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## ABSTRACT

This paper deals with the behavior of a gas in a cylindrical pipe of infinite length. It is assumed that the gas is at rest under atmospheric pressure when an explosion takes place within the pipe. The problem is to determine the velocity and pressure of the gas at any instant after the explosion and at any point in the pipe. The pressure is assumed to depend only on density, and motion is assumed to take place only along straight lines parallel to the axis of the pipe. Under these assumptions the problem reduces to the solution of the hydrodynamical equations for one-dimensional motion.

The mathematical formulation of this problem is given by Weber, who was able to predict the behavior of the gas for a limited time interval after the explosion. The contribution of this paper is an extension of the time interval over which behavior of the gas may be predicted.

The mapping of the velocity-density plane upon the distance-time plane, which is equivalent to a solution of the problem, is governed by a second order partial differential equation due to Riemann. A particular solution of this equation makes possible the extension of the time interval mentioned above. The behavior of the gas over a greater time interval is found to depend on an ordinary differential equation with algebraic coefficients. No solution to this equation is given.

The results of this investigation are summarized in graphic form in the last section, where the variation of the pressure along the pipe is indicated for selected instants of time.

# THE RECTILINEAR MOTION OF A GAS

# SUBSEQUENT TO

# AN INTERNAL EXPLOSION

By

Robert Rand

Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy

1943

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THE STATE OF A GAS SUBSEQUENT TO AN INTERNAL EXPLOSION

1. Introduction. In this paper we treat mathematically the physical problem of determining the behavior of a gas inside a cylindrical pipe of infinite length subsequent to the instantaneous ignition of a stick of explosive. We neglect effects of the walls of the pipe and of temperature changes, assuming the relation between pressure $p$ and density $\rho$ is given everywhere by the adiabatic law

$$p = k^2 \rho^\gamma, \qquad 1 < \gamma < 3, \tag{1.1}$$

where $k$ and $\gamma$ denote constants.

We consider motion to be one-dimensional and assume that the gas particles move on straight lines parallel to a fixed axis under no external forces and that the velocity $u$ and density $\rho$ of the gas are constant within any plane perpendicular to that axis, these quantities being permitted to vary from plane to plane. In the main we restrict our attention to values of $\gamma$ given by

$$\gamma = \frac{2n+3}{2n+1}, \qquad (n = 1, 2, ---). \tag{1.2}$$

These values agree with the values given by classical theory[1] for gases in which the number of degrees of freedom of the molecule is odd, monatomic and diatomic gases falling under this classification.

The problem we consider may be formulated as follows:[2] to determine

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[1] $\gamma$ actually varies slightly with the temperature and assumes the value given only at one particular temperature.

[2] This problem has been formulated, but not solved, by H. Weber in Die Partiellen Differential-gleichungen der mathematischen Physik (Braunschweig: Freidr. Vieweg und Sohn, 1919) Vol. 2, pp. 549-550.

the subsequent states of a gas which initially is at rest everywhere and whose initial density is a constant $\rho_2$ everywhere except in a finite interval in which it assumes a constant value $\rho_0 > \rho_2$. Thus if $x$ denotes distance along an axis and $t$ denotes the time, our problem is to determine the functions $u = u(x,t)$, $\rho = \rho(x,t)$ subject to the initial conditions

$$u(x,0) = 0\,;\ \rho(x,0) = \rho_2 \quad \text{for } |x| > 1,\ \rho(x,0) = \rho_0 \quad \text{for } |x| < 1.$$

We divide the investigation into four parts. The first part contains the necessary mathematical and physical foundations and includes Riemann's[3] work together with other known results. Part II contains a solution to the problem for any time $t < 1/G_0$, where $G_0$ is the velocity of sound at density $\rho_0$. The variation of $\rho$ with $x$, shown for $t = 0$ in Figure A, is illustrated in Figure B for a value of $t$ between $0$ and $1/G_0$.

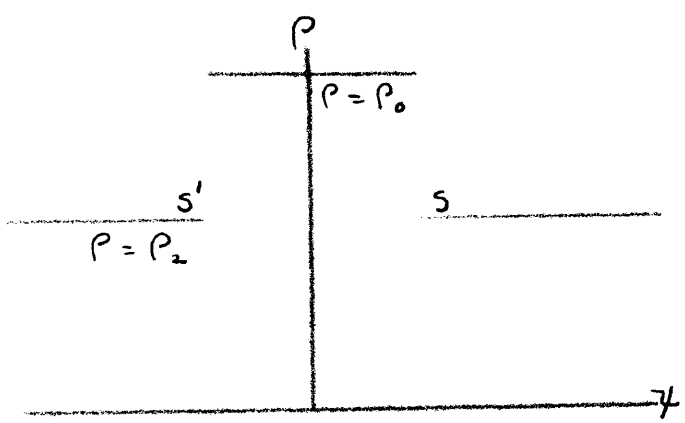


Figure A

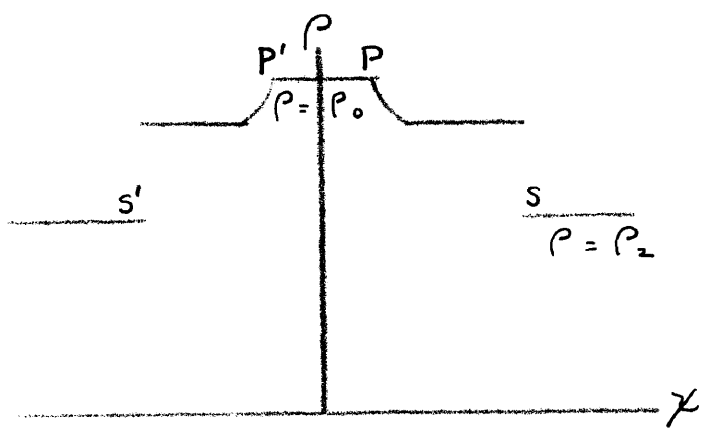


Figure B

The abrupt increase of density at S and S' is due to a shock wave arising from the explosion. We find in Part II that the segment PP' decreases linearly in length as $t$ increases, and vanishes when $t = 1/G_0$. In seeking the prolongation of the solution for $t > 1/G_0$ we are led to what we term

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[3] B. Riemann, Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite, Werke, 2nd Ed. (Leipzig, 1892), pp 156-175.

the <u>first initial value problem</u>. In the first section of Part III we divide this problem into three cases dependent upon the value of $\rho_0$. The remainder of Part III we devote to a further investigation of the first initial value problem and find a solution valid for $t > 1/\sigma_0$ until $t$ reaches a value $t = T$. The variation of $\rho$ with $x$ when $t$ lies between $1/\sigma_0$ and $T$ is shown in Figure C.

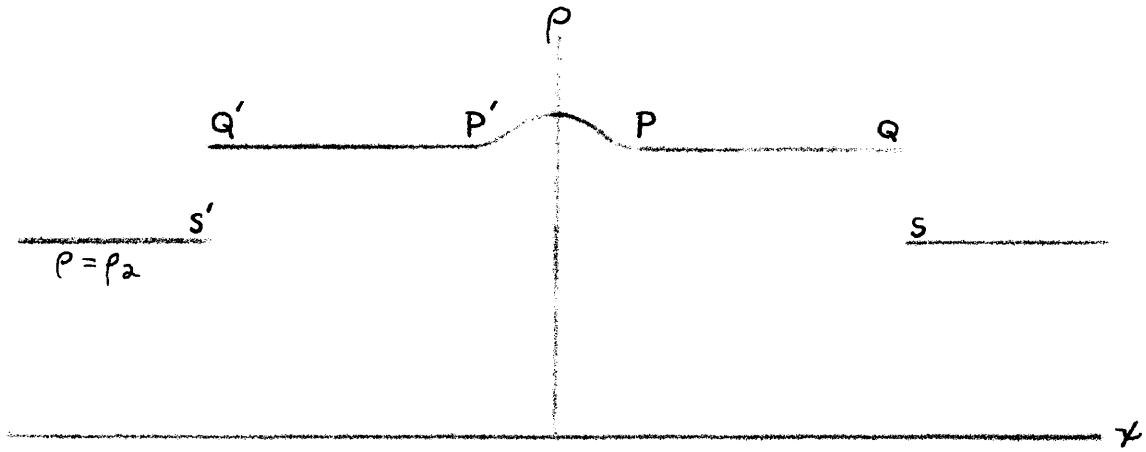


Figure C

The curve joining P and P' differs in the three cases mentioned above but in each case we find the segments PQ and P'Q' first increase, then decrease in length linearly with increasing $t$ and reduce to points when $t = T$, leading us to the <u>second initial value problem</u>. In Part IV we reduce this problem to the solution of an ordinary differential equation with algebraic coefficients, but are unable to obtain this solution. The final section is devoted to a discussion, including all cases, of the variation of $\rho$ with $x$ for $t < T$.

# PART I

## THE MATHEMATICAL AND PHYSICAL FOUNDATIONS OF THE PROBLEM

2. Riemann's treatment of the hydrodynamical equations. Under the assumptions of § 1 the state of a gas in one-dimensional motion is governed by the well-known hydrodynamical equations

(2.1) $u_t + u_x u + \frac{dp}{d\rho}\frac{\rho_x}{\rho} = 0, \quad \rho_t + \rho_x u + \rho u_x = 0,$

in which $p$ and $\rho$ are connected by the so-called equation of state

(2.2) $$p = p(\rho).$$

Riemann[4] has reduced these equations to two linear partial differential equations by the method given below.

Introducing two functions $G, v$ of the density $\rho$

(2.3) $$G = \left(\frac{dp}{d\rho}\right)^{1/2}, \quad v = \int G \frac{d\rho}{\rho},$$

$G$ being the local velocity of sound, it is apparent that

(2.4) $$v_x = G\frac{\rho_x}{\rho}, \quad v_t = G\frac{\rho_t}{\rho},$$

and hence that equations (2.1) may be written in the form

(2.5) $u_t + u_x u + G v_x = 0, \quad v_t + v_x u + G u_x = 0.$

Addition and subtraction of these two equations yields

(2.6) $r_t + (u + G) r_x = 0, \quad s_t + (u - G) s_x = 0,$

wherein we have placed

(2.7) $u = r + s, \quad v = r - s; \quad 2r = u + v, \quad 2s = u - v.$

A solution of (2.6) is equivalent to a solution of (2.1) and henceforth we shall restrict our attention to equations (2.6). Since $G$ and $v$ are both functions of $\rho$, it follows that $G$ is a function of $v$, that is, of $r - s$. Hence from (2.7) we have

(2.8) $$G_r = -G_s, \quad u_r = u_s = 1.$$

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[4] B. Riemann, loc. cit.

A solution $r = r(\psi, t)$, $s = s(\psi, t)$ of (2.6) defines a mapping of the $(\psi, t)$-plane upon the $(r, s)$-plane. Using (2.6) the Jacobian of the transformation reduces to $2G r_\psi s_\psi$. Assuming that a region in the $(r, s)$-plane is in one-to-one correspondence with a region of the $(\psi, t)$-plane[5], the inverse transformation $\psi = \psi(r, s)$, $t = t(r, s)$ satisfies the partial differential equations

(2.9) $$\psi_r - (u - G)\, t_r = 0, \quad \psi_s - (u + G)\, t_s = 0 ;$$

and the Jacobian $\dfrac{\partial(\psi, t)}{\partial(r, s)}$ takes the form

(2.10) $$J = -2G\, t_r\, t_s .$$

Since $u$ and $G$ are functions of $r$ and $s$ we see that equations (2.9) are linear in the variables $\psi$ and $t$, whereas (2.6) were not linear in the variables $r$ and $s$. Using (2.8) and (2.9) it is easily verified that the line integral

$$w = \int \left\{ (\psi - [u + G]\, t)\, dr + (\psi - [u - G]\, t)\, ds \right\}$$

is independent of the path, and thus defines a function $w(r, s)$ for which

(2.11) $$w_r = \psi - (u + G)\, t, \quad w_s = \psi - (u - G)\, t .$$

Subtracting these two equations and differentiating the second with respect to $r$ we are led, upon elimination of $t$ to the second order partial differential equation

(2.12) $$w_{rs} + \frac{G_r - 1}{2G}\,(w_r - w_s) = 0 .$$

After obtaining a solution of (2.12) one solves (2.11) for $\psi$, $t$ to obtain the above-mentioned inverse transformation in the form

(2.13) $$\psi = \frac{w_r + w_s}{2} - (r + s)\,\frac{w_r - w_s}{r - s}, \qquad t = -\frac{w_r - w_s}{2G}; \quad G = G(r - s),\ w = w(r, s).$$

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[5]This assumption is false for the important case in which $u$ and $\rho$ are constant throughout a region of the $(\psi, t)$-plane. The mapping in the $(r, s)$-plane of such a region is a point.

The problem of determining a solution of (2.1) has thus been reduced to solving (2.12), provided that the mapping of the $(x,t)$-plane upon the $(r,s)$-plane permits of an inverse transformation as assumed above. Preassigned values of $u, \rho$ on a curve in the $(x,t)$-plane determine, by virtue of (2.11), values of $w_r, w_s$ on the corresponding curve in the $(r,s)$-plane. A solution of (2.12) which satisfies these initial values will lead, as shown above, to a solution of the initial value problem in the $(x,t)$-plane.

If the adiabatic assumption (1.1) is used in (2.3) these equations take the form

$$(2.14) \quad G = k\, r^{\frac{1}{2}} \rho^{\frac{\gamma-1}{2}}, \quad v = \frac{2k\, r^{\frac{1}{2}}}{\gamma - 1} \rho^{\frac{\gamma-1}{2}},$$

from which we see that

$$(2.15) \quad G = \frac{\gamma-1}{2} v = \frac{\gamma-1}{2}(r-s).$$

Using (2.15) we see that (2.12) reduces to

$$(2.16) \quad w_{rs} - \lambda \frac{w_r - w_s}{r-s} = 0,$$

where we have put

$$(2.17) \quad \lambda = \frac{3-\gamma}{2(\gamma-1)} > 0, \quad \gamma = \frac{2\lambda+3}{2\lambda+1}.$$

Using (2.17) in (2.15) we have

$$(2.18) \quad G = \frac{v}{2\lambda+1} = \frac{r-s}{2\lambda+1}$$

and, upon using this result in (2.13), the mapping of the $(r,s)$-plane upon the $(x,t)$-plane becomes

$$(M) \quad \begin{aligned} x &= \frac{w_r + w_s}{2} - \frac{2\lambda+1}{2} \frac{w_r - w_s}{r-s}(r+s), \\ t &= -\frac{2\lambda+1}{2} \frac{w_r - w_s}{r-s}, \quad w = w(r,s). \end{aligned}$$

It should be noted for future reference that $p$, $v$, and $G$ are monotonic increasing functions of $\rho$. This fact is easily established from (1.1) and (2.14).

3. <u>Propagation of state</u>. If there exists in the $(x,t)$-plane a locus L of points upon which a given state $(\bar{u},\bar{\rho})$ of the gas remains unchanged,

we say the state $(\bar{u},\bar{\rho})$ is <u>propagated</u> along L. Since the mapping in the $(r,s)$-plane of such a locus is a single point, the correspondence between the $(x,t)$-plane and the $(r,s)$-plane is no longer one-to-one. Solutions of (2.6) for which $r$ or $s$ is constant are characterized by lines of propagation, and the results needed here may be summed up in the following theorem.[6]

THEOREM 1. <u>If on a curve C in the $(x,t)$-plane the states $(u,\rho)$ of the gas satisfy the relation</u>

(3.1) $2r = u + v = \text{const.}\ [2s = u - v = \text{const.}]$

<u>then states of the gas on the curve C are propagated along straight lines of slope</u> $(u - G)^{-1}\ [(u + G)^{-1}]$.

We shall restrict the proof to the case in which $r$ is constant. We note first that the condition (3.1) reduces equations (2.6) to the single equation

(3.2) $$\alpha_t + \alpha\alpha_x = 0$$

on placing $\alpha = u - G$, inasmuch as $\alpha_t = \frac{d\alpha}{ds} s_t$, $\alpha_x = \frac{d\alpha}{ds} s_x$.

If we observe that

$$\frac{\partial(x - \alpha t, \alpha)}{\partial(x,t)} = \alpha_t + \alpha\alpha_x$$

it is apparent that the general solution of (3.2) is given by

(3.3) $$x - \alpha t = F(\alpha)$$

where $F(\alpha)$ denotes an arbitrary function of $\alpha$. After $F(\alpha)$ has been assigned, equation (3.3) may be solved for $\alpha(x,t)$, and the velocity and density of the gas for any $x$ and $t$ is determined from the pair of equations

(3.4) $$u - G(\rho) = \alpha(x,t),\ u + v(\rho) = 2r = \text{const.}$$

It is clear from (3.4) that $\alpha$ = const. is a sufficient condition for states to remain constant, and from (3.3) we see that $\alpha$ is constant

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[6] H. Weber, <u>op. cit.</u> pp. 518-519.

along lines of slope $1/\alpha$, proving the theorem. It should be noted that in general the domain of validity of such a solution is limited by the envelope of the lines of propagation.

4. Shock waves and buffer regions. Let us assume that as one proceeds in the $(x,t)$-plane from left to right on the line $t = t_0$, one encounters at some point $x = \xi$ an abrupt drop in density and a change in the velocity of the gas. In such an event we shall say that a shock is present at $x = \xi$, and the locus in the $(x,t)$-plane of the shock we shall term a shock wave. It is known[7] that the state $(u,\rho)$ of the gas to the left of the shock and the state $(u_2,\rho_2)$ to the right of the shock are not independent, but are connected with the velocity of propagation $\frac{d\xi}{dt}$ of the shock wave by the conditions:

1. COMPATIBILITY CONDITION: $\rho V = \rho_2 V_2$.
2. IMPULSE CONDITION: $p(\rho) - p(\rho_2) = \rho_2 V_2^2 - \rho V^2$,

where $V = u - \frac{d\xi}{dt}$, $V_2 = u_2 - \frac{d\xi}{dt}$.

In the case of the so-called progressive condensation shock waves these conditions reduce to[8]

$$(4.1)\quad u - u_2 = \left(\frac{(p-p_2)(\rho-\rho_2)}{\rho\rho_2}\right)^{1/2}, \quad \rho > \rho_2, \; p_2 = p(\rho_2);$$

$$(4.2)\quad \frac{d\xi}{dt} = u + \left(\frac{\rho_2}{\rho}\,\frac{p-p_2}{\rho-\rho_2}\right)^{1/2} = u_2 + \left(\frac{\rho}{\rho_2}\,\frac{p-p_2}{\rho-\rho_2}\right)^{1/2}.$$

Equations (4.1) and (4.2) establish the following lemma:

LEMMA 1. If on the line $t = 0$ there exist the constant states

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[7] H. Weber, op. cit. pp. 510–511.

[8] H. Weber, op. cit. pp 513–514

$$u = u_1 , \quad \rho = \rho_1 \qquad \text{for} \quad x < 1 ,$$
$$u = u_2 , \quad \rho = \rho_2 \qquad \text{for} \quad x > 1 ,$$

$$\rho_1 > \rho_2 \quad ;$$

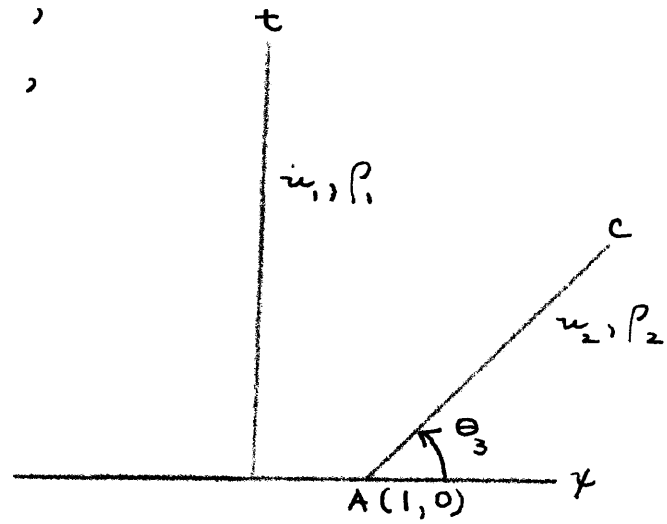


Figure 1

then a necessary and sufficient condition that a progressive shock wave emanate from (1,0) is given by

$$(4.3) \quad u_1 - u_2 = \left( \frac{(p_1 - p_2)(\rho_1 - \rho_2)}{\rho_1 \ \rho_2} \right)^{1/2} , \quad p_i = p(\rho_i) ,$$

and if (4.3) is satisfied the velocity of propagation of the shock wave is

$$(4.4) \quad \cot \theta_3 = u_1 + \left( \frac{\rho_2}{\rho_1} \ \frac{p_1 - p_2}{\rho_1 - \rho_2} \right)^{1/2} = u_2 + \left( \frac{\rho_1}{\rho_2} \ \frac{p_1 - p_2}{\rho_1 - \rho_2} \right)^{1/2} .$$

Let us consider now the following constant states on the line $t = 0$ as shown in Figure 2.

$$(4.5) \quad \begin{aligned} u = u_0 , \ \rho = \rho_0 \ \text{for} \quad x < 1 , \\ u = u_1 , \ \rho = \rho_1 \ \text{for} \quad x > 1 , \end{aligned}$$

$$\rho_0 > \rho_1 \ ;$$

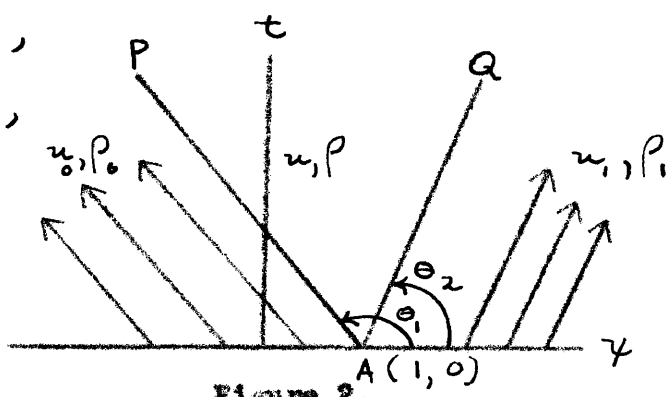


Figure 2.

where $u_0, \rho_0; u_1, \rho_1$ satisfy the relation

(4.6) $2r = u_0 + v_0 = u_1 + v_1 = \text{const.}, \; v_i = v(\rho_i).$

By Theorem 1 states will be propagated from the segment $x > 1, t = 0$ along lines of slope $1/[u_1 - G(\rho_1)]$, and from the segment $x < 1, t = 0$ along lines of slope $1/[u_0 - G(\rho_0)]$. From (4.6) we see that $u_1 - u_0 > 0$ since $v$ increases monotonically with $\rho$, hence $u_1 - G(\rho_1) > u_0 - G(\rho_0)$ and $\theta_1 > \theta_2$. To define the solution in the region PAQ we refer to (3.3) and choose $F(x) = \frac{x-1}{t}$. From (3.4) we have then in region PAQ

(4.7) $u - G = \frac{x-1}{t}, \; u + v = 2r.$

It is easily seen that the solution thus defined is continuous on the boundary lines PA and QA. A region in which the solution is of the type (4.7) is called a <u>buffer region</u>, the lines PA and QA are termed the <u>boundaries</u>, and cot $\theta_1$ and cot $\theta_2$ are the <u>velocities of propagation</u> of these boundaries. In particular, since cot $\theta_1 < u_0$ the region is called a <u>retrogressive buffer region</u>. We have shown that (4.6) is sufficient for the emanation from (1,0) of a buffer region. To prove it is also necessary we observe that if (4.7) is satisfied states are constant on the line $x - (u-G)t = 1$, hence $r_t + (u-G)r_x = 0$. But from (2.6) we have $r_t + (u+G)r_x = 0$; so $r_t = r_x = 0$ and (4.6) is satisfied. We summarize this result in Lemma 2.

LEMMA 2. <u>If on the line $t = 0$ there exist the constant states (4.5), then the necessary and sufficient condition for the emanation from (1,0) of a retrogressive buffer region is given by (4.6). If (4.6) is satisfied the velocities of propagation of the left and right boundaries of the buffer region are</u>

(4.8) $\cot \theta_1 = u_0 - G_0, \; \cot \theta_2 = u_1 - G_1, \; G_i = G(\rho_i),$

<u>and the states inside the buffer region are given by</u> (4.7).

If we ask that both a shock wave and buffer region emanate from

(1,0) we are led to the following theorem[9].

THEOREM II. *If on the line $t=0$ there exist the constant states*

$$u=u_0,\ \rho=\rho_0 \text{ for } x<1,$$
$$u=u_2,\ \rho=\rho_2 \text{ for } x>1,$$

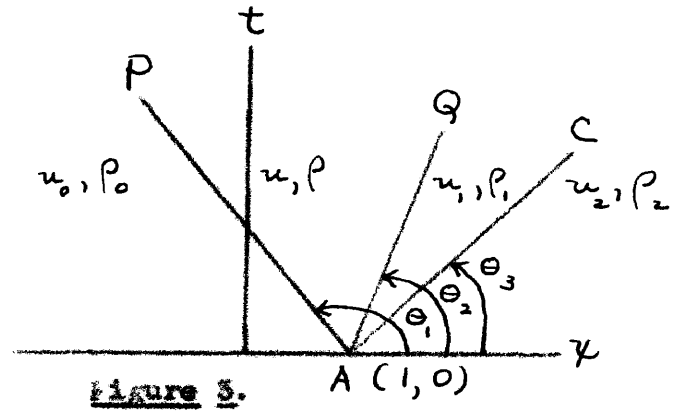


Figure 3.

*then the necessary and sufficient condition for the emanation from (1,0) of a progressive shock wave and a retrogressive buffer region is*

$$(4.9)\quad v_2 - v_0 < u_0 - u_2 < \left(\frac{(p_0-p_2)(\rho_0-\rho_2)}{\rho_0\,\rho_2}\right)^{1/2},\ \rho_2<\rho_0.$$

*If this condition is satisfied the state $(u_1,\rho_1)$ in the region bounded by the buffer region and shock wave is found from the equations*

$$(4.10)\quad \begin{aligned} u_0 - u_2 &= v_1 - v_0 + \left(\frac{(p_1-p_2)(\rho_1-\rho_2)}{\rho_1\,\rho_2}\right)^{1/2},\ \rho_2<\rho_1<\rho_0, \\ u_0 - u_1 &= v_1 - v_0 . \end{aligned}$$

*The velocities of propagation of the left and right boundaries of the buffer region are given by (4.8) and the velocity of propagation of the shock wave by (4.4). The state $(u,\rho)$ inside the buffer region is determined from (4.7).*

We see from Lemmas 1 and 2 that (4.3) and (4.6) must both be satisfied. Writing (4.6) in the form

$$u_0 - u_1 = v_1 - v_0$$

and adding to (4.3) we establish (4.10). If we write the first equation

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[9] H. Weber, op. cit. pp 528-530

of (4.10) in the form

$$u_0 - u_2 = v_1 - v_0 + \left((p_1 - p_2)\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\right)^{\frac{1}{2}}$$

it becomes clear that $u_0 - u_2$ is a monotonic increasing function of $\rho_1$, but since $\rho_2 < \rho_1 < \rho_0$ we see that $u_0 - u_2$ must satisfy (4.9).

To prove the sufficiency we observe that if (4.9) is satisfied the first equation of (4.10) will uniquely determine a value of $\rho_1$ lying between $\rho_2$ and $\rho_0$. $u_1$ may then be found from the second equation of (4.10), and the angles $\theta_1$, $\theta_2$, and $\theta_3$ are determined by (4.8) and (4.4). From (4.8)

$$\cot \theta_2 - \cot \theta_1 = v_0 - v_1 + G_0 - G_1 > 0,$$

and also from (4.4) and (4.8)

$$\cot \theta_3 - \cot \theta_2 = \left(\frac{\rho_2}{\rho_1} \; \frac{p_1 - p_2}{\rho_1 - \rho_2}\right)^{\frac{1}{2}} + G_1 > 0.$$

The relative magnitudes of $\theta_1$, $\theta_2$, and $\theta_3$ are then as shown in Figure 3, and the proof of the theorem is completed.

# PART II

## SOLUTION OF THE PROBLEM FOR $t < 1/G_0$.

**States of the gas for** $t < 1/G_0$

5. We assume for $t = 0$ the following states:

$$u = 0,\ \rho = \rho_2 \quad \text{for} \quad |x| > 1,$$

$$u = 0,\ \rho = \rho_0 \quad \text{for} \quad |x| < 1,\ \rho_0 > \rho_2 .$$

If in Theorem 2 we put $u_0 = u_2 = 0$, we see that (4.9) is satisfied since $\rho_0 > \rho_2$, hence a progressive condensation shock wave and a retrogressive buffer region emanate from (1,0), as shown in Figure 4.

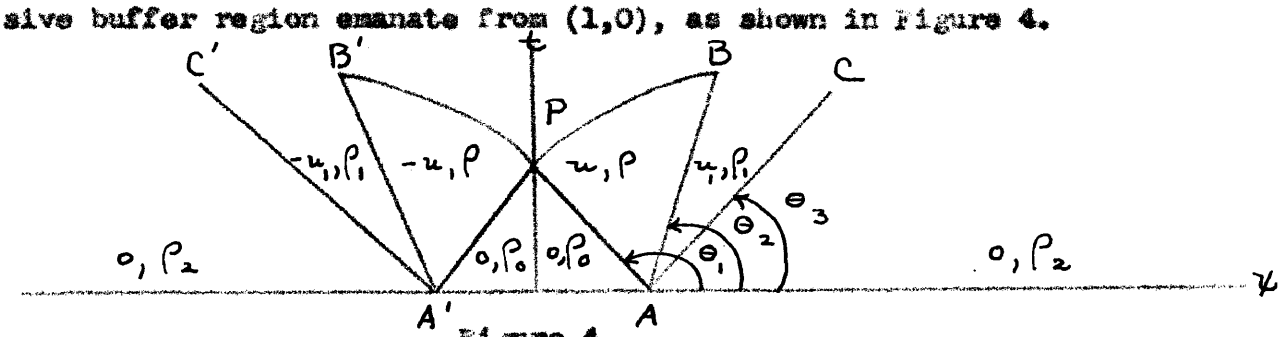


**Figure 4**

From (4.10) the state $(u_1,\rho_1)$ of the gas between the shock wave and the buffer region is determined by

$$v_0 = v_1 + \left( \frac{(p_1 - p_2)(\rho_1 - \rho_2)}{\rho_1 \rho_2} \right)^{\frac{1}{2}}, \quad \rho_2 < \rho_1 < \rho_0 ,$$

(5.1)

$$u_1 = v_0 - v_1 .$$

The velocities of propagation of the boundaries of the buffer region are from (4.8)

(5.2) $\cot \theta_1 = -G_0 , \quad \cot \theta_2 = u_1 - G_1 ,$

and the velocity of propagation of the shock wave is given by (4.4). The state $(u,\rho)$ of the gas inside the buffer region is determined from the equations

(5.3) $u - G = \dfrac{x-1}{t} , \quad u + v = v_0 ,$

by virtue of (4.7).

It is clear from (5.2) that the angle $\theta_1$ is obtuse and from (4.4) that $\theta_3$ is acute. The angle $\theta_2$, however, may be either acute or obtuse depending only on the value of $\rho_0$ as will be shown by the following lemma:

LEMMA 3. <u>As $\rho_0$ varies from $\rho_2$ to $+\infty$, $\theta_2$ decreases monotonically from $\cot^{-1} - G_2$ to $0$. In particular there exists $\rho_0^* > \rho_2$ for which $\theta_2 = \frac{\pi}{2}$.</u>

If we write (5.1) in the form

$$v_0 = v_1 + \left( (p_1 - p_2)\left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \right)^{\frac{1}{2}}$$

it is clear that $\rho_1$ is a monotonically increasing, single-valued function of $\rho_0$. Since $\rho_1 = \rho_2$ when $\rho_0 = \rho_2$ we see that $\rho_1$ <u>varies monotonically from $\rho_2$ to $+\infty$ as $\rho_0$ varies between the same limits.</u> Substituting the value of $u_1$ given in (5.1) into (5.2) we find

$$\cot \theta_2 = v_0 - v_1 - G_1$$

which we may write in the form

$$\cot \theta_2 = \left( (p_1 - p_2)\left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \right)^{\frac{1}{2}} - G_1 .$$

Substituting for $p$ and $G$ from (1.1) and (2.14) we are led to the relation

$$\cot \theta_2 = k \rho_1^{\frac{r}{2}} \left\{ \left( 1 - \left( \frac{\rho_2}{\rho_1} \right)^r \right)^{\frac{1}{2}} \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right)^{\frac{1}{2}} - \left( \frac{r}{\rho_1} \right)^{\frac{1}{2}} \right\} .$$

Since we have shown above that $\rho_1$ increases monotonically with $\rho_0$ from $\rho_2$ to $+\infty$, it is evident that $\theta_2$ decreases monotonically from $\cot^{-1} - G_2$ to $0$ as $\rho_0$ increases from $\rho_2$ to $+\infty$. Inasmuch as $G_2 > 0$ and $\cot \theta_1$ is a single-valued function of $\rho_1$ we conclude that there is a unique value of $\rho_0$ for which $\theta_2 = \frac{\pi}{2}$.

By appealing to symmetry we see that states for $\psi < 0$ will differ from those for $\psi > 0$ only in that the velocities will be the negative of those for $\psi > 0$. Accordingly we have a shock wave and buffer region

emanating from (-1,0). The state to the left of the shock wave will be $(0, \rho_2)$. Between the shock wave and buffer region we have the state $(-u_1, \rho_1)$, and inside the buffer region we find, replacing $v$ by $-v$ and $u$ by $-u$ in (5.3)

(5.4) $$u + G = \frac{v+1}{t}, \quad u - v = -v_0 .$$

The solution thus obtained[10] for the state of the gas is valid until the boundaries of the buffer regions meet at time $t = 1/G_0$.

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[10] See H. Weber, op. cit. pp. 549-550, in which the solution is considered only in the interval $0 \leq t \leq 1/G_0$.

# PART III

## THE FIRST INITIAL VALUE PROBLEM

6. The fundamental square. The solution defined in Part II inside the buffer regions PAB and PA'B' will be valid until the two buffer regions meet at P. Let the curves PB and PB' in Figure 4 represent the loci of the points at which the solutions (5.3) and (5.4) become invalid. These solutions determine initial values on the arcs PB and PB' for a new solution of (2.6). It was shown in §2, however, that an initial value problem for (2.6) in the $(x,t)$-plane leads to an initial value problem for (2.16) in the $(r,s)$-plane. It is this latter problem we now formulate explicitly.

On the arc PB we have from (5.3)

$$r = \frac{u+v}{2} = \frac{v_0}{2}, \quad x - (u-G)t = 1;$$

and on the arc PB' we have from (5.4)

$$s = \frac{u-v}{2} = -\frac{v_0}{2}, \quad x - (u+G)t = -1.$$

The initial conditions thus imposed on the solution of (2.16) become, on using (2.11) and writing $\frac{v_0}{2} = r_0$,

(6.1) $\quad w_r = -1 \quad$ on $s = -r_0$, $\qquad w_s = +1 \quad$ on $r = r_0$.

A solution of (2.16) satisfying these initial conditions will be termed a fundamental solution. We next determine the region in the $(r,s)$-plane to be mapped onto the $(x,t)$-plane by means of the mapping (M) when $w$ is a fundamental solution.

Clearly the points B,B' correspond to the points $(r_0, s_1)$, $(r_1, -r_0)$, where $-r_1 = s_1 = \frac{u_1 - v_1}{2}$. Thus we are led to the square

$$r_1 < r < r_0, \quad -r_0 < s < s_1$$

in the $(r,s)$-plane which we shall term the fundamental square, and we propose to study the mapping of this square upon the $(x,t)$-plane by the

fundamental solution as a means of prolonging our solution into the region $t > 1/G_0$ of the $(y,t)$-plane. Before doing so it is essential to study the variation of the fundamental square with the value of $\rho_0$.

Inasmuch as $-r_1 = s_1$, the fundamental square is completely determined by the point $(r_0, s_1)$. As $\rho_0$ varies from $\rho_2$ to $+\infty$ the point $(r_0, s_1)$ will trace out a curve which we shall call $\sigma$.

THEOREM 3. <u>Both coordinates of the point $(r_0, s_1)$ on the curve $\sigma$ are monotonic increasing functions of $\rho_1$.</u>

Using (1.1) and (2.14) in (5.1) we find as the parametric equations of $\sigma$

$$(6.2)\qquad \begin{aligned} r_0 &= \frac{u_1+v_1}{2} = \frac{k}{2}\left\{\left(\frac{(\rho_1^\gamma-\rho_2^\gamma)(\rho_1-\rho_2)}{\rho_1\rho_2}\right)^{\frac{1}{2}} + \frac{2}{\gamma-1}\gamma^{\frac{1}{2}}\rho_1^{\frac{\gamma-1}{2}}\right\}, \\ s_1 &= \frac{u_1-v_1}{2} = \frac{k}{2}\left\{\left(\frac{(\rho_1^\gamma-\rho_2^\gamma)(\rho_1-\rho_2)}{\rho_1\rho_2}\right)^{\frac{1}{2}} - \frac{2}{\gamma-1}\gamma^{\frac{1}{2}}\rho_1^{\frac{\gamma-1}{2}}\right\}. \end{aligned}$$

It is evident that $\frac{dr_0}{d\rho_1} > 0$ for $\rho_1 > \rho_2$. The inequality $\frac{ds_1}{d\rho_1} > 0$ reduces to

$$(6.3)\qquad \frac{\gamma\rho_1^{\gamma-1}(\rho_1-\rho_2)}{\rho_1\rho_2} + \frac{1}{\rho_1^2}(\rho_1^\gamma-\rho_2^\gamma) > 2\gamma^{\frac{1}{2}}\rho_1^{\frac{\gamma-3}{2}}\left(\frac{(\rho_1^\gamma-\rho_2^\gamma)(\rho_1-\rho_2)}{\rho_1\rho_2}\right)^{\frac{1}{2}}.$$

Since twice the geometric mean of the two terms of the left member is seen to be the right member, (6.3) is clearly satisfied for $\rho_1 > \rho_2$ completing the proof of the theorem.

COROLLARY 1. <u>Both coordinates of the point $(r_0, s_1)$ are monotonic increasing functions of $\rho_0$ and there exists a unique value $\bar{\rho}_0$ of $\rho_0$ for which $s_1 = 0$.</u>

Referring to the proof of Lemma 5 we see that it is necessary only to prove that $s_1$ changes sign as $\rho_1$ increases from $\rho_2$ to $+\infty$. Writing $s_1$ in the form

$$(6.4)\qquad s_1 = \frac{k}{2}\rho_1^{\frac{\gamma}{2}}\left\{\left[1-\left(\frac{\rho_2}{\rho_1}\right)^\gamma\right]^{\frac{1}{2}}\left[\frac{1}{\rho_2}-\frac{1}{\rho_1}\right]^{\frac{1}{2}} - \frac{2}{\gamma-1}\left(\frac{\gamma}{\rho_1}\right)^{\frac{1}{2}}\right\},$$

it is clear that $s_1 < 0$ when $\rho_1 = \rho_2$ and that $s_1 > 0$ for $\rho_1$ sufficiently large.

When $\rho_0 = \rho_1 = \rho_2$ the point $(r_0, s_1)$ coincides with the point $r_2 = \frac{v_2}{2}$, $s_2 = -\frac{v_2}{2}$, which corresponds to the state $(0, \rho_2)$. By symmetry the locus of the points $(r_1, -r_0)$ will be a curve $\sigma'$ symmetric to $\sigma$ in the line $r + s = 0$ and sharing with $\sigma$ the point $(r_2, s_2)$. From (2.7) and (2.8) we find

(6.5) $$u - G = \frac{2}{2\lambda+1}\left[\lambda r + (\lambda+1)s\right], \quad u + G = \frac{2}{2\lambda+1}\left[(\lambda+1)r + \lambda s\right].$$

We investigate now the variation of the fundamental square with $\rho_0$, listing three cases.

CASE I. $\rho_2 < \rho_0 < \rho_0^*$. By (5.2) and Lemma 3 $u_1 < G_1$ and the fundamental square is as shown in Figure 5a.

CASE II. $\rho_0^* \leq \rho_0 < \bar{\rho}_0$. By Lemma 3 and Corollary 1 $u_1 > G_1$ and $s_1 < 0$ as shown in Figure 5b. For the case $\rho_0 = \rho_0^*$ the points $(r_0, s_1)$ and $(r_1, -r_0)$ will lie on the transforms (6.5) of $u - G = 0$ and $u + G = 0$ respectively.

CASE III. $\rho_0 \geq \bar{\rho}_0$. Here $s_1 > 0$ by Corollary 1. From (6.4) we see that $s_1$ may be arbitrarily large, but from (5.1) we see that $s_1 < r_0$. For the case $\rho_0 = \bar{\rho}_0$, $s_1 = 0$. Case III is shown in Figure 5c.

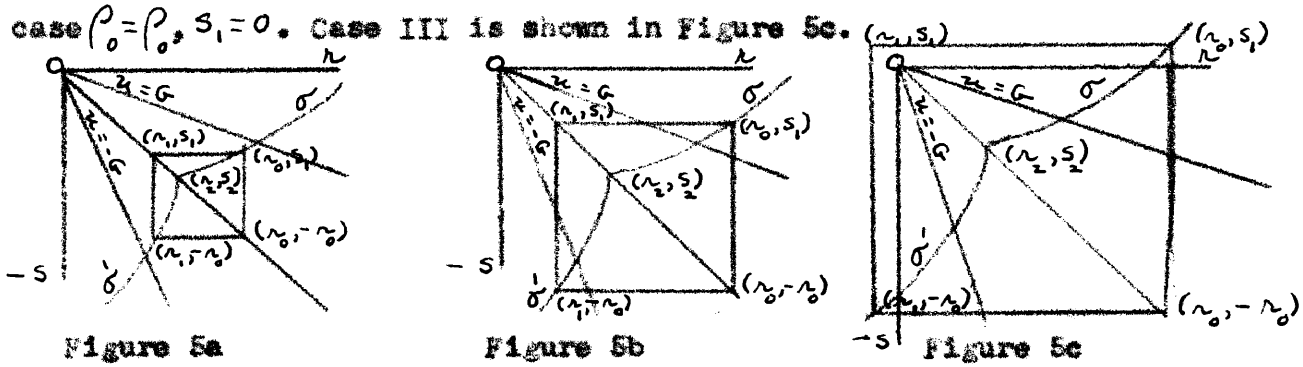


Figure 5a Figure 5b Figure 5c

7. <u>The fundamental solution for arbitrary positive</u> $\lambda$. The solution to the first initial value problem, i. e., the fundamental solution, is [11]

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[11] From a paper by M. Martin entitled <u>The Rectilinear Motion of a Gas</u> to be published in the American Journal of Mathematics.

$$w = (r_0 - r)\left(\frac{2r_0}{r_0 - s}\right)^{\lambda} F_1\left(1+\lambda, -\lambda, \lambda; 2; \frac{r - r_0}{-2r_0}, \frac{r - r_0}{s - r_0}\right)$$

$$(7.1) \quad + (s + r_0)\left(\frac{2r_0}{r + r_0}\right)^{\lambda} F_1\left(1+\lambda, -\lambda, \lambda; 2; \frac{s + r_0}{2r_0}, \frac{s + r_0}{r + r_0}\right),$$

where

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, n)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n, \quad (\alpha, r) = \alpha(\alpha+1)\cdots(\alpha+r-1),$$

is Appel's hypergeometric function of two variables. This solution is valid for all real $\lambda$ and for all points $(r, s)$ in the fundamental square.

We turn now to the investigation of some properties of the fundamental solution and of the functions $x$ and $t$ determined from the solution by the mapping (M). We shall prove the following three properties:

PROPERTY I. <u>The fundamental solution $w$ is symmetric with respect to the line $r + s = 0$.</u>

PROPERTY II. <u>The functions $t(r, s)$, $x(r, s)$ in (M) are respectively symmetric and antisymmetric in the line $r + s = 0$.</u>

PROPERTY III. <u>The transforms of the lines $r = r_0$, $s = -r_0$, in the $(x, t)$-plane are respectively</u>

$$(7.2) \quad x = 1 + 2\left(\frac{2r_0}{r_0 - s}\right)^{\lambda}\left(\frac{\lambda r_0 + (\lambda+1)s}{r_0 - s}\right), \quad t = \frac{2\lambda+1}{r_0 - s}\left(\frac{2r_0}{r_0 - s}\right)^{\lambda};$$

$$(7.3) \quad x = -1 + 2\left(\frac{2r_0}{r + r_0}\right)^{\lambda}\left(\frac{(\lambda+1)r - \lambda r_0}{r + r_0}\right), \quad t = \frac{2\lambda+1}{r + r_0}\left(\frac{2r_0}{r + r_0}\right)^{\lambda}.$$

The first property is apparent from inspection of (7.1). To prove the second property we use the first property to obtain the results

$$w_r(r, s) = -w_s(-s, -r), \quad w_s(r, s) = -w_r(-s, -r), \quad w_{rs}(r, s) = w_{rs}(-s, -r);$$

while from (M) we have

$$x(-s, -r) = \frac{-w_s - w_r}{2} - \left(\frac{2\lambda+1}{2}\right)\left(\frac{-w_s + w_r}{r - s}\right)(-r - s) = -x(r, s),$$

$$t(-s, -r) = -\left(\frac{2\lambda+1}{2}\right)\left(\frac{-w_s + w_r}{r - s}\right. = t(r, s),$$

which proves Property 2.

To establish (7.2) we observe that on the side $r = r_0$ of the fundamental square, $w_s = +1$ and $w_r$ is the solution

(7.4) $$w_r(r_0, s) = 1 - 2\left(\frac{2r_0}{r_0 - s}\right)^{\lambda}$$

of

$$(r - s)\frac{dw_r}{ds} - \lambda w_r = -\lambda,$$

satisfying the condition: $w_r = -1$ for $s = -r_0$. Putting $w_s = +1$ (7.2) follows from (7.4) and (M). (7.3) then follows from Property 2.

8. <u>The fundamental solution for positive integral</u> $\lambda$. For the case in which $\lambda$ is a positive interger the fundamental solution can be found from the following approach which lends itself more readily to our needs.

The general solution of (2.16) when $\lambda = n$ is given by[12]

(8.1) $$w = \frac{\partial^{2n-2}}{\partial r^{n-1}\partial s^{n-1}}\left(\frac{R + S}{r - s}\right)$$

where $R$ is an arbitrary function of $r$ and $S$ an arbitrary function of $s$. Setting

$$R = (r^2 - r_0^2)^n, \qquad S = (s^2 - r_0^2)^n,$$

we observe that

(8.2) $$w = -\frac{1}{n!\,(n-1)!}\,\frac{\partial^{2n-2}}{\partial r^{n-1}\partial s^{n-1}}\left\{\frac{(r^2 - r_0^2)^n + (s^2 - r_0^2)^n}{r - s}\right\}$$

is a solution of (2.16). Writing (8.2) in the form

(8.3) $$w = -\frac{1}{n!}\left\{\frac{\partial^{n-1}}{\partial r^{n-1}}\left(\frac{r^2 - r_0^2}{r - s}\right)^n + (-1)^{n-1}\frac{\partial^{n-1}}{\partial s^{n-1}}\left(\frac{s^2 - r_0^2}{r - s}\right)^n\right\}$$

we see that the second term on the right vanishes for $s = -r_0$ since that is a zero of order $n$ of $\left(\frac{s^2 - r_0^2}{r - s}\right)^n$. Hence

$$w(r, -r_0) = -\frac{1}{n!}\,\frac{\partial^{n-1}}{\partial r^{n-1}}(r - r_0)^n = -(r - r_0),$$

and similarly

$$w(r_0, s) = \frac{1}{n!}\,\frac{\partial^{n-1}}{\partial s^{n-1}}(s + r_0)^n = s + r_0,$$

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[12] Cf. G. Darboux, <u>Leçons sur la Théorie générale des Surfaces</u>, vol. 2 (Paris, 1915), p. 65.

proving (8.2) is a fundamental solution. For monatomic gases $n = 1$ and the fundamental solution reduces to

$$w = -\frac{r^2 - r_0^2 + s^2 - r_0^2}{r - s}$$

Using the fundamental solution (8.2) we investigate the mapping of the fundamental square onto the $(\gamma, t)$-plane by the mapping (M).

THEOREM 4. <u>The functions $t, t_s$ in the mapping (M) are finite and positive inside and on the boundary of the fundamental square when $\rho_0 < \bar{\rho}_0$.</u>

For the proof of this theorem we shall need three lemmas.

LEMMA 4. <u>For $n \geq 1$ and $p + q \geq 2$,</u>

$$\frac{\partial^{2n-2+p+q}}{\partial r^{n-1+p}\,\partial s^{n-1+q}} \frac{(r^2 - r_0^2)^n}{r - s} = \frac{\partial^{2n-2+p+q}}{\partial r^{n-1+p}\,\partial s^{n-1+q}} \frac{(s^2 - r_0^2)^n}{r - s}. \tag{8.4}$$

To prove this lemma we observe that the rational function

$$P(r, s) = \frac{(r^2 - r_0^2)^n - (s^2 - r_0^2)^n}{r - s}$$

is an integral polynomial of degree $2n - 1$, and hence that

$$\frac{\partial^{2n-2+p+q}}{\partial r^{n-1+p}\,\partial s^{n-1+q}} P$$

vanishes if $p + q \geq 2$, proving the lemma.

LEMMA 5. <u>Let the function $F(r, s)$ be defined by</u>

$$F(r, s) = \frac{s^2 - r_0^2}{r - s} \tag{8.5}$$

<u>Then for</u> $-r < s < r < r_0$, $\dfrac{\partial^k F}{\partial s^k} < 0$; $\dfrac{\partial^{k+1} F}{\partial s^k \partial r} > 0$, $(k = 1, 2, ---)$.

From (8.5)

$$\frac{\partial F}{\partial s} = -\left[1 + \frac{r_0^2 - r^2}{(r - s)^2}\right], \quad \frac{\partial^k F}{\partial s^k} = -k!\,\frac{r_0^2 - r^2}{(r - s)^{k+1}}, \quad k > 1,$$

and it is apparent that $\dfrac{\partial^k F}{\partial s^k} < 0$. Also from (8.5) we have

$$\frac{\partial F}{\partial r} = \frac{r_0^2 - s^2}{(r - s)^2}, \quad \frac{\partial^{k+1} F}{\partial r\,\partial s^k} = k!\left\{\frac{1}{(r - s)^k} + \frac{(r_0^2 - s^2) + k(r_0^2 - r^2)}{(r - s)^{k+2}}\right.,$$

from which we see that $\dfrac{\partial^{k+1} F}{\partial r\,\partial s^k} > 0$, proving Lemma 5.

LEMMA 6. <u>For</u> $-r_0 < s < r < r_0$, $(-1)^n \dfrac{\partial^k F^n}{\partial s^k} > 0$

$$\text{and } (-1)^n \frac{\partial^{k+1} F^n}{\partial r\, \partial s^k} < 0.$$

Observe that $\dfrac{\partial^k F^n}{\partial s^k}$ is a polynomial of degree $n$ in the variables $F, \dfrac{\partial F}{\partial s}, \cdots \dfrac{\partial^k F}{\partial s^k}$ with positive coefficients. By virtue of Lemma 5 each of the variables is negative, hence $(-1)^n \dfrac{\partial^k F^n}{\partial s^k} > 0$. If now each of the terms of this polynomial be differentiated with respect to $r$, each will contain one positive factor and $n-1$ negative factors, hence $(-1)^n \dfrac{\partial^{k+1} F^n}{\partial s^k \partial r} < 0$, which completes the proof.

To prove the theorem we use Lemma 4 and (8.5) in (8.2) to obtain

$$(8.6)\quad w_{rs} = (-1)^n \frac{2}{n!} \frac{\partial^{n+1} F^n}{\partial r\, \partial s^n}, \quad w_{rss} = (-1)^n \frac{2}{n!} \frac{\partial^{n+2} F^n}{\partial r\, \partial s^{n+1}},$$

and from (M) we have, on using (2.16)

$$(8.7)\quad t = -\frac{2n+1}{2n} w_{rs}, \quad t_r = -\frac{2n+1}{2n} w_{rrs}, \quad t_s = -\frac{2n+1}{2n} w_{rss}.$$

When $\rho_0 < \bar{\rho}_0$ it follows from Corollary 1 that $s_1 < 0$, hence $s < 0$ and $r > s$ inside and on the boundary of the fundamental square. Then by Lemma 6 $w_{rs} < 0$ and $w_{rss} < 0$. Since the only singularities of $\dfrac{\partial^k F^n}{\partial s^k}$ are on the line $r = s$, $w_{rs}$ and $w_{rss}$ remain finite, and the theorem follows from (8.7).

Theorem 4 has the following important corollaries.

COROLLARY 2. <u>The function</u> $t_s(r, s)$ <u>given by</u> (M) <u>is positive on the side</u> $r = r_0$ <u>of the fundamental square and the function</u> $t_r(r, s)$ <u>is negative on the side</u> $s = s_1$.

From Property 2 of the mapping (M), $t_r(r, s) = -t_s(-s, -r)$, and the corollary follows from Theorem 4.

COROLLARY 3. <u>The function</u> $\psi_s(r, s)$ <u>given by</u> (M) <u>is positive on the line</u> $r = r_0$. <u>In Case I</u> $\psi_r$ <u>is positive on the side</u> $s = s_1$, <u>while in Case II</u> $\psi_r$ <u>changes monotonically from negative to positive on</u> $s = s_1$ <u>as</u> $r$ <u>decreases from</u> $r_0$ <u>to</u> $r_1$.

Referring to (2.9) and Corollary 2 we see it is necessary to determine only the sign of $u+G$ on the side $r=r_0$ and of $u-G$ on the side $s=s_1$. Referring to Figures 5a and 5b we see that $u+G>0$ on $r=r_0$ and that $u-G<0$ on $s=s_1$ in Case I, while $u-G$ changes sign once in Case II. If $\rho_0=\rho_0^*$, $\psi_r=0$ at $(r_0,s_1)$.

Appealing to symmetry for the transforms of $s=-r_0$ and $r=r_1$, we find the mapping of the fundamental square onto the $(\psi,t)$-plane for Cases I and II is as shown in Figures 6a and 6b.

COROLLARY 4. <u>The mapping (M) of the fundamental square onto the $(\psi,t)$-plane is one-to-one if $\rho_0<\bar{\rho}_0$.</u>

By Corollaries I and II it is clear that the boundary of the fundamental square is in one-to-one continuous correspondence with a simply-closed regular curve in the $(\psi,t)$-plane. Also, since $t_r(r,s)=-t_s(-s,-r)$, it follows from Theorem 4 that the Jacobian (2.10) is positive and finite throughout the fundamental square. These two conditions are sufficient[13] to insure that the mapping (M) be one-to-one.

We turn now to the investigation of Case III.

THEOREM 5. <u>If $\rho_0>\bar{\rho}_0$, the functions $t(r,s)$, $t_s(r,s)$ are positive in the fundamental square underneath the line $r=s$ and become positively infinite as the point $(r,s)$ approaches the line $r=s$.</u>

This theorem follows from Lemma 6 in view of (3.6) and (3.7).

For points above the line $r=s$ we see from (2.7) that $v(\rho)<0$ and such points will not be considered in view of (2.14).

COROLLARY 5. <u>If $\rho_0\geq\bar{\rho}_0$ the slope in the $(\psi,t)$-plane of the transform of $s=s_1$ $(r=r_1)$ is a monotonic increasing (decreasing) function of $\psi$ tending towards a finite or infinite limit according as $\rho_0\geq\bar{\rho}_0$.</u>

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[13] Cf. G. A. Bliss, <u>Fundamental Existence Theorems</u>, (The Princeton Colloquium, 1909) Part I, p 42

From (2.9) $\frac{d\psi}{dt} = (u-G)^{-1}$ and since $u-G$ decreases as $r$ ranges from $r_0$ to $r_1$ the first part of the theorem is obvious. From (6.5) we see that when $S_1 = 0$, $u-G$ decreases to zero as $r$ ranges from $r_0$ to $0$, hence in this case $\frac{d\psi}{dt}$ becomes infinite.

The mapping of the fundamental square for Case III is shown in Figure 6c.

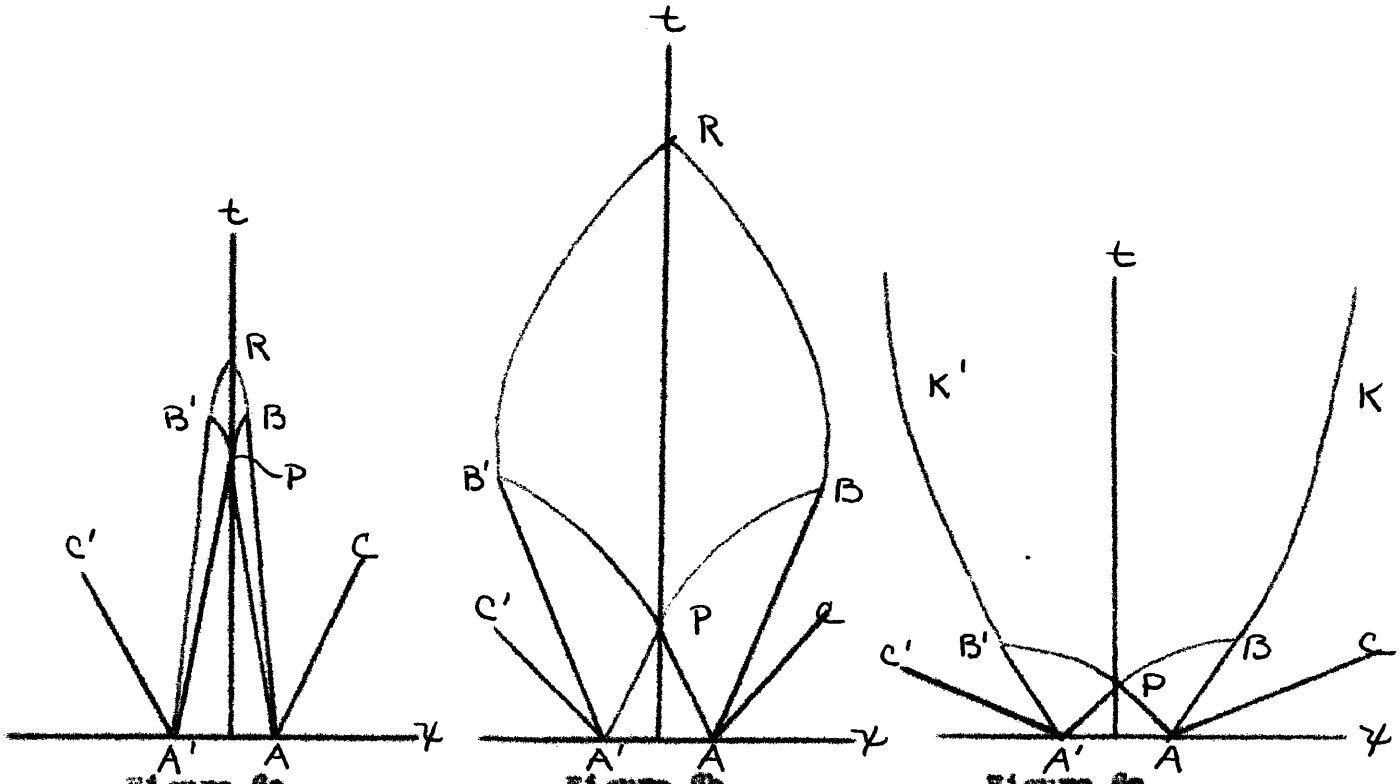


Figure 6a Figure 6b Figure 6c

# PART IV

## THE SECOND INITIAL VALUE PROBLEM

9. Prolongation of the solution. Let us agree to call the region in the $(x,t)$-plane covered by the mapping of the fundamental square the fundamental region. In Figures 6a and 6b the fundamental region is PBRB', while in Figure 6c it is the infinite region bounded by PBK and PBK'. Since states of the gas are symmetrical about the line $x = 0$ we confine our attention in the future to the half-plane $x > 0$. The following theorem may be used to determine states of the gas in a limited region adjoining the fundamental region.

THEOREM 6. The states of the gas along the arc BR (BK) are propagated from this arc along lines of the family

$$x - (u + G)t = w_r, \quad r_0 \geq r \geq r_1, \quad s = s_1; \tag{9.1}$$

where $w$ is the fundamental solution. The extent of this solution is limited by the intersection of the propagation line from B with the shock wave emanating from A.

For convenience we shall restrict our attention to Case I, but the proof will be valid for Cases II and III as well. Referring to Figure 8a and recalling that the quantity $2s_1 = u - v$ is constant on the arc BR, it follows from Theorem I that states on this arc are propagated along lines of slope $(u+G)^{-1}$. By (2.11) we see that points on the arc BR satisfy (9.1), and since the slope of each member of this family is $(u+G)^{-1}$ it follows that the states on BR are propagated as stated, thus prolonging the solution above the fundamental region. To show that the solution thus defined is not limited above the fundamental region by the envelope of the family (9.1) we shall need the following lemma.

LEMMA 7. The quantity $w_{rr}$ is positive for $-r_0 < s < r < r_0$.

Using Lemma 4 and (8.5) in the fundamental solution (8.2) we find

$$w_{ss} = (-1)^n \frac{2}{n!} \frac{\partial^{n+1}}{\partial s^{n+1}} F^n.$$

Then from Lemma 6 we see that $w_{ss} > 0$ and from the symmetry property of $w$ we have

$$w_{rr}(r,s) = + w_{ss}(-s,-r) > 0,$$

establishing Lemma 7.

From (6.5) and the above lemma we see that for a fixed $s$, $u + G$ and $w_r$ are monotonic increasing functions of $r$. Then from (9.1) it is evident that the slopes of the lines of this family decrease as the $\psi$ intercepts increase, eliminating the possibility of an envelope in the region $t > 0$. Since no two members of (9.1) intersect in the region $t > 0$, no member of (9.1) intersects the arc BR a second time.

Finally we show that the propagation line from B intersects the shock wave emanating from A at a point C (see Figure 7).

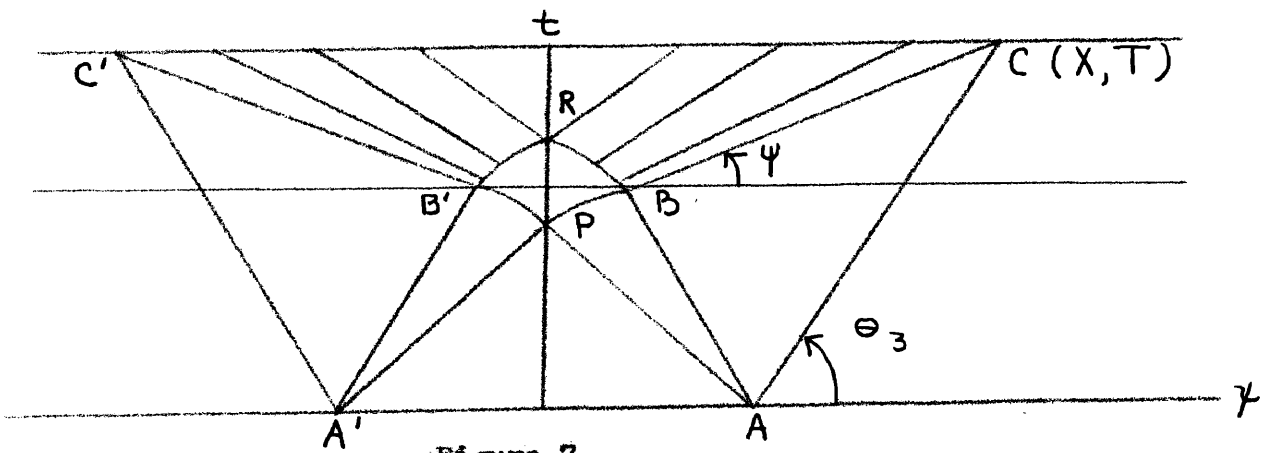


Figure 7

From (9.1) we have

$$\cot \psi = u(r_0, s_1) + G(r_0, s_1) = u_1 + G_1.$$

Using (2.14) and (4.4) the inequality

(9.2) $\cot \theta_3 < u_1 + G_1$

reduces to the inequality

$$r\rho_1^{r+1} - (r+1)\rho_1^r \rho_2 + \rho_2^{r+1} > 0$$

which is easily verified for $\rho_1 > \rho_2$ inasmuch as the first member vanishes for $\rho_1 = \rho_2$ and has a positive derivative with respect to $\rho_1$ for $\rho_1 > \rho_2$. This completes the proof.

If in (7.2) we put $s = s_1$ and use (6.5) we find as the coordinates of the point B

$$(9.3) \qquad x_1 = 1 + \left(\frac{v_0}{v_1}\right)^{\lambda} \frac{u_1 - G_1}{G_1}, \quad t_1 = \frac{1}{G_1}\left(\frac{v_0}{v_1}\right)^{\lambda}.$$

Upon solving the equation of the line BC simultaneously with that of the shock wave AC we find as the coordinates $(X, T)$ of the point C,

$$(9.4) \qquad X = 1 + \left(\frac{v_0}{v_1}\right)^{\lambda} \frac{2 \cot \theta_3}{u_1 + G_1 - \cot \theta_3}, \quad T = \left(\frac{v_0}{v_1}\right)^{\lambda} \frac{2}{u_1 + G_1 - \cot \theta_3}.$$

For $t > T$ the state to the left of the shock wave is no longer constant, the velocity of the shock wave varying as the shock is propagated through the gas.

For the region bounded by the propagation line emanating from the point B and the $t$-axis in Cases I and II we choose the constant solution $r = r_1$, $s = s_1$ of (2.6), which joins continuously to the propagated solution.

10. <u>The second initial value problem.</u> The intersection at C in Figure 7 of the propagation line from B and the shock wave AC will affect the states of the gas for $t > T$. Let us denote by L the locus of points in the $(x, t)$-plane on which the solution propagated from BR becomes invalid, and by S the path in the $(x, t)$-plane of the shock wave for $t > T$. We seek a solution of (2.6) for which states on L are given by Theorem 6 and for which the slope of S at any point is found from (4.2). Following §2 we transform this problem to the $(r, s)$-plane and seek a solution $\bar{w}$ to (2.16) for which the initial values are determined as follows.

From Theorem 6 we see that at every point of L, $x - (u+G)t = w_r$, where

$w$ is the fundamental solution. From (2.11) we have, since L must pass through the point $(X, T)$,

(10.1) $\bar{w}_r(r, s_1) = w_r(r, s_1), \quad \bar{w}_s(r_0, s_1) = X - (u_1 - G_1)T,$

as initial conditions on the solution $\bar{w}$ of (2.18). The parametric equations of the transform in the $(r, s)$-plane of the curve S may be obtained by using (1.1), (4.1), and (2.14) in (2.7), and it may be easily verified that this transform is the curve $\sigma$ given by (6.2), where $\rho$, rather than $\rho_1$, is now the parameter. If we write

$$\frac{d\psi}{dt} = \frac{\psi_r \dot{r} + \psi_s \dot{s}}{t_r \dot{r} + t_s \dot{s}},$$

where a dot indicates differentiation with respect to $\rho$, and use (2.9) we obtain for the slope of S

$$\text{(10.2)} \qquad \frac{d\psi}{dt} = \frac{(u - G)t_r \dot{r} + (u + G)t_s \dot{s}}{t_r \dot{r} + t_s \dot{s}},$$

in which $u$ and $\rho$ are connected by (4.1). Solving (10.2) for $\frac{t_s}{t_r}$ we find

$$\text{(10.3)} \qquad \frac{t_s}{t_r} = \frac{\dot{r}}{\dot{s}} \quad \frac{u - G - \frac{d\psi}{dt}}{\frac{d\psi}{dt} - (u + G)}.$$

But since the velocity $\frac{d\xi}{dt}$ of the shock wave is equal to the value of $\frac{d\psi}{dt}$ on S, we may use (4.2) and (4.1) in (10.3) to obtain $\frac{t_s}{t_r}$ as a function of $\rho$ alone. If from (6.2) we eliminate the parameter $\rho$ we have finally

$$\text{(10.4)} \qquad \frac{t_s}{t_r} = \alpha(r, s), \quad \alpha = \frac{dr}{ds} \quad \frac{u - G - \frac{d\xi}{dt}}{\frac{d\xi}{dt} - (u + G)}.$$

From (8.7) we obtain the condition for the solution $\bar{w}$ on the curve $\sigma$,

$$\text{(10.5)} \qquad \bar{w}_{rss} - \alpha \bar{w}_{rrs} = 0, \quad \alpha = \alpha(r, s).$$

If (10.5) is satisfied on the curve $\sigma$ and if (10.1) is satisfied we shall call $\bar{w}$ a <u>secondary solution</u>. In order to determine the region in the

$(r, s)$-plane to be mapped under the transformation (M) we investigate the sign of the Jacobian at the point $(r_0, s_1)$.

THEOREM 7. <u>The Jacobian of the transformation (M) with $w = \bar{w}$ is negative at $(r_0, s_1)$.</u>

We observe from (2.10) that it is sufficient to prove that $t_r(r_0, s_1)$ is finite and different from zero and that $\frac{t_s}{t_r} > 0$. From (M) we find, using (8.7),

$$(r_0 - s_1)\, t_r(r_0, s_1) = -(n+1)\,T - \tfrac{1}{2}(2n+1)\, w_{rr}(r_0, s_1),$$

hence $t_r(r_0, s_1) < 0$ by virtue of (9.4), (10.1), and Lemma 7. To show $\frac{t_s}{t_r} > 0$ we use (10.3) and replace $u$ by $u_1$, $G$ by $G_1$, and $\frac{d\psi}{dt}$ by $\cot \Theta_3$. By Theorem 5 $\frac{\dot{r}}{\dot{s}} > 0$, by (4.4) $u_1 - G_1 - \cot \Theta_3 < 0$, and from (9.2) $\cot \Theta_3 - (u_1 + G_1) < 0$, completing the proof.

Since L is the transform of $s = s_1$ and the Jacobian of (M) is negative at $(r_0, s_1)$, the curve S will be the transform of the portion of $\sigma$ included between $(r_2, s_2)$ and $(r_0, s_1)$. We define as the <u>secondary</u> $(r, s)$-<u>region</u> in Cases I and II that region bounded by the lines $s = s_1$, $r = r_1$, $s = s_2$ and the curve $\sigma$. In Case III the line $r = s$ also bounds the secondary region (see Figure 8). From (2.9) we note that the slope of L at a point at which the state is $(u, \rho)$ is identical with the slope of BR at the point which bears the same state. From (2.9) we see also that the slope of the transform of $r = r_1$ is positive at the point M (Cases I and II) as illustrated in Figure 9. Further information about the mapping of the secondary $(r, s)$-region requires a knowledge of the properties of $\bar{w}$.

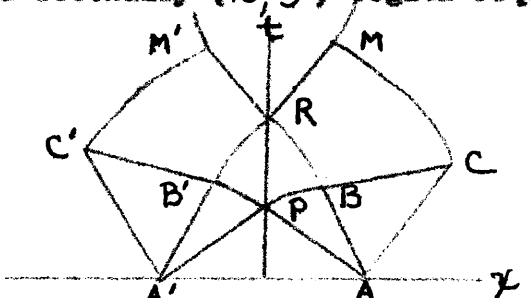

Figure 9

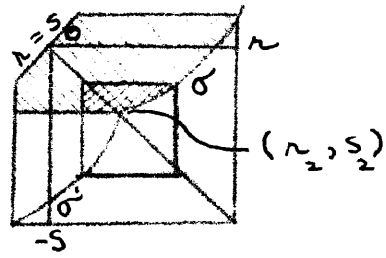

Figure 8

11. <u>The ordinary differential equation associated with the second initial value problem for positive integral</u> $\lambda$. According to (8.1) the secondary solution for the case $\lambda = n$ has the form

$$\bar{w} = -\frac{1}{n!\,(n-1)!}\,\frac{\partial^{2n-2}}{\partial r^{n-1}\partial s^{n-1}}\,\frac{\bar{R}+\bar{S}}{r-s}$$

where $\bar{R}$ is a function of $r$ and $\bar{S}$ a function of $s$. If we choose $\bar{R} = (r^2 - r_0^2)^n$ we see by the first condition of (10.1), upon using (8.2) and equating coefficients of $(r-s)^{-1}$

$$(11.1)^{14}\quad \begin{aligned} &\bar{S}(s_1) = (s_1^2 - r_0^2)^n, \\ &\bar{S}'(s_1) = \left[(s^2 - r_0^2)^n\right]'_{s=s_1}, \\ &- - - - - - - - - \\ &- - - - - - - - - \\ &\bar{S}^{(n-1)}(s_1) = \left[(s^2 - r_0^2)^n\right]^{(n-1)}_{s=s_1}; \end{aligned}$$

while the second member of (10.1) imposes the condition

$$(11.2)\quad -\frac{1}{n!\,(n-1)!}\,\frac{\partial^{2n-1}}{\partial r^{n-1}\partial s^{n}}\left\{\frac{(r^2-r_0^2)^n+\bar{S}}{r-s}\right\}_{\substack{r=r_0\\ s=s_1}} = X-(u_1-G_1)T.$$

Thus the value of $\bar{S}$ and its first $n$ derivatives at $s = s_1$ are determined by (10.1). From (10.5) we find also

$$(11.3)\quad \frac{\partial^{2n+1}}{\partial r^{n}\partial s^{n+1}}\left\{\frac{(r^2-r_0^2)^n+\bar{S}}{r-s}\right\} - \alpha\,\frac{\partial^{2n+1}}{\partial r^{n+1}\partial s^{n}}\left\{\frac{(r^2-r_0^2)^n+\bar{S}}{r-s}\right\} = 0,$$

where $r$ is a function of $s$ by virtue of (6.2). The function $\bar{S}$ is accordingly a solution of an ordinary differential equation of order $n+1$ subject to the initial conditions (11.1) and (11.2). To throw the equation into a more convenient form we note from (8.4) that

$$\frac{\partial^{2n+1}}{\partial r^{n}\partial s^{n+1}}\left\{\frac{(r^2-r_0^2)^n}{r-s}\right\} = \frac{1}{2}\,\frac{\partial^{2n+1}}{\partial r^{n}\partial s^{n+1}}\left\{\frac{(r^2-r_0^2)^n+(s^2-r_0^2)^n}{r-s}\right\}$$
$$= -\frac{n!\,(n-1)!}{2}\,w_{rss},$$

---

[14] In (11.1), primes indicate differentiation with respect to $s$.

and similarly

$$\frac{\partial^{2n+1}}{\partial r^{n+1}\partial s^{n}}\left\{\frac{(r^2-r_0^2)^n}{r-s}\right\} = -\frac{n!\,(n-1)!}{2}\,w_{rrs},$$

$w$ being the fundamental solution. Finally from (11.5) and (8.7) we obtain

$$(11.4)\quad \frac{\partial^{n+1}}{\partial s^{n+1}}\frac{\bar{S}}{(r-s)^{n+1}} + (n+1)\alpha\frac{\partial^{n}}{\partial s^{n}}\frac{\bar{S}}{(r-s)^{n+2}} = \frac{(-1)^{n+1}n!}{2n+1}(t_s-\alpha t_r)$$

where $r$ is to be evaluated as a function of $s$ after partial differentiation, and $t$ is given by (4) with $w$ given by (8.2).

12. <u>The variation of</u> $\rho$ <u>with</u> $x$ <u>for</u> $t < T$. From Figure 9 it is evident that horizontal lines drawn through the points P, B, and C will divide the $(x, t)$-plane into regions in which the graphs of $\rho$ vs. $x$ for constant $t$ will be essentially different. Since the arc PB and the segment BC have positive slope for all $\rho_0 > \rho_2$, the variation of $\rho$ with $x$ will be of the same nature in Cases I, II, and III unless there exist values of $\rho_0 > \rho_2$ for which the ordinate of C is greater than that of R, in which case a horizontal line through R adds a fourth region to be considered (see Figure 10). We shall show that such values of $\rho_0$ do exist, and a study of Regions I, II, III, IV in Figure 10 will be sufficient to include all cases.

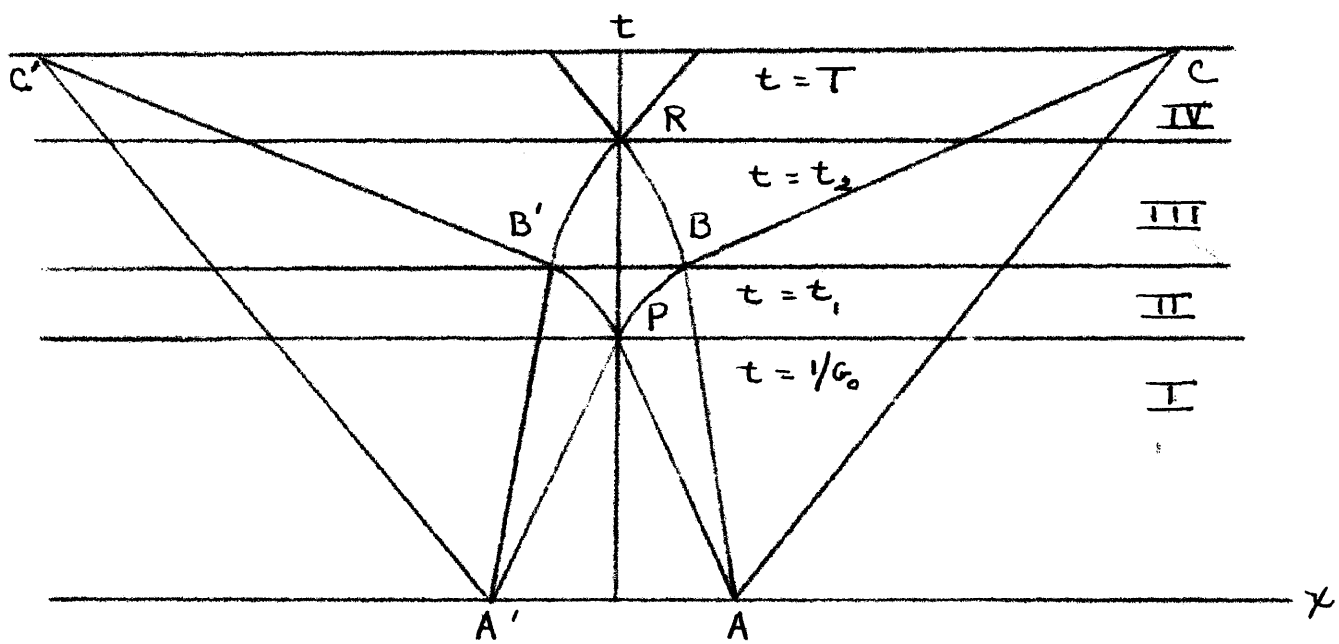


Figure 10

From (9.4) and (4.4),

$$\frac{1}{T} = \frac{1}{2}\left(\frac{v_1}{v_0}\right)^{\lambda}\left(G_1 - \left[\frac{\rho_2}{\rho_1}\,\frac{p_2 - p_1}{\rho_2 - \rho_1}\right]^{1/2}\right)$$

and using (2.3) we find

$$\lim_{\rho_1 \to \rho_2}\left[\frac{\rho_2}{\rho_1}\,\frac{p_1 - p_2}{\rho_1 - \rho_2}\right]^{\frac{1}{2}} = \left(\frac{dp_1}{d\rho_1}\right)^{\frac{1}{2}}_{\rho_1 = \rho_2} = G_2 = \lim_{\rho_1 \to \rho_2} G_1 .$$

As $\rho_1$ decreases toward $\rho_2$, $\rho_0$ also decreases toward $\rho_2$, hence $T$ is arbitrarily large if $\rho_0 - \rho_2$ is sufficiently small. By Theorem 4 the ordinate $t_2$ of B remains finite when $\rho_0 < \bar{\rho}_0$ and the case shown in Figure 10 thus exists. We investigate first Region I.

As one proceeds from right to left on a horizontal line for which $t < 1/\sigma_0$ one encounters an abrupt increase in density in crossing the shock wave, the new density $\rho_1$ then remaining constant until the segment AB is reached. Inside the buffer region ABP the quantity $2r_0 = u + v$ is constant while $u - v$ decreases from $2S_1$ to $-2r_0$, hence $v$ (and $\rho$) increases as shown in Figure B of § 1.

In Region II the shock wave, the constant state $(u_1, \rho_1)$, and the buffer region are again encountered as in Region I. To determine the variation of $\rho$ inside the fundamental region, we restrict ourselves to a monatomic gas ($\gamma = \frac{5}{3}$, $n = 1$) and show that inside the fundamental region

(12.1) $$v_\psi < 0 \text{ for } \psi > 0; \quad v_\psi > 0 \text{ for } \psi < 0.$$

By virtue of (M) the line $t =$ const. defines a curve in the $(r, s)$-plane for which

(12.2) $$t_r \frac{dr}{ds} + t_s = 0, \quad r = r(s).$$

On this curve we find from (M) and (2.7)

$$\frac{d\psi}{ds} = \psi_r \frac{dr}{ds} + \psi_s, \quad \frac{dv}{ds} = \frac{dr}{ds} - 1,$$

which, upon using (2.9) and (12.2), lead us to the relation

$$\frac{dv}{d\psi} = \frac{t_r + t_s}{-2\sigma t_r t_s}$$

Putting $n = 1$ in (8.7) and using (8.2) we have

$$t = 6\,\frac{r_0^2 - rs}{(r-s)^3}, \quad t_r + t_s = -6\,\frac{r+s}{(r-s)^3},$$

from which we obtain

$$\frac{dv}{d\psi} = -6\,\frac{r+s}{-2G\,t_r t_s (r-s)^3}\,.$$

But the Jacobian $-2G\,t_r\,t_s$ was shown in § 8 to be positive and the region $r+s>0$ of the fundamental square maps, under (M), onto the region $\psi>0$ of the fundamental region, hence (12.1) is established. The variation of $\rho$ with $\psi$ in Region II is illustrated in Figure C of § 1.

In Region III after passing through the constant state $(u_1, \rho_1)$ one encounters the states propagated from the line BR. On BR the quantity $2s_1 = u - v$ is constant while $2r = u+v$ decreases from $r_0$ to $r_1$, hence $v$ decreases with $\psi$ in Region III. In the fundamental region $v$ again increases, and the variation of $\rho$ with $\psi$ is as shown in Figure D.

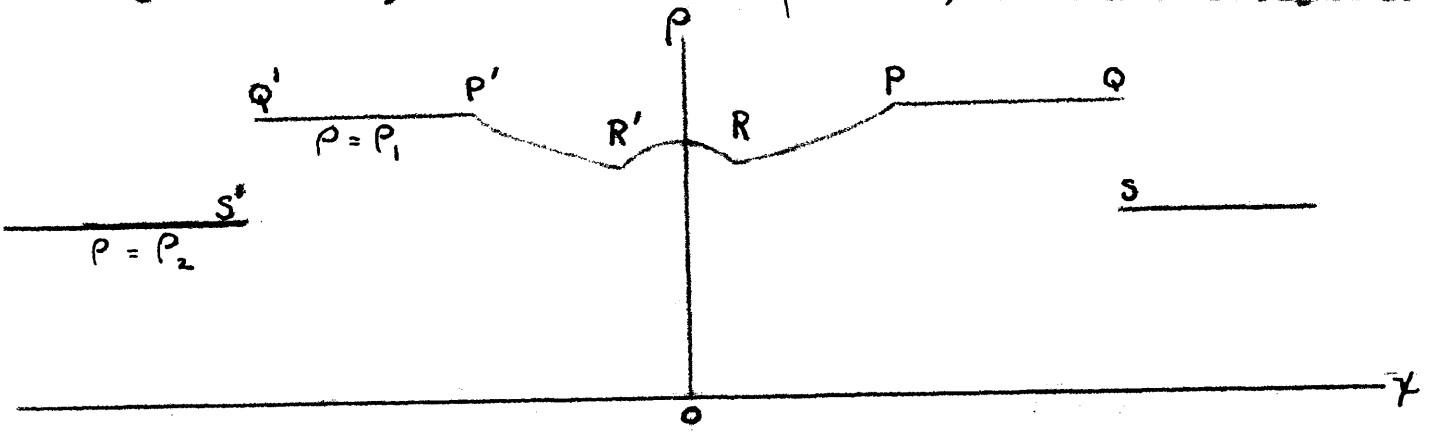


Figure D

The value of $\rho$ on all or a portion of RR' may be less than $\rho_2$ if the constant value of $t$ is sufficiently near $t_2$.

In Region IV after emerging from the region of propagated states one encounters the constant state in which the function $v$ has the value $2r_1$. Since $2r_1 < 2r_2$ and $2r_2 = v_2$, we see that the value of $\rho$ in this region is less than $\rho_2$ as illustrated in Figure E.

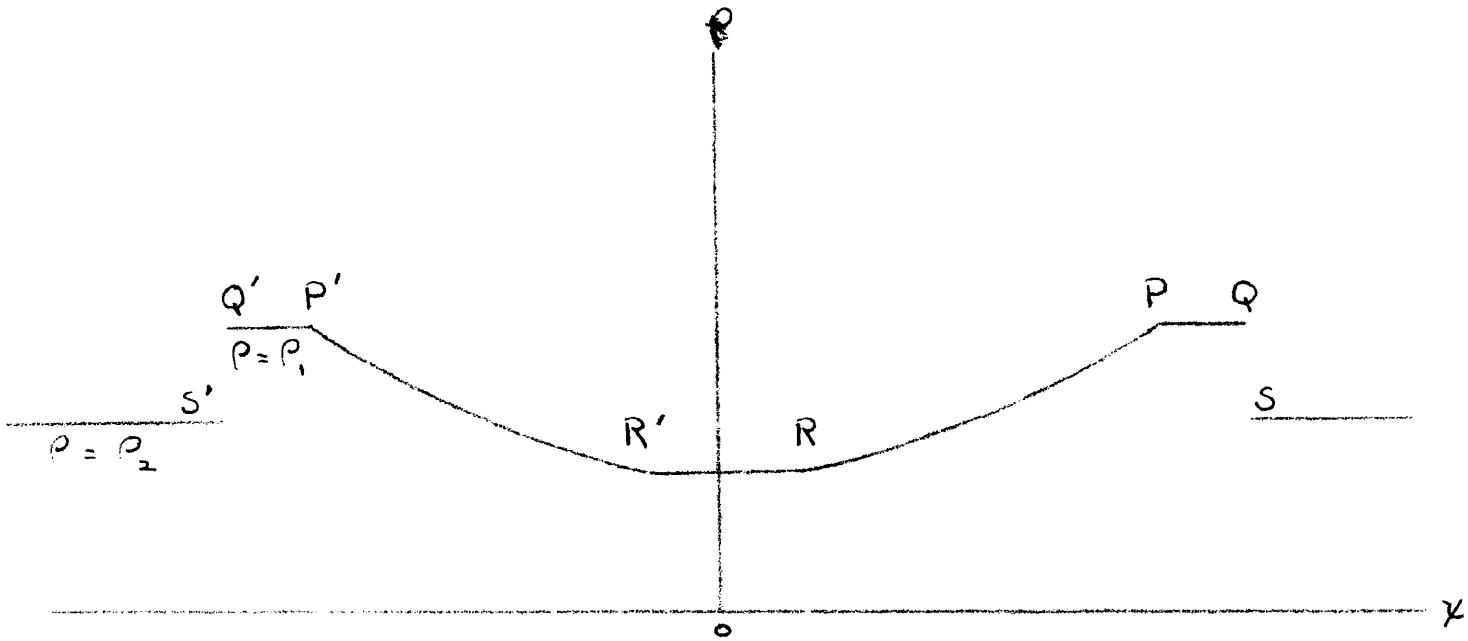


Figure 5

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