Finding the Value of Information About a State Variable in a Markov Decision Process[1]

Gilvan C. Souza
The Robert H. Smith School of Business, The University of Maryland, College Park, MD, 20742
Phone: 301-405-0628; fax: 301-405-8655; email: gsouza@umd.edu

ABSTRACT

In this paper we present a mixed–integer programming formulation that computes the optimal solution for a certain class of Markov decision processes with finite state and action spaces, where a state is comprised of multiple state variables, and one of the state variables is unobservable to the decision maker. Our approach is a much simpler modeling alternative to the theory of partially observable Markov decision processes (POMDP), where an information and updating structure about the decision variable needs to be defined. We illustrate the approach with an example of a duopoly where one firm's actions are not immediately observable by the other firm, and present computational results. We believe that this approach can be used in a variety of applications, where the decision maker wants to assess the *value* of information about an additional decision variable.

Keywords: Markov decision processes, mixed-integer linear programming, partially observable Markov decision processes

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1. Introduction

Consider a Markov decision process (MDP) where the state is a vector comprised of multiple state variables. The decision maker is interested in knowing the *value* of a particular state variable. That is, how worse off would he or she be if s/he employed the best policy that ignores the information provided by that state variable, compared to the optimal policy computed using all state variables? In a practical situation, this would occur when a state variable is unobservable to the decision maker, or it can be observed but at a cost.

In this paper we propose a simple approach to answer this question for a particular class of MDP's where state and action spaces are finite. This class of MDPs was motivated by a problem that arose from the study of competition from a decision–theoretic perspective [9]. Our starting point is the (dual) linear programming formulation of an MDP [6], see §2. We define additional binary variables and additional constraints that force the actions for a set of states to be identical. The result is a mixed–integer programming (MIP) formulation that can be solved using branch and bound. Our approach is appropriate where the action space is relatively small, so that the resulting MIP can be solved in a reasonable amount of time.

A similar problem has been analyzed in the literature: a partially observable MDP, or POMDP [2, 4, 5, 7]. In a POMDP, the state of the system $\mathbf{s}$ is unobservable, and an action $\mathbf{a}$ taken at period t results in an unobservable state j at the period $t + 1$, as well as an outcome y, which is observable with probability $q\left(y \mid \mathbf{s}, \mathbf{a}, j\right)$. There are important differences with respect to our approach. First, in our approach we do not define an observation process, and thus there is no need to define the process y with its associated probabilities $q\left(y \mid \mathbf{s}, \mathbf{a}, j\right)$. This provides a significant simplification in the formulation, as well as in the data to be collected. Second, our approach is only valid when a state is comprised of multiple state variables, and the goal is find the *value* of a state variable. In contrast, a POMDP formulation models uncertainty for the complete state. So, our formulation has a more limited applicability. The simplicity and computational efficiency of the approach, however, makes it very suitable for the situations where is applicable. Moreover, our approach can be implemented very quickly since MIP solvers are readily available. To illustrate, we solve a numerical example for a duopoly, where firms compete on the introduction of new products (adapted from [9]) using Excel Solver. On a PC Pentium IV, we are able to obtain results for a reasonably sized problem in just two seconds.

This paper is organized as follows. In §2 we provide our general integer programming formulation. In §3, we present the example and computational results. We conclude in §4.

2. The Mixed–Integer Programming Formulation

Consider an infinite–horizon Markov decision process with a finite state space **S** and finite action space **A**. For presentation purposes, we consider the discounted reward case with discount factor α where $0 \le \alpha < 1$; the unichain–average reward case is easily extended from it. Denote the one–period expected reward at state **s** if action **a** is taken by $r(\mathbf{s}, \mathbf{a})$, which we assume to be bounded. Let $\pi: \mathbf{S} \to \mathbf{A}$ denote a policy that induces a bivariate discrete time reward process $\{(X_t, r(X_t, Y_t); t = 1, 2, \ldots\}$, where the first component X_t represents the state of the system at time t and the second component represents the bounded reward of using action Y_t at state X_t. The decision maker wants to find an optimal policy π^* that maximizes $f_\alpha^\pi(\mathbf{s}) = E_\mathbf{s}^\pi \left\{\sum_{t=1}^\infty \alpha^{t-1} r(X_t, Y_t)\right\}$, where $E_\mathbf{s}^\pi\{\cdot\}$ denotes the expectation under policy π, starting at $t = 1$ in state **s**. It is well–known that a stationary policy that takes an action $\mathbf{a}(\mathbf{s}) \in \mathbf{A}$ for each state $\mathbf{s} \in \mathbf{S}$ is optimal [6]. Denote the transition probabilities for the Markov chain that results from this stationary policy by $p\left(\mathbf{s}' \mid \mathbf{s}, \mathbf{a}\right)$. The optimality equations are:

$$f(\mathbf{s}) = \max_{\mathbf{a} \in \mathbf{A}} \left\{ r(\mathbf{s}, \mathbf{a}) + \alpha \sum_{s' \in S} p\left(\mathbf{s}' \mid \mathbf{s}, \mathbf{a}\right) f(\mathbf{s}') \right\}, \qquad (1)$$

where we define, for simplicity, $f(\mathbf{s}) \equiv f_\alpha^{\pi^*}(\mathbf{s})$.

Further, consider a state **s** that is comprised of multiple state variables. To simplify exposition, consider a state comprised of two state variables *i* and *j*, i.e., **s** = (*i*, *j*), where $i \in \mathbf{I}$, $j \in \mathbf{J}$, **I** and **J** are finite sets. Suppose that the decision maker does not *consider* the state variable *j* when making a decision. In a practical situation, this would occur if *j* is *unobservable* (or costly to observe). Alternatively, the decision maker may deliberately pursue a policy that *ignores* the information provided by *j*—for example, Medtronic pursues a strategy of introducing new products at regular time intervals ("train scheduling") regardless of competitors' actions [1]. By finding the best policy that does not consider *j*, and comparing its performance against the optimal policy (the solution of (1)), one is able to assess how important is the contribution of the state variable *j* to the optimal policy.

We address this issue by modifying the MDP formulation as follows. Consider the LP formulation ([6], p. 223) of the discounted-reward MDP (1):

$$\begin{aligned}
&\min \quad \sum_{\mathbf{s}\in\mathbf{S}} \gamma(\mathbf{s}) f(\mathbf{s}) \\
&\text{subject to} \\
&f(\mathbf{s}) - \alpha \sum_{\mathbf{s}'\in\mathbf{S}} p(\mathbf{s}' \mid \mathbf{s},\mathbf{a}) f(\mathbf{s}') \geq r(\mathbf{s},\mathbf{a}), \qquad \mathbf{s}\in\mathbf{S}, \mathbf{a}\in\mathbf{A} \\
&f(\mathbf{s}) \text{ unrestricted for all } \mathbf{s}\in\mathbf{S}.
\end{aligned} \tag{2}$$

The optimal solution of (2) is the same for any positive vector $\gamma(\mathbf{s})$ ([6], p. 228), but as we see below, it is more interesting to think of it as an initial probability distribution on the state space. (In our numerical study for the example in §3, we use $\gamma(\mathbf{s}) = 1/|\mathbf{S}|$ for all **s**, but for the exposition here we maintain $\gamma(\mathbf{s})$ general.) The dual of (2) is given by:

$$\begin{aligned}
&\max \quad \sum_{\mathbf{s}\in\mathbf{S}} \sum_{\mathbf{a}\in\mathbf{A}} r(\mathbf{s},\mathbf{a}) w(\mathbf{s},\mathbf{a}) \\
&\text{subject to} \\
&\sum_{\mathbf{a}\in\mathbf{A}} w(\mathbf{s}',\mathbf{a}) - \alpha \sum_{\mathbf{s}\in\mathbf{S}} \sum_{\mathbf{a}\in\mathbf{A}} p(\mathbf{s}' \mid \mathbf{s},\mathbf{a}) w(\mathbf{s},\mathbf{a}) = \gamma(\mathbf{s}'), \quad \mathbf{s}'\in\mathbf{S} \\
&w(\mathbf{s},\mathbf{a}) \geq 0, \quad \mathbf{s}\in\mathbf{S}, \mathbf{a}\in\mathbf{A}.
\end{aligned} \tag{3}$$

In (3), $w(\mathbf{s}, \mathbf{a})$ represents the total discounted joint probability under initial-state distribution $\gamma(\mathbf{s})$ that the system occupies state **s** and chooses action **a**, and it is therefore bounded by $1/(1-\alpha)$. The solution of either (2) or (3) yields a deterministic policy: for all transient states $w(\mathbf{s}, \mathbf{a}) = 0$ and for each non-transient state **s,** there is only one $\mathbf{a} \in \mathbf{A}$ such that $w(\mathbf{s}, \mathbf{a}) > 0$.

If the state j is unobservable to the decision maker, or if s/he does not consider j to make a decision, then his/her objective is that for each i, the action **a** in state (i, j) be the same for all $j \in \mathbf{J}$. To accomplish this, define binary variables $b(i, \mathbf{a})$ for each $\mathbf{a} \in \mathbf{A}$ such that $b(i, \mathbf{a}) = 1$ if **a** is optimal for state variable i and 0 otherwise. That is, add to (3) the following constraints:

$$
\begin{aligned}
& w(\mathbf{s},\mathbf{a}) \le (1-\alpha)^{-1} b(i,\mathbf{a}), && \mathbf{s} = (i,j) \in \mathbf{S}, \mathbf{a} \in \mathbf{A} \\
& \textstyle\sum_{\mathbf{a}\in\mathbf{A}} b(i,\mathbf{a}) = 1, && i \in \mathbf{I} \\
& b(i,\mathbf{a}) \in \{0,1\}, && i \in \mathbf{I}, \mathbf{a} \in \mathbf{A}.
\end{aligned}
\tag{4}
$$

In (4), the first set of constraints ensures that if $b(i, \mathbf{a}) = 0$ then $w(\mathbf{s}, \mathbf{a}) = 0$, and the second set of constraints ensures that only one action is taken for each i. This procedure results in an MIP that can be solved using branch and bound, provided that the size of the state space $|\mathbf{S}|$ and action space $|\mathbf{A}|$ are not too large, since our MIP formulation introduces $|\mathbf{I}|\cdot|\mathbf{A}|$ integer variables. In the next section, we present an example and computational study for a duopoly, where we study the influence of one firm's strategy on the other firm's.

3. Example: A Duopoly

Consider a competitive market comprised of two firms, A and B. As an example, consider the market for DVD drives. The planning horizon is infinite and divided into time periods of equal length, say quarters. At the start of each period, firms A and B decide, simultaneously and without collusion, whether to introduce a new product in the following period. Their decisions are known to each other at the start of the following period. When a firm introduces a product, it takes its old product out of the market, such that, at any point in time, the market comprises only two products–

–one for each firm. Production occurs quickly enough to be available for demand in the current period.

We formulate the decision problem from firm A's perspective. In every period, the state of the process is represented by (i, j), where i is the age of firm A's product, and j is the age of firm B's product. We assume that both i and j are integers between 1 and n, where 1 is the age of a new product, and n is the maximum age of a product. To optimize its product introduction decision, firm A needs firm B's strategy p_{ij}^B, the probability that firm B has a new product next period at state (i, j), where $0 \le p_{ij}^B \le 1$. Denote the matrix of p_{ij}^B by $\mathbf{p}^B$. The matrix $\mathbf{p}^B$ is a *contingent* strategy; it represents firm B's plan of action for *every* state (i, j), even though some states may never be realized. Thus, firm B's strategy actually reflects firm A's strategy. We assume that firm A knows $\mathbf{p}^B$, presumably from firm B's historic behavior. For example, if firm B introduces a product every three periods, regardless of firm A's product age, then, $p_{ij}^B = 1$ if $j = 3$, and 0 otherwise, $\forall i$.

Firm A's only decision is whether or not to introduce a product next period at a fixed cost of K. We use z to denote the decision variable, and let $z = 1$ if a product introduction is made next period and 0 otherwise.

We formulate firm A's optimization problem as a Markov decision process (MDP). The state space is $\mathbf{S} = \{1,\ldots,n\} \times \{1,\ldots,n\}$. The action space is $\mathbf{A} = \{0,1\}$. The one period profit is $R_{ij} - Kz$, where R_{ij} denotes firm A's net–revenue at state (i, j). Let $f(i, j)$ denote firm A's optimal expected discounted profit in an infinite horizon at state (i, j). The extremal equation is

$$f(i,j) = \max \begin{Bmatrix} R_{ij} - K + \alpha\left[p_{ij}^B f(1,1) + \left(1 - p_{ij}^B\right) f\left(1, j+1\right)\right] \\ R_{ij} + \alpha\left[p_{ij}^B f(i+1,1) + \left(1 - p_{ij}^B\right) f\left(i+1, j+1\right)\right] \end{Bmatrix}, \tag{5}$$

where $0 \le \alpha < 1$ is the discount factor.

Firm A wants to know how important it is to consider the competitor's actions when formulating its own product introduction strategy. That is, firm A wants to answer the question: how worse off is firm A if it introduces products at its own schedule—regardless of firm B's actions? For example, firm A may be implementing a "train scheduling" strategy similar to Medtronic's, as discussed previously. To answer this question, we perform a numerical study using our integer programming formulation (3) and (4) to assess the importance of competition (state variable j) on firm A's optimal product introduction strategy and profit.

We assume that $R_{ij} = \left\{1+(i/j)^{\delta}\right\}^{-1}$ where δ is a parameter; note that $0 < R_{ij} < 1$; this function is non–increasing in i and non–decreasing in j; $R_{ij} = 0.5$ if $i = j$. The maximum age of a product is $n = 8$. We propose a full factorial experimental design for the numerical study on the factors δ, K, and $\mathbf{p}^B$ as shown in Table 1, which is based loosely on [9], and explained as follows.

Table 1: Experimental design

Parameter	Levels
δ	0.25, 0.50, 0.75, 1.00
K	0.25, 0.50, 0.75, 1.00
$\mathbf{p}^B$	$\begin{bmatrix} .1 & .2 & \dots & .7 & 1 \\ .2 & .3 & \dots & .8 & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ .8 & .9 & \dots & 1 & 1 \end{bmatrix}, \begin{bmatrix} .3 & .4 & \dots & .9 & 1 \\ .4 & .5 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}, \begin{bmatrix} .5 & .6 & \dots & 1 & 1 \\ .6 & .7 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$ (type 1)
	$\begin{bmatrix} 0 & .15 & .3 & \dots & .9 & 1 \\ 0 & .15 & .3 & \dots & .9 & 1 \\ \vdots & \vdots & \vdots & \ddots & .9 & 1 \\ 0 & .15 & .3 & \dots & .9 & 1 \end{bmatrix}, \begin{bmatrix} .2 & .35 & .5 & \dots & 1 & 1 \\ .2 & .35 & .5 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 & 1 \\ .2 & .35 & .5 & \dots & 1 & 1 \end{bmatrix}, \begin{bmatrix} .4 & .55 & .7 & \dots & 1 & 1 \\ .4 & .55 & .7 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 & 1 \\ .4 & .55 & .7 & \dots & 1 & 1 \end{bmatrix}$ (type 2)
	$\begin{bmatrix} 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \dots & 0 & 1 & 1 & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & 1 & 1 & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 1 & \dots & 1 \end{bmatrix}$ (type 3)

The four levels of δ capture different intensities of the decrease in net revenues with product age. The four levels of K capture different levels of fixed product introduction costs; from half (K = 0.25) to twice (K = 1) each firm's net revenue per period when both products are identical ($i = j$). There are nine possibilities for $\mathbf{p}^B$, separated into three distinct types of firm B's behavior. For each behavioral type, there are three levels of $\mathbf{p}^B$, representing an increasingly aggressive competitor. In type 1, firm B's product introduction probability is non–decreasing in both i and j; in type 2 firm B's product introduction probability is non–decreasing in j but independent of i; in type 3 firm B follows a periodic product introduction strategy (introduce a product every 7, 5 and 3 periods, irrespective of firm A). We use a discount factor $\alpha = 0.9756$; this number represents a 10% annual interest rate when each period is a thought of as a quarter. Souza ([8]) shows that α does not significantly impact the firm's optimal product introduction strategy, when kept at reasonable levels. As a result, our experimental design has 4x4x9 = 144 experimental cells.

We solve the MDP (5) using the linear programming formulation (2), and we also compute the solution to the MIP (3) and (4), which provides firm A's optimal policy *constrained* to firm B's actions being ignored or unobservable. We use the term "constrained optimal policy" for the MIP solution. The MIP has 120 nonnegative variables, 190 constraints, and 16 binary variables. For each experimental cell, both solutions are obtained using Excel Solver; solution times are less than two seconds on a Pentium IV 2.8GHz with 512 MB of memory. For this example, the difference in computational times between finding the optimal policy and constrained optimal policy is negligible.

For each experimental cell, a policy (optimal or constrained optimal) generates a discrete–time Markov chain [for details see 6]. The chain's stationary probability distribution can be computed readily [3]. For a given stationary probability distribution associated with a policy, we use standard Markov chain techniques to compute equivalent average expected profit per period over an unbounded horizon [for details see 6], denoted by Profit, and the stationary probabilities of time between product introductions by firm A. A policy results in a deterministic product–introduction decision for each state (i, j) over an unbounded horizon. The time between consecutive product introductions by firm A, however, can be probabilistic because firm A's optimal product introduction decision may depend on j, which may evolve according to a probabilistic process [for details see 8]. The expected value of this stationary probability distribution is denoted as expected time between product introductions (ETBP).

The detailed results are presented in the Appendix; we summarize the results here. For the optimal policy across 144 cells, the mean value of Profit is 0.35, with a standard deviation of 0.08; the mean value of ETBP is 5.56, with a standard deviation of 2.14. The high standard deviation for ETBP provides quick evidence that our experimental design results in a wide variety of product introduction strategies by firm A, from introducing a product very frequently (e.g., ETBP = 2) to very infrequently (ETBP = 8). The constrained optimal policy found through our MIP formulation performs very well with respect to the optimal policy; across all cells the decrease in Profit averages only 0.23%, with a maximum of 6.32% and a 90^{th} percentile of 0.55%. (These numbers are slightly different for the objective function of (2), which computes the total expected discounted profit with an equal probability for each state, at 0.24%, 7.69% and 0.62%.) Thus, the answer to our original question—how worse off is firm A if it introduces products at its own schedule, regardless of firm B's actions—is 0.23% on average. Consequently, under the assumptions of our numerical study, ignoring the competitor's actions to formulate an optimal strategy does not result in a significant impact in profit. From a practical perspective, we find that the "train scheduling" product introduction strategy found by our MIP formulation, similar to the Medtronic example discussed before, performs very well.

4. Conclusion

This note presents a MIP formulation that computes an optimal strategy for a finite state and action space Markov decision process, where a state is comprised of multiple state variables, to find the *value* of a particular state variable to the optimization. We provide a numerical example to illustrate the implementation and computational efficiency of our approach.

REFERENCES

[1]. Christensen, C., *We've Got Rhythm! Medtronic Corporation's Cardiac Pacemaker Business.* Harvard Business School Case # 9-698-004, 1997.

[2]. Costa, P.C. and D.M. Buede, *Dynamic Decision Making: A Comparison of Approaches.* Journal of Multi-Criteria Decision Anaylsis, 2000. **9**(6): p. 243-262.

[3]. Kulkarni, V., *Modeling and Analysis of Stochastic Systems*. 1995, London: Chapman & Hall.

[4]. Lin, A., J. Bean, and C. White III, *A Hybrid Genetic/Optimization Algorithm for Finite Horizon Partially Observed Markov Decision Processes.* Informs Journal on Computing, 2004. **forthcoming**.

[5]. Monahan, G., *A Survey of Partially Observable Markov Decision Processes: Theory, Models and Algorithms.* Management Science, 1982. **28**(1): p. 1-16.

[6]. Puterman, M., *Markov Decision Processes*. 1994, New York, NY: John Wiley & Sons Inc.

[7]. Serin, Y., *A nonlinear programming model for partially observable Markov decision processes.* European Journal of Operational Research, 1995. **86**: p. 549-564.

[8]. Souza, G.C., *A Theory of Product Introduction under Competition*. 2000: Doctoral Dissertation, The University of North Carolina at Chapel Hill.

[9]. Souza, G.C., B.L. Bayus, and H.M. Wagner, *New Product Strategy and Industry Clockspeed.* Management Science, 2004. **50**(4): p. 537-549.

Appendix: Detailed results for experimental design of table 1

				Profit					Profit					Profit		
δ	K	p^B	ETBP	Optimal	MIP	%Differ.	p^B	ETBP	Optimal	MIP	%Differ.	p^B	ETBP	Optimal	MIP	%Differ.
0.25	0.25	type 1, 1	6.0	0.4275	0.4275	0.00%	type 2, 1	6.0	0.4420	0.4420	0.00%	type 3, 1	4.1	0.4279	0.4279	0.00%
0.25	0.50	type 1, 1	8.0	0.3925	0.3925	0.00%	type 2, 1	8.0	0.4071	0.4071	0.00%	type 3, 1	8.0	0.3827	0.3827	0.00%
0.25	0.75	type 1, 1	8.0	0.3612	0.3612	0.00%	type 2, 1	8.0	0.3758	0.3758	0.00%	type 3, 1	8.0	0.3514	0.3514	0.00%
0.25	1.00	type 1, 1	8.0	0.3300	0.3300	0.00%	type 2, 1	8.0	0.3446	0.3446	0.00%	type 3, 1	8.0	0.3202	0.3202	0.00%
0.50	0.25	type 1, 1	3.0	0.4174	0.4174	0.00%	type 2, 1	3.0	0.4453	0.4453	0.00%	type 3, 1	3.0	0.4290	0.4290	0.00%
0.50	0.50	type 1, 1	6.0	0.3572	0.3572	0.00%	type 2, 1	6.0	0.3849	0.3849	0.00%	type 3, 1	4.3	0.3563	0.3552	0.29%
0.50	0.75	type 1, 1	8.0	0.3191	0.3191	0.00%	type 2, 1	8.0	0.3473	0.3473	0.00%	type 3, 1	6.0	0.3076	0.3076	0.00%
0.50	1.00	type 1, 1	8.0	0.2878	0.2878	0.00%	type 2, 1	8.0	0.3161	0.3161	0.00%	type 3, 1	8.0	0.2699	0.2699	0.00%
0.75	0.25	type 1, 1	3.0	0.4204	0.4201	0.07%	type 2, 1	2.2	0.4606	0.4605	0.03%	type 3, 1	2.0	0.4479	0.4479	0.00%
0.75	0.50	type 1, 1	4.0	0.3400	0.3400	0.00%	type 2, 1	4.3	0.3817	0.3816	0.04%	type 3, 1	3.2	0.3512	0.3512	0.01%
0.75	0.75	type 1, 1	6.0	0.2904	0.2904	0.00%	type 2, 1	6.0	0.3296	0.3293	0.11%	type 3, 1	5.0	0.2859	0.2678	6.31%
0.75	1.00	type 1, 1	7.5	0.2512	0.2504	0.30%	type 2, 1	7.5	0.2915	0.2908	0.27%	type 3, 1	6.0	0.2372	0.2372	0.00%
1.00	0.25	type 1, 1	2.0	0.4333	0.4333	0.00%	type 2, 1	2.0	0.4838	0.4838	0.00%	type 3, 1	2.0	0.4678	0.4678	0.00%
1.00	0.50	type 1, 1	3.0	0.3389	0.3389	0.00%	type 2, 1	3.4	0.3873	0.3858	0.37%	type 3, 1	3.0	0.3560	0.3560	0.00%
1.00	0.75	type 1, 1	6.0	0.2696	0.2693	0.13%	type 2, 1	4.8	0.3252	0.3221	0.96%	type 3, 1	4.1	0.2787	0.2786	0.02%
1.00	1.00	type 1, 1	6.0	0.2280	0.2220	2.61%	type 2, 1	6.1	0.2774	0.2753	0.75%	type 3, 1	5.1	0.2196	0.2195	0.05%
0.25	0.25	type 1, 2	5.0	0.4502	0.4497	0.12%	type 2, 2	6.0	0.4288	0.4288	0.00%	type 3, 2	5.0	0.4142	0.4142	0.00%
0.25	0.50	type 1, 2	8.0	0.4147	0.4147	0.00%	type 2, 2	8.0	0.3939	0.3939	0.00%	type 3, 2	8.0	0.3716	0.3716	0.00%
0.25	0.75	type 1, 2	8.0	0.3835	0.3835	0.00%	type 2, 2	8.0	0.3627	0.3627	0.00%	type 3, 2	8.0	0.3404	0.3404	0.00%
0.25	1.00	type 1, 2	8.0	0.3522	0.3522	0.00%	type 2, 2	8.0	0.3314	0.3314	0.00%	type 3, 2	8.0	0.3091	0.3091	0.00%
0.50	0.25	type 1, 2	3.0	0.4604	0.4604	0.00%	type 2, 2	3.0	0.4193	0.4193	0.00%	type 3, 2	3.0	0.4017	0.4017	0.00%
0.50	0.50	type 1, 2	5.0	0.4018	0.3999	0.48%	type 2, 2	6.0	0.3592	0.3592	0.00%	type 3, 2	5.0	0.3302	0.3302	0.00%
0.50	0.75	type 1, 2	8.0	0.3621	0.3621	0.00%	type 2, 2	8.0	0.3220	0.3220	0.00%	type 3, 2	7.0	0.2830	0.2830	0.00%
0.50	1.00	type 1, 2	8.0	0.3308	0.3308	0.00%	type 2, 2	8.0	0.2907	0.2907	0.00%	type 3, 2	8.0	0.2487	0.2487	0.00%
0.75	0.25	type 1, 2	2.5	0.4834	0.4822	0.25%	type 2, 2	2.1	0.4232	0.4232	0.00%	type 3, 2	2.0	0.4069	0.4069	0.00%
0.75	0.50	type 1, 2	5.0	0.4057	0.4036	0.51%	type 2, 2	4.1	0.3440	0.3439	0.01%	type 3, 2	3.3	0.3114	0.3101	0.44%
0.75	0.75	type 1, 2	5.0	0.3557	0.3508	1.35%	type 2, 2	6.1	0.2924	0.2924	0.02%	type 3, 2	5.0	0.2497	0.2497	0.00%
0.75	1.00	type 1, 2	7.5	0.3135	0.3117	0.56%	type 2, 2	8.0	0.2549	0.2549	0.00%	type 3, 2	6.3	0.2032	0.2032	0.02%
1.00	0.25	type 1, 2	2.0	0.5114	0.5114	0.00%	type 2, 2	2.0	0.4366	0.4366	0.00%	type 3, 2	2.0	0.4157	0.4157	0.00%
1.00	0.50	type 1, 2	3.8	0.4155	0.4141	0.36%	type 2, 2	3.3	0.3387	0.3376	0.30%	type 3, 2	3.0	0.3048	0.3048	0.00%
1.00	0.75	type 1, 2	5.0	0.3621	0.3621	0.00%	type 2, 2	4.8	0.2756	0.2748	0.32%	type 3, 2	4.1	0.2298	0.2298	0.01%
1.00	1.00	type 1, 2	5.0	0.3121	0.3027	3.00%	type 2, 2	6.5	0.2301	0.2281	0.87%	type 3, 2	5.1	0.1734	0.1733	0.04%
0.25	0.25	type 1, 3	5.6	0.4660	0.4658	0.04%	type 2, 3	6.0	0.4171	0.4171	0.00%	type 3, 3	5.0	0.4036	0.4036	0.00%

δ	K	p^B	ETBP	Profit Optimal	Profit MIP	Profit %Differ.	p^B	ETBP	Profit Optimal	Profit MIP	Profit %Differ.	p^B	ETBP	Profit Optimal	Profit MIP	Profit %Differ.
0.25	0.50	type 1, 3	8.0	0.4308	0.4308	0.00%	type 2, 3	8.0	0.3822	0.3822	0.00%	type 3, 3	8.0	0.3639	0.3639	0.00%
0.25	0.75	type 1, 3	8.0	0.3996	0.3996	0.00%	type 2, 3	8.0	0.3510	0.3510	0.00%	type 3, 3	8.0	0.3326	0.3326	0.00%
0.25	1.00	type 1, 3	8.0	0.3683	0.3683	0.00%	type 2, 3	8.0	0.3197	0.3197	0.00%	type 3, 3	8.0	0.3014	0.3014	0.00%
0.50	0.25	type 1, 3	3.0	0.4913	0.4913	0.00%	type 2, 3	3.0	0.3961	0.3961	0.00%	type 3, 3	3.0	0.3787	0.3787	0.00%
0.50	0.50	type 1, 3	7.0	0.4317	0.4313	0.10%	type 2, 3	6.0	0.3365	0.3365	0.00%	type 3, 3	5.0	0.3096	0.3096	0.00%
0.50	0.75	type 1, 3	7.0	0.4000	0.3934	1.66%	type 2, 3	8.0	0.2996	0.2996	0.00%	type 3, 3	8.0	0.2652	0.2652	0.00%
0.50	1.00	type 1, 3	8.0	0.3621	0.3621	0.00%	type 2, 3	8.0	0.2684	0.2684	0.00%	type 3, 3	8.0	0.2340	0.2340	0.00%
0.75	0.25	type 1, 3	2.3	0.5273	0.5238	0.66%	type 2, 3	2.0	0.3896	0.3896	0.01%	type 3, 3	2.0	0.3718	0.3718	0.00%
0.75	0.50	type 1, 3	4.2	0.4487	0.4482	0.11%	type 2, 3	4.2	0.3105	0.3105	0.01%	type 3, 3	4.0	0.2783	0.2783	0.00%
0.75	0.75	type 1, 3	7.0	0.3999	0.3959	1.00%	type 2, 3	6.2	0.2599	0.2599	0.00%	type 3, 3	5.0	0.2199	0.2199	0.00%
0.75	1.00	type 1, 3	7.0	0.3642	0.3565	2.11%	type 2, 3	8.0	0.2236	0.2236	0.00%	type 3, 3	8.0	0.1769	0.1769	0.00%
1.00	0.25	type 1, 3	2.3	0.5643	0.5624	0.33%	type 2, 3	2.0	0.3934	0.3934	0.00%	type 3, 3	2.0	0.3704	0.3704	0.00%
1.00	0.50	type 1, 3	3.5	0.4745	0.4675	1.46%	type 2, 3	3.5	0.2944	0.2942	0.05%	type 3, 3	3.0	0.2614	0.2614	0.00%
1.00	0.75	type 1, 3	5.3	0.4102	0.4069	0.80%	type 2, 3	5.2	0.2330	0.2329	0.03%	type 3, 3	4.1	0.1891	0.1890	0.02%
1.00	1.00	type 1, 3	7.0	0.3718	0.3598	3.23%	type 2, 3	7.2	0.1886	0.1885	0.08%	type 3, 3	6.0	0.1357	0.1357	0.00%