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Incomplete Variational Preferences

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Abstract

We examine incomplete preference structures in a framework that allows for various relaxations of the independence axiom. We derive preference representations in terms of willingness-to-pay measures, and demonstrate how these representation can be used to determine preference incompleteness and to elicit preferences empirically.

Decision makers faced with an array of choices that are poorly understood or difficult to compare may experience "...sensations of indecision or vacillation, which we may be reluctant to identify with indifference" (Savage 1954). In such settings, a number of authors (Aumann 1962, Kannai 1963, Fishburn 1965, Bewley 1986 and 2002, Dubra, Maccheroni, and Ok 2001, Mandler 2004, Baucells and Shapley 2008, and Galabaatar and Karni 2013) have argued that it may be inappropriate to impose the completeness axiom on rational choice behavior. Relaxing the completeness axiom while maintaining the other standard axioms, including transitivity and 'irrelevance of independent alternatives', yields a decision criterion that requires one alternative to dominate another for a set of probability measures before it can be regarded as preferred (Aumann 1962, Bewley 1986 and 2002).

Even for complete preferences, it is widely understood that the independence axiom can be unduly restrictive. A large number of generalizations of expected utility theory have been proposed, and nearly all weaken the independence axiom (for example, Dekel 1986, Schmeidler 1989, Gilboa and Schmeidler 1989, Gul and Lantto 1990, Maccheroni, Marinacci, and Rustichini 2006).

Further difficulties arise when preferences are incomplete. Standard arguments for independence, such as Savage's 'sure-thing principle' require that decision makers rank, not only the choices open to them, but all possible choices. In particular, the idea of 'compound independence' inherent in the sure-thing principle requires that decision makers can evaluate a given act by considering arbitrary compound lotteries yielding the same probability distribution over outcomes. By contrast, the appeal of the transitivity axiom is, at least for the case of individual decisions, enhanced in the case of incompleteness. Because no requirement then exists to rank alternatives that may be difficult to compare, there is less risk that combining two choices using transitivity will yield a ranking incompatible with actual preferences.

Regardless of which axioms are maintained, very few attempts have been made to examine incomplete preference structures empirically. We are aware of only two such studies (Danan and Ziegelmeyer 2006 and Cettolin and Riedl 2016). We conjecture that one reason for this empirical paucity is the lack of empirically implementable tests possessing a solid theoretical foundation on which empirical analysis of incomplete preferences can be based.

This paper examines incomplete preference structures in an axiomatic framework that relaxes independence. Our particular focus is on developing representation results and predictions that can be easily tested in a laboratory or field-experiment setting. That concern causes us to depart from the usual tradition in decision theory that focuses on obtaining "utility function" representations. Instead we focus our analysis on preference representations in terms of willingness-to-pay measures that both fully characterize preferences and can be easily elicited empirically.

In what follows, we first discuss briefly the existing theoretical literature on preference incompleteness in the absence of independence and the empirical literature on examining preference incompleteness. Then we introduce the basic decision set up and introduce and characterize the willingness-to-pay measures (Proposition 1). These measures are then shown to characterize preferences exhaustively (Proposition 2). Empirical tests for and measures of the comparability of two gambles and for preference completeness are then developed (Proposition 3).

The next step is to introduce five alternatives to or weakening of independence (weak-certainty independence, radial independence, certainty independence, betweenness, and dominance independence). We then identify the structural restrictions for the willingness-to-pay measures implied by the imposition of the different alternatives to independence (Proposition 6). The consequence of these restrictions for empirical testing are then discussed and compared.

The next section studies local approximations to incomplete preference structures. In particular, it is shown that any incomplete preference structure satisfying our fundamental axioms generates two local multiple-prior preference functionals that evaluate differential adjustments in gambles as would an individual with a multiple prior preference function. The essential idea is to extend Machina's (1982) notion of a local-utility function to our set up. The sets of priors defining these local multiple-prior preference functionals characterize the individual's local perception of ambiguity. Computationally, they are derived from the Clarke superdifferential correspondence applied to our willingness-to-pay measures that characterize preferences. Once these local multiple-prior preference functionals are derived, the structural consequences for them of different alternatives to independence are derived (Proposition 13).

After these local multiple-prior preference functionals are developed, Fenchel conjugation is used to induce incomplete multiple-prior functionals that globally represent "hulls" of arbitrary incomplete preferences that are consistent with, respectively, uncertainty aversion and mixture domination. The motivation for these developments follows from the fact that Bewley's (1986 and 2002) incomplete preference model and Faro's (2015) extension of that model maintain both uncertainty aversion and mixture domination. Our penultimate section then treats Bewley preferences and shows their specific relation to the different alternatives to independence that we consider. The final section concludes.

1 Previous Work

Only a handful of studies both abandon the completeness axiom and weaken the independence axiom (Maccheroni 2004, Safra 2014, Zhou 2014, Faro 2015 and Karni and Zhou 2017). Maccheroni (2004) drops the completeness axiom and replaces the independence axiom by comonotonic independence. Using Yaari's dual characterization, he demonstrates that when preferences are incomplete there exists a set of probability transformation functions such that one prospect is preferred to another if and only if the former's rank-dependent expected value is larger for every probability transformation function in that set of transformations. Safra (2014) examines incomplete preferences that satisfy the betweenness axiom. He derives a multi-betweenness functional representation for these preferences. Safra (2014) also demonstrates that the same representation holds when the transitivity axiom is replaced by a weaker requirement of dominance. Zhou (2014) and Karni and Zhou (2017) substitute the independence axiom with an analogue of weak substitution axiom (Chew and MacCrimmon 1979, Chew 1989) to obtain a multiple weighted utility representation which accommodates two sources of preference incompleteness. The first pertains to the incompleteness of tastes and it is represented by multiple utility function. The second captures the incompleteness of perception and it is characterized by multiple weight functions. Faro (2015) considers the case of a policymaker who must combine the judgments of advisers of differering credibility. In this context, as in other models of collective decisionmaking, transitivity may break down. Faro (2015) presents a generalized variational Bewley model, with the standard model arising as a special case when either transitivity or independence is imposed.

Danan and Ziegelmeyer (2006) develop an experimental procedure where subjects choose between two alternatives and have the ability to postpone their choice at a small cost. The authors argue that subjects who choose postponement reveal indecisiveness. Cettolin and Riedl (2016) conduct an experiment where subjects face a sequence of choices that include a risky and an ambiguous prospects. In addition to these two options, subjects can choose an indifference option in which case one of the two options is selected by a randomization device. The authors argue that when the same subject chooses indifference on multiple occasions she reveals preference incompleteness. Our procedure (Proposition 3) is different from the ones pursued in these studies. It relies on willingess-to-pay measures, applies to any setting, and allows one to elicit a function representation of decision-makers' preferences in addition to determining whether these preferences are incomplete.

2 Axioms, Cardinal Representation Results, and Comparability and Completeness Tests

Uncertainty is represented by a finite state space, S, and states are indexed with a slight abuse of notation by $(1, 2, ..., S)$. To focus attention on empirically measurable outcomes, we treat the case where consequences (outcomes) are restricted to a compact convex subset of the real numbers $\mathcal{A}$. Acts, therefore, can be identified with elements of $\mathcal{A}^S$. $\Delta \subset \mathcal{A}^S$ represents the unit simplex identifying the probability measures over S. $X \subset \mathcal{A}^S$ denotes the constant acts and we write $x \in X$ to denote the constant act taking the same real value, x, in each state of Nature.

Preferences are represented by a binary relation defined on $\mathcal{A}^S$ and denoted by $\succcurlyeq$ where $f \succcurlyeq g$ is to be read as $f \in \mathcal{A}^S$ is weakly preferred to $g \in \mathcal{A}^S$. We will say that acts f and g are *comparable* if $f \succcurlyeq g$ or $g \succcurlyeq f$. A preference relation is *complete* if and only if all pairs of acts are comparable. We define an indifference relation $\sim$ corresponding to $\succcurlyeq$ as follows: $f \sim g$ if and only if $f \succcurlyeq g$ and $g \succcurlyeq f$. The strict preference relation $\succ$ corresponding to $\succcurlyeq$ is defined as follows: $f \succ g$ if and only if $f \succcurlyeq g$ but $(g \succcurlyeq f)^c$, where superscript c denotes the negation operator. Note that the preference relation $\succcurlyeq^n$, defined as $f \succcurlyeq^n g \Leftrightarrow (g \succ f)^c$, is in general different from the preference relation $\succcurlyeq$ (Karni, 2011).

The *least-as-good correspondence* $P : \mathcal{A}^S \rightrightarrows \mathcal{A}^S$, associated with the upper contour sets of $\succcurlyeq$, is defined as

$$P(g) = \left\{ f \in \mathcal{A}^S : f \succcurlyeq g \right\},$$

the *no-better-than* correspondence $N : \mathcal{A}^S \rightrightarrows \mathcal{A}^S$ associated with the lower contour sets of $\succcurlyeq$ is defined as

$$N(g) = \left\{ f \in \mathcal{A}^S : g \succcurlyeq f \right\},$$

and their intersection $I : \mathcal{A}^S \rightrightarrows \mathcal{A}^S$ is defined

$$I(g) = \left\{ f \in \mathcal{A}^S : f \in N(g) \cap P(g) \right\}.$$

P and N are lower inverses (in the sense of Berge 1963) of one another, that is, for example

$$N(g) = \left\{ f \in \mathcal{A}^S : g \in P(f) \right\}.$$

We impose the following axioms on $\succcurlyeq$:

(**A.1**) (Reflexivity) $\forall f \in \mathcal{A}^S$, $f \succcurlyeq f$.

(**A.2**) (Transitivity) For all $h, f, g \in \mathcal{A}^S$, $h \succcurlyeq f$ and $f \succcurlyeq g$ implies $h \succcurlyeq g$.

(**A.3**) (Strict Monotonicity) For all $f \in \mathcal{A}^S$ and all $g \in \mathbb{R}^S_+ / \{0\}$, $f + g \succ f$

(**A.4**) (Continuity) For all $g \in \mathcal{A}^S$, both $P(g)$ and $N(g)$ are closed.

Axioms (A.1) through (A.4) are standard. Only A.2 requires comment. As noted in the introduction, transitivity is at least as appealing a condition for individual choice in the absence of completeness as in the case of complete preferences. However, as in Faro (2015) group rankings may display both incompleteness and intransitivity.

Because of our emphasis on testing for the presence of incompleteness and other structural restrictions, we consider two cardinal representations of $\succcurlyeq$, originally due to Blackorby and Donaldson (1980) and extended by Luenberger (1992), that can be directly elicited in empirical or experimental settings.

The *upper translation function* $t : \mathcal{A}^S \times \mathcal{A}^S \to \bar{\mathbb{R}}$ is defined as

$$t(f, g) \equiv \max \left\{ \beta \in \mathbb{R} : f - \beta \succcurlyeq g \right\}, \tag{1}$$

if there exists $\beta \in X$ such that $f - \beta \succcurlyeq g$ and $-\infty$ otherwise. Given A.4, the maximum in the definition of the translation function is well defined. The *lower translation function* $b : \mathcal{A}^S \times \mathcal{A}^S \to \bar{\mathbb{R}}$ is defined as

$$b(f, g) \equiv \max \{\beta \in \mathbb{R} : g \succcurlyeq f + \beta\}$$

if there exists $\beta \in X$ such that $g \succcurlyeq f + \beta$ and $-\infty$ otherwise.

In contrast to utility-based characterizations of preferences, where the functional representation only depends on a single act being evaluated, both translation functions depend on two acts. Intuitively, f and g can be thought as the decision-maker's "new" and "original" (*status quo*) positions, respectively. The upper translation function, $t(f, g)$, measures the decision-maker's willingness to pay in units of the riskless asset, $1 \in X$, for the variation in the gamble $(f - g) \in \mathcal{A}^S$ from g. $b(f, g)$ measures the corresponding willingness to accept (sell). Thus, the former characterizes $\succcurlyeq$ from the perspective of $P(g)$ and the latter from the perspective of $N(g)$. Empirical and experimental studies of choice under uncertainty routinely elicit versions of $t(f, g)$ or $b(f, g)$ (see, for example, Eisenberger and Weber 1995, Harbaugh, Krause and Vesterlund 2010).

We have:[1]

Proposition 1 [2,3] *Suppose that* $\succcurlyeq$ *satisfies* $(A.1) - (A.4)$. *Then,* t *satisfies*

(a) *(Indication) for all* $f, g \in \mathcal{A}^S$, $f \succcurlyeq g \Leftrightarrow t(f, g) \geq 0$ *and* $t(g, g) = 0$;

(b) *(Translatability in the numeraire)* $t(f + x, g) = t(f, g) + x$ *for all* $f, g \in \mathcal{A}^S$, $x \in X$;

(c) *(Monotonicity) for all* $f, f^o, g, g^o \in \mathcal{A}^S$, $(f^o, -g^o) \geq (f, -g) \Rightarrow t(f^o, g^o) \geq t(f, g)$ *with strict inequality if* $f^o - f \in \mathbb{R}^S_+ \setminus \{0\}$ *or* $g - g^o \in \mathbb{R}^S_+ \setminus \{0\}$; *and*

(d) *(Lipschitz in* f*) for all* $f, g \in \mathcal{A}^S$, $t(f, g)$ *is Lipschitz in* $f \in \mathcal{A}^S$ *and continuous in* g.

Suppose that $\succcurlyeq$ *satisfies* $(A.1) - (A.4)$. *Then,* b *satisfies*

(a') *(Indication) for all* $f, g \in \mathcal{A}^S$, $g \succcurlyeq f \Leftrightarrow b(f, g) \geq 0$ *and* $b(g, g) = 0$;

(b') *(Translatability in the numeraire)* $b(f + x, g) = b(f, g) - x$ *for all* $f, g \in \mathcal{A}^S$, $x \in X$;

[1] The proofs of all results are relegated to Appendix.

[2] Parts (b) and (f) of the Proposition require only axiom A.4.

[3] Analogues to (a) through (d) of Proposition 1 were established in Chambers (2014) in the context of a strict but potentially incomplete preference order. Chambers and Quiggin (2007) establish similar results for complete preference structures.

(c') *(Monotonicity) for all* $f, f^o, g, g^o \in \mathcal{A}^S$, $(f, -g) \geq (f^o, -g^o) \Rightarrow b(f^o, g^o) \geq b(f, g)$ *with strict inequality if* $f^o - f \in \mathbb{R}^S_+ \setminus \{0\}$ *or* $g - g^o \in \mathbb{R}^S_+ \setminus \{0\}$; *and*
(d') *(Lipschitz in* f*) for all* $f, g \in \mathcal{A}^S$, $b(f, g)$ *is Lipschitz in* $f \in \mathcal{A}^S$ *and continuous in* g.

Our next result establishes equivalence between the axioms imposed on $\succcurlyeq$ and the properties of t :

Proposition 2 *Preferences* $\succcurlyeq$ *satisfy* $(A.1)$ *through* $(A.4)$ *if and only if* t *satisfies parts (a)-(d) of Proposition 1.*

Indication ensures that either condition in Proposition 2 is equivalent to requiring that b satisfies parts (a')-(d') of Proposition 1. Our central result on testing for comparability and completeness is:

Proposition 3 *For all* $f, g \in \mathcal{A}^S$,
(a) $\succcurlyeq$ *is complete if and only if* $b(f, g) + t(f, g) = 0$ *for all* $f, g \in \mathcal{A}^S$;
(b) f *and* g *are comparable if and only if* $b(f, g) \cdot t(f, g) \leq 0$; *and*
(c) $\min_{s \in S} \{f_s - g_s\} - \max_s \{f_s - g_s\} \leq t(f, g) + b(f, g) \leq 0$.

Figure 1 illustrates Proposition 3. Geometrically, incompleteness permits the existence of "gaps" between $P(g)$ and $N(g)$. In Figure 1, all acts in the union of the areas AgC and BgD are non-comparable to act g. Acts falling in these gaps are not comparable to g because they do not fall in either g's least-as-good set CgD or its no-better-than set AgB. Similarly, all acts in the union of the areas GfE and HfF are non-comparable to act f. Note that acts f and g in Figure 1 are non-comparable. The value of the upper translation function evaluated at (f, g) is given by the negative of the length of the line segment fI while the value of the lower translation function evaluated at (f, g) is given by the negative of the length of the line segment fJ. Hence, we have that $b(f, g) \cdot t(f, g) > 0$ which confirms Proposition 3.b.

The largest gap between the least-as-good set $P(g)$ and no-better-than set $N(g)$ will occur when $P(g)$ consists of only those points that weakly dominate g in a vector sense and $N(g)$ consists of points weakly dominated by g. Intuitively, this is the case of *maximal* *preference incompleteness.* The first inequality in Proposition 3.c uses this scenario to place a lower bound on $t(f,g)+b(f,g)$. The upper bound shows that the translation of f in the direction of the riskless asset 1 that makes, say, f at least as good as g can never be any larger than the translation that makes f no better than g. And so, for example, because $f-t(f,g)\succcurlyeq g$, g can never be strictly preferred to $f-t(f,g)$ and, hence, $b(f-t(f,g),g)=t(f,g)+b(f,g)\leq 0$. The first part of the Proposition shows what is required to always translate a gamble, f, to a common frontier, $I(g)$, regardless of whether the lower or the upper notion of translation is used. When both translations bring the act to a common point for all possible f, the preference structure is complete.

An example illustrates the force of Proposition 3. Under completeness, the (upper) *certainty equivalent* for a gamble g

$$\inf\{\gamma:\gamma\succcurlyeq g\}=-t(0,g)$$

provides a complete function representation of $\succcurlyeq$ in the sense that $f\succcurlyeq g\Leftrightarrow -t(0,f)\geq -t(0,g)$. But without completeness, the implication runs only one way, that is

$$0-t(0,f)\succcurlyeq f\succcurlyeq g$$

and indication and translation establish that $t(0-t(0,f),g)=t(0,g)-t(0,f)\geq 0$.

The converse, however, is not generally true. Another (lower) certainty equivalent exists,

$$\max\{\beta:g\succcurlyeq\beta\}=b(0,g),$$

which (Proposition 3.c) satisfies

$$b(0,g)\leq -t(0,g).$$

When this inequality is strict, certainty equivalents falling in the interval $(b(0,g),-t(0,g))$ are not comparable to g. Hence, complete ordering by certainty equivalents is not possible.

Completeness eliminates this possibility. Proposition 3.a then ensures that the "gap" between the upper and lower certainty equivalents disappears. Hence,

$$0+b(0,g)=0-t(0,g)\in I(g),$$

and

$$0 + b(0, f) = 0 - t(0, f) \in I(f),$$

and thus $t(0, g) - t(0, f) \geq 0$ implies

$$f \sim 0 - t(0, f) \geq 0 - t(0, g) \sim g.$$

Monotonicity and transitivity axioms then imply $f \succcurlyeq g$ as required. For latter reference, we summarize this example in corollary form as

Corollary 4 *If $\succcurlyeq$ satisfies A.1-A.4 and is complete, the following statements are equivalent:*
(a) $f \succcurlyeq g$;
(b) $t(0, g) - t(0, f) \geq 0$; *and*
(c) $b(0, f) - b(0, g) \geq 0$.

Example 5 *To illustrate the testing procedure in Proposition 3, consider an hypothetical experimental setting where subjects are confronted in a two-state setting with three assets:*

Asset	*Payout*
f	$(0, 2)$
g	$(1, 1)$
h	$(2, 0)$

.

Also presume that a subject has been endowed with g and that his or her elicited willingness-to-pay and williness-to-sell measures are

Asset	$t(\cdot, g)$	$b(\cdot, g)$
f	*-.5*	*-.5*
g	*0*	*0*
h	*0*	*-.5*

.

The evidence indicates that preferences are incomplete, that act g is comparable to itself and act h (because $b(h, g) \cdot t(h, g) = 0$), but that g not comparable to f because $b(f, g) \cdot t(f, g) =$ $.25 > 0$.

The most conservative approximation to $P(g)$ that rationalizes these choices under A.1 to

A.4 is given by the free disposal hull of the points $f - t(f,g) = (.5, 2.5)$, $g = (1,1)$, *and* $h - t(h,g) = (2,0)$:

$$\hat{P}(g) = \left\{ f \in \mathcal{A}^S : f \geq (.5, 2.5), \; f \geq (1,1), \; \textit{and } f \geq (2,0) \right\}.$$

For $N(g)$, *the corresponding conservative approximation is the free disposal hull (from below) of the points* $f + b(f,g) = (-.5, 1.5)$, $g = (1,1)$, *and* $h + b(h,g) = (1.5, -.5)$:

$$\hat{N}(g) = \left\{ f \in \mathcal{A}^S : f \leq (-.5, 1.5), f \leq (1,1), \; \textit{and } f \leq (1.5, -.5) \right\}.$$

Notice that versions of $\hat{P}$ *and* $\hat{N}$ *can be constructed for any data set in the above form and that they will satisfy A.1-A.4. Thus, strictly speaking these axioms are not falsifiable in an empirical context.*

3 Structural Restrictions

The results in the previous section apply for arbitrary monotonic preference structures, whether complete or not. Decision-theoretic models, however, routinely consider more restrictive axiomatic structures. We now consider how various relaxations of the "independence" axiom manifest themselves as structural restrictions on $t(f,g)$ and $b(f,g)$. We consider five such structural restrictions: weak-certainty independence, radial independence, certainty independence, betweeness, and dominance independence.

Maccheroni, Marinacci, and Rustichini (2006) proposed *weak-certainty independence* as a replacement for Gilboa and Schmeidler's (1989) certainty independence criterion.

(**A.5.1**) (Weak Certainty Independence) For all $f, g \in \mathcal{A}^S, x, y \in X, \alpha \in (0,1)$, $\alpha f + (1-\alpha) x \succcurlyeq \alpha g + (1-\alpha) x \Leftrightarrow \alpha f + (1-\alpha) y \succcurlyeq \alpha g + (1-\alpha) y$.

Weak certainty independence requires that if convex combinations of an act f and a riskless act x dominate convex combinations of another act g and x, then all such convex combinations of f and g with riskless acts preserve the preference ordering. Thus, mixing riskless assets with arbitrary acts does not distort the ordering of the acts by $\succcurlyeq$.

The next restriction replaces invariance to mixing with riskless acts to radial expansions or contractions of acts being compared.

(**A.5.2**) (Radial Invariance) For all $f, g, \in \mathcal{A}^S$, $\alpha \in (0,1)$, $f \succcurlyeq g \Leftrightarrow \alpha f \succcurlyeq \alpha g$.

Axiom (A.5.2) requires rankings of acts to be independent of scaling. Figures 2 and 3 illustrate the difference between (A.5.1) and (A.5.2). Figure 2 depicts preferences that satisfy (A.5.1). By weak certainty independence, better-than and no-better than sets are translates of one another in the direction of the sure thing. In Figure 2, the cone with a vertex at point $g+x$, which represents the better-than set at $g+x$, can be obtained by sliding the cone with a vertex at point g, which represents the better-than set at g, along the line connecting g and $g+x$. A similar correspondence holds for acts $\hat{g}$ and $\hat{g}+x$. However, the cones at points g and $\hat{g}$, which lie along different lines parallel to the certainty line, are in general different.

Axiom (A.5.2), on the other hand, requires that better-than and no-better than sets are radial blow ups of one another. In Figure 3, the better-than set at αg can be obtained by sliding the better-than set at g from g to αg along the ray that passes through αg and g. Hence, the amount of ambiguity perceived by a decision maker and characterized by the size of the kink at the original act g remains constant when g is scaled down. This property of better-than sets, which provides the motivation for the name of axiom (A.5.2), also holds for acts $\hat{g}$ and $\hat{g}+x$ in Figure 3 that also lie along the same ray from the origin. However, the cones at points g and $\hat{g}$, which lie along different rays from the origin, are different. This reflects the fact that axiom (A.5.2) does not restrict the relationship between better-than sets of acts which are not scaled up or down versions of each other.

Gilboa and Schmeidler's (1989) certainty independence requires both weak-certainty independence and radial invariance. And so in terms of Figures 2 and 3, it requires that the better-than and worse-than sets are preserved along both rays from the origin and rays that parallel X.

(**A.5.3**) (Certainty independence) For all $f, g, \in \mathcal{A}^S$, $\alpha \in (0,1)$, $x \in X$, $f \succcurlyeq g \Leftrightarrow \alpha f + (1-\alpha) x \succcurlyeq \alpha g + (1-\alpha) x$.

Our next axiom requires that both $P(g) - g$ and $N(g) - g$ be *absorbing sets*, so that shrinking any act that dominates or is dominated by g towards g maintains the preference ordering.

(**A.5.4**) (Betweenness) For all $f, g, \in \mathcal{A}^S$, $\alpha \in (0,1)$, $f \succcurlyeq g \Leftrightarrow \alpha f + (1-\alpha) g \succcurlyeq g$.

Our final axiom is due to Faro (2015).

(**A.5.5**) (Dominance Independence) For all $f, g, h, i \in \mathcal{A}^S$, $\alpha \in (0,1)$, $f \succcurlyeq g$ and $h \succcurlyeq i \Rightarrow \alpha f + (1-\alpha) h \succcurlyeq \alpha g + (1-\alpha) i$.

Dominance independence requires that the correspondences P and N satisfy the following convexity requirements for all $\alpha \in (0,1)$, $g^o, g^* \in A^S$

$$\alpha P(g^o) + (1-\alpha) P(g^*) \subset P(\alpha g^o + (1-\alpha) g^*),$$

and

$$\alpha N(g^o) + (1-\alpha) N(g^*) \subset N(\alpha g^o + (1-\alpha) g^*).$$

As a consequence, both $P(g)$ and $N(g)$ are closed convex sets for all $g \in A^S$. Behaviorally, convexity of $P(g)$ translates into uncertainty aversion in the sense of Gilboa and Schmeidler (1989). Convexity of $N(g)$, on the other hand, translates into mixture domination so that if g dominates two outcomes, f^o and f^*, it also dominates all mixtures of those acts. Geometrically, therefore, dominance independence strengthens betweenness (A.5.4). In particular, it requires sets of the form $P(g) - g$ must be convex absorbing sets.

Uncertainty aversion is a common feature of complete preference structures for decision makers facing uncertain choices. Mixture domination, on the other hand, which requires uncertainty-loving behavior in the neighborhood of g is not commonly maintained. And, in fact, for a complete preference structure, it is only consistent with uncertainty aversion if the preference functional is linear (see, for example, Proposition 15 below). Nevertheless, dominance independence weakens the standard independence criterion which also requires both mixture domination and uncertainty aversion (see, for example, Galabaatar and Karni 2013). The difference between independence and dominance independence is that the implication defining the latter only runs in one direction, while independence requires it to run in both directions. By taking $i = h$, it is easy to see that dominance independence and reflexivity imply a weakened (one way) version of independence.

We detail the implications of these axioms for the translation functions in the following:

Proposition 6 *Suppose that preferences $\succcurlyeq$ satisfy $(A.1)$ through $(A.4)$ so that t satisfies parts (a)-(d) of Proposition 1, and $b(f,g)$ satisfies parts (a')-(d'). Then,*

(e.1) *(Translation Invariance) Preferences satisfy $(A.5.1)$ if and only if all $f, g \in \mathcal{A}^S$ and*

$x \in X$, $t(f,g) = b(g,f)$.

(e.2) *(Radial Invariance) Preferences satisfy* $(A.5.2)$ *if and only if all* $f, g \in \mathcal{A}^S$ *and* $\mu > 0$, $t(\mu f, \mu g) = \mu t(f,g)$ *and* $b(\mu f, \mu g) = \mu b(f,g)$.

(e.3) *(Radial Invariance and Translation Invariance) Preferences satisfy* $(A.5.3)$ *if and only if preferences satisfy (e.1) and (e.2).*

(e.4) *(Betweeness) Preferences satisfy* $(A.5.4)$ *if and only if for all* $f, g \in \mathcal{A}^S$ *and* $\alpha \in (0,1)$, $\frac{t(f+\alpha(g-f),g)-t(f,g)}{\alpha} \geq -t(f,g)$ *and* $\frac{b(f+\alpha(g-f),g)-b(f,g)}{\alpha} \geq -b(f,g)$.

By Proposition 6.e.1 and e.2 and Proposition 3, the weak-certainty independent and certainty-independent cases are particularly tractable frameworks in which to conduct empirical tests of comparability and completeness because they only require looking at either t or b. We have:

Corollary 7 *If preferences satisfy* $(A.5.1)$ *or* $(A.5.3)$, *then*

(a) f *and* g *are comparable if and only if* $t(g,f) \cdot t(f,g) \leq 0$ *(or, equivalently,* $b(g,f) \cdot b(f,g) \leq 0$*); and*

(b) $\succcurlyeq$ *is complete if and only if* $t(g,f) + t(f,g) = 0 = b(g,f) + b(f,g)$ *for all* $f, g \in \mathcal{A}^S$.

The structural consequence of radial invariance is that radially shrinking or expanding gambles does not change their relative preference ranking. Instead, it simply renormalizes the units in which our willingness-to-pay measures are counted. Betweenness requires that "differential" valuations (for example, $\frac{t(f+\alpha(g-f),g)-t(f,g)}{\alpha}$) towards the status quo, g, from an aribtrary act are always bound by the discrete valuation.

Faro (2015) has established that if $\succcurlyeq$ satisfies (A.1)-(A.4) and dominance independence, the model reduces to Bewley's set up with independence replacing (A.5.5). On the other hand, if (A.2) (transitivity) is not satisfied then dominance independence does not imply independence and one obtains what he refers to as the *Bewley variational model.* To state his result in our terms, define the conjugates $t^* : \Delta \times A^S \to \mathbb{R}$ and $b^* : \Delta \times A^S \to \mathbb{R}$, of the functions t and b as

$$t^*(\pi, g) = \inf_f \{\pi' f - t(f,g)\}, \tag{2}$$

and

$$-b^{*}(\pi,g)=\sup_{f}\{\pi' f+b(f,g)\}, \tag{3}$$

respectively.[4]

Proposition 8 *(Faro 2015) If preferences satisfy* (A.1), (A.3), (A.4), *and* (A.5.5), *then*

$$t(f,g)=\inf_{\pi\in\Delta}\{\pi' g-t^{*}(\pi,g)\},$$

with $t^{*}(\pi,g)$ *closed concave in* π, *nondecreasing and convex in* g, *and* $t^{*}(\pi,g)-\pi' g\leq 0$. *If preferences satisfy* (A.1), (A.3), (A.4), *and* (A.5.5), *then*

$$-b(f,g)=\sup_{\pi\in\Delta}\{\pi' g+b^{*}(\pi,g)\},$$

with $b^{*}(\pi,g)$ *closed concave in* π, *nonincreasing and convex in* g, *and* $b^{*}(\pi,g)-\pi' g\geq 0$.

The first multiple-prior preference functional in Proposition 8 extends both the Maccheroni, Marinacci, and Rustichini (2006) variational preference class and the Chambers and Quiggin (2007) dual multiple-prior preference relation to encompass incomplete preference structures. By Corollary 4 (also see Section 4.2 below), the Maccheroni, Marinacci, and Rustichini (2006) variational representation requires, in our notation, that

$$-t(0,g)=\inf_{\pi\in\Delta}\{\pi' g-c(\pi)\}$$

where c is a closed, concave function that indexes ambiguity attitudes.

The representation in Proposition 8 extends the Maccheroni, Marinacci, and Rustichini (2006) representation in several ways. There is no requirement for completeness. The index of ambiguity attitudes, $t^{*}(\pi,g)$ in our notation, depends on the status quo g. And that index is nondecreasing and convex in g. The last reflects the dominance-independence requirement that $P(g)$ and $N(g)$ be intersecting (tangent to one another at g) convex sets. Bewley preferences also require $P(g)$ and $N(g)$ to be convex but they must also be conical.

Dominance independence, therefore, represents a strong smoothness criterion. The multiple-prior form of the representations ensure that $t(f,g)$ and $b(f,g)$ are closed concave functions

[4] For more on the conjugate representation, see Section 4.2.

of f, and the convexity of t^* and b^* in g, in turn, ensure concavity in g. Thus, both representations will be differentiable in f and g almost everywhere on the relative interiors of their effective domains.[5] And where they are not differentiable, superdifferentials in the sense of convex analysis exist. Hence, except in pathological cases, local behavior in the face of uncertainty will be characterizable in differential terms under dominance independence.

That smoothness, in turn, ensures the existence of a closed convex set of probability measures conditioned by the status quo g, $D(g) \subset \Delta$, such that for all $f, g \in A^S$

$$f \succcurlyeq g \Leftrightarrow \pi'(f-g) \geq 0 \text{ for all } \pi \in \mathcal{D}(g).$$

In words, under dominance independence f is preferrred to g if and only if every probability measure in $D(g)$ assigns a higher expected value to f than to g.[6] It is natural in this context to interpret $D(g)$ as a set of local beliefs that the decision maker will entertain as possible representations of the likelihood of different outcomes $s \in S$ in an ambiguous decision setting. And thus, as Faro (2015) points out, it is natural to interpret dominance independence as leading to an extended version of Bewley's incomplete preference structure that he refers to as Bewley variational preferences.

Example 9 *Recall the structure of the elicited $t(f,g)$ and $b(f,g)$ measures in Example 5*

Asset	$t(\cdot,g)$	$b(\cdot,g)$
f	*-.5*	*-.5*
g	*0*	*0*
h	*0*	*-.5*

.

Proposition 6.e.1 shows that (A.5.1) requires that $t(f,g) = b(g,f)$ *so that* $t(f,g+1) = b(g+1,f) = b(g,f) - 1 = t(f,g) - 1$. *Similarly,* $b(f,g+1) = t(g+1,f)$ *imples* $b(f,g+1) =$

[5] The effective domain of $t(f,g)$ (in f) is defined by

$$\{f : t(f,g) > -\infty\}.$$

[6] When $t(f,g)$ is differentiable at $f = g$, $\mathcal{D}(g) = \{\nabla_f t(g,g)\}$ where $\nabla_f t(f,g)$ represents the gradient of t in f. When $t(f,g)$ is not differentiable at $f = g$, $\mathcal{D}(g) = \partial_f t(g,g) \subset \Delta$ where $\partial_f t(f,g)$ is the superdifferential of t in f. Chambers (2014, Lemma 1) establishes that $\partial_f t(f,g) \subset \Delta$ for all f, g.

$b(f,g)+1$. *Hence, if (A.5.1) is to hold then changing the status quo position from* g *to* $g+1$ *requires that*

Asset	$t(\cdot, g+1)$	$b(\cdot, g+1)$
f	*-1.5*	*.5*
$g+1$	*0*	*0*
h	*-1*	*.5*

,

while consistency with (A.5.2) requires that for any $\mu > 0$

Asset	$t(\cdot, \mu g)$	$b(\cdot, \mu g)$
μf	*-.5*μ	*-.5*μ
μg	*0*	*0*
μh	*0*	*-.5*μ

.

For (A.5.3) both of these response patterns must hold. Evidence against any of these response patterns would constitute empirical evidence against maintenance of the respective structural restriction.

4 Multiple-Prior Representations of Incomplete Preferences

Dominance independence and independence both induce multiple-prior preference representations. As the preceding arguments have shown, these characteristics emerge from the smoothing requirements that dominance independence and independence impose upon individual behavior. Smoothness can fail in those set ups, but only on sets of measure zero that are associated with "kinked" indifference curves. And those kinks, which imply behavioral inertia in the neighborhood of g, are the behavioral consequence of perceived ambiguity.

More generally, multiple-prior preference representations have proven an important, if not dominant, class of decision-theoretic models that accommodate uncertain and ambiguous decision settings. The next subsection shows that all incomplete preference structures satisfying our basic axioms have local multiple-prior preference representations. This finding considerably extends the practical applicability of multiple-prior models well beyond existing structures to ones that only require extremely modest restrictions on rational behavior.

It accomplishes this extension by developing two local multiple-prior preferences functionals that characterize the local behavior of any preference structure. Once that is done, the structural consequences of the various alternatives to independence for the local multiple-prior preference functionals are derived. The subsection that then follows shows how either uncertainty aversion or mixture domination can be used to develop multiple-prior preference functionals that generalize both the Bewley variational structures and the Bewley structures.

Some basic results from variational analysis are needed. For $m : \mathcal{A}^S \to \mathbb{R}$ define its (one-sided) *Clarke directional derivative,* $m^f : \mathcal{A}^S \times \mathcal{A}^S \to \mathbb{R}$ in the direction of $h \in \mathcal{A}^S$ as

$$m^f(f;h) = \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{m(f^o + \lambda h) - m(f^o)}{\lambda}.$$

If m is Lipschitz of order K at f, then m^f is closed superlinear in f. The *Clarke superdifferential correspondence,* $\partial^f m : \mathcal{A}^S \to \mathcal{A}^S$ is defined as

$$\partial^f m(f) = \left\{ q \in \mathcal{A}^S : q'h \geq m^f(f;h) \text{ for all } h \in \mathcal{A}^S \right\}.$$

It is nonempty, convex, and satisfies

$$m^f(f;h) = \inf_h \left\{ q'h : q \in \partial^f m(f) \right\},$$

and

$$\partial^f m(f) = co \left\{ \lim \nabla m\left(f^i\right) : f^i \to f, f^i \notin L, f^i \notin \Omega \right\}, \tag{4}$$

where co denotes the convex hull, ∇m denotes the usual gradient, $L \subset \mathbb{R}^S$ is a set of measure zero, and Ω is the set of points at which m is not differentiable.

When m is concave, $\partial^f m(f)$ is equal to the gradient $\{\nabla m(f)\}$ whenever m is differentiable (almost everywhere) and equal to the superdifferential in the sense of convex analysis, $\partial_f m(f)$, everywhere else (Clarke 1983), where

$$\partial_f m(f) = \{ q : q'(f' - g) \geq m(f') - m(f) \text{ for all } f' \}.$$

4.1 Local Multiple-Prior Preferences

The Lipschitz property of both t and b ensure that they typically admit Clarke directional derivatives in f. For t and b, let superscript f denote the Clarke derivative with respect to the first argument and superscript g denote the Clarke derivative (when it exists) in the second argument. We have:

Proposition 10 *If $\succcurlyeq$ satisfies (A.1) through (A.4), then for all $f, g \in \mathcal{A}^S$ and $x \in X$*

$$t^f(f, g; x) = x \text{ and } b^f(f, g; x) = -x,$$

and $\partial^f t(f, g) \subset \Delta$ and $\partial^f b(f, g) \subset -\Delta$.

Both $\partial^f t(f, g)$ and $-\partial^f b(f, g)$ have natural interpretations as, respectively, sets of "subjective beliefs" and "risk-neutral probabilities" that partially characterize preferences in the neighborhood of f. This connection between the Clarke superdifferential and the belief structure mirrors in the incomplete preference case the connection between the Clarke superdifferential of the preference functional and "revealed ambiguity" established by Ghirardato, Maccheroni, and Marinacci (2004) in the complete preference case. That contribution, in turn, generalized Gilboa and Schmeidler's (1989) identification of perceived ambiguity with the set of priors characterizing their maximin expected utility model. That maximin set of priors is the (ordinary) superdifferential of the preference function in the neighborhood of the sure thing.

Intuitively speaking, the "size" of $\partial^f t(f, g)$ and $\partial^f b(f, g)$ communicate information about the decisionmaker's perceived ambiguity when comparing gambles f and g. Each consists of a set of "possible probabilistic scenarios" that must be examined in comparing these two acts. When either consists of a singleton set, the corresponding representation is smooth, and the decisionmaker's local perceptions can be approximated by those of an expected-utility maximizer characterized by those subjective probabilities. When these sets of priors are not singleton sets, the resulting local nonsmoothness in the preference representation captures the decisionmaker's ambiguity perceptions looking upward from the status-quo gamble in the case of $\partial^f t(f, g)$ and downward in the case of $\partial^f b(f, g)$. The bigger the set, the larger the number of probabilistic scenarios that must be considered in evaluating alternatives. These beliefs are local and will vary over gambles.

From the observation that $t(g,g)=0$, it follows immediately

$$t^f(g,g;h-g)=\inf\left\{\pi'(h-g):\pi\in\partial^f t(g,g)\right\} \tag{5}$$

can be interpreted as a local multiple-prior preference functional in the sense that differential adjustments from g in the direction $h-g\in A^S$ are judged similarly to an individual with the multiple-prior preference function defined by a perceived ambiguity of $\partial^f t(g,g)$. So instead of the linear local utility preference functions derived by Machina (1982), we obtain *superlinear* local preference functionals, where superlinearity is characterized by the range of subjective beliefs that the decision maker will entertain about the true state of the world.

Similarly,

$$b^f(g,g;h-g)=-\sup\left\{\pi'(h-g):-\pi\in\partial^f b(g,g)\right\}, \tag{6}$$

is interpretable as a *local multiple-prior preference functional* that judges differential adjustments from g as would an individual with the multiple-prior preference function defined by $-\partial^f b(g,g)$. Notice, however, that $t^f(g,g;h-g)\geq 0$ is interpretable as a local preference for h over g and $b^f(g,g;h-g)\geq 0$ as a local preference for g over h.

Both t^f and b^f are closed superlinear functions. The former evaluates differential departures from g from the perspective of $P(g)$, and the latter from $N(g)$. Because, for example, $h\notin P(g)$ does not imply $h\in N(g)$, both perspectives are needed to examine local behavior.

Nevertheless, even though these local multiple-prior preference functionals describe different manifestations of preferences, they are not independent of one another because they have been derived from a common $\succcurlyeq$. A clear way to grasp the innate nature of the connection is to recall that Indication (Proposition 1.a) implies

$$b(f,g)=\max\{\beta:t(g,f+\beta)\geq 0\}.$$

The generalized envelope theorem then ensures that $\partial^f b(g,g)$ is closely connected with $\partial^g t(g,g)$ (assuming that the latter is well-defined). As we shall demonstrate below, this correspondence is exact under (A.5.1) and (A.5.3).

So while there can be disagreement between the two local multiple-prior preference functionals on "subjective" beliefs, that disagreement has its limits:

Proposition 11 *For all* $g, h \in \mathcal{A}^S$,

$$0 \in \partial^f t(g,g) + \partial^f b(g,g),$$

and

$$\min_{s \in S} \{f_s - g_s\} - \max_s \{f_s - g_s\} \le t^f(g,g;h) + b^f(g,g;h) \le 0.$$

The first part of Proposition 11 requires that belief sets overlap. Beliefs derived from a common preference structure cannot be totally disparate. $P(g)$ and $N(g)$ must be supported by a common prior and that prior must separate some (but not all) elements of the former from the latter. In particular, axiom (A.3) ensures that this common prior will separate $\{g\} + \mathcal{A}^S_{++}$ from $\{g\} - \mathcal{A}^S_{++}$ because the former consists of points that strictly dominate g and the latter of points strictly dominated by g.

Figure 4, which is drawn for the case where $P(f) = P(g)$ and strictly convex, helps illustrate. The set $P(f) = P(g)$ is given by the area above the curve $CfgD$ while the set $N(f) = N(g)$ is given by the area below the curve $AfgB$. At point f, $\partial^f b(f,g)$ consists of the set of supporting hyperplanes to $N(f) = N(g)$ at f illustrated by the lines between KK' and LL'. Similarly, $\partial^f t(f,g)$ is illustrated by the (singleton) hyperplane tangent to $P(f) = P(g)$ at f (not drawn). These supporting hyperplanes do not agree, but as f moves towards g that difference will disappear. A single tangent will emerge. Local beliefs are common from above and from below.

Notice, however, that this is only one perspective. The hyperplane tangent to $P(g)$ at g (not drawn) portrays the variation in g (and not f) that balances $t(f,g)$, and it is more naturally associated with $\partial^g t(f,g)$, which in this nicely smooth case exists. For the special case drawn where $t(f,g) = t(g,f)$, we naturally have locally that $\partial^g t(f,g) = \partial^f t(g,f)$. More generally, however, this will not be true, and the information captured by these two sets of local probabilities will be different, just as will the local information communicated by $\partial^g t(f,g) = \partial^f t(g,f)$.

The second part of Proposition 11 establishes that $t^f(g,g;h) + b^f(g,g;h)$ must respect the same bounds as $t(f,g) + b(f,g)$. The upper bound will be attained, for example, when the differential adjustment is from g to an element of $I(g)$. But if a differential adjustment carries the individual into the interior of either $P(g)$ or $N(g)$ or to a point not comparable to g, the sum of the local multiple-prior preference functionals will be negative.

When preferences are complete, these types of discrepancies disappear. For example, in the case of complete beliefs $N(g)$ corresponds to everything falling below the convex indifference curve $CfgD$, and not the area below the curve $AfgB$. There is no longer any disagreement between the hyperplanes supporting the boundaries. This occurs whether the boundary to $P(g)$ possesses "kinks" or not. More formally,

Proposition 12 *If $\succcurlyeq$ is complete, then for all $f, g \in \mathcal{A}^S$*

$$0 \in \partial^f t(f,g) + \partial^f b(f,g).$$

The local multiple-prior preference functionals in (5) and (6) have been derived using only axioms (A.1)-(A.4). It is apparent, however, that each of the axioms (A.5.1)-(A.5.5) impose additional structure upon $t^f(g,g;h-g)$ and $b^f(g,g;h-g)$:

Proposition 13 *Assume $(A.1)-(A.4)$.*
(a) *If $\succcurlyeq$ satisfies $(A.5.1)$, then for all $f,g,h \in \mathcal{A}^S$, $t^f(f+x,g+x;h) = t^f(f,g;h)$ for all $x \in X$, $t^f(f,g;h) = b^g(g,f;h)$, $t^g(f+x,g+x;h) = t^g(f,g;h)$ for all $x \in X$, and $t^g(f,g;h) = b^f(g,f;h)$;*
(b) *If $\succcurlyeq$ satisfies $(A.5.2)$, then for all $f,g,h \in \mathcal{A}^S$ and all $\alpha > 0$, $t^f(\alpha f, \alpha g; \alpha h) = \alpha t^f(f,g;h)$, $b^f(\alpha f, \alpha g; \alpha h) = \alpha b^f(f,g;h)$;*
(c) *If $\succcurlyeq$ satisfies $(A.5.3)$, then both (a) and (b) apply;*
(d) *If $\succcurlyeq$ satisfies $(A.5.4)$, then for all $f,g,h \in \mathcal{A}^S$, $t(f,g) + t^f(f,g;g-f) \geq 0$ and $b(f,g) + b^f(f,g;g-f) \geq 0$; and*
(e) *Assume $(A.1)$, $(A.3)$, $(A.4)$, and $(A.5.5)$. For all f,g in the relative interior of the effective domain of t, the Clarke directional derivatives t^f and t^g exist with*

$$t^f(f,g;h) = \nabla_f t(f,g)' h$$

and

$$t^g(f,g;h) = \nabla_g t(f,g)' h,$$

almost everywhere.

Assume (*A*.1)*,* (*A*.3)*,* (*A*.4)*, and* (*A*.5.5)*. For all* f, g *in the relative interior of the effective domain of* b*, the Clarke directional derivatives* b^f *and* b^g *exist with*

$$b^f(f, g; h) = \nabla_f b(f, g)' h$$

and

$$b^g(f, g; h) = \nabla_g b(f, g)' h,$$

almost everywhere.

Proposition 13 establishes the implications of common structural restrictions for the Clarke directional derivatives. These in turn have direct implications for the associated local multiple-prior preference functionals and their belief sets.

From part (a), it follows that for all x

$$\partial^f t(f + x, g + x) = \partial^f t(f, g),$$

and

$$\partial^f b(f + x, g + x) = \partial^f b(f, g),$$

so that both local multiple-prior preference functionals and their belief sets are invariant to mixing g with riskless acts. Moreover, it follows immediately that

$$\partial^f t(f, g) = \partial^g b(g, f).$$

Part (b) shows that radial invariance requires that local subjective beliefs are invariant to radial transformations of f and g because for $\alpha > 0$

$$\begin{aligned} \partial^f t(\alpha f, \alpha g) &= \left\{q : q'\alpha h \geq t^f(\alpha f, \alpha g; \alpha h) \text{ for all } \alpha h\right\} \\ &= \left\{q : q'h \geq t^f(f, g; h) \text{ for all } h\right\} \\ &= \partial^f t(f, g). \end{aligned}$$

Certainty independence requires that subjective beliefs be both radially invariant and translation invariant. And finally, betweenness requires that for all f, g

$$f + t^f(f, g; g - h) \succcurlyeq g$$

because

$$t(f,g) + t^f(f,g;g-h) \geq 0 \Rightarrow t\left(f + t^f(f,g;g-h), g\right) \geq 0,$$

by translatability. Similar comments apply for b^f.

Dominance independence has particularly strong differential implications for both t and b. Concavity ensures differentiability in the usual sense almost everywhere on the relative interior of the effective domain of the function in question. Differentiability in turn allows us to replace the Clarke differentials defining the local-multiple prior preference functions with derivatives interpretable as local probability measures almost everywhere. On those sets of measure zero where differentiability fails, beliefs are captured by superdifferentials just as they are in the Gilboa and Schmeidler (1989) maximin case. And, thus, with the exception of pathological cases, we can generate well-defined superlinear local-multiple prior preferences functions everywhere from two perspectives, both above and below.

Example 14 *Recall that in Example 5, the response pattern can be rationalized by a $P(g)$ and $N(g)$ corresponding to*

$$\hat{P}(g) = \left\{f \in \mathcal{A}^S : f \geq (.5, 2.5), f \geq (1,1), \text{ and } f \geq (2,0)\right\},$$

and

$$\hat{N}(g) = \left\{f \in \mathcal{A}^S : f \leq (-.5, 1.5), f \leq (1,1), \text{ and } f \leq (1.5, -.5)\right\},$$

respectively. For these approximations, $\hat{t}((1,1),(1,1)) = 0$ and $\hat{b}((1,1),(1,1))$ with $\partial^f \hat{t}((1,1),(1,1)) = \Delta = -\partial^f \hat{b}((1,1),(1,1))$ (using (4)) and

$$\hat{t}^f((1,1),(1,1);h-(1,1)) = \min_{s \in S}\{h_s - 1\}$$

and

$$\hat{b}^f((1,1),(1,1);h-(1,1)) = -\max_{s \in S}\{h_s - 1\},$$

reflecting the complete ambiguity that is associated with $\succcurlyeq$ defined by $\geq$.

4.2 Conjugating t and b

Proposition 8 shows that dominance independence ensures that preferences not only manifest a local multiple-prior representation but a global one as well. The same, of course, is true for independence. That smooth structure is an immediate consequence of both versions of independence requiring that preferences exhibit uncertainty aversion and mixture domination. But satisfying both of these restrictions is not required to ensure that $\succcurlyeq$ has a multiple-prior representation. Maintaining either one alone is sufficient.

The importance of this realization is underlined by the implications of maintaining both for complete preference structures. Uncertainty aversion, with little doubt, is the more conventional assumption. Mixture domination, which implies a preference for gambling under completeness, is less conventional. Complete preference structures, therefore, can maintain both only at the expense of inducing linearity in the preference maps.

Proposition 15 *If $P(g)$ and $N(g)$ are convex, preferences are complete if and only*

$$t(f,g) = \sum_s \pi(f_s - g_s).$$

We, therefore, turn our attention to alternative axiomatic structures that continue to support multiple-prior representations. Fenchel conjugation of t and b provides a natural approach. Recall that for t and b consistent with (A.1)-(A.4), their conjugates are defined by (2) and (3), respectively. Proposition 10.a (Indication) ensures that $t^*(\pi,g) \leq \pi' g$ and $-b^*(\pi,g) \geq \pi' g$. Moreover, it is well-known (Rockafellar, 1970, Section 12) that t^* and b^* are closed concave functions of π.

The conjugates of the conjugates are defined by applying the conjugacy mapping to t^* and b^*:

$$t^{**}(f,g) = \inf_{\pi \in \Delta} \{\pi' f - t^*(\pi,g)\}$$

and

$$-b^{**}(f,g) = \sup_{\pi \in \Delta} \{\pi' f + b^*(\pi,g)\}. \tag{7}$$

The following results are well-known (for example, Rockafellar 1970, Theorem 12.2):

$$\begin{aligned} t^{**}(f,g) &\geq t(f,g), \quad -b^{**}(f,g) \geq -b(f,g), \\ t^*(\pi,g) &= \inf_f \{\pi' f - t^{**}(f,g)\} \end{aligned}$$

and

$$-b^*(\pi,g) = \sup_f \{\pi' f + b^{**}(f,g)\}.$$

Under our maintained assumptions, $t^{**}(f,g)$ and $b^{**}(f,g)$ are closed concave functions of f. The former is non-decreasing in f and non-increasing in g, and the latter is non-increasing in f and non-decreasing in g. Moreover, both are translatable in the direction X. In other words, these are closed concave functions satisfying (b)-(d) and (b')-(d') of Proposition 1. Proposition 2 thus implies that the preference structure $\succcurlyeq^*$ induced by

$$f \succcurlyeq^* g \Leftrightarrow t^{**}(f,g) \geq 0,$$

and the preference structure $\succcurlyeq_*$ induced by

$$g \succcurlyeq_* f \Leftrightarrow b^{**}(f,g) \geq 0,$$

will satisfy (A.1)-(A.4).

Preference structure $\succcurlyeq^*$ is not equivalent to $\succcurlyeq$, but it is true that $f \succcurlyeq g \Rightarrow f \succcurlyeq^* g$. $P^*(g)$ induced by

$$P^*(g) = \{f : f \succcurlyeq^* g\}$$

is the convex hull of $P(g)$. Similarly, $\succcurlyeq_*$ is not equivalent to $\succcurlyeq$, but it is true that $g \succcurlyeq f \Rightarrow g \succcurlyeq_* f$ and that

$$N_*(g) = \{g : g \succcurlyeq_* f\}$$

is the convex hull of $N(g)$.

Preference structure $\succcurlyeq^*$ is characterized by uncertainty aversion. Preference structure $\succcurlyeq_*$, on the other hand, is characterized by mixture domination. Thus, these conjugate representations extend the variational representation of Maccheroni, Marinacci, and Rustichini (2006) and the dual representation of Chambers and Quiggin (2007) by generating multiple-prior representations for incomplete structures. They also extend the Bewley variational preference structure and the Bewley structure by not requiring simultaneous satisfaction of uncertainty aversion and mixture domination. In particular, note that there is now no requirement that $t^*(\pi,g)$ or $b^*(\pi,g)$ satisfy particular curvature conditions in g.

Example 16 *Any $\hat{P}(g)$ or $\hat{N}(g)$ developed as in Example 5 can be made consistent with (A.5.5) by constructing its free disposal convex hull. By Carathéodory's Theorem, imposing convexity solely upon $\hat{P}(g)$ requires that the convex hull of* $\{(.5,2.5),(1.1),\ (2,0)\}$,

$$co\{(.5,2.5),(1,1),\ (2,0)\} = \left\{f \in \mathcal{A}^2 : f = \lambda_1(.5,2.5) + \lambda_2(1.1) + \lambda_3(2,0), \lambda \in \Delta^3\right\},$$

where Δ^3 is the unit simplex for $\mathbb{R}^3$, belong to the new-least-as good set. It is then evident that the following least-as-good set satisfying (A.1)-(A.4) and (A.5.5),

$$\hat{P}^*(g) = \left\{ f \in \mathcal{A}^2 : f \geq \lambda_1 (.5, 2.5) + \lambda_2 (1,1) + \lambda_3 (2,0), \lambda \in \Delta^3 \right\},$$

is consistent with these data. Moreover, the corresponding upper translation function for arbitrary f can be calculated via

$$t^*(f,g) = \max \left\{ \beta : f - \beta \geq \lambda_1 (.5, 2.5) + \lambda_2 (1,1) + \lambda_3 (2,0), \lambda \in \Delta^3 \right\},$$

which is a simple linear program. For this represesentation $\partial^f t^*(g,g) = \partial_f t^*(g,g)$ is the convex hull of the vectors normal to $(.5, 2.5) + \alpha [(1,1) - (.5, 2.5)]$ and $(2,0) + \alpha [(1,1) - (2,0)]$ for $\alpha \in (0,1)$.

Example 17 *Exactly parallel arguments demonstrate how to construct a no-better-than set consistent with (A.5.5) for any observed data set. It follows immediately that*

$$f \succcurlyeq_*^* g \Leftrightarrow \left\{ \begin{array}{c} (f,g) : f \geq \lambda_1 (.5, 2.5) + \lambda_2 (1,1) + \lambda_3 (2,0), \\ g \leq \lambda_1 (-.5, 1.5) + \lambda_2 (1,1) + \lambda_3 (1.5, -.5), \lambda \in \Delta^3 \end{array} \right\}$$

is consistent with (A.1)-(A.4) and (A.5.5). And thus a dominance independent preference order can be constructed that will rationalize any observed data set.

Their closely parallel structure ensures that developments made for t^* translate readily into developments for b^*. Therefore, in what follows attention is concentrated on the former.

Borrowing Maccheroni, Marinacci, and Rustichini's (2006) terminology, we refer to $t^*(\pi, g)$ as a (local) *uncertainty aversion index.* (Implicitly, uncertainty neutrality is defined as $\pi' g$.) The main difference between their construct and ours is that ours is reference-dependent. Consequently, where the Maccheroni, Marinacci, Rustichini (2006) analogue to $t^*(\pi, g)$ is always "grounded" or bounded by zero, ours is grounded by the expected value of the status-quo gamble $\pi' g$.

This difference reflects the fact that complete preference structures, as Corollary 4 demonstrates, require three comparisons to make a single pairwise comparison. So, for example, f is ranked relative to $0 \in X$, g is ranked relative to $0 \in X$, and the two are then compared to arrive at a relative ranking. In our set up, gambles are compared directly.

The interpretation of t^* as an index of local uncertainty aversion reflects its ability to capture how agent behavior departs from that of an individual who exhibits maximal preference incompleteness. An individual with maximal preference incompleteness willingly adopts only alternatives that at least weakly dominate g in all states $s \in S$. This reflects a complete inability to attach any information to the relative likelihood of alternative states occuring. Behaviorally, such individuals always prepare for the worst. When confronted with any chance to gamble at odds π, the individual would anchor himself or herself to g and realize an expected return of $\pi' g$ without encountering any marginal risk.

We thus define the *index of local absolute uncertainty aversion,* $A : \Delta \times \mathcal{A}^S \to \mathbb{R}_-$ as

$$A(\pi, g) = t^*(\pi, g) - \pi' g \leq 0.$$

We then say that preference structure a exhibits *more local absolute uncertainty aversion* than b if

$$A^a(\pi, g) \geq A^b(\pi, g) \text{ for all } \pi \in \Delta, g \in \mathcal{A}^S,$$

where $A^a(\pi, g) \equiv t^{*a}(\pi, g) - \pi' g$, $A^b(\pi, g) \equiv t^{*b}(\pi, g) - \pi' g$, $t^{*a}(\pi, g)$ denotes the conjugate function for preference structure a, and $t^{*b}(\pi, g)$ denotes the conjugate function for preference structure b. In other words, the closer the individual sticks to not accepting any marginal gambles from g, the more uncertainty averse the individual is. Because for all $f, g \in \mathcal{A}^S$, $f \succcurlyeq g$ implies $f \succcurlyeq^* g$, but not the other way around, the latter preference relation is never more uncertainty averse than the former.

Proposition 18 ***(a)*** *Preference order* $\succcurlyeq^*_a$*satisfying* $(A.1) - (A.4)$ *is more local uncertainty averse than preference order* $\succcurlyeq^*_b$ *if and only if*

$$t^{**b}(f, g) \geq t^{**a}(f, g) \text{ for all } f \in \mathcal{A}^S.$$

(b) *Preference order* $\succcurlyeq_a$ *satisfying* $(A.1)-(A.4)$ *and uncertainty aversion is more uncertainty averse at* g *than preference order* $\succcurlyeq_b$ *if and only if*

$$t^b(f, g) \geq t^a(f, g) \text{ for all } f \in \mathcal{A}^S.$$

Restricting attention to the case where $f, g \in \mathcal{A}^S_{++}$, an exactly parallel argument shows that defining a local *index of relative uncertainty aversion* by

$$R(\pi, g) = \frac{t^*(\pi, g)}{\pi' g},$$

and defining *more relatively uncertainty averse* in a parallel fashion yields a proposition that parallels Proposition 18 directly. To conserve space, we leave the details to the reader.

Our next result establishes the consequences of imposing versions of A.5 on $\succcurlyeq^*$.

Proposition 19 *Assume* $(A.1) - (A.4)$*. Then,*

(a) *If* $\succcurlyeq^*$ *satisfies* $(A.5.1)$*, then for all* $g \in \mathcal{A}^S, x \in X,\ A(\pi, g + x) = A(\pi, g)$;

(b) *If* $\succcurlyeq^*$ *satisfies* $(A.5.2)$*, then for all* $g \in \mathcal{A}^S_{++}, \mu > 0,\ R(\pi, \mu g) = R(\pi, g)$*;*

(c) *If* $\succcurlyeq^*$ *satisfies* $(A.5.3)$*, then both parts (a) and (b) apply;*

(d) *If* $\succcurlyeq^*$ *satisfies* $(A.5.4)$*, then for all* $g \in \mathcal{A}^S, \alpha \in (0,1)$, $A\left(\frac{\pi}{\alpha}, g\right) \leq (1-\alpha) A\left(\frac{\pi}{1-\alpha}, g\right)$ *;and*

(e) *Assume* $(A.1)$*,* $(A.3)$*,* $(A.4)$*, and* $(A.5.5)$*. Then for all* $g^0, g' \in \mathcal{A}^S, \alpha \in (0,1)$, $A(\pi, \alpha g^0 + (1-\alpha) g') \leq \alpha A(\pi, g^0) + (1-\alpha) A(\pi, g')$.

Thus, (A.5.1) implies constant absolute uncertainty aversion. (A.5.2) implies constant relative uncertainty aversion. (A.5.3) implies both constant absolute uncertainty aversion and constant relative uncertainty aversion, which might be thought of as constant uncertainty aversion by analogy with constant risk aversion (Safra and Segal 1998, Quiggin and Chambers 1998). Betweeness limits the effect of rescaling of priors on uncertainty aversion.

Dominance independence requires that absolute uncertainty aversion be convex in g and thus will be minimized where

$$0 \in \partial_g t^*(\pi, g) - \pi,$$

which is satisfied at g where the subdifferential in the sense of convex analysis of $\partial_g t^*(\pi, g)$ in g is the prior in question. Standard results from convex analysis (Rockafellar 1970), however, ensure that

$$\pi \in \partial t_f(f, g) \Leftrightarrow f \in \partial_\pi t^*(\pi, g),$$

so that uncertainty aversion is minimized at the g where the priors supporting $t(f, g)$ at f are the priors supporting $t^*(\pi, g)$ at g.

5 Bewley, Variational, and Maximin Preferences

This section demonstrates the relationships between the well-known Bewley preference structure and our results. Bewley preferences require independence, which strengthens (A.5.1), (A.5.2), (A.5.3), (A.5.4), and (A.5.5).

(A.5.6) (Independence) For all $f, g, h \in \mathcal{A}^S$, $\alpha \in (0,1)$, $f \succcurlyeq g \Leftrightarrow \alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h$.

Independence requires that the ranking of acts not change even after mixing with arbitrary acts. By contrast, (A.5.1) and (A.5.3) only require that mixing with constant acts preserve rankings. Independence also requires radial independence as a special case. Finally, independence strengthens dominance independence by requiring that for all

$$f, g, h \in \mathcal{A}^S, \alpha \in (0,1), \alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h \Rightarrow f \succcurlyeq g$$

Under independence, both $P(g)$ and $N(g)$ are convex for all acts g (see, for example, Galabataar and Karni 2013). Under (A.1)-(A.4) and (A.5.6), a straightforward extension of Proposition 6 reveals that for all $\alpha > 0$ and all $f, g, h \in \mathcal{A}^S$

$$t(\alpha f + h, \alpha g + h) = \alpha t(f, g),$$

whence

$$t(f, g) = t(f - g, 0)$$

and

$$t(\alpha(f-g), 0) = \alpha t(f-g, 0).$$

Because convexity of $P(g)$ ensures concavity of $t(f,g)$, positive homogeneity of $t(f-g,0)$ in $f-g$ implies superlinearity. Superlinearity in $f-g$ ensures that (Rockafellar 1970, Corollary 13.2.1)

$$t(f-g, 0) = \inf\left\{\pi'(f-g) : \pi \in D\right\}, \tag{8}$$

where $D \subset \Delta$ is a closed and convex set given by

$$\begin{aligned} D &= \left\{\pi \in \Delta : \pi'(f-g) \geq t(f-g,0) \text{ for all } f-g \in \mathcal{A}^S\right\} \\ &= \partial t_f(0,0) \\ &= \partial^f t(0,0). \end{aligned}$$

Expression (8) corresponds to Bewley's (1986) representation result.

Maintaining independence, Corollary 7 implies

$$\begin{aligned} b(f,g) &= t(g,f) = \inf\{\pi'(g-f) : \pi \in D\} \\ &= -\sup\left\{\pi'(f-g) : \pi \in D\right\}. \end{aligned}$$

so that completeness occurs if and only if for all f, g

$$\inf\left\{\pi'(f-g) : \pi \in D\right\} - \sup\left\{\pi'(f-g) : \pi \in D\right\} = 0.$$

That is, D must be a singleton set so that the Minkowski set difference $D - D$ is singleton set $\{0\}$ (Schneider 1993). And so, as previously established by Rigotti and Shannon (2005), Bewley preferences are complete if and only if they are consistent with those of an expected-utility maximizer.

The key behavioral difference between dominance independence, (A.5.5), and independence, (A.5.6), is that the implication in the definition of the former only runs in one direction (Faro 2015). Suppose instead that dominance independence were strengthened to require for all $f, g, h, i \in A^S$, $\alpha \in (0,1)$,

$$f \succcurlyeq g \text{ and } h \succcurlyeq i \Leftrightarrow \alpha f + (1-\alpha) h \succcurlyeq \alpha g + (1-\alpha) i.$$

Taking $0 \succcurlyeq 0$, reflexivity and this strengthened version of dominance independence requires that for all $f, g \in A^S$, $\alpha \in (0,1)$

$$f \succcurlyeq g \Leftrightarrow \alpha f \succcurlyeq \alpha g,$$

whence choosing f and g such that

$$\frac{f}{\alpha} \succcurlyeq \frac{g}{\alpha} \Leftrightarrow f \succcurlyeq g$$

for $\alpha \in (0,1)$, which implies

$$f \succcurlyeq g \Leftrightarrow \mu f \succcurlyeq \mu g$$

for all $f, g \in A^S$ and $\mu > 0$.

Because $\frac{h}{(1-\alpha)} \succcurlyeq \frac{h}{(1-\alpha)}$ with h arbitrary, it therefore follows that the strengthened version of dominance independence requires

$$f \succcurlyeq g \Leftrightarrow \mu f + h \succcurlyeq \mu g + h,$$

for all $f, g, h \in A^S$ and $\mu > 0$, which is equivalent to independence. Proceeding as before then yields the Bewley case.

Weak certainty independence, (A.5.1), relaxes (A.5.5). When weak certainty independence is coupled with uncertainty aversion (which is equivalent to convexity of $P(g)$), the multiple-prior representation requires that

$$t(f,g) = \inf_{\pi \in \Delta} \{\pi' f - t^*(\pi, g)\}$$

with $t^*(\pi, g)$ and closed concave in π and $t^*(\pi, g + x) = t^*(\pi, g) - x$ for all $g \in A^S, x \in X$. By Corollary 7, completeness of $\succcurlyeq$ requires that

$$t(f,g) = -t(g,f),$$

for all f, g, and specifically $t(0,g) = -t(g,0)$. Using Corollary 4 then gives

$$\begin{array}{ccc} f & \succcurlyeq & g \\ & \Updownarrow & \\ \inf_{\pi \in \Delta} \{\pi' f - t^*(\pi, 0)\} & \geq & \inf_{\pi \in \Delta} \{\pi' g - t^*(\pi, 0)\} \end{array}$$

with $t^*(\pi, x) = t(x, 0) + x$, which corresponds to Maccheroni, Marinacci, and Rustichini's (2006) variational model.

By Proposition 6, certainty independence requires that for all $\alpha > 0$

$$t(f, \alpha g) = \alpha t\left(\frac{f}{\alpha}, g\right),$$

and thus

$$\begin{aligned} t^*(\pi, \alpha g) &= \inf \left\{\pi' f - \alpha t\left(\frac{f}{\alpha}, g\right)\right\} \\ &= \alpha t^*(\pi, g). \end{aligned}$$

Thus, certainty independence when coupled with uncertainty aversion requires that the following multiple-prior representation characterize preferences

$$t(f,g) = \inf_{\pi \in \Delta} \{\pi' f - t^*(\pi, g)\},$$

with $t^*(\pi, g)$ closed concave in π and positively homogeneous and translatable in g.

Imposing completeness (Corollary 7) and taking $f = 0$, we obtain that for all $\alpha > 0$

$$t(0, \alpha g) = \alpha t(0, g) \Leftrightarrow t(\alpha g, 0) = \alpha t(g, 0).$$

Uncertainty aversion implies that $t(g, 0)$ is concave in g. Positive homogeneity then implies that $t(g, 0)$ is superlinear and the support function for $\partial t(0, 0)$. Hence,

$$t(g, 0) = \inf \{\pi' g : \pi \in \partial t(0, 0)\},$$

which corresponds to the maximin preference structure (Gilboa and Schmeidler 1989).

6 Conclusion

The standard model of decision theory is that of an unboundedly rational agent, who has well-specified preferences over every conceivable prospect (completeness) and who obeys the axioms of expected utility theory. This model has long been recognized to be impossibly demanding for real human agents. In addition to its normative appeal the dominance reflects in part the availability of tools to derive representations of preferences from observational data

In this paper, we have developed a characterization of incomplete preferences in terms of upper and lower translation functions, observable from choice data. We have shown how the properties of these functions relate to axiomatic conditions on preferences. For local properties, we have developed the connection between the Clarke superdifferential and the belief structure, and shown the relationship to various multiple priors models.

Our result have a number of practical implications. The most notable are the following two mechanisms. First, we have developed a procedure that uses the reports of willingness to pay and willingness to accept to determine whether decision-makers can rank different alternatives. Second, we have outlined how to use these reports to construct approximations of least-as-good and no-better-than sets. The next natural step in this research agenda is to take this methodology to the lab and naturally occurring data.

7 Appendix

Proof of Proposition 1: (a) Consider arbitrary $f, g \in \mathcal{A}^S$ with $f \succcurlyeq g$. The latter ranking implies that $0 \in \{\beta \in \mathbb{R} : f - \beta \succcurlyeq g\}$. Hence, by the definition of the upper translation function, $t(f,g) \geq 0$. Conversely, consider $f, g \in \mathcal{A}^S$ for which $t(f,g) \geq 0$. It then follows from the definition of the upper translation function that $f - t(f,g) \succcurlyeq g$. Using monotonicity (axiom (A.3)) and transitivity (axiom (A.2)), we obtain $f \succcurlyeq g$. Finally, strict monotonicity of $\succcurlyeq$ implies that $t(g,g) = 0$ for arbitrary $g \in \mathcal{A}^S$.
(b) For arbitrary $f \in \mathcal{A}^S$ and $x \in X$, we have

$$\begin{aligned} t(f+x,g) &= \max\{\beta \in \mathbb{R} : f + x - \beta \succcurlyeq g\} \\ &= x + \max\{\beta - x \in \mathbb{R} : f - (\beta - x) \succcurlyeq g\} \\ &= t(f,g) + x. \end{aligned}$$

(c) Consider arbitrary $f, f^o, g \in \mathcal{A}^S$ with $f^o \geq f$. If $f - \beta \succcurlyeq g$, then by monotonicity (axiom (A.3)) and transitivity (axiom (A.2)) we have that $f^o - \beta \succcurlyeq g$. Hence, $\{\beta \in \mathbb{R} : f - \beta \succcurlyeq g\} \subseteq \{\beta \in \mathbb{R} : f^o - \beta \succcurlyeq g\}$, which implies that $t(f^o, g) \geq t(f,g)$. To establish the second part, note that $f^o - t(f^o, g) \succcurlyeq g \succcurlyeq g^o$ by the definition of t, monotonicity, and transitivity. Hence, $t(f^o, g^o) \geq t(f^o, g) \geq t(f,g)$.
(d) Translatability in the numeraire requires that

$$t(f+x,g) = t(f,g) + x \text{ for any } x \in X.$$

For $f, f' \in \mathcal{A}^S$, choose $K \in \mathcal{A}$ so that $f \leq f' + K\|f' - f\|$. By monotonicity and translatability, we have $t(f,g) \leq t(f' + K\|f'-f\|, g) = t(f',g) + K\|f' - f\|$, whence $t(f,g) - t(f',g) \leq K\|f' - f\|$. Switching the roles of g and g' in the last inequality gives

$$|t(f,g) - t(f',g)| \leq K\|f' - f\|,$$

establishing the Lipschitz property. Continuity in g follows immediately from continuity of $\succcurlyeq$. This establishes (d). Parts (a')-(d') of the present Proposition are derived in an analogous manner. ■

Proof of Proposition 2: Parts (a)-(d) of Proposition 1 provide one direction of the proposition. We now turn to proving the reverse direction.

It follows from the definition of the upper translation function and Proposition 1.b (translatability in the numeraire) that $t(f, f) = 0$. Axiom (A.1) (reflexivity) then follows by applying Proposition 1.a (indication).

From Proposition 1.c, for all $f \in \mathcal{A}^S$ and all $g \in \mathcal{A}^S / \{0\}$, $t(f+g, f) \geq t(f, f) = 0$, where the last equality follows from the definition of the upper translation function and Proposition 1.b (translatability in numeraire). From the definition of the translation function, $f + g - t(f+g, f) \succcurlyeq f$. The latter ordering and $t(f+g, f) \geq 0$ imply, by Proposition 1.a (indication), that $f + g \succcurlyeq f$. It also follows by the definition of the preference structure that $(f \succcurlyeq f+g)^c$. Thus, $\succcurlyeq$ satisfies axiom (A.3) (monotonicity).

To prove transitivity, take arbitrary $h, f, g \in \mathcal{A}^S$ and suppose that $h \succcurlyeq f$ and $f \succcurlyeq g$. By the definition of the upper translation function, $h - t(h, f) \succcurlyeq f$ and $f - t(f, g) \succcurlyeq g$. By the definition of the upper translation function and axiom (A.3) (monotonicity), these two rankings imply $t(h, f) \geq 0$ and $t(f, g) \geq 0$. By Proposition 1.c (monotonicity), $h \succcurlyeq f$ implies $t(h, g) \geq t(f, g)$. Combining the last inequality with $t(f, g) \geq 0$, we obtain that $t(h, g) \geq 0$. Hence, by 1.a (indication), $f \succcurlyeq g$. Thus, $\succcurlyeq$ satisfies axiom (A.2) (transitivity).

That $P(g)$ is closed follows by Proposition 1.a (indication) and the closed concavity of $t(f, g)$ in f. Lipschitz continuity of $t(f, g)$ in g establishes that $t(f, g)$ is uniformly continuous in g and that $N(g)$ is closed.

Consider arbitrary $f, g, \in \mathcal{A}^S$ and $x \in X$. By Proposition 1.e, $t(f+x, g+x) = t(f, g)$. Proposition 1.a (indication) and the latter equality then imply, by monotonicity, that $f \succcurlyeq g$ is equivalent to $f + x \succcurlyeq g + x$. ■

Proof of Proposition 3: We first prove part (a), then part (c), and finally part (b).

(a) $\Rightarrow$ Assume that $b(f, g) + t(f, g) = 0$. For this equality to be satisfied, either both of the functions on the left-hand-side have to be zero, implying comparability, or they have to have opposite signs. Without loss of generality, suppose $b(f, g)$ is strictly positive so that $g \succ f$. The presumed equality establishes that $t(f + b(f, g), g) = 0$ so that $g \succcurlyeq f + b(f, g)$ and $f + b(f, g) \succcurlyeq g$ and thus $f + b(f, g) \in I(g)$. $\Leftarrow$ Assume completeness and suppose that $t(f, g) + b(f, g) < 0$, then it must be true that $b(f - t(f, g), g) < 0$ so that $g \succ f - t(f, g) \succcurlyeq g$ which violates axiom (A.3). Hence, $t(f, g) + b(f, g) = 0$.

(c) We begin with the second inequality in Proposition 3.c; $b(f, g) + t(f, g) \leq 0$. If f and

g are not comparable, Proposition 1.a and a' establish that $t(f,g) < 0$ and $b(f,g) < 0$ to satisfy the second inequality inequality in Proposition 3.c. Now consider the case where f and g are comparable. Comparability requires that either $f \succcurlyeq g$ or that $g \succcurlyeq f$. Without loss of generality, suppose the latter. Proposition 1.a' requires that $b(f,g) \geq 0$ and that $g \succcurlyeq f + b(f,g)$. Proposition 1.b implies $t(f,g) + b(f,g) = t(f + b(f,g), g)$. If $t(f,g) + b(f,g)$ was strictly positive then, by axiom (A.3), we would have $f + b(f,g) \succ g$, which is a contradiction. Hence, $0 \geq b(f,g) + t(f,g)$.

Consider now the first inequality in Proposition 3.c. Axiom (A.3) ensures that for all g

$$g + \mathcal{A}_+^S \subset P(g).$$

Define

$$\begin{aligned}
\hat{t}(f,g) &= \max\left\{\beta : f - \beta \in g + \mathcal{A}_+^S\right\} \\
&= \max\left\{\beta : f_s - g_s \geq \beta \text{ for all } s \in S\right\} \\
&= \min_{s \in S}\left\{f_s - g_s\right\}
\end{aligned}$$

and note that

$$f - \hat{t}(f,g) \in g + \mathcal{A}_+^S$$

for all f, g. Thus, $f - \hat{t}(f,g) \in P(g)$ for all g, whence $t(f,g) \geq \hat{t}(f,g)$ for all f, g. Similarly, axiom (A.3) also ensures that $g - \mathcal{A}_+^S \subset N(g)$. Define

$$\begin{aligned}
\hat{b}(f,g) &= \max\left\{\beta : f + \beta \in g - \mathcal{A}_+^S\right\} \\
&= -\max_{s \in S}\left\{f_s - g_s\right\},
\end{aligned}$$

and note that $f - \max_{s \in S}\{f_s - g_s\} \in g - \mathcal{A}_+^S$ to establish that $b(f,g) \geq -\max_{s \in S}\{f_s - g_s\}$ for all f, g, and thus

$$b(f,g) + t(f,g) \geq \min_{s \in S}\{f_s - g_s\} - \max_{s \in S}\{f_s - g_s\}.$$

(b) $\Rightarrow$ Without loss of generality suppose that $b(f,g) \geq 0$. Then it must be true by part (c) of the present proposition that $t(f,g) \leq 0$, and the desired inequality follows. $\Leftarrow$To go the other way, suppose, first, that $t(f,g)\, b(f,g) = 0$. There are three possibilities; either

$b(f,g)$ alone, or $t(f,g)$ alone, or both are zero. All of these possibilities imply comparability by Proposition 1. Now suppose that $t(f,g)\,b(f,g) < 0$. For this inequality to hold, either $t(f,g)$ or $b(f,g)$ must be strictly positive and either condition implies comparability. ■

Proof of Proposition 6: For each part of the proposition, $\Leftarrow$ direction follows by indication (Proposition 1.a and a'). We now prove the reverse direction for each of the parts.

(e.1) $\Rightarrow$ Setting $x = 0$, axiom (A.5.1) can be written as

$$\alpha f \succcurlyeq \alpha g \Leftrightarrow \alpha f + (1-\alpha)\,y \succcurlyeq \alpha g + (1-\alpha)\,y, \tag{9}$$

for all $\alpha \in (0,1)$, $f, g \in \mathcal{A}^S$, and $y \in X$. Because $y \in X$ is arbitrary, by taking $z \equiv (1-\alpha)\,y$, condition 9 requires that for all $z \in X$

$$\alpha f \succcurlyeq \alpha g \Leftrightarrow \alpha f + z \succcurlyeq \alpha g + z,$$

which can be rewritten as

$$f' \succcurlyeq g' \Leftrightarrow f' + z \succcurlyeq g' + z$$

for all $z \in X$ and $f', g' \in \mathcal{A}^S$. Thus,

$$\begin{aligned} t(f,g) &= \max\{\beta : f - \beta \succcurlyeq g\} \\ &= \max\{\beta : f \succcurlyeq g + \beta\} \\ &= b(g,f). \end{aligned}$$

(e.2) $\Rightarrow$ By radial invariance, for all $f, g, \in \mathcal{A}^S$, $\alpha \in (0,1)$,

$$\frac{f}{\alpha} \succcurlyeq \frac{g}{\alpha} \Leftrightarrow f \succcurlyeq g \Leftrightarrow \alpha f \succcurlyeq \alpha g,$$

so that $f \succcurlyeq g \Leftrightarrow \mu f \succcurlyeq \mu g$ for all $\mu > 0$. Hence,

$$\begin{aligned} t(\mu f, \mu g) &= \max\{\beta \in \mathbb{R} : \mu - \beta \succcurlyeq \mu g\} \\ &= \mu \max\left\{\frac{\beta}{\mu} \in \mathbb{R} : f - \frac{\beta}{\mu} \succcurlyeq g\right\} \\ &= t(f,g). \end{aligned}$$

Similarly,

$$\begin{aligned} b(\mu f, \mu g) &= \max\{\beta \in \mathbb{R} : \mu g \succcurlyeq \alpha f + \beta\} \\ &= \mu \max\left\{\frac{\beta}{\alpha} \in \mathbb{R} : g \succcurlyeq f + \frac{\beta}{\alpha}\right\}. \end{aligned}$$

(e.3) Follows from (e.1) and (e.2).

(e.4) For all $f, g \in \mathcal{A}^S$

$$f - t(f,g) \succcurlyeq g$$

Betweenness thus requires that for all $\alpha \in (0,1)$

$$\begin{aligned} f - t(f,g) + \alpha (g - f + t(f,g)) &\succcurlyeq g \\ f + \alpha (g - f) - (1-\alpha) t(f,g) &\succcurlyeq g, \end{aligned}$$

and hence

$$\begin{aligned} t(f + \alpha (f - g), g) &= \max \{\beta : f + \alpha (g - f) - \beta \succcurlyeq g\} \\ &\geq (1-\alpha) t(f,g). \end{aligned}$$

Similarly,

$$g \succcurlyeq f + b(f,g),$$

whence

$$g \succcurlyeq f + \alpha (g - f) + (1-\alpha) b(f,g),$$

which implies

$$b(f + \alpha (f - g), g) \geq (1 - a) b(f,g).$$

Proof of Proposition 8: By definition for all $f, g, h, i \in \mathcal{A}^S$,

$$\begin{aligned} f - t(f,g) &\succcurlyeq g, \\ h - t(h,i) &\succcurlyeq i. \end{aligned}$$

Dominance independence requires that for all $\alpha \in (0,1)$

$$\alpha f + (1-\alpha) h - \alpha t(f,g) - (1-\alpha) t(h,i) \succcurlyeq \alpha g + (1-\alpha) i,$$

which implies that

$$t(\alpha f + (1-\alpha) h, \alpha g + (1-\alpha) i) \geq \alpha t(f,g) + (1-\alpha) t(h,i).$$

Thus, $t(f,g)$ is concave in f. Define the concave conjugate of t by $t^* : \Delta \times \mathcal{A}^S \to \bar{\mathbb{R}}$ by

$$t^*(\pi, g) = \inf_f \{\pi' f - t(f,g)\}.$$

t^* is closed convex and by Theorem 12.2 and Corollary 12.2.1 of Rockafellar (1970) it forms a dual pair with $t(f,g)$ in the sense that

$$t(f,g) = \inf_{\pi \in \Delta} \{\pi' f - t^*(\pi, g)\}.$$

This establishes the general structural result. Reflexivity (A.1) establishes that $t^*(\pi, g) \leq \pi' g$ and monotonicity establishes that $t^*(\pi, g)$ is nondecreasing in g. To establish the desired convexity note that:

$$\begin{aligned} t^*(\pi, \alpha g^o + (1-\alpha) g') &= \inf\{\pi'(\alpha f^o + (1-\alpha) f') - t(\alpha f^o + (1-\alpha) f', \alpha g^o + (1-\alpha) g')\} \\ &\leq \inf\{\pi'(\alpha f^o + (1-\alpha) f') - \alpha t(f^o, g^o) + (1-\alpha) t(f', g')\} \\ &= \alpha t^*(\pi, g^o) + (1-\alpha) t^*(\pi, g'). \end{aligned}$$

The proof for b is completely parallel and thus not repeated. ■

Proof of Proposition 10: By the Lipschitz property, both t and b possess the Clarke directional derivatives (Clarke 1983). To establish the first part, note that indication requires $t(f+x, g) = t(f,g) + x$ for all $f, g \in \mathcal{A}^S$ $x \in X$. Hence,

$$\begin{aligned} t^f(f, g; x) &= \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{t(f^o + \lambda x, g) - t(f,g)}{\lambda} \\ &= \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{t(f^o, g) + \lambda x - t(f,g)}{\lambda} \\ &= x. \end{aligned}$$

The proof that $b^f(f, g; x) = -x$ is virtually identical. Thus, $q \in \partial^f t(f,g)$ only if for all $x \in X$

$$q' x \geq x \text{ and } -q' x \leq -x$$

so that $q' x = 1$. That $\partial^f t(f,g) \subset \Delta$ and $\partial^f b(f,g) \subset -\Delta$ now follows from Proposition 1. ■

Proof of Proposition 11: Observe that

$$g \in \arg\max_f \{t(f,g) + b(f,g)\}$$

from which it follows (Clarke 1983, Proposition 2.3.2) that

$$0 \in \partial^f t(g,g) + \partial^f b(g,g),$$

which establishes the first part. For the second, note that

$$\begin{aligned} 0 &= t(g,g) + b(g,g) \\ &\geq t(f,g) + b(f,g) \end{aligned}$$

for all $f, g \in \mathcal{A}^S$ so that $t(f,g) + b(f,g)$ attains its global maximum. Hence, marginal departures from g in all directions $h \in \mathcal{A}^S$ must be nonpositive, whence

$$\begin{aligned} \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{t(f^o + \lambda h, g) - t(f,g) + b(f^o + \lambda h, g) - b(f,g)}{\lambda} &= \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{t(f^o + \lambda h, g) - t(f,g)}{\lambda} \\ &\quad + \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{b(f^o + \lambda h, g) - b(f,g)}{\lambda} \\ &\leq 0 \end{aligned}$$

and thus $t^f(g, g; h) + b^f(g, g; h) \leq 0$. ■

Proof of Proposition 12: If $\succcurlyeq$ is complete, Proposition 3 establishes that $t(f,g)+b(f,g) = 0$ for all f, g and thus the upper bound established in Proposition 3 implies

$$\mathcal{A}^S \subset \arg\max_f \{t(f,g) + b(f,g)\}$$

for all g. Proposition 2.3.2 in Clarke (1983) then implies

$$0 \in \partial^f t(f,g) + \partial^f b(f,g)$$

for all $f \in \mathcal{A}^S$. ■

Proof of Proposition 13 (a) Proposition 6 requires $t(f,g) = b(g,f)$. Hence, both $t(f,g)$ and $b(f,g)$ are Lipschitz in g establishing that t^g and b^g exist. Thus,

$$\begin{aligned} t^f(f+x, g+x; h) &= \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{t(f^o + x + \lambda h, g + x) - t(f^o + x, g + x)}{\lambda} \\ &= \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{t(f^o + \lambda h, g + x) - t(f^o, g + x)}{\lambda} \\ &= \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{b(g + x, f^o + \lambda h) - b(g + x, f^o)}{\lambda} \\ &= \lim \inf_{f^o \to f, \lambda \downarrow 0} \frac{b(g, f^o + \lambda h) - b(g, f^o)}{\lambda} \\ &= b^g(g, f; h) \\ &= t^f(f, g; h), \end{aligned}$$

where the second equality follows from Proposition 1.b, the third from $t(f,g) = b(g,f)$, the fourth from Proposition 1.b', and the fifth and sixth from $t(f,g) = b(g,f)$. This demonstrates the first and second equalities of this part. The third and fourth equalities follow similarly.

(b) By Proposition 6 for all $f, g \in \mathcal{A}^S$ and all $\mu > 0$, $t(\mu f, \mu g) = \mu t(f,g)$. Hence,

$$\begin{aligned} t^f(\mu f, \mu g; \mu h) &= \lim_{f^o \to f, \lambda \downarrow 0} \inf \frac{t(\mu f^o + \mu\lambda h, \mu g) - t(\mu f^o, \mu g)}{\lambda} \\ &= \mu \lim_{f^o \to f, \lambda \downarrow 0} \inf \frac{t(f^o + \lambda h, g) - t(f^o, g)}{\lambda} \\ &= \mu t^f(f, g; h). \end{aligned}$$

A parallel proof applies for b^f.

(c) Part (c) follows from parts (a) and (b).

(d) Proposition 6.e.4 requires that for all $f, g \in \mathcal{A}^S$ and $\alpha \in (0,1)$

$$t(f + \alpha(g - f), g) \geq (1 - \alpha) t(f, g),$$

and thus for all $\alpha \in (0,1)$

$$\frac{t(f + \alpha(g - f), g) - t(f, g)}{\alpha} \geq -t(f, g).$$

Hence,

$$\begin{aligned} t^g(f, g; g - f) &= \lim_{f^o \to f, \lambda \downarrow 0} \inf \frac{t(f^o + \lambda(g - f^o), g) - t(f^o, g)}{\lambda} \\ &\geq -t(f, g), \end{aligned}$$

establishing the result. A parallel proof applies for b^f.

(e) Proposition 8 establishes that t and b are concave in their arguments. Theorem 23.4 of Rockafellar (1970) establishes that the one-sided directional derivative (in the sense of convex analysis) exists and is the support function for its superdifferential. Proposition 2.2.6 and 2.2.7 of Clarke (1983) establish that the Clarke directional derivative corresponds to the one-sided directional derivative from convex analysis when t or b are concave. ■

Proof of Proposition 15⇒Monotonicity ensures that $ri(P(g)) \cap ri(N(g)) = \emptyset$. If both are convex, then the Finite-Dimensional Separating Hyperplane Theorem (see, for example, Aliprantis and Border 2006) ensures the existence of a $\pi \in \Delta$ such that

$$f \in P(g)$$

implies

$$\pi' f \geq \pi' g$$

and $f \in N(g)$ requires

$$\pi' g \geq \pi' f.$$

To show that $\pi' f \geq \pi' g$ implies $f \in P(g)$, suppose to the contrary that there exists an f such that $\pi' f \geq \pi' g$ but $f \notin P(g)$. Completeness then ensures that $f \in N(g)$ which leads to a contradiction unless $\pi' f = \pi' g$. $\Leftarrow$ Trivial.

Proof of Proposition 19: **(a)** By Proposition 6, axiom (A.5.1) implies that $t(f, g+x) = t(f, g) - x$ for all $x \in X$ and, hence,

$$\begin{aligned} t^*(\pi, g+x) &= \inf_f \{\pi' f - t(f, g+x)\} \\ &= \inf_f \{\pi' f - t(f, g) + x\} \\ &= t^*(\pi, g) + x \text{ for all } x \in X. \end{aligned}$$

Hence,

$$\begin{aligned} A(\pi, g+x) &= t^*(\pi, g+x) - \pi'(g+x) \\ &= t^*(\pi, g) - \pi' g \\ &= A(\pi, g) \ \text{ for all } x \in X. \end{aligned}$$

(b) By Proposition 6, axiom (A.5.2) requires that $t(f, \mu x) = \mu t\left(\frac{f}{\mu}, x\right)$ for all $\mu > 0$, whence

$$\begin{aligned} & t^*(\pi, \mu g) \\ =\ & \inf_f \{\pi' f - t(f, \mu g)\} \\ =\ & \mu \inf_f \left\{\pi' \frac{f}{\mu} - t\left(\frac{f}{\mu}, g\right)\right\} \\ =\ & \mu t^*(\pi, g) \ \text{ for all } \mu > 0. \end{aligned}$$

Hence,

$$R(\pi, \mu g) = \frac{t^*(\pi, \mu g)}{\mu \pi' g} = \frac{\mu t^*(\pi, g)}{\mu \pi' g} = R(\pi, g).$$

(c) Follows from parts (b) and (c).

(d) By Proposition 6, axiom (A.5.4) requires that $t(\alpha f + (1-\alpha) g, g) \geq (1-\alpha) t(f, g)$ for all $\alpha \in (0,1)$, whence

$$\begin{aligned}
& \inf \{\pi' f - t(\alpha f + (1-\alpha) g, g)\} \\
= \; & \inf \left\{ \frac{\pi}{\alpha}' [\alpha f + (1-\alpha) g - (1-\alpha) g] - t(\alpha f + (1-\alpha) g, g) \right\} \\
= \; & t^* \left(\frac{\pi}{\alpha}, g \right) - \frac{1-\alpha}{\alpha} \pi' g \\
\leq \; & \inf \{\pi' f - (1-\alpha) t(f, g)\} \\
= \; & (1-\alpha) \inf \left\{ \frac{\pi}{(1-\alpha)}' f - t(f, g) \right\} \\
= \; & (1-\alpha) t^* \left(\frac{\pi}{(1-\alpha)}, g \right) \text{ for all } \alpha \in (0,1).
\end{aligned}$$

Hence,

$$\begin{aligned}
A\left(\frac{\pi}{\alpha}, g\right) & = t^* \left(\frac{\pi}{\alpha}, g \right) - \frac{\pi}{\alpha}' g \\
& \leq (1-\alpha) t^* \left(\frac{\pi}{(1-\alpha)}, g \right) + \frac{1-\alpha}{\alpha} \pi' g - \frac{\pi}{\alpha}' g \\
& = (1-\alpha) \left(t^* \left(\frac{\pi}{(1-\alpha)}, g \right) - \frac{\pi}{(1-\alpha)}' g \right) \\
& = (1-\alpha) A \left(\frac{\pi}{1-\alpha}, g \right).
\end{aligned}$$

(e) Follows immediately from the convexity of $t^*(\pi, g)$. ■

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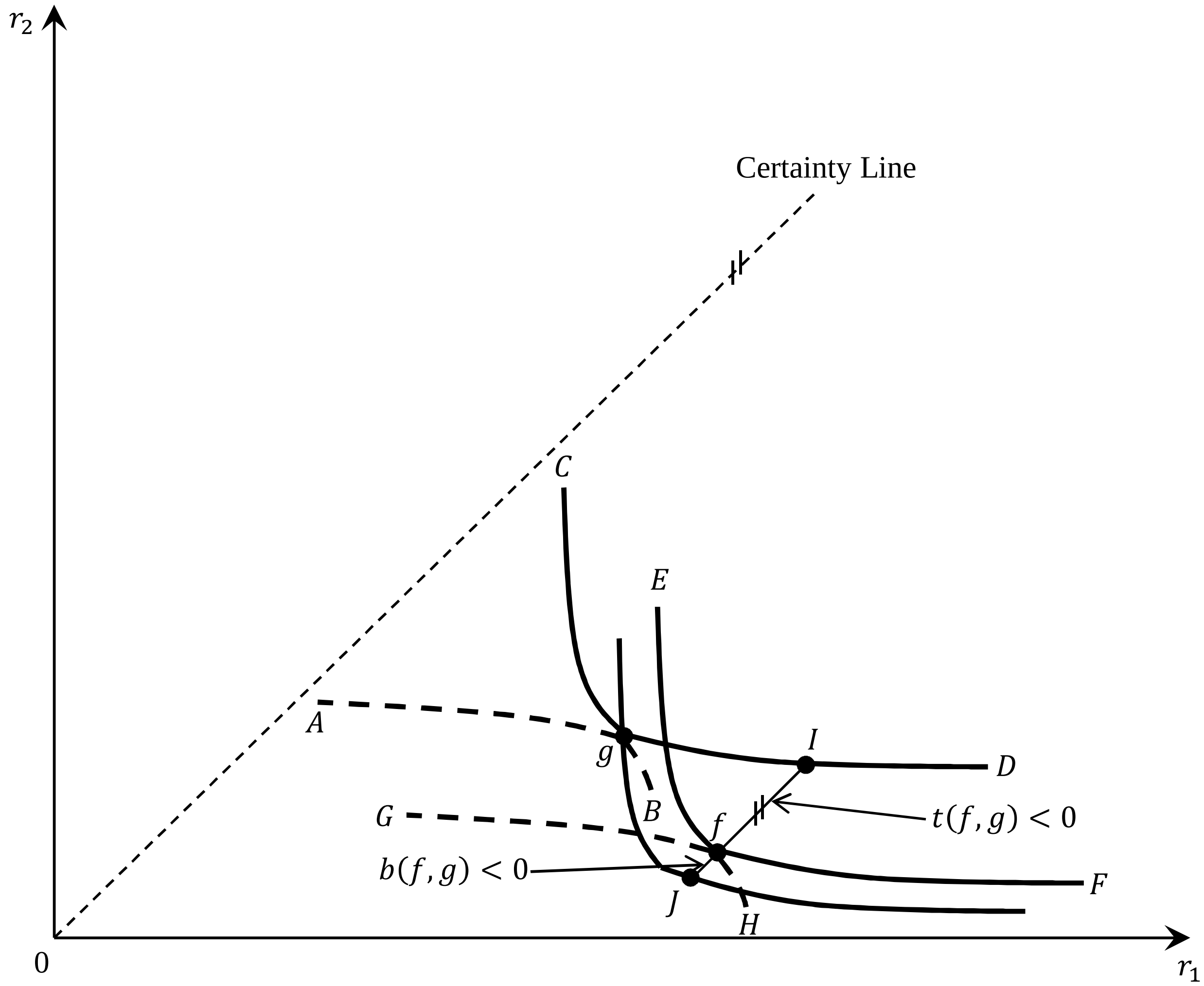

Figure 1. Illustration of Proposition 3

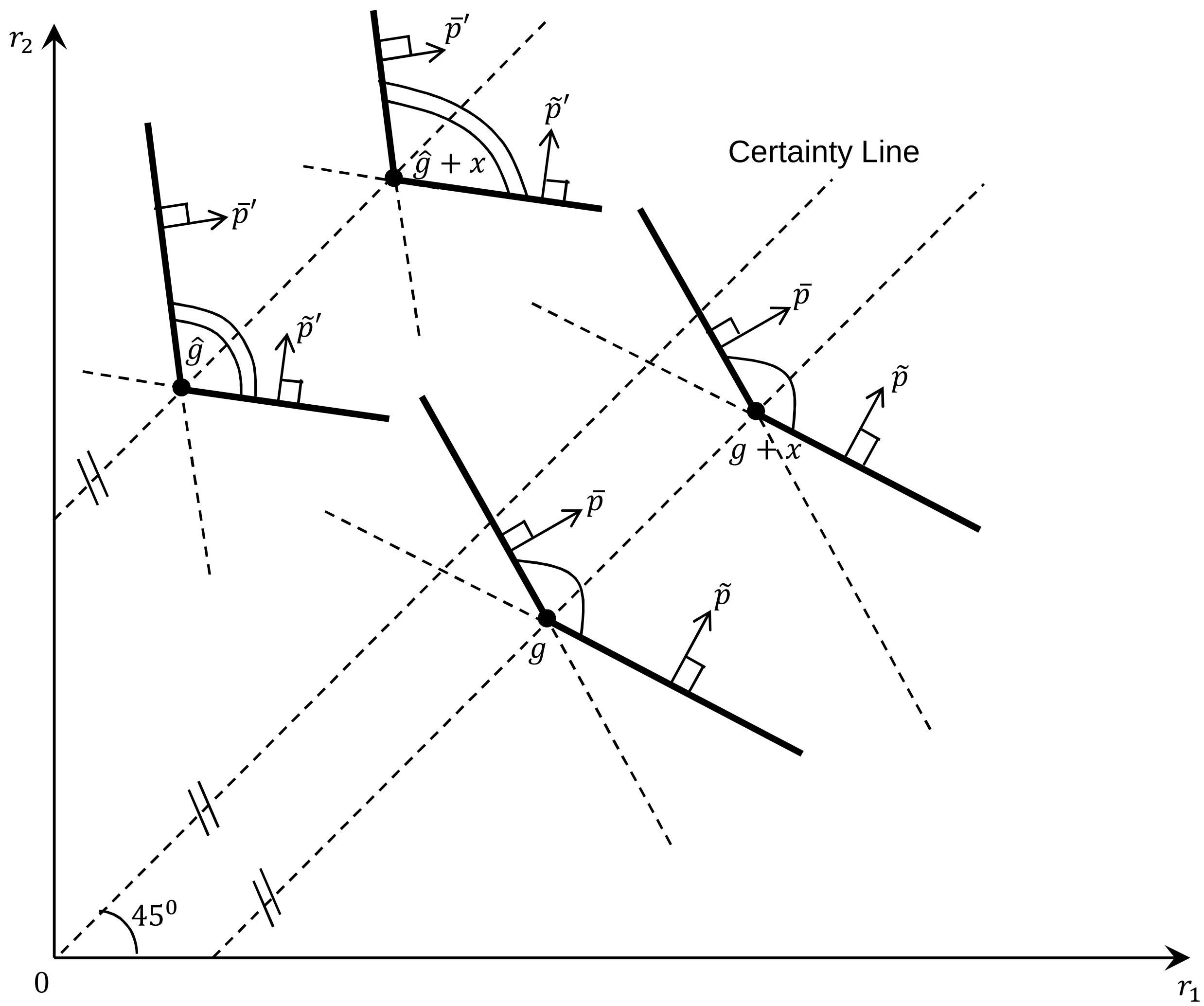

Figure 2. Incomplete Certainty Independent Preferences

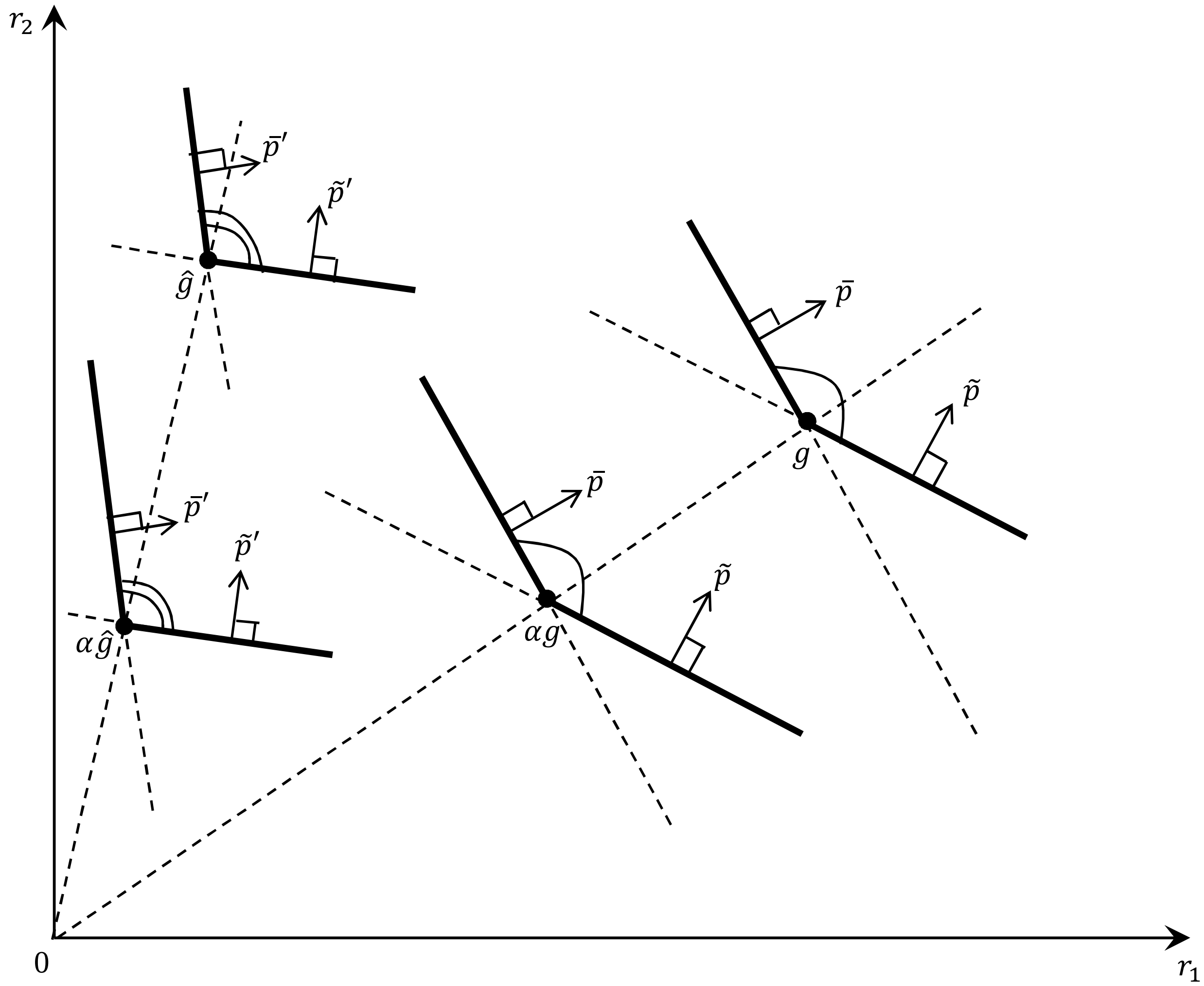

Figure 3. Incomplete Preferences Satisfying Radial Invariance

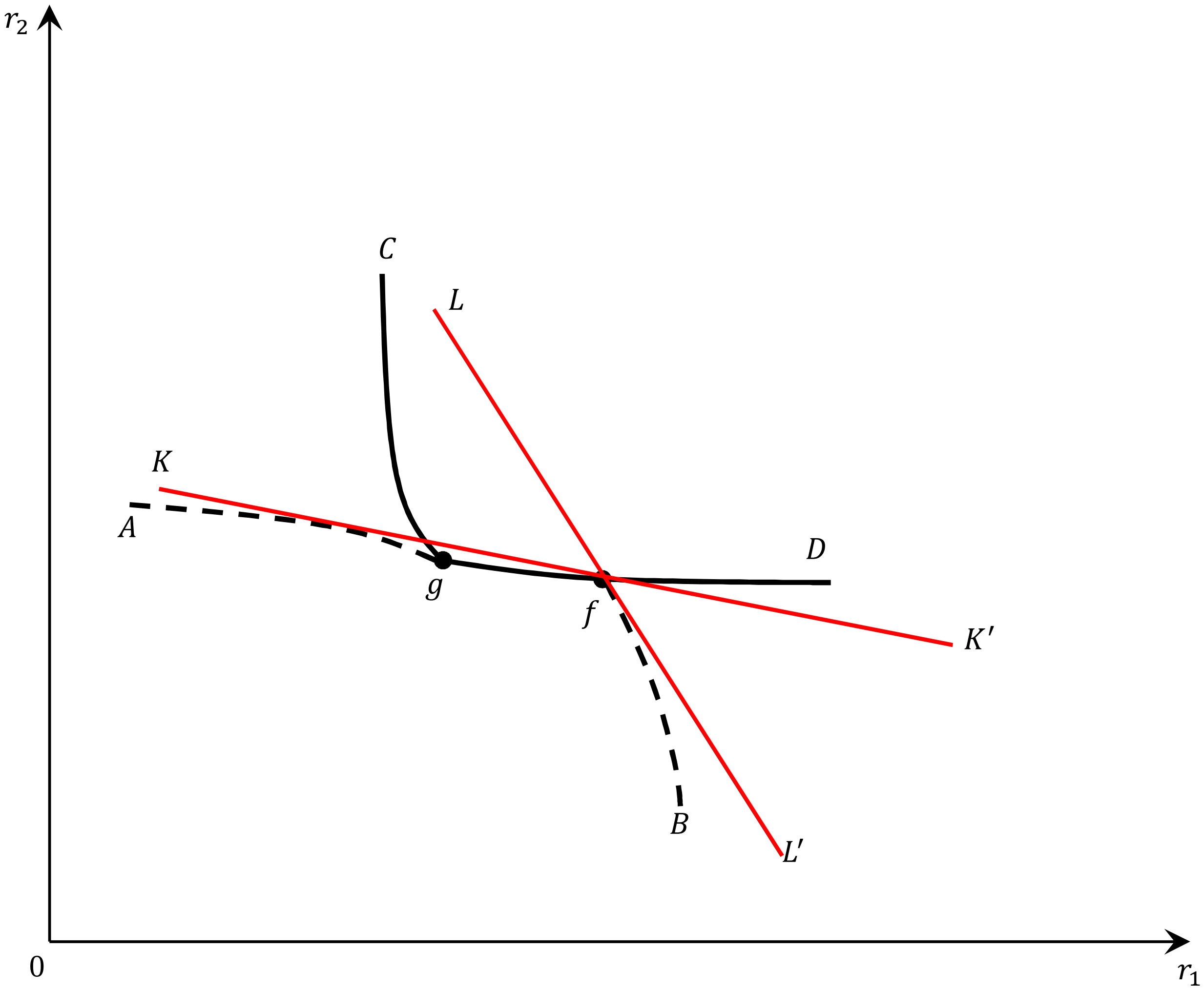

Figure 4. Subjective Beliefs