v^o(s) = \Phi_v(s,\varphi_v^o)$, this section is K_v-invariant on the right, so that the method of Gindiken-Karpelevich may be applied.

LEMMA 7.4.1.

$$M_v(s)\Phi_v^o(s) = 2^n \frac{a_v(n,s)}{b_v(n,s)} \Phi_v^o(-s).$$

Sketch of Proof. For $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \in SL(2,\mathbb{C})$, we may write

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} a & \bar{z}a \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a & -\bar{z}a \\ za & a \end{pmatrix},$$

where $a = a(z) \stackrel{def}{=} (z\bar{z}+1)^{-\frac{1}{2}} \in \mathbf{R}$, noting that the last matrix above lies in $Sp(1) = SU(2)$. Referring to the notation of §5.2, and recalling that $\chi_v \equiv 1$, we have

$$Z_{\alpha_0}(s) = \int\limits_{k_v} |a_1(z)^{\Lambda(\mu+\frac{\alpha_0}{2})}|_v \, dz,$$

where we may take $a_1(z) = \varphi_{\alpha_0}\begin{pmatrix} a(z) & \\ & a(z)^{-1} \end{pmatrix}$ by the above. As in Tate, note that $|z|_v = z\bar{z}$ and $dz = 2dx\,dy$ for $z - x + iy$. Simplifying the expression above,

$$Z_{\alpha_0}(s) = \begin{cases} \int (z\bar{z}+1)^{-(s+\rho_n+i+1)} dz & \text{if } \alpha_0 = 2x_i, \\ \int (z\bar{z}+1)^{-(2(s+\rho_n)+i+j+1)} dz & \text{if } \alpha_0 = x_i + x_j \ (i<j). \end{cases}$$

But now by an easy computation,

$$\int\limits_{\mathbf{C}} (z\bar{z}+1)^{-\alpha-1} \, dz = \frac{2\pi}{\alpha} \qquad \text{for } \operatorname{Re}(\alpha) > 0 ,$$

and so

$$\begin{aligned} Z_{\alpha_0}(s) &= \frac{c \cdot 2\pi}{2(s+\rho_n)+i+j} = c \cdot \frac{2\pi\Gamma(2(s+\rho_n)+i+j)}{\Gamma(2(s+\rho_n)+i+j+1)} \\ &= c \cdot \frac{\zeta_v(2(s+\rho_n)+i+j)}{\zeta_v(2(s+\rho_n)+i+j+1)} \end{aligned}$$

for $\alpha_0 = x_i + x_j \ (i \le j)$ and $c = \begin{cases} 2, & i = j \\ 1, & i \ne j \end{cases}$. See Definition 6.1.1. We are then finished by the proof of Theorem 5.1.1. □

The analogue of Proposition 7.3.1 holds again in this case, and so for a K_v -finite Weil-Siegel section Φ_v

$$\min_{k \in K_v} \operatorname{ord}_{s_o} M_v(s)\Phi_v(k,s) \ge \operatorname{ord}_{s_o} M_v(s)\Phi_v^o(1,s),$$

which gives us

PROPOSITION 7.4.2. *For a complex place* $v \in \Sigma_k$, *let* Φ_v *be a* K_v *-finite Weil-Siegel section. Then*

$$\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s)$$

is holomorphic at $s_o(m,n)$.

Proof. Lemma 7.4.1 and the preceding remark. □

§8.1 The intertwining operators. By equation (6.1.1) and the work of the last three chapters, we have compiled fairly complete information about the order of the poles of the global intertwining operator

$$M(s) : I(s)_{\widetilde{K}} \longrightarrow I(-s)_{\widetilde{K}}$$

when applied to Weil-Siegel sections $\Phi \in I_{\widetilde{K}}$. This will be analysed in chapter 9.

Now we turn our attention to the middle terms in the sum (4.4.1). We must study the analytic properties of the terms

$$E^r_{\widetilde{P}_n}(g, s, \Phi) = \sum_{\gamma \in Q_r \backslash M_n} \Phi_r(\gamma g, s)$$

where $1 \leq r \leq n-1$ and

$$\Phi_r(g, s) = \int_{N'_{n-r}(\mathbb{A})} \Phi(w_r n g, s)\, dn.$$

First of all, consider the standard parabolic subgroup given by $P = Q_r \cdot N_n$, and notice that Φ_r satisfies:

LEMMA 8.1.1. *For* $n \in N_n(\mathbb{A})$ *and* $m = \left[m\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix}, \epsilon\right] \in \widetilde{Q}_r(\mathbb{A})$,

$$\Phi_r(nmg, s) = \chi(m)\ |a_1|^{s+\rho_n}\ |a_2|^{-s-\rho_n+r+1}\ \Phi_r(g, s).$$

Let $I_r(s)$ be the space of smooth functions on $\widetilde{G}(\mathbb{A})$ satisfying the equality of the lemma. This is actually a $\widetilde{G}(\mathbb{A})$ -space induced from the parabolic

$\widetilde{P}(\mathbb{A})$. As with the operator $M(s)$ studied previously, we may define a $\widetilde{G}(\mathbb{A})$ - intertwining operator $M_r(s) : I(s) \to I_r(s)$ for $\mathrm{Re}(s) > \rho_n$ by

$$M_r(s)\Phi(g,s) \stackrel{def}{=} \int_{N'_{n-r}(\mathbb{A})} \Phi(w_r n g, s)\, dn,$$

so that $M_r(s)\Phi(s) = \Phi_r(s)$.

To analyse the analytic properties of $E^r_{\widetilde{P}_n}(s)$, we first consider the properties of the operator $M_r(s)$ above, which will be obtained easily from those of $M(s)$. We will then find the poles introduced by the Eisenstein series on $\widetilde{M}_n(\mathbb{A})$. It will be possible to derive information on this last item by tensoring $E^r_{\widetilde{P}_n}(m,s)$ with a character, which will allow it to be considered as an Eisenstein series on $M_n(\mathbb{A}) \cong GL(n,\mathbb{A})$. Such series are studied carefully in section 5 of [K-R1].

As explained in [K-R1], if we consider a $\widetilde{K}$ -finite Weil-Siegel section Φ , and write

$$\rho(k)\Phi(s) = \sum_j c_j(k)\Phi^j(s)$$

for a finite number of $\widetilde{K}$ -finite standard Weil-Siegel sections Φ^j (ρ represents right action by $\widetilde{K}$), then the analytic properties of the *sections* $M_r(s)\Phi(s)$ are determined by those of the *functions* $M_r(s)\Phi^j(1,s)$. For the rest of this section, write $G^r(\mathbb{A}) = Sp(r,\mathbb{A})$, and add superscripts r and n to indicate objects associated to $G^r(\mathbb{A})$ or $G^n(\mathbb{A})$. The map

$$i_o : G^r(\mathbb{A}) \longrightarrow G^n(\mathbb{A})$$
$$i_o \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1_{n-r} & & & \\ & a & & b \\ & & 1_{n-r} & \\ & c & & d \end{pmatrix}$$

described in section 4.4 of [K-R1] admits an extension to $G^r(\mathbb{A})$.

LEMMA 8.1.2. *The map* $i : \widetilde{G}^r(\mathbb{A}) \to \widetilde{G}^n(\mathbb{A})$ *defined by* $i([g, \epsilon]) = [i_o(g), \epsilon]$ *is an injective homomorphism. It induces* $i^* : I^n(s) \to I^r(s')$, *where* $s' = s + \rho_n - \rho_r$. *The corresponding local statements also hold.*

Proof. We must show that $\beta^r(g_1, g_2) = \beta^n(i_o(g_1), i_o(g_2))$ for all $g_1, g_2 \in G^r(\mathbb{A})$. Omitting superscripts where no confusion will result, for any place $v \in \Sigma_k$,

$$c_v(g_1, g_2) = c_v(i_o(g_1), i_o(g_2)) \quad \text{for all } g_1, g_2 \in G^r_v$$

by Theorem 1.4.5. It is routine to check that

$$i_o(K^r_v) \subset K^n_v, \quad i_o(\Omega^r_j) \subset \Omega^n_j, \quad i_o(N^r_{r,v}) \subset N^n_{n,v}, \quad \text{and} \quad i_o(M^r_{r,v}) \subset M^n_{n,v}$$

and so, that $\beta_v(g_1, g_2) = \beta_v(i_o(g_1), i_o(g_2))$ for all places v . This gives the local result, and the global result follows by taking $\beta = \prod \beta_v$.

Next, let $n \in N^r_r(\mathbb{A})$, $m = [m(a), \epsilon] \in \widetilde{M}^r_r(\mathbb{A})$, and $g \in \widetilde{G}^r(\mathbb{A})$. Then for a section $\Phi(s) \in I^n(s)$,

$$\Phi(i(nmg), s) = \chi^n(i(m))\ |a(i(m))|^{s+\rho_n}\ \Phi(i(g), s)$$
$$= \chi(\det(a), \epsilon)\ |a|^{s+\rho_n}\ \Phi(i(g), s)$$
$$= \chi^r(m)\ |a(m)|^{s'+\rho_r}\ \Phi(i(g), s)$$

and so we may define a section $i^*\Phi$ by $(i^*\Phi)(g, s') = \Phi(i(g), s)$, where $s' = s + \rho_n - \rho_r$. $\square$

Given the above, the operators $M_r^n(s)$ and $M_r^r(s')$ are related by the following commutative diagram (see [K-R1]):

$$\begin{array}{ccc} I^n(s) & \xrightarrow{M_r^n(s)} & I_r^n(s) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ I^r(s') & \xrightarrow[M_r^r(s')]{} & I^r(-s') \end{array}.$$

Hence

$$M_r^n(s)\Phi(1, s) = M_r^r(s')(i^*\Phi)(1, s'), \tag{8.1.1}$$

where the analytic properties of the right-hand side are well-known by the work of the previous three chapters. It should also be noted that $i^* : I^n(s) \to I^r(s')$ takes Weil-Siegel sections to Weil-Siegel sections on the smaller group $\widetilde{G}^r(\mathbb{A})$, and that the special value $s = s_o(m, n)$ corresponds to $s' = s_o(m, r)$, which is the value at which we have full knowledge of $M_r^r(s')$ applied to Weil-Siegel sections. The appropriate local analogue of (8.1.1) also holds.

§8.2 Degenerate Eisenstein series. Since the previous section gives us information about the order of the term $\Phi_r(s)$ at $s_o(m,n)$, we now concentrate on the degenerate Eisenstein series

$$E^r_{\widetilde{P}_n}(g,s,\Phi) = \sum_{\gamma\in Q_r\backslash M_n} \Phi_r(\gamma g,s), \quad g\in \widetilde{G}(\mathbb{A}).$$

Notice that if $n\in N_n(\mathbb{A})$, $m\in \widetilde{M}_n(\mathbb{A})$, and $k\in\widetilde{K}$, then

$$E^r_{\widetilde{P}_n}(nmk,s,\Phi) = E^r_{\widetilde{P}_n}(m,s,\rho(k)\Phi).$$

Using the $\widetilde{K}$-finiteness of the section $\Phi(s)$ again, the analytic properties of $E^r_{\widetilde{P}_n}(s,\Phi)$ may be determined from the restriction of $E^r_{\widetilde{P}_n}(s,\Phi^j)$ to $\widetilde{M}_n(\mathbb{A})$, letting Φ^j vary over a finite collection of "nice" sections (as before).

It appears, therefore, that we need to know about Eisenstein series on $\widetilde{M}_n(\mathbb{A})\cong\widetilde{GL}(n,\mathbb{A})$ (the latter cover being defined via the Hilbert symbol). But it is easily seen that any poles or zeros of $E^r_{\widetilde{P}_n}(m,s,\Phi)$ are preserved by the functions on $M_n(\mathbb{A})\cong GL(n,\mathbb{A})$ given by

$$m\longmapsto \chi([m,1]^{-1})\, E^r_{\widetilde{P}_n}([m,1],s,\Phi), \quad m\in M_n(\mathbb{A}).$$

We develop notation for this. Let

$$\widetilde{I}_r(s) = \{\Psi : M_n(\mathbb{A})\to\mathbb{C} \text{ smooth } \mid \Psi(mm') = |a_1|^{s+\rho_n}\,|a_2|^{-s-\rho_n+r+1}\,\Psi(m')$$
$$\text{for all } m'\in M_n(\mathbb{A}) \text{ and } m = m\begin{pmatrix} a_1 & x \\ & a_2\end{pmatrix}\in Q_r(\mathbb{A})\}$$

and consider the map

$$I_r(s)\longrightarrow \widetilde{I}_r(s)$$
$$\Phi_r(s)\longmapsto\widetilde{\Phi}_r(s)$$

which is defined by (i) restricting to $\widetilde{M}_n(\mathbb{A})$, (ii) tensoring with χ^{-1} , and (iii) further restricting to $M_n(\mathbb{A}) \times \{1\} \subset \widetilde{M}_n(\mathbb{A})$. Then setting

$$\widetilde{E}^r_{\widetilde{P}_n}(m,s,\Phi) \stackrel{def}{=} \sum_{\gamma \in Q_r \backslash M_n} \widetilde{\Phi}_r(\gamma m, s), \quad m \in M_n(\mathbb{A}),$$

the following lemma holds:

LEMMA 8.2.1. *The analytic properties of* $E^r_{\widetilde{P}_n}(s,\Phi)$ *may be obtained by studying those of* $\widetilde{E}^r_{\widetilde{P}_n}(s,\Phi)$. *Furthermore, this last is an Eisenstein series on* $M_n(\mathbb{A}) \cong GL(n,\mathbb{A})$ *of exactly the type studied in section 5 of [K-R1].*

Unfortunately, the properties of these Eisenstein series are known not in terms of the *standard* sections of $\widetilde{I}_r(s)$ (those whose restriction to the standard maximal compact of $GL(n,\mathbb{A})$ is s-independent), but in terms of certain special sections $F(s) \in \widetilde{I}_r(s)$.

Specialized to our situation, we briefly describe the facts concerning these sections. Define quasi-characters of $\mathbb{A}^\times/k$ via $\mu_1 = |\,|^{s-\frac{r}{2}+\rho_n}$, $\mu_2 = |\,|^{-s+\rho_r}$, and $\omega = \mu_1\mu_2^{-1}|\,|^{\frac{n}{2}} = |\,|^{2s+n-r}$. Let $f \in \mathcal{S}(M(r,n,\mathbb{A}))$ be a Schwartz function in the usual sense. Then identifying $M_n(\mathbb{A})$ with $GL(n,\mathbb{A})$, we may define a section $F(s) \in \widetilde{I}_r(s)$ via

$$F(g,s,f) = \mu_1(g)\,|g|^{\frac{r}{2}} \int_{GL(r,\mathbb{A})} f(h^{-1}(0,1_r)g)\,\omega^{-1}(h)\,d^\times h$$

for $g \in GL(n,\mathbb{A})$ and $s > r - \frac{n}{2}$. Note the fact that our space $\widetilde{I}_r(s)$ is exactly the same as that defined in (5.29) of [K-R1], except that our μ_1 and

μ_2 involve no characters other that those of the form $||^z$. This has no effect on the results: see section 5 of [K-R1] for details. The following proposition describes the situation:

PROPOSITION 8.2.2 [K-R1]. *The series*

$$E(m,s,f) \stackrel{def}{=} \sum_{\gamma \in Q_r \backslash M_n} F(\gamma m, s, f), \quad m \in M_n(\mathbb{A}),$$

converges for $s > \frac{r}{2}$, *and has a meromorphic analytic continuation. Writing* $s' = s + \rho_n - \rho_r$, *the only possible poles occur at the points*

$$2s' = 0, 1, \ldots, r-1 \qquad \textit{(ascending poles)}$$

and

$$2s' = n, n-1, \ldots, n-r+1 \qquad \textit{(descending poles)}$$

counted with multiplicities if the two series overlap. Furthermore, suppose that $f = \otimes_v f_v$, *and that there is a place* $v \in \Sigma_k$ *such that*

$$supp(f_v) \subset \{x \in M(r,n,k_v) \mid rank(x) = r\}.$$

Then the ascending poles do not occur.

Remark. This is a straightforward combination of Proposition 5.3 and Lemma 6.3 of [K-R1]. There is nothing new to our situation, and virtually nothing to be checked.

In view of Lemma 8.2.1 and Proposition 8.2.2, the goal now is to express the section $\widetilde{\Phi}_r(s)$ as a linear combination (with meromorphic coefficients) of sections of the form $F(s, f^j)$ for $f^j \in \mathcal{S}(M(r, n, \mathbb{A}))$. First of all, it is natural to express the space $\widetilde{I}_r(s)$ as a restricted tensor product of similarly defined local spaces $\widetilde{I}_{r,v}(s)$ with respect to the spherical sections $\Psi_v^o(s) \in \widetilde{I}_{r,v}(s)$ defined for $v < \infty$ by

$$\Psi_v^o(g, s) = 1 \qquad \text{for all } g \in M_n(\mathcal{O}_v).$$

Now fix $\varphi = \otimes_v' \varphi_v \in \mathcal{S}(V(\mathbb{A})^n)_{\widetilde{K}}$, and let $S \supset S_k$ be a finite set of places such that if $v \notin S$, then $\varphi_v = \varphi_v^o$ and χ_v is unramified. Setting $\Phi(s) = \Phi(s, \varphi)$, we see that $v \notin S$ implies that $\Phi_v(s) = \Phi_v^o(s)$, and also that the following two properties hold:

(1) for $m \in M_{n,v}(\mathcal{O}_v)$,

$$\begin{aligned}\widetilde{\Phi}_{r,v}^o(m, s) &= \chi_v([m, 1])^{-1} \, M_{r,v}(s)\Phi_v^o([m, 1], s) \\ &= M_{r,v}(s)\Phi_v^o(1, s) = \widetilde{\Phi}_{r,v}^o(1, s), \quad \text{and}\end{aligned}$$

(2)

$$\begin{aligned}\widetilde{\Phi}_{r,v}^o(1, s) &= M_{r,v}^n(s)\Phi_v^{o,n}(1, s) = M_{r,v}^r(s')\,(i^*\Phi_v^{o,n})\,(1, s') \\ &= M_{r,v}^r(s')\Phi_v^{o,r}(1, s') = \frac{a_v(r, s')}{b_v(r, s')}.\end{aligned}$$

This last is by Theorem 5.1.1 and equation (8.1.1). These two facts imply that

$$\widetilde{\Phi}_{r,v}^o(s) = \frac{a_v(r, s')}{b_v(r, s')}\Psi_v^o(s).$$

On the other hand, any $f = \otimes_v f_v \in \mathcal{S}(M(r,n,\mathbb{A}))$ must have $f_v = f_v^o$ for almost all v, where $f_v^o = \mathrm{Char}(M(r,n,\mathcal{O}_v))$ for $v < \infty$. Let

$$F_v^o(s) \stackrel{def}{=} F_v(s, f_v^o) \in \widetilde{I}_{r,v}(s).$$

Then a trivial modification of Lemma 5.4 of [K-R1] shows that

$$F_v^o(s) = c_v(r,s)\Psi_v^o(s),$$

where

(8.2.1)
$$c_v(r,s) = \prod_{k=0}^{r-1} \zeta_v(2s+n-r-k).$$

Hence we see that

(8.2.2)
$$\widetilde{\Phi}_r(s) = \frac{1}{c(r,s)} \frac{a(r,s')}{b(r,s')} \left(\bigotimes_{v\in S} c_v(r,s) \frac{b_v(r,s')}{a_v(r,s')} \widetilde{\Phi}_{r,v}(s) \right) \otimes F^{S,o}(s)$$

where a, b, and c are the products of all the local factors, and $F^{S,o}(s) = \otimes_{v \notin S} F_v^o(s)$.

So it remains to express a finite number of factors

$$c_v(r,s) \frac{b_v(r,s')}{a_v(r,s')} \widetilde{\Phi}_{r,v}(s), \qquad v \in S$$

in terms of sections $F(s, f^j) \in \widetilde{I}_{r,v}(s)$. This is done by a straightforward modification of Propositions 5.7, 5.10, and 5.11 of [K-R1]. We will limit ourselves to stating the collected results.

PROPOSITION 8.2.3. *For each place $v \in S$, there exists a finite collection of functions $\{f_v^j\} \subset \mathcal{S}(M(r,n,k_v))$, and corresponding meromorphic functions $\beta_v^j(s)$ which are holomorphic at $s_o(m,n)$, such that*

$$c_v(r,s)\frac{b_v(r,s')}{a_v(r,s')}\widetilde{\Phi}_{r,v}(s) = \sum_j \beta_v^j(s)F_v(s,f^j).$$

We have the following additional facts:

(1) *if v is a finite place and $\frac{b_v(r,s')}{a_v(r,s')}M_{r,v}(s)\Phi_v(s)$ has a zero at $s_o(m,n)$, then either $\beta_v^j(s_o)=0$ for all j (this will always be the case if $m < r+1$ or $m > 2r$), or each f_v^j is supported in the matrices of rank r in $M(r,n,k_v)$ (c.f. Proposition 8.2.2).*

(2) *If v is a real place, then each β_v^j may be written as*

$$\beta_v^j(s) = \left(d_{r,v}(s',l)\frac{b_v(r,s')}{a_v(r,s')}\right)\beta_v^{j,o}(s)$$

where each $\beta_v^{j,o}(s)$ is holomorphic at $s_o(m,n)$. The notation is as in §7.3, and the order at s_o of the expression in parentheses is given by Proposition 7.3.4.

If v is a complex place, then nothing further may be said.

§**9.1 The global intertwining operator.** Finally, we collect all the information in chapters 5 through 8 together and find lower bounds for the order at $s_o(m,n)$ of each summand in the expression for the constant term of $E(g,s,\Phi)$:

$$E_{\widetilde{P}_n}(g,s,\Phi) = \sum_{r=0}^{n} E^r_{\widetilde{P}_n}(g,s,\Phi)$$

The bookkeeping here is much easier than in [K-R1] because our $a(n,s)$ and $b(n,s)$ involve only products of ordinary zeta functions, and not zeta functions depending on a character, as in that paper. For the remainder of this chapter, fix a function $\varphi = \otimes_v \varphi_v \in \mathcal{S}(V(\mathbb{A})^n)_{\widetilde{K}}$, and let $\Phi(s) = \Phi(s,\varphi)$.

We begin by solving for the order at s_o of the factor

(9.1.1)
$$E^n_{\widetilde{P}_n}(s,\Phi) = \Phi_n(s) = M(s)\Phi(s) = \frac{a(n,s)}{b(n,s)}\left(\bigotimes_{s\in S}\frac{b_v(n,s)}{a_v(n,s)}M_v(s)\Phi_v(s)\right)\otimes\Phi^o_S(-s)$$

(see §4.4 and equation (6.1.1)). While we have investigated the term in parentheses completely, we have yet to write down the contribution of the factor $\frac{a(n,s)}{b(n,s)}$.

LEMMA 9.1.1.

$$\operatorname*{ord}_{s=s_o}\frac{a(n,s)}{b(n,s)} = \begin{cases} +1 & \text{if } 1\le m < n+1, \\ 0 & \text{if } m = n+1, \\ -1 & \text{if } n+1 < m \le 2n+1, \\ 0 & \text{if } 2n+3 \le m. \end{cases}$$

Sketch of Proof. Note first that the global zeta function $\zeta(s)$ defined in Definition 6.1.1 has no zeros at integer points, and its only poles are simple poles at $s = 0$ and 1. Hence $\zeta(2s + l)$ will have no zeros at half-integer points for any $l \in \mathbf{Z}$. The rest is a simple counting argument using the definitions of a and b in Theorem 5.1.1. □

It was proven in Corollary 6.6.3 and Propositions 7.3.4 and 7.4.2 that the term

$$\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s)$$

is at least holomorphic at s_o for all $v \in S$. Hence by the lemma above, it might appear that $\Phi_n(s)$ has a pole when m and n fall in the range $n + 1 < m \leq 2n + 1$. To show that this is not the case, we must use the fact that $(V(k), (\,,\,))$ is anisotropic. Recall the following standard facts about quadratics modules.

LEMMA 9.1.2. *Let $(V(k), q)$ be a non-degenerate quadratic module of dimension m over a global number field k. (Here $q(x) = (x, x)$ is a quadratic form.) Then*

(1) *(Hasse-Minkowski) $V(k)$ is isotropic (represents 0 non-trivially) if and only if $V_v = V(k_v)$ is isotropic for every place $v \in \Sigma_k$.*

(2) *Suppose that $m = 3$ and that V_v is isotropic for all places except at most one. Then all V_v are isotropic, and so $V(k)$ is isotropic.*

(3) *Suppose that $m \geq 5$. Then $V(k)$ is isotropic if and only if V_v is*

isotropic for all places v such that $k_v \cong \mathbf{R}$.

For proofs of these facts, see Serre [Se], or any reference on quadratic forms. We now consider two cases:

Case I: $m > 1$ and there exists a real place v_o for which V_{v_o} is anisotropic.

Case II: Either m=1 (so that V_v is anisotropic at *every* place), or $m > 1$ and there exists no anisotropic real place. If $m > 1$, then by Lemma 9.1.2 we see that $m = 3$ and that there must be at least two non-archimedean places at which V_v is anisotropic.

Suppose now that we are in case I. Without loss of generality, assume that the signature of V_{v_o} is $(p,q) = (m,0)$ since the results of Proposition 7.3.4 were symmetric in p and q . Then by that proposition,

$$\operatorname*{ord}_{s_o} \frac{b_{v_o}(n,s)}{a_{v_o}(n,s)} M_{v_o}(s)\Phi_{v_o}(s) \geq \begin{cases} 0 & \text{if } m \leq n+1, \\ \left[\frac{m-n}{2}\right] & \text{if } n+1 < m < 2n, \\ \left[\frac{n+1}{2}\right] & \text{if } 2n < m. \end{cases}$$

Assuming only that the other $v \in S$ terms in (9.1.1) are holomorphic at $s = s_o$, we combine the above with Lemma 9.1.1, yielding

$$\operatorname*{ord}_{s_o} \Phi_n(s) \geq \begin{cases} +1 & \text{if } 1 \leq m < n+1, \\ 0 & \text{if } m = n+1, \\ \left[\frac{m-n}{2}\right] - 1 & \text{if } n+1 < m < 2n, \\ \left[\frac{n+1}{2}\right] - 1 & \text{if } m = 2n+1, \\ \left[\frac{n+1}{2}\right] & \text{if } 2n+3 \leq m. \end{cases} \tag{9.1.2}$$

We will organize this in a moment.

Suppose we are in case II with $m = 1$. Then Lemma 9.1.1 gives us $\operatorname{ord}_{s_o} \frac{a(n,s)}{b(n,s)} = +1$, and this is all we need to show that $\operatorname{ord}_{s_o} \Phi_n(s) \geq +1$.

Next, suppose that $m = 3$. If $m < n+1$, then we again obtain $\operatorname{ord}_{s_o} \Phi_n(s) \geq +1$ from the global $\frac{a(s)}{b(s)}$. If $m = 3 = n+1$, then $\operatorname{ord} \frac{a(s)}{b(s)} = 0$, and we get no assistance from Proposition 6.6.2, so the best we can do is $\operatorname{ord}_{s_o} \Phi_n(s) \geq 0$. In the range $n+1 < m \leq 2n+1$, we see that $m = 3$ only occurs when $n = 1$, and obviously $2n+1 \leq m = 3$ cannot occur at all. So if $m = 3, n = 1$, then since we have at least two non-archimedean places v_1 and v_2 for which V_{v_i} is anisotropic, we may use $l = 0$ in Proposition 6.6.2 for each of these places. This yields $h_{v_i}(m,n,0) = 0$ for $i = 1,2$, and so by Corollary 6.6.3,

$$\begin{aligned}\operatorname*{ord}_{s_o} \Phi_1(s) &\geq \operatorname*{ord}_{s_o} \left[\frac{a(1,s)}{b(1,s)} \prod_{i=1}^{2} \frac{b_{v_i}(1,s)}{a_{v_i}(1,s)} M_{v_i}(s) \Phi_{v_i}(s) \right] \\ &\geq -1+1+1 = +1\end{aligned}$$

in this case. We collect these results:

LEMMA 9.1.3.

$$\operatorname*{ord}_{s_o} \Phi_n(s) \geq \begin{cases} +1 & \text{if } 1 \leq m < n+1, \\ 0 & \text{if } n+1 \leq m \leq 2n+1 \text{ and } m = n+1,\ n+2, \text{ or } n+3, \\ & \qquad \text{except } m = 3,\ n = 1, \\ +1 & \text{if } n+1 \leq m \leq 2n+1 \text{ and } m = n+4,\ n+5, \\ & \qquad \text{or } m = 3,\ n = 1, \\ +2 & \text{if } n+1 \leq m \leq 2n+1 \text{ and otherwise}, \\ +1 & \text{if } 2n+1 < m \text{ and } n = 1 \text{ or } 2, \\ +2 & \text{if } 2n+1 < m \text{ and otherwise}. \end{cases}$$

Note that in all cases $\Phi_n(s)$ is holomorphic at $s_o(m,n)$.

§9.2 The middle terms. As noted in Lemma 8.2.1, the analytic properties of $E^r_{\widetilde{P}_n}(s)$ for $1 \le r \le n-1$ may be obtained by examining

$$\widetilde{E}^r_{\widetilde{P}_n}(m,s,\Phi) = \sum_{\gamma \in Q_r \backslash M_n} \widetilde{\Phi}_r(\gamma m, s) \quad m \in M_n(\mathbb{A}).$$

Now by equation (8.2.2) and Proposition 8.2.3, we may write

$$\widetilde{E}^r_{\widetilde{P}_n}(m,s,\Phi) = \frac{1}{c(r,s)} \frac{a(r,s')}{b(r,s')} \sum_j \beta^j(s) E(m,s,f^j) \tag{9.2.1}$$

(see also Proposition 8.2.2). Now we know that all the $\beta^j(s)$ are holomorphic at $s = s_o$, but we will need the extra information given in Proposition 8.2.3 in many cases.

First of all, note that by definition (see (8.2.1)) and a simple calculation,

$$\operatorname*{ord}_{s=s_o(m,n)} c(r,s)^{-1} = \operatorname*{ord}_{s'=s_o(m,r)} \prod_{k=0}^{r-1} \zeta(2s'-k)^{-1} = \begin{cases} 0 & \text{if } 1 \le m < r+1, \\ +1 & \text{if } m = r+1, \\ +2 & \text{if } r+1 < m < 2r, \\ +1 & \text{if } m = 2r+1, \\ 0 & \text{if } 2r+1 < m. \end{cases}$$

It is also easy to see from Proposition 8.2.2 that the Eisenstein series $E(s,f^j)$ has a possible ascending pole at s_o only if

$$r+1 \le m < 2r$$

and a possible descending pole at s_o only if

$$n+2 \le m \le n+r+1. \tag{9.2.2}$$

For convenience, we record the order of $c(r,s)^{-1}$ times a possible ascending pole:

$$\operatorname*{ord}_{s_o} \frac{(\text{asc. pole})}{c(r,s)} \ge \begin{cases} 0 & \text{if } 1 \le m \le r+1, \\ +1 & \text{if } r+1 < m \le 2r+1, \\ 0 & \text{if } 2r+1 < m \end{cases} \tag{9.2.3}$$

Now suppose we are in case I. Then by Proposition 8.2.3, every function $\beta^j(s)$ will equal

$$d_{r,v_o}(s',m)\frac{b_{v_o}(r,s')}{a_{v_o}(r,s')}\cdot\beta_*^j(s),$$

where $\beta_*^j(s)$ is holomorphic at $s_o(m,n)$, and v_o is our anisotropic real place (here assumed to be positive definite without loss). Hence in this case, $\frac{a(r,s')}{b(r,s')}\beta^j(s)$ has the same estimate on its order that we obtained for $\Phi_n(s)$ in equation (9.1.2), with r in place of n (since $s=s_o(m,n)$ corresponds to $s'=s_o(m,r)$). Combining this estimate with equation (9.2.3),

$$\operatorname*{ord}_{s_o}\frac{(\text{asc. pole})}{c(r,s)}\frac{a(r,s')}{b(r,s)}\beta^j(s)\geq\begin{cases}+1 & \text{if } 1\leq m<r+1,\\ 0 & \text{if } r+1\leq m\leq 2r+1 \text{ and } m=r+1,\\ +1 & \text{if } r+1\leq m\leq 2r+1 \text{ and } m=r+2,\\ & \qquad\qquad\qquad\qquad\qquad\qquad r+3,\\ +2 & \text{if } r+1\leq m\leq 2r+1 \text{ and } m\geq r+4,\\ +1 & \text{if } 2r+1<m \text{ and } r=1,\,2,\\ +2 & \text{if } 2r+1<m \text{ and } r\geq 3.\end{cases}$$

We write $e(m,n,r)$ for $\operatorname{ord}_{s_o}E^r_{\widetilde{P}_n}(s,\Phi)$. By equation (9.2.1), all we have left to consider is the possible descending pole from $E(s,f^j)$. Since a descending pole may occur only if $m>r+2$ ($r<n\Longrightarrow r+2<n+2\leq m$), we see from the chart above that $e(m,n,r)\geq 0$ for all m, n, and r. The only combinations which allow the possibility that $e(m,n,r)=0$ in case I are the

following:

$$m = r + 1,$$

$$m = n + 2 \text{ and } r = 1,\ 2, \text{ or } n - 1,$$

$$m = n + 3 \text{ and } r = 2. \tag{9.2.4}$$

(the only overlap being $m = n + 2$, $r = 2 = n - 1 \implies (m, n, r) = (5, 3, 2)$).

Next, suppose we are in case II with $m = 1$. Then by Lemma 9.1.1, $\operatorname{ord}_{s_o} \frac{a(r,s')}{b(r,s')} = +1$. The term $\beta^j(s)$ has no pole, and by equations (9.2.2) and (9.2.3), we hit neither poles of $E(s, f^j)$ nor zeros of $c(r, s)^{-1}$. Hence in this case, $e(1, n, r) \geq +1$.

Finally, suppose we are in case II with m=3, so that there are at least two anisotropic non-archimedean places v_1 and v_2.

(1) If $m = 3 < r + 1$ (i.e. $r > 2$), then we again have $e(3, n, r) \geq +1$ from Lemma 9.1.1 as in the $m = 1$ case above.

(2) If $m = 3 = r + 1$ (i.e. $r = 2$), then one may check that Proposition 6.6.2 fails to help. We hit no descending pole by (9.2.2), and (9.2.3) yields

$$\operatorname*{ord}_{s_o} (\text{asc. pole})\, c(r, s)^{-1} \geq 0,$$

so the best we can do is to note that $\operatorname{ord}_{s_o} \frac{a(r,s')}{b(r,s')} \beta^j(s) \geq 0 \implies e(3, n, 2) \geq 0$.

(3) If $m = 3 > r + 1$, then only $r = 1$ occurs. But now $m = 2r + 1$, and so by Proposition 6.6.2, each place v_i ($i = 1, 2$) has $h_{v_i}(m, n, 0) =$

0 , and so Proposition 8.2.3 assures us that $\beta^j(s)$ has a double zero. Equations (9.2.2) and (9.2.3) contribute another zero, while $\frac{a(r,s')}{b(r,s')}$ has a pole. So finally $e(3,n,1) \geq +2$.

Note that in all cases, we have established that $E^r_{\widetilde{P}_n}(s,\Phi)$ with $1 \leq r \leq n-1$ is holomorphic at $s_o(m,n)$.

§9.3 Diagrams. Before organizing the preceding material into a chart to show the vanishing of various terms, we must mention a factor which we have ignored until now: the matching of the central characters of the terms $E^r_{\widetilde{P}_n}(m,s,\Phi)$, considered as functions on $\widetilde{M}_n(\mathbb{A})$. It is clear that $E^r_{\widetilde{P}_n}(m,s,\Phi)$ has the same central character as $\Phi_r(m,s)$, and by Lemma 8.1.1, this is given by

$$(9.3.1) \qquad t = [m\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}, \epsilon] \longmapsto \chi(t)\ |a|^{(n-2r)(s+\rho_n)+r(r+1)}.$$

Then Lemma 3.2 of [K-R1] applies:

LEMMA 9.3.1 [K-R1]. *For $0 \leq r < r' \leq n$, the central characters of Φ_r and $\Phi_{r'}$, considered as functions on $\widetilde{M}_n(\mathbb{A})$, coincide at the critical value $s_o(m,n)$ if and only if $r + r' = m - 1$.*

In the following chart, the top row of each case lists the value of r , $0 \leq r \leq n$, while the bottom row contains either 0 or $*$ to represent the vanishing or possible non-vanishing of $E^r_{\widetilde{P}_n}(g,s,\Phi)$ at $s_o(m,n)$. Possibly non-vanishing terms $E^r_{\widetilde{P}_n}$ whose central characters match at s_o will be marked in pairs as $*'$ $*'$ and $*''$ $*''$.

Proposition 9.3.2. *The vanishing of the terms* $E^{r}_{\widetilde{P}_n}(g,s,\Phi)$ *at* $s_o(m,n)$ *is given by*

$$m=1 \qquad \begin{array}{cccc} 0 & 1 & \cdots & n \\ * & 0 & \cdots & 0 \end{array}$$

$$m=3 \quad n=1 \qquad \begin{array}{cc} 0 & 1 \\ * & 0 \end{array}$$

$$n\geq 2 \qquad \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & n \\ *' & 0 & *' & 0 & \cdots & 0 \end{array}$$

$$m=5 \quad n=1 \qquad \begin{array}{cc} 0 & 1 \\ * & 0 \end{array}$$

$$n=2 \qquad \begin{array}{ccc} 0 & 1 & 2 \\ * & 0 & 0 \end{array}$$

$$n=3 \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ * & *' & 0 & *' \end{array}$$

$$n\geq 4 \qquad \begin{array}{cccccccc} 0 & 1 & \cdots & m-2 & m-1 & m & \cdots & n \\ *' & 0 & \cdots & 0 & *' & 0 & \cdots & 0 \end{array}$$

$$m\geq 7 \quad n+3<m \qquad \begin{array}{cccc} 0 & 1 & \cdots & n \\ * & 0 & \cdots & 0 \end{array}$$

$$m=n+3 \qquad \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & n-1 & n \\ * & 0 & *' & 0 & \cdots & 0 & *' \end{array}$$

$$m=n+2 \qquad \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ * & *'' & *' & 0 & \cdots & 0 & *' & *'' \end{array}$$

$$1\leq m\leq n+1 \qquad \begin{array}{cccccccc} 0 & 1 & \cdots & m-2 & m-1 & m & \cdots & n \\ *' & 0 & \cdots & 0 & *' & 0 & \cdots & 0 \end{array}$$

Proof. All the required information for the cases $m\neq 5$ is contained in Lemma 9.1.3 and in §9.2. If $m=5$, then the cases $n=1$ and $n\geq 4$ are also finished. If $m = 5, n = 2$, then we must prove that $\Phi_n = \Phi_2 = 0$, while if $m = 5, n = 3$, we need to show that $\Phi_r = \Phi_2 = 0$.

Fixing $m = 5$, by Lemma 9.1.2 we know that there must by at least one real place v_o for which V_{v_o} is anisotropic. We claim the following: either

(1) there exists *another* anisotropic real place v_1 , or

(2) there exists some other place v_1 (finite or not) such that the dimension l_{v_1} equals 1.

We will write $\epsilon_v(Q)$ for the local Hasse invariant of the matrix Q (choosing a basis for $V(k)$, etc., as in §2.1) to avoid confusion with $h_v(m,n,l)$. Suppose now that $l_v = 2$ for all places $v \neq v_o$ (notice that $l = 0, 1$, or 2 are the only possibilities by considering a Witt decomposition of V_v). Then for $v \neq v_o$, the matrix Q is similar to

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & a & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}$$

over k_v for some $a \in k_v^\times$ by splitting off 2 hyperbolic planes. For a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_m)$, the local Hasse invariant is given by $\epsilon_v(D) = \prod_{i<j}(d_i, d_j)_v$, hence for Q we have

$$\epsilon_v(Q) = (a, -1)_v^2 (-1, -1)_v = (-1, -1)_v$$

for $v \neq v_o$ (see [Se]). Notice that since v_o is a real place at which $(p, q) = (5, 0)$ or $(0, 5)$, we have

$$\epsilon_{v_o}(Q) = (-1)^{\frac{q(q-1)}{2}} = +1$$

in either case. But now by the product formula for ϵ_v ,

$$1 = \prod_{v \in \Sigma_k} \epsilon_v(Q) = \prod_{v \neq v_o} \epsilon_v(Q) = \prod_{v \neq v_o} (-1,-1)_v$$

since Q has coefficients in k , which implies that

$$(-1,-1)_{v_o} = \prod_{v \in \Sigma_k} (-1,-1)_v = 1$$

by the product formula for the Hilbert symbol. But this is a contradiction, since $(-1,-1)_{v_o} = -1$ at a real place. Hence there exists some other place v_1 with $l_{v_1} = 0$ or 1. If $l_{v_1} = 0$, then v_1 is a real anisotropic place by Lemma 9.1.2, and so the claim is proven.

Now suppose that $m = 5$ and $n = 2$. By Lemma 9.1.1, $\mathrm{ord}_{s_o} \frac{a(n,s)}{b(n,s)} = -1$, and so we need at least two zeros coming from terms of the form

$$T_v(s) = \frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Phi_v(s)$$

in equation (9.1.1). If v is a real anisotropic place, then supposing that V_v has signature $(5,0)$, we have $\mathrm{ord}_{s_o} T_v(s) \geq [(n+1-q)/2] = [3/2] = +1$. If v is a real place with $l_v = 1$, then without loss of generality $(p,q) = (4,1)$, and so $\mathrm{ord}_{s_o} T_v(s) \geq [(n+1-q)/2] = [2/2] = +1$ again. Finally, if v is a finite place with $l_v = 1$, then $h_v(m,n,l) = 0$ by Proposition 6.6.2, and so $\mathrm{ord}_{s_o} T_v(s) \geq +1$ by Corollary 6.6.3.

Next, suppose that $m = 5, n = 3$, so that we are concerned with the $r = 2$ term. We have a possible descending pole by (9.2.2), and $\mathrm{ord}_{s_o} \frac{\text{(asc. pole)}}{c(r,s)} \geq +1$

by (9.2.3). The term $\frac{a(r,s')}{b(r,s')}$ also contributes a pole by Lemma 9.1.1. Hence we need two "local" zeros to show that the $r = 2$ term is zero. By the claim proven above, a similar analysis to that just completed shows that each place v_o and v_1 contributes a zero to the total. This finishes the proof □

COROLLARY 9.3.3. *For all* $\varphi \in \mathcal{S}(V(\mathbb{A})^n)_{\widetilde{K}}$,

(1) *the constant term* $E_{\widetilde{P}_n}(s, \varphi)$ *is holomorphic at* $s_o(m, n)$, *and*

(2) *Theorem 3.3.1 (the Weil-Siegel formula) has been proven for all cases in which only a single non-zero term appears in the chart of Proposition 9.3.2.*

Proof. See Theorem 4.3.4 and the proposition above.

§10.1 The unitary case. Notice that all the remaining cases of Theorem 3.3.1 lie in the range $m \leq n+3$. The strategy for these cases (still following [K-R1] closely) is to begin by proving the case $m = n+1$, and then use an induction on n (with the space V, and hence m, fixed) to step up and down the n range, using the central $m = n+1$ case to prove the others.

If $m = n+1$, notice that $s_o(m,n) = \frac{m}{2} - \frac{n+1}{2} = 0$, and so the action of $\widetilde{G}(\mathbb{A})$ on $I(0)$ is unitary and

$$M(0) : I(0) \longrightarrow I(0)$$

is a unitary operator satisfying $M(0)^2 = 1$ by [A] and [M]. At this point we will begin quoting results from [K-R1] freely, going into detail where the bookkeeping differs significantly, and trying to explain the outline of the proof as well.

PROPOSITION 10.1.1 [K-R1]. *If $m = n+1$ and if $\Phi(s) \in I(s)_{\widetilde{K}}$ is a Weil-Siegel section, then $M(0)\Phi(0) = \Phi(0)$.*

Remark. The proof hinges on showing that $M_v(0)\Phi_v(0) = \mu_v \cdot \Phi_v(0)$ for some constant μ_v and for all places $v \in \Sigma_k$, noting that $\mu_v = 1$ whenever $\Phi_v = \Phi_v^o$. The details of the proof carry over almost unchanged.

COROLLARY 10.1.2. *Theorem 3.3.1 is proven in all cases for which $m = n+1$.*

Proof. By Proposition 9.3.2, if $m = n+1$ then

$$E_{\widetilde{P}_n}(s,\Phi)|_{s=0} = \Phi(s) + M(0)\Phi(0) = 2\Phi(0).$$

□

§10.2 The cases $m = n+2$ and $m = n+3$. Referring back to the notation of §4.1, consider the parabolic P_1, which has a Levi factor $M_1 \cong GL(1) \times Sp(n-1)$. The idea in this section is to compute the constant term of $E(g,s,\Phi)$ with respect to the parabolic subgroup $\widetilde{P}_1(\mathbb{A}) \cap \widetilde{P}_n(\mathbb{A})$ in two ways: one may first take the constant term with respect to $\widetilde{P}_1(\mathbb{A})$ (integrating over $N_1(\mathbb{A})$) and then with respect to the maximal (Siegel) parabolic $\widetilde{P}^{n-1}_{n-1}(\mathbb{A}) \subset \widetilde{Sp}(n-1,\mathbb{A})$, or one may take the constant term with respect to $\widetilde{P}_n(\mathbb{A})$ first, and then with respect to the parabolic $\widetilde{Q}_{n-1}(\mathbb{A}) \subset \widetilde{M}_n(\mathbb{A})$. See the following diagram (the diamond of [K-R1]):

(10.2.1)

$$\widetilde{Sp}(n)$$

$$\text{(Levi of } \widetilde{P}_1\text{)} \quad \widetilde{GL(1) \times Sp(n-1)} \qquad\qquad \widetilde{GL}(n) \quad \text{(Levi of } \widetilde{P}_n\text{)}$$

$$\begin{gathered} GL(1) \times \widetilde{GL}(n-1) \\ \text{(Levi of } \widetilde{P}_1 \cap \widetilde{P}_n\text{)} \end{gathered}$$

We begin with the $\widetilde{P}_1(\mathbb{A})$ side of the diamond, reverting to the notational convention of §8.1 in which a superscript of n or $n-1$ is used to denote whether an object is associated to $Sp(n)$ or $Sp(n-1)$, respectively.

LEMMA 10.2.1 [K-R1]. *Let* $i : \widetilde{Sp}(n-1,\mathbb{A}) \to \widetilde{Sp}(n,\mathbb{A})$ *be the map given in Lemma 8.1.2, and fix a section* $\Phi \in I_{\widetilde{K}}$. *Then for* $g \in \widetilde{Sp}(n-1,\mathbb{A}) \subset \widetilde{M}_1(\mathbb{A})$,

$$(10.2.2) \qquad E_{\widetilde{P}_1}(i(g),s,\Phi) = E^{n-1}(g,s+\tfrac{1}{2},i^*\Phi) + E^{n-1}(g,s-\tfrac{1}{2},i^*(U(s)\Phi))$$

for $Re(s)$ *large, where* $U(s)$ *is the operator defined by*

$$U(s)\Phi(g,s) = \int_{N^u(\mathbb{A})} \Phi(wng,s)\,dn.$$

Here

$$N^u(\mathbb{A}) = \left\{ \begin{pmatrix} 1 & x & y & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -{}^t x & 1_{n-1} \end{pmatrix} \mid y \in \mathbb{A},\ x \in \mathbb{A}^{n-1} \right\} \subset N_B(\mathbb{A}) \hookrightarrow \widetilde{Sp}(n,\mathbb{A})$$

with

$$w = \begin{pmatrix} 0 & & 1 & \\ & 1_{n-1} & & \\ -1 & & 0 & \\ & & & 1_{n-1} \end{pmatrix} \in Sp(n,k)$$

and E^{n-1} *denotes the Eisenstein series on* $\widetilde{Sp}(n-1,\mathbb{A})$ *with respect to the parabolic* $\widetilde{P}^{n-1}_{n-1}(\mathbb{A})$.

Remark. The proof is straightforward by the comments in [K-R1], and will be omitted.

Now from [A] and [M], the functional equation of $E(g,s,\Phi)$ is given by

$$E(g,s,\Phi) = E(g,-s,M(s)\Phi(s)),$$

by which we mean that the meromorphic analytic continuation of $E(g,s,\Phi)$ equals the continuation of the series

$$\sum_{\gamma \in P_n \backslash G} M(s)\Phi(\gamma g,s).$$

For $Sp(n-1)$, this says that

$$E^{n-1}(g,s,\Psi)=E^{n-1}(g,-s,M^{n-1}(s)\Psi(s)),$$

and if $U(s)$ can be continued to a meromorphic section, then using this equality on the last term of (10.2.2) would yield:

$$E^{n-1}(g,\tfrac{1}{2}-s,M^{n-1}(s-\tfrac{1}{2})\circ i^*\circ U(s)\ \Phi) \tag{10.2.3}$$

Again, from [K-R1] we have:

LEMMA 10.2.2 [K-R1].

(1) $U(s)$ *has a meromorphic analytic continuation.*

(2) $M^{n-1}(s-\frac{1}{2})\circ i^*\circ U(s)=i^*\circ M^n(s)$.

Using these two facts, equation (10.2.2) then becomes

$$E_{\widetilde{P}_1}(i(g),s,\Phi)=E^{n-1}(g,s+\tfrac{1}{2},i^*\Phi)+E^{n-1}(g,\tfrac{1}{2}-s,i^*\circ M^n(s)\Phi). \tag{10.2.4}$$

Now if $m=n+1$, then writing $n'=n-1$, we have $m=n'+2$, and $s_o(m,n')=\frac{1}{2}$. Hence we wish to evaluate the right-hand side of (10.2.4) at $s=0$. The proof of the proposition used to simplify the right-hand side of (10.2.4) in [K-R1] depends on the following conjecture, which has not yet been proven in the metaplectic case:

CONJECTURE 10.2.3. *Fix any constant* $s_o=\frac{m}{2}-\frac{n+1}{2}\geq 0$, *and let* v *be a finite place of* k . *For a non-degenerate symmetric space* V_v *of odd dimension*

m over k_v, let $R(V_v^n)$ be the image of the map

$$\mathcal{S}(V_v^n) \longrightarrow I_v(s_o) = I(\chi_v|\,|^{s_o})$$

$$\varphi \longmapsto \{g \mapsto \omega_v(g)\varphi(0)\}$$

where $I_v(s_o)$ is the induced space on $\widetilde{Sp}(n,k_v)$ introduced in §4.4. Then

$$I_v(s_o) = \sum_{\dim(V_v)=m} R(V_v^n)$$

where V_v is allowed to vary over all spaces of dimension m. In other words, for a fixed s_o as above, the space $I_v(s_o)$ will be generated by Weil-Siegel sections coming from different symmetric spaces.

Given this conjecture, the proof of the following proposition works just as in [K-R1].

PROPOSITION 10.2.4. *Suppose that Conjecture 10.2.3 is true. If $m \geq 5$ and $m = n+1$, then for a Weil-Siegel section $\Phi \in I_{\widetilde{K}}$, we have*

$$E^{n-1}(g, \tfrac{1}{2} - s, i^* \circ M^n(s)\Phi)|_{s=0} = E^{n-1}(g, \tfrac{1}{2}, i^*\Phi),$$

for all $g \in \widetilde{Sp}(n-1, \mathbb{A})$. In particular, by equation (10.2.4),

$$E_{\widetilde{P}_1}(i(g), 0, \Phi) = 2E^{n-1}(g, \tfrac{1}{2}, i^*\Phi).$$

Using the $n' = n-1$ notation, we then see that if we take the constant term of $E_{\widetilde{P}_1}$ with respect to $P_{n'} \subset Sp(n')$, we have

$$E_{\widetilde{P}_1 \cap \widetilde{P}_n}(i(g), 0, \Phi) = 2E^{n'}_{\widetilde{P}_{n'}}(g, \tfrac{1}{2}, i^*\Phi).$$

Now taking the constant term of E along the other side of the diamond (10.2.1), by the Weil-Siegel formula for $m = n+1$ we have

$$E_{\widetilde{P}_1 \cap \widetilde{P}_n}(i(g), 0, \Phi) = 2\omega(i(g))\varphi(0)$$

since $\Phi(g, 0, \varphi)$ is invariant on the left by the unipotent radical of $\widetilde{Q}_{n-1}(\mathbb{A})$. Hence

$$E^{n'}_{\widetilde{P}_{n'}}(g, \tfrac{1}{2}, i^*\Phi) = \omega(i(g))\varphi(0)$$

for $g \in \widetilde{Sp}(n-1, \mathbb{A})$. But now, given a $\widetilde{K}^{n-1}$-finite function $\phi \in \mathcal{S}(V(\mathbb{A})^{n-1})$, we may choose $\varphi \in \mathcal{S}(V(\mathbb{A})^n)_{\widetilde{K}^n}$ such that

$$\omega^n(i(g))\varphi(0) = \omega^{n-1}(g)\phi(0)$$

by Proposition 1.3.3, and so we see that

$$E^{n'}_{\widetilde{P}_{n'}}(g, \tfrac{1}{2}, \phi) = \omega(g)\phi(0).$$

This then gives us:

PROPOSITION 10.2.5. *Given Conjecture 10.2.3, Theorem 3.3.1 holds for* $m = n+2$.

Trying to prove Weil-Siegel for $m = n' + 3$, we see that $m = 5, n' = 2$ is the first case of this sort with m odd, and it has been done by Proposition 9.3.2. As mentioned in the statement of Theorem 3.3.1, the case $(m, n) = (7, 4)$ remains unproven, so we now assume that $m > 7$ with $m = n+2$. For convenience, we still write $n' = n-1$, so that $m = n' + 3$.

Returning now to equation (10.2.4), write E_1 for $E^{n-1}(g, s+\frac{1}{2}, i^*\Phi)$ and E_1' for $E^{n-1}(g, \frac{1}{2}-s, i^* \circ M^n(s)\Phi)$ as in [K-R1]. Further taking the constant term with respect to $\widetilde{P}_{n'} \subset \widetilde{Sp}(n')$, $(E_1)_{\widetilde{P}_{n'}}$ splits up into a sum of $n'+1 = n$ terms, each having central character with repect to $\widetilde{M}_{n'}(\mathbb{A})$ given by

$$\chi|\,|^{(n-1-2r)(s'+\rho_{n-1})+r(r+1)}$$

where $s' = s+\frac{1}{2}$ and $0 \leq r \leq n-1$ (see equation (9.3.1)). At $s_o(m,n)$, the exponent becomes

$$(n-1-2r)\frac{m}{2} + r(r+1) \qquad ((E_1)_{\widetilde{P}_{n'}} \text{ exponent})$$

Similarly, $(E_1')_{\widetilde{P}_{n'}}$ is a sum of n terms with exponents

$$(n-1-2r)(\tfrac{1}{2} - s + \rho_{n-1}) + r(r+1)$$

which at $s_o(m,n)$ yields

$$(n-1-2r)(n+1-\frac{m}{2}) + r(r+1) \qquad ((E_1')_{\widetilde{P}_{n'}} \text{ exponent}).$$

On the other hand, taking the constant term along the right-hand side of the diamond, we have

$$E_{\widetilde{P}_n}(i(g), s_o(m,n), \Phi) = \omega(i(g))\varphi(0)$$

by the Weil-Siegel formula for $m = n+2$, and so again we obtain

$$E_{\widetilde{P}_1 \cap \widetilde{P}_n}(i(g), s_o(m,n), \Phi) = \omega(i(g))\varphi(0), \tag{10.2.5}$$

which has central character $\chi\,||^{(n-1)\frac{m}{2}}$ by Proposition 2.4.2. Now we will use the fact that functions on $\widetilde{M}_{n'}(\mathbb{A})$ transforming with different central characters are linearly independent (in a suitable space). By Proposition 9.3.2, we now notice that $(E_1)_{\widetilde{P}_{n'}}$ has only 3 possibly non-zero terms: these are given by $r = 0, 2$, and n. The $r = 0$ term has a central character with exponent $(n-1)\frac{m}{2}$ at $s_o(m,n)$, which matches that in equation (10.2.5) from the right-hand side of the diamond. The other two terms $r = 2$ and $r = n$ have exponent $\frac{n^2}{2} - \frac{3}{2}n + 1$. If we can show that no term in $(E_1')_{\widetilde{P}_{n'}}$ has this exponent, then the $r = 2$ and $r = n$ terms must cancel each other at the value $s = s_o(m,n)$. There will be a matching exponent from $(E_1')_{\widetilde{P}_{n'}}$ if and only if

$$\frac{n^2}{2} - \frac{3}{2}n + 1 = (n-1-2r)(n+1-\frac{m}{2}) + r(r+1)$$

has a solution for r with $0 \le r \le n-1$. This amounts to asking for an integer solution to

$$r^2 + (1-n)r + (n-1) = 0.$$

Unfortunately, for $n = 5$ (i.e. $(m,n') = (7,4)$), this equation factors as $(r-2)^2 = 0$, and so we do get matching, which accounts for the gap in our proof of the $(7,4)$ case. However, if $m > 7$, $m = n+2$, then

$$\begin{aligned}
&11 < 2n \text{ and } 5 < 9 \\
&\Longrightarrow n^2 - 8n + 16 < n^2 - 6n + 5 < n^2 - 6n + 9 \\
&\Longrightarrow n - 4 < \sqrt{(n-1)(n-5)} < n - 3 \\
&\Longrightarrow \sqrt{(n-1)(n-5)} = \sqrt{(1-n)^2 - 4(n-1)} \notin \mathbf{Q}
\end{aligned}$$

and so the quadratic equation above has no solution in integers. Hence the $r = 2$ term in $(E_1)_{\widetilde{P}_{n'}}$ cancels with the $r = n$ term at $s = s_o(m,n)$, which tells us that

$$E^{n'}_{\widetilde{P}_{n'}}(g, s_o(m,n'), i^*\Phi) = (i^*\Phi)(g, s_o(m,n')).$$

Finally, an argument like that preceding Proposition 10.2.5 gives us:

PROPOSITION 10.2.6. *Given Conjecture 10.2.3, the* $m = n+3$ *case of Theorem 3.3.1 holds for* $(m,n) \neq (7,4)$.

§10.3 The cases $3 < m < n+1$. Now we suppose that $3 < m < n+1$, so that by Proposition 9.3.2,

$$E_{\widetilde{P}_n}(g, s_o, \Phi) = \Phi(g, s_o, \Phi) + E^{(m-1)}(g, s_o, \Phi), \tag{10.3.1}$$

using parentheses around the superscript $m-1$ to indicate that $E^{(m-1)}$ is the $r = m-1$ term in the expression for the constant term. The goal here is obviously to show that

$$\Phi(g, s_o, \Phi) = E^{(m-1)}(g, s_o, \Phi) \tag{10.3.2}$$

for all $g \in \widetilde{Sp}(n, \mathbb{A})$. First of all, note that since both of these are left $N(\mathbb{A})$ - invariant, by an argument using the $\widetilde{K}$ -finiteness condition, it suffices to prove this equality for all $g \in \widetilde{M}_n(\mathbb{A}) \cong \widetilde{GL}(n, \mathbb{A})$ and all $\widetilde{K}$ -finite $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$. The plan is to prove that the constant terms of each side of equation (10.3.2), considered as functions on $\widetilde{GL}(n, \mathbb{A})$ and taken with respect to $\widetilde{Q}_{n-1}$, are equal.

Set $k = n + 1 - m > 0$, and assume that the Weil-Siegel formula has been proven for all smaller k , where we are fixing m and V and allowing n to vary. The starting point for the induction, $k = 0$, has already been proven by Corollary 10.1.2. Using the E_1 , E_1' notation of the last section, by our induction assumption, at $s_o(m, n)$ we have

$$\begin{aligned}\left((E_1)_{\widetilde{P}_{n-1}}\right)|_{s=s_0} &= 2(i^*\Phi)(g, s + \tfrac{1}{2})|_{s=s_o} \\ &= 2\Phi(i(g), s_o)\end{aligned}$$

for all $g \in \widetilde{GL}(n-1)$ (recall the shift caused by i^* from Lemma 8.1.2). Suppose for now that we have proven

LEMMA 10.3.1. *For* $5 \leq m < n + 1$, *the function*

$$E_1' = E^{n-1}(g, \tfrac{1}{2} - s, i^* M^n(s)\Phi(s))$$

has a zero at $s_o(m, n)$.

This will be the subject of the next section. Then

$$(E_{\widetilde{P}_n})_{\widetilde{Q}_{n-1}}(g, s_o, \Phi) = \left((E_1)_{\widetilde{P}_{n-1}}\right)|_{s=s_0}$$

$$= 2\Phi(i(g), s_o)$$

for all $g \in \widetilde{GL}(n-1)$ by equating the results from both sides of the diamond. On the other hand, by equation (10.3.1), $(E_{\widetilde{P}_n})_{\widetilde{Q}_{n-1}}$ also equals

$$\Phi(i(g), s_o) + (E^{(m-1)}_{\widetilde{P}_n})_{\widetilde{Q}_{n-1}}(g, s_o, \Phi),$$

and so

$$(10.3.3) \qquad E^{(m-1)}_{\widetilde{P}_n}(g, s_o, \Phi) \quad \text{and} \quad \Phi(g, s_o) \qquad g \in \widetilde{GL}(n)$$

have constant terms with respect to $\widetilde{Q}_{n-1}$ which are equal. Notice here that if $g \in \widetilde{GL}(n) = \widetilde{M}_n$, then $\Phi(g, s_o(m,n)) = \chi(g)\,|g|^{\frac{m}{2}}\,\varphi(0)$.

But now one of the terms in (10.3.3) is a multiple of a character of $\widetilde{GL}(n, \mathbb{A})$, while the other is an Eisenstein series on $\widetilde{GL}(n, \mathbb{A})$ associated to a character of a Levi factor of $\widetilde{Q}_{n-1}(\mathbb{A})$. It is a basic fact that characters are always concentrated on the Borel, while this follows for the Eisenstein series by an analogue of Proposition 4.3.2 . The analogue of Lemma 4.2.5 then implies that

$$E^{(m-1)}_{\widetilde{P}_n}(g, s_o, \Phi) = \Phi(g, s_o)$$

for all $g \in \widetilde{GL}(n)$, and so modulo Lemma 10.3.1, we have proven:

PROPOSITION 10.3.2. *Theorem 3.3.1 has been proven for* $5 \leq m < n+1$.

§10.4 The vanishing of E_1' **.** As in (7.27) of [K-R1], write

$$M^n(s)\Phi(s) = \sum_j \gamma_j(s)\Psi^j(-s),$$

where $\Psi^j(s) \in I(s)$ are $\widetilde{K}$ -finite standard sections and the $\gamma_j(s)$ are meromorphic functions. This is possible by the work of §4.4. By Lemma 9.1.3, since $\Phi_n(s) = M^n(s)\Phi(s)$ vanishes at $s_o(m,n)$ for $m < n+1$, we know that $\gamma_j(s_o) = 0$ for all j (choosing the Ψ^j to be linearly independent as functions on $\widetilde{K}$). Now note that setting $s'' = \frac{1}{2} - s$,

$$E^{n-1}(g, \tfrac{1}{2}, i^*M^n(s)\Phi(s)) = \sum_j \gamma_j(s) \cdot E^{n-1}(g, s'', i^*\Psi(s'' - \tfrac{1}{2})),$$

and so we will have proven Lemma 10.3.1 if we show that each of the Eisenstein series $E^{n-1}(g, s'', i^*\Psi(s'' - \frac{1}{2}))$ is holomorphic at $s_o'' = \frac{n+2}{2} - \frac{m}{2}$.

Following the conventions of section 8 of [K-R1], we now renormalize: where we had an n before, we now use an $n+1$, setting $k = n+2-m \geq 1$. We also let $m' = n+2+k = 2n+4-m$, so that $s_o(m', n) = \frac{n+3}{2} - \frac{m}{2}$ corresponds to the special value s_o'' above (with $n+1$ in place of n). Write $\Psi(s) \in I^n(s)$ for any of the sections given by

$$i^*\Psi^j(s - \tfrac{1}{2}).$$

PROPOSITION 10.4.1. *For* $m \geq 5$ *, and with all notation as above, the Eisenstein series*

$$E(g, s, \Psi(s)), \qquad g \in \widetilde{Sp}(n, \mathbb{A})$$

are holomorphic at the special value $s_o(m', n)$. *Hence Lemma 10.3.1 holds true by the work above.*

Proof. The proof follows the proof of Proposition 8.1 of [K-R1] in its essentials, differing mainly in the details of balancing poles and zeros of the various terms. First of all, noting that $m' = m+2k$ with $k \geq 1$, we add k hyperbolic planes to the space V , writing

$$V' = V \oplus W$$

for a split form W of dimension $2k$. Then V' is a non-degenerate symmetric space of odd dimension m' . We define a mapping

$$\mathcal{S}(V_v^n) \longrightarrow \mathcal{S}((V_v')^n) \cong \mathcal{S}(V_v^n) \otimes \mathcal{S}(W_v^n)$$

$$\varphi_v \longmapsto \varphi_v'$$

for any $v \in \Sigma_k$ with v archimedean or $v \notin S_k$ as follows. Let $\widetilde{Sp}(n, k_v)$ act on $\mathcal{S}(W_v^n)$ by composing the Weil representation of $Sp(n, k_v)$ ($\dim(W)$ is even here: see §2.2) with the projection $\widetilde{Sp}(n, k_v) \to Sp(n, k_v)$. We may then define a function φ_v^{oo} which is $\widetilde{K}_v$ -invariant (K_v -invariant under the Weil representation) by setting φ_v^{oo} equal to a Gaussian for v archimedean, or the characteristic function of a self-dual lattice for v finite, $v \notin S_k$. The mapping $\varphi_v \mapsto \varphi_v'$ is then given by $\varphi_v' = \varphi_v \otimes \varphi_v^{oo}$, and is easily seen to be $\widetilde{K}_v$ -intertwining.

We assume now that Φ is factorizable, as well as all the Ψ^j . In the proof of Proposition 7.6 of [K-R1] (Proposition 10.2.3 of this paper), there is an argument which asserts that for any place v which is archimedean, or such that χ_v is unramified, $\Psi_v(s)$ will be a Weil-Siegel section coming from $\mathcal{S}(V_v^n)$. This is equally valid in our case. Lemma 8.2 of [K-R1] then asserts that for a function $\phi_v \in \mathcal{S}(V_v^n)$, the standard section $\Phi_v(s, \phi_v) \in I_v(s)$ equals the standard section $\Phi_v(s, \phi'_v) \in I_v(s)$. Notice that this makes sense: the mapping

$$\mathcal{S}((V'_v)^n) \longrightarrow I_v(s)$$

$$\phi'_v \longmapsto \Phi_v(s, \phi'_v)$$

is well-defined, since the character χ_v given in Lemma 2.2.4 depends on V only in the factor $(-1)^{\frac{m-1}{2}} \det(V)$, and we have

$$(-1)^{\frac{m-1}{2}} \det(V) = (-1)^{\frac{m'-1}{2}} \det(V').$$

Hence we see that $\Psi_v(s) = \Phi_v(s, \phi'_v)$ for some $\phi'_v \in \mathcal{S}((V'_v)^n)$ for all places except those finite places at which χ_v is ramified. Now the proof that $E(g, s, \Psi)$ is holomorphic at $s_o(m', n)$ follows along the same lines as §9.1 and 9.2. First we must consider $\Psi_n(s) = M(s)\Psi(s)$, writing

$$\Psi_n(s) = \frac{a(n,s)}{b(n,s)} \left(\bigotimes_{s \in S} \frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Psi_v(s) \right) \otimes \Psi_S^o(-s)$$

for an appropriate finite set of places S . Notice that $m' = n + 2 + k$ and $k = n + 2 - m$ with $k \geq 1$, so that $n + 3 \leq m' \leq 2n - 1$ is easily shown. By Lemma 9.1.1, this tells us that

$$\operatorname*{ord}_{s_o(m',n)} \frac{a(n,s)}{b(n,s)} = -1.$$

Now if $v \in S$ is archimedean, then $\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Psi_v(s)$ is holomorphic at s_o, since $\Psi(s)$ is Weil-Siegel here. On the other hand, if $v \in S$ is finite, then $\mathrm{ord}_{s_o}\, b_v(n,s) = 0$ by Lemma 6.6.1, and so $\frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Psi_v(s)$ is holomorphic at *any* value of s by Theorem 6.1.2. Since we are assuming that $m \geq 5$, we know that V has an archimedean place v for which V_v is anisotropic. Assuming that V_v is positive definite, the signature of V_v' is then $(m+k,k) = (p',q')$. Proposition 7.3.4 then yields

$$\operatorname*{ord}_{s_o} \frac{b_v(n,s)}{a_v(n,s)} M_v(s)\Psi_v(s) \geq \left[\frac{p'-n}{2}\right] = \left[\frac{m+k-n}{2}\right] = \left[\frac{2}{2}\right] = +1.$$

Combining all this information, we have shown that $\Psi_n(s)$ is holomorphic at $s_o(m',n)$.

For the terms $E^r_{\tilde{P}_n}$ with $1 \leq r \leq n-1$, we refer to equation (9.2.1). We have

$$\operatorname*{ord}_{s_o} \frac{(\text{asc. pole})}{c(r,s)} \geq \begin{cases} +1 & \text{if } r+1 < m' \leq 2r+1, \\ 0 & \text{if } 2r+1 < m' \end{cases}$$

by equation (9.2.3) (these are the only possible cases), and

$$\operatorname*{ord}_{s=s_o} \frac{a(r,s)}{b(r,s)} = \begin{cases} -1 & \text{if } r+1 < m' \leq 2r+1, \\ 0 & \text{if } 2r+1 < m', \end{cases}$$

by Lemma 9.1.1, so that taken together, these two terms contribute no poles in any case. By equation (9.2.2), a descending pole may occur if

$$n+2 \leq m' \leq n+r+1$$

$$\iff k \leq r-1,$$

but since $m \geq 5$, as above we have an anisotropic archimedean place v which gives us an extra factor of

$$d_{r,v}(s',l)\frac{b_v(r,s')}{a_v(r,s')}.$$

Here as before, we take V_v to be positive definite, so that V_v' has signature $p' = m+k$, $q' = k$. The archimedean factor then has

$$\operatorname*{ord}_{s_o} \geq \begin{cases} \left[\frac{p'-r}{2}\right] = \left[\frac{n+2-r}{2}\right] & \text{if } m' < 2r \\ \left[\frac{r+1-q'}{2}\right] = \left[\frac{r+1-k}{2}\right] & \text{if } 2r < m'. \end{cases}$$

Now if $k \leq r-1$, then $2 \leq r+1-k$, implying that $\left[\frac{r+1-k}{2}\right] \geq +1$. Also, the fact that $r \leq n_1$ implies that $3 \leq n+2-r$, and so $\left[\frac{n+2-r}{2}\right] \geq +1$. In either case, we get a zero to balance the descending pole of the Eisenstein series, and so the proof is finished. □

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