ABSTRACT

Title of dissertation: CLASSICAL INVARIANTS OF PRINCIPAL SERIES AND ISOMORPHISMS OF ROOT DATA

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We develop some new techniques to calculate the Schur indicator for self-dual irreducible Langlands quotients of the principal series representations. Using these techniques we derive some new formulas for the Schur indicator and the real-quaternionic indicator.

We make progress towards developing an algorithm to decide whether or not two root data are isomorphic. When the derived group has cyclic center, we solve the isomorphism problem completely. An immediate consequence is a clean and precise classification theorem for connected complex reductive groups whose derived groups have cyclic center.

CLASSICAL INVARIANTS OF PRINCIPAL SERIES
AND
ISOMORPHISMS OF ROOT DATA

by

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Dedication

To Annie McLean and Kai McLean for all of their patience.

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$\mathbb{C}$	complex numbers
$\mathbb{R}$	real numbers
$\mathbb{Z}$	integers
$\mathbb{N}$	natural numbers
$\mathbb{F}^\times$	multiplicative group of the field $\mathbb{F}$
S^1	$\{z \in \mathbb{C}^\times : \|z\| = 1\}$
i	$\sqrt{-1}$
$\Re(\nu)$	real part of $\nu \in \mathbb{C}^n$
G_0	identity component of the Lie group G
$\mathrm{Lie}(G)$	Lie algebra of the Lie group G
$\mathfrak{g}_0(\mathbb{C})$	complexification of a real Lie algebra $\mathfrak{g}_0$
$\mathfrak{g}$	complex Lie algebra of a complex group or the complexification of a real Lie algebra $\mathfrak{g}_0$
V^*	dual of a vector space V
$N_K(A)$	normalizer of A in K
$Z_K(A)$	centralizer of A in K
χ_π	central character of a representation π
id_S	identity function on the set S
$\mathrm{diag}(d_1, d_2, \ldots, d_n)$	$n \times n$ diagonal matrix D with d_j as the coefficient in row j column j, $j = 1, \ldots, n$
$\widehat{G}$	irreducible representations of G

Chapter 1: Classical Invariants for Principal Series

1.1 Introduction

Let V be a complex vector space and let G be a Lie group. We denote a representation $\pi : G \to \mathrm{Aut}(V)$ by the pair (π, V).

Let V be a finite dimensional complex vector space. The dual of V is defined to be $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Let $\langle f, v\rangle$ be the natural pairing: $\langle f, v\rangle = f(v)$, $f \in V^*$, $v \in V$. Let (π, V) be a representation of G. The dual of the representation (π, V) is the representation (π^*, V^*) defined by

$$\langle \pi^*(x) f, w\rangle = \left\langle f, \pi(x)^{-1} v\right\rangle, \text{ for } f \in V^*,\ v \in V,\ x \in G.$$

The dual is defined so that the natural pairing pairing $\langle\ ,\ \rangle : V^* \times V \to \mathbb{C}$ is G-invariant, which means that

$$\langle \pi^*(x) f, \pi(x) v\rangle = \langle f, v\rangle, \text{ for } f \in V^*,\ v \in V, \text{ and } x \in G.$$

A representation (π, V) is called self-dual if (π, V) is isomorphic to (π^*, V^*).

Let $\phi : V \to W$ be a linear map of finite dimensional complex vector spaces. The transpose of ϕ is the linear map ${}^T\phi : W^* \to V^*$ defined by

$$\left\langle {}^T\phi f, v\right\rangle = \langle f, \phi v\rangle, \quad \text{for } f \in W^*,\ v \in V.$$

Suppose (π, V) is irreducible and self-dual and let $\phi : (\pi, V) \to (\pi^*, V^*)$ be an isomorphism. After identifying (π^{**}, V^{**}) with (π, V), we see that ${}^T\phi : (\pi, V) \to (\pi^*, V^*)$ is also an isomorphism. By Schur's lemma, ${}^T\phi = \epsilon(\pi)\,\phi$, for some $\epsilon(\pi) \in \mathbb{C}^\times$. Taking the transpose again yields $\phi = \epsilon(\pi)^2\,\phi$, so

$$\epsilon(\pi) = \pm 1.$$

The sign $\epsilon(\pi)$ is an invariant of π. The isomorphism $\phi : (\pi, V) \to (\pi^*, V^*)$ defines a nondegenerate bilinear form $(\ ,\ )$ on V via $(v, w) = \langle \phi v, w \rangle$, which is G-invariant. That is,

$$(\pi(x)\,v, \pi(x)\,w) = (v, w)\,, \text{ for all } v, w \in V,\ x \in G.$$

The sign $\epsilon(\pi)$ determines whether the G-invariant form is symmetric or skew-symmetric:

$$(v, w) = \epsilon(\pi)\,(w, v)\,, \quad \text{for } v, w \in V.$$

Definition 1.1.1. Let (π, V) be an irreducible self-dual representation of a Lie group G on a finite dimensional complex vector space V and let $(\ ,\ )$ be a nondegenerate G-invariant bilinear form on V. The Schur indicator is the sign $\epsilon(\pi)$ that determines whether the bilinear form $(\ ,\ )$ is symmetric or skew-symmetric:

$$(v, w) = \epsilon(\pi)\,(w, v)\,, \quad \text{for } v, w \in V.$$

The Schur indicator, in particular its application to real and quaternionic representations, is a classic topic in the representation theory of finite groups and compact Lie groups and is covered in most introductory textbook on these subjects.

For example, see [12], Chapter 23 , for finite groups and [7], Chapter VI, Section 4, for compact groups.

Definition 1.1.2. A real Lie group G satisfies condition A if G is the set of points fixed by some conjugate-linear involution of $G(\mathbb{C})$, where $G(\mathbb{C})$ is some connected reductive complex group.

Equivalently, G satisfies condition A if G is the set of real points of a connected reductive linear algebraic group defined over $\mathbb{R}$. See [18], Chapter III, Theorem 2.1.

Let G be a real Lie group that satisfies condition A and let (π, V) be a representation of G on a finite dimensional complex vector space V. Let $H \subseteq G$ be a Cartan subgroup and let $\mathfrak{h}$, $\mathfrak{g}$ be their complexified Lie algebras, respectively. Let $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ be the roots and let $\Delta^+ \subseteq \Delta$ be a set of positive roots. For $\alpha \in \Delta$, let $\alpha^\vee \in \mathfrak{h}$ be the corresponding coroot. Let $\rho^\vee = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha^\vee$. In [6], Chapter IX, Section 7, Proposition 1, there is a formula for the Schur indicator in terms of $\rho^\vee$ and the highest weight λ of (π, V):

$$\epsilon(\pi) = (-1)^{\lambda(2\rho^\vee)}. \tag{1.1}$$

In [1], Adams interprets this formula in terms of the central character of π, χ_π, evaluated on the central element $z_{\rho^\vee} = \exp(2\pi i \rho^\vee)$ and extends this formula to some infinite dimensional representations.

Theorem 1.1.3. *(Adams) [1], Theorem 1.9. Let G be a real Lie group that satisfies condition A. Suppose every irreducible representation of G is self-dual. Then, for any irreducible representation π of G,*

$$\epsilon(\pi) = \chi_\pi(z_{\rho^\vee}).$$

Following the proof of Theorem 1.8 in [1], Adams lists all simple groups with the property that every irreducible representation is self-dual and therefore Theorem 1.1.3 applies to calculate the Schur indicator. The list is long and it's surprising that this assumption, which seems quite strong, applies to so many simple groups. There are some notable exceptions, though. Here is a nonexhaustive list of noncompact simple groups for which Adams' theorem *does not apply*:

(1) A_n: All real forms for $n > 1$. Also, $SL(2, \mathbb{R})$.

(2) B_n: $Spin(4p + 2, 2q + 1)$.

(3) C_n: $Sp(2n, \mathbb{R})$.

(4) D_{2n+1}: All real forms.

(5) D_{2n}: $Spin(4p + 2, 4q)$, $Spin(4p + 2, 4q + 2)$.

(6) E_6: All real forms.

(7) E_7: The nonadjoint split real form.

In this chapter, we present two new techniques for calculating the Schur indicator for some infinite dimensional representations of real reductive linear groups. (See Defintion 1.3.4.) Using these techniques we obtain some new results and some new formulas. Before we state the results, we introduce some basic notation and terminology.

Let G be a real Lie group and let $\mathfrak{g}$ represent the complexified Lie algebra of G. Let θ be a Cartan involution of G. (See [21] Definition 0.1.3.) Extend θ to a linear involution of $\mathfrak{g}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. A root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is called real if $\theta\alpha = -\alpha$, imaginary if $\theta\alpha = \alpha$, and complex otherwise. Define $\mathfrak{k} = \mathfrak{g}^{\theta} = \{X \in \mathfrak{g} : \theta(X) = X\}$ and $\mathfrak{p} = \mathfrak{g}^{-\theta} = \{X \in \mathfrak{g} : \theta(X) = -X\}$ so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. A Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is split if $\mathfrak{h} \subseteq \mathfrak{p}$. Let $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$. In this case the roots $\Delta(\mathfrak{g}, \mathfrak{h})$ are real and the pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ is a split Lie algebra according to [6], Chapter 8, Section 2. A real Lie group G is split if $\mathfrak{g}$ has a split Cartan subalgebra. If G satisfies condition A and G is split then G is called a split real form for $G(\mathbb{C})$. A Lie group G is of equal rank if any of the following equivalent conditions hold: G has a compact Cartan subgroup; $\operatorname{rank}(K) = \operatorname{rank}(G)$; the Cartan involution θ is an inner automorphism of $G(\mathbb{C})$, where $G(\mathbb{C})$ is the connected complex reductive group with Lie algebra $\mathfrak{g}$, [1], Section 3. The principal series and Langlands quotients are defined in Section 1.2.

Theorem 1.1.4. *Let G be an equal rank split real reductive linear group that satisfies condition A. Let π be the Langlands quotient of a principal series representation of G. If π is irreducible then π is self-dual and*

$$\epsilon(\pi) = \chi_{\pi}(z_{\rho^{\vee}}).$$

The discrete series representations of $SL(2, \mathbb{R})$ are not self-dual so Theorem 1.1.3 does not apply to $SL(2, \mathbb{R})$. However, Theorem 1.1.4 does apply to $SL(2, \mathbb{R})$ and may be used to calculate the indicator for the Langlands quotients of the principal series representations. See Table 1.3.6, which lists more groups to which Theorem 1.1.4 applies.

Theorem 1.1.5. *Let G be a split real reductive linear group that satisfies condition A and suppose $-1 \in W(K_0, T_0)$, where $T_0 \subseteq K_0$ is a Cartan subgroup of the identity component K_0 of the maximal compact subgroup $K \subseteq G$. Let π be the Langlands quotient of a principal series representation of G. If π is self-dual and irreducible then*

$$\epsilon(\pi) = 1.$$

Let G be a group and let $g \in G$. Let $\mathrm{int}(g) : G \to G$ be the automorphism of G defined by $\mathrm{int}(g)(x) = gxg^{-1}$, $x \in G$. Let $\mathrm{Int}(G) = \{\mathrm{int}(g) : g \in G\}$. The group $\mathrm{Int}(G)$ is called the group of inner automorphisms of G.

The main result is a formula for the Schur indicator for irreducible Langlands quotients of principal series representations of a real reductive linear group.

Theorem 1.1.6. *Let G be a real reductive linear group that satisfies condition A. Let $J_{MAN}(\sigma, \nu)$ be the Langlands quotient of the principal series representation $Ind_{MAN}^{G}(\sigma \otimes \nu \otimes 1)$ and suppose that $J_{MAN}(\sigma, \nu)$ is irreducible. Let $w \in W(G, A)$ such that $w\sigma \cong \sigma^*$, $w\nu = -\nu$, and $w^2 = 1$. Let $s \in N_K(A)$ be a representative of w such that $int(s)$ is distinguished. Then*

$$\epsilon(J_{MAN}(\sigma, \nu)) = \chi_\sigma\left(s^2\right)\chi_\sigma\left(z_{\rho_M^\vee}\right),$$

where $\rho_M^\vee$ is $1/2$ the sum of the positive coroots of M.

See Theorem 1.6.17 for a more precise statement of the theorem above. For more details about distinguished automorphisms, see Section 1.6.1 and the main application which is in Proposition 1.6.10.

One interesting thing to note regarding the formula for the Schur indicator in Theorem 1.1.6 is that if σ is self-dual, then $\epsilon(\sigma) = \chi_\sigma\left(z_{\rho_M^\vee}\right)$, which is Adams' interpretation of the Schur indicator formula (1.1). However, the formula in Theorem 1.1.6 holds whether σ is self-dual or not!

As mentioned previously, one of the applications of the Schur indicator in the case of compact Lie groups, is to determine whether a representation is of real or quaternionic type. The notion of real/quaternionic representations applies to self-conjugate representations. A representation (π, V) is self-conjugate if it is isomorphic to its conjugate $(\overline{\pi}, V)$.

Definition 1.1.7. Let V be a complex vector space and let (π, V) be an irreducible self-conjugate representation of G. There exists a conjugate-linear map T unique up to scalar that commutes with π. It satisfies $T^2 = \lambda \mathrm{id}_V$, $\lambda \in \mathbb{R}^\times$. Define the δ indicator[1]of π to be the sign of T^2:

$$\delta(\pi) = \mathrm{sgn}(\lambda).$$

An irreducible self-conjugate representation π is said to be real if $\delta(\pi) = 1$ or quaternionic if $\delta(\pi) = -1$. Consequently, the δ indicator may sometimes also be called the real-quaternionic indicator. For more details see Section 1.5.

In the case that G is compact, it's well-known that the δ indicator coincides with the Schur indicator. But when G is not compact and the δ indicator and the

[1]There seems to be no standard terminology for the indicator that determines real or quaternionic type. In [7], Chapter VI, Section 4, it's called the "index." In many sources its simply referred to as "the indicator," but this terminology is not precise enough for this paper.

Schur indicator both exist, they may not be equal. Lemma 1.1.8 below demonstrates a close relationship between the δ indicator and the Schur indicator.

Let V be a finite dimensional complex vector space. The Hermitian dual of V, denoted V^h, is defined to be $V^h = \{f : V \to \mathbb{C} \mid f \text{ is conjugate-linear}\}$. Let (π, V) be a representation. The Hermitian dual of (π, V), denoted $\left(\pi^h, V^h\right)$ is the representation defined by

$$\left\langle \pi^h (x) f, w \right\rangle = \left\langle f, \pi (x)^{-1} v \right\rangle, \text{ for } f \in V^*, \, v \in V, \, x \in G.$$

Here, $\langle\ ,\ \rangle : V^h \times V \to \mathbb{C}$ is the natural sesquilinear function $\langle f, v\rangle = f(v)$, which is bilinear in the first variable and conjugate-linear in the second. A representation (π, V) is Hermitian if $\pi \cong \pi^h$. The Hermitian dual of a $(\mathfrak{g}, K)$ module is defined similarly. See Definition 1.5.4.

Lemma 1.1.8. *(See [9].) Let π be an irreducible Hermitian representation of a (possibly noncompact) Lie group G on a complex vector space V. Then*

(i) π is self-dual if and only if π is self-conjugate.

(ii) If π is unitary and self-dual, or if π is unitary and self-conjugate, then

$$\epsilon(\pi) = \delta(\pi).$$

Lemma 1.1.8 is well-known in the classical case that G is a compact group. In Lemma 1.1.8 above, note that G is not assumed to be compact and therefore Hermitian does not imply unitary. Lemma 1.1.8 *(i)* is true because $\overline{\pi} \cong \pi^{*h} \cong \pi^{h*}$. See Section 1.5.1 and Proposition 1.5.5 for details. The proof of Lemma 1.1.8 *(ii)* is not hard. For a generalization, see Theorem 1.6.14 in Section 1.6.2.

In the case that π is Hermitian but not unitary, the Schur indicator and the δ indicator, assuming they both exist, may not agree. Cui [9] has studied the relationship in Lemma 1.1.8 *(ii)* for representations π of real Lie groups, both in the case that π is finite dimensional and also when π is an admissible $(\mathfrak{g}_0, K)$ module.

Lemma 1.1.8 has important implications for unitarity. If π is Hermitian and self-dual but the Schur indicator and the δ indicator do not agree, then π is *not unitary.*

Proposition 1.1.9. *(Nonunitarity Criterion) Let G be a split real reductive linear group that satisfies condition A. Let $J(\sigma, \nu)$ be the Langlands quotient of the principal series representation $Ind_{MAN}^G(\sigma \otimes \nu \otimes 1)$. Suppose that ν is real and there exists $w \in W(G, A)$ such that $(w\sigma, w\nu) = (\sigma, -\nu)$ and $w^2 = 1$. If $J(\sigma, \nu)$ is irreducible and*

$$\chi_\sigma \left(\exp\left(\pi i \left(\rho^\vee - w\rho^\vee\right)\right)\right) = -1$$

then $J(\sigma, \nu)$ is nonunitary.

For closely related nonunitarity results, see [14], Proposition 16.8, and [5], Proposition 4.6.

One of the techniques used to calculate the Schur indicator is general enough to apply to the δ indicator.

Theorem 1.1.10. *Let G be a real reductive linear group with that satisfies condition A. Let $J_{MAN}(\sigma, \nu)$ be the Langlands quotient of the principal series representation $Ind_{MAN}^G(\sigma \otimes \nu \otimes 1)$ and suppose that $J_{MAN}(\sigma, \nu)$ is irreducible. Let $w \in W(G, A)$ such that $w\sigma \cong \overline{\sigma}$, $w\nu = \overline{\nu}$, and $w^2 = 1$. Let $s \in N_K(A)$ be a representative of w*

such that $int(s)$ is distinguished. Then

$$\delta\left(J_P\left(\sigma,\nu\right)\right)=\chi_\sigma\left(s^2\right)\chi_\sigma\left(z_{\rho_M^\vee}\right).$$

See Theorem 1.6.19 for a more precise statement of the theorem above.

1.2 Dual $(\mathfrak{g},K)$-modules

For a real Lie group G, let $\mathfrak{g}_0$ represent the real Lie algebra of G and let $\mathfrak{g}=\mathfrak{g}_0\left(\mathbb{C}\right)$ be the complexification. Let θ be a Cartan involution for G and let $K=G^\theta$ be the maximal compact subgroup of G. Throughout this paper, the term involution always means automorphism of order 1 or 2.

Definition 1.2.1. [21], Definition 8.5.1. If X is a $(\mathfrak{g},K)$-module[2] , set

$$X^*=\left\{f:X\to\mathbb{C}:\dim U\left(\mathfrak{k}\right)\cdot f<\infty\right\};$$

here $\mathfrak{k}$ acts by $\left(Z\cdot f\right)(x)=f\left(-Z\cdot x\right)$, $x\in X$, $Z\in\mathfrak{k}$ and X^* is the dual module of X.

By definition, the natural pairing $\langle\ ,\ \rangle:X^*\times X\to\mathbb{C}$ is $\mathfrak{g}$-invariant and K-invariant.

Let X be a $(\mathfrak{g},K)$-module and let $v\in X$. If v satisfies the condition that $\dim U\left(\mathfrak{k}\right)\cdot v<\infty$, then v is called K-finite.

Here are some more facts regarding the dual $(\mathfrak{g},K)$-module:

Proposition 1.2.2. *[21], Lemma 8.5.2 and Corollary 8.5.3. Suppose X is a $(\mathfrak{g},K)$-module.*

[2]For an introduction to $(\mathfrak{g},K)$-modules, including the definition, see [21], Chapter 0, Section 3.

1. *Then X^* is a $\mathfrak{g}$-stable subspace of the algebraic dual of X. If*

$$X = \bigoplus_{\mu \in \widehat{K}} \bigoplus_{\alpha \in I_\mu} V_\alpha^\mu,$$

with I_μ an index set and V_α^μ an irreducible $\mathfrak{k}$-module of type μ, then

$$X^* = \bigoplus_{\mu \in \widehat{K}} \prod_{\alpha \in I_\mu} \left(V_\alpha^\mu\right)^*,$$

with $\left(V_\alpha^\mu\right)^*$ an irreducible $\mathfrak{k}$-module belonging to the contragredient of μ. In particular, X^* is a $(\mathfrak{g}, K)$-module.

2. *If X has $\mathfrak{k}$-finite multiplicities, then $X^{**} \cong X$, canonically. So X is irreducible if and only if X^* is.*

For this paper, the $(\mathfrak{g}, K)$-modules of interest are the irreducible Langlands quotients of principal series, especially those of the Langlands classification [14], Theorem 14.92.

The principal series are representations induced from a minimal parabolic subgroup $P \subseteq G$. The minimal parabolic subgroup P may be constructed as follows. Let $\mathfrak{a}_0 \subseteq \mathfrak{p}_0 = \{X \in \mathfrak{g}_0 : \theta(X) = -X\}$ be a maximal abelian subalgebra and let $A = \exp(\mathfrak{a}_0)$. Let $\Delta = \Delta(\mathfrak{a}_0, \mathfrak{g}_0)$, which is called the set of restricted roots. Fix a positive system Δ^+ in Δ and let $\mathfrak{n}_0$ be the span of the root spaces for roots in Δ^+. Let $N = \exp(\mathfrak{n}_0)$ and let $M = Z_K(A)$. Define $P = MAN$ to be a minimal parabolic subgroup of G. The decomposition $P = MAN$ is called the Langlands decomposition.

Let (σ, V_σ) be an irreducible unitary representation of M on the complex vector space V_σ and let $\nu \in \mathfrak{a}^*$. Since M is compact then V_σ is finite dimensional. Let

$\rho_{\text{res}} = \frac{1}{2}\sum_{\alpha\in\Delta^+}(\dim(\alpha))\,\alpha$, where $\dim(\alpha)$ is the dimension of the α-weight space in $\mathfrak{g}_0$. Let $\mathfrak{t}_0 \subseteq \mathfrak{m}_0$ be a maximal abelian subalgebra and let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$. Then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra. Choose a positive system $\Delta^+(\mathfrak{t},\mathfrak{m})$ and extend it to a positive system $\Delta^+(\mathfrak{h},\mathfrak{g})$ by defining $\alpha \in \Delta(\mathfrak{h},\mathfrak{g})$ be positive if $\alpha\mid_{\mathfrak{a}_0} \in \Delta^+(\mathfrak{a}_0,\mathfrak{g}_0)$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{h},\mathfrak{g})}\alpha$. Since $\rho_{\text{res}} = \rho\mid_{\mathfrak{a}_0}$, we choose to use ρ in place of ρ_{res} when defining the principal series below.

Following [14], Chapter VII, Sections 1-3, or [17] Chapter XI, Section 1, let

$$\operatorname{Ind}_P^G(\sigma\otimes\nu\otimes 1) = \left\{f : G \to V_\sigma \text{ continuous} \mid f(xman) = a^{-(\nu+\rho)}\sigma(m)^{-1}f(x)\right\}.$$

The notation $a^{-(\nu+\rho)}$ means $e^{-(\nu+\rho)\log a}$. The action of G on this space is the left action, denoted $\pi(\sigma,\nu)(g)f(x) = f(g^{-1}x)$, for $g, x \in G$. If it is unnecessary to emphasize the parameters σ and ν, then the action of $g \in G$ may be denoted by simply $\pi(g)$. Let

$$I_P^G(\sigma,\nu) = \left\{f \in \operatorname{Ind}_P^G(\sigma\otimes\nu\otimes 1) : f \text{ is } K\text{-finite.}\right\}$$

This space is invariant under the action of $\mathfrak{g}_0$ and K and is an admissible $(\mathfrak{g}_0, K)$-module. The real Lie algebra $\mathfrak{g}_0$ may be complexified and the resulting action of $\mathfrak{g}$ and K on $I_P^G(\sigma,\nu)$ is a $(\mathfrak{g}, K)$-module.

The representation $\operatorname{Ind}_P^G(\sigma\otimes\nu\otimes 1)$ is called the principal series with parameters (σ,ν) and $I_P^G(\sigma,\nu)$ is the associated $(\mathfrak{g}, K)$-module or $(\mathfrak{g}_0, K)$-module, depending on the context. If the groups G and/or P are clear from context, then we may sometimes drop the superscripts and/or subscripts.

Integration over K defines a nondegenerate G-invariant pairing:

$$\langle\ ,\ \rangle : I(\sigma^*, -\nu) \times I(\sigma, \nu) \to \mathbb{C}, \quad \langle f_1, f_2 \rangle = \int_K \langle f_1(k), f_2(k) \rangle_\sigma \, dk, \tag{1.2}$$

where $\langle\ ,\ \rangle_\sigma$ is the natural pairing $V_\sigma^* \times V_\sigma \to \mathbb{C}$. (For a reference, see [3], Section 7.) Fix arbitrary $f_1 \in V_\sigma^*$, $f_2 \in V_\sigma$, and define

$$F : G \to \mathbb{C}, \quad F(x) = \langle f_1(x), f_2(x) \rangle_\sigma .$$

The G-invariance of the pairing in (1.2) is due to the fact that F is a density on G/P. (See [17], Chapter 11, Section 1.) This situation is analogous to that of the Hermitian dual, which is a concept that is fundamentally important to the problem of classifying the unitary dual. For the details, see [17] Chapter 11, Section 1, specifically Proposition 11.26.

Let $\overline{N} = \theta(N)$ and let $\overline{P} = MA\overline{N}$. The subgroup $\overline{P}$ is also a minimal parabolic subgroup and is sometimes called the opposite parabolic of P. Let $\nu \in \mathfrak{a}^*$ and let $\Re(\nu)$ be the real part of ν. Assume $\Re(\nu)$ is in the closed positive Weyl chamber. The Langlands quotient of $I_P(\sigma, \nu)$, denoted $J_P(\sigma, \nu)$, is the quotient of $I_P(\sigma, \nu)$ by the kernel of a certain intertwing operator

$$A\left(\overline{P} : P : \sigma : \nu\right) : I_P(\sigma, \nu) \to I_{\overline{P}}(\sigma, \nu) . \tag{1.3}$$

For the definition of this operator, see Section 1.4. For a reference, see [14] Theorem 14.92 and the remarks immediately after. This operator of (1.3) imbeds $J_P(\sigma, \nu)$ into $I_{\overline{P}}(\sigma, \nu)$ and its image, $I_{\overline{P}}(\sigma, \nu)_{\text{sub}}$, is called the Langlands submodule. On the other hand, since $\Re(\nu)$ is in the closed positive Weyl chamber relative to N, then

$-\Re(\nu)$ is in the closed positive Weyl chamber relative to $\overline{N} = \theta(N)$. Therefore

$$A\left(P : \overline{P} : \sigma : \nu\right) : I_{\overline{P}}(\sigma^*, -\nu) \to I_P(\sigma^*, -\nu)$$

maps $J_{\overline{P}}(\sigma^*, -\nu)$ isomorphically onto $I_P(\sigma^*, -\nu)_{\text{sub}}$.

Lemma 1.2.3. *The bilinear form* (1.2) *defines an embedding* $I_P(\sigma^*, -\nu)_{sub} \hookrightarrow J_P(\sigma, \nu)^*$.

To prove this, it is convenient to use a calculation for the transpose of the intertwining operator that we don't state until Proposition 1.4.3, in Section 1.4.1. The algebraic approach to calculating the Schur indicator in Section 1.3 doesn't require any knowledge of intertwining operators at all, so we delay introducing the analytic details until Section 1.4.

Proof. Let $f_1 \in I_P(\sigma^*, -\nu)_{\text{sub}}$, and $[f_2] \in J_P(\sigma, \nu)$ be given. Define

$$B(f_1)([f_2]) = \langle f_1, f_2 \rangle_{I_P(\sigma^*, -\nu) \times I_P(\sigma, \nu)},$$

where $\langle\ ,\ \rangle_{I_P(\sigma^*, -\nu) \times I_P(\sigma, \nu)}$ is the pairing from (1.2). The only thing to check is that this is well-defined. Let $[f_2'] = [f_2]$ so that $k = f_2' - f_2 \in \ker A\left(\overline{P} : P : \sigma : \nu\right)$. Since $f_1 \in I_P(\sigma^*, -\nu)_{\text{sub}}$, there exists $f_1' \in I_{\overline{P}}(\sigma, \nu)$ such that $A\left(P : \overline{P} : \sigma^* : -\nu\right) f_1' = f_1$.

Now,

$$
\begin{aligned}
B\left(f_1,[f_2']\right) &= \langle f_1, f_2'\rangle_{I_P(\sigma^*:-\nu)\times I_P(\sigma,\nu)} \\
&= \left\langle A\left(P:\overline{P}:\sigma^*:-\nu\right) f_1', f_2'\right\rangle_{I_P(\sigma^*:-\nu)\times I_P(\sigma,\nu)} \\
&= \left\langle f_1', {}^{T}A\left(P:\overline{P}:\sigma^*:-\nu\right) f_2'\right\rangle_{I_{\overline{P}}(\sigma^*:-\nu)\times I_{\overline{P}}(\sigma,\nu)} \\
&= \left\langle f_1', A\left(\overline{P}:P:\sigma:\nu\right) f_2'\right\rangle_{I_{\overline{P}}(\sigma^*:-\nu)\times I_{\overline{P}}(\sigma,\nu)} \qquad \text{(by Proposition 1.4.3)} \\
&= \left\langle f_1', A\left(\overline{P}:P:\sigma:\nu\right) (f_2+k)\right\rangle_{I_{\overline{P}}(\sigma^*:-\nu)\times I_{\overline{P}}(\sigma,\nu)} \\
&= \left\langle f_1', A\left(\overline{P}:P:\sigma:\nu\right) f_2\right\rangle_{I_{\overline{P}}(\sigma^*:-\nu)\times I_{\overline{P}}(\sigma,\nu)} \\
&= \left\langle {}^{T}A\left(\overline{P}:P:\sigma:\nu\right) f_1', f_2\right\rangle_{I_{\overline{P}}(\sigma^*:-\nu)\times I_{\overline{P}}(\sigma,\nu)} \\
&= \left\langle A\left(P:\overline{P}:\sigma^*:-\nu\right) f_1', f_2\right\rangle_{I_P(\sigma^*:-\nu)\times I_P(\sigma,\nu)} \qquad \text{(by Proposition 1.4.3)} \\
&= \langle f_1, f_2\rangle_{I_P(\sigma^*:-\nu)\times I_P(\sigma,\nu)} \\
&= B\left(f_1,[f_2]\right)
\end{aligned}
$$

□

The Weyl group $W(G,A)$ is defined to be

$$W(G,A) = N_K(A)/Z_K(A).$$

Since P is minimal, the restricted roots $\Delta(\mathfrak{a}_0,\mathfrak{g}_0)$ form an abstract root system and the Weyl group of $\Delta(\mathfrak{a}_0,\mathfrak{g}_0)$ is isomorphic to $W(G,A)$, [14] Theorem 5.17.

Proposition 1.2.4. *Suppose $J(\sigma,\nu)$ is irreducible Langlands quotient of a principal series. Then $J(\sigma,\nu)$ is self-dual if and only if there exists $w \in W(G,A)$ such that $w\sigma \cong \sigma^*$ and $w\nu = -\nu$.*

Proof. Suppose there exists $w \in W(G, A)$ such that $w\sigma \cong \sigma^*$ and $w\nu = -\nu$. There is an intertwining operator $A : J(\sigma, \nu) \to I(w\sigma, -\nu)_{\text{sub}}$ which is an isomorphism. Since $J(\sigma, \nu)$ is irreducible then by Proposition 1.2.2, 2., $J(\sigma, \nu)^*$ is irreducible. Since $J(\sigma, \nu)^*$ is irreducible, then Lemma 1.2.3 implies $I(\sigma^*, -\nu)_{\text{sub}}$ is irreducible and $I(\sigma^*, -\nu)_{\text{sub}} \cong J(\sigma, \nu)^*$. Therefore,

$$J(\sigma, \nu) \cong I(w\sigma, -\nu)_{\text{sub}} \cong I(\sigma^*, -\nu) \cong J(\sigma, \nu)^* .$$

On the other hand if

$$J(\sigma, \nu) \cong J(\sigma, \nu)^* \cong I(\sigma^*, -\nu)_{\text{sub}}$$

then there exists $w \in W(G, A)$ such that $w\sigma \cong \sigma^*$ and $w\nu = -\nu$. □

1.3 Schur Indicators for Split Groups

This chapter contains two approaches to calculating the Schur indicator for self-dual, irreducible Langlands quotients of principal series representations of split groups. The first approach is more algebraic and uses extensions of groups. The second is more analytic and uses the classic intertwining operators developed in [15] and [16]. Even though the second, more analytic approach seems to generalize the easiest, the approach using group extensions is quite different and interesting because it requires little or no knowledge of the classic intertwining operators.

The algebraic approach is illustrated by Example 1.3.1 below.

Example 1.3.1. Let $G = SL(2, \mathbb{R})$, $P = MAN \subseteq G$ be the Langlands decomposition for the minimal parabolic subgroup of upper triangular matrices, where

$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R} \right\}$, $M = \langle -I \rangle$ and $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\sigma \in \widehat{M}$ and $\nu \in \mathfrak{a}^*$. In this example, the Langlands quotient $J_P(\sigma, \nu)$ is always self-dual, even when it is reducible. The only time $J_P(\sigma, \nu)$ is reducible is when $\nu = 0$ and $\sigma(-I) = -1$. In this case, $J_P(\sigma, 0)$ is the direct sum of the limit of discrete series which are dual to each other. Let $\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The automorphism $\gamma = \mathrm{int}(\gamma_1)$ is an involution of G since $\gamma_1^2 = -I \in Z(G)$. A consequence of this is that $\langle G, \gamma_1 \rangle$ forms a group which is an extension of G. If

$$\begin{array}{ll} (i) & J_p(\sigma, \nu) \text{ extends to an irreducible representation of } \langle G, \gamma_1 \rangle \text{, and} \\ (ii) & \text{the extended representation remains self-dual, then} \end{array} \tag{1.4}$$

the same technique from [1] in the proof of Lemma 5.2. may be used to calculate the Schur indicator. We will demonstrate this technique in a moment.

Let $(\ ,\ )$ be a G-invariant bilinear form on $J_P(\sigma, \nu)$. The G-invariant bilinear form is also K-invariant and therefore pairs the K-types with their duals. In this example the K-types are 1-dimensional and are generated by the functions:

$$f_\lambda \mid_K (R_\psi) = e^{\lambda i \psi}, \quad \text{for } R_\psi = \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix}, \text{ and } \psi \in \mathbb{R}, \text{ where}$$

$$\lambda = \begin{cases} 2n & \text{for } n \in \mathbb{Z} \text{ if } \sigma \text{ is trivial, or} \\ 2n+1 & \text{for } n \in \mathbb{Z} \text{ if } \sigma \text{ is nontrivial.} \end{cases}$$

The K-type of weight $\lambda \in \mathbb{Z}$ is denoted μ_λ and

$$\mu_\lambda = \mathbb{C} \langle f_\lambda \rangle .$$

Let λ be the highest weight of a lowest K-type of $J_P(\sigma,\nu)$. Since $J_P(\sigma,\nu)$ is self-dual, then $-\lambda$ is also the highest weight of a lowest K-type. There are two cases, the first of which is easy.

Case 1. If σ is trivial, then $\lambda = 0$, so $\mu_0 = \mathbb{C}\langle f_0 \rangle$, and the unique lowest K-type, μ_0, is the trivial K-representation. Furthermore, since μ_0 is the trivial representation then μ_0 is self-dual and $(\ ,\ )$ restricts to a nondegenerate bilinear form on μ_0, which must be symmetric. Therefore,

$$\sigma \text{ is trivial } \Rightarrow \epsilon\left(J_P(\sigma,\nu)\right) = 1.$$

Case 2. If σ is nontrivial, $\lambda = \pm 1$, so there are two lowest K-types $\mu_{\pm 1}$, which are *not self-dual*, but are dual to each other. Since γ_1 acts by inversion on the compact Cartan subgroup $T = \{R_\psi : \psi \in \mathbb{R}\}$, then $\gamma_1 \cdot f_1$ has weight -1. Therefore, $(f_1, \gamma_1 \cdot f_1) \neq 0$. Assuming both (i) and (ii) from (1.4) hold, the G-invariant bilinear form $(\ ,\ )$ extends to $\langle G, \gamma_1 \rangle$ and

$$\begin{aligned}(f_1, \gamma_1 \cdot f_1) &= \left(\gamma_1 \cdot f_1, \gamma_1^2 \cdot f_1\right) \\ &= \chi_{J_P(\sigma,\nu)}(-I)\left(\gamma_1 \cdot f_1, f_1\right), \quad \text{since } \gamma_1^2 = -I \in Z(G).\end{aligned}$$

Therefore,

$$\epsilon\left(J_P(\sigma,\nu)\right) = \chi_{J_P(\sigma,\nu)}(-I) \tag{1.5}$$

Actually, Equation (1.5) holds in both cases. The rest of this section is dedicated to developing this idea in some generality and describing situations in which conditions (i) and (ii) from (1.4) hold.

To make a final comment, the calculation in Example 1, Case 1 may be generalized slightly. Let G be a real reductive linear group and let $J(\sigma,\nu)$ be a Langlands quotient of a principal series representation. Suppose that $J(\sigma,\nu)$ is irreducible and self-dual. If $J(\sigma,\nu)$ has a self-dual lowest K-type μ, then $\epsilon(J(\sigma,\nu)) = \epsilon(\mu)$. (In fact, when $J(\sigma,\nu)$ is self-dual and has a unique lowest K-type then this lowest K-type is necessarily self-dual.) These observations are fairly obvious, but they can be useful. For example, when $G = SL(4,\mathbb{R})$ this type of reasoning demonstrates that the indicator formula of Adams in Theorem 1.1.3 does not apply to certain irreducible Langlands quotients of principal series representations of G. For a theorem that describes the indicator in this case, see Theorem 1.3.21.

1.3.1 Extended Groups

Extended groups play an important role in the development of an algorithm which can detect whether a representation is unitary or not, [11]. In [11], the authors first develop an algorithm for equal rank groups, which are groups in which the Cartan involution θ is an inner automorphism of the associated complex group $G(\mathbb{C})$. In order to apply their results to unequal rank groups, the authors extend the complex group $G(\mathbb{C})$ by θ, thereby forcing θ to be inner in the extended group.

What we seek is an involution of a real group G which pairs the K-types with their duals. Instead of extending the complex group, we will extend the real group. Since the application here is a little different, the definitions and ideas from [11] are modified slightly when necessary in order to meet different needs.

Definition 1.3.2. (Compare to Definition 12.3, [11]) Let γ be an automorphism of the real Lie group G that commutes with θ and satisfies $\gamma^m = \operatorname{int}(z)$, for some $z \in Z(G) \cap K$ and some $m \in \mathbb{N}$. The extended group, denoted ${}^{\gamma,z}G$, is the set $G \cup \left\{\gamma_1^k : k \in \mathbb{Z}\right\}$ with relations:

1. $\gamma_1 x \gamma_1^{-1} = \gamma(x)$, for all $x \in G$, and

2. $\gamma_1^m = z \in Z(G) \cap K$.

The differential of γ, also denoted γ, acts on $\mathfrak{g}$. Since γ commutes with θ, γ preserves the maximal compact subgroup $K \subseteq G$. Furthermore, $\gamma_1^m \in Z(K)$ so it makes sense to extend K and the pair $(\mathfrak{g}, K)$. The extended pair is denoted $(\mathfrak{g}, {}^{\gamma,z}K)$. In [11], this is Definition 8.12.

The group ${}^{\gamma,z}G$ is an extension since it fits in the short exact sequence

$$1 \to G \xrightarrow{i} {}^{\gamma,z}G \xrightarrow{p} \mathbb{Z}/m\mathbb{Z} \to 0$$

where i is inclusion and p is quotient map $q : {}^{\gamma,z}G \to {}^{\gamma,z}G/G$ followed by mapping $[\gamma_1] \mapsto 1$. If the central element $z = \operatorname{int}(\gamma_1^m)$ is clear from context or unnecessary to the discussion, then the extended group may simply be denoted ${}^{\gamma}G$.

The extended group ${}^{\gamma}G$ is a semidirect product if and only if there exists $x \in G$ such that

$$1 = (x\gamma_1)^m = x\gamma(x)\gamma^2(x)\cdots\gamma^{m-1}(x)\gamma_1^m.$$

For example, if $\gamma_1^m = 1$, then the short exact sequence splits and ${}^{\gamma}G$ is a semidirect product. (In the equation above, simply choose $x = 1$.)

Let γ be an involution with $\gamma^2 = \operatorname{int}(z_\gamma)$, $z_\gamma \in Z(G) \cap K$. Let ξ be another involution of G with $\xi^2 = \operatorname{int}(z_\xi)$, $z_\xi \in Z(G) \cap K$. The extended group ${}^{\xi}G$ is isomorphic to ${}^{\gamma}G$ if and only if there exists a homomorphism $\phi : {}^{\xi}G \to {}^{\gamma}G$ such that

1. $\phi(G) = G$, and

2. $\phi(\xi_1) = x\gamma_1$ for some $x \in G$ with $x\gamma(x) \in Z(G)$.

Example 1.3.3. Let $G = SL(2, \mathbb{R})$, and let $\gamma = \operatorname{int}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$, which is an involution of G. Let I be the identity matrix and let $-I$ be the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. From the previous discussion, ${}^{\gamma,I}G \cong G \rtimes \mathbb{Z}/2\mathbb{Z}$. However, the group ${}^{\gamma,-I}G$ is not a semidirect product. The short exact sequence for the extended group ${}^{\gamma,-I}G$ splits if and only if there exists $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ such that $I = (x\gamma_1)^2 = x\gamma(x)\gamma_1^2 \Leftrightarrow -I = x\gamma(x)$. The product

$$x\gamma(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} a^2 - bc & b(d-a) \\ c(a-d) & d^2 - bc \end{pmatrix}$$

is diagonal if and only if $d = a$. But if $d = a$, then $x\gamma(x) = I$. This demonstrates that the short exact sequence does not split and therefore that ${}^{\gamma,-I}G \ncong {}^{\gamma,I}G$.

Definition 1.3.4. [21], 0.1.2 a)-f). A real reductive linear group is a real Lie group G, a maximal compact subgroup K of G, and an involution θ of $\mathfrak{g}_0$, satisfying

a) $\mathfrak{g}_0$ is a real reductive Lie algebra;

b) If $g \in G$, the automorphism $\operatorname{Ad}(g)$ of $\mathfrak{g}$ is inner (for the corresponding complex connected group);

c) The fixed point set of θ is $\mathfrak{k}_0$;

d) Write $\mathfrak{p}_0$ for the -1 eigenspace of θ; then the map $K \times \mathfrak{p}_0 \to G$, $(k, X) \mapsto k \cdot \exp(X)$ is a diffeomorphism;

e) G has a faithful finite dimensional representation;

f) Let $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ be a Cartan subalgebra, and let H be the centralizer of $\mathfrak{h}_0$ in G. Then H is abelian.

The extended group ${}^{\gamma}G$ is a reductive group if and only if $\gamma = \mathrm{int}(x)$ for some $x \in G(\mathbb{C})$ where $G(\mathbb{C})$ is the connected complex Lie group whose Lie algebra is $\mathfrak{g} = \mathfrak{g}_0(\mathbb{C})$. Refer back to the Example 1.3.3 above. In this example $\gamma = \mathrm{int}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$. But it's also true that $\gamma = \mathrm{int}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)$ so both ${}^{\gamma,I}G$ and ${}^{\gamma,-I}G$ are reductive Lie groups.

1.3.2 Representations of Extended Groups

Let π be a representation of G on a complex vector space V. Let γ be an automorphism of G. The twist of π by γ, denoted (π^{γ}, V), is the representation on the same vector space V defined by $\pi^{\gamma}(x)v = \pi(\gamma(x))v$, $x \in G$.

Definition 1.3.5. [11], Definition 8.12. Let (π, V) be a $(\mathfrak{g}, K)$-module and let γ be an involution of G that commutes with θ, so that γ acts on K. The twist of π by γ, denoted (π^{γ}, V), is the $(\mathfrak{g}, K)$-module with the same underlying vector space V, but with the new action of $\mathfrak{g}$ and K, denoted π^{γ}, defined by

$$\pi^{\gamma}(X)v = \pi(\gamma(X))v, \quad \pi^{\gamma}(k)v = \pi(\gamma(k))v, \quad X \in \mathfrak{g}, k \in K, v \in V.$$

Proposition 1.3.6. *[11], Proposition 8.13 5. Let γ be an order d automorphism of G that commutes with θ, and satisfies $\gamma^d = int(z)$ for some $z \in Z(G) \cap K$. (See Definition 1.3.2.) The irreducible $(\mathfrak{g}, K)$-module (π, V) extends to a $(\mathfrak{g}, {}^{\gamma} K)$-module if and only if $(\pi, V) \cong (\pi^{\gamma}, V)$. In this case there are exactly d inequivalent extensions.*

Lemma 1.3.7. *Let γ be an automorphism of G. Precomposition by γ^{-1}, $D(f) = f\ \gamma^{-1}$, defines an isomorphism*

$$D : I_{MAN}(\sigma, \nu) \xrightarrow{\cong} I_{\gamma(M)\gamma(A)\gamma(N)}(\gamma\sigma, \gamma\nu).$$

and this factors through the Langlands quotient to give an isomorphism

$$D : J_{MAN}(\sigma, \nu) \xrightarrow{\cong} J_{\gamma(M)\gamma(A)\gamma(N)}(\gamma\sigma, \gamma\nu).$$

The statement of Lemma 1.3.7 is really just a formality. Nevertheless the proof is included below.

Proof. Recall

$$\mathrm{Ind}_P(\sigma \otimes \nu \otimes 1) = \left\{ f : G \to V_\sigma \text{ continuous} \mid f(xman) = a^{-(\nu+\rho)}\sigma(m)^{-1} f(x) \right\}$$

and the action of G is the left action $\pi(g) f(x) = f(g^{-1}x)$. Let $F = D(f) = f\gamma^{-1}$.

$$\begin{aligned} F(x\gamma(m)) &= f\ \gamma^{-1}(x\gamma(m)) \\ &= f\left(\gamma^{-1}(x)\, m\right) \\ &= \sigma(m)^{-1} f\left(\gamma^{-1}(x)\right) \\ &= \gamma\sigma(\gamma(m))^{-1}(F(x)). \end{aligned}$$

The calculation demonstrating that F transforms properly with respect to $\gamma(a)$ and $\gamma(n)$ is exactly the same. D is injective and surjective. Also, D is an intertwiner:

$$\begin{aligned}
D(\pi(g)(f))(x) = \pi(g)(f)\left(\gamma^{-1}(x)\right) &= f\left(g^{-1}\gamma^{-1}(x)\right) \\
&= f\left(\gamma^{-1}\left(\gamma\left(g^{-1}\right)x\right)\right) \\
&= D(f)\left(\gamma(g)^{-1}x\right) \\
&= \pi^{\gamma}(g)(D(f))(x).
\end{aligned}$$

For any set $Y \subseteq G$, let $Y_0 = \gamma(Y)$, and let $\overline{P}_0 = M_0A_0\overline{N}_0$. Since

$$\begin{aligned}
&A(P_0' : P_0 : \gamma\sigma : \gamma\nu)(D(f))(x) \\
&= \int_{\overline{N}_0} D(f)(x\overline{n}_0)\,d\overline{n}_0 \\
&= \int_{\overline{N}_0} f\left(\gamma^{-1}(x)\gamma^{-1}(\overline{n}_0)\right)d\overline{n}_0 \\
&= c\left(\gamma^{-1}\right)\int_{\overline{N}} f\left(\gamma^{-1}(x)\overline{n}\right)d\overline{n}, \qquad \text{c.o.v. } v = \gamma^{-1}(v_0) \text{ where } c\left(\gamma^{-1}\right) \in \mathbb{C}^{\times} \text{ is the Jacobian factor.} \\
&= c\left(\gamma^{-1}\right)\ A\left(\overline{P} : P : \sigma : \nu\right)(f)\left(\gamma^{-1}(x)\right) \\
&= c\left(\gamma^{-1}\right)\ D\left(A\left(\overline{P} : P : \sigma : \nu\right)(f)\right)(x)
\end{aligned}$$

then if $[f] \in J_P(\sigma, \nu)$,

$$[D(f)] = 0 \Leftrightarrow [f] = 0.$$

Therefore, the map D factors through the quotient map giving isomorphism of Langlands quotients. $\square$

Proposition 1.3.8. *Let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup of a real reductive linear group G. Let γ be an order d automor-*

phism of G such that (i) γ commutes with θ, (ii) $\gamma^d = int(z)$, $z \in Z(G) \cap K$ and (iii) $\gamma(x) = x$, for all $x \in MA$. If the Langlands quotient $J_P(\sigma, \nu)$ is irreducible, then $J_P(\sigma, \nu)$ extends to exactly d inequivalent irreducible $(\mathfrak{g}, {}^{\gamma}K)$-modules.

Proof. Condition *(i)* implies that γ preserves K. Condition *(i)* and *(ii)* imply that γ also preserves M, A and N and therefore P. Furthermore, *(i)* and *(ii)* imply $\pi \cong \pi^{\gamma}$, using Proposition 1.3.7. The statement now follows from Proposition 1.3.6. $\square$

Let $P = MAN$ be a minimal parabolic subgroup of the real reductive linear group G. Let $d \in \mathbb{N}$ and let γ be an order d automorphism of G with the following properties:

$$\begin{aligned} &\text{(i)} \quad \text{the differential of } \gamma \text{ on } \mathfrak{g} \text{ is inner,} \\ &\text{(ii)} \quad \gamma\theta = \theta\gamma, \\ &\text{(iii)} \quad \gamma(x) = x, \text{ for all } x \in MA, \text{ and} \\ &\text{(iv)} \quad \gamma^d = \mathrm{int}(z), \; z \in Z(G) \cap M. \end{aligned} \tag{1.6}$$

Then ${}^{\gamma}G$ is a real reductive linear group with minimal parabolic subgroup ${}^{\gamma}P = ({}^{\gamma}M)AN$. Let (σ, V) be an irreducible representation of M. Given properties (iii) and (iv) it now makes sense to form the extended group ${}^{\gamma}M = \langle M, \gamma_1 \rangle$ and also to discuss the extensions of σ to ${}^{\gamma}M$. Let $\widetilde{\sigma}$ be an extension of σ to ${}^{\gamma}M$. Since $\sigma^{\gamma} = \sigma$ then Schur's lemma implies that $\widetilde{\sigma}(\gamma_1)$ is a scalar. Therefore, we may write $\widetilde{\sigma}(\gamma_1) = \chi_{\widetilde{\sigma}}(\gamma_1)\,\mathrm{id}_V$, where $\chi_{\widetilde{\sigma}}(\gamma_1) \in S^1$. Fix a d-th root ζ of $\chi_{\sigma}(z)$, $\zeta^d = \chi_{\sigma}(z)$. There are exactly d inequivalent extensions of σ, denoted σ_k, which are defined by the formula

$$\chi_{\sigma_k}(\gamma_1) = e^{k2\pi i/d}\zeta, \quad k = 0, 1, 2, \ldots, d-1.$$

If X is a $(\mathfrak{g}, {}^{\gamma}K)$-module, let $\mathrm{Res}_{(\mathfrak{g},K)}X$ be the resulting $(\mathfrak{g}, K)$-module.

Lemma 1.3.9. *Let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup of a real reductive linear group G. Let γ be an automorphism of G satisfying conditions (1.6) (i)-(iv). Let (σ, V) be an irreducible representation of M and let $(\widetilde{\sigma}, V)$ be an extension of (σ, V) to ${}^{\gamma}M$. Let $J_P(\sigma,\nu)$ be a Langlands quotient. Then*

$$J_P(\sigma,\nu) \cong \mathit{Res}_{(\mathfrak{g},K)} J_{{}^{\gamma}P}(\widetilde{\sigma},\nu). \tag{1.7}$$

As a consequence, $J_{{}^{\gamma}P}(\widetilde{\sigma},\nu)$ is irreducible if and only if $J_P(\sigma,\nu)$ is irreducible.

Lemma 1.3.9 above is fairly obvious, but the proof is included below for completeness.

Proof. The two representations $\mathrm{Res}_G \mathrm{Ind}_{{}^{\gamma}MAN}^{{}^{\gamma}G}(\widetilde{\sigma}\otimes\nu\otimes 1)$ and $\mathrm{Ind}_{MAN}^{G}(\sigma\otimes\nu\otimes 1)$ are naturally isomorphic. Let $\iota : G \hookrightarrow {}^{\gamma}G$ be the natural inclusion. For $F \in \mathrm{Ind}_{{}^{\gamma}MAN}^{{}^{\gamma}G}(\widetilde{\sigma}\otimes\nu\otimes 1)$, the restriction map $\mathrm{res}_G(F) = F\ \iota$ is an isomorphism:

$$F \mapsto F\ \iota : \mathrm{Res}_G \mathrm{Ind}_{{}^{\gamma}MAN}^{{}^{\gamma}G}(\widetilde{\sigma}\otimes\nu\otimes 1) \xrightarrow{\cong} \mathrm{Ind}_{MAN}^{G}(\sigma\otimes\nu\otimes 1) \tag{1.8}$$

It's injective, since if $F\ \iota \equiv 0$, then for $x \in G$,

$$F(x\gamma_1) = \widetilde{\sigma}\left(\gamma_1^{-1}\right)(F(x)) = 0.$$

Therefore $F \equiv 0$. If ϕ is a map, let $\mathrm{Im}(\phi)$ be its image. If $f \in \mathrm{Im}(\mathrm{res}_G)$ then f uniquely extends to a function $F : {}^{\gamma}G \to V$ by setting $F(x\gamma_1) = \widetilde{\sigma}\left(\gamma_1^{-1}\right)(f(x))$, for $x \in G$. The map res_G is just precomposition by ι and so it commutes with the intertwining integrals and therefore factors through the quotient map to a map on the Langlands quotients. Here is the explicit calculation:

$$\begin{aligned}
\operatorname{res}_G \left(A\left({}^{\gamma}\overline{P} : {}^{\gamma}P : \widetilde{\sigma} : \nu\right)(F)\right)(x) &= A\left({}^{\gamma}\overline{P} : {}^{\gamma}P : \widetilde{\sigma} : \nu\right)(F)\left(\iota\left(x\right)\right) \\
&= \int_{\overline{N}} F\left(\iota\left(x\right)\overline{n}\right) d\overline{n} \\
A\left(\overline{P} : P : \sigma : \nu\right)\left(\operatorname{res}_G\left(F\right)\right)(x) &= \int_{\overline{N}} \operatorname{res}_G\left(F\right)\left(x\overline{n}\right) d\overline{n} \\
&= \int_{\overline{N}} F\left(\iota\left(x\overline{n}\right)\right) d\overline{n} \\
&= \int_{\overline{N}} F\left(\iota\left(x\right)\overline{n}\right) d\overline{n}
\end{aligned}$$

Therefore,

$$J_P\left(\sigma, \nu\right) \cong \operatorname{Res}_{(\mathfrak{g},K)} J_{{}^{\gamma}P}\left(\widetilde{\sigma}, \nu\right).$$

Any nonzero subrepresentation of $J_P\left(\sigma, \nu\right)$ extends to a subrepresentation of $J_{{}^{\gamma}P}\left(\widetilde{\sigma}, \nu\right)$. On the other hand any nonzero subrepresentation of $J_{{}^{\gamma}P}\left(\widetilde{\sigma}, \nu\right)$ restricts to a subrepresentation of $\operatorname{Res}_{(\mathfrak{g},K)} J_{{}^{\gamma}P}\left(\widetilde{\sigma}, \nu\right)$. Therefore $J_P\left(\sigma, \nu\right)$ is irreducible if and only if $J_{{}^{\gamma}P}\left(\widetilde{\sigma}, \nu\right)$ is irreducible.

□

Proposition 1.3.10. *Suppose γ is an order d automorphism of G satisfying properties (i)-(iv) in (1.6). Let $P = MAN \subseteq G$ be a minimal parabolic subgroup. Fix an extended group ${}^{\gamma}G$. Let $J_P\left(\sigma, \nu\right)$ be an irreducible Langlands quotient of a principal series representation of G and let $\widetilde{J_P\left(\sigma, \nu\right)}$ be an extension of $J_P\left(\sigma, \nu\right)$. Then*

$$\widetilde{J_P\left(\sigma, \nu\right)} \cong J_{{}^{\gamma}P}\left(\widetilde{\sigma}, \nu\right)$$

where $\widetilde{\sigma}$ is an extension of σ to ${}^{\gamma}M$.

Proof. The representation σ of M extends to d representations of ${}^{\gamma}M$, denoted σ_k, $k = 0, 1, \ldots, d-1$. By Lemma 1.3.9, each $J_{{}^{\gamma}P}\left(\sigma_k, \nu\right)$ is an extension of $J_P\left(\sigma, \nu\right)$ and we know by Proposition 1.3.8 that there are d extensions of $J_P(\sigma,\nu)$. Therefore the extensions of $J_P(\sigma,\nu)$ must coincide with the induced representations $J_{\gamma P}(\sigma_k,\nu)$.

$\square$

When G is a split real reductive linear group, these same ideas can be used to show that the irreducible Langlands quotients restrict irreducibly down to $(\mathfrak{g},K_0)$-modules, where $K_0 = G_0^\theta$ is connected and $G_0 \subseteq G$ is the identity component of G. This idea will be convenient later.

Lemma 1.3.11. *Let G be a split real reductive linear group and let $J_P(\sigma,\nu)$ be an irreducible Langlands quotient of a principal series representation of G. Then $Res_{(\mathfrak{g},K_0)}J_P(\sigma,\nu)$ is isomorphic to an irreducible Langlands quotient of G_0.*

Proof. Let $P_0 = M_1AN$, where $M_1 = Z_{K_0}(A)$. Suppose first that $|G/G_0| = 2$. That is, $M = \langle M_1, x\rangle$ where $x \in M\backslash M_1$. Let $\gamma = \operatorname{int}(x)$ so that $G = \langle G_0, x\rangle \cong {}^{\gamma}G_0$. Note that γ is finite order because x is finite order. Also, $\gamma\mid_{M_1A} = \mathrm{id}_{M_1A}$ since $x \in M$ and M is abelian. Therefore, by Lemma 1.3.9,

$$J_{P_0}(\sigma\mid_{M_1},\nu) \cong \mathrm{Res}_{(\mathfrak{g},K_0)}J_{\gamma P}(\sigma,\nu),$$

where $J_{P_0}(\sigma\mid_{M_1},\nu)$ is irreducible. Apply this argument inductively if $|G/G_0| > 2$.

$\square$

Example 1.3.12. Let $SL_\pm(2,\mathbb{R}) = \{x \in GL(2,\mathbb{R}) : \det(x) = \pm 1\}$. Note that $SL_\pm(2,\mathbb{R}) = \left\langle SL(2,\mathbb{R}), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\rangle$ so $SL_\pm(2,\mathbb{R}) \cong {}^{\gamma,I}SL(2,\mathbb{R})$, where $\gamma = \operatorname{int}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$. Let $J(\sigma,\nu)$ be an irreducible Langlands quotient of a principal series representation of $SL_{\pm}(2,\mathbb{R})$. By Proposition 1.3.11, $J(\sigma,\nu)$ restricts irreducibly to $SL(2,\mathbb{R})$. On the other hand, the discrete series of $SL(2,\mathbb{R})$ are not Langlands quotients of principal series representations and they do not extend to $SL_{\pm}(2,\mathbb{R})$. See [21] Chapter 1, Section 4, specifically Definition 1.4.7. In fact, the discrete series for $SL_{\pm}(2,\mathbb{R})$ are induced from the discrete series for $SL(2,\mathbb{R})$. The discrete series for $SL_{\pm}(2,\mathbb{R})$, when restricted to $SL(2,\mathbb{R})$, split into a direct sum of two irreducible discrete series.

1.3.3 Self-dual Representations of Extended Groups

Let G be a real reductive linear group and let $P = MAN \subseteq G$ be the Langlands decomposition of a minimal parabolic subgroup. We would like to use infinitesimal equivalence provided by the action of the Weyl group $W(G,A)$ to determine self-duality for representations of extended groups. The first step is to show that the action of $W(G,A)$ extends uniquely to an action on the Cartan subgroup in the extended group.

Proposition 1.3.13. *Let γ be an involution of G that satisfies (ii)-(iv) of (1.6). If $s \in N_K(A)$, then $s\gamma(s)^{-1} \in M$.*

Proof. Recall that, by definition, $M = Z_K(A)$. It's clear that $s\gamma(s)^{-1} \in K$, so it's sufficient to show that it commutes with A. Let $a \in A$ be arbitrary. Now,

$$\begin{aligned} s\gamma(s)^{-1} a\gamma(s) s^{-1} &= s\gamma\left(s^{-1}\gamma(a) s\right) s^{-1} \\ &= s\gamma\left(s^{-1}as\right) s^{-1}, \quad \text{since } \gamma \mid_A = \mathrm{id}, \\ &= ss^{-1}ass^{-1}, \quad \text{also since } \gamma \mid_A = \mathrm{id}, \\ &= a. \end{aligned}$$

Therefore $s\gamma(s)^{-1} \in M$. $\square$

Proposition 1.3.14. *Let γ be an involution of G that satisfies (ii)-(iv) of (1.6). There is a unique, well-defined action of $W(G, A)$ on ${}^{\gamma}M = \langle M, \gamma_1 \rangle$. The action of $w \in W(G, A)$ satisfies*

$$w(\gamma_1) = s\gamma(s)^{-1}\gamma_1$$

where $s \in N_K(A)$ is a representative of w.

Proof. Let s be a representative of $w \in W(G, A)$. Working in the extended group ${}^{\gamma}M$, let

$$w(\gamma_1) = s\gamma_1 s^{-1} = s\gamma_1 s^{-1}\gamma_1^{-1}\gamma_1 = s\gamma\left(s^{-1}\right)\gamma_1.$$

By Proposition 1.3.13, $s\gamma(s)^{-1} \in M$, so the action of w, if well-defined, maps ${}^{\gamma}M$ to ${}^{\gamma}M$. Let s_1 be another representative for w so that $s^{-1}s_1 = m \in M$.

$$\begin{aligned} s_1\gamma(s_1)^{-1} &= sm\gamma(sm)^{-1} \\ &= sm\gamma(m)^{-1}\gamma(s)^{-1} \\ &= s\gamma(s)^{-1} \qquad \text{(using } \gamma \mid_M = \mathrm{id}_M\text{)}. \end{aligned}$$

$\square$

Proposition 1.3.15. *Let G be a split real reductive linear group and let $J_P(\sigma,\nu)$ be an irreducible self-dual Langlands quotient of a principal series representation of G. Let γ be an involution of G satisfying (1.6). The $(\mathfrak{g},K)$-module $J_P(\sigma,\nu)$ extends to an irreducible $(\mathfrak{g},{}^{\gamma,z}K)$-module. The extended module is self-dual if and only if there exists $w\in W(G,A)$ such that $(w\sigma,w\nu)=(\sigma^*,-\nu)$ and*

$$\chi_\sigma\left(s^{-1}\gamma(s)\,z\right)=1,$$

where $s\in N_K(A)$ is a representative of w. As a result, if one of the extended representations is self-dual, then so is the other one.

Proof. Using Proposition 1.3.10, suppose that the extension of $J_P(\sigma,\nu)$ is isomorphic to $J_{\gamma,zP}(\widetilde{\sigma},\nu)$, where $\widetilde{\sigma}\mid_M=\sigma$. For $w\in W(G,A)$, choose an arbitrary representative $s\in N_K(A)$ of w. By Proposition 1.3.14,

$$w\widetilde{\sigma}(\gamma_1)=\widetilde{\sigma}\left(w^{-1}(\gamma_1)\right)=\widetilde{\sigma}\left(s^{-1}\gamma(s)\,\gamma_1\right)\tag{1.9}$$

Since G is split, M is abelian, and the assumptions on γ imply that the extended group ${}^{\gamma,z}M$ is abelian. Therefore $(\widetilde{\sigma})^*(m)=\widetilde{\sigma}(m)^{-1}$ and $J_{\gamma,z}(\widetilde{\sigma},\nu)$ will be self-dual if and only if there exists $w\in W(G,A)$ such that $w\sigma=\sigma$, $w\nu=-\nu$ and

$$w\widetilde{\sigma}(\gamma_1)=\widetilde{\sigma}(\gamma_1)^{-1}.\tag{1.10}$$

Combining equations (1.9) and (1.10), we see that the extended module is self-dual if and only if there exists $w\in W(G,A)$ such that $w\sigma=\sigma$, $w\nu=-\nu$ and

$$\chi_\sigma\left(s^{-1}\gamma(s)\,z\right)=\chi_\sigma\left(s^{-1}\gamma(s)\,\gamma_1^2\right)=1.$$

This condition depends only on the extended group ${}^{\gamma,z}G$ and σ, so it is independent of the choice of extended module, equivalently, extended character of σ.

$\square$

1.3.4 Application to the Schur Indicator

Theorem 1.3.16. *Let G be a split real reductive linear group and let $\pi = J_P(\sigma, \nu)$ be an irreducible self-dual Langlands quotient. Let γ be an involution of G that satisfies (1.6) where $\gamma^2 = int(z)$, for $z \in Z(G) \cap M$. Assume further that γ acts by -1 on a Cartan subalgebra of $\mathfrak{k} = Lie(K)$. Then π extends to an irreducible $(\mathfrak{g}, {}^{\gamma,z}K)$ module $\widetilde{\pi}$. (See Proposition 1.3.10.) If $\widetilde{\pi}$ is self-dual, then*

$$\epsilon(\pi) = \chi_\pi(z).$$

Note that Proposition 1.3.15 provides neccessary and sufficient conditions for $\widetilde{\pi}$ to be self-dual.

Proof. Consider the $(\mathfrak{g}, K)$-module $\pi = J_P(\sigma, \nu)$ and let $\widetilde{\pi}$ be a $(\mathfrak{g}, {}^{\gamma,z}K)$-module that extends π. By assumption, $\widetilde{\pi}$ is self-dual so there exists a ${}^{\gamma,z}G$-invariant bilinear form $(\ ,\ )$ on $\widetilde{\pi}$.

Also, by assumption there exists a Cartan subgroup $H_K \subseteq K_0$, such that γ acts on H_K by -1. Choose a lowest K_0-type $\mu \subseteq \mathrm{Res}_{(\mathfrak{g},K_0)}\pi$ and let λ be the highest weight in μ with nonzero λ-weight vector v_λ. Define $v_{-\lambda} = \widetilde{\pi}(\gamma_1{}^{-1})\, v_\lambda$. For

$Z \in \mathrm{Lie}\,(H_K)$,

$$\begin{aligned}
\widetilde{\pi}(Z)\, v_{-\lambda} &= \widetilde{\pi}\left(\gamma_1^{-1}\right) \widetilde{\pi}(\gamma_1)\, \widetilde{\pi}(Z)\, \widetilde{\pi}\left(\gamma_1^{-1}\right) v_\lambda \\
&= \widetilde{\pi}\left(\gamma_1^{-1}\right) \widetilde{\pi}(\gamma(Z))\, v_\lambda \\
&= \widetilde{\pi}\left(\gamma_1^{-1}\right) \widetilde{\pi}(-Z)\, v_\lambda \\
&= -\lambda(Z)\, v_{-\lambda}
\end{aligned}$$

so $v_{-\lambda} = \pi(\gamma_1)\, v_\lambda$ has weight $-\lambda$, as the notation suggests. By Lemma 1.3.11, $\mathrm{Res}_{(\mathfrak{g},K_0)}\pi$ is irreducible, so the lowest K_0-types are unique and the extremal weights of the lowest K_0-types are unique among the lowest K_0-types. Therefore,

$$(\widetilde{\pi}(\gamma_1)\, v_\lambda, v_\lambda) \neq 0.$$

Now, to calculate the Schur indicator, just apply the technique used in the proof of Lemma 5.2 in [1].

$$\begin{aligned}
(\widetilde{\pi}(\gamma_1)\, v_\lambda, v_\lambda) &= \left(\widetilde{\pi}\left({\gamma_1}^2\right) v_\lambda, \widetilde{\pi}(\gamma_1)\, v_\lambda\right) \qquad (\text{using the } {}^{\gamma,z}G\text{-invariance of the form}), \\
&= (\chi_{\widetilde{\pi}}(z)\, v_\lambda, \widetilde{\pi}(\gamma_1)\, v_\lambda) \qquad (\text{since } z = \gamma_1^2 \in Z(G)), \\
&= \chi_{\widetilde{\pi}}(z)\, (v_\lambda, \widetilde{\pi}(\gamma_1)\, v_\lambda)
\end{aligned}$$

Therefore $\epsilon(\widetilde{\pi}) = \chi_\pi(z)$. Of course the bilinear form remains G-invariant upon restriction to $\pi = \mathrm{Res}_{(\mathfrak{g},K)}\widetilde{\pi}$. Therefore

$$\epsilon(\pi) = \chi_\pi(z)\,.$$

$\square$

1.3.5 Schur Indicator for $SL(4n+2,\mathbb{R})$

We begin by introducing notation. Let $G = SL(4n+2,\mathbb{R})$ and let $P \subseteq G$ be the minimal parabolic subgroup of upper triangular matrices with Langlands decomposition $P = MAN$.

Let

$$M_{\pm} = \{\operatorname{diag}(\eta_1,\ldots,\eta_{4n+2}) : \eta_j = \pm 1\} \subseteq GL(4n+2,\mathbb{R})$$

so that

$$M = \{m \in M_{\pm} : \det(m) = 1\} \subseteq SL(4n+2,\mathbb{R}).$$

For $\lambda \in (\mathbb{Z}/2\mathbb{Z})^{4n+2}$, define

$$\widetilde{\sigma_\lambda} : M_{\pm} \to \{\pm 1\} : \widetilde{\sigma_\lambda}(\operatorname{diag}(\eta_1,\ldots,\eta_{4n+2})) = \prod_{j=1}^{4n+2} \eta_j^{\lambda_j}.$$

The map

$$(\mathbb{Z}/2\mathbb{Z})^{4n+2} \to \widehat{M_{\pm}} : \lambda \mapsto \widetilde{\sigma_\lambda}$$

is an isomorphism of groups. Given $\widetilde{\sigma} \in \widehat{M_{\pm}}$, let $\sigma = \widetilde{\sigma}\,|_M$. Note that $\sigma_\lambda \equiv 1$ if and only if $\lambda \in \langle (1,1,\ldots,1)\rangle$. Therefore, define

$$\sigma_{[\lambda]} = \sigma_\lambda \in \widehat{M}, \tag{1.11}$$

which is a well-defined character of M and

$$(\mathbb{Z}/2\mathbb{Z})^{4n+2} / \langle (1,1,\ldots,1)\rangle \to \widehat{M} : [\lambda] \mapsto \sigma_{[\lambda]}.$$

is an isomorphism of groups. The action of the Weyl group $W(G,A) = N_K(A)/M$ on $\sigma_{[\lambda]}$ is

$$w\sigma_{[(\lambda_1,\ldots,\lambda_{4n+2})]} = \sigma_{[(\lambda_{p_w(1)},\ldots,\lambda_{p_w(4n+2)})]}, \qquad w \in W(G,A)$$

where p_w is a permutation on the set $\{1, 2, \ldots, 4n+2\}$ determined by w. Given $w \in W(G, A)$, let $w(\lambda_1, \ldots, \lambda_{4n+2}) = \left(\lambda_{p_w(1)}, \ldots, \lambda_{p_w(4n+2)}\right)$, so that $w[\lambda] = [w\lambda]$ and

$$w\sigma_{[\lambda]} = \sigma_{[w\lambda]}.$$

Now $w\sigma_{[\lambda]} = \sigma_{[\lambda]}$ if and only if $w[\lambda] = [\lambda]$ if and only if

1. $w\lambda = \lambda$, or,

2. $w\lambda = \lambda + (1, 1, \ldots, 1)$.

Theorem 1.3.17. *Using the notation for the characters in (1.11), let $J_P\left(\sigma_{[\lambda]}, \nu\right)$ be the Langlands quotient of a principal series representation of $SL(4n+2, \mathbb{R})$, for $n = 0, 1, 2, \ldots$ Suppose $J_P\left(\sigma_{[\lambda]}, \nu\right)$ is irreducible and self-dual. Let $w \in W(G, A)$ such that $w[\lambda] = [\lambda]$ and $w\nu = -\nu$. Then*

$$\epsilon\left(J_P\left(\sigma_{[\lambda]}, \nu\right)\right) = \begin{cases} 1 & \text{if } w\lambda = \lambda, \text{ or} \\ -1 & \text{if } w\lambda = \lambda + (1, 1, \ldots, 1). \end{cases}$$

Proof. Let

$$H_K = \bigoplus_{j=1}^{2n+1} R_{\psi_j} \qquad \text{with } \psi_j \in \mathbb{R}, \qquad \text{(using notation from Example 1.3.1)}$$

which is a Cartan subgroup of K. We will extend G by an involution that acts by -1 on H_K. There are two cases to consider and each case utilizes a different extension of G.

Case 1: $w\lambda = \lambda$.

Let $\gamma_1 = \bigoplus_{j=1}^{2n+1} \operatorname{diag}(1, -1)$ and $\gamma = \operatorname{int}(\gamma_1)$. Consider the extended group

$$^{\gamma, I}G \cong \langle SL(4n+2, \mathbb{R}), \gamma_1 \rangle.$$

Using the notation above, let $\widetilde{\sigma_\lambda} \in \widehat{M_\pm}$ such that

$$\sigma_{[\lambda]} = \widetilde{\sigma_\lambda} \mid_M .$$

Of course there are a couple of choices for the extension of σ to $M_\pm$, but the choice doesn't matter.

The assumption that $w\lambda = \lambda$ means that

$$w\widetilde{\sigma_\lambda} = \widetilde{\sigma_\lambda}$$

Since $M_\pm$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{4n+2}$, then

$$w\widetilde{\sigma_\lambda} = \widetilde{\sigma_\lambda} = \left(\widetilde{\sigma_\lambda}\right)^{-1} = \left(\widetilde{\sigma_\lambda}\right)^* .$$

This implies that the extended representation is self-dual. Since γ acts by -1 on H_K and $z = \gamma_1^2 = I$, then by Theorem 1.1.6

$$\epsilon\left(J_P\left(\sigma_{[\lambda]}, \nu\right)\right) = \chi_{J_P\left(\sigma_{[\lambda]}, \nu\right)}(I) = 1.$$

Case 2: $w\lambda = \lambda + (1, 1, \ldots, 1)$. Then λ must have exactly $(2n+1)$ even coefficients and $2n+1$ odd coefficients and w swaps each even coefficient with an odd coefficient. Let ν be in the closed positive Weyl chamber with $\nu \neq 0$. Without loss of generality, the parameter ν has the following form:

$$(\nu_1, \ldots, \nu_k, 0, \ldots, 0, -\nu_k, \ldots, -\nu_1)$$

with $\Re(\nu_j) > 0$ for $1, \ldots, k$ where $0 < k \leq 2n+1$. First, assume that λ has the following form:

$$(\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}, \lambda_{2n+1} + 1, \ldots, \lambda_2 + 1, \lambda_1 + 1) . \tag{1.12}$$

In this case $w = w_0$, where w_0 is the long element of $W(G, A)$. Now, let $\gamma_1 = \bigoplus_{j=1}^{2n+1} \operatorname{diag}(i, -i)$, $\gamma = \operatorname{int}(\gamma_1)$ and consider the extended group ${}^{\gamma,-I}G \cong \langle SL(4n+2, \mathbb{R}), \gamma_1 \rangle$. Since γ acts trivially on $H_s = MA$ and $w_0(\gamma_1) = \gamma_1^{-1}$, then Theorem 1.1.6 implies

$$\epsilon(\pi) = \chi_\pi\left(\gamma_1^2\right) = \chi_\pi(-I) = -1 \qquad \text{(since } \lambda \text{ has } 2n+1 \text{ minus one's).}$$

For general λ with $w\lambda = \lambda + (1, 1, \ldots, 1)$, modify $\left(\sigma_{[\lambda]}, \nu\right)$ by an element of $W(G, A)^\nu$ to obtain $\left(\sigma_{[\lambda']}, \nu\right)$ such that λ' satisfies the form in (1.12).

$\square$

1.3.6 Schur Indicators for Split Equal Rank Groups

Let G be a split real reductive linear group that satisfies condition A. That is, G is the set of real points of some connected reductive complex group $G(\mathbb{C})$. Let θ be a Cartan involution of G and extend θ to an analytic involution of $G(\mathbb{C})$. Let θ also represent the action of the differential of θ on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g}^\theta$ and $\mathfrak{p} = \mathfrak{g}^{-\theta} = \{X \in \mathfrak{g} : \theta(X) = -X\}$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace and let $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is nilpotent. The corresponding Iwasawa decomposition for $\mathfrak{g}_0$ is $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ where $\mathfrak{k}_0$, $\mathfrak{a}_0$, and $\mathfrak{n}_0$ are the real subspaces of $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively, fixed by the real form of G.

The Iwasawa decomposition of G is $G = KAN$, where $A = \exp(\mathfrak{a}_0)$, $N = \exp(\mathfrak{n}_0)$, and $K = G^\theta$. Let $M = Z_K(A)$. The subgroup $P = MAN$ is a minimal parabolic subgroup of G that contains the split Cartan subgroup $H_s = MA$. A fundamental Cartan subgroup $H_f \subseteq G$ is a Cartan subgroup with minimal split rank. Fundamental Cartan subgroups have no real roots and are sometimes called "maximally compact." They are unique up to conjugation by elements of G, [1], Section 1.

Let $\mathfrak{h}_s = \operatorname{Lie}(H_s(\mathbb{C}))$ and let $\mathbf{d}$ be the Cayley transform on $\mathfrak{g}$ that maps $\mathfrak{h}_s \to \mathfrak{h}_f$, where $\mathfrak{h}_f = \operatorname{Lie}(H_f(\mathbb{C}))$ is the complexified Cartan subalgebra of a fundamental Cartan subgroup $H_f \subseteq G$. This induces a map on $G(\mathbb{C})$, also called $\mathbf{d}$, that maps $H_s(\mathbb{C}) \to H_f(\mathbb{C})$. Define

$$\gamma = \mathbf{d}\ \theta\ \mathbf{d}^{-1}.$$

Proposition 1.3.18. *(i) The involution γ acts by -1 on $\mathfrak{h}_f$. (ii) If G is equal rank then γ acts by 1 on $\mathfrak{h}_s$.*

Proof (i). Let $Z \in \mathfrak{h}_f$. Then $\mathbf{d}^{-1}Z \in \mathfrak{h}_s$ and $\theta(\mathbf{d}^{-1}Z) = -\mathbf{d}^{-1}Z$, so

$$\gamma(Z) = \mathbf{d}\left(\theta\left(\mathbf{d}^{-1}Z\right)\right) = -Z.$$

One way to prove (ii) is to use the definition of the Cayley transform. Let α be a real root of some Cartan subgroup $\mathfrak{h} \subseteq \mathfrak{g}$ and let $Z_\alpha, X_\alpha, \theta X_\alpha \in \mathfrak{g}$ such that

$$[Z_\alpha, X_\alpha] = 2X_\alpha, \quad [Z_\alpha, \theta X_\alpha] = -2\theta X_\alpha, \quad [X_\alpha, \theta X_\alpha] = -Z_\alpha.$$

The Cayley transform in the real root α is

$$\mathbf{d}_\alpha = \operatorname{Ad}(g_\alpha), \text{ where } g_\alpha = \exp i\frac{\pi}{4}(\theta X_\alpha - X_\alpha) \in G(\mathbb{C}).$$

The Cayley transform $\mathbf{d}_\alpha$ maps $\mathfrak{h}$ to another Cartan subgroup $\mathfrak{h}' \subseteq \mathfrak{g}$. The image of the real root α, $\mathbf{d}_\alpha\alpha$, is a noncompact imaginary root of $\mathfrak{h}'$. Since $\theta g_\alpha = g_\alpha^{-1}$, then

$$\theta\ \mathbf{d}_\alpha^{-1} = \mathbf{d}_\alpha\ \theta.$$

Two roots α and β are said to be strongly orthogonal if $\alpha \pm \beta$ is not a root. The Cayley transform that maps $H_s \to H_f$ is $\mathbf{d} = \mathrm{Ad}\,(g)$, where $g = g_{\alpha_n} \cdots g_{\alpha_1}$ and α_j is a maximal set of strongly orthogonal real roots. The strong orthogonality implies that the g_{α_j} commute, so

$$\theta\,(g) = g_{\alpha_n}^{-1} \cdots g_{\alpha_1}^{-1} = g_{\alpha_1}^{-1} \cdots g_{\alpha_n}^{-1} = g^{-1},$$

and

$$\theta\ \mathbf{d}^{-1} = \mathbf{d}\ \theta.$$

Proof (ii). Let $Z \in \mathfrak{h}_s$. Then $\mathbf{d}Z \in \mathfrak{h}_f$ and

$$\theta\,(\mathbf{d}Z) = \mathbf{d}^{-1}\,(\theta Z) = -\mathbf{d}^{-1}Z.$$

The equal rank assumption implies $\mathbf{d}Z \in \mathfrak{h}_f \subseteq \mathfrak{k}$, so

$$\mathbf{d}Z = \theta\,(\mathbf{d}Z) = -\mathbf{d}^{-1}Z \quad \Rightarrow \quad \mathbf{d}^2 Z = -Z.$$

Therefore

$$\gamma\,(Z) = \mathbf{d}\ \theta\ \mathbf{d}^{-1}\,(Z) = \mathbf{d}^2\ \theta\,(Z) = \mathbf{d}^2\,(-Z) = Z.$$

$\square$

As a consequence, γ fixes H_s, and therefore must preserve the α root spaces $\mathfrak{g}_\alpha \in \mathfrak{n}$. Here is an explicit description of the action of γ on $\mathfrak{n}$, in the case that G is equal rank. Let α be any real root of $\mathfrak{h}_s$ with root vector X_α. Let $\beta = \mathbf{d}\alpha$ and $X_\beta = \mathbf{d}X_\alpha$. Since $\mathfrak{h}_f$ is compact (because G is equal rank), β is imaginary. If β is compact, then $\theta X_\beta = X_\beta$, or if β is noncompact, then $\theta X_\beta = -X_\beta$. In any case,

$$\pm\mathbf{d}X_\alpha = \pm\, X_\beta = \theta X_\beta = \theta \mathbf{d} X_\alpha = \mathbf{d}^{-1}\theta X_\alpha \quad \Leftrightarrow \quad \mathbf{d}^2 X_\alpha = \pm\theta X_\alpha.$$

Therefore,

$$\gamma\left(X_{\alpha}\right)=\mathbf{d}^{2}\,\theta\left(X_{\alpha}\right)=\begin{cases} X_{\alpha} & \text{if } \mathbf{d}\alpha \text{ is compact, or} \\ -X_{\alpha} & \text{if } \mathbf{d}\alpha \text{ is noncompact.}\end{cases}$$

Theorem 1.3.19. *(Theorem 1.1.4) Let G be an equal rank split real reductive linear group that satisfies condition A. Let π be the Langlands quotient of a principal series representation of G. If π is irreducible then π is self-dual and*

$$\epsilon\left(\pi\right)=\chi_{\pi}\left(z_{\rho^{\vee}}\right).$$

Proof. Since G is equal rank, $\theta=\operatorname{int}\left(\theta_{1}\right)$ for $\theta_{1}\in G\left(\mathbb{C}\right)\cap K\left(\mathbb{C}\right)$ and $\theta_{1}^{2}\in Z\left(G\left(\mathbb{C}\right)\right)\cap K\left(\mathbb{C}\right)\subseteq H_{s}\left(\mathbb{C}\right)\cap K\left(\mathbb{C}\right)=M$.

Let $\mathbf{d}=\operatorname{int}\left(g\right):H_{s}\left(\mathbb{C}\right)\to H_{f}\left(\mathbb{C}\right)$ be the Cayley transform where $H_{f}\subseteq G$ is a fundamental Cartan subgroup. Define $\gamma_{1}=g_{0}\theta_{1}g_{0}^{-1}\in G\left(\mathbb{C}\right)$ and let $\gamma=\operatorname{int}\left(\gamma_{1}\right)$. Of course, $\gamma_{1}^{2}=\theta_{1}^{2}$, so $\gamma^{2}=\operatorname{int}\left(\gamma_{1}^{2}\right)=\operatorname{int}\left(\theta_{1}^{2}\right)$ and $\theta_{1}^{2}\in Z\left(G\right)\cap M$.

By Proposition 1.3.18, γ fixes H_{s} so

(i) the $(\mathfrak{g},K)$-module π extends to a $(\mathfrak{g},{}^{\gamma}K)$-module $\widetilde{\pi}$, and

(ii) $\gamma_{1}\in H_{s}\left(\mathbb{C}\right)$ which means there is already a natural action of $W\left(G,A\right)$ on γ_{1} in the extended group ${}^{\gamma}G$.

In particular, $s_{0}\gamma_{1}s_{0}^{-1}=\gamma_{1}^{-1}$. Now, by Theorem 1.3.16,

$$\epsilon\left(\pi\right)=\chi_{\pi}\left(\gamma_{1}^{2}\right)=\chi_{\pi}\left(\theta_{1}^{2}\right).$$

Since θ_{1} acts by -1 on $\mathfrak{h}_{s}$, then $\theta_{1}\in\operatorname{Norm}_{K(\mathbb{C})}\left(H_{s}\left(\mathbb{C}\right)\right)$. Since G is split, there is a natural isomorphism between $W\left(G,A\right)$ and the complex Weyl group

$W(G(\mathbb{C}), H_s(\mathbb{C})) = N_{G(\mathbb{C})}(H_s(\mathbb{C}))/H_s(\mathbb{C})$. Consequently, there is a representative $s_0 \in N_K(A)$ that acts by -1 on $\mathfrak{a}_0$. Since s_0, θ_1 define the same class in $N_{K(\mathbb{C})}(H_s(\mathbb{C}))/H_s(\mathbb{C})$ then

$$\theta_1^{-1} s_0 \in H_s(\mathbb{C}) \cap K(\mathbb{C}) = M,$$

and

$$\theta_1^2 = \theta_1^2 \left(\theta_1^{-1} s_0 \theta_1^{-1} s_0\right) = \theta(s_0) s_0 = s_0^2.$$

Therefore

$$\epsilon(\pi) = \chi_\pi\left(\gamma_1^2\right) = \chi_\pi\left(\theta_1^2\right) = \chi_\pi\left(s_0^2\right).$$

The analysis using the Tits group in [3], Section 5 applies to $W(G(\mathbb{C}), H_s(\mathbb{C}))$. Since s_0 acts by $-1 \in W(H_s(\mathbb{C}), G(\mathbb{C}))$, then $s_0^2 = z_{\rho^\vee}$ and

$$\epsilon(\pi) = \chi_\pi(z_{\rho^\vee}).$$

$\square$

Theorem 1.3.19 applies to $Sp(2n, \mathbb{R})$ (including $SL(2, \mathbb{R})$), $Spin(n+1, n)$, $SO(n+1, n)$, $Spin(2n, 2n)$, $SO(2n, 2n)$, and the split real forms of E_7 and E_8. Table 1.3.6 depicts the groups which are covered by Theorem 1.3.19, but not covered by [1], Theorem 1.9.

1.3.7 Schur Indicators: Extension Not Required

Let (π, V) be an irreducible self-dual $(\mathfrak{g}, K)$-module for a real reductive linear group G. Let $\mu \subseteq V$ be a lowest K-type and suppose μ is self-dual. Since μ has

Type	Select Groups to which Theorem 1.3.19 applies
A_n	$SL(2,\mathbb{R})$
B_n	$SO(n+1,n), \forall n \in \mathbb{N}$ $Spin(n+1,n)$, for $n \equiv 1,2,3 \pmod 4$
C_n	$Sp(2n,\mathbb{R})$
$D_{2(2n+1)}$	$Spin(2(2n+1), 2(2n+1))$
E_7	nonadjoint split real form

Table 1.1: Select Groups to which Theorem 1.3.19 Applies.

multiplicity one, the nondegenerate G-invariant bilinear form on V restricts to a nondegenerate bilinear form on μ. Therefore,

$$\epsilon(\pi) = \epsilon(\mu). \tag{1.13}$$

This line of reasoning is particularly applicable when G is connected and $-1 \in W(K,T)$, where $T \subseteq K$ is a Cartan subgroup of K. Since $-1 \in W(K,T)$, then every K-type is self-dual and Equation (1.13) holds. This is one of the motivating ideas behind Theorem 1.3.21.

Let G be a connected split real reductive linear group and assume G satisfies condition A. That is, assume G is the set of real points of a connected reductive complex group $G(\mathbb{C})$. Let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup of G. Then $H_s = MA$ is a split Cartan subgroup of G. Let $\{\alpha_1, \ldots, \alpha_r\}$ be a maximal set of strongly orthogonal real roots for $(\mathfrak{g}, \mathfrak{h}_s)$. Let $\mathbf{d} = \mathbf{d}_{\alpha_r} \cdots \mathbf{d}_{\alpha_1}$, be the Cayley transform in this set of strongly orthogonal real roots such that $\mathbf{d} : \mathfrak{h}_s \to \mathfrak{h}_f$, where $\mathfrak{h}_f = \mathrm{Lie}(H_f(\mathbb{C}))$ is the complexified Cartan subalgebra of a fundamental Cartan subgroup $H_f \subseteq G$. Let $H_K = H_f^{\theta}$ $(= H_f \cap K)$, which is a Cartan subgroup of K.

Proposition 1.3.20. *There exists an element $s' \in N_{G(\mathbb{C})}(H_f(\mathbb{C}))$ such that $Ad(s')$*

(i) acts by -1 on $\mathfrak{h}_K$, and

(ii) acts by 1 on $\mathfrak{h}_s$.

Of course *(ii)* implies that $s' \in H_s(\mathbb{C})$.

Proof. This essentially the same proof as Proposition 1.3.18, just slightly more general. Again, the proof will use the Cayley transform. Since α_j is real, choose an isomorphism with $\mathfrak{sl}(2, \mathbb{R})$: $Z_\alpha, X_\alpha, \theta X_\alpha \in \mathfrak{g}_0$ such that

$$[Z_\alpha, X_\alpha] = 2X_\alpha, \quad [Z_\alpha, \theta X_\alpha] = -2X_\alpha, \quad [X_\alpha, \theta X_\alpha] = -Z_\alpha$$

This choice defines an embedding $\mathfrak{sl}(2, \mathbb{R}) \hookrightarrow \mathfrak{g}_0$ via

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto Z_\alpha, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_\alpha, \quad \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \mapsto \theta X_\alpha$$

and therefore an embedding $SL(2,\mathbb{R}) \hookrightarrow G$. Let

$$g_{\alpha_j} = \exp\left(\frac{\pi i}{4}\left(\theta X_{\alpha_j} - X_{\alpha_j}\right)\right) \in \mathfrak{g} \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \in SU(2), \text{ and}$$

$$s_{\alpha_j} = \exp\left(\frac{\pi}{2}\left(\theta X_{\alpha_j} + X_{\alpha_j}\right)\right) \in \mathfrak{g}_0 \quad \longleftrightarrow \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SO(2,\mathbb{R}).$$

Here is a useful formula:

$$s_{\alpha_j} g_{\alpha_j} s_{\alpha_j}^{-1} = g_{\alpha_j}^{-1}. \tag{1.14}$$

Let $H_K(\mathbb{C}) = H_f(\mathbb{C})^{\theta}$, $A'(\mathbb{C}) = H_f(\mathbb{C})^{-\theta} \subseteq H_s(\mathbb{C})$, and $\mathfrak{a}' = \mathrm{Lie}(A'(\mathbb{C})) \subseteq \mathfrak{h}_s$. There is the orthogonal decomposition of $\mathfrak{h}_f$:

$$\mathfrak{h}_f = \mathfrak{h}_K \oplus \mathfrak{a}' = \mathbb{C}\langle X_{\alpha_1} + \theta X_{\alpha_1}, \ldots, X_{\alpha_r} + \theta X_{\alpha_r}\rangle \oplus \bigcap_{j=1}^{r} \ker(\alpha_j).$$

which corresponds to the following orthogonal decomposition of $\mathfrak{h}_s$:

$$\mathfrak{h}_s = \mathbb{C}\langle Z_{\alpha_1}, \ldots, Z_{\alpha_r}\rangle \oplus \bigcap_{j=1}^{r} \ker(\alpha_j).$$

Let $s = s_{\alpha_r} \cdots s_{\alpha_1}$, $g = g_{\alpha_r} \cdots g_{\alpha_1}$. Define $s' = gsg^{-1}$ so that

$$\mathrm{Ad}(s') = \mathrm{Ad}\left(gsg^{-1}\right) = \mathbf{d}\ \mathrm{Ad}(s)\ \mathbf{d}^{-1},$$

where $\mathbf{d} = \mathrm{Ad}(g)$ is the Cayley transform.

Proof of (i). Suppose that $Z \in \mathfrak{h}_K$ which implies that $\mathbf{d}^{-1}Z \in \mathbb{C}\langle Z_{\alpha_1}, \ldots, Z_{\alpha_r}\rangle$. Since $\{\alpha_1, \ldots, \alpha_r\}$ are strongly orthogonal, the reflections s_{α_j} all commute and satisfy

$$s_{\alpha_j} Z_{\alpha_k} s_{\alpha_j}^{-1} = (-1)^{\delta_{jk}} Z_{\alpha_k}, \quad \text{where } \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\mathrm{Ad}(s)(\mathbf{d}^{-1}Z) = -\mathbf{d}^{-1}(Z)$ and

$$\mathrm{Ad}(s')(Z) = \mathbf{d}\left(\mathrm{Ad}(s)\left(\mathbf{d}^{-1}Z\right)\right) = \mathbf{d}\left(-\left(\mathbf{d}^{-1}(Z)\right)\right) = -Z.$$

Proof of (ii). Since s_{α_j}'s commute because of strong orthogonality, then using formula (1.14),

$$\mathrm{Ad}(s') = \mathbf{d}^2 \ \mathrm{Ad}(s). \tag{1.15}$$

Also, by definition of g_{α_j},

$$\theta \ \mathbf{d} = \mathbf{d}^{-1} \ \theta. \tag{1.16}$$

It's sufficient to check two cases. Case 1. Let $Z \in \mathbb{C}\langle Z_{\alpha_1}, \ldots, Z_{\alpha_r}\rangle$. Then $\mathbf{d}Z \in \mathfrak{h}_K$. Using formula (1.16)

$$\mathbf{d}Z = \theta(\mathbf{d}Z) = \mathbf{d}^{-1}\theta(Z) = -\mathbf{d}^{-1}Z \ \Rightarrow \mathbf{d}^2 Z = -Z.$$

Combining this with (1.15) results in

$$\mathrm{Ad}(s')(Z) = \mathbf{d}^2 \ \mathrm{Ad}(s)(Z) = \mathbf{d}^2(-Z) = Z.$$

Case 2. $Z \in \bigcap_{j=1}^{r} \ker(\alpha_j)$. In this case, $[Z, X_{\alpha_j}] = 0$ for all $j = 1, 2, \ldots, r$. Therefore $\mathrm{Ad}(g)(Z) = Z$ and $\mathrm{Ad}(s)(Z) = Z$ and

$$\mathrm{Ad}(s')(Z) = Z.$$

$\square$

Theorem 1.3.21. *(Theorem 1.1.5) Let G be a split real reductive linear group that satisfies condition A and suppose $-1 \in W(K_0, T_0)$, where $T_0 \subseteq K_0$ is a Cartan subgroup of the identity component K_0 of the maximal compact subgroup $K \subseteq G$.*

Let π be the Langlands quotient of a principal series representation of G. If π is self-dual and irreducible then

$$\epsilon(\pi) = 1.$$

Proof. First, we reduce to the case that G is connected. Let π be the Langlands quotient of a principal series representation of G and suppose π is irreducible and self-dual. By Lemma 1.3.11, $\mathrm{Res}_{(\mathfrak{g},K_0)}\pi$ is irreducible, so

$$\epsilon(\pi) = \epsilon\left(\mathrm{Res}_{(\mathfrak{g},K_0)}\pi\right).$$

Without loss of generality, assume G is connected. Let H_f be the fundamental Cartan subgroup of G obtained from $H_s = MA$ by a sequence of Cayley Transforms. Let $H_K = H_f^\theta = H_f \cap K$, which is a Cartan subgroup of K. Let $s_1 \in N_K(H_K)$ such that s_1 is a representative of $-1 \in W(K, H_K)$.

Let s' be the element of $N_{G(\mathbb{C})}(H_f(\mathbb{C}))$ constructed in Proposition 1.3.20. Using the same notation from above, s' is the image via the Cayley transform of the specially constructed element in $s \in N_K(A)$. By Proposition 1.3.20 *(ii)*, $s' \in H_s(\mathbb{C})$, since s' acts by 1 on $\mathfrak{h}_s$. By Proposition 1.3.20 *(i)*, s' acts by -1 on $H_K(\mathbb{C})$, so the element $s_1 s'$ acts by 1 on $H_K(\mathbb{C})$. Therefore

$$s_1 s' \in Z_{G(\mathbb{C})}(H_K(\mathbb{C})) = H_f(\mathbb{C}).$$

Let $s_1 s' = t_{\mathbb{C}} a'_{\mathbb{C}}$, where $t_{\mathbb{C}} \in H_K(\mathbb{C})$ and $a'_{\mathbb{C}} \in A'(\mathbb{C}) \subseteq H_s(\mathbb{C})$. Then

$$s'(a'_{\mathbb{C}})^{-1} = s_1^{-1} t_{\mathbb{C}} \in H_s(\mathbb{C}) \cap K(\mathbb{C}) = M.$$

As a consequence there exists $a'_{\mathbb{C}} \in A'(\mathbb{C}) \subseteq H_s(\mathbb{C})$ such that $s'(a'_{\mathbb{C}})^{-1} \in M$. Define $\gamma_1 = s'(a'_{\mathbb{C}})^{-1} \in M$. Since $\gamma_1 \in M \cong (\mathbb{Z}/2\mathbb{Z})^m$, then $\gamma_1^2 = 1$ and $\gamma = \mathrm{Ad}(\gamma_1)$ is an inner involution of G that acts by -1 on $\mathfrak{h}_K$ and also fixes $\mathfrak{h}_s$. Let π be the Langlands quotient of a principal series representation. If π is irreducible and self-dual then the same technique used in the proof of Lemma 5.2 in [1] applies to calculate the Schur indicator:

$$\epsilon(\pi) = \chi_\sigma\left(\gamma_1^2\right) = \chi_\sigma(1) = 1.$$

□

Example 1.3.22. For $G = SL(2n+1,\mathbb{R})$, $SL(4n,\mathbb{R})$, or $SO(2n+1,2n+1)$, let $T_0 \subseteq K_0$ be a Cartan subgroup of the identity component K_0 of the maximal compact subgroup $K \subseteq G$. Let π be the Langlands quotient of a principal series representation of G. Suppose π is self-dual and irreducible. Since $-1 \in W(K_0, T_0)$, then by Theorem 1.3.21,

$$\epsilon(\pi) = 1.$$

See Table 1.2 for a (nonexhaustive) list of simple groups to which the Schur indicator calculations presented thus far are applicable. The last column indicates whether the result is "new." If the last column has a "**Y**", this means the corresponding result is new and it is not covered by the the techniques in [1]. If the last column has an "**S**", then the corresponding result is "semi-new." These results correspond to nonadjoint split groups in which every irreducible representation is self-dual. These results are not new in the sense that the Schur indicator was calculated in [1] and satisifies $\epsilon(\pi) = \chi_\pi(z_{\rho^\vee})$. However, when the column has an **S**, I am asserting something stronger, namely that the indicator is 1. If the last column has an "N", then the corresponding result was calculated already in [1].

Split Group	Type	Type of K_0	$-1 \in W(K_0)$?	$=$ rk?	$\epsilon(\pi)$	New?
$SL(2n+1,\mathbb{R})$	A_{2n}	B_n	**Y**	N	1	**Y**
$SL(4n,\mathbb{R})$	A_{4n-1}	D_{2n}	**Y**	N	1	**Y**
$SL(4n+2,\mathbb{R})$	A_{4n+1}	D_{2n+1}	N	N	See Thm 1.3.17	**Y**
$SO(4n+1,4n)$	B_{4n}	$B_{2n} \times D_{2n}$	**Y**	**Y**	1	N
$SO(4n+2,4n+1)$	B_{4n+1}	$D_{2n+1} \times B_{2n}$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)=1$	N
$SO(4n+3,4n+2)$	B_{4n+2}	$B_{2n+1} \times D_{2n+1}$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)=1$	N
$SO(4n+4,4n+3)$	B_{4n+3}	$D_{2(n+1)} \times B_{2n+1}$	**Y**	**Y**	1	N
$Spin(4n+1,4n)$	B_{4n}	$B_{2n} \times D_{2n}$	**Y**	**Y**	1	**S**
$Spin(4n+2,4n+1)$	B_{4n+1}	$D_{2n+1} \times B_{2n}$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)$	**Y**
$Spin(4n+3,4n+2)$	B_{4n+2}	$B_{2n+1} \times D_{2n+1}$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)$	**Y**
$Spin(4n+4,4n+3)$	B_{4n+3}	$D_{2(n+1)} \times B_{2n+1}$	**Y**	**Y**	1	**S**
$Sp(2n,R)$	C_n	A_n	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)$	**Y**
$SO(4n,4n)$	D_{4n}	$D_{2n} \times D_{2n}$	**Y**	**Y**	1	**S**
$SO(4n+2,4n+2)$	D_{4n+2}	$D_{2n+1} \times D_{2n+1}$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)=1$	**Y**
$SO(2n+1,2n+1)$	D_{2n+1}	$B_n \times B_n$	**Y**	N	1	**Y**
$Spin(4n,4n)$	D_{4n}	$D_{2n} \times D_{2n}$	**Y**	**Y**	1	**S**
$Spin(4n+2,4n+2)$	D_{4n+2}	$D_{2n+1} \times D_{2n+1}$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)$	**Y**
$Spin(2n+1,2n+1)$	D_{2n+1}	$B_n \times B_n$	**Y**	N	1	**Y**
split real form	E_6	$C_2\ (\mathfrak{sp}(4))$	**Y**	N	1	**Y**
nonadjoint split real form	E_7	$A_7\ (\mathfrak{su}(8))$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)$	**Y**
split real form	E_8	$D_8\ (\mathfrak{so}(16))$	**Y**	**Y**	1	N
split real form	F_4	$C_3 \times A_1$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)$	N
split real form	G_2	$A_1 \times A_1$	N	**Y**	$\chi_\pi\left(z_{\rho^\vee}\right)$	N

Table 1.2: Schur Indicators for Irreducible Self-dual Langlands Quotients of Principal Series for Select Split Groups

1.4 Schur Indicators: An Analytic Perspective

In [16], the authors study intertwining integrals for parabolically induced representations. The authors study the (Hermitian) adjoint of the intertwining integrals and show that the adjoint is itself equal to an intertwining integral. The goal in this chapter is to compute the transpose, rather than the adjoint, and the methods and techniques are borrowed from [16] without much modification. First we introduce the intertwining operators; in the following section we study the transpose.

Let $P_1 = MAN_1$ and $P_2 = MAN_2$ be minimal parabolics of G and let $V_j = \theta N_j$, for $j = 1, 2$. For $f \in I_{P_1}(\sigma, \nu)$, define

$$A(P_2 : P_1 : \sigma : \nu) f(x) = \int_{\overline{N}_1 \cap N_2} f(x\overline{n})\, d\overline{n}, \tag{1.17}$$

When $\Re(\nu)$ in the open positive Weyl chamber, this integral converges for K-finite vectors, maps $I_{P_1}(\sigma, \nu) \to I_{P_2}(\sigma, \nu)$, and intertwines the respective $(\mathfrak{g}, K)$-actions, [16]. By analytic continuation, the intertwining integral in (1.17) extends to an operator defined on $I_{P_1}(\sigma, \nu)$ for any ν, [16].

For $s \in N_K(A)$, define

$$R(s) f(x) = f(xs).$$

Let $P \subseteq G$ be a parabolic subgroup and let $s \in N_K(A)$. The standard intertwining operator $A_P(s, \sigma, \nu) : I_P(\sigma, \nu) \to I_P(s\sigma, s\nu)$ is defined to be

$$A_P(s, \sigma, \nu) = R(s) A\left(s^{-1}Ps : P : \sigma : \nu\right).$$

Explicitly, the formula is

$$A_P(s,\sigma,\nu)(f)(x) = \int_{\overline{N}\cap s^{-1}Ps} f(xs\overline{n})\, d\overline{n},$$

for $f \in I_P(\sigma,\nu)$. This operator factors through the Langlands quotient and gives an isomorphism $J_P(\sigma,\nu) \xrightarrow{\cong} I_P(s\sigma, s\nu)_{\text{sub}}$. For justification, see [14], specifically the proof of Theorem 16.6.

1.4.1 The Transpose of Standard Intertwining Operators

In this section we study the transpose of standard intertwining integrals. The transpose of a linear map is implicitly defined relative to pairings or bilinear forms. In the case of principal series representations, we define the transpose of a linear map to be the transpose relative to the pairings defined in Equation (1.2).

Definition 1.4.1. The transpose of a linear map

$$A : I_{P_1}(\sigma_1,\nu_1) \to I_{P_2}(\sigma_2,\nu_2),$$

is the linear map

$${}^{T}A : I_{P_2}(\sigma_2^*, -\nu_2) \to I_{P_1}(\sigma_1^*, -\nu_1)$$

defined by

$$\left\langle {}^{T}Af_2, f_1 \right\rangle = \langle f_2, Af_1 \rangle, \qquad \begin{array}{l} \text{for } f_2 \in I_{P_2}(\sigma_2^*, -\nu_2), \text{ and} \\ f_1 \in I_{P_1}(\sigma_1,\nu_1). \end{array}$$

The parings $\langle\ ,\ \rangle$ used above are the G-invariant pairings defined in Equation (1.2).

Compare Proposition 1.4.3 below to [16], Proposition 7.1 and 7.8(iii.). The only difference is that in [16] the pairing is conjugate-linear and here the pairing is bilinear. Lemma 1.4.2 also has an analog in [16].

Lemma 1.4.2. *Let $s \in N_K(A)$. ${}^T R(s) = R(s^{-1})$.*

Proof. Let $f_1 \in I_P(\sigma^*, -\nu)$, $f_2 \in I_P(\sigma, \nu)$.

$$\begin{aligned}
\left\langle {}^T R(s) f_1, f_2 \right\rangle &= \langle f_1, R(s) f_2 \rangle \\
&= \int_K \langle f_1(k), f_2(ks) \rangle_{V_\sigma} dk \\
&= \int_K \left\langle f_1\left(k' s^{-1}\right), f_2(k') \right\rangle_{V_\sigma} dk', \qquad \text{using the c.o.v. } k' = ks \\
&= \int_K \left\langle R\left(s^{-1}\right) f_1(k'), f_2(k') \right\rangle_{V_\sigma} dk' \\
&= \left\langle R\left(s^{-1}\right) f_1, f_2 \right\rangle
\end{aligned}$$

□

Proposition 1.4.3. *The operators below, when interpreted K-type by K-type satisfy*

(i) ${}^T A(P_2 : P_1 : \sigma : \nu) = A(P_1 : P_2 : \sigma^* : -\nu)$, *and*

(ii) ${}^T A_P(s, \sigma, \nu) = A_P(s^{-1}, s\sigma^*, -s\nu)$.

The picture for (1.4.3) *(i)* is that the transpose of

$$I_{P_1}(\sigma, \nu) \xrightarrow{A(P_2 : P_1 : \sigma : \nu)} I_{P_2}(\sigma, \nu),$$

is a map

$$I_{P_1}(\sigma^*, -\nu) \xleftarrow{{}^T A(P_2 : P_1 : \sigma : \nu)} I_{P_2}(\sigma^*, -\nu).$$

The obvious guess is

$$^{T}A\left(P_2:P_1:\sigma:\nu\right)=A\left(P_1:P_2:\sigma^*:-\nu\right)$$

and this turns out to be correct.

Proof. For *(i)*, the proof is essentially the same, line by line, as [16], the proof of Proposition 7.1 (iv). The only modification is substituting the Hermitian form for the natural G-invariant bilinear pairing $\langle\ ,\ \rangle : V_\sigma^* \times V_\sigma \to \mathbb{C}$.

For *(ii)*,

$$\begin{aligned}
{}^{T}A_P\left(s,\sigma,\nu\right) &= {}^{T}A\left(s^{-1}Ps:P:\sigma:\nu\right)\ {}^{T}R\left(s\right)\\
&= A\left(P:s^{-1}Ps:\sigma^*:-\nu\right)R\left(s^{-1}\right), \qquad \text{using } \textit{(i)} \text{ and Lemma 1.4.2}\\
&= R\left(s^{-1}\right)A\left(sPs^{-1}:P:s\sigma^*:-s\nu\right)\\
&= A_P\left(s^{-1},s\sigma^*,-s\nu\right).
\end{aligned}$$

$\square$

1.4.2 Schur Indicators for Quasisplit Groups

Let G be quasisplit and suppose there exists $s \in N_K\left(A\right)$ such that $\left(s\sigma, s\nu\right) = \left(\sigma^*, -\nu\right)$. The standard intertwining operator $A_P\left(s,\sigma,\nu\right)$ maps $J_P\left(\sigma,\nu\right) \overset{\cong}{\to} I_P\left(\sigma^*,-\nu\right)_{\mathrm{sub}}$ and

$$\left(f_1,f_2\right)=\left\langle A_P\left(s,\sigma,\nu\right)f_1,f_2\right\rangle \tag{1.18}$$

defines a nondegenerate $\left(\mathfrak{g},K\right)$-invariant bilinear form on $J_P\left(\sigma,\nu\right)$. Calculating the transpose of this form is equivalent to calculating the transpose of $A_P\left(s,\sigma,\nu\right)$ and this is the technique we use to caluclate the Schur indicator in Theorem 1.4.4.

When G is not quasisplit, it's sufficient to require that there exists $s \in N_K(A)$ such that $s\sigma \cong \sigma^*$ and $s\nu = -\nu$. However, this introduces an extra complication; the isomorphism $s\sigma \cong \sigma^*$ must also be taken into account. We first assume G is quasisplit and this allows us to safely ignore this complication. In the case that G is quasisplit, M is abelian, σ is 1-dimensional and $\sigma^* = \sigma^{-1}$. The theory for general real reductive linear groups is the topic of Section 1.6.

Theorem 1.4.4. *Let G be a quasisplit real reductive linear group and let $J(\sigma,\nu)$ be the Langlands quotient of a principal series representation of G. Suppose there exists $w \in W(G,A)$ such that $w^2 = 1$ and $(w\sigma, w\nu) = (\sigma^{-1}, -\nu)$. If $J(\sigma,\nu)$ is irreducible, then*

$$\epsilon(J(\sigma,\nu)) = \chi_\sigma\left(s^2\right),$$

where $s \in N_K(A)$ is any representative of w.

Note that this formula is expressed in terms of the central character of σ, as opposed to the previous formulas which were expressed in terms of the central character of $\pi = J(\sigma,\nu)$. In this formula, $s^2 \in M$, so in general $s^2 \notin Z(G)$.

Proof. Since $s^2 \in M$, then $\chi_\sigma(s^2) = \zeta \in S^1$. Since

$$\zeta = \chi_\sigma\left(s^2\right) = \chi_{w\sigma}\left(s^2\right) = \chi_{\sigma^{-1}}\left(s^2\right) = \chi_\sigma\left(s^2\right)^{-1} = \zeta^{-1}$$

then $\zeta = \pm 1$. Since M is abelian and $s^2 \in M$, then $s\sigma = \sigma^{-1} \Leftrightarrow s\sigma^{-1} = \sigma$. Using the formula for the transpose in Proposition 1.4.3,

$$\begin{aligned}
{}^{T}A_P(s,\sigma,\nu) &= A_P\left(s^{-1}, s\sigma^{*}, -s\nu\right) \\
&= A_P\left(s^{-1}, s\sigma^{-1}, \nu\right) \\
&= A_P\left(s^{-1}, \sigma, \nu\right) \\
&= \chi_\sigma\left(s^2\right) A_P(s,\sigma,\nu).
\end{aligned}$$

The last step is a change of variable for the integral:

$$\begin{aligned}
A_P\left(s^{-1},\sigma,\nu\right)(f)(x) &= \int_{\overline{N}\cap sPs^{-1}} f\left(xs^{-1}\overline{n}\right) d\overline{n} \\
&= \int_{\overline{N}\cap sPs^{-1}} f\left(xs\left(s^{-2}\overline{n}s^{2}\right)s^{-2}\right) d\overline{n} \\
&= \int_{\overline{N}\cap sPs^{-1}} \sigma\left(s^2\right)\left(f\left(xs\left(s^{-2}\overline{n}s^{2}\right)\right)\right) d\overline{n} \\
&= \chi_\sigma\left(s^2\right)\int_{\overline{N}\cap sPs^{-1}} f\left(xs\left(s^{-2}\overline{n}s^{2}\right)\right) d\overline{n} \\
&= \chi_\sigma\left(s^2\right)\int_{\overline{N}\cap s^{-1}Ps} f\left(xs\overline{n}_0\right) d\overline{n}_0, \qquad \text{c.o.v. } \overline{n}_0 = s^{-2}\overline{n}s^{2}, \\
&= \chi_\sigma\left(s^2\right) A_P(s,\sigma,\nu).
\end{aligned}$$

Since $s^{-2}\in M\subseteq K$ the Jacobian factor for $\mathrm{Ad}\left(s^{-2}\right)$ is 1. See [13], Corollary 8.30 (b).

The last item to check is that the formula is independent of choice of representative s of w. Let $s_1 = sm$ for some $m\in M$.

$$\begin{aligned}
\sigma\left(s_1^2\right) = \sigma(smsm) &= \sigma\left(s^2s^{-1}msm\right) \\
&= \sigma\left(s^2\right) s\sigma(m)\,\sigma(m) \\
&= \sigma\left(s^2\right)\sigma(m)^{-1}\sigma(m) \\
&= \sigma\left(s^2\right)
\end{aligned}$$

□

Corollary 1.4.5. *In the context of Theorem 1.4.4, if G is split, then*

$$\epsilon(\pi) = \chi_\sigma\left(\exp\left(\pi i\left(\rho^\vee - w\rho^\vee\right)\right)\right),$$

where $\rho^\vee$ is 1/2 the sum of the positive coroots of G. If $w = w_0$ is the long element of the Weyl group, then

$$\epsilon(\pi) = \chi_\pi\left(z_{\rho^\vee}\right).$$

Proof. Let $H_s = MA$ be a split Cartan subgroup of G. Since G is split, the roots of $\Delta(\mathfrak{h}_s, \mathfrak{g})$ are all real, so there is a natural isomorphism $W(G, A) \cong W(\mathfrak{h}_s, \mathfrak{g})$. The analysis using the Tits group in [3], Section 5 applies to $W(\mathfrak{h}_s, \mathfrak{g})$. Since $w^2 = 1$, by [3], Lemma 5.4, there exists a representative $s \in N_K(A)$ of w such that $s^2 = \exp(\pi i(\rho^\vee - w\rho^\vee))$. If $w = w_0$, the long element of the Weyl group, then

$$s^2 = \exp\left(\pi i\left(\rho^\vee - w_0\rho^\vee\right)\right) = \exp\left(2\pi i\rho^\vee\right) = z_{\rho^\vee} \in Z(G).$$

□

1.5 The δ Indicator

1.5.1 Conjugate Representations

For a reference that treats conjugate representations in a similar way to this section, see [22], Section 4.1. For an additional reference for Definition 1.5.1, see [17], Chapter VI, Section 2.

Definition 1.5.1. A real form for a complex vector space V is a conjugate-linear involution $\phi : V \to V$.

Definition 1.5.2. Let $A : V \to W$ be an isomorphism of complex vector spaces and let ϕ_V and ϕ_W be real forms on V and W, respectively. The conjugate of A relative to the real forms ϕ_V and ϕ_W is

$$\overline{A} = \phi_W A \phi_V.$$

Of course this definition satisfies $\overline{\overline{A}} = A$. Two conjugates of A defined with respect to different real forms on V and W are related by change of basis.

Suppose V has a real structure ϕ. Then

$$V = V^{\phi} \oplus V^{-\phi} = V^{\phi} \oplus iV^{\phi}$$

where $V^{\phi} = \{v \in V : \phi v = v\}$ and $V^{-\phi} = \{v \in V : \phi(v) = -v\}$ are real vector spaces. The direct sum decomposition is

$$v = \frac{1}{2}(v + \phi(v)) + \frac{1}{2}(v - \phi v), \qquad \text{for } v \in V,$$

which is a standard decomposition for an involution of a module over any field $\mathbb{F}$, as long as the characteristic $\mathbb{F}$ is not 2.

Certainly any finite dimensional complex vector space V has a real structure. Let $\dim(V) = n$. To get a real structure for V, just choose an isomorphism $\psi : V \to \mathbb{C}^n$ and then pull back the typical real structure γ on $\mathbb{C}^n$, $\phi = \psi^{-1}\gamma\psi$, where

$$\gamma\left(\sum_{j=1}^{n} \alpha_j e_j\right) = \sum_{j=1}^{n} \overline{\alpha_j} e_j,$$

and $e_1 = (1, 0, \ldots, 0)$,$e_2 = (0, 1, \ldots, 0)$,…,$e_n = (0, \ldots, 0, 1)$.

Definition 1.5.3. Let G be a real Lie group and let ϕ be a real form on the complex vector space V.

(i) Let (π, V) be a representation of G. The conjugate of (π, V) is the representation $(\overline{\pi}, V)$ defined by

$$\overline{\pi}(g) = \overline{\pi(g)} = \phi\pi(x)\phi, \quad \text{for } g \in G.$$

(ii) Now let (π, V) be a $(\mathfrak{g}_0, K)$-module. (It's important to note that $\mathfrak{g}_0$ is the real Lie algebra of G.) The conjugate of (π, V) is $(\overline{\pi}, V)$ where

$$\overline{\pi}(X) = \overline{\pi(X)} = \phi\pi(X)\phi,$$

$$\overline{\pi}(k) = \overline{\pi(k)} = \phi\pi(k)\phi, \quad \text{for } X \in \mathfrak{g}_0, \text{ or } k \in K.$$

Clearly, the definition of the conjugate representation is dependent on the choice of real structure on V. There is no way to avoid this. However, the isomorphism class is *not* dependent on the real structure. Let $\overline{\pi}_\phi$, $\overline{\pi}_{\phi_0}$ be conjugates defined relative to two different real structures ϕ and ϕ_0. Then $\overline{\pi}_{\phi_0}$ and $\overline{\pi}_\phi$ are isomorphic via

$$\phi\phi_0 : (\overline{\pi}_{\phi_0}, V) \xrightarrow{\cong} (\overline{\pi}_\phi, V).$$

Definition 1.5.4. Let (π, V) be a $(\mathfrak{g}_0, K)$-module and let

$$V^h = \{f : V \to \mathbb{C} \mid f \text{ is conjugate-linear and } K\text{-finite.}\}$$

The Hermitian dual of (π, V) is the representation $\left(\pi^h, V^h\right)$, where

$$\left\langle \pi^h(X)(f), v\right\rangle = \langle f, \pi(-X)(v)\rangle,$$

$$\left\langle \pi^h(k)(f), v\right\rangle = \left\langle f, \pi(k)^{-1}(v)\right\rangle, \quad \text{for } f \in V^h,\ v \in V,\ X \in \mathfrak{g}_0,\ k \in K.$$

A $(\mathfrak{g}_0, K)$-module is Hermitian if it is isomorphic to its Hermitian dual. In this case (π, V) admits a nondegenerate G-invariant Hermitian form. If G is not compact, this form may not be unitary.

Proposition 1.5.5. *Let V be a complex vector space with a real form ϕ and let (π, V) be a $(\mathfrak{g}_0, K)$ module. Then*

$$(\overline{\pi}, V) \cong \left(\pi^{*h}, V^{*h}\right) \cong \left(\pi^{h*}, V^{h*}\right).$$

Proof. Define the conjugate of π via ϕ: $\overline{\pi} = \phi\pi\phi$. By definition,

$$V^{*h} = \{F : V^* \to \mathbb{C} : \ F \text{ is conjugate-linear and } f \text{ is } K\text{-finite}\}$$

Define $\iota_\phi : V \to V^{*h}$, $\iota_\phi (v) (f) = \overline{f (\phi (v))}$. Given $v \in V$, the map $\iota_\phi (v)$ is conjugate-linear:

$$\iota_\phi (v) (z \ f) = \overline{z \ f (\phi (v))} = \overline{z} \ \overline{f (\phi (v))} = \overline{z} \ \iota_\phi (v) (f).$$

The map ι_ϕ is linear since f is linear and ϕ is conjugate-linear:

$$\iota_\phi (z \ v) (f) = \overline{f (\phi (z \ v))} = z \ \overline{f (\phi (v))} = z \ \iota_\phi (v) (f).$$

Furthermore, ι_ϕ intertwines $\overline{\phi}$ and π^{*h}, since

$$\iota_\phi (\phi \ \pi (X) \ \ \phi (v)) (f) = \overline{f (\pi (X) (\phi (v)))}, \qquad X \in \mathfrak{g}_0$$

and

$$\begin{aligned} \pi^{*h} (X) (\iota_\phi (v)) (f) &= \iota_\phi (v) (\pi^* (-X) f) \\ &= \pi^* (-X) \overline{f (\phi (v))} \\ &= \overline{f (\pi (X) (\phi (v)))}, \qquad X \in \mathfrak{g}_0. \end{aligned}$$

To demonstrate the isomorphism of $(\overline{\pi}, V)$ with $\left(\pi^{h*}, V^{h*}\right)$, define $\iota_\phi : V \to V^{h*}$ via $\iota_\phi(v)(f) = f(\phi(v))$. In this case $f \in V^h$ and is therefore conjugate-linear, so precomposition with the conjugate-linear ϕ makes ι_ϕ linear. The rest of the proof is straightforward and omitted.

□

Recall that Definition 1.5.3 defines the conjugate of a $(\mathfrak{g}_0, K)$-module. In practice, it's often more convenient to work with $(\mathfrak{g}, K)$-modules rather than $(\mathfrak{g}_0, K)$-modules, and in this case it's necessary to modify Definition 1.5.3 slightly. For $X \in \mathfrak{g}$, let $\overline{X}$ represent conjugation with respect to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Let V be a complex vector space with a real form ϕ. Define the conjugate of a $(\mathfrak{g}, K)$-module (π, V) to be $(\overline{\pi}, V)$, where

$$\overline{\pi}(X)(v) = \phi\pi\left(\overline{X}\right)\phi(v) = \overline{\pi\left(\overline{X}\right)}(v), \ X \in \mathfrak{g}, \ v \in V.$$

1.5.2 Structure Maps and Self-Conjugate Representations

Everything in this section is stated for $(\mathfrak{g}_0, K)$-modules. After making the modifications, if necessary, analogous definitions and propositions hold for $(\mathfrak{g}, K)$-modules and also finite dimensional representations of Lie groups.

Definition 1.5.6. (This definition corresponds to [4], Definition 3.2.) Let V be a complex vector space with a real form ϕ and let (π, V) be a $(\mathfrak{g}_0, K)$-module. A structure map is a conjugate-linear map $j : V \to V$ that satisfies

$$
\begin{aligned}
j\pi(X) &= \pi(X)\,j, \\
j\pi(k) &= \pi(k)\,j, \quad \text{for } X \in \mathfrak{g}_0,\ k \in K, \text{ and} \\
j^2 &= \pm \mathrm{id}_V.
\end{aligned}
$$

Definition 1.5.7. Suppose there exists a structure map for the $(\mathfrak{g}_0, K)$-module (π, V).

(i) If $j^2 = \mathrm{id}_V$ then j defines a real structure on V, with $V = V^j \oplus iV^j$, and π acts on the real vector space V^j. In this case (π, V) is called real.

(ii) If $j^2 = -\mathrm{id}_V$ then j defines a quaternionic structure on V which is compatible with the action of π. In this case π is called quaternionic.

Proposition 1.5.8. *Let π be an irreducible, admissible $(\mathfrak{g}_0, K)$-module on a complex vector space V. Then $\pi \cong \overline{\pi}$ if and only if there exists a structure map on V.*

Proof. Let ϕ be a real structure on V and let $\overline{\pi} = \phi\pi\phi$. If there exists a structure map $j : V \to V$ that commutes with π, then

$$\pi(X)\,j\phi = j\pi(X)\,\phi = j\phi\overline{\pi}(X)\,, \text{for } X \in \mathfrak{g}_0,$$

and similarly for the action of K. Therefore $j\phi : (\pi, V) \to (\overline{\pi}, V)$ is an isomorphism.

If $(\overline{\pi}, V) \cong (\pi, V)$, there exists a linear intertwining map $T : (\pi, V) \to (\overline{\pi}, V)$. Let $j = \phi T$; then $j : (\pi, V) \to (\pi, V)$ is a conjugate-linear map that intertwines π. Furthermore, $j^2 : (\pi, V) \to (\pi, V)$ is a linear intertwining map, so by Schur's lemma, $j^2 = \lambda\ \mathrm{id}_V$, for $\lambda \in \mathbb{C}^\times$. Actually, since $j^2 = \lambda\ \mathrm{id}_V$ and j is conjugate-linear, then

$\lambda \in \mathbb{R}^{\times}$. Scaling j by any complex number z results in scaling j^2 by the positive real number $|z|^2$. Therefore, j may be normalized so that $j^2 = \pm \mathrm{id}_V$ and j is a structure map. $\square$

A $(\mathfrak{g}_0, K)$-module (π, V) is self-conjugate if it is isomorphic to its conjugate. Irreducible self-conjugate $(\mathfrak{g}_0, K)$-modules are either real or quaternionic. The δ indicator of a self-conjugate representation π (See Definition 1.1.7) is the sign $\delta(\pi)$ that has value $\delta(\pi) = 1$ when π is real, or $\delta(\pi) = -1$ when π is quaternionic.

Here is another way to think about self-conjugate $(\mathfrak{g}_0, K)$-modules, which will be useful later. Suppose $(\pi, V) \cong (\overline{\pi}, V)$, where the conjugate is defined via the real form ϕ on V. Let $T : (\pi, V) \to (\overline{\pi}, V)$ be an intertwining operartor. The conjugate of T is an isomorphism mapping in the opposite direction:

$$\overline{T} = \phi T \phi : (\overline{\pi}, V) \to (\pi, V).$$

Assuming T is normalized appropriately, then

$$\overline{T} \quad = \delta(\pi) T^{-1}.$$

That is, given an isomorphism $T : (\pi, V) \to (\overline{\pi}, V)$, calculating the δ indicator for (π, V) is the same as calculating the difference in sign between $\overline{T}$ and T^{-1}.

1.5.3 The Conjugate Principal Series

Let G be a real reductive linear group and let $P = MAN \subseteq G$ be the Langlands decomposition of a minimal parabolic subgroup. Let σ be an irreducible representation of M on a complex vector space V_σ and let $\nu \in \mathfrak{a}^*$. Let ϕ be a real structure on V_σ and let $\overline{\sigma} = \phi\sigma\phi$. The real structure ϕ on V_σ induces a conjugate-linear map, which we will also call ϕ,

$$\phi : \mathrm{Ind}_P^G (\sigma \otimes \nu \otimes 1) \to \mathrm{Ind}_P^G (\overline{\sigma} \otimes \overline{\nu} \otimes 1), \quad \phi(f)(x) = \phi(f(x)),$$

which intertwines the G-actions:

$$\phi(\pi(\sigma, \nu, g)(f))(x) = \pi(\overline{\sigma}, \overline{\nu}, g)(\phi(f))(x), \quad f \in \mathrm{Ind}_P^G (\sigma \otimes \nu \otimes 1), \ x \in G.$$

This leads to the identity

$$\phi\pi(\sigma, \nu, x)\phi = \pi(\overline{\sigma}, \overline{\nu}, x), \quad \text{for } x \in G.$$

By definition, the conjugate of $\mathrm{Ind}_P^G (\sigma \otimes \nu \otimes 1)$ is $\overline{\pi(\sigma, \nu)} = \phi\pi(\sigma, \nu)\phi$. Therefore,

$$\overline{\pi(\sigma, \nu)} = \pi(\overline{\sigma}, \overline{\nu})$$

and the conjugate of the principal series $\mathrm{Ind}_P^G (\sigma \otimes \nu \otimes 1)$ *is* $\mathrm{Ind}_P^G (\overline{\sigma} \otimes \overline{\nu} \otimes 1)$ and the conjugate of the $(\mathfrak{g}_0, K)$-module $I_P(\sigma, \nu)$ *is* $I_P(\overline{\sigma}, \overline{\nu})$.

Lemma 1.5.9. *Let $P = MAN$ be the Langlands decomposition of a minimal parabolic P with opposite parabolic $P_{op} = MAV_{op}$, $V_{op} = \theta(N)$. Let $(\sigma, V_\sigma) \in \widehat{M}$ and let $\nu \in \mathfrak{a}^*$. Let ϕ be a real form on V_σ and define $\overline{\sigma} = \phi\sigma\phi$. The induced map $\phi(f)(x) = \phi(f(x))$ satisfies:*

$$\phi\ A(P_{op} : P : \sigma : \nu) = A(P_{op} : P : \overline{\sigma} : \overline{\nu})\phi$$

Note that if $\Re(\nu)$ is in the closed positive Weyl chamber, then so is $\Re(\overline{\nu})$.

Proof. Since P is minimal and σ is irreducible, then V_σ is finite dimensional. For $f \in I_P(\sigma, \nu)$, let $f(x) = \sum_{j=1}^r f_j(x)\beta_j$, where $\{\beta_1, \ldots, \beta_r\}$ is a basis for V_σ and

$f_j : G \to \mathbb{C}$.

$$\begin{aligned}
\phi\left(A\left(P_{op} : P : \sigma : \nu\right)\right)(f)(x) &= \phi\left(\int_{N_{op}} f\left(xn_{op}\right) d_{op}\right) \\
&= \sum_{j=1}^{r}\left(\overline{\int_{N_{op}} f_j\left(xn_{op}\right) dn_{op}}\right)\phi\left(\beta_j\right) \\
A\left(P_{op} : P : \overline{\sigma} : \overline{\nu}\right)\left(\phi\left(f\right)\right)(x) &= \int_{N_{op}} \phi\left(f\left(xn_{op}\right)\right) dn_{op} \\
&= \sum_{j=1}^{r}\left(\int_{N_{op}} \overline{f_j\left(xn_{op}\right)} dn_{op}\right)\ \phi\left(\beta_j\right)
\end{aligned}$$

Since "bar" is a bounded *linear* map $\mathbb{C} \to \mathbb{C}$, then for each $j = 1, \ldots, r$,

$$\overline{\int_{N_{op}} f_j\left(xn_{op}\right) dn_{op}} = \int_{N_{op}} \overline{f_j\left(xn_{op}\right)} dn_{op}.$$

□

The main point of Lemma 1.5.9 is that $\phi : I_P\left(\sigma, \nu\right) \to I_P\left(\overline{\sigma}, \overline{\nu}\right)$ factors through the Langlands quotient to give a well defined bijective conjugate-linear map: $J_P\left(\sigma, \nu\right) \to J_P\left(\overline{\sigma}, \overline{\nu}\right)$ that intertwines the respective $\left(\mathfrak{g}_0, K\right)$-actions. Therefore,

$$\overline{J\left(\sigma, \nu\right)} = J\left(\overline{\sigma}, \overline{\nu}\right).$$

Lemma 1.5.10. *Let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup of G and let $s \in N_K\left(A\right)$. Let ϕ be a real form on V_σ the representation space of σ. Then*

$$\overline{A_P\left(s, \sigma, \nu\right)} = A_P\left(s, \overline{\sigma}, \overline{\nu}\right).$$

Proof. Let ϕ also represent the induced map $\phi\left(f\right)(x) = \phi\left(f\left(x\right)\right)$, for $f \in J_P\left(\sigma, \nu\right)$. Since ϕ commutes with $R\left(s\right)$, then the statement follows by Lemma 1.5.9. □

Lemma 1.5.11. *Let $J_P(\sigma,\nu)$ be the irreducible Langlands quotient of a principal series representation of a real reductive linear group. If σ is self-conjugate and ν is real, then $J_P(\sigma,\nu)$ is self-conjugate and*

$$\delta\left(J_P(\sigma,\nu)\right)=\delta(\sigma).$$

Proof. Consider $J_P(\sigma,\nu)$ as a $(\mathfrak{g}_0,K)$-module. Let V_σ be the complex vector space on which the representation σ acts. Suppose that $\sigma\cong\overline{\sigma}$ and let $j:V_\sigma\to V_\sigma$ be a structure map. Since ν is real, j commutes with the representation $(\sigma\otimes\nu\otimes 1,V_\sigma)$ of P so j is also a structure map for $(\sigma\otimes\nu\otimes 1,V_\sigma)$. This induces to a structure map on $J_P(\sigma,\nu)$, denoted j', which is simply precomposition by j. Therefore $\operatorname{sgn}\left((j')^2\right)=\operatorname{sgn}(j^2)$ and

$$\delta\left(J_P(\sigma,\nu)\right)=\delta(\sigma\otimes\nu\otimes 1)=\delta(\sigma).$$

$\square$

Suppose $J_P(\sigma,\nu)$ is irreducible and ν is real and lies in the open positive Weyl chamber. Then $J_P(\sigma,\nu)$ is self-conjugate if and only if σ is self-conjugate. If ν is real and lies in the open positive Weyl chamber, then

$$J_P(\sigma,\nu)\cong\overline{J_P(\sigma,\nu)}=J_P(\overline{\sigma},\nu)\quad\Leftrightarrow\quad\overline{\sigma}\cong\sigma,$$

by the uniqueness of the Langlands classification. (See [14], Theorem 14.92.)

Example 1.5.12. Let $\gamma_1=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$, $G=SL(2,\mathbb{R})$, and $P=MAN$ be the Langlands decomposition of the minimal parabolic subgroup of upper triangular matrices. Suppose the Langlands quotient $J_P(\sigma,\nu)$ is irreducible. By Propositions

1.3.8, 1.3.10 and 1.3.15, $J_P(\sigma,\nu)$ extends to a self-dual $(\mathfrak{g}, {}^{\gamma}K)$-module isomorphic to $J_{{}^{\gamma}P}(\widetilde{\sigma},\nu)$, where $\widetilde{\sigma}$ is an extension of σ to ${}^{\gamma}M$. Suppose that $\chi_{\sigma}(-I)=-1$ and $\nu>0$ is real. Then $\widetilde{\sigma}$ is not self-conjugate; in this case conjugation exchanges the two nonisomorphic extensions of σ to ${}^{\gamma}M$. By Lemma 1.5.11, $J_{{}^{\gamma}P}(\widetilde{\sigma},\nu)$ cannot be self-conjugate either. But $J_{{}^{\gamma}P}(\widetilde{\sigma},\nu)$ is self-dual which means that it also cannot be Hermitian. This last statement is not hard to see directly. Since $\nu>0$, then only the nontrivial Weyl group element w_0 satisfies $w_0\nu=-\nu$. However, $w_0\widetilde{\sigma}(\gamma_1)=-\widetilde{\sigma}(\gamma_1)\neq\widetilde{\sigma}(\gamma_1)$.

1.5.4 A Nonunitarity Criterion for Split Groups

Lemma 1.5.11 has the following easy corollary:

Corollary 1.5.13. *Let G be a split real reductive linear group that satisfies condition A and let $J_P(\sigma,\nu)$ be the Langlands quotient of a principal series representation of G. Suppose ν is real. If $J_P(\sigma,\nu)$ is irreducible, then*

$$\delta\left(J_P(\sigma,\nu)\right)=1.$$

Corollary 1.5.13 may be clear without having to appeal to any previous results. Since σ and ν are clearly real, then $\sigma\otimes\nu\otimes 1$ (and therefore $\mathrm{Ind}_P^G(\sigma\otimes\nu\otimes 1)$) could have been defined on a real vector space from the beginning.

Proof. Since G has satisfies condition A, $M\cong(\mathbb{Z}/2\mathbb{Z})^n$. Therefore $\sigma\cong\overline{\sigma}$ and $\delta(\sigma)=1$, so by Proposition 1.5.11,

$$\delta\left(J_P(\sigma,\nu)\right)=1.$$

□

Proposition 1.5.14. *(Proposition 1.1.9, Nonunitarity Criterion) Let G be a split real reductive linear group that satisfies condition A and let $J(\sigma,\nu)$ be the Langlands quotient of a principal series representation of G. Suppose that ν is real and there exists $w \in W(G,A)$ such that $(w\sigma, w\nu) = (\sigma, -\nu)$ and $w^2 = 1$. If $J(\sigma,\nu)$ is irreducible and*

$$\chi_\sigma\left(\exp\left(\pi i\left(\rho^\vee - w\rho^\vee\right)\right)\right) = -1$$

then $J(\sigma,\nu)$ is nonunitary.

Proof. By hypothesis, $J(\sigma,\nu)$ is Hermitian, self-dual, and self-conjugate. By Corollary 1.4.5,

$$\epsilon\left(J(\sigma,\nu)\right) = \chi_\sigma\left(\exp\left(\pi i\left(\rho^\vee - w\rho^\vee\right)\right)\right) = -1.$$

If $J(\sigma,\nu)$ were unitary, then by Lemma 1.1.8, $\delta\left(J(\sigma,\nu)\right) = -1$. But this is impossible because it contradicts Corollary 1.5.13. □

There are nonunitarity criteria in the literature which are very similar to Proposition 1.1.9. See [5], Section 4.3, Proposition 4.6 and [14] Proposition 16.8. Both of these propositions are proved by showing that a Hermitian intertwining operator cannot be unitary because it is $+1$ on some lowest K-types and -1 on other lowest K-types. In Proposition 1.1.9, the Schur indicator is detecting exactly this same phenomenon.

1.6 Indicators for Principal Series of Real Reductive Linear Groups

We begin with a couple of auxiliary sections in which we develop some theory to address problems that arise when we try to extend our previous results to groups that are not quasisplit.

The main theorems and results are in Section 1.6.3.

1.6.1 Distinguished Involutions of Real Reductive Lie Groups

A distinguished automorphism of a connected complex reductive group $G(\mathbb{C})$ is one that preserves a *pinning*, which is a triple

$$\left(H(\mathbb{C}), B(\mathbb{C}), \{X_\alpha : \alpha \in \Pi\}\right),$$

where $B(\mathbb{C})$ is a Borel subgroup, $H(\mathbb{C}) \subseteq B(\mathbb{C})$ is a Cartan subgroup, Π is the associated set of simple roots, and $X_\alpha \in \mathrm{Lie}(B(\mathbb{C}))$ is a root vector in the α-weight space for $\alpha \in \Pi$. For some background on the concept of pinnings in the context of complex reductive groups, relevant theorems, and further references see [3] Section 2, and [1], Section 2.

The definitions and results of this section are for real reductive linear groups.

Definition 1.6.1. Let G be a real reductive linear group and let $H \subseteq G$ be a Cartan subgroup. Let $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ be the roots and let $\Delta^+ = \Delta^+(\mathfrak{h}, \mathfrak{g})$ be a positive system for the roots. Let $\Pi \subseteq \Delta^+$ be the set of simple roots. For $\alpha \in \Pi$, let $X_\alpha \neq 0 \in \mathfrak{g}_\alpha$. A pinning of a real reductive linear group is an ordered triple,

$$\mathscr{P} = \left(H, \Delta^+, \{X_\alpha : \alpha \in \Pi\}\right).$$

That is, a pinning is an ordered triple consisting of a choice of Cartan subgroup, a choice of positive system, and choice of simple root vectors. Let $\mathscr{P}$ be a pinning. An automorphism γ of G preserves $\mathscr{P}$ if γ acts on all three sets defining $\mathscr{P}$. If γ preserves $\mathscr{P}$, then γ is called $\mathscr{P}$-distinguished.

Recall that if $\tau \in \mathrm{Int}(G)$, then $\tau = \mathrm{int}(g)$, for some $g \in G$, where $\mathrm{int}(g)(x) = gxg^{-1}$, for all $x \in G$.

Proposition 1.6.2. *Let τ be an automorphism of the real reductive linear group G. Suppose that*

(i) τ is $\mathscr{P}$-distinguished, for some pinning $\mathscr{P}$, and

(ii) $\tau \in \mathit{Int}(G)$.

Then $\tau = \mathit{int}(z)$, for some $z \in Z_G(G_0)$. Furthermore, if G is connected, then $\tau = 1$.

If G satisfies condition A, then this seems to be a trivial consequence of [3], Theorem 2.2. For completeness, we provide a proof below that does not assume G satisfies condition A.

Proof. Let H be a Cartan subgroup of G and let $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ be the root system with positive roots Δ^+ and simple roots $\Pi \subseteq \Delta^+$. Let $\mathscr{P} = (H, \Delta^+, \{X_\alpha : \alpha \in \Pi\})$ and suppose $\tau = \mathrm{int}(g)$, $g \in G$, is $\mathscr{P}$-distinguished. There is an embedding

$$N_G(H)/H \hookrightarrow W(\mathfrak{h}, \mathfrak{g}) = W(\Delta)$$

so there exists $w \in W(\Delta)$ such that $w = [g]$. Since g preserves Δ^+ then so must w, but this implies $w = 1$ and $g \in H$. But if $g \in H$ and $\mathrm{int}(g)$ is distinguished, then $\operatorname{int}(g)$ fixes $\mathfrak{g}$. Consequently, $gxg^{-1} = x$ for all $x \in G_0$, so $g \in Z_G(G_0)$. If G is connected, then $g \in Z(G)$ and $\tau = \operatorname{int}(g) = 1$. □

We now turn to pinnings for equal rank groups, which are groups in which $\operatorname{rank}(K) = \operatorname{rank}(G)$, where $K = G^\theta \subseteq G$ is the maximal compact subgroup relative to the Cartan involution θ. A Lie group G has equal rank if and only if it has a compact Cartan subgroup. For equal rank groups, it will sometimes be convenient to fix a pinning in which the Cartan is compact and the root vectors satisfy certain equations.

Let G be an equal rank real reductive linear group. We fix a compact Cartan subgroup $T \subseteq G$, a set of positive roots $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ and the associated simple roots $\Pi \subseteq \Delta^+$. Now we choose a collection of simple root vectors that satisfy certain conditions. For $X \in \mathfrak{g}$, let $\overline{X}$ represent conjugation with respect to $\mathfrak{g}_0 = \operatorname{Lie}(G)$. Let $X_\alpha \neq 0 \in \mathfrak{g}_\alpha$. Since $T \subseteq G$ is a compact Cartan subgroup, then all roots of T are imaginary and $\overline{X_\alpha} \in \mathfrak{g}_{-\alpha}$. After scaling X_α, we may assume that

$$\begin{aligned} &\text{(i)} \quad [X_\alpha, -\overline{X_\alpha}] = \alpha^\vee, \quad \text{if } \alpha \in \Pi \text{ is compact, or} \\ &\text{(ii)} \quad [X_\alpha, \overline{X_\alpha}] = \alpha^\vee, \quad \quad \text{if } \alpha \in \Pi \text{ is noncompact.} \end{aligned} \tag{1.19}$$

For (i), see [13], the Remark after Theorem 4.54. For (ii), see [13], Chapter IV, Section 7.

Proposition 1.6.3. *Let G be a real reductive linear group and suppose $T \subseteq G$ is a compact Cartan subgroup with roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$. Fix a choice of positive roots $\Delta^+ \subseteq \Delta$ and a pinning $\mathscr{P} = (T, \Delta^+, \{X_\alpha : \alpha \in \Pi\})$ with root vectors satisfying (1.19). Let γ be an automorphism of G that commutes with θ and preserves Δ^+.*

There exists $t \in T$ and a $\mathscr{P}$-distinguished automorphism τ such that

$$\gamma = int(t)\ \tau.$$

The hypothesis in Proposition 1.6.3 that T is a compact Cartan subgroup is critical. Here is an example. Fix an arbitrary $z \in \mathbb{C}^{\times}$. Let $G = SL(2, \mathbb{R})$, $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^{\times} \right\}$, $X_{\alpha} = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$, and $\mathscr{P} = (H, \{\alpha\}, \{X_{\alpha}\})$. In this case if τ is $\mathscr{P}$-distinguished, then τ *fixes* the pinning, including the positive root $\{\alpha\}$. Therefore any $\mathscr{P}$-distinguished involution τ induces the identity map on $\mathscr{P}$. Let $\gamma = \mathrm{int}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$. Then $\gamma(X_{\alpha}) = -X_{\alpha}$, but

$$\mathrm{int}\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right)(X_{\alpha}) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} X_{\alpha} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = a^2 X_{\alpha}, \quad \text{for any } a \in \mathbb{R}^{\times},$$

so there does not exist $h \in H$ such that $\gamma = \mathrm{int}(h)\, \tau$.

Proof. By hypothesis γ preserves Δ^{+}. As a consequence, γ also preserves

(i) the simple roots Π,

(ii) the positive coroots, $\Delta^{\vee +}$, and

(iii) the simple coroots, $\Pi^{\vee} \subseteq \Delta^{\vee +}$.

The coroots are unique, so for $\alpha^{\vee} \in \Delta^{\vee}$,

$$(\gamma\alpha)^{\vee} = \gamma(\alpha^{\vee}).$$

Let $\alpha \in \Pi$ and let $X_{\alpha} \neq 0 \in \mathfrak{g}_{\alpha}$. Since γ acts on T, then

$$\gamma(X_{\alpha}) = c_{\alpha} X_{\gamma\alpha}, \quad \text{for some } c_{\alpha} \in \mathbb{C}^{\times}.$$

Let $\mathscr{P} = (T, \Delta^+, \{X_\alpha : \alpha \in \Pi\})$ be a pinning with root vectors that satisfy (1.19). If α is compact, then $\gamma\alpha$ remains compact, since γ commutes with θ. Now, apply γ to both sides of (1.19) (i) above:

$$\begin{aligned}
(\gamma\alpha)^\vee &= \gamma\left(\alpha^\vee\right) \\
&= \gamma\left(\left[X_\alpha, -\overline{X_\alpha}\right]\right) \\
&= \left[\gamma\left(X_\alpha\right), -\overline{\gamma\left(X_\alpha\right)}\right] \\
&= \left[c_\alpha X_{\gamma\alpha}, -\overline{c_\alpha X_{\gamma\alpha}}\right] \\
&= \left|c_\alpha\right|^2 \left[X_{\gamma\alpha}, -\overline{X_{\gamma\alpha}}\right] \\
&= \left|c_\alpha\right|^2 (\gamma\alpha)^\vee .
\end{aligned}$$

We conclude that $\left|c_\alpha\right|^2 = 1$ and $c_\alpha \in S^1$. When α is noncompact, the same conclusion holds, using the same argument with (1.19) (ii) instead of (1.19) (i) above.

Let $\{\alpha_1, \ldots, \alpha_m\}$ be the simple roots of $\mathscr{P}$ and let $c_{\alpha_j} = e^{iy_j}$ for $y_j \in \mathbb{R}$, $j = 1, \ldots, m$. Let $x_1, \ldots, x_m \in \mathbb{R}$ be the solution to linear system

$$\alpha_j\left(\sum_{k=1}^{m} x_k \alpha_k^\vee\right) = y_j, \quad \text{for } j = 1, \ldots, m.$$

Define

$$t = \exp\left(-\sum_{k=1}^{m} \alpha_k^\vee\left(x_k i\right)\right).$$

Note that $\overline{\alpha^\vee} = -\alpha^\vee$, so the multiplication by i in the definition of t forces $\overline{t} = t$, so $t \in T$. Let X_{α_j} be a simple root vector in the pinning $\mathscr{P}$. Then

$$\mathrm{Ad}\,(t)\, X_{\alpha_j} = e^{\alpha_j(\log t)} X_{\alpha_j} = e^{-iy_j} X_{\alpha_j} = c_{\alpha_j}^{-1} X_{\alpha_j}.$$

Define $\tau = \gamma \ \mathrm{int}\,(t)$. The automorphism τ acts on T and

$$\tau\left(X_{\alpha_j}\right) = \gamma \ \mathrm{int}\,(t)\left(X_{\alpha_j}\right) = \gamma\left(c_{\alpha_j}^{-1} X_{\alpha_j}\right) = X_{\gamma\alpha_j}.$$

Therefore τ preserves $\{X_\alpha : \alpha \in \Pi\}$ and is $\mathscr{P}$-distinguished. Finally,

$$\gamma = \tau \text{ int}\left(t^{-1}\right) = \text{int}\left(\tau\left(t^{-1}\right)\right)\tau.$$

$\square$

Lemma 1.6.4 will not be needed until the proof of Theorem 1.1.6.

Lemma 1.6.4. *Let G be a real reductive linear group with a compact Cartan T. Fix a set of positive roots $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ and a corresponding set of simple roots $\Pi \subseteq \Delta^+$. Let $\mathscr{P} = \{T, \Delta^+, \{X_\alpha : \alpha \in \Pi\}\}$ be a pinning with root vectors that satisfy (1.19). If $\mathscr{P}_0 = \{T, \Delta^+, \{X^0_\alpha : \alpha \in \Pi\}\}$ is another pinning with root vectors satisfying (1.19), then there exists $t \in T$ s.t. $X^0_\alpha = int(t)(X_\alpha)$, for all $\alpha \in \Pi$.*

Proof. Let $X^0_\alpha = c_\alpha X_\alpha$, for some $c_\alpha \in \mathbb{C}^\times$. Using condition (1.19),

$$\alpha^\vee = [X^0_\alpha, \pm\overline{X^0_\alpha}] = [c_\alpha X_\alpha, \pm\overline{c_\alpha}\overline{X_\alpha}] = |c_\alpha|^2 [X_\alpha, \pm\overline{X_\alpha}] = |c_\alpha|^2 \alpha^\vee.$$

Therefore $|c_\alpha|^2 = 1$ and $c_\alpha = e^{iy_\alpha}$ for some $y_\alpha \in \mathbb{R}$. The same linear algebra argument from the proof of Proposition 1.6.3 may be used to construct $t \in T$ such that for all $\alpha \in \Pi$, $X^0_\alpha = \text{int}(t)(X_\alpha)$. $\square$

1.6.2 Twisted Indicators

Let M be a real reductive linear group and let σ be an irreducible representation of M on a complex vector space V and let ω be an involution of M. Let σ^ω represent precomposition by ω, so that $\sigma^\omega(m) = \sigma(\omega(m))$, for each $m \in M$. In the terminology of Adams, et al, in [11], Definition 8.11, σ^ω is called *the twist of σ by*

ω. Suppose that $\sigma^{\omega} \cong \sigma^{*}$ and let

$$\phi : (\sigma^{\omega}, V) \xrightarrow{\cong} (\sigma^{*}, V^{*})$$

be an isomorphism. By definition, the transpose of ϕ is the map

$${}^{T}\phi : (\sigma, V) \xrightarrow{\cong} \left((\sigma^{\omega})^{*}, V^{*}\right) = \left((\sigma^{*})^{\omega}, V^{*}\right),$$

where * commutes with the twist of σ by ω. But since $\omega^{2} = 1$, then

$${}^{T}\phi : (\sigma^{\omega}, V) \xrightarrow{\cong} (\sigma^{*}, V^{*}).$$

By Schur's lemma, ${}^{T}\phi = \epsilon(\omega, \sigma)\,\phi$, for $\epsilon(\omega, \sigma) \in \mathbb{C}^{\times}$. Since ${}^{T}\left({}^{T}\phi\right) = \phi$, then $\epsilon(\omega, \sigma) = \pm 1$.

Definition 1.6.5. Let ω be an involution of the real reductive linear group M. Let σ be an irreducible representation of M on a complex vector space V.

(i) If $\sigma^{\omega} \cong \sigma^{*}$, define the ω-twisted ϵ-indicator to be $\epsilon(\omega, \sigma)$, where $\phi : (\sigma^{\omega}, V) \rightarrow (\sigma^{*}, V^{*})$ is an isomorphism and ${}^{T}\phi = \epsilon(\omega, \sigma)\,\phi$.

(ii) Suppose that M is a subgroup of G and $s \in N_{G}(M)$, with $s^{2} \in Z(M)$ and let $\omega = \mathrm{int}(s)$. The usual left action of s on σ, defined by $s\sigma = \sigma \mathrm{int}(s^{-1})$ agrees with the twist of σ by ω:

$$s\sigma = \sigma \mathrm{int}\left(s^{-1}\right) = \sigma \mathrm{int}(s) = \sigma^{\omega}, \qquad \text{since } s^{2} \in Z(M).$$

In this case, we may write $\epsilon(s, \sigma) = \epsilon(\omega, \sigma)$.

Example 1.6.6. Let ω be an involution of a compact group M and let σ be a representation of M. Suppose that $\sigma^{\omega} \cong \sigma^{*}$ so that the twisted indicator $\epsilon(\omega, \sigma)$ is defined. In this example, we emphasize that $\epsilon(\omega,\sigma)$ is *not a well-defined invariant* for the class of involutions that are *inner* to ω. That is, just because ω' is inner to ω, it does *not* mean that $\epsilon(\omega',\sigma)=\epsilon(\omega,\sigma)$.

Consider the standard representation σ of $SU(2)$. The dual of the standard representation is ${}^t x^{-1}$, $x\in SU(2)$. The matrix $B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ gives an isomorphism between the standard representation and it's dual. This is expressed by the following formula:

$$Bx={}^t x^{-1}B.$$

Since $\epsilon(\sigma)B={}^T B=-B$, then the Schur indicator is $\epsilon(\sigma)=-1$. The Schur indicator is also a twisted indicator, only the twist is trivial. So, if $\omega'=1$, then

$$\epsilon(\omega',\sigma)=-1.$$

Now let $\omega=\mathrm{int}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$, so that $\omega(x)=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}x\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$, $x\in SU(2)$. Take the diagonal matrices of $SU(2)$ as the Cartan subgroup, so that ω is just a twist by an element of the Cartan subgroup and ω is inner. We will show that $\epsilon(\omega,\sigma)=1\neq\epsilon(\omega',\sigma)$. There is an isomorphism:

$$\sigma^{\omega}\longrightarrow\sigma\longrightarrow\sigma^{*}$$

and since ω is an involution, then there is a twisted indicator $\epsilon(\omega,\sigma)$. Let A be a matrix that gives the isomorphism $A:\sigma^{\omega}\longrightarrow\sigma^{*}$. Up to scalar in $\mathbb{C}^{\times}$,

$$A=B\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

Here is the check that A intertwines the representations σ^{ω} and σ^{*}:

$$\begin{aligned} A\omega\left(x\right) &= A \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} x \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= Bx \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= {}^{t}x^{-1}B \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= {}^{t}x^{-1}A \end{aligned}$$

Since A is symmetric, the twisted indicator satisifies:

$$\epsilon\left(\omega,\sigma\right) = 1.$$

Let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup $P \subseteq G$. Then M is compact reductive but not necessarily connected. Assume now that G satisfies condition A. In this case, $M = M_0F$, where M_0 is a connected compact group and $F \subseteq Z\left(M\right)$ with $x^2 = 1$, for all $x \in F$. See [13], Chapter VII, Section 5. The highest weight theory applies to M_0 and some important aspects of this theory also apply to M. In particular, if σ is an irreducible representation of M, then σ has a highest weight λ with a 1-dimensional λ-weight space.

Lemma 1.6.7. *Let M be a connected compact Lie group. Let $T \subseteq M$ be a Cartan subgroup and let $w \in W\left(M,T\right)$ such that $w^2 = 1$. There exists a representative $s_w \in N_M\left(T\right)$ of w such that $s_w^2 = \exp\left(\pi i\left(\rho_M^{\vee} - w\rho_M^{\vee}\right)\right)$.*

Proof. For $X \in \mathfrak{m}$ let $\overline{X}$ denote conjugation with respect to the real form that defines M. The theorems and results of [3], Section 5 are for complex groups and show that there exists $s_w \in N_{M(\mathbb{C})}(T(\mathbb{C}))$ such that $s_w^2 = \exp(\pi i(\rho_M^\vee - w\rho_M^\vee))$. By selecting root vectors $X_\alpha \in \mathfrak{g}_\alpha$ appropriately, then the element s_w constructed in [3], Section 5 will actually be in M. Choose a positive set $\Delta_M^+ \subseteq \Delta(\mathfrak{m}, \mathfrak{t})$ and let $\Pi_M \subseteq \Delta_M^+$ be the simple roots. Because M is compact, then we may fix a pinning $\mathscr{P} = \left(T, \Delta_M^+, \{X_\alpha : \alpha \in \Pi_M\}\right)$ with root vectors satisfying (1.19) (i). Using the root vectors in $\mathscr{P}$, let $s_\alpha = \exp\left(\pi/2\left(X_\alpha + \overline{X}_\alpha\right)\right)$ and observe that $s_\alpha \in N_M(T)$. Apply the same construction in [3] Proposition 5.2 using $\{s_\alpha : \alpha \in \Pi_M\}$ and then apply [3], Lemma 5.5 to obtain $s_w \in N_M(T)$ such that $s_w^2 = \exp(\pi i(\rho_M^\vee - w\rho_M^\vee))$. $\square$

Let $y_0 \in N_M(T)$ be a representative of the long element $w_{M,0} \in W(M,T)$ such that $y_0^2 = z_{\rho_M^\vee} \in Z(M)$. (This element exists by Lemma 1.6.7 and [3], Section 5.) Let $w \in W(G,A)$ such that $w^2 = 1$. By [14] Theorem 14.48, there exists a representative $s \in N_K(A)$ of w such that $\operatorname{int}(s^{-1})$ preserves the set $\Delta^+(\mathfrak{m}, \mathfrak{t})$. This implies that $\operatorname{int}(s^{-1})$ must also preserve the set of simple roots in $\Delta^+(\mathfrak{m}, \mathfrak{t})$, but it may not fix y_0. In general, $y_0 s y_0^{-1} s^{-1} = t$, for some $t \in T$. Define $[y_0, s] = y_0 s y_0^{-1} s^{-1}$, which is typical notation for the commutator (or its inverse).

Theorem 1.6.8. *Let G be real reductive linear group that satisfies condition A and let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup. Let σ be an irreducible representation of M on the complex vector space V_σ. Let $s \in N_K(A)$ such that $s\sigma \cong \sigma^*$, s preserves Δ_M^+, and $s^2 \in Z(M)$. Let λ be the highest weight of σ and let y_0 be a representative of the long element of $W(M,T)$*

such that $y_0^2 = z_{\rho_M^\vee}$. *The twisted* ϵ*-indicator satisfies*

$$\epsilon(s,\sigma) = \Lambda([y_0,s])\,\chi_\sigma\left(z_{\rho_M^\vee}\right),$$

where $\Lambda(t) = e^{\lambda(\log(t))}$, *for* $t \in T$.

Proof. The method of proof is to use the highest weight theory to calculate the twisted ϵ-indicator. Let λ be the highest weight of σ and let v_λ generate the highest weight space. Let $w_{M,0}$ be the long element of $W(M,T)$ and let $y_0 \in N_M(T)$ be a representative of $w_{M,0}$ such that $y_0^2 = z_{\rho_M^\vee}$. Here is a table to keep track of the highest weights of the various representations involved:

representation	highest weight
σ	λ
$s\sigma$	$s\lambda$
σ^*	$-w_{M,0}\lambda$
$s\sigma^*$	$-w_{M,0}s\lambda$

Note that $-sw_{M,0}\lambda = -w_{M,0}s\lambda$, since sy_0 and y_0s differ only by an element of the Cartan subgroup $T \subseteq M$. Let $\phi : (s\sigma, V_\sigma) \to (\sigma^*, V_\sigma^*)$ be an isomorphism. Define a bilinear form on V_σ by

$$(u,v) = \langle \phi u, v\rangle, \tag{1.20}$$

where the pairing on the right hand side is the natural pairing $V_\sigma^* \times V_\sigma \to \mathbb{C}$. (As a reminder, the natural pairing is M-invariant:

$$\langle \sigma^*(m)f, \sigma(m)v\rangle = \langle f, v\rangle, \text{ for } f \in V_\sigma^*,\ v \in V_\sigma.$$

If $f_{-\lambda} \in V_\sigma^*$ generates the $-\lambda$ weight space in (σ^*, V_σ^*) and if v_λ generates the λ-weight space in (σ, V_σ), then $\langle f_{-\lambda}, v_\lambda \rangle \neq 0$.)

Since $s^2 \in Z(M)$, the map ϕ is also an isomorphism:

$$\phi : (\sigma, V_\sigma) \to (s\sigma^*, V_\sigma^*) .$$

Since s preserves the set of positive roots, the highest weights of the two isomorphic representations must match, so

$$-\lambda = s w_{M,0} \lambda.$$

Now, consider again $\phi : (s\sigma, V_\sigma) \to (\sigma^*, V_\sigma^*)$. For $H \in \mathfrak{t}$,

$$\begin{aligned} s\sigma(H)\,\sigma(y_0)\,v_\lambda &= \sigma\left(s^{-1} H s y_0\right) v_\lambda \\ &= \sigma(y_0)\,\sigma\left(y_0^{-1} s^{-1} H s y_0\right) v_\lambda \\ &= s w_{M,0} \lambda(H)\,\sigma(y_0)\,v_\lambda \\ &= -\lambda(H)\,\sigma(y_0)\,v_\lambda \end{aligned}$$

so $\sigma(y_0)\,v_\lambda \in V_\sigma$ has $-\lambda$ weight under the action of $s\sigma$. The isomorphism ϕ preserves the extremal weights, so define $f_{-\lambda} = \phi(\sigma(y_0)\,v_\lambda) \in V_\sigma^*$, which is a nonzero vector of weight $-\lambda$. Therefore

$$\begin{aligned} (\sigma(y_0)\,v_\lambda, v_\lambda) &= \langle \phi(\sigma(y_0)\,v_\lambda), v_\lambda \rangle \\ &= \langle f_{-\lambda}, v_\lambda \rangle \\ &\neq 0. \end{aligned}$$

The bilinear form (1.20) satisfies a "twisted" M-invariance property:

$$(w\sigma(m)\,u, \sigma(m)\,v) = (u, v), \text{ for } m \in M,\ u, v \in V_\sigma. \tag{1.21}$$

Therefore,

$$\begin{aligned}
(\sigma(y_0)v_\lambda, v_\lambda) &= \left(s\sigma\left(sy_0s^{-1}\right)v_\lambda, v_\lambda\right) \\
&= \left(v_\lambda, \sigma\left(sy_0^{-1}s^{-1}\right)v_\lambda\right), \qquad \text{(by “twisted” } M\text{-invariance)} \\
&= \left(v_\lambda, \sigma\left(y_0^{-1}\right)\sigma\left(y_0sy_0^{-1}s^{-1}\right)v_\lambda\right) \\
&= \Lambda([y_0, s])\left(v_\lambda, \sigma\left(y_0^{-1}\right)v_\lambda\right) \\
&= \Lambda([y_0, s])\left(v_\lambda, \sigma\left(y_0^3\right)v_\lambda\right), \quad \left(\text{since } y_0^4 = z_{\rho_M^\vee}^2 = 1\right) \\
&= \Lambda([y_0, s])\,\chi_\sigma\left(z_{\rho_M^\vee}\right)(v_\lambda, \sigma(y_0)v_\lambda),
\end{aligned}$$

since $y_0^2 = z_{\rho^\vee} \in Z(M)$.

The last item to check is that the formula really is independent of the choice of representative y_0 of $w_{M,0}$ such that $y_0^2 = z_{\rho_M^\vee}$. Let y_0' be another representative of $w_{M,0}$ such that $(y_0')^2 = z_{\rho_M^\vee}$. Then $y_0' = hy_0$ for some $h \in T$. Now,

$$z_{\rho_M^\vee} = (y_0')^2 = hy_0hy_0 = hw_{M,0}(h)\,y_0^2 = hw_{M,0}(h)\,z_{\rho_M^\vee},$$

so

$$w_{M,0}(h) = h^{-1}. \tag{1.22}$$

Therefore

$$\begin{aligned}
\Lambda\left(\left[y_0', s\right]\right) &= \Lambda\left(\left[hy_0, s\right]\right) \\
&= \Lambda\left(hy_0 s\left(y_0^{-1}h^{-1}\right)s^{-1}\right) \\
&= \Lambda\left(h\left(y_0 s y_0^{-1}s^{-1}\right)\left(sh^{-1}s^{-1}\right)\right) \\
&= \Lambda\left(h[y_0, s]\left(sh^{-1}s^{-1}\right)\right) \\
&= \Lambda\left(h\right)s\Lambda\left(h^{-1}\right)\Lambda\left(\left[y_0, s\right]\right) \\
&= \Lambda\left(h\right)w_{M,0}\Lambda\left(h\right)\Lambda\left(\left[y_0, s\right]\right) && \text{using } -s\lambda = w_{M,0}\lambda \\
&= \Lambda\left(\left[y_0, s\right]\right) && \text{using (1.22).}
\end{aligned}$$

□

The factor $\Lambda\left(\left[y_0, s\right]\right)$ really is a sign factor. To simplify notation, let $t = [y_0, s]$. Note that $y_0\left(t\right) = t^{-1}$ and $s\left(t\right) = t^{-1}$. Therefore $sy_0\left(t\right) = t$ and since

$$\Lambda\left(t\right)^{-1} = sy_0\Lambda\left(t\right) = \Lambda\left(sy_0\left(t\right)\right) = \Lambda\left(t\right)$$

then $\Lambda\left(t\right)^2 = 1$.

Example 1.6.9. In the setting of Example 1.6.6, we use the formula in Theorem 1.6.8 to calculate the twisted indicator. Recall σ is the standard representation of $SU\left(2\right)$. Let $\omega_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\omega = \mathrm{int}\left(\omega_1\right)$. Take the diagonal matrices of $SU\left(2\right)$ as the Cartan subgroup. Let $y_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which is a representative of the long element of the Weyl group such that $y_0^2 = z_{\rho^\vee} = -I$. The highest weight for the standard representation σ is $\Lambda \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} = e^{ix}$, and since

$$[\omega_1, y_0] = \omega_1 y_0 \omega_1^{-1} y_0^{-1} = \omega_1^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$\Lambda\left([\omega_1, y_0]\right) = \Lambda \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1.$$

Using the twisted indicator formula from Theorem 1.6.8,

$$\begin{aligned} \epsilon\left(\omega, \sigma\right) &= \Lambda\left([\omega_1, y_0]\right) \chi_\sigma\left(z_{\rho^\vee}\right) \\ &= \Lambda\left([\omega_1, y_0]\right) \epsilon\left(\sigma\right) \\ &= (-1)(-1) \\ &= 1. \end{aligned}$$

This agrees with the calculation in Example 1.6.6.

Let $w \in W(G, A)$ such that $w^2 = 1$. [14], Thereom 14.48 guarantees that there exists a representative $s \in N_K(A)$ of w such that s preserves $\Delta^+(\mathfrak{m}, \mathfrak{t})$ and therefore $s^2 \in T$. In fact you can modify s to obtain a representative s_1 of w such that ${s_1}^2 \in Z(M)$. This is the content of Proposition 1.6.10 (and it's proof) below. As a corollary, we show that certain choices of representatives s_1 of w result in a simpler formula for the s_1-twisted indicator.

Proposition 1.6.10. *Let G be a real reductive linear group that satisfies condition A and let $P = MAN$ be the Langlands decomposition of a minimal parabolic subgroup* *of G. Let $w \in W(G,A)$ such that $w^2 = 1$ be given. Fix a Cartan subgroup $T \subseteq M$, a set of positive roots $\Delta_M^+ = \Delta^+(\mathfrak{m}, \mathfrak{t})$ and corresponding simple roots $\Pi_M \subseteq \Delta_M^+$, and let $\mathscr{P} = \left\{T, \Delta_M^+, \{X_\alpha : \alpha \in \Pi_M\}\right\}$ be a pinning with root vectors satisfying (1.19) (i). There exists a representative $s \in N_K(A)$ of w such that $int(s^{-1})$ is $\mathscr{P}$-distinguished. As a consequence $s^2 \in Z(M)$.*

Proof. Let $T \subseteq M$ be a Cartan subgroup, which is compact because M is compact, and let $\Delta_M^+ = \Delta^+(\mathfrak{m}, \mathfrak{t})$ be a positive system and let $\Pi_M \subseteq \Delta_M^+$ be the simple roots. Let $w \in W(G,A)$ such that $w^2 = 1$. By [14] Theorem 14.48, there exists a representative $s \in N_K(A)$ of w such that $\operatorname{int}(s)$ preserves Δ_M^+. Since $T \subseteq M$ is compact, let $\mathscr{P} = \left\{T, \Delta_M^+, \{X_\alpha : \alpha \in \Pi_M\}\right\}$ be a pinning with root vectors that satisfy (1.19) (i). By Proposition 1.6.3, there exists $t \in T$ such that

$$\operatorname{int}(s) = \operatorname{int}(t)\ \tau$$

where τ is $\mathscr{P}$-distinguished. Define

$$s_1 = t^{-1}s.$$

Then s_1 is a representative of w and since

$$\operatorname{int}(s_1) = \operatorname{int}\left(t^{-1}s\right) = \tau$$

then $\operatorname{int}(s_1)$ is $\mathscr{P}$-distinguished. Now,

$$\tau^2 = \operatorname{int}\left(s_1^2\right)$$

and since $s_1^2 \in T$, then τ^2 is a distinguished inner automorphism of M. By Proposition 1.6.2, $s_1^2 \in Z_M(M_0) = Z(M)$. $\square$

In the setting of Proposition 1.6.10, let $\mathscr{P} = \left(T, \Delta_M^+, \{X_\alpha : \alpha \in \Pi_M\}\right)$ and let s be a $\mathscr{P}$-distinguished representative of $w \in W(G, A)$. Let y_0 be the representative of the long element $w_{M,0} \in N_M(T)/T$ constructed according to [3], Lemma 5.3, using $\mathscr{P}$. Then $y_0^2 = z_{\rho_M^\vee}$. Furthermore, since $\mathrm{int}(s)$ is $\mathscr{P}$-distinguished, then also by [3], Lemma 5.3, $sy_0s^{-1} = y_0$. Therefore,

$$y_0^2 = z_{\rho_M^\vee}, \text{ and } [y_0, s] = 1. \tag{1.23}$$

This proves Corollary 1.6.11 below.

Corollary 1.6.11. *Suppose we are in the context of Theorem 1.6.8. Let $\mathscr{P}$ be a pinning of M with root vectors that satisfy (1.19).*

(a) If the representative s is chosen such that $\mathrm{int}(s)$ is $\mathscr{P}$-distinguished (which is possible by Proposition (1.6.10)) then

$$\epsilon(s, \sigma) = \chi_\sigma\left(z_{\rho_M^\vee}\right).$$

(b) If σ is self-dual and $\mathrm{int}(s)$ is chosen to be $\mathscr{P}$-distinguished, then

$$\epsilon(s, \sigma) = \epsilon(\sigma).$$

Proposition 1.6.12. *Let M be a connected compact group. Fix a pinning $\mathscr{P}$ and let ω be a distinguished involution of M. Let σ be an irreducible self-dual representation of M and assume $\sigma^\omega \cong \sigma$. Then σ extends to a self-dual representation of ${}^{\omega,z}M$ if and only if $\chi_\sigma(z) = 1$.*

Proof. Consider the extended group ${}^{\omega,z}M = \langle M, \omega_1\rangle$, subject to the relations $\omega = \mathrm{int}(\omega_1)$ and $\omega_1^2 = z \in Z(M)$. Let $\phi : \sigma^\omega \to \sigma$ and $B : \sigma \to \sigma^*$ be isomorphisms.

Scale ϕ such that $\phi^2 = 1$. Since σ is self-dual, then ${}^T B = \epsilon(\sigma) B$. Since $\sigma^\omega \cong \sigma$, σ extends to a representation $\widetilde{\sigma}$ of ${}^{\omega,z}M$ in exactly two ways, the choice of which does not matter. Without loss of generality, fix a square root of $\chi_\sigma(z)$ and let

$$\widetilde{\sigma}(\omega_1) = \sqrt{\chi_\sigma(z)}\phi.$$

Now

$$\begin{aligned} B\phi &= \epsilon(\sigma)\ {}^T(B\phi), && \text{(Corollary 1.6.11 (iii))} \\ &= \epsilon(\sigma)\ {}^T\phi\ {}^T B \\ &= \epsilon(\sigma)^2\ {}^T\phi B, && \text{(since } \sigma \text{ is self-dual)} \\ &= {}^T\phi B. \end{aligned} \tag{1.24}$$

The extended representation $\widetilde{\sigma}$ is self-dual if and only if

$$B\widetilde{\sigma}(\omega_1) = B\widetilde{\sigma}(\omega_1)^* \quad \Leftrightarrow \quad {}^T\widetilde{\sigma}(\omega_1)\, B\widetilde{\sigma}(\omega_1) = B. \tag{1.25}$$

Using Equation (1.24) and the fact that $\phi^2 = 1$, the self-duality condition on the right side of (1.25) implies

$$B = {}^T\widetilde{\sigma}(\omega_1)\, B\widetilde{\sigma}(\omega_1) = \chi_\sigma(z)\, {}^T\phi B\phi = \chi_\sigma(z)\, B.$$

Therefore the extended representation $\widetilde{\sigma}$ is self-dual if and only if $\chi_\sigma(z) = 1$. $\square$

Let M be a connected compact group and let ω be an involution of M. Let σ be an irreducible representation of M on the (finite dimensional) complex vector space V. Suppose $\sigma^\omega \cong \overline{\sigma}$ and let

$$\phi : (\sigma^\omega, V) \xrightarrow{\cong} (\overline{\sigma}, V)$$

be an isomorphism. By definition,

$$\overline{\phi} : (\overline{\sigma^{\omega}}, V) = ((\overline{\sigma})^{\omega}, V^{*}) \overset{\cong}{\to} (\sigma, V).$$

Since $\omega^2 = 1$, then

$$\overline{\phi} : (\overline{\sigma}, V) \overset{\cong}{\to} (\sigma^{\omega}, V).$$

By Schur's lemma, $\overline{\phi} = \delta(\omega, \sigma)\, \phi^{-1}$, for $\delta(\omega, \sigma) \in \mathbb{C}^{\times}$. Since

$$\delta(\omega, \sigma) = \overline{\phi}\ \phi = \phi\ \overline{\phi} = \overline{\overline{\phi}\ \phi} = \overline{\delta(\omega, \sigma)},$$

then $\delta(\omega, \sigma) \in \mathbb{R}^{\times}$. Scaling ϕ by any multiple $z \in \mathbb{C}^{\times}$ has the effect of scaling $\delta(\omega, \sigma)$ by $|z|^2 > 0$. After scaling ϕ appropriately, $\delta(\omega, \sigma) = \pm 1$.

Definition 1.6.13. Let ω be an involution of M and let σ be an irreducible representation of M on a complex vector space V.

(i) If $\sigma^{\omega} \cong \overline{\sigma}$, define the *$\omega$-twisted δ-indicator* to be $\delta(\omega, \sigma)$, where $\phi : (\sigma^{\omega}, V) \to (\overline{\sigma}, V)$ is an appropriately scaled isomorphism that satisfies $\overline{\phi} = \delta(\omega, \sigma)\, \phi^{-1}$ and $\delta(\omega, \sigma) = \pm 1$.

(ii) Suppose that M is a subgroup of G, there exists $s \in N_G(M)$, with $s^2 \in Z(M)$ and let $\omega = \mathrm{int}(s)$. The usual left action of s on σ, defined by $s\sigma = \sigma\ \mathrm{int}(s^{-1})$ agrees with the twist of σ by ω:

$$s\sigma = \sigma\ \mathrm{int}\left(s^{-1}\right) = \sigma\ \mathrm{int}(s) = \sigma^{\omega}, \qquad \text{since } s^2 \in Z(M).$$

In this case, we may write $\delta(s, \sigma) = \delta(\omega, \sigma)$.

Theorem 1.6.14. *Let M be a real reductive linear group and let ω be an involution of M. Let σ be an irreducible Hermitian representation of M on a complex vector space. Then*

(i) $\sigma^\omega \cong \overline{\sigma} \Leftrightarrow \sigma^\omega \cong \sigma^*$.

(ii) If σ is unitary and $\sigma^\omega \cong \overline{\sigma}$, then

$$\delta(\omega, \sigma) = \epsilon(\omega, \sigma).$$

Proof. The method used in this proof was motivated by the proof of Lemma 3.35 in [10].

Let $\phi : (\sigma^\omega, V) \to (\overline{\sigma}, V)$ be an isomorphism. Let ψ be the real form on V that defines $\overline{\sigma}$: $\overline{\sigma} = \psi\sigma\psi$. Since σ is Hermitian, let H be an M-invariant Hermitian form on V. Define an M-invariant bilinear form B on V via

$$B(u, v) = H(u, \phi\psi v).$$

This form satisfies the "twisted" M-invariance property, namely,

$$\begin{aligned} B(\sigma^\omega(m)u, \sigma(m)v) &= H(\sigma^\omega(m)u, \phi\psi\sigma(m)v) \\ &= H\left(u, \sigma^\omega\left(m^{-1}\right)\phi\psi\sigma(m)v\right) \\ &= H\left(u, \phi\overline{\sigma}\left(m^{-1}\right)\psi\sigma(m)v\right) \\ &= H\left(u, \phi\psi\sigma\left(m^{-1}\right)\sigma(m)v\right) \\ &= H(u, \phi\psi v) \\ &= B(u, v). \end{aligned}$$

This means that $(\sigma^{\omega}, V) \to (\sigma^{*}, V^{*}) : u \mapsto B(u, \cdot)$ is an isomorphism. Therefore

$$B(u,v) = \epsilon(\omega,\sigma) B(v,u).$$

Since

$$\begin{aligned} B(v, \phi\psi u) &= H\left(v, (\phi\psi)^{2} u\right) \\ &= H\left(v, \left(\phi\overline{\phi}\right) u\right) \\ &= \delta(\omega,\sigma) H(u,v), \end{aligned}$$

then

$$H(\phi\psi u, \phi\psi v) = B(\phi\psi u, v) = \epsilon(\omega,\sigma) B(v, \phi\psi u) = \epsilon(\omega,\sigma)\delta(\omega,\sigma) H(u,v).$$

If H is positive definite, then

$$\delta(\omega,\sigma) = \epsilon(\omega,\sigma).$$

□

1.6.3 Indicator Formulas for Langlands Quotients of Principal Series

Proposition 1.6.15. *Let $P = MAN$ be a minimal parabolic subgroup of the real reductive linear group G and let (σ, V_{σ}) and (τ, V_{τ}) be isomorphic irreducible representations of M. Let $\phi : (\sigma, V_{\sigma}) \to (\tau, V_{\tau})$ be an isomorphism. Define $\phi(f)(x) = \phi(f(x))$, for $x \in G$. For $s \in N_K(A)$*

$$A_P(s,\sigma,\nu) = \phi^{-1} A_P(s,\tau,\nu)\phi.$$

This is essentially [14], Proposition 14.21, (b). This statement is true in the more general case when P is standard cuspidal and σ is in the discrete series, but this is not needed for this paper. In the case that P is minimal, M is compact, so V_σ and V_τ are finite dimensional; every linear operator is bounded and the integrals are linear on elements of V_σ and V_τ.

Proof. This statement is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccc} I_P(\sigma,\nu) & \xrightarrow{A_P(s,\sigma,\nu)} & I_P(s\sigma,s\nu) \\ \phi\downarrow & & \downarrow\phi \\ I_P(\tau,\nu) & \xrightarrow{A_P(s,\tau,\nu)} & I_P(s\tau,s\nu) \end{array}$$

Let $n = \dim V_\sigma$ and let $f \in I_P(\sigma,\nu)$. Fix a basis $\beta_1,\ldots,\beta_n$ for V_σ and let $f(x) = \sum_{i=1}^n f_i(x)\beta_i$, where $f_i(x)$ are complex-valued functions such that $f_i(x)\beta_i \in I_P(\sigma,\nu)$, for $i = 1,\ldots,n$. Here is the calculation moving around the diagram in one way:

$$\begin{aligned} \phi(A_P(s,\sigma,\nu)(f))(x) &= \phi(A_P(s,\sigma,\nu)(f)(x)) \\ &= \phi\left(\int_{\overline{N}\cap s^{-1}Ns}\left(\sum_{i=1}^n f_i(xs\overline{n})\beta_i\right)d\overline{n}\right) \\ &= \phi\left(\sum_{i=1}^n\left(\int_{\overline{N}\cap s^{-1}Ns} f_i(xs\overline{n})\,d\overline{n}\right)\beta_i\right) \\ &= \sum_{i=1}^n\left(\int_{\overline{N}\cap s^{-1}Ns} f_i(xs\overline{n})\,d\overline{n}\right)\phi\beta_i. \end{aligned}$$

Moving around the diagram in the other way produces the same result:

$$\begin{aligned} A_P(s,\tau,\nu)(\phi(f))(x) &= \int_{\overline{N}\cap s^{-1}Ns} \phi(f)(xs\overline{n})\, d\overline{n} \\ &= \int_{\overline{N}\cap s^{-1}Ns} \phi(f(xs\overline{n}))\, d\overline{n} \\ &= \int_{\overline{N}\cap s^{-1}Ns} \left(\sum_{i=1}^{n} f_i(xs\overline{n})\, \phi\beta_i \right) d\overline{n} \\ &= \sum_{i=1}^{n} \left(\int_{\overline{N}\cap s^{-1}Ns} f_i(xs\overline{n})\, d\overline{n} \right) \phi\beta_i. \end{aligned}$$

□

Here is an outline of the method used to calculate the Schur indicator for self-dual principal series representations of G, when G is not quasisplit. Let $\pi = J(\sigma,\nu)$ be irreducible and suppose there exists $w \in W(G,A)$ such that $w\sigma \cong \sigma^*$, $w\nu = -\nu$, and $w^2 = 1$. Let $s \in N_K(A)$ be a representative of w and let $\phi : I(s\sigma,-\nu) \to I(\sigma^*,-\nu)$ be an isomorphism. The composition $B = \phi\, A(w,\sigma,\nu)$

$$B : I(\sigma,\nu) \xrightarrow{A(s,\sigma,\nu)} I(s\sigma,-\nu) \xrightarrow{\phi} I(\sigma^*,-\nu) \tag{1.26}$$

factors through to an isomorphism between π and its dual and leads to a G-invariant bilinear form. The Schur indicator satisfies

$$^{T}B = \epsilon(\pi)\, B.$$

Assuming we can choose s such that $s^2 \in Z(M)$, then the indicator $\epsilon(\pi)$ factors into two signs, one coming from ϕ and the other coming from $A(s,\sigma,\nu)$. The sign coming from ϕ is a twisted indicator and can be calculated using Theorem 1.6.8. The sign coming from $A(s,\sigma,\nu)$ can be calculated using the formula for the transpose and then applying a change of variable, just as in the proof of Theorem 1.4.4. The details are in the proof of Lemma 1.6.16 below.

Lemma 1.6.16. *Let G be a real reductive linear group that satisfies condition A. Let $J(\sigma, \nu)$ be the Langlands quotient of the principal series $Ind_{MAN}^{G}(\sigma \otimes \nu \otimes 1)$ and suppose that $J(\sigma, \nu)$ is irreducible. Let $w \in W(G, A)$ such that $w\sigma \cong \sigma^*$, $w\nu = -\nu$, and $w^2 = 1$. Suppose there exists $s \in N_K(A)$ such that s is a representative of w, s preserves a set of positive roots of M, and $s^2 \in Z(M)$. Then*

$$\epsilon(J(\sigma, \nu)) = \epsilon(s, \sigma)\, \chi_\sigma\left(s^2\right). \tag{1.27}$$

The right hand side of the formula above is independent of the representative s as long as s satisfies the hypotheses of the lemma.

Proof. Fix a Cartan $T \subseteq M$ and a set of positive roots Δ_M^+. Let s be a representative of w such that s preserves Δ_M^+ and $s^2 \in Z(M)$. Let $\phi : s\sigma \to \sigma^*$ be an isomorphism.

$$\begin{aligned}
{}^T[\phi A(s, \sigma, \nu)] &= {}^T A(s, \sigma, \nu)\ {}^T\phi \\
&= A\left(s^{-1}, s\sigma^*, \nu\right)\ {}^T\phi \qquad \text{(using Proposition 1.4.3 (ii) and } -s\nu = \nu) \\
&= {}^T\phi\ A\left(s^{-1}, \sigma, \nu\right) {}^T\phi^{-1}\ {}^T\phi \qquad \text{(by Proposition 1.6.15)} \\
&= {}^T\phi\ A\left(s^{-1}, \sigma, \nu\right) \\
&= \epsilon(s, \sigma)\, \phi\ A\left(s^{-1}, \sigma, \nu\right) \qquad \text{(the twisted indicator)} \\
&= \epsilon(s, \sigma)\, \chi_\sigma\left(s^2\right) \phi\ A(s, \sigma, \nu) \qquad \text{(by change of variable.)}
\end{aligned}$$

Let $s_0 = ts \in N_K(A)$ be another representative of w that preserves Δ_M^+ such that $s_0^2 \in Z(M)$. Then $t \in T$ and

$$s_0^2 = t\ \mathrm{int}(s)(t)\, s^2 \Rightarrow t\ \mathrm{int}(s)(t) \in \mathbb{Z}(M).$$

Now recall the formula for the twisted indicator from Theorem 1.6.8 and recall that y_0 is a representative of the long element $w_{M,0}$ of $N_M(T)/T$ such that $y_0^2 = z_{\rho_M^\vee}$. We have

$$\begin{aligned}
\Lambda([s_0, y_0]) &= \Lambda\left(t[s, y_0]\ \mathrm{int}(w_{M,0})\left(t^{-1}\right)\right) \\
&= \Lambda\left(t\ \mathrm{int}(w_{M,0})\left(t^{-1}\right)\right)\Lambda([s, y_0]) \\
&= \Lambda(t\ \mathrm{int}(s)(t))\Lambda([s, y_0]) \qquad\qquad (\text{using } -w_{M,0}\lambda = s\lambda) \\
&= \chi_\sigma(t\ \mathrm{int}(s)(t))\Lambda([s, y_0]).
\end{aligned}$$

So

$$\begin{aligned}
\epsilon(s_0, \sigma)\chi_\sigma\left(s_0^2\right) &= \Lambda([s_0, y_0])\chi_\sigma\left(z_{\rho_M^\vee}\right)\chi_\sigma\left(s_0^2\right) \\
&= \chi_\sigma(t\ \mathrm{int}(s)(t))\Lambda([s, y_0])\chi_\sigma\left(z_{\rho_M^\vee}\right)\chi_\sigma\left(t\ \mathrm{int}(s)(t)\, s^2\right) \\
&= \chi_\sigma(t\ \mathrm{int}(s)(t))^2\Lambda([s, y_0])\chi_\sigma\left(z_{\rho_M^\vee}\right)\chi_\sigma\left(s^2\right) \\
&= \Lambda([s, y_0])\chi_\sigma\left(z_{\rho_M^\vee}\right)\chi_\sigma\left(s^2\right) \qquad\qquad (1.28) \\
&= \epsilon(s, \sigma)\chi_\sigma\left(s^2\right)
\end{aligned}$$

This completes the demonstration that the choice of representative of w does not matter, as long as the hypotheses of the thoerem are satsified.

We take a moment to explain how we arrived at the equality in (1.28) above, in case it is not clear. Define

$$z = t\ \mathrm{int}(s)(t) \in Z(M).$$

The third equality uses the fact that $\chi_\sigma(z)^2 = 1$. To see this, recall the bilinear form (1.20) which satisfies the "twisted" M-invariance condition (1.21). Let V_σ be the representation space of σ. For any $v, w \in V_\sigma$

$$\begin{aligned}(v,w) &= (s\sigma(z)v, \sigma(z)w)\\ &= (\sigma(z)v, \sigma(z)w) \qquad\qquad \text{(since by definition } \operatorname{int}(s)(z) = z\text{)}\\ &= (\chi_\sigma(z)v, \chi_\sigma(z)w)\\ &= \chi_\sigma(z)^2(v,w).\end{aligned}$$

We conclude that $\chi_\sigma(z)^2 = 1$. □

Theorem 1.6.17. *(Theorem 1.1.6) Let G be a real reductive linear group that satisfies condition A. Let $J(\sigma,\nu)$ be the Langlands quotient of the principal series representation $Ind_{MAN}^G(\sigma \otimes \nu \otimes 1)$ and suppose that $J(\sigma,\nu)$ is irreducible. Let $w \in W(G,A)$ such that $w\sigma \cong \sigma^*$, $w\nu = -\nu$, and $w^2 = 1$. Let $\mathscr{P}$ be a pinning of M with root vectors satisfying (1.19). Let $s \in N_K(A)$ be a representative of w such that $int(s)$ is $\mathscr{P}$-distinguished. (Such a representative exists by Proposition 1.6.10.) Then*

$$\epsilon(J(\sigma,\nu)) = \chi_\sigma(s^2)\,\chi_\sigma\left(z_{\rho_M^\vee}\right), \tag{1.29}$$

where $\rho_M^\vee$ is $1/2$ the sum of the positive coroots of M.

The right hand side of (1.29) is independent of the choice of pinning $\mathscr{P}$ with root vectors satisfying (1.19) and independent of the choice of s such that $\operatorname{int}(s)$ is $\mathscr{P}$-distinguished.

Proof. By [14] Theorem 14.48 and Proposition 1.6.10 there exists a representative $s \in N_K(A)$ of w such that $\operatorname{int}(s)$ is $\mathscr{P}$-distinguished. Because we also assume that $w^2 = 1$, then $s^2 \in Z(M)$. Let $\phi : s\sigma \to \sigma^*$ be an isomorphism. The formula for the indicator follows from Lemma 1.6.16 and Corollary 1.6.11(i). We will now check that the right hand side of (1.29) is independent of the choice of pinning and the representative s.

Let $\mathscr{P} = \left(T, \Delta_M^+, \{X_\alpha : \alpha \in \Pi_M\}\right)$ be a pinning satisfying (1.19) and let s be a representative of w such that $\operatorname{int}(s)$ is $\mathscr{P}$-distinguished and $s^2 \in Z(M)$. Let $\mathscr{P}_0$ be another pinning of M satisfying (1.19) and let s_0 be a representative of w such that $\operatorname{int}(s_0)$ is $\mathscr{P}_0$-distinguished and $s_0^2 \in Z(M)$. After conjugating by M, we may assume that $\mathscr{P}_0 = \left(T, \Delta_M^+, \{X_\alpha^0 : \alpha \in \Pi_M\}\right)$. By Lemma 1.6.4, there exists $t \in T$ such that $X_\alpha^0 = tX_\alpha t^{-1}$, for all $\alpha \in \Pi$. Let $s_0 = us$ for some $u \in M$. Since s and s_0 both preserve Δ_M^+, then $u \in T$. Define $z = t^{-1}usts^{-1} = ut^{-1}\operatorname{int}(s)(t) \in T$. We will show that $z \in Z(M)$. For each $\alpha \in \Pi$,

$$\begin{aligned}
zX_\alpha z^{-1} &= t^{-1}ust\left(s^{-1}X_\alpha s\right)t^{-1}s^{-1}u^{-1}t \\
&= t^{-1}(us)\left(tX_{s^{-1}\alpha}t^{-1}\right)\left(s^{-1}u^{-1}\right)t && \text{(using int}\left(s^{-1}\right)\text{ is }\mathscr{P}\text{-distinguished)} \\
&= t^{-1}\left(s_0\left(tX_{s^{-1}\alpha}t^{-1}\right)s_0\right)t && \text{(using } s_0 = us\text{)} \\
&= t^{-1}\left(tX_\alpha t^{-1}\right)t && \text{(using } s_0 \text{ is } \mathscr{P}_0\text{-distinguished and } s_0 = us\text{)} \\
&= X_\alpha.
\end{aligned}$$

Consequently, $\operatorname{int}(z)$ is an inner involution that is distinguished. By Lemma 1.6.2,

$z \in Z(M)$. Now,

$$\begin{aligned}
s_0^2 &= usus \\
&= u \text{ int}(s)(u)\, s^2 \\
&= zt \text{ int}(s)\left(t^{-1}\right) \text{int}(s)\left(zt \text{ int}(s)\left(t^{-1}\right)\right) s^2 \qquad \left(\text{using } u = zt \text{ int}(s)\left(t^{-1}\right)\right) \\
&= zt \text{ int}(s)\left(t^{-1}\right) \text{int}(s)(zt)\, t^{-1} s^2 \\
&= z \text{ int}(s)(z)\, s^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\chi_\sigma\left(s_0^2\right) &= \chi_\sigma(z \text{ int}(s)(z))\, \chi_\sigma\left(s^2\right) \\
&= \chi_\sigma(z)\, \chi_{s\sigma}(z)\, \chi_\sigma\left(s^2\right) \\
&= \chi_\sigma(z)\, \chi_{w_{M,0}\sigma}\left(z^{-1}\right) \chi_\sigma\left(s^2\right) \qquad (\text{using } s\sigma \cong \sigma^*) \\
&= \chi_\sigma\left(s^2\right) \qquad (\text{using } z \in Z(M).)
\end{aligned}$$

□

Corollary 1.6.18. *Suppose the hypotheses of Theorem 1.1.6 hold. If in addition σ is self-dual then*

$$\epsilon\left(J_P(\sigma, \nu)\right) = \chi_\sigma\left(s^2\right) \epsilon(\sigma).$$

Suppose the hypotheses of Theorem 1.1.6 hold and that σ is self-dual. Let $\omega = \text{int}(s)$ be a distinguished involution, so $z = s^2 \in Z(M)$. If $\chi_\sigma\left(s^2\right) = 1$ then by Proposition 1.6.12 the representation σ of M on V_σ extends to a self-dual representation $\widetilde{\sigma}$ of ${}^{\omega,z}M$. But since $s\nu = -\nu$, the representation $\sigma \otimes e^\nu \otimes 1$ of P on V_σ (which is not self dual if $\nu \neq 0$) extends to the self-dual representation $\widetilde{\sigma} \otimes e^\nu \otimes 1$

of the extended group ${}^{\omega,z}P$. As a consequence,

$$\epsilon\left(J_P\left(\sigma,\nu\right)\right)=\epsilon\left(\widetilde{\sigma}\otimes e^{\nu}\otimes 1\right)=\epsilon\left(\widetilde{\sigma}\right)=\epsilon\left(\sigma\right). \tag{1.30}$$

So, what does happen is exactly what should happen, heuristically speaking. The representation $J_P\left(\sigma,\nu\right)$ is the irreducible quotient of the representation induced from ${}^{\omega,z}P$ and so one should expect the indicators to match. The point of the second and third equalities in Equation (1.30) is that the indicator of $\widetilde{\sigma}\otimes e^{\nu}\otimes 1$ is easy to calculate.

Theorem 1.6.19. *(Theorem 1.1.10) Let G be a real reductive linear group that satisfies condition A. Let $J\left(\sigma,\nu\right)$ be the Langlands quotient of the principal series representation $Ind_P^G\left(\sigma\otimes\nu\otimes 1\right)$ and suppose that $J\left(\sigma,\nu\right)$ is irreducible. Let $w\in W\left(G,A\right)$ such that $w\sigma\cong\overline{\sigma}$, $w\nu=\overline{\nu}$, and $w^2=1$. Let $\mathscr{P}$ be a pinning of M with root vectors satisfying (1.19). Let $s\in N_K\left(A\right)$ be a representative of w such that $int(s)$ is $\mathscr{P}$-distinguished. (Such a representative exists by Proposition 1.6.10.) Then*

$$\delta\left(J\left(\sigma,\nu\right)\right)=\chi_\sigma\left(s^2\right)\chi_\sigma\left(z_{\rho_M^\vee}\right). \tag{1.31}$$

Proof. By [14] Theorem 14.48 and Proposition 1.6.10 there exists a representative $s\in N_K\left(A\right)$ such that $s^2\in Z\left(M\right)$ and $\left[y_0,s\right]=1$, where y_0 is a representative of the long element of $W\left(M,T\right)$ that satisfies $y_0^2=z_{\rho_M^\vee}$.

Let ψ be the real form on V_σ defining $\overline{\sigma}$ and let ψ also represent the map on

$I_P(\sigma, \nu)$ defined by $\psi(f)(x) = \psi(f(x))$. Let $\phi : s\sigma \to \overline{\sigma}$ be an isomorphism.

$$\begin{aligned}
\overline{\phi\ A(s, \sigma, \nu)} &= \overline{\phi}\ \ \overline{A(s, \sigma, \nu)} \\
&= \overline{\phi}\ A(w, \overline{\sigma}, \overline{\nu}) \\
&= \overline{\phi}\ \phi\ A_P(s, s\sigma, s\nu)\ \ \phi^{-1} \qquad \text{by Proposition 1.6.15} \\
&= \delta(s, \phi)\, A_P(s, s\sigma, s\nu)\ \ \phi^{-1} \qquad \text{(b/c this is an } s\text{-twisted } \delta\text{-indicator)} \\
&= \chi_\sigma\left(z_{\rho_M^\vee}\right)\ A(s, s\sigma, s\nu)\ \ \phi^{-1} \qquad \text{(by Theorem 1.6.14)} \\
&= \chi_\sigma\left(s^2\right)\chi_\sigma\left(z_{\rho_M^\vee}\right)\ A\left(s^{-1}, s\sigma, s\nu\right)\ \ \phi^{-1} \qquad \text{(by a change of variable)} \\
&= \chi_\sigma\left(s^2\right)\chi_\sigma\left(z_{\rho_M^\vee}\right)(\phi\ A(s, \sigma, \nu))^{-1}
\end{aligned}$$

The argument that the right hand side of (1.31) is independent of the choice of pinning and distinguished representative is the same as the argument in the proof of Theorem 1.6.17

□

Chapter 2: Isomorphisms of Root Data

We change some of the notation used in the previous chapter. In this chapter G represents a connected complex reductive linear group.

2.1 Root Systems and Root Data

The definitions and basic notions in this section are all from [20], Chapter 7, Section 4.

Definition 2.1.1. [20], Definition 7.4.1. A root datum is a quadruple $\Psi = (X, R, X^\vee, R^\vee)$, where

(a) X and $X^\vee$ are free abelian groups of finite rank, in duality by a pairing $X \times X^\vee \to \mathbb{Z}$, denoted by $\langle\ ,\ \rangle$;

(b) R and $R^\vee$ are finite subsets of X and $X^\vee$, and we are given a bijection $\alpha \mapsto \alpha^\vee$ of R onto $R^\vee$.

For $\alpha \in R$ we define endomorphisms s_α and $s_\alpha^\vee$ of X and $X^\vee$ by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha,\ s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee \ \ (x \in X,\ y \in X^\vee).$$

The following axioms are imposed.

(RD1) If $\alpha \in R$ then $\langle \alpha, \alpha^\vee \rangle = 2$;

(RD2) If $\alpha \in R$ then $s_\alpha R = R$, $s_\alpha^\vee (R^\vee) = R^\vee$.

The Weyl group $W = W(\Psi)$ is the group of automorphisms of X generated by the s_α, $\alpha \in R$.

The set R is called the roots and $R^\vee$ is called the coroots. Let Q be the subset of X generated by R and let $V' = \mathbb{R} \otimes Q$. If $R \neq \emptyset$, then R is a root system in V'.

Definition 2.1.2. [20], Section 7.4. Let V' be a finite dimensional real vector space. A subset $R \subseteq V'$ is a root system in V' if R satisfies the following axioms:

(RS1) R is nonempty, finite, $0 \notin R$, and R spans V';

(RS2) If $\alpha \in R$ there is an $\alpha^\vee$ in the dual of V' such that $\langle \alpha, \alpha^\vee \rangle = 2$ and the endomorphism s_α stabilizes R;

(RS3) If $\alpha \in R$ then $\alpha^\vee (R) \subseteq \mathbb{Z}$.

Let R be a root system in V' and fix a set of positive roots $R^+ \subseteq R$. A base, or a set of simple roots, denoted $\Pi \subseteq R^+$, is the set of positive roots that cannot be written as the sum of two elements of R^+, [13], Chapter II, Section 5. A base Π generates R and is also a basis for V'. If two root systems R and R_1 are isomorphic and $\phi : R_1 \to R_1$ is an isomorphism, then ϕ extends uniquely to a vector space isomorphism $\phi : V' \to V_1'$.

An isomorphism of root datum $\Psi = (X, R, X^\vee, R^\vee)$ and $\Psi_1 = (X_1, R_1, X_1^\vee, R_1^\vee)$, is an isomorphism $\phi : X \to X_1$ mapping R onto R_1 such that its transpose, ${}^T\phi$, maps $R_1^\vee$ onto $R^\vee$, [20], Section 9.6.

Theorem 2.1.3. *[20], Theorem 9.6.2 [Isomorphism Theorem]. Let G and G_1 be two connected, reductive linear algebraic groups over an algebraically closed field k, with maximal tori T, T_1 and corresponding root data $\Psi = (X, R, X^\vee, R^\vee)$, $\Psi_1 = (X_1, R_1, X_1^\vee, R_1^\vee)$. Let f be an isomorhpism of Ψ_1 onto Ψ. There exists an isomorphism of algebraic groups $\phi : G \to G_1$ with $\phi T = T_1$ such that ϕ induces f. If ϕ' is an other isomorphism with these properties there is $t \in T$ such that $\phi'(g) = \phi(tgt^{-1})$, for $g \in G$.*

The questions we are interested in run converse to Theorem 2.1.3. Given a pair of root data Ψ and Ψ_1, how can you tell if they are isomorphic? If they are isomorphic, can you find an isomorphism?

Definition 2.1.4. Let $(X, R, X^\vee, R^\vee)$ be a root datum and let $\Pi \subseteq R$ be a set of simple roots with corresponding simple coroots $\Pi^\vee \subseteq R^\vee$. A based root datum is a quadruple $(X, \Pi, X^\vee, \Pi^\vee)$. An ordered based root datum (OBRD) is a based root datum in which an ordering is specified on the roots and coroots. Typically, the ordering will be specified by a pair of matrices A, B whose columns are the simple roots, simple coroots, respectively, such that ${}^TAB$ is a Cartan matrix. In this case, transpose ^{T} is interpreted as transpose relative to the pairing $\langle\ ,\ \rangle : X \times X^\vee \to \mathbb{Z}$.

Let G be a connected complex reductive linear group with rank m and semisimple rank n. Let $\Psi = (X, \Pi, X^\vee, \Pi^\vee)$ be a based root datum for G. Choose a basis for X, meaning an isomorphism $\phi : X \to \mathbb{Z}^m$ and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$. Define A to be the matrix with coefficients in $\mathbb{Z}$ where A_{ij} is the i-th coordinate of α_j in the chosen basis ϕ. Similarly, define B to be the matrix with coefficients in $\mathbb{Z}$ where B_{ij} is i-th coordinate of $\alpha_j^\vee$ in the chosen basis ϕ. Now $(\mathbb{Z}^m, A, \mathbb{Z}^m, B)$ is an OBRD of G. The duality pairing for the OBRD $\langle\ ,\ \rangle$ coincides with the usual dot product on $\mathbb{Z}^m$ and ${}^T AB$ is a Cartan matrix.

Definition 2.1.5. Let $\Psi = (\mathbb{Z}^m, A, \mathbb{Z}^m, B)$ and $\Psi' = (\mathbb{Z}^m, A, \mathbb{Z}^m, B)$ be OBRD. Then $\Psi \cong \Psi'$ if and only if there exists $\phi \in GL(m, \mathbb{Z})$ such that $\phi A = A'$ and ${}^T\phi^{-1}B = B'$.

Example 2.1.6. Suppose semisimple rank$(G) =$ rank$(G) = 2$ and suppose G has Lie type A_2. There are exactly two OBRD's, up to isomorphism. The matrices of roots and coroots, respectively, are shown in the table below.

$SL(3,\mathbb{C})$	$PGL(3,\mathbb{C})$
$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Example 2.1.7. Suppose now that semisimple rank$(G) = 1$, rank$(G) = 2$, and G has Lie type A_1. There are exactly three OBRD's, up to isomorphism. The matrices of roots and coroots, respectively, are shown in the table below.

$SL(2,\mathbb{C}) \times \mathbb{C}^\times$	$PGL(2,\mathbb{C}) \times \mathbb{C}^\times$	$GL(2,\mathbb{C})$
$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Some obvious necessary conditions for OBRD Ψ and Ψ' to be isomorphic are

1. Ψ and Ψ' must correspond to groups with the same semisimple rank n, and

2. the Cartan matrices must be *equal*: ${}^T AB = {}^T A'B'$.

If Ψ and Ψ' are isomorphic as OBRD's, then of course the corresponding based root data and root data are isomorphic. However, the converse is not true because it may be necessary to account for a permutation on the simple roots (simple coroots) to make the Cartan matrices equal, or a permuation on factors of simple Lie types with multiplicity greater than one. The based root data and root data, for Ψ and Ψ' are isomorphic if and only if

1. there exists a permutation matrix P such that ${}^{T}P\ {}^{T}ABP = {}^{T}A'B'$, and

2. there exists $\phi \in GL\,(m, \mathbb{Z})$ such that $\phi AP = A'$ and ${}^{T}\phi^{-1}BP = B'$.

Finding a permutation matrix P such that ${}^{T}P\ {}^{T}ABP = {}^{T}A'B'$ is possible by analyzing the Cartan matrices ${}^{T}AB$ and ${}^{T}A'B'$. If the Lie algebras have simple types with multiplicites greater than one, then it may be necessary to do an outer exhaust over the orderings of these factors. If there are a lot of factors of the same Lie type, then the exhaust could be expensive, but at least there are only a finite number of orderings to check.

Our main focus is on finding isomorphisms of OBRD, or testing a pair of OBRD to determine if they are isomorphic or not. Given a pair of OBRD, just deciding whether the pair is isomorphic or not seems to be generally an interesting problem. However, when the OBRD correspond to semisimple data, the decision problem seems fairly easy.

Proposition 2.1.8. *Let $\Psi = (\mathbb{Z}^m, A, \mathbb{Z}^m, B)$ and $\Psi' = (\mathbb{Z}^m, A', \mathbb{Z}^m, B')$ be OBRD for semisimple groups of rank m with ${}^{T}AB = {}^{T}A'B'$. Then Ψ and Ψ' are isomorphic if and only if $\phi = A'A^{-1} \in GL\,(m, \mathbb{Z})$. Furthermore, $A'A^{-1} \in GL\,(m, \mathbb{Z})$ if and only* *if A and A' have the same Hermite normal form.*

In the statement above, $A^{-1} \in GL(m, \mathbb{Q})$ because A (and also A') are square matrices with full rank. Note that we are *not* asserting that $A^{-1} \in GL(m, \mathbb{Z})$.

Proof. Since G, G' are semisimple, then the rank m equals the semisimple rank n. As a consequence, A,B, A', B' are $m \times m$ full rank matrices with integer coefficients. Define $\phi = A'A^{-1}$. Then $\phi A = A'$ and

$$^{T}\phi^{-1}B = \left({}^{T}(A')^{-1}\,{}^{T}A\right)\left({}^{T}A^{-1}\,{}^{T}A'B'\right) = B'.$$

At this point, ϕ is an isomorphism of root systems and ϕ is unique! If ϕ has integer coefficients and $\det(\phi) = \pm 1$, then $\phi \in GL(m, \mathbb{Z})$ and ϕ is an isomorphism of OBRD. The final assertion is just a typical fact regarding uniqueness of the Hermite normal form. See Appendix B for more information about the Hermite normal form. □

2.2 Towards a Normal Form for OBRD

One way to decide whether a pair of OBRD are isomorphic would be to find a normal form for OBRD, a canonical form analogous to the Hermite normal form, that would be obtained via row reductions by matrices in $GL(m, \mathbb{Z})$. The goal of this section is to try to make progress in this direction.

Let $\Psi = (\mathbb{Z}^m, A, \mathbb{Z}^m, B)$, $\Psi' = (\mathbb{Z}^m, A', \mathbb{Z}^m, B')$ be OBRD with semisimple rank n, such that ${}^{T}AB = {}^{T}A'B'$. When $m > n$ and $\Psi \cong \Psi'$ then an isomorphism $\phi : \Psi \to \Psi'$ may no longer be unique. Compare this to the semisimple case, especially the proof of Proposition 2.1.8.

Let $\begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ be the Hermite normal form for B. That is, for some $u \in GL(m, \mathbb{Z})$, which is not unique when $m > n$, $uB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$. Let $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = {}^T u^{-1} A$ so that

$$\left(\mathbb{Z}^m, \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \right) \tag{2.1}$$

is an OBRD isomorphic to Ψ.

Given another OBRD $\Psi' = (\mathbb{Z}^m, A', \mathbb{Z}^m, B')$, let $\begin{pmatrix} B_1' \\ 0 \end{pmatrix} = u'B'$ be the Hermite normal form of B', where $u' \in GL(m, \mathbb{Z})$, and let $\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = {}^T (u')^{-1} A'$. This results in the OBRD

$$\left(\mathbb{Z}^m, \begin{pmatrix} A_1' \\ A_2' \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1' \\ 0 \end{pmatrix} \right),$$

isomorphic to Ψ'. The Hermite normal forms B_1' and B_1 are unique and if they are not equal, then Ψ and Ψ' cannot be equivalent OBRD. So assume that $B_1' = B_1$. Since ${}^T A_1' B_1' = {}^T A' B' = {}^T AB = {}^T A_1 B_1$, then $A_1' = A_1$.

Definition 2.2.1. An OBRD of the form $\Psi = \left(\mathbb{Z}^m, \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \right)$, where B_1 is in Hermite normal form, is called Hermite reduced. In this form, A_1, B_1 are canonical but A_2 is not.

To summarize, we've reduced the problem of deciding if Ψ and Ψ' are isomorphic to finding an equivalence between Hermite reduced OBRD:

$$\Psi \cong \left(\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \right), \qquad \Psi' \cong \left(\mathbb{Z}^m, \begin{pmatrix} A_1 \\ A_2' \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \right).$$

Finding an equivalence between Hermite reduced OBRD is the same as finding

$$U = \begin{pmatrix} I_n & 0 \\ X & v \end{pmatrix}, \quad \text{where } X \text{ is an } m-n \times n \text{ integer matrix, } v \in GL\left(m-n, \mathbb{Z}\right)$$

such that

$$U \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2' \end{pmatrix}.$$

Of course, since

$${}^T U^{-1} = \begin{pmatrix} I_n & -{}^T X \ {}^T v^{-1} \\ 0 & {}^T v^{-1} \end{pmatrix}$$

then ${}^T U^{-1} \begin{pmatrix} B_1 \\ 0 \end{pmatrix} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$.

At this point, we might as well mention one more easy case in deciding whether Ψ and Ψ' are isomorphic. If $m = n+1$, then the only possibility for v is $v = \begin{pmatrix} \pm 1 \end{pmatrix}$. In this case, it's easy to just check both possibilities for v. Once $v \in GL\left(m-n, \mathbb{Z}\right)$ chosen, then X is determined,

$$X = \left(A_2' - vA_2\right) A_1^{-1}$$

and $U = \begin{pmatrix} I_n & 0 \\ X & v \end{pmatrix} \in GL\left(m, \mathbb{Z}\right)$ if and only if X has integer coefficients.

2.3 Hermite Reduced OBRD and Canonical Subgroups

One may construct any connected complex reductive group in the following way. Choose

$$\begin{array}{rl} \text{(i)} & \text{a connected semisimple group } G_{ss}, \\ \text{(ii)} & \text{a finite subgroup } A \subseteq Z(G_{ss}) \text{ with } R \text{ invariant factors,} \\ \text{(iii)} & \text{a torus } T \text{ of rank } r \geq R, \\ \text{(iv)} & \text{an embedding } j : A \to T, \end{array} \tag{2.2}$$

and construct the quotient

$$G = G_{ss} \times T / i \times j(A),$$

where $i : A \hookrightarrow Z(G_{ss})$ is the natural inclusion. This is the basic construction utilized by Adams, et al, in the Atlas of Lie Groups and Representations Project. Another good reference for this type of construction is the theorem in Section 6.9 in [19].

The basic strategy that we employ to decide if two OBRD are isomorphic is kind of like "inverting" the construction above in an attempt to get back to the original "ingredients" in (2.2). Given OBRD for a connected complex reductive group G, we identify some canonical subgroups corresponding to (2.2). See Proposition 2.3.1 below.

Let $Z = Z(G)$ represent the center of G. The radical of G, denoted $R(G)$, is the identity component of the center of G: $R(G) = Z_0$. If G is semisimple, the radical is trivial. If G has rank $m > n$, where n is the semisimple rank, then $R(G)$ is a torus of rank $r = m - n$. The derived group of G, denoted G_d, is the commutator subgroup, $G_d = [G, G] \subseteq G$. Define $Z_1 = G_d \cap Z_0$. The three subgroups (G_d, Z_0, Z_1) are canonical and determine determine G up to isomorphism.

Proposition 2.3.1. *Let* $\Psi = \left(\mathbb{Z}^m, \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1 \\ 0 \end{pmatrix}\right)$ *be a Hermite reduced OBRD for a connected complex reductive linear group* G *with rank* m *and semisimple rank* n*. Then,*

1. *Lie* $(Z_0) \cong \left\{ \begin{pmatrix} \left(-{}^T A_1^{-1}\right) \left({}^T A_2\right) \\ I_r \end{pmatrix} x : x \in \mathbb{C}^r \right\}$,
2. G_d *has OBRD isomorphic to* $(\mathbb{Z}^n, A_1, \mathbb{Z}^n, B_1)$*, and*
3. $Z_1 \cong \left\{ \left[\begin{pmatrix} \left({}^T A_1^{-1}\right) \left({}^T A_2\right) \\ 0 \end{pmatrix} x \right] : x \in \mathbb{Z}^r \right\}$*, where* $[X]$ *is the class of* X *in* $\mathbb{Q}^m / \mathbb{Z}^m$*, for* $X \in \mathbb{Q}^m$*.*

Proof. Essentially, this is just an application of [20] Proposition 8.1.8 and Corollary 8.1.9

1. Let $H \subseteq G$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the torus and roots corresponding to the OBRD. Since $\mathrm{Lie}(Z_0) = \bigcap_{\alpha \in \Delta} \ker \alpha$, then

$$\mathrm{Lie}(Z_0) \cong \left\{ \begin{pmatrix} \left(-{}^T A_1^{-1}\right) \left({}^T A_2\right) \\ I_r \end{pmatrix} x : x \in \mathbb{C}^r \right\}.$$

2. Let

$$R = \left\{ \begin{pmatrix} A_1 \\ \\ A_2 \end{pmatrix} x : x \in \mathbb{Z}^n \right\},$$

$$R^\vee = \left\{ \begin{pmatrix} B_1 \\ \\ 0 \end{pmatrix} x : x \in \mathbb{Z}^n \right\}$$

be the root lattice and coroot lattice, respectively. Let $\mathbb{Z}^m = \mathbb{Z}^n \times \mathbb{Z}^r$. Then

$$(R^\vee)^\perp = \{0\} \times \mathbb{Z}^r$$

$$\left((R^\vee)^\perp\right)^\perp = (\{0\} \times \mathbb{Z}^r)^\perp = \mathbb{Z}^n \times \{0\}$$

Let $\mathrm{Im}_q(X)$ denote the image of $X \subseteq \mathbb{Z}^m$ under the quotient map $q : \mathbb{Z}^m \mapsto \mathbb{Z}^m / (R^\vee)^\perp = \mathbb{Z}^n \times \mathbb{Z}^r / \{0\} \times \mathbb{Z}^r \cong \mathbb{Z}^n$. By Corollary 8.1.9 in [20], the commutator subgroup[1], G_d, has root datum

$$\left(\mathbb{Z}^m / (R^\vee)^\perp, \mathrm{Im}_q(R), \left((R^\vee)^\perp\right)^\perp, R^\vee\right).$$

Therefore the OBRD for G_d is isomorphic to

$$(\mathbb{Z}^n, A_1, \mathbb{Z}^n, B_1).$$

3. Let $Z_1 = G_d \cap Z_0 = Z(G_d) \cap Z_0$.

$$Z(G_d) \cong \left\{ \left[\begin{pmatrix} {}^T A_1^{-1} \\ \\ 0 \end{pmatrix} x \right] : x \in \mathbb{Z}^n \right\}.$$

Let $y \in \mathrm{Lie}(Z_0)$.

$$\exp(2\pi i y) \in Z(G_d) \Leftrightarrow y \in \left\{ \begin{pmatrix} \left(-{}^T A_1^{-1}\right)\left({}^T A_2\right) \\ \\ I_r \end{pmatrix} x : x \in \mathbb{Z}^r \right\}.$$

[1] In [20], the commutator subgroup is denoted (G, G).

Therefore,

$$Z_1 \cong \left\{ \left[\begin{pmatrix} \left({}^T A_1^{-1}\right) \left({}^T A_2\right) \\ 0 \end{pmatrix} x \right] : x \in \mathbb{Z}^r \right\}.$$

$\square$

Recall the ingredients (2.2) in the construction of complex reductive groups, as described at the beginning of this section. One interesting question is: How sensitive is the isomorphism class of G to the choice of embedding $j : A \to T$? Theorem 2.3.2 below provides an answer to this question, in the case that the derived group G_d has cyclic center. It's interesting to observe that once the rank of the radical is sufficiently large, the choice of embedding does not matter at all!

Theorem 2.3.2. *Fix an integer $r \geq 0$ and a semisimple group G_{ss} with cyclic center $Z(G_{ss}) \cong \mathbb{Z}/k\mathbb{Z}$, where k is some positive integer. Let G be a connected complex reductive linear group with derived group $G_d \cong G_{ss}$ and $r = \dim\left(Z(G)_0\right)$.*

1. *If $r = 0$, there is only one group G up to isomorphism.*

2. *If $r = 1$, there are $\lfloor k/2 \rfloor$ isomorphism classes.*

3. *If $r > 1$, the isomorphism classes are determined by the cyclic subgroups of $\mathbb{Z}/k\mathbb{Z}$. That is, there is one isomorphism class for each divisor of k.*

Proof. If $r = 0$, then the statement is clear, so suppose $r > 0$. Let G and G' be two groups whose derived groups G_d, G'_d, respectively, are isomorphic. Suppose the rank of G and G' is m and let $n = m - r$ be the rank of G_d, G'_d. Since G_d and G'_d

are isomorphic, Proposition 2.3.1 implies that there exists OBRD Ψ and Ψ', for G and G', respectively that have the following form:

$$\Psi = \left(\mathbb{Z}^m, \begin{pmatrix} A_1 \\ \\ A_2 \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1 \\ \\ 0 \end{pmatrix}\right) \qquad \Psi' = \left(\mathbb{Z}^m, \begin{pmatrix} A_1 \\ \\ A_2' \end{pmatrix}, \mathbb{Z}^m, \begin{pmatrix} B_1 \\ \\ 0 \end{pmatrix}\right)$$

For convenience, apply the same reduction that puts A_1 in Smith normal form, to both matrices of roots. Now, both sets of roots have the top $n \times n$ matrix in the form

$$A_1 = \operatorname{diag}(1, \ldots, 1, k).$$

By applying elementary row reductions using only elements in $GL(m, \mathbb{Z})$ we may replace the first $n-1$ columns of both A_2 and A_2' with zeros. After applying these reductions (which of course correspond to isomorphisms of OBRD), we have

$$A_2 = \begin{pmatrix} 0 & \cdots & 0 & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_r \end{pmatrix} \qquad A_2' = \begin{pmatrix} 0 & \cdots & 0 & a_1' \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_r' \end{pmatrix}.$$

If $r = 1$, then the only remaining reduction is to multiply the first rows of A_2 and A_2' by ± 1 and reduce mod k to produce

$$A_2 = \begin{pmatrix} 0 & \cdots & 0 & a \end{pmatrix} \qquad A_2' = \begin{pmatrix} 0 & \cdots & 0 & a' \end{pmatrix}, \quad 0 \leq a, a' \leq \lfloor k/2 \rfloor.$$

Now it's clear that $G \cong G' \Leftrightarrow a = a'$.

If $r > 1$, we next reduce A_2 and A_2' to Hermite normal form and obtain:

$$A_2 = \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad A_2' = \begin{pmatrix} 0 & \cdots & 0 & b' \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

There is one final reduction which is interesting. Using an elementary row operation in $GL(m, \mathbb{Z})$, add k into the second row of A_2 and A_2', just under b and b'. Now reducing A_2 and A_2' to Hermite normal form replaces the submatrices

$$\begin{pmatrix} b \\ k \end{pmatrix} \text{ with } \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ where } a = \gcd(b, k), \text{ and}$$
$$\begin{pmatrix} b' \\ k \end{pmatrix} \text{ with } \begin{pmatrix} a' \\ 0 \end{pmatrix} \text{ where } a' = \gcd(b', k).$$

After applying this reduction,

$$A_2 = \begin{pmatrix} 0 & \cdots & 0 & a \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad A_2' = \begin{pmatrix} 0 & \cdots & 0 & a' \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{with } a, a' > 0 \text{ and } a, a' \mid k.$$

Let $Z_1 = Z(G)_0 \cap G_d$ and $Z_1' = Z(G')_0 \cap G_d'$. In this case Z_1 and Z_1' have canonical generators $a, a' > 0$ with $a, a' \mid k$. In this case $G \cong G'$ if and only if $a = a'$ and the isomorphism classes are determined by the cyclic subgroups of $\mathbb{Z}/k\mathbb{Z}$. $\square$

Given a pair of OBRD Ψ and Ψ', one can use the reductions in the proof above to determine if Ψ and Ψ' are isomorphic. If they are isomorphic, then by tracking the reductions in each step, one obtains a matrix $\phi \in GL\,(m, \mathbb{Z})$ that maps Ψ isomorphically onto Ψ'.

Appendix A: Special Case of the Schur Indicator Calculation

Theorem A.1 below is a special case of Theorem 1.1.6 when σ is self-dual. In this case you could use the *adjusted intertwining operators* of [14], Proposition 14.23. Suppose that σ is self-dual and there exists $w \in W(G, A)$ such that $w\sigma \cong \sigma$, $w\nu = -\nu$, and $w^2 = 1$. Let $s \in N_K(A)$ be any representative of w. Since $s\sigma \cong \sigma$ then σ extends to a representation $\widetilde{\sigma}$ of $\langle M, s\rangle$ and the adjusted operator is defined to be $\widetilde{\sigma}(s) A_P(s, \sigma, \nu)$. The adjusted operators have all the same properties of the original operators only now they are actually independent of the choice of representative s of $w \in W(G, A)$. For details, see [14] Chapter XIV, Section 6, especially Proposition 14.23.

Theorem A.1. *Let $P = MAN$ be a minimal parabolic subgroup of the real reductive linear group G and suppose (σ, V_σ) is an irreducible, self-dual representation of M. Let $J_P(\sigma, \nu)$ be the Langlands quotient of a principal series representation of G and suppose $J_P(\sigma, \nu)$ is irreducible and self-dual. Let $w \in W(G, A)$ such that $w\sigma \cong \sigma^*$, $w\nu = -\nu$ and $w^2 \in Z(M)$. Let $\mathscr{P}$ be a pinning of M with root vectors satisfying (1.19). Let $s \in N_K(A)$ be a representative of w such that $int(s)$ is $\mathscr{P}$-distinguished. (Such a representative exists by Proposition 1.6.10.) Then,*

$$\epsilon(J(\sigma, \nu)) = \chi_\sigma\left(s^2\right)\epsilon(\sigma).$$

As is implied, the formula on the right hand side of the equation is independent of the chosen s as long as s satisfies the hypotheses stated in the theorem.

Proof. Under the assumption that σ is self-dual, let $B : (\sigma, V_\sigma) \to (\sigma^*, V_\sigma^*)$ and let B also represent the induced map:

$$B(f)(x) = B(f(x)) : I_P(\sigma, -\nu) \to I_P(\sigma^*, -\nu).$$

Since σ is self-dual, ${}^T B = \epsilon(\sigma) B$.

Let $s \in N_K(A)$ a representative of w such that $\operatorname{int}(s)$ is $\mathscr{P}$-distinguished. Let $\phi : (s\sigma, V_\sigma) \to (\sigma, V_\sigma)$ be an isomorphism. Extend σ to a representation $\widetilde{\sigma}$ of $\langle M, s\rangle$ by scaling ϕ so that $\phi^2 = 1$ and defining[1]:

$$\widetilde{\sigma}(s) = \sqrt{\chi_\sigma(s^2)}\phi.$$

Then

$$I_P(\sigma, \nu) \overset{\widetilde{\sigma}(s)A_P(s,\sigma,\nu)}{\longrightarrow} I_P(\sigma, -\nu) \overset{B}{\longrightarrow} I_P(\sigma^*, -\nu)$$

and calculating the indicator is the same as calculating the difference between this map and it's transpose.

$$\begin{aligned}
{}^T\left[B\widetilde{\sigma}(s) A(s,\sigma,\nu)\right] &= {}^T A(s,\sigma,\nu)\ {}^T\widetilde{\sigma}(s)\ {}^T B \\
&= A\left(s^{-1}, s\sigma^*, -s\nu\right)\ {}^T\widetilde{\sigma}(s)\ {}^T B \\
&= \widetilde{\sigma}^*\left(s^{-1}\right) A\left(s^{-1}, \sigma^*, \nu\right) \left[\widetilde{\sigma}^*(s)\ {}^T\widetilde{\sigma}(s)\right] {}^T B \\
&= \widetilde{\sigma}^*\left(s^{-1}\right) B\ A\left(s^{-1}, \sigma, \nu\right) B^{-1}\ {}^T B \\
&= \epsilon(\sigma)\left[\widetilde{\sigma}^*\left(s^{-1}\right) B\right] A\left(s^{-1}, \sigma, \nu\right).
\end{aligned}$$

[1]Here, there is a choice of square root. One must be chosen, but it doesn't matter which. The argument and the result is independent of this choice.

The final step involves a twisted indicator calculation:

$$(s\sigma, V_\sigma) \xrightarrow{\phi} (\sigma, V_\sigma) \xrightarrow{B} (\sigma^*, V_\sigma^*) .$$

$$\begin{aligned}
{}^T\phi B &= \epsilon(\sigma)\ {}^T\left({}^T\phi B\right) && \text{by [the twisted indicator result]} \\
&= \epsilon(\sigma)\ {}^T B\phi \\
&= \epsilon(\sigma)^2 B\phi && \text{(since } \sigma \text{ is self-dual)} \\
&= B\phi.
\end{aligned}$$

Finally,

$$\begin{aligned}
\widetilde{\sigma}^*\left(s^{-1}\right) B &= {}^T\widetilde{\sigma}(s) B \\
&= \left(\sqrt{\chi_\sigma(s^2)}\ {}^T\phi\right) B \\
&= \sqrt{\chi_\sigma(s^2)}\left({}^T\phi B\right) \\
&= \chi_\sigma\left(s^2\right)\left(\frac{1}{\sqrt{\chi_\sigma(s^2)}}\right)\left({}^T\phi B\right) \\
&= \chi_\sigma\left(s^2\right)\left(\frac{1}{\sqrt{\chi_\sigma(s^2)}}\right)(B\phi) \\
&= \chi_\sigma\left(s^2\right) B\left(\frac{1}{\sqrt{\chi_\sigma(s^2)}}\phi\right) \\
&= \chi_\sigma\left(s^2\right) B\ \widetilde{\sigma}\left(s^{-1}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
{}^T\left[B\widetilde{\sigma}(s) A(s,\sigma,\nu)\right] &= \chi_\sigma\left(s^2\right)\epsilon(\sigma)\left[B\widetilde{\sigma}\left(s^{-1}\right) A\left(s^{-1},\sigma,\nu\right)\right] \\
&= \chi_\sigma\left(s^2\right)\epsilon(\sigma)\left[B\widetilde{\sigma}(s) A(s,\sigma,\nu)\right].
\end{aligned}$$

The final equality is due to the fact that $w^2 = 1$ and $\widetilde{\sigma}(s) A(s, \sigma, \nu)$ is independent of the choice of representative s of $w \in W(G, A)$. $\square$

Appendix B: Hermite Normal Form

Let A be an $n \times n$ integer matrix. If $\det(A) = \pm 1$ then A is called unimodular.

The original reference for the idea of the Hermite normal form is the following theorem.

Theorem B.1. *(Hermite, [11]) Given a nonsingular $n \times n$ integer matrix A, there exists an $n \times n$ unimodular matrix v such that Av is lower triangular with positive diagonal elements. Furthermore, each off-diagonal element of Av is nonpositive and strictly less in absolute value than the diagonal elements in its row.*

The matrix Av is called the Hermite normal form.

We prefer an equivalent version of the Hermite normal form, which is obtained via elementary row operations over $\mathbb{Z}$ instead of elementary column operations. See [8].

Definition B.2. Let A be an $m \times n$ integer matrix with rank n. The matrix A is in Hermite normal form if

1. $A_{ij} = 0$, $i > j$,
2. $A_{ii} > 0$, and
3. $A_{ii} > A_{ij} \geq 0$, $i > j$.

Let A' be another $m \times n$ matrix of rank n. The matrices A and A' have the same Hermite normal form if and only there exists a matrix $u \in GL(m, \mathbb{Z})$ s.t. $A' = uA$. Again, see [8].

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