AN ABSOLUTE MAGNETOMETER FOR TERRESTRIAL FIELD DETERMINATION

By

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TABLE OF CONTENTS

INTRODUCTION

The intensity of a magnetic field may be determined by referring the various properties of the field to the fundamental physical dimensions, length, mass and time. The laws of Coulomb, Ampere and Faraday serve to define a magnetic field in terms of other measurable parameters. If these parameters be measured in terms of fundamental physical standards, the field will have been determined absolutely. Previous techniques for determining the absolute field intensity have employed the torsion magnetometer,[1] the Cotton balance, the sine galvanometer[2] and the electromagnetic inductor.[3]

When a complete determination of a space vector field, such as the terrestrial magnetic field, is desired, the direction as well as the magnitude of the field must be specified. The terrestrial field has been by convention specified by its vertical and horizontal components. In addition to these components, the direction with respect to a fixed object of the plane containing those components must be stated. Various other combinations of three parameters may be used to define a special magnetic field. In measurements

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[1]Karl F. Gauss, Intensitas vis magneticae ad mensuram absolutum revocata, Comment. Soc. Reg. Gottingen, (1833).

[2]W. Watson, Phil. Trans. Roy. Soc. A., 198, 431 (1902).

[3]M. Faraday, Experimental Researches in Electricity vol. 1 (1831).

of the terrestrial field it has been common practice to measure the horizontal component, the inclination and the declination of the field. The vertical intensity may then be computed from the horizontal intensity and the dip. Such a technique, however, leads to a result no more precise than the least precisely measured parameter. The dip is, in general, most difficult to measure. A preferred method for defining the Earth's field consists in measuring the horizontal and vertical components and the declination. Until quite recently, such measurements were impossible because there existed no magnetometer for accurately measuring the vertical component of the Earth's field. In England, Dye[4] has developed an absolute magnetometer adapted for observatory measurements of the vertical component. An ideal instrument for terrestrial magnetic measurements should be capable of measuring both components of the vector field, as well as the declination of the field. A universal observatory magnetometer of this type is being constructed by the Carnegie Institution of Washington.[5] This instrument uses an electromagnetic inductor operating at a comparatively low speed. In the development of a similar

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[4] D. W. Dye, A Magnetometer for the Measurement of the Earth's Vertical Field Intensity in C. G. S. Measure. Proc. Roy. Soc. A, 117, 434 (1928).

[5] E. A. Johnson, A Primary Standard for Measuring the Earth's Magnetic Vector, Terr. Mag. and Atmos. Elec. 44, 1, 29, 1939

magnetometer, the subject of this dissertation, the author chose to use an extremely high speed inductor in order to gain a sufficiently high sensitivity in the rather weak Earth's field.

In its simplest form, the magnetic inductor consists of a coil which is rotated about an axis perpendicular to the magnetic field whose magnitude is desired. Faraday's law states the relation between the induced voltage and the rate of cutting lines of induction in the form

$$e = -\frac{dN}{dt} \times 10^{-8}$$

For an inductor of n turns, of cross-section A, rotating in a field H at an angular speed ω the induced voltage at any time t becomes in a medium of magnetic permeability unity,

$$e = nAH\omega \sin \omega t \times 10^{-8} \qquad \text{Equation 1.}$$

A critical study of equation 1 shows that not only must the induced sinusoidal emf be measured with the same precision as is required in the final field determination, but also during the measurement, the angular velocity must be known to the same ultimate precision. When better than 1% precision is desired, this method cannot be used because present day alternating current voltmeters are not absolute standards. An immediate modification presents itself in the form of the null technique.

If around the inductor is placed a current-carrying coil whose axis coincides with the magnetic vector under measurement, a field can be set up in the coil of such a sense as to oppose the magnetic vector. The magnitude of the coil's field can be adjusted by varying the current flowing through it. When the field of the coil is exactly equal and opposite in sense to that of the magnetic vector being measured, the voltage induced in the inductor is zero. The value of the magnetic vector is now defined in terms of the current in the coil and in terms of the physical dimensions of the coil. Both of these factors must be determined to precision desired in the field measurements. By careful construction and by the use of a material of low coefficient of thermal expansion for supporting the coil, the coil will have a permanent calibration constant determinable to high precision. Although a direct correlation between the current and the fundamental dimensions of time and mass may be obtained by the silver deposition method, a precise value of the instantaneous current can be obtained by interpolating the voltage drop produced by the current through a standard resistance in terms of the voltage of a standard cell. This interpretation may be executed by means of a precision potentiometer.

Since the rotating inductor and its circuital elements are used merely as a null detector, only the sensitivity of detecting the equality of the opposing fields is affected by the physical characteristics of the null detecting constituents. From equation 1 we see that the sensitivity is given by

$$\frac{de}{dH} = nA\omega \times 10^{-8} \qquad \text{Equation 2.}$$

Now, if the fineness of measurement must be made a maximum, the sensitivity of the measurements of the voltage "e", as well as of the product "$n A \omega$" must be increased to the limit of experimental technique. Because the most sensitive A.C. voltmeters are those preceded by several stages of electron tube amplification, this problem was attacked by the use of such a detector. On casual thought, one might attempt extremely high amplification. There occur, however, in vacuum tube circuits, certain random noises[6] called thermal agitation and shot noises whose effects set a limit to the amplification of the small voltages. The product "$n A \omega$" shows that an increase in any one of the factors n, A, or ω produces improved sensitivity by increasing the induced emf per unit field intensity, which is, by equation 2, the sensitivity. The dimensions of the inductor, which determine A, are, however, limited by the area over which the field at the center of the coil can be said to be uniform, while the number of turns on the inductor is also limited by the physical dimensions of the coil as well as by the electrical characteristics of the input circuit of the amplifier. The latter condition will be studied more minutely in the following pages.

In the preceding paragraph the area of the inductor was said to be limited by the region over which the null field is uniform. If the null field is not uniform it is possible to adjust the magnitude of the field so that the resultant

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[6] J. B. Johnson, Phys. Review, 32, 97, (1928).

of the null field and field being measured is zero but the gradient is not zero. The gradient field results in the generation of harmonic voltages in the inductor. These harmonic voltages, if large enough, set a limit to the amplification which can be used without overloading the amplifier. In addition, these harmonics may be large enough to overload the output meter, thereby reducing the sensitivity of detection. The degree of uniformity of the null field thus sets a limit to the ultimate precision obtainable in the final measurements. While there are many configurations of current-carrying elements which yield a sufficiently uniform field, there are but two forms which have been adopted for terrestrial magnetic instruments, namely the Gaugain-Helmholtz coil and the long solenoid. The long solenoid possesses one feature in addition to a uniform field at its center, which makes it particularly suitable for use as an absolute instrument, - its calibration depends only on one length measurement. The coil constant of a long solenoid can be determined to a high order of precision by comparing its length with that of a primary length standard. The radius of the solenoid enters into the calculation of the coil constant only as a weak parameter. The selection of a long solenoid for the magnetometer described in the paper was dictated by these considerations.

Referring again to equation 2, it will be seen that an increase in the angular speed of the inductor gives by far the best promise of improving the sensitivity of the null detector. With an appreciable increase in rotational speeds, there occur mechanical problems of increased imposed stresses and greatly enhanced bearing wear. Despite these difficulties, it was possible to make continuous measurements with the inductor rotating at 1000 revolutions per second. At these speeds, were one to convey the induced voltages by the usual slip ring and brushes, one would encounter high contact noises. Localized heat developed at the sliding surfaces causes fluctuations in the velocities of the free electrons in the conductors. This phase of the problem was solved by a novel technique suggested to the author by Dr. F. W. Lee. The prosecution of this suggestion led to the development of the "magnetic mirror" to be described in the following pages. Briefly, the "magnetic mirror" comprises a method for inductively coupling the rotating inductor to a stationary coil, thereby eliminating brushes and slip rings. Previous to this work, J. J. Jakosky[7] had developed a comparison magnetometer, (not an absolute instrument), in which the induced voltage was transferred by induction from a rotating coil to a stationary coil.

In the foregoing introduction it was pointed out that magnetic field intensities are precisely defined in terms of primary physical standards by the use of the null technique. In the following sections the conditions yielding a maximum null detection sensitivity will be analyzed. Experimental

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[7] J. J. Jakosky, Exploration Geophysics, Times Mirror Press, 1st. Ed. (1940).

apparatus developed for the verification of this analysis will be described and experimental results of the tests will be given. Theoretically, it appeared that the ultimate limit of precision was determined by the statistical voltages and the goal of this research was to approach this limit, subordinating all other limitations by the careful selection of conflicting parameters. From this study a portable magnetometer capable of measuring both components of the Earth's field to a high order of precision was developed.

STATISTICAL VOLTAGE FLUCTUATIONS

In a properly designed high-gain amplifier, the limit of amplification is determined by the background noises developed in the amplifier input circuit. These noises are characteristic of thermal agitation, being non-periodic and having a distribution over the complete frequency spectrum from the lowest audible frequencies to the ultra-high radio frequencies. Theory attributes this noise to the random motion of the free electrons in a conductor. As in gases, where the random motion of molecules is described by the Maxwell-Boltzman distribution law, the free electrons obey a distribution law which defines the number of electrons having a certain range of velocities. Since the motion of the electrons constitutes a current, this current flowing through the resistance of the conductor develops a voltage. The thermal fluctuation of the motion of the electrons causes the randomness in the periodicity and magnitude of the voltages. When the audible spectrum of these noises is selected and studied through a sound reproducer, frying, hissing sounds are heard. The magnitude of this voltage can be obtained from the relation due to Johnson.[8]

Equation 3.

$$E^2 = 4KT\int_{f_1}^{f_2} Z\,df$$

Equation 3.

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[8]J. B. Johnson, <u>loc. cit.</u>

where E is the effective voltage in the frequency of band width f_2-f_1, K is Boltzman's constant, T is the absolute temperature of the conductor in which the voltage is developed and Z is the impedence of the conductor over an incremental frequency band width df.

If the impedence Z is constant over the band width f_2-f_1, this relation becomes

$$E^2 = 4KTZ(f_2-f_1) \qquad \text{Equation 4.}$$

It is apparent that if an impedence is connected to the input grid of a high-gain amplifier, the noise voltages will be amplified along with any signal voltage that may be developed across that impedence. In this research, the impedence connected to the grid of the first stage of the amplifier is the inductor in which is induced the voltage resulting from cutting the magnetic field under measurement. Now, if the signal voltage is made to approach zero (by adjusting the current in the solenoid), the output meter of the amplifier will approach not zero, but a certain minimum determined by the thermal agitation voltages. This minimum reading of the output meter represents the background below which contrast in signal does not exist. Another serious handicap is the fact that this background limits the attainable amplification.

On examining equation 4, one sees that there are several possibilities for reducing the magnitude of these statistical voltages. The use of only a narrow frequency band within which measurements are made is the most practical attack. Band pass filters can be obtained to limit the spectrum f_2-f_1 to any desired width. There are, however, certain considerations which preclude any extensive gains by this band narrowing. Other possible reductions result from lowering the temperature of the conductor by means of liquid air or oxygen or from making the impedence in the input circuit smaller. With the lower impedence inductor there will occur the attendant reduction in signal voltage, since the signal is the result of a voltage induced in a certain number of turns of wire. The effects of statistical voltages upon the present experiment will be demonstrated in the section on the "magnetic mirror".

THEORY AND OPERATION OF THE MAGNETIC MIRROR

If a coil of one turn having a very low resistance is rotated in a magnetic field so as to cut lines of field, there is induced in this coil an emf, $e = nA\omega \sin \omega t$. This emf can now accelerate the free electrons in the metal of the coil and thus a current is caused to flow in the coil. By Ampere's law, this current "i" gives rise to a magnetic field,

$$H = \oint \frac{i\,dl \times \vec{r}}{r^3}$$ (in vector notation).

Because the current is varying with time, the field resulting from this current will induce an emf in another coil, (the secondary), placed over the inductor so as to intercept the inductor's field. The justification for these multiple transformations of field to current, current to field and, finally, to an emf, is that the secondary need not revolve with the inductor. By a practical utilization of this action it is possible to transfer the voltage induced in the inductor to the amplifier without the use of sliding contacts. Thermal agitation and contact noises between brushes and slip rings are therefore entirely eliminated.

In actual practice, the inductor can be set at an angle of 45° to its axis of rotation. Coaxial with the shaft upon which the inductor rotates is a stationary secondary in which is induced the voltage resulting from the inductor's action in cutting lines of field. Figure 1 illustrates the mechanical arrangement used. Because this experimental device bends the original field through a definite angle, 2α, it has been termed a "magnetic mirror" after its optical analog.

Referring to Figure 1, the flux through the inductor is given by

$$N = AH\cos\alpha \sin\omega t \quad \text{Equation 5.}$$

where α is the angle the plane of the inductor makes with the axis of rotation and μ is the magnetic permeability of the medium surrounding the inductor. The other symbols of equation 5 are those defined in equation 1.

The emf induced in the inductor is

$$e_p = -\dot{N} = \mu\omega AH\cos\alpha\cos\omega t \quad \text{Equation 6.}$$

The current flowing through the inductor is by Ohm's law,

$$i_p = \frac{\mu\omega AH\cos\alpha\cos\omega(t-\phi)}{Z} \quad \text{Equation 7.}$$

where Z is the impedence of the inductor and ϕ is the phase angle.

If the mutual inductance between inductor and the secondary is M*, the emf induced in the secondary will be

$$e_s = \frac{\mu\omega^2 AH\cos\alpha\sin\omega(t-\phi)}{Z} \quad \text{Equation 8.}$$

Although an integral for the calculation of the mutual

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*See appendix

inductance is given in the appendix, it has been found most expedient to measure the constant $\frac{M}{Z}$ in place by simply measuring the voltage developed in the secondary when the inductor operates in a known field.

Figure 2 shows the inductor of the magnetic mirror as well as the turbine impeller upon which the inductor is installed. The inductor is machined from a solid rod of high conductivity copper. As an aid in preserving the dynamic balance of the rotor, the inductor is made symmetrical about its axis by machining a second similar coil integral with the first coil, but with its circuit opened by a narrow slit. This is shown clearly in Figure 2. After the inductor is slipped over the impeller, the whole is wound tightly with linen thread and then given a coat of strong cement.

Despite careful machining, the rotor must now be dynamically balanced in a machine which was developed specifically for balancing these small 12 gram rotors.

The necessity for carefully balancing the rotor is demonstrated by the fact that on the periphery of a rotor of one centimeter radius the radial acceleration at 1000 r.p.s. is 40,000 times that of gravity. An unbalanced mass of one milligram, therefore, results in a force of 40 grams. The apparatus for balancing these small rotors is shown in Figure 3. Its operation depends upon the transformation of the mechanical vibration of the elastic bearing supports into an electrical vibration by means of the crystal phonograph pickups. The voltage from each bearing causes a deflection in each of the two orthogonal directions of the cathode ray beam in the oscillograph. The relative phase of the rotor unbalance at the bearings and the magnitude of the unbalance, is indicated by the Lissajou figure on the screen of the cathode-ray oscillograph.

The voltage from either of the bearings may be amplified and fed to the ignitor electrode of a stroboscope. At each peak of the bearing vibration the stroboscope flashes a pulse of light, causing the rotor to appear stationary. A reference mark in the rotor will therefore appear in one particular position with respect to the position of the unbalanced mass of the rotor. Thus, the phase as well as the magnitude of the unbalanced masses may be determined and corrections made by subtracting weight from the ends of the rotor. With this technique, the vibration due to an unbalance rotor may be reduced to an imperceptible value.

Over the rotor is placed the secondary of the magnetic mirror. The secondary was wound with the finest wire available (No. 46 B. & S.). By its use it was possible to wind 8,525 turns in the small volume available. It was found necessary to shield the secondary from the inductor with an electrostatic shield made by painting the secondary with a conducting coating of colloidally dispersed graphite (Aquadag). This shield, as well as the rotating inductor coil, was grounded.

THE HIGH SPEED TURBINE

High angular speeds can be produced by the use of electric motors, gearing systems, or other mechanical transformers or by the use of turbines. For magnetic measurements, electric motors with their attendant external fields must obviously be excluded. Gearing systems, in addition to being mechanically wasteful of power, are apt to set up high amplitudes of vibration which might cause spurious voltages to be generated in the inductor. It was by the consideration of these various factors that a turbine for driving the inductor was selected. High speed turbines may be driven by air, steam, or oil. For a device which is to be used in the field, steam and oil are not highly adaptable. A turbine driven by compressed air was therefore designed for this work.

Beams[9] has shown in his work on ultra-centrifuges, that speeds as high as 2000 r.p.s. can be attained with air driven turbines of the impulse type. While it might have been possible to follow the principles laid down by Beams, it was decided to design an entirely different type of unit in which the impeller and inductor were combined in one integral rotor.

The turbine rotor is machined from a plastic, "Plexiglas", possessing relatively high tensile strength and a

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[9]See bibliography.

density of only 1.25 gms. per c.c. These properties fulfill conditions developed by Biscoe, Pickels and Wyckoff[10] for a turbine rotor to withstand the bursting stresses due to centrifugal forces.

Reaction turbine blades were cut on a milling machine directly into a cylinder of Plexiglas. The shape of the blades was made to approximate an aircraft wing section having a high coefficient of maximum lift. The high lift coefficient improves the performance of the turbine at low air pressures.

The turbine body as well as the injector and exhaust ports were made of Plexiglas (see Figures 4 and 5). The injector consists of nine holes inclined at 30° to the rotor axis. The injectors are laid out on a circle of radius such that the turbine blades advance directly into the air jets of the injectors. It may not be amiss here to point out that to eliminate powerful mechanical resonance, the number of holes in the injector must be odd if the number of blades on the impeller is even. It has, in fact, been found to improve the smoothness of a turbine if the number of blades is prime to the number of injectors. The exhaust ports are not critical in design, but should be of ample size to reduce the resistance of these jets to the flow of the expanded air. In the experimental turbine the exhaust ports were so placed as to allow the tangentially escaping air

[10] Rev. Sci. Instr., 7, 246, (1936).

from the rotor to flow directly into the exhaust ports.

A number of experimenters have described high speed bearings of the air cushion type,[11] of the oil cushion type,[12] and of the magnetically suspended type.[13]. Because the orientation of the inductor within the solenoid must be accurately maintained, it was impossible to use any of the above types of bearings. A series of experiments on cone bearings showed low friction only under pressure lubrication. Cone bearings have an inherent tendency of throwing the lubricant away from the sliding surfaces, thereby becoming dry, and as a result, enough heat is generated at these speeds to melt the bearing metals. Though mechanically efficient at high speeds, ultra-precision ball bearings had to be discarded when it was found that they became magnetized and induced a strong voltage in the coil. These strong voltages were of such a character as to make impossible their elimination by a counter voltage or by filtering. Since it was impossible to obtain non-magnetic ball bearings, another attack on the bearing problem became necessary. A simple but effective approach to this problem consisted in choosing a bearing made of plain silver babbit sleeves against a phosphor bronze shaft. Lubricated with colloidal graphite dispersed in mineral oil, these bearings

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[11] W. Beams. Reference in bibliography.

[12] E. G. Pickels, Rev. Sci. Instr. 9, 358 (1938).

[13] F. T. Holmes, Phys. Rev. 51, 689 (1937).

have withstood speeds of 1000 r.p.s. for as long as fifty hours.

Figure 6 shows the performance of the turbine just described. At an operating speed of 900 r.p.s. the turbine consumes 2.7 cubic feet of air per minute. When an investigation of the speed stability of this unit was made, certain short period fluctuations were discovered. These fluctuations, having a range of 100 cycles per second, made null detection rather tedious. The fluctuations were suspected to be due to air pressure variations, and to turbulence developed in the impeller. The installation of a pressure regulator later reduced these speed variations considerably, though they were not entirely eliminated. The electrical effects of those variations in speed were, however, eliminated by the technique of using the filter characteristic to counteract the resultant voltage changes.

THE HIGH GAIN LOW NOISE LEVEL AMPLIFIER

Where voltages of the order of 10^{-6} volts are to be amplified, considerable care must be exercised to prevent the amplification of stray voltages, as well as to retain the functions of an amplifier by preventing intercoupling in the various stages. For this research, a three stage amplifier was constructed, in which the components of each stage were in a separate shield box. Resistance-capacity coupling was selected to minimize inductive coupling through the usual interstage transformers. Complete battery operation was used with the batteries in the same box as the amplifier, - a further precaution against stray noise pickup.

Voltages from the magnetic inductor are brought on a concentric cable directly to the grid of the first stage. In conventional amplifiers, the input to the first grid is obtained from an attenuator connected to the voltage source. However, in this amplifier it was necessary to prevent the noises usually developed in the variable resistance of the attenuator from being amplified in the succeeding stages. The attenuator was therefore placed in the second stage. A special low noise tube is used in the first stage as a screen grid amplifier. The constants of the circuit in the first stage are so selected as to keep fluctuation noises at a low level and still retain a relatively high amplification calculated to be about 80. A further precaution against small noises being developed here and being amplified by the successive stages is the use of a high grade wire-wound plate coupling resistor.

The output from the first stage is fed through a concentric shielded line to the attenuator of the second stage. The concentric line shields the voltage to the second stage from external strays and also prevents intercoupling and its resultant oscillation. It is possible in the second stage to obtain as high a gain as 165 with a pentode amplifier, R.C.A. 1620.

From the second stage, the signal is impressed upon the grid of a pentode having its screen and plate connected in parallel. This connection, according to the R.C.A. Tube Handbook HB-3, gives an amplification factor of 20 with a plate resistance of 9000 ohms.

Since the plate voltages of the three stages of this amplifier are supplied from a common plate battery, certain precautions must be taken to prevent regeneration through a common plate impedence.[14] In the circuit diagram (Figure 7) the resistors R_{11}, R_{12}, R_{13} and the condensers C_8, C_{10}, C_{11} decouple the voltages acting through the common plate impedence. With careful decoupling, it was possible to obtain a power gain of 16×10^{10} without setting up oscillations or causing frequency distortion as a result of regeneration.

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[14] F. E. Terman, Radio Engineering, pp. 238-246, 2nd ed.

The condensers C_{12} and C_{13} shunting the plate resistors of the first two stages, introduce in the amplifier response a falling frequency characteristic beyond 400, (see Figure 9). It was found necessary to select empirically the value of these condensers so that the response curve of the amplifier and band pass filter would possess a frequency range over which the output was proportional to the inverse square of the frequency. Without this property, the detector circuit output would be affected by speed variations of the turbine. This feature will be described in detail under the heading "Detector Circuit".

THE DETECTOR CIRCUIT

In the schematic circuit of the null magnetometer, Figure 8, the detector circuit is seen to consist of the turbo-inductor, the high-gain amplifier, the band pass filter and the vacuum tube voltmeter. The amplifier and the turbo-inductor have already been described. The actual detecting instruments will be discussed here.

For detecting small voltages a desirable instrument characteristic is an increasing sensitivity as the null point is approached. This behavior permits a large voltage range as well as a very high null detection sensitivity. A non-linear instrument of this type functions well as a null indicator, even though its precision of calibration may be poor. A calibration is, in fact, not needed, although a scale is convenient for reference. A logarithmic vacuum tube voltmeter possessing the above-mentioned characteristics proved to be an ideal null detector. In some cases the voltage generated by the turbo-inductor cannot be brought to zero with the null field. Under this condition, it is necessary to detect the minimum voltage. The ability of an instrument to work against such a background is a further requisite of a magnetometer null detector. With the vacuum tube voltmeter it was possible to detect a minimum with a background 20 db. above the signal level.

In spite of its many good features, the vacuum tube voltmeter cannot be conveniently used for declination measurements. The technique used for declination measurements requires the null indicator to be near the magnetometer where it can be observed as the magnetometer is slowly rotated about a vertical axis. The magnet in the voltmeter would distort the Earth's field in the neighborhood of the magnetometer. An additional requirement of a null detector is, therefore, that it be non-magnetic. For declination work either a cathode-ray oscilloscope or a piezoelectric telephone receiver may be used. A telephone receiver in connection with the human ear has a logarithmic response and therefore possesses the property of high sensitivity near the null voltage point. In the range of frequencies used in this work (500-1000 cycles per second) the ear happens to be most sensitive. The ear, being logarithmic in amplitude response,[15] is not well suited to detecting a null in the presence of random background noises. However, in the presence of harmonic voltages, it is capable of selectively detecting null in the fundamental frequency.

The cathode ray oscilloscope has a linear voltage response, but because it cannot be overloaded by a high voltage, it serves almost as well as a logarithmic device. With it, one can work against random background voltages of nearly the same magnitude as the vacuum tube voltmeter. This instrument can be made non-magnetic and may, therefore, be

[15] H. Fletcher, Speech and Hearing, 1st ed.

used in declination measurements. The cathode-ray oscilloscope has one disadvantage in that it requires high auxiliary voltages. Aside from this, it fulfills the functions of a null detector admirably. In concluding the study of null detectors, it may be well to mention the rectifier instruments. All of these devices display a parabolic response possessing an increasing sensitivity with increasing voltage. Such a behavior makes null detection tedious, - any small voltage increment near the top of the scale throws the needle of the galvanometer violently off the scale. Furthermore, the sensitivity of detection near the null point approaches zero. Instruments possessing parabolic response are, therefore, undesirable as null detectors.

In a preceding section, mention was made of a technique for compensating the output of the turbo-inductor for speed variation. This technique will now be described.

From equation 4, the mean square thermal agitation voltage is seen to be proportional to the frequency band width. Theoretically, this voltage could be reduced to a very small magnitude if a sharply tuned filter passing only few cycles were used. The inherent speed instability of our turbines, however, sets a mechanical limitation to carrying this reduction very far. Speed fluctuations in turbines occur as a result of aerodynamic turbulence in the blades of the rotor, or of variations in bearing friction or of pressure variations in the air supply line. A band between 500 and 1000 cycles per second was selected as being most feasible for a measuring instrument of this type.

The filter which passes the above-named band of frequencies, consists of two sections connected in tandem, one a low pass unit cutting off at 1000 cycles per second, and the other a high pass unit attenuating all frequencies below 800 cycles. Following these filters is a terminating resistance which is used to adjust the characteristics of the band pass filter response curves. It is generally known that the response of any impedence network for various frequencies is affected by the terminating impedences. In an experimental analysis it was found that by varying the terminating resistance as well as the shunt condensers C_{12}, C_{13} (Figure 7) in the amplifier, a response curve could be found such that the sensitivity of the complete detector becomes independent of the speed of rotation of the inductor over a rather wide range of frequencies.

Curves in Figure 9 show several filter response curves obtained by impressing a constant sinusoidal voltage from an electronic oscillator upon the amplifier input and reading the output meter. These readings are repeated for various frequencies yielding the response curves illustrated. If now this same network be connected to the turbo-inductor and the response at different speeds be desired, the response for a particular speed w is given by

$$E = R(w)\, e_s$$

where R(w) is the response coefficient of the filter network as shown by the curves in Figure 10 and "e_s" is the emf induced in the inductor. The output in root mean square volts is given by

$$E = \frac{AHM\omega^2 R(\omega)\cos\alpha}{\sqrt{2}\; Z \times 10^8} \qquad \text{Equation 9.}$$

$$\text{or } \log E = \log K + 2\log\omega + \log R(\omega) \qquad \text{See Equation 8.}$$

Now, for "E" to be constant over a range of ω it will be sufficient for log E = k where k is a constant.

Therefore,

$$\log R(\omega) = K' - 2\log\omega$$

The curves of Figure 10 are plotted with $R(\omega)$ as a function of ω on log-log coordinates. For the output of the detector to remain constant over a range of ω, it will merely be necessary to select a portion of the $R(\omega) - \omega$ curves having a slope of -2. It will be seen that the response curve for the amplifier and band pass with a 25,000Ω terminating resistance displays a range 200 - 830 cycles having a slope of -2. In this range the overall response of the detector circuit should be independent of speed. Upon measuring the overall response of inductor, amplifier and band pass filter, it was discovered that the range between 600 - 800 cycles per second possessed an even flatter characteristic than could be expected from the curves in Figure 11. This behavior can probably be attributed to the inductance in the secondary of the "magnetic mirror".

During the experimental work involved in this research, the speed of the turbine was measured by an oscilloscope comparing the inductor voltage with a known frequency. This was accomplished by introducing the known frequency onto the horizontal plates and the inductor frequency onto the vertical plates of the cathode-ray oscilloscope. By varying the known frequency, a one-to-one correspondence as evidenced by the corresponding Lissajou figure could be obtained. When using the magnetometer as a portable field instrument, it became necessary to determine the operating frequency without resorting to an oscillator and an oscilloscope. Since the exact frequency need not be known, it suffices for one to merely adjust the frequency to some value within the operating range 500-1000 cycles. By making use of the output response characteristic, it is possible to place the operating range in the center of the plateau, Figure 11. The technique of this operation consists in accelerating the inductor until the output meter comes to a maximum, drops off slightly and holds this minimax as the frequency increases. If it should happen that the turbine exceed the 1000 cycle upper limit, the operator is warned of this fact by a sudden decrease in output beyond 1030 cycles per second. By this relatively simple technique, one is easily able to operate the turbo-inductor at the operating frequency of 750 cycles.

THE NULL FIELD SOLENOID

Except in areas of geological discontinuity, the magnetic field of the Earth is extremely uniform. S. Chapman[16] has computed the gradients of the field and found the maximum gradient to be $\frac{\partial H_r}{\partial r} = 2.8 \times 10^{-4} \cos \theta$ gammas per centimeter. In order to obtain an absolutely null field in the region of the inductor, it is necessary that the null field coil be designed to produce a very uniform field. If the null field is not uniform over the area of the inductor, it will be possible to bring the resultant field to zero only at the center of the inductor. On each side of the center there will exist a small field due to the gradient of the null field. The rotation of an inductor in a gradient field results in the generation of harmonic voltages. These harmonic voltages introduce a background signal in the detecting circuit, which limits the available amplification by overloading the amplifier. By narrowing the band of frequencies passed by the detector circuit, it is possible to reduce these harmonics in the output by a factor of 40 db. If the ratio of fundamental harmonics happens to be as large as 40 db. the amplification will be limited to 50 or 60 db. If the highest precision of detection is desired, it is

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[16] S. Chapman, The Space Gradients of the Earth's Magnetic Field, Terr. Mag. and Atmos. Elec. 41, 2, 127 (1936).

therefore necessary to reduce the harmonic generation by operating the inductor in a uniform field.

In the introduction to this paper, two circuit configurations yielding very uniform fields were mentioned. These magnetic elements were the Gaugain-Helmholtz coil and the long solenoid. In many of the magnetometers described in the literature, the Gaugain-Helmholtz coil was used to produce the field. In some cases the selection of the double coil was dictated by the mechanical features of the inductor drive; in other cases the logic in selecting the double coil is not obvious. When constructing a coil from whose physical dimensions the field is to be computed, a great deal of effort is expended on adjusting and measuring each of the dimensions which enters in the final computation of that field. In the case of the Gaugain-Helmholtz coil having a finite length of winding at least four dimensions must be known to primary accuracy. Not only must the dimensions be determined, but if there are variations in the dimensions, corrections for the field must be applied for each of the variations. It takes no great imagination for one to foresee the large expenditure of time and effort to construct a Gaugain-Helmholtz coil to a precision of 1 part in 10,000 or better.

In the case of a long solenoid, however, the number of dimensions required to primary accuracy is only one, the overall length of the winding. If there should happen to exist constructional inaccuracies, it is merely necessary to measure the pitch variations of the wires and correct the field for these variations.

CONSTRUCTION OF THE PRECISION SOLENOID

For solenoids of high precision the most generally used material for the supporting form has been marble. It possesses a low coefficient of thermal expansion, it is easily worked and has a high dielectric strength. Further, the magnetic susceptibility of selected marble is very low. Marble is, however, subject to plastic flow over a long time period. It is for this reason that magnetic field standards are now being constructed of a more stable material.

For the solenoid used in this research, a piece of commercial Pyrex pipe, originally intended for piping in breweries, was ground on the outside to a nearly cylindrical shape. Although it has a coefficient of expansion nearly three times that of the best marble, and despite the difficulty of machining it, Pyrex does possess a high dielectric strength and a consistently low magnetic susceptibility.

Into the cylinder of Pyrex was turned a shallow V-shaped groove. The groove was approximately 0.012 cm. wide and of such a depth that the 0.0072 cm. wire used for the solenoid was about 0.002 cm. below the surface of the glass cylinder. This construction was chosen to permit the level used for orienting the solenoid in the Earth's field to rest on the glass and not on the wires. The grooving process proved somewhat tedious because the diamond tool used in the work occasionally broke off, requiring regrinding. In connection with this thread, it may be mentioned that a very fine pitch was chosen for the solenoid in order to reduce the radial components of the field at the center. These components are given by Smythe as

$$H_x = i\int_{\phi_1}^{\phi_2} \frac{\tan\beta(\cos\phi + \phi\sin\phi)\,d\phi}{a(1+\phi^2\tan^2\beta)^{3/2}}$$

$$H_y = i\int_{\phi_1}^{\phi_2} \frac{\tan\beta(-\phi\cos\phi + \sin\phi)\,d\phi}{a(1+\phi^2\tan^2\beta)^{3/2}}$$

It is readily seen that both H_x and H_y approach zero as β approaches zero. Another reason for selecting a small pitch lies in the fact that in precision coils there often exists an uncertainty in the true radius of the coil. This effect is especially important in those coils wherein the radius must be known to primary precision. The uncertainty in determining the geometric radius may be small, but it is the radius on which the current flow occurs which must be used in computing the field. There are two factors which cause a difference in the geometric radius and electrical radius. The first results from the technique used in winding such coils. When a wire is bent around a radius the area outside the neutral axis of bending is elongated and the area inside the axis is compressed. This asymmetrical distortion results in a difference in conductivity of the copper wire on each side of the neutral axis. Dr. Frank Wenner[17] of the Bureau of Standards states that the asymmet-

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[17] Private communication.

rical conductivity can be as high as 4% of the conductivity of the copper. The second cause for the divergence in the geometric radius and the electrical radius of the coil results from the repulsion of electrons by the field of the neighboring turns. This force is such as to drive the electrons toward the outer edge of the wire.

The two effects tend, therefore, to cancel each other, though they cannot in general be expected to annul each other completely. It may be possible by the proper choice of parameters to bring about a substantial reduction in the effect for one particular current flow in the coil. In long solenoids, however, the radius of the coil is a very weak parameter in the field calculation, and the effects of asymmetrical conductivity are therefore of small importance. This uncertainty has been further reduced by using a fine pitch and winding the coil of a fine wire. Since this wire is 0.0075 centimeters in diameter, the uncertainty in the radius cannot be greater than this value.

The wire used in winding this solenoid was made of a beryllium copper alloy. Its properties, particularly adapting it to the construction of magnetic field standards, are its high elastic limit and its low magnetic susceptibility. It is desirable to wind the wire under a relatively high stress in order that the differential expansion between the coil form and the wire should not become so great as to allow the wire to leave its grooves.

The difficulty of working Pyrex required a compression chuck type of mounting at each end of the solenoid. On one of these compression chucks was mounted the base plate and its levels. The top chuck of the solenoid is fitted for mounting the turbine assembly. This mounting is shown in Figure 12. Bands at each end of the solenoid serve to fasten the ends of the solenoid wires.

The turbo-inductor mounting is shown in the photograph (Figure 13). The turbo-inductor is fitted to a split block of plastic having a cylindrical hole cut into it. Into this hole is fitted the inductor unit. To permit the inductor axis to be adjusted perpendicular to the axis of the solenoid, the plastic block is made adjustable on a rocker. The two thumb screws at the top of the mounting serve as a fine motion adjustment of the orientation. It may be stated here, that this orientation may best be accomplished by using the electrical null circuit itself as an indicator of perpendicularity of the axis. When the two axes are perpendicular to each other, the voltage generated by the inductor is a minimum.

Another problem resulting from differential temperature expansion occurs in the mounting of the turbo-inductor within the solenoid. Because the field within the solenoid is so uniform, the inductor need not be centered in the solenoid to high precision. As a matter of fact, axial variations of as much as one centimeter cause only changes of one gamma in the measured field. For this reason, the inside of the solenoid did not require precision grinding. The inductor is positioned within the solenoid by three phosphor bronze springs shown in Figure 13. These springs permit differential expansion of the plastic tube used for the turbine support and the Pyrex solenoid. These springs permit axial expansion as well as radial expansion of the mounting tube.

The complete magnetometer unit on a tripod arranged for vertical intensity measurements is shown in Figure 12. The tripod is rotatable about a vertical axis. On this axis the magnetometer can be rotated until the axis of the inductor is in the plane of the magnetic vector. With this orientation only the vertical component of the Earth's magnetic field is effective in inducing a voltage in the inductor. This voltage may be brought to a minimum by adjusting the current in the solenoid.

SOLENOID MEASUREMENTS AND COMPUTATIONS

The pitch of the solenoid being a primary parameter for the calculation of the field in a solenoid, it was necessary to make a direct comparison of the pitch against a linear standard of length.

The technique developed for the measurement on the precision solenoid consisted in mounting the solenoid on the axis of a lathe. The solenoid was leveled and centered in the lathe. One element of the solenoid was selected for the measurements. For extremely high precision one should measure the pitch along more than one element of the solenoid, but for this coil where a precision of only 1 : 20,000 was desired, the fine pitch made unnecessary the investigation of the drunkedness in the thread.

The actual measurements were made by a micrometer microscope, mounted on the lathe carriage as shown in Figure 14. Since the microscope has a screw of only 5.0 centimeter length it was necessary to measure the position of the wires on the solenoid in a stepwise fashion. The actual measurements consisted in measuring the position of each side of the wire. The pitch variation was measured for every 20 turns. These variations are plotted in Figure 15. The total length of the solenoid was obtained by summing the 5.0 centimeter intervals. By this method, not only were the variations and total length obtained, but also the number of turns in the solenoid.

The micrometer microscope was then calibrated against a Geneva Society International Meter Bar. Corrections for the error in the microscope were applied to the length of the solenoid.

The results of three independent total length measurements, (Table I), agreed to 0.0006 cms. This value represents the maximum deviation rather than the probable error existing in the length measurement. The uncertainty in the total length is represented by the deviation given above.

The radius of the solenoid was measured with a screw micrometer, since it is not required to primary precision.

The measured constants of the solenoid are given by

Radius, a	4.351 $\pm$ 0.003 cms.
Length at 28° C.	61.012 $\pm$ 0.003 cms.
Turns	1201

From the above values the field at the center of the solenoid may be computed from the well-known relation

$$H = 2\pi n i (\cos\phi_2 - \cos\phi_1)$$

ϕ is the angle subtended at the center by the end coils of the solenoid.

The constant for the coil in gammas per milliampere of current resulting from the calculation is 2448.86. This result must be corrected for the effect of pitch variations $\triangle$ and for temperature.

The coefficient of linear expansion of Pyrex is given by the Corning Company as 3.6×10^{-6} per degree centigrade. Using this value for the temperature coefficient of the solenoid, the coil constant at any temperature t becomes

$$H = \left[2448.86 + \Delta\right] \left[1 - 3.6 \times 10^{-6} \, (t - 28)\right]$$

or at 20° C.

$$H = \left[2448.94 + \Delta\right] \quad \text{gammas per milliampere.}$$

CORRECTION OF THE FIELD AT THE CENTER OF A SOLENOID HAVING VARIATIONS IN PITCH

The technique for lapping solenoid to a uniform pitch is a standard practice in precision measurements.[18] Where extreme precision is not required, a solenoid is constructed by turning a thread into a cylinder of an insulating material and winding into the thread a wire through which the current passes. In this method one may rely on the lead screw of the lathe in which the thread is turned, or if a more precise result is desired, the pitch variations may be measured on a comparator. The average pitch for the solenoid may be measured to high precision by measuring the total length of the winding and counting the turns on the solenoid. Where there exist pitch variations in the solenoid, the magnetic field computed from the average pitch is not the correct field existing in the solenoid. For this purpose, where the pitch is not uniform, there has been developed in this research a mathematical device for computing the correction to the field. In particular, the technique has been used to compute the correction to the field at the center of a long solenoid.

Suppose the pitch on the solenoid be so small that the solenoid may be considered to be made of a large number of

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[18] H. L. Curtis, McGraw-Hill, 1937.

circular coils. The field on the axis of one of these coils is

$$H = \frac{2\pi i a^2}{(a^2+x^2)^{3/2}}$$ Equation 10.

where a is the radius of the coil and x is the distance to the plane of the coil from field point.

The technique depends on computing the change in the axial field when the coil is displaced a small distance δx, that is, when there exists a pitch error of δx in that coil, the pitch error being defined as the displacement of the coil from the position computed from the average pitch for the solenoid. The change in the field is to be given by differentiating Equation 10

$$\frac{\partial H}{\partial x} = \frac{-6\pi i a^2 x}{(a^2+x^2)^{\frac{5}{2}}}$$

Equation 11.

If these variations in axial field be summed for all the coils possessing pitch variations there results

$$\Delta H = \frac{-6\pi i a^2 n}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{x\,\delta x\,dx}{(a^2+x^2)^{\frac{5}{2}}} = \frac{-6\pi i n}{a^2 L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\frac{x}{a}\,\delta x\,dx}{\left[1+\left(\frac{x}{a}\right)^2\right]^{\frac{5}{2}}}$$

Equation 12.

The value of the above integral may be readily obtained graphically to sufficient accuracy if the pitch variation be plotted against the position on the solenoid. The pitch variations for the solenoid used in this work are shown in Figure 15. If the resulting pitch variation be multiplied by the ordinates of the function,

$$f\left(\frac{x}{a}\right) = \frac{x/a}{\left[1+\left(\frac{x}{a}\right)^2\right]^{\frac{5}{2}}}$$

each ordinate will represent the function in the integrand above. See Figures 16 and 17. The area under the represented curve will be the value of the required integral. For the coil used in this work whose pitch variations have been illustrated the area amounts to 3.8×10^{-3} cm.^{2} Using the values given previously for the coil constant the correction to the solenoid constant is -0.85 gammas per milliampere. The corrected coil constant at 20° C. is then 2448.09 gammas per milliampere.

Since the pitch was not measured for each turn on the solenoid, the correction may be somewhat in error. It cannot, however, be in error by a very large amount because the total correction to the coil is quite small. In this connection, it may be well to point out that only the coils in the immediate vicinity of the center of the solenoid contribute to the correction, the effect of more distant coils being negligible. It might be well, therefore, to emphasize the importance of measurements in the neighborhood of the center. For high precision work it is necessary, therefore, that only the pitch near the center be uniform. It is, in fact, at the center of a long solenoid that the pitch becomes most uniform after just a small amount of lapping.

THE EFFECT OF VARIATIONS OF THE RADIUS UPON THE FIELD

The axial field at a point x from the plane of a circular coil of radius a is given by

$$H = \frac{2\pi i a^2}{(a^2+x^2)^{\frac{3}{2}}}$$

The change in the field resulting from a variation in radius is obtained by differentiating the above

$$\frac{\partial H}{\partial a} = 2\pi i a \frac{(2x^2-a^2)}{[a^2+x^2]^{\frac{5}{2}}}$$

If the radius of the solenoid varies from point to point, the field at the center of the solenoid will differ from the field computed on the basis of a uniform solenoid by an amount

$$\Delta H = 2\pi i \int_{-\lambda}^{\lambda} \frac{2x^2-a^2}{[a^2+x^2]^{\frac{5}{2}}} \, a n \delta a \, dx$$

where λ is the half-length of the solenoid and n is the number of turns per unit length of the solenoid. n dx is the number of turns possessing a variation from the mean radius of δa. The field correction may be written

$$\Delta H = 2\pi n i \int_{-\lambda}^{\lambda} \frac{2(\frac{x}{a})^2-1}{[1+(\frac{x}{a})^2]^{\frac{5}{2}}} \frac{\delta a}{a^2} dx$$

The above integral may be evaluated by the same technique as that used to determine the correction for pitch variations. The function

$$g\left(\frac{x}{a}\right) = \frac{2\left(\frac{x}{a}\right)^2 - 1}{\left[1 + \left(\frac{x}{a}\right)^2\right]^{\frac{5}{2}}}$$

may be used as a weighting factor to multiply the radius variations. If each radius variation be weighted by the value of $g\left(\frac{x}{a}\right)$ corresponding to the distance of the radius variation from the center of the solenoid, the area under the curve of the weighted radius variations will be the value of the required integral.

If the radius variations are so small that the correction term will be negligible, it will suffice for the magnitude of the correction to be computed. The function is seen to be an even function. In general, when a cylinder is turned in a lathe the cause of variations in the radius from point to point lies in the fact that the cylinder was not parallel to the axis along which the cutting tool moves. Such an error of alignment would result in a cone being developed. The pitch variation of a cone is linear. Consequently, the integral expressing the field correction will consist of an even function multiplying an odd function. The value of such an integral between the same limits $-\lambda$ and λ will be zero. It may happen that the source of error in cutting the thread for the solenoid lies in other phases of the cutting process which may lead to an even function for the radius variations. One possible source of error arises in the deflection of the solenoid form by the cutting tool forces. These forces were necessarily low because the diamond cutting tool required extremely small cuts. The variation in radius of the present solenoid was not greater than 0.0001 cm. Consequently, the correction term due to long wave variations (extending over several centimeters of solenoid length) can be said to be less than $\frac{0.0001}{a^2} = 1:200,000$. Short wave variation would tend to cancel, since the value of $g\left(\frac{x}{a}\right)$ is nearly equal over the short wave.

It must not be inferred from this analysis that the radius must be known to the precision of the variations in radius. On the contrary, the radius need not be known to better than 1/100 of the precision required of the final field measurement for a solenoid having a slenderness ratio of 1/7.

COMPUTATIONS OF THE HETEROGENEITY OF THE NULL FIELD

In a preceding section, it was stated that the non-uniformity of a magnetic field in which an inductor rotates determines the harmonic content in the induced voltage. The harmonic voltages limit the attainable amplification by overloading the amplifier. A uniform field not only reduces harmonic generation, but also permits a larger tolerance in the position of the field at the center of the solenoid.

The axial field within a solenoid is given at any field point by

$$H = 2\pi n i (\cos\phi_2 - \cos\phi_1)$$

where ϕ_1 and ϕ_2 are the angles subtended at the field point by the ends of the solenoid.

The field at the center is given by

$$H_0 = \frac{4\pi n i}{\sqrt{1 + \frac{a^2}{\lambda^2}}}$$

where a is the radius and λ the half-length of the solenoid.

Suppose it is desired to know the field at a field point displaced distance δ from the center

$$H_\delta = 2\pi n i \left\{ \frac{1}{\sqrt{1 + \left(\frac{a}{\lambda - \delta}\right)^2}} + \frac{1}{\sqrt{1 + \left(\frac{a}{\lambda + \delta}\right)^2}} \right.$$

Equation 13.

If Equation 13 be expanded by the binomial theorem, remembering that $\frac{a}{\lambda} = \frac{1}{7}$ for the solenoid used in this work,

$$H_0 = 4\pi ni \left\{ 1 - \frac{1}{2}\left(\frac{a}{\lambda}\right)^2 + \frac{3}{4}\left(\frac{a}{\lambda}\right)^4 + \cdots \right\}$$

$$H_\delta = 2\pi ni \left\{ 1 - \frac{1}{2}\left(\frac{a}{\lambda-\delta}\right)^2 + \frac{3}{4}\left(\frac{a}{\lambda-\delta}\right)^4 + \cdots \right. \left. 1 - \left(\frac{a}{\lambda+\delta}\right)^2 + \frac{3}{4}\left(\frac{a}{\lambda+\delta}\right)^4 + \cdots \right\}$$

Equations 14 and 15.

The change in the magnetic field in moving the inductor from the center to a point δ on the axis is

$$\Delta H = H_0 - H_\delta = \pi ni \frac{a^2}{\lambda^2}\left[6\left(\frac{\delta}{\lambda}\right)^2 + 10\left(\frac{\delta}{\lambda}\right)^4 + \cdots\right]$$

Equation 16.

For small displacements δ from the center only the first term in the series need be used for estimating the variation of the field,

$$\Delta H = 6\pi ni \left(\frac{a}{\lambda}\right)^2 \left(\frac{\delta}{\lambda}\right)^2$$

Equation 17.

No great error is made in computing the relative change in the field if the expression for an infinitely long solenoid is used

$$\frac{\Delta H}{H} = \frac{6\pi ni \frac{a^2}{\lambda^2} \frac{\delta^2}{\lambda^2}}{4\pi ni} = \frac{3}{2} \frac{a^2}{\lambda^2} \frac{\delta^2}{\lambda^2}$$

Equation 18.

Suppose the relative change in field in a solenoid having a half-length of 30 centimeters and a slenderness ratio $\frac{a}{\lambda} = \frac{1}{7}$ is desired for a point one centimeter off the center

$$\frac{\Delta H}{H} = \frac{3}{2}\left(\frac{1}{49}\right)\left(\frac{1}{900}\right) = \frac{3}{10^5}$$

This calculation shows that the field is very uniform even as far as one centimeter from the center. For even as large an error as one millimeter in axial misplacement of the inductor the error in the field measurement is only 3 parts in 10^7. In the construction of a magnetometer using a solenoid for a null field, it is, therefore, unnecessary to precisely center the inductor.

The effect of inhomogeneity of the field upon the generation of harmonics will now be investigated. The general form of Faraday's law will be used $e = \vec{v} \times \vec{B} = \vec{v} \times \vec{H}$ for a permeability of unity.

The field H at the center of the solenoid will be considered adjusted to zero. The fields remaining in the neighborhood of the center will be given by

$$\Delta H = 6\pi n i \left(\frac{a}{\lambda}\right)^2 \left(\frac{\delta}{\lambda}\right)^2$$ Equation 19.

, where $\delta = R \cos \omega t$

When the radius of the inductor is R, the area A and the angle of the inductor with the field ωt, the velocity of cutting the field is given by

$$v = \omega R \cos \omega t$$

The electromotive force induced in one side of the inductor will be

$$e = A\omega \cos \omega t \cdot 6\pi n i \frac{a^2}{\lambda^2} \frac{R^2 \cos^2 \omega t}{\lambda^2}$$

There will, of course, be a similar contribution from the other side of the inductor. The ratio of harmonic to fundamental peak voltage is then

$$\frac{e}{E} = \frac{3\pi n i A R^2 \omega}{4\pi n i A \omega} \left(\frac{a}{\lambda}\right)^2 \left(\frac{1}{\lambda^2}\right) = \frac{3a^2}{\lambda^4}$$ Equation 20.

The largest ratio of harmonic to fundamental is, therefore, 6 parts in 10^5 for an inductor of one centimeter radius. For an inductor having the form of a circle, the ratio will be even less than this value where the coil was assumed to have a rectangular cross section. The ratio computed above corresponds to a harmonic 116 db. below the fundamental. Such a ratio permits an amplifier having a gain of 120 db. to be used for amplifying the induced voltages.

In addition to axial gradients, the magnetic field in a solenoid also possesses a gradient in the radial direction. The general relation for the axial field in a solenoid is given by[19]

$$H = 2\pi n i \left[\frac{\delta + \lambda}{r_1} = \frac{\delta - \lambda}{r} + \frac{3}{4} a^2 y^2 \left\{ \frac{\delta + \lambda}{r_1{}^5} - \frac{\delta - \lambda}{r_2{}^5} \right\} + \cdots \right]$$

Equation 21.

For a point at the center of the solenoid $\delta = 0$ and

$$H_0 = 4\pi n i \left[\frac{\lambda}{r} + \frac{3}{4} a^2 y^2 \frac{\lambda}{r^5} + \cdots \right]$$

Equation 22.

And the change in axial field for a radial displacement is

$$\Delta H = 3\pi n i \, a^2 y^2 \frac{\lambda}{r^5}$$

Equation 23.

For a long solenoid $\lambda = r$ approximately and the

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[19]Andrew Gray, Absolute Measurements in Electricity and Magnetism, Macmillan, 2nd ed. (1921).

relative change in field becomes

$$\frac{\Delta H}{H} = \frac{3}{4}\,\frac{a^2 y^2}{\lambda^4}$$

which is one half the previously computed variation in the axial direction. The limits of construction and the harmonic to fundamental voltage ratios will be correspondingly reduced. The field can be said to be of such a homogeneity that no great precision is required when the inductor is being mounted within the solenoid. In addition, the solenoid's field is such as to generate harmonic voltages at least 100 db. below the fundamental voltages.

The generous tolerances permitted by the uniform field of a solenoid were a distinct advantage in that the mounting of the turbo-inductor did not require compensation for temperature effects.

EFFECT OF LEAD WIRES

In any precision solenoid the effect of the lead wires connecting to the ends of the solenoid must be considered in the field computation. One method for connecting to a solenoid consists in bringing one lead wire from one end of the solenoid to the other end and connecting a twisted cable to the two leads at the one end. The lead wire from the far end is brought to the cable end parallel to the axis of the solenoid. Consequently, no correction need be added to the coil constant. There is, however, a component of the magnetic field from the lead wire which will be cut by the inductor unless the inductor axis is perpendicular to the plane containing the lead wire and the solenoid axis. If this condition does not obtain, the field which is cut by the inductor is

$$H = \frac{2i}{r} \sin \theta \qquad \text{Equation 24.}$$

θ = error in setting the inductor perpendicular to the plane of the lead wire and solenoid axis.

The effect of the lead wires may be still further reduced if there is introduced another lead wire diametrically opposite the first having the same resistance as the first. The two wires would consequently carry an equal current in the same direction. Theoretically, the field at the center of the inductor would be zero. However, if the wires were not of exactly the same resistance or symmetrically spaced from the axis of the solenoid, there will be a residual field given by

$$H = 2\left(\frac{i_1}{r_1} - \frac{i_2}{r_2}\right)\sin\theta$$

Equation 25.

Comparing this equation with equation 24 it will be seen that the residual field is very much smaller and can be made even by the crudest construction to be of negligible proportions. This device was used to permit connecting a cable to the magnetometer developed as part of this research. It was found experimentally that the two lead wires were symmetrical enough so that θ could approach 30° before a background voltage could be detected on the null detector.

ORIENTATION OF THE MAGNETOMETER IN THE EARTH'S FIELD

To measure the vertical component of the Earth's field the magnetometer is fitted with a base which mounts on a leveling tripod. Figure 12 shows the magnetometer complete with levels for orienting the solenoid vertical. The tripod is constructed to permit rotation of the solenoid about a vertical axis. With the level vials on the instrument, it is easily possible to set the axis of rotation vertical to 5". In operation, the magnetometer is rotated on the vertical axis until the inductor's axis is in the magnetic meridian. This azimuth is determined electrically to a high precision by the minimum in induced voltage when the inductor cuts only the lines of the vertical component. The current required to bring the induced voltage to an absolute minimum is measured and the instrument turned through 180° about the vertical axis and a second value of current is determined. The average of the two currents is a measure of the vertical component of the field. For a small error ϕ in orientation the measured value of the vertical component is

$$Z_m + Z \cos\phi + H \sin\phi$$

or $Z_m = Z + H\phi$

where Z = 50,000 gammas and H = 20,000 gammas the error in resulting from an orientation error of 5" amounts to 0.4 gammas.

No provision has been made in the present instrument for using it as a horizontal component instrument, but it has been tested as such with the cooperation of the United States Coast and Geodetic Survey. This work will be described later in the paper. When used for horizontal measurements, the instrument is rotated about its axis until the inductor axis is vertical. The fulfillment of this condition is determined by a minimum induced voltage in the null detector. The precision of leveling required for a certain precision of final result is obtained from the relation

$$H_m = H\cos\phi + Z\sin\phi$$

or

$$H_m = H + Z\phi$$

In the fields stated in the example above, the value of O must be kept to 3" if the error is to be kept to one gamma.

For making horizontal measurements, one has the choice of taking a current reading, rotating the solenoid 180° on its axis repeating the current, or of taking two current readings with the solenoid rotated through 180° on a vertical axis between readings. The latter technique is to be preferred for precision work, since errors in leveling the solenoid tend to cancel. However, where the true magnetic axis of the solenoid is determined, rotation about the solenoid's axis yields good results and at the same time permits a more economical design.

THE CURRENT MEASURING AND CONTROLLING CIRCUIT

The schematic circuit of the magnetometer is shown in Figure 8. The current flowing through the solenoid causes a voltage drop in the precision resistor. The voltage drop across the resistor is compared with the potential of a standard cell by means of a potentiometer having a precision of 0.01%. In field practice it would be advisable to use three standard cells in a heat-insulated container. By using three cells, one would have the means for determining an error existing in any one of the standard cells.

To control the current in the solenoid the use of a single rheostat must be avoided. Its hyperbolic behavior is such that it becomes exceedingly sensitive to adjust at high currents. No better current control has been found than a standard four decade resistance box. Course adjustments are made on the high resistances, and the final adjustment on the unit resistance decade.

The resistance of the solenoid is 6,000 ohms requiring 180 volts to cause 30 milliamperes to flow through its windings. Since a field of 50,000 gammas requires about 20 milliamperes, a radio plate supply battery is ample for the source of current. In future modifications of the instrument, the same battery can be used to furnish the plate current for the amplifier and tubes.

The voltage used for the null field may lead one to suspect leakage of current between turns of the solenoid. The voltage difference existing between turns is, however, only $\frac{180}{1201}$ or 0.15 volts. The spacing between turns being 0.05 cm., it is readily seen that the glass insulation between turns is subject to a gradient of 3 volts per centimeter. It is important though, that the surface of the solenoid be kept clean. It would be well to provide an outside wover for the solenoid to protect it from dirt and moisture and to prevent damage to the lead wires.

LIMIT OF FIELD MEASUREMENTS AND SOURCES OF ERROR

In a previous research[20] the author had determined the limit of sensitivity of null detector as set by the statistical voltage fluctuations in the input circuit of the amplifier. The results of that work showed that the smallest detectable field would have a value of approximately one gamma. Since the same inductor is being used in the present problem as was used in the work referred to above, the same limit of measurement can be expected in the present instrument.

In order to experimentally determine the limit of sensitivity, the output voltage of the null detector was determined for various increments of current through the solenoid. This data is plotted in Figure 18. The average slope of the curve is 0.01 $\frac{\text{ma}}{\text{volt}}$.

The sensitivity in terms of gammas per volt may be obtained from the slope by using the coil constant of the solenoid 2448.1 gammas per milliampere. The resulting sensitivity is about 25 gammas per volt. On the vacuum tube voltmeter used it was possible to adjust the voltage to within one-tenth of a volt. The limit of field measurement of a single observation is therefore 2.5 gammas.

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[20] Master's thesis, University of Maryland, 1940.

There are a number of factors which may affect the above limit of field measurement of this instrument. If the background voltages are so high that the amplification must be reduced in order to permit the null detector to remain on scale, the sensitivity will be reduced in a degree corresponding to the reduced amplification. The background voltages consist of harmonics generated by a gradient in the null field, of voltages generated electrostatically when the inductor becomes charged and rotates past the connections of the secondary of the "magnetic mirror", and of random voltages due to wobble of the inductor in its bearings.

The first of the above-mentioned background voltages has been studied and shown to be small. The contribution of the electrostatically generated voltages to the background is quite large and amounts to about one volt in the output unless special precautions are taken to ground the inductor and to shield the secondary. Grounding the inductor and shielding the secondary reduces the electrostatically generated voltages to less than 0.1 volt in output. It would be possible to entirely eliminate this source of background by winding the secondary in two symmetrical banks so that the electrostatic voltages annul themselves in the secondary of the "magnetic mirror".

The third source of background is one which results from wear in the bearings. For an absolute minimum to be set in the detector voltage requires that the inductor rotate in the plane of the magnetic vector and that its axis be perpendicular to the solenoid axis. If the bearings do not constrain the inductor to rotate on a definite axis, there will be voltages generated which follow the random oscillations of the inductor on its axis. These voltages are particularly annoying when making measurements, because they tend to obscure the null point. In the experimental work connected with this paper, it was noticed that the random voltages in the output amounted to 0.2 volt when the bearings showed some wear. The solution to this particular phase of the problem would include an improved bearing, and possibly some method for decreasing the amplitude of the fluctuations of these voltages. Increasing the moment of inertia of the inductor about its axis would decrease the amplitude by introducing a gyroscopic restoring torque. Another possible approach would be to average the fluctuations with a long period null detector.

It may be mentioned that if there is enough wear in the bearings of the inductor to permit the inductor to rotate during the period of measurement on an axis not perpendicular to the solenoid axis, there will result an error in that measurement. In horizontal field measurements if the inductor axis is tilted toward the magnetic vector more current will be required for a null than if the axis is vertical. It is for this reason that stress should be placed upon the bearing design of future models of the instrument.

In some positions where the Earth's field is being measured there may exist in addition to the constant field a fluctuating magnetic field having a frequency of the local alternating power supplies. This alternating field also contributes to the background, but it is possible to eliminate this effect by turning off devices drawing current through the power lines in the vicinity of the measuring point. In future models of this instrument, one could introduce an electromagnetic shield around the turbo-inductor to shield it from the alternating fields.

In the preceding paragraph the error in field measurement due to displacement of the axis of rotation of the inductor was mentioned. This error not only results from a displacement of the rotor in its bearings, but also from a misalignment of the turbo-inductor. It was discovered during the Cheltenham tests to be described later, that the technique of adjusting the angle between the magnetic axis of the solenoid and turbo-inductor axis until a minimum in output voltage is reached is not sufficiently sensitive for precise work. It was found that with the solenoid horizontal there resulted a difference in field of 200 gammas when the coil was rotated on its axis through 180°. The technique finally used was to adjust the turbo-inductor axis until the difference in measured values of field upon rotation of the solenoid through 180° was reduced to a very small value. When this was done, the measured values of field agreed quite well with those measured by the magnetographs.

TESTS AT THE CHELTENHAM MAGNETIC OBSERVATORY

OF THE U. S. COAST AND GEODETIC SURVEY

These tests were made possible by a cooperative agreement between the Section of Geophysics of the United States Geological Survey and Division of Geomagnetics and Seismology of the United States Coast and Geodetic Survey. The work at the Magnetic Observatory of the Coast and Geodetic Survey consisted of three phases, horizontal and vertical intensity determinations and declination sensitivity tests.

For the horizontal measurements the null magnetometer was mounted on a 30 centimeter large circle. The mount consisted of a rigid framework of duralumin with two brass Y's for supporting the coil at points corresponding to the Airy points for minimum bending of the solenoid under its own weight. The brass Y's had the edges in contact with the solenoid covered with plate glass strips. A leveling adjustment was provided on one of the Y's. For the actual leveling a precision level having a sensitivity of 1.3" per scale division was converted by a suitable structure into a straddle level which rested directly on the Pyrex solenoid form. Because this solenoid form is not a true cylinder, a position on the solenoid was chosen for the level where the diameters of the solenoid at the straddle points were equal. Such a technique would, of course, reduce errors in determining the geometric axis of the solenoid, but would not compensate for deviations in the geometric and magnetic axes of the solenoid. Such deviations result from disturbance of the magnetic field by magnetic components or from a bent solenoid. If such a deviation existed it would have been possible to detect it by rotating the solenoid through 180° on its axis, taking null current readings at the two orientations. Differences in the current did exist, but it was found they could be reduced to a negligible value by eliminating several magnetic parts and by adjusting the inductor axis to exact perpendicularity with the solenoid's magnetic axis.

The actual procedure for leveling consisted of first adjusting the vertical axis of the great circle in a true vertical. A precision of plus or minus one half of a scale division corresponding to 1.8" of arc was attained in this setting. The solenoid was then leveled by the adjustment on one of the Y's. This leveling could be depended to only 3". The precision of leveling can be estimated from the disagreement of field measurements taken by averaging readings obtained by rotation on the solenoid's axis with those obtained by rotating through 180° on the vertical axis (large circle axis).

The leveling accomplished, there was left only for the solenoid to be oriented in the magnetic meridian and for the inductor axis to be adjusted in the plane of the magnetic meridian. The latter adjustment was made roughly by rotating the solenoid and the turbo-inductor about the solenoid's axis until a minimum in the output voltage was reached. The solenoid was then rotated on the vertical axis of the long circle until the output voltage was reduced to a lower minimum. The current was then varied for another minimum. Another adjustment of the inductor axis and the declination axis sufficed to permit the output voltage to be brought to an absolute minimum by current adjustments only. It may be mentioned here that lacking a slow motion adjustment for rotation about the solenoid axis, the rather rough technique of gently tapping the corner of the base had to be used for securing small angular increments. This procedure was highly unsatisfactory and should be rectified in future developments of this instrument. The declination adjustment was made simple and precise by a tangent screw on the large circle.

With the magnetometer properly orientated in the magnetic field, null current readings were made with the Leeds and Northrup type K potentiometer. A resistor of 100 ohms having a precision of .01% was used to secure the voltage drop measured by the potentiometer. In each set ten readings were taken, each one of which was made with an independent setting of the output voltage to an absolute minimum. The probable error in these current readings multiplied by the coil constant therefore represents the probable error in null field detection.

In Table II are shown the results of the measurement of H by the magnetometer developed in the author's research.

The measurements obtained by the subject magnetometer are compared with the values of H furnished by the Observatory's recording instruments. These instruments have their base line corrected periodically by comparison with the Carnegie Institute of Washington sine galvanometer.[21] This instrument is the present standard for horizontal field intensity. Its coil constant is known to a precision of 3 parts in 100,000.

During the sets 1, 2, 3, and 4 the magnetogram showed a rather calm period in magnetic activity. The results obtained during the time of the calm period can therefore be considered quite dependable. Analyzing the readings by sets, it is seen that the first set has the largest probable error amounting to 0.0004 milliamperes in current, or to plus or minus one gamma in field uncertainty. It is interesting to note that the probable error is ± 0.0003 ma in a single observation. The error in field measurement is then approximately 0.75 gammas. This value agrees quite well with the limit of field measurement determined theoretically from statistical voltage considerations.

Returning again to the first of four sets of readings, it should be pointed out that the agreement of the field measurements obtained from averaging the null currents with the solenoid turned through 180° on the vertical axis indicates that the leveling error amounts to less than one

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[21] S. J. Barnett, A sine galvanometer, cf. Bibliography.

gamma. The averages of sets 1-4 and 2-3 deviate from the averages of sets 1-2 and 3-4 by one gamma at most.

On examining sets 5 and 6, it will be noted that there exists a large divergence in the two average currents. This divergence is positive indication that the axis of the inductor is not perpendicular to the magnetic axis of the solenoid. Adjustment of this axis brought the large divergence of 0.03 ma down to 0.01 ma. In the final computation of the results the readings on sets 5 and 6 should be discarded, because they diverge between themselves, and because set 6 has a large probable error.

Surveying the whole of the horizontal intensity measurements, there are found the following deviations between the observatory values and those measured by the null magnetometer described in this report: 1, 1, 1, 2, -1, and 2 gammas. The mean deviation is one gamma, an error of one part in 18,200. Although the error in the average pitch measurement of the solenoid amounts to no more than one part in 60,000, there does, however, exist a larger uncertainty in the correction for pitch variations. It would be well to redetermine the pitch variations measuring every turn and recomputing the correction of these variations to the coil constant. This measurement need only extend over a length of two diameters of the solenoid on each side of the center.

In Table III are given the data taken with the magnetometer oriented for vertical measurements. Preceding this group of measurements the inductor axis was carefully adjusted until reading taken with the solenoid rotated 180° on its axis and in its normal position agreed very closely. Excepting the first two sets 1-1 and 1-2, the group agrees fairly well among themselves. Admitting the result obtained from the first set to be divergent enough to be rejected, the mean of the observed deviations between the Observatory vertical intensity and that obtained with the null magnetometer is five gammas. On the basis of the error apparently existing in the null magnetometer coil constant as determined in the horizontal intensity measurements, there should be a deviation in the vertical comparison of three gammas. There remain two gammas to be accounted for. The Observatory obtains its vertical intensity from the horizontal value based on the sine galvanometer and from the value of the dip measured with an earth inductor. Since the observation is not a direct one, depending as it does upon two parameters, the two gamma difference might easily be ascribed to an error in the base line of the Observatory. It is noteworthy to remark that this work has independently confirmed the Observatory base line to a precision of five gammas and that assuming the sine galvanometer to be a true standard of horizontal field, the vertical field intensity base line was determined to be correct to 2 gammas with a probable error of one gamma.

The third phase of the work at the Cheltenham consisted in taking a series of declination readings against an arbitrary index. It was the purpose of this experiment to determine the sensitivity of the null magnetometer in the horizontal orientation. The instrument would have been more sensitive as a declinometer if it had been vertical, measuring the direction of the magnetic meridian by the position of null when the inductor axis is in the magnetic meridian. However, the uncertainty in determining the inductor axis is much larger than the uncertainty in the solenoid axis. With the solenoid horizontal the instrument will yield a more precise measurement of declination.

In summary, it must be emphasized that these tests at the Cheltenham Observatory have yielded in addition to the comparison of field values, much data upon which future designs of the null magnetometer can be based. During these tests, in addition to collecting numerical data, considerable thought was given to improving the operation of the instrument. It was discovered that the inductor should be shielded electromagnetically to reduce the background voltage to 3 volts. An improved bearing is needed. The principal change which the author recommends is a redesign of the turbine mount to permit a finer adjustment of the axis of the inductor and such that the adjustment is permanent and not affected by temperature. During the measurements, much time and effort could have been spared the observers if it were possible for the observer making the adjustments to see the output meter. A sensitive non-magnetic indicator is required. Another suggested change would require the tubes in the amplifier to be mounted on a resilient base. It was discovered at Cheltenham that the amplifier responded to vibration and noises.

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APPENDIX

The Calculation of the Mutual Inductance of the Magnetic Mirror

Consider the general case of the mutual inductance between a helix and another helix[a] whose equation is given below parametrically. The axes of the helixes are coincident.

The mutual inductance of any two circuits is given by Neumann's relation[b]

$$M = \frac{1}{c}\int \frac{dl_1 \cdot dl_2}{r_{1,2}} \qquad \text{Equation 1a.}$$

where dl_1 represents a current carrying element on circuit 1

dl_2 represents a current carrying element on circuit 2

$r_{1,2}$ is the distance between the two elements.

The coordinates of circuit 1, the helix, are

$$x = a\,\theta$$
$$y = b\cos\theta$$
$$z = b\sin\theta$$

where "a" is the pitch of the helix

b is the radius of the cylinder on which the helix is wound

θ is the azimuthal angle

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[a] Commonly called a drunken helix.

[b] For setting up the integral for this general case, the author is indebted to Dr. Irwin Roman.

The coordinates of the circuit 2 are

$$\xi = c \cos \theta \cot \alpha + h \phi$$

$$\eta = c \sin \theta$$

$$\zeta = c \cos \theta$$

where

h is the pitch of the second helix

c is the radius of its cylinder

is the angle of drunkenness

ϕ is the azimuthal angle

The distance $r_{1,2}$ is then given by

$$r_{1,2}^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$$

$$r_{1,2}^2 = h^2\phi^2 + a^2\theta^2 + 2ah\phi\theta + 2(h\phi - a\theta)\, c \cos \phi \cot \alpha$$
$$+ c^2\cos^2\phi \cot^2\alpha$$
$$+ c^2\sin^2\phi - 2cb \sin \phi \cos \theta + b^2\cos^2\theta$$
$$c^2\cos^2\phi - 2\, cb \sin \theta \cos \phi + b^2\sin^2\theta$$
$$= h^2\phi^2 + a^2\theta^2 - 2ah\phi\theta + 2(h\phi - a\theta)\, c \cos \phi \cot \alpha +$$
$$c^2\cos^2\phi \cot \alpha +$$
$$c^2 + b^2 - 2bc \sin(\theta + \phi)$$

$$dl_1 = ad\theta,\ -b \sin \theta d\theta,\ b \cos \theta d\theta$$

$$dl_2 = [h - c \sin \phi \cot\alpha,\ c \cos \phi,\ c \sin \phi]\ d\phi$$

$$dl_1 \cdot dl_2 = [ah - ac \sin \phi \cot\theta - bc \sin (\theta + \phi) \quad d\theta d\phi$$

$$M = \int_0^{n(2\pi)} \int_0^{m(2\pi)} \frac{ah - ac \sin \phi \cot\alpha - bc \sin(\theta+\phi)\ d\theta d\phi}{\left[h^2\phi^2 + a^2\theta^2 - 2ah\phi\theta + 2(h\phi - a\theta)\, c \cos \phi \cot\alpha + c^2\cos^2\phi \cot^2\alpha + b^2 + c^2 + bc \sin (\theta+\phi)\right]^{\frac{1}{2}}} \qquad \text{Equation 2a}$$

where n is the number of turns on helix 1 and m is number of turns on helix 2.

This integral cannot readily be evaluated for the general case. It can, however, be reduced to a more simple form by introducing the actual parameters of the magnetic mirror. The pitch of the helix $f(x,y,z)$ was very small, and therefore a can be made $= 0$ to a high degree of accuracy. In actual construction $h = 0$ and the distance $(hO - aO) = K$. Also, introducing these relations into Equation 2a

$$M = \int_0^{2n\pi}\int_0^{2m\pi} \frac{-\,bc\ \sin(\theta+\phi)\ d\theta d\phi}{\left[K^2+2Kc\ \cos\phi\ \cot\alpha + c^2\cos^2\phi\ \cot^2\alpha + b^2 + c^2 + bc\ \sin(\theta+\phi)\right]^{\frac{1}{2}}}$$

also $\cot\alpha = 1$ and $b = c$ approximately.

$$M = \int_0^{2n\pi}\int_0^{2m\pi} \frac{-b^2\ \sin\ (\theta+\phi)\ d\theta d\phi}{\left[(K + b\ \cos\ \phi)^2 + b^2 \left\{2\ + \sin\ (\theta+\phi)\right\}\right]^{1/2}}$$

The above integral can be evaluated by the use of one of two different techniques. The integral may be reduced to an elliptic integral or it may be expanded in a series. Neither of these attacks, however, is as direct as the physical measurement of the mutual inductance described in a preceding section.

Table II

Horizontal Field Measurements
at the
Cheltenham Magnetic Observatory

Date of Test--March 12, 1942

Coil Constant at 20°C. 2448.08 gammas per milliampere.

Null Current Readings, Milliamperes.

Set No.	1	2	3	4
	7.4313	7.4389	7.4294	7.4384
	4300	4389	4292	4376
	4339	4389	4295	4383
	4290	4398	4299	4383
	4306	4395	4312	4376
	4290	4388	4292	4384
	4290	4395	4290	4390
	4288	4388	4298	4381
	4288	4392	4295	4382
	4286	4375	4295	4381
Average	7.4299	7.4390	7.4296	7.4382
Prob. error	.0004	.0001	.0001	.0001

Set No. 1 Top of solenoid to North; turbine inlet up.
Set No. 2 Top of solenoid to North, solenoid turned 180° on its axis, inlet down
Set No. 3 Top of solenoid to South, corresponding set 2, inlet down.
Set No. 4 Top of solenoid to South, corresponding set 1, inlet up.

Set No.	1	2	3	4
Time start	14:54	15:00	15:15	15:45
75 GMT end	14:56	15:03	15:18	15:48
H from observatory magnetograph	18,199	18,199	18,198	18,197

Sets	1-2	3-4	1-4	2-3
Average current	7.4344	7.4339	7.4340	7.4343
H measured by U.S.G.S. Magnetometer	18,200	18,199	18,199	18,200
H from observatory magnetograph	18,199	18,198	18,198	18,198
Deviation	1	1	1	2

TABLE III

Vertical Field Measurements
at the
Cheltenham Magnetic Observatory

Date--March 16, 1942. Null current readings, milliamperes.

Inductor 124 cm. above Bronze Magnetic Station.

Set No.	11	12	13	14	15	16
	22.035	22.006	22.035	22.006	22.037	22.008
	22.032	22.002	22.035	22.006	22.037	22.006
	22.031	22.006	22.040	22.010	22.037	22.011
	22.031	22.006	22.036	22.008	22.037	22.008
	22.033	22.007	22.036	22.009	22.035	22.007
	22.031	22.007	22.038	22.007	22.037	22.007
	22.031	22.004	22.038	22.008	22.037	22.010
	22.033	22.006	22.035	22.008	22.036	22.011
	22.030	22.005	22.035	22.008	22.036	22.010
	22.031	22.009	22.035	22.008	22.036	22.007
Average	22.0318	22.0058	22.0353	22.0068	22.0365	22.0087
Prob. error	0.0002	0.0002	0.0002	0.0002	0.0002	0.0004

Set No.	11	12	13	14	15	16
Orientation of inlet	N	S	N	S	N	S
Time start	11:29	11:47	12:12	12:05	12:31	12:44
Time end	11:33	11:50	12:15	12:08	12:34	12:47
H, from observatory magnetograph	53,904	53,903	53,905	53,905	53,909	53,909

Sets	11-12	13-14	15-16
Average current ma.	22.0188	22.0210	22.0226
H, U.S.G.S. magnetometer	53904	53909	53913
H, Magnetograph	53904	53905	53909
Deviation	0	+4	+4

TABLE III (CONTINUED)

Vertical Field Measurements
at the
Cheltenham Magnetic Observatory

Date--March 16, 1942. Inductor 124 cm. above Bronze Magnetic Station.

Set No.	17	18	19	20
	22.035	22.010	22.040	22.010
	22.040	22.009	22.040	22.009
	22.039	22.009	22.038	22.011
	22.035	22.011	22.041	22.010
	22.039	22.007	22.039	22.014
	22.036	22.010	22.041	22.010
	22.033	22.010	22.041	22.014
	22.038	22.010	22.038	22.010
	22.037	22.010	22.041	22.010
	22.036	22.010	22.038	22.011
Average	22.0368	22.0096	22.0397	22.0109
Prob. error	0.0005	0.0002	0.0003	0.0003

Set No.	17	18	19	20
Orientation of inlet	N	S	N	S
Time start	12:54	13:11	13:52	13:35
Time end	12:57	13:13	13:54	13:38
H, from observatory magnetograph	53,909	53,910	53,914	53,913

Sets	17-18	19-20
Average current	22.0232	22.0253
H, U.S.G.S. magnetometer	53914	53920
H, magnetograph	53909	53914
Deviation	+5	+6

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