AN ARC PROBLEM

By

Jean M. Boyer

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AN ARC PROBLEM

A topological space A is said to be embedded topologically in a topological space B if there exists a subset A' of the space B and a transformation T(A) = A' which is one-to-one and continuous in both directions. Under these conditions the set A' is said to be homeomorphic to A, and the transformation is called a homeomorphism.

The question as to whether a given topological space A can be embedded topologically in a topological space B is an unsolved and apparently extremely difficult problem. It may be approached from at least two points of view. First, one may require that there exist a subset A' of B and a single valued continuous mapping T(A) = A', and then seek conditions on this mapping T to insure that it is a homeomorphism. Such an approach has been used by J. F. Wardwell.[1] It can be considered an analytic approach to the problem.

The other approach (the one which will interest us here) attempts to solve the problem from a structure-theoretic standpoint. In other words, one attempts to solve the problem by placing additional hypotheses on the structure of the space A rather than on a mapping from this space into the space B.

The problem can be made more meaningful, perhaps, if we restrict

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[1] J. F. Wardwell, "Continuous transformations preserving all topological properties", American Journal of Mathematics, vol.58 (1936), pp. 709-726.

the spaces A and B in our discussion to spaces having well known properties. Consider, for example, the case where A is an arbitrary compact locally connected continuum (a Peano space) and where B is the two dimensional sphere $x^2 + y^2 + z^2 = 1$. The set A is said to be <u>skew</u> if it cannot be embedded topologically in the set B.

Kuratowski has introduced the following two sets, which have become quite famous in topology.[2]

<u>A primitive skew curve</u> is said to be of type 1 if it consists of six distinct points P_1, P_2, P_3, Q_1, Q_2, Q_3, and nine arcs P_1Q_1, P_1Q_2, ... , P_3Q_3 with end points as indicated and with the common part of two of these arcs that intersect each other being an end point of each.

<u>A primitive skew curve</u> is said to be of type 2 if it consists of five distinct points P_1, P_2, P_3, P_4, P_5, and ten arcs P_1P_2, P_1P_3, ... , P_4P_5 with end points as indicated and with the common part of two of these arcs that intersect each other being an end point of each.

Kuratowski has proved that a skew Peano space containing only a finite number of simple closed curves must contain a primitive skew curve of type 1 or type 2.[3] Claytor showed that a cyclic Peano space can be skew only if it contains one of these two types

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[2] C. Kuratowski, "Sur le probleme des courbes gauches en topologie", <u>Fundamenta Mathematicae</u>, vol. 15 (1930), pp. 271-283.

[3] Kuratowski, <u>loc. cit.</u>

of curves.[4]

R. H. Bing has recently stated the following two theorems.[5] The proofs of these theorems were not published because of their lengths.

Theorem A: Suppose that P_1, P_2, P_3, Q_1, Q_2, Q_3 are six distinct points and that P_1Q_1, P_1Q_2, ... , P_3Q_3 are nine arcs with end points as indicated and such that two of these arcs intersect each other only if they have an end point in common. Then the sum of these arcs is not homeomorphic to any plane set.

Theorem B: Suppose that P_1, P_2, P_3, P_4, P_5 are five distinct points and that P_1P_2, P_1P_3, ... , P_4P_5 are ten arcs with end points as indicated and such that two of these arcs intersect each other only if they have an end point in common. Then the sum of these arcs is not homeomorphic to any plane set.

The reader is cautioned to note carefully the difference between the set described in Theorem A and a primitive skew curve of type 1; also the difference between the set described in Theorem B and a primitive skew curve of type 2. It is evident that either of these theorems could be proved by constructing in the set described a primitive skew curve of either type 1 or type 2.

The discussion of the previous paragraph indicates, and it can be proved rather easily, that both Theorem A and Theorem B

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[4] S. Claytor, "Topological immersion of Peanian continua in a spherical surface", Annals of Mathematics, vol. 35 (1934), pp. 809-835.

[5] R. H. Bing, "Skew Sets", American Journal of Mathematics, vol. 69 (1947), pp. 493-498.

would follow at once if it were known that the following more general question of Bing could be answered in the affirmative:[6]

Question: _Is the following statement true? If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arcs two of which intersect only if they have an end point in common, then there exist arcs $\beta_1, \beta_2, \ldots, \beta_n$ such that_

i) _two of these arcs intersect only if they have an end point in common;_

ii) _the common part of two of these arcs that intersect is connected;_

iii) β_i $(i = 1,2,\ldots,n)$ _is an arc in_ $\bigcup_{j=1}^{n} \alpha_j$ _having the same end points as_ α_i.

In this paper we answer this question in the affirmative in each of the following cases:

a) At least n-2 of the arcs have a common end point.

b) No three arcs have a common end point, if n is even.

c) The integer n is less than five.

We also establish the truth of the statement in certain special cases when n = 5,6.

It is found in the proof that, for small values of n at least, the problem may be subdivided into the consideration of certain simple geometric configuations. It is hoped that the methods developed here can be extended eventually to answer the general question in the affirmative.

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[6]Bing, _loc. cit._ p

By modifying the conclusion of the statement slightly we obtain a theorem valid for all values of n which is easily proved. This general result is stated as the first theorem of the paper.

Throughout the paper the notation $x = (rs:K)$ will be used to denote the first point of the closed set K on the simple arc rs in the order from r to s. The end points of the arc α_i will be denoted by a_i, b_i.

The following theorem differs from the problem of Bing in that two intersecting arcs β_i, β_j are not required to have an end point in common.

__Theorem 1: If $\alpha_1, \alpha_2, \dots, \alpha_n$ are arcs two of which intersect only if they have an end point in common, then there exist arcs $\beta_1, \beta_2, \dots, \beta_n$ such that__

a) __the common part of two of these arcs is connected or empty;__

b) β_i $(i = 1,2,\dots,n)$ __is an arc in $\bigcup_{j=1}^{i} \alpha_j$ having the same end points as α_i;__

c) __the set $B_n = \bigcup_{i=1}^{n} \beta_i$ contains no simple closed curve;__

Proof: The proof will be by induction on the number of arcs. The theorem is obviously true for $n = 1$. We shall assume it is true for $n = k-1$ and prove it is true for $n = k$.

Choose any k-1 of the given arcs as $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$. We can find arcs $\beta_1, \beta_2, \dots, \beta_{k-1}$ satisfying the required conditions. Let $B_{k-1} = \bigcup_{i=1}^{k-1} \beta_i$. Every component of B_{k-1} which intersects α_k contains at least one of the points a_k, b_k. Thus there exist at most two such components.

If $a_k \not\in B_{k-1}$ and $b_k \not\in B_{k-1}$ then α_k and B_{k-1} are disjoint. Thus we can define $\beta_k = \alpha_k$.

If $a_k \in B_{k-1}$, and $b_k \not\in B_{k-1}$, let $p = (\overrightarrow{b_k a_k} : B_{k-1})$. Then a_k and p must lie in the same component B' of B_{k-1}. There is an arc γ in B' from a_k to p since B' is arcwise connected. Define $\beta_k =$ $(\gamma) \cup (\widehat{pb_k}$ of $\alpha_k)$.

If a_k and b_k are in the same component B' of B_{k-1} there is an arc γ in B' from a_k to b_k. Define $\beta_k = \gamma$.

If a_k and b_k are in distinct components A' and B' respectively of B_{k-1}, let $q = (\overrightarrow{b_k a_k} : A')$ and let $r = (\overrightarrow{qb_k} : B')$. Then there is an arc γ_1 in A' from a_k to q and an arc γ_2 in B' from r to b_k. Define $\beta_k = (\gamma_1) \cup (\widehat{qr}$ of $\alpha_k) \cup (\gamma_2)$.

The reader may note that if $\beta_k \cap \beta_i$ is not connected for some $i = 1,2,\ldots,k-1$ then there is a subarc σ of β_k spanning β_i and lying in B_{k-1}.[7] Hence the union of β_i and σ contains a simple closed curve. This contradicts the inductive hypothesis.

If we define $B_k = \bigcup_{i=1}^{k} \beta_i$ it is easy to see that B_k contains no simple closed curve, and the arcs β_i $(i = 1,2,\ldots,k)$ satisfy the required conditions. Hence Theorem 1 is established.

An example will show that this theorem is not true if the additional restriction is imposed that two intersecting arcs β_i, β_j

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[7] A nondegenerate arc σ spans an arc δ if σ has exactly its end points on δ. An arc σ with distinct end points r,s is said to span from a set A to a set B provided $r \in A$, $s \in B$, and $(\sigma - r - s) \cap (A \cup B) = 0$.

have a common end point. Let

$$\alpha_i \cap \alpha_{i+1} = b_i = a_{i+1}, \ (i = 1,2,3),$$
$$\alpha_4 \cap \alpha_1 \quad = b_4 = a_1.$$

To satisfy the additional restriction we must choose $\beta_i = \alpha_i$, $(i = 1,2,3,4)$. This choice gives us a simple closed curve in $\bigcup_{i=1}^{4} \beta_i$ which contradicts condition c).

In discussing the problem in its original form we shall call n the order of the set $\bigcup_{i=1}^{n} \alpha_i$. We can assume $\bigcup_{i=1}^{n} \alpha_i$ is connected, since if not, the problem can be reduced to several cases of lower order. Also, if $a_k = a_j$ and $b_k = b_j$ for any k,j we can choose $\beta_k = \beta_j$ since ii) and iii) are obviously satisfied and β_k may intersect anything that β_j is permitted to intersect. Thus we can assume $\alpha_k = \alpha_j$ and reduce the order of the set by one. We shall henceforth consider the order of our sets has been reduced as much as possible in this manner.

For $n = 1$ the solution to the problem is trivially in the affirmative. Hence we can assume $n > 1$, and need not establish a particular case in each inductive argument.

<u>Theorem 2</u>: <u>The question of Bing can be answered in the affirmative under any one of the following conditions</u>:

a) <u>All the arcs have a common end point. In this case we can number the arcs in any order and so choose β_i, $(i = 1,2,\ldots,n)$ that $\bigcup_{i=1}^{n} \beta_i$ contains no simple closed curve. Moreover, $\beta_j \subset \bigcup_{i=1}^{j} \alpha_i$ for</u> $j = 1,2,\ldots,n$.

b) <u>Exactly</u> n-1 <u>of the arcs have a common end point. In this case $\bigcup_{i=1}^{n} \beta_i$ contains at most one simple closed curve.</u>

c) No three arcs have a common end point., if n is even.

d) Exactly n−2 of the arcs have a common end point.

e) The integer n is less than five.

Proof: a) This statement is a corollary to Theorem 1.

b) Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ denote the arcs with a common end point $a_1 = a_2 = \ldots = a_{n-1}$. Since the set is connected, there is an integer $t \leq n-1$ such that $b_n = b_t$ or $a_n = b_t$. No generality is lost if we assume that the former is the case. We shall number the arcs α_i, $i = 1,2,\ldots,n-1$ in such a manner that $t = n-1$. By a) the arcs $\beta_1, \beta_2, \ldots, \beta_{n-1}$ may be found satisfying the requirements i), ii), and iii) and such that $\beta_j \subset \bigcup_{i=1}^{j} \alpha_i$ for $j = 1,2,\ldots,n-1$.

If $a_n \neq b_j$ for $j = 1,2,\ldots,n-2$, we let $c_1 = (\widehat{a_n b_n} : \bigcup_{i=1}^{n-1} \beta_i)$. Note that c_1 must lie on β_{n-1} and on no other β_i since α_n meets no arc α_i except α_{n-1} and $\beta_j \subset \bigcup_{i=1}^{j} \alpha_i$ for $j = 1,2,\ldots,n-2$. Since c_1 is not in any β_j, $j \leq n-2$, then no point of $\widehat{c_1 b_n}$ in β_{n-1} is in any β_j, $j \leq n-2$ by ii). We define β_n to be the union of $\widehat{a_n c_1}$ in α_n and $\widehat{c_1 b_n}$ in β_{n-1}.

If $a_n = b_t$ for $t \leq n-2$ we number the arcs α_i, $i = 1,2,\ldots,n-2$ in such a manner that $t = n-2$. Let $c_1 = (\widehat{b_n a_n} : \beta_{n-2})$, $c_2 = (\widehat{c_1 b_n} : \beta_{n-1})$. Define β_n as contained in the union of $\widehat{b_n c_2}$ of β_{n-1}, $\widehat{c_2 c_1}$ of α_n, and $\widehat{c_1 a_n}$ of β_{n-2}. Condition iii) is immediate. Also $\beta_n \cap \beta_{n-1} = \widehat{b_n c_2}$ which is connected; $\beta_n \cap \beta_{n-2} = \widehat{c_1 a_n}$ which is connected; and $\beta_n \cap \beta_j = 0$ for $j = 1,2,\ldots,n-3$ since $\beta_j \subset \bigcup_{i=1}^{n-3} \alpha_i$ and α_n does not meet $\bigcup_{i=1}^{n-3} \alpha_i$. Hence neither c_1 nor c_2 is on any β_j, $j \leq n-3$, and therefore neither $\widehat{c_1 a_n}$ on β_{n-2} nor $\widehat{b_n c_2}$ on β_{n-1} meet β_j, $j = 1,2,\ldots,n-3$. It is evident that in either case $\bigcup_{i=1}^{n} \beta_i$ contains at most one simple closed curve. Hence Part b) is established.

All cases of order $n \leq 3$ are covered by a) and b). Therefore we shall henceforth assume $n \geq 4$.

c) If there exists an α with a free end, let any such be α_1. If not, let any α be α_1. Since $\bigcup_{i=1}^{n} \alpha_i$ is connected, the notation may be chosen in such a way that $a_2 = b_1$. Similarly if α_{k-1} has been designated, we may choose the notation such that $a_k = b_{k-1}$. Define

$$c_{11} = (\widehat{b_2a_2} : \alpha_1)$$

$$c_{1i} = (\widehat{b_{i+1}a_{i+1}} : \widehat{b_ic_{1(i-1)}} \text{ of } \alpha_i), \quad (i = 2,3,\ldots,n-1).$$

If $b_n = a_1$ we define in addition

$$c_{1n} = (\widehat{c_{1(n-1)}b_n} \text{ of } \alpha_n : \alpha_1), \quad d_1 = (\widehat{b_1a_1} : \widehat{b_nc_{1n}} \text{ of } \alpha_n).$$

If $b_n \neq a_1$ we can reach the desired conclusion by defining $\beta_1 = \alpha_1$, $\beta_i = (\widehat{a_ic_{1(i-1)}} \text{ of } \alpha_{i-1}) \cup (\widehat{c_{1(i-1)}b_i} \text{ of } \alpha_i)$, $(i = 2,3,\ldots,n)$.

The remaining case, $b_n = a_1$ is more troublesome. If the order on α_1 is $b_1c_{11}d_1a_1$ we define $\beta_1 = (\widehat{a_1d_1} \text{ of } \alpha_n) \cup (\widehat{d_1b_1} \text{ of } \alpha_1)$, $\beta_i = (\widehat{a_ic_{1(i-1)}} \text{ of } \alpha_{i-1}) \cup (\widehat{c_{1(i-1)}b_i} \text{ of } \alpha_i)$, $(i = 2,3,\ldots,n)$. See Figure 1.

If the order on α_1 is $b_1d_1c_{11}a_1$ (see Figure 2) we let

$$c_{21} = (\widehat{b_2a_2} : \widehat{b_1d_1} \text{ of } \alpha_1),$$

$$c_{2i} = (\widehat{b_{i+1}a_{i+1}} : \widehat{b_ic_{2(i-1)}} \text{ of } \alpha_i), \quad (i = 2,3,\ldots,n-1),$$

$$c_{2n} = (\widehat{c_{2(n-1)}b_n} \text{ of } \alpha_n : \alpha_1), \quad d_2 = (\widehat{b_1a_1} : \widehat{b_nc_{2n}} \text{ of } \alpha_n).$$

Since n is even we note that the order on α_1 is $b_1d_2d_1a_1$. If the order on α_1 is $b_1c_{21}d_2a_1$ we define the β's in a way similar to that above. If the order on α_1 is $b_1d_2c_{21}a_1$ we note that $d_1 \neq d_2$, hence $c_{1(n-1)} \neq c_{2(n-1)}$ and $c_{2n} \neq c_{1n}$. Hence α_n contains at least three disjoint subarcs which span from α_1 to α_{n-1}, namely $c_{2(n-2)}c_{2n}$, $c_{1(n-1)}c_{1n}$, and one contained in the arc $c_{1(n-1)}c_{2n}$.

If c_{ki} $(i = 1,2,\ldots,n)$, and d_k have been defined, and the β's are not yet defined we proceed as follows. If the order on a_1 is $b_1 d_k c_{k1} a_1$ we let

$$c_{(k+1)1} = (\widehat{b_2 a_2} : \widehat{b_1 d_k} \text{ of } \alpha_1),$$
$$c_{(k+1)i} = (\widehat{b_{i+1} a_{i+1}} : \widehat{b_i c_{(k+1)(i-1)}} \text{ of } \alpha_1),\ (i = 2,3,\ldots,n-1),$$
$$c_{(k+1)n} = (\widehat{c_{(k+1)(n-1)} b_n} : a_1),$$
$$d_{k+1} = (\widehat{b_1 a_1} : \widehat{b_n c_{(k+1)n}} \text{ of } \alpha_n).$$

If the order on α_1 is $b_1 c_{(k+1)1} d_{k+1} a_1$ we define the β_i's as follows:

$$\beta_1 = (\widehat{a_1 d_{k+1}} \text{ of } \alpha_n) \cup (\widehat{d_{k+1} b_1} \text{ of } \alpha_1),$$
$$\beta_i = (\widehat{a_i c_{(k+1)(i-1)}} \text{ of } \alpha_{i-1}) \cup (\widehat{c_{(k+1)(i-1)} b_i} \text{ of } \alpha_i),$$

$(i = 2,3,\ldots,n)$.

If the order on a_1 is $b_1 d_{k+1} c_{(k+1)1} a_1$ we note that α_n contains at least 2k-1 disjoint subarcs which span from a_1 tp a_{n-1}.

As we continue in this manner it is clear that since α_n contains at most a finite number of disjoint subarcs which span from a_1 to a_{n-1},[8] we will eventually reach a point where we can define the β_i's. This concludes the proof of Part c).

d) In tabulating the geometric configurations for more complicated cases we have made use of a convenient diagramatical representation in which the arcs are all drawn as nonintersecting except at the end points. If exactly n-2 arcs have a common end point, the figure must represent one of the following cases.

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[8]This follows since there exists a positive distance from a_1 to a_{n-1}, and a_n like every simple arc has property S. A set which has property S can be written as the sum of a finite number of connected sets each of diameter less than ε for any preassigned positive ε.

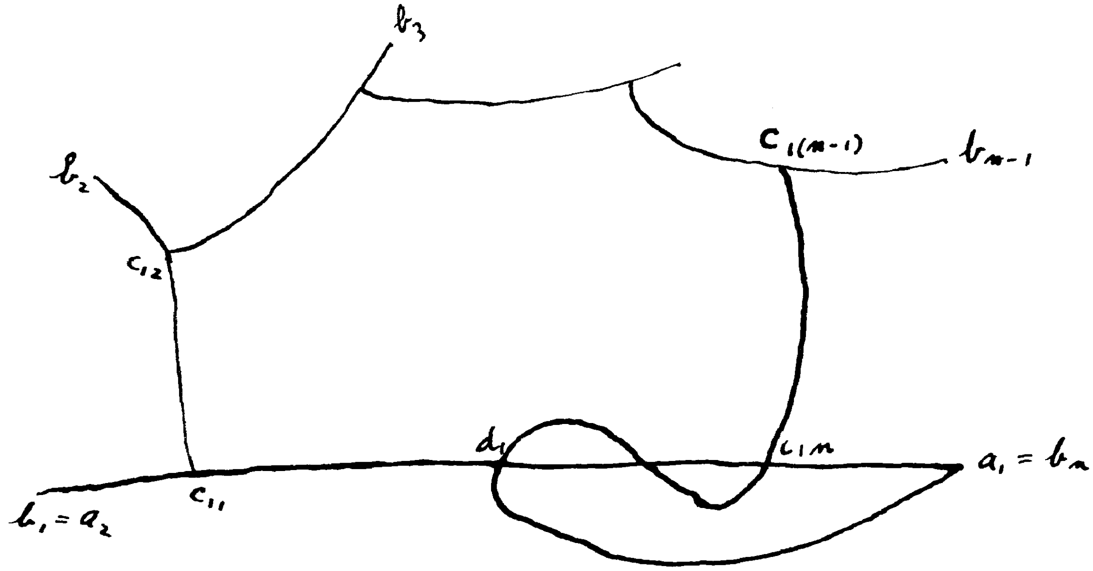


FIGURE 1

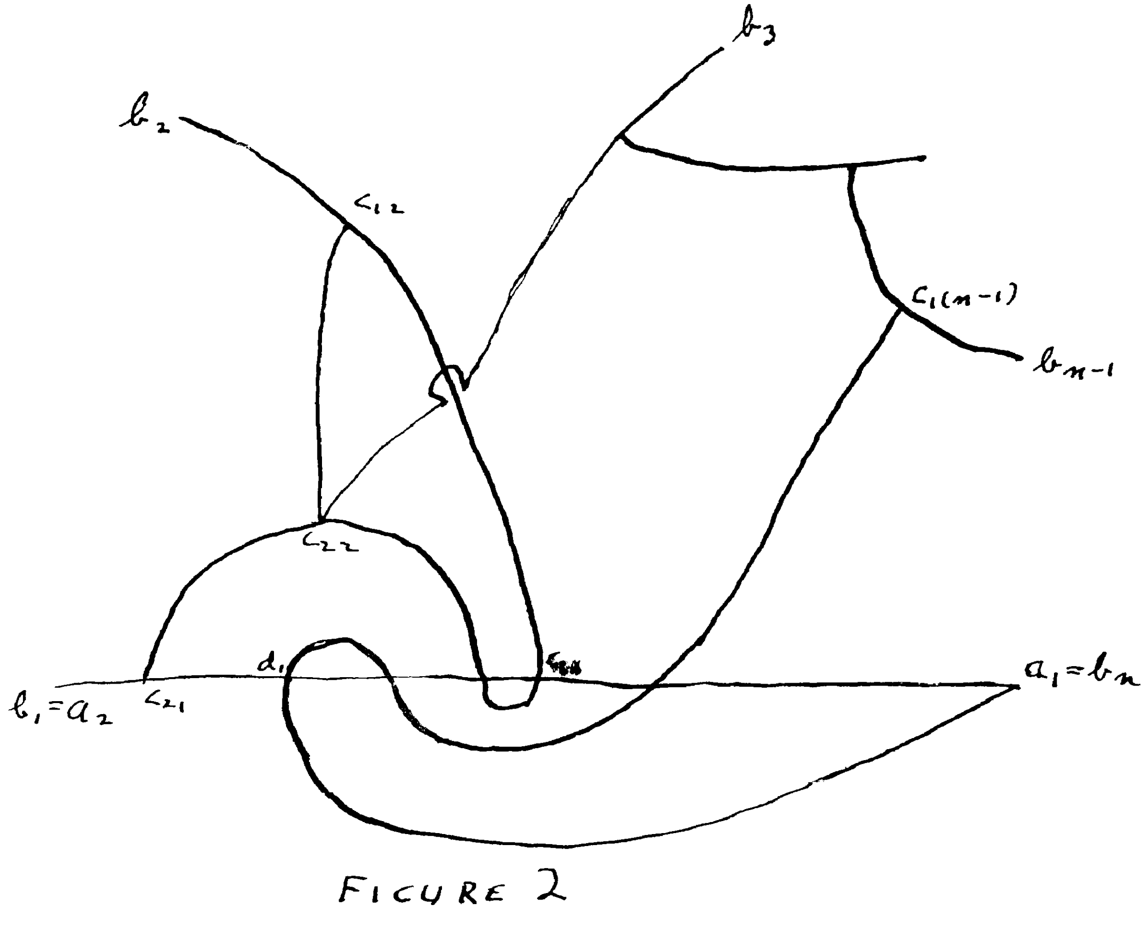
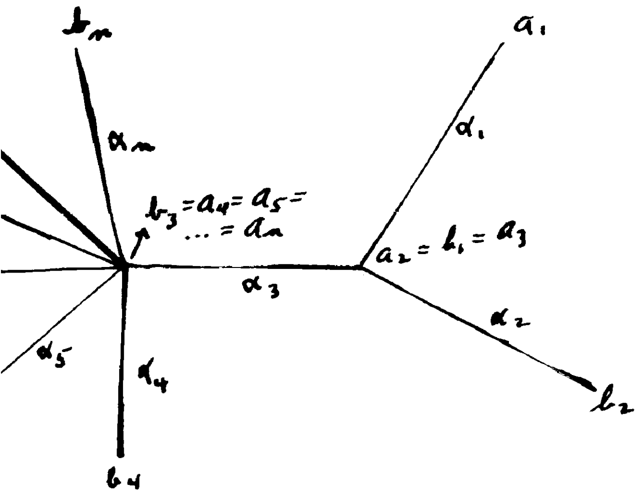


FIGURE 2

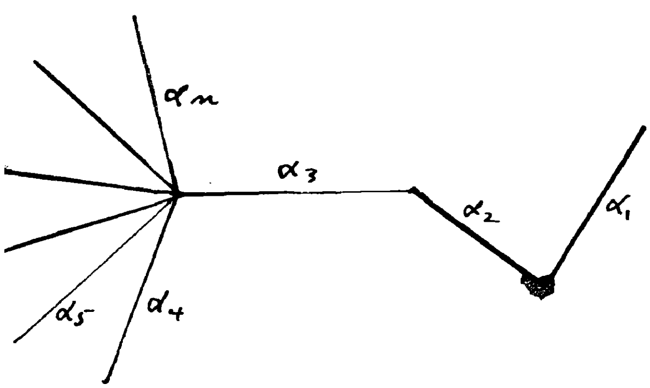
Case I:



Define $\beta_1 = \alpha_1$. For $j = 2,3,\ldots,n$ let $c_{j-1} = (\widehat{b_j a_j} : \bigcup_{i=1}^{j-1} \beta_i)$. Define $\beta_j = (\widehat{b_j c_{j-1}}$ of $\alpha_j) \cup (\widehat{c_{j-1} a_j}$ of $\bigcup_{i=1}^{j-1} \beta_i)$.

Figure 3

Case II:



This follows by a slight modification of Case I.

Figure 4

Case III:

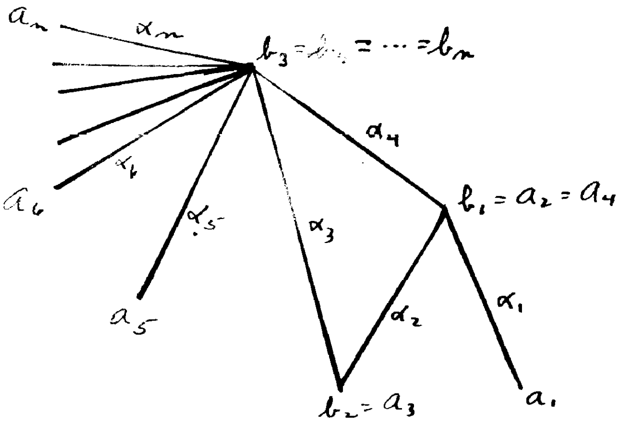


Figure 5

Define $\beta_1 = \alpha_1$. Let $c_1 = (b_2a_2 : \beta_1)$.

Define $\beta_2 = (\overset{\frown}{b_2c_1} \text{ of } \alpha_2) \cup (\overset{\frown}{c_1a_2} \text{ of } \beta_1)$.

Let $c_2 = (b_3a_3 : \bigcup_{i=1}^{2} \beta_i)$. Define

$\beta_3 = (\overset{\frown}{b_3c_2} \text{ of } \alpha_3) \cup (\overset{\frown}{c_2a_3} \text{ of } \bigcup_{i=1}^{2} \beta_i)$.

Let $c_3 = (b_4a_4 : \beta_1 \cup \overset{\frown}{c_2a_2} \text{ of } \beta_2)$. Let

$c_4 = (c_3b_4 : \beta_3)$. Define

$\beta_4 = (\overset{\frown}{b_4c_4} \text{ of } \beta_3) \cup (\overset{\frown}{c_4c_3} \text{ of } \alpha_4) \cup$

$(\overset{\frown}{c_3a_4} \text{ of } \bigcup_{i=1}^{2} \beta_i)$. For $j = 5, 6, \ldots, n$, let

$c_j = (a_jb_j : \bigcup_{i=1}^{j-1} \beta_i)$. Define

$\beta_j = (\overset{\frown}{a_jc_j} \text{ of } \alpha_j) \cup (\overset{\frown}{c_jb_j} \text{ of } \bigcup_{i=1}^{j-1} \beta_i)$.

Case IV:

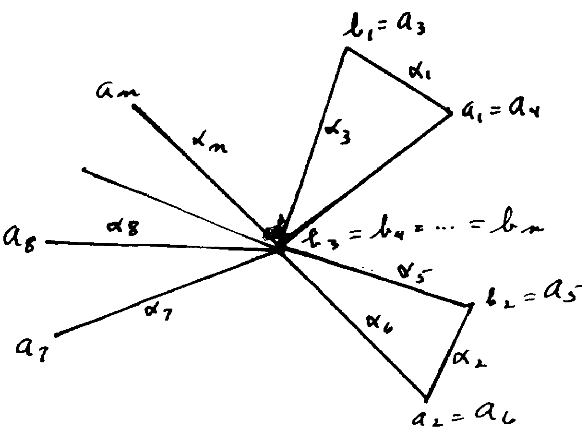


Figure 6

Define $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$. Let

$c_1 = (b_3a_3 : \beta_1)$. Define

$\beta_3 = (\overset{\frown}{b_3c_1} \text{ of } \alpha_3) \cup (\overset{\frown}{c_1a_3} \text{ of } \beta_1)$. Define

$\beta_4 = (\overset{\frown}{b_4c_1} \text{ of } \alpha_3) \cup (\overset{\frown}{c_1a_4} \text{ of } \beta_1)$. Let

$c_2 = (b_5a_5 : \beta_2)$. Let $c_3 = (c_2b_5 : \beta_3 \cup \beta_4)$.

Define $\beta_5 = (\overset{\frown}{b_5c_3} \text{ of } \beta_3) \cup (\overset{\frown}{c_3c_2} \text{ of } \alpha_5)$

$(\overset{\frown}{c_2a_5} \text{ of } \beta_2)$. Define

$\beta_6 = (\widehat{b_6c_3}$ of $\beta_3) \cup (\widehat{c_3c_2}$ of $\alpha_5) \cup (\widehat{c_2a_6}$ of $\beta_2)$. For $j = 7,8,\ldots,n$ let $c_j = (\widehat{a_jb_j} : \bigcup_{i=1}^{j-1} \beta_i)$. Define $\beta_j = (\widehat{a_jc_j}$ of $\alpha_j) \cup (\widehat{c_jb_j}$ of $\bigcup_{i=1}^{j-1} \beta_i)$.

Cases V and VI:

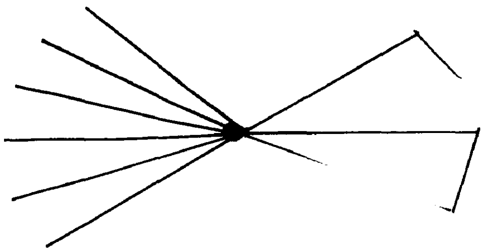


Figure 7

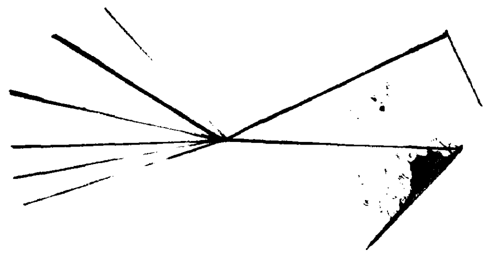


Figure 8

These can be solved by obvious dimplifications of the above solution.

Case VII:

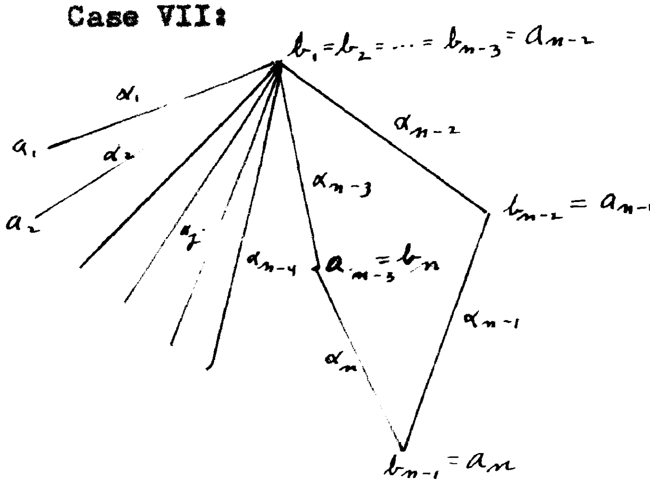


Figure 9

Define $\beta_1 = \alpha_1$. For $j = 2,3,\ldots,n-4$ let $c_j = (\widehat{a_jb_j} : \bigcup_{i=1}^{j-1} \beta_i)$. Define $\beta_j = (\widehat{a_jc_j}$ of $\alpha_j) \cup (\widehat{c_jb_j}$ of $\bigcup_{i=1}^{j-1} \beta_i)$. Let $c_0 = (a_{n-3}b_{n-3} : \bigcup_{i=1}^{n-4} \beta_i)$. Let

$c_{1(n-3)} = (b_{n-2}a_{n-2} : \widehat{a_{n-3}c_0}$ of $\alpha_{n-3} \cup \bigcup_{i=1}^{n-4} \beta_i)$.

For $i = n-2, n-1$ let

$c_{1i} = (b_{i+1}a_{i+1} : b_ic_{1(i-1)}$ of $\alpha_i)$.

Let $c_{1n} = (\overset{\frown}{c_{1(n-1)}b_n}$ of $\alpha_n : \overset{\frown}{c_o a_{n-3}}$ of $\alpha_{n-3})$. Define $d_1 = (\overset{\frown}{c_o a_{n-3}} : \overset{\frown}{b_n c_{1n}}$ of $\alpha_n)$. If $c_{1(n-3)}$ does not lie on $\overset{\frown}{c_o a_{n-3}}$ of α_{n-3} or if the order on α_1 is $c_o c_{1(n-3)} d_1 a_{n-3}$ we define $\beta_{n-3} = (\overset{\frown}{a_{n-3}d_1}$ of $\alpha_n) \cup (\overset{\frown}{d_1 c_o}$ of $\alpha_{n-3}) \cup (\overset{\frown}{c_o b_{n-3}}$ of $\bigcup_{i=1}^{n-4} \beta_i)$; for $i = n-2, n-1, n$ we define $\beta_i = (\overset{\frown}{a_i c_{1(i-1)}}$ of $\alpha_{i-1}) \cup (\overset{\frown}{c_{1(i-1)} b_i}$ of $\alpha_i)$. If the order on α_1 is $c_o d_1 c_{1(n-3)} a_{n-3}$ we continue as in part c.

Case VIII:

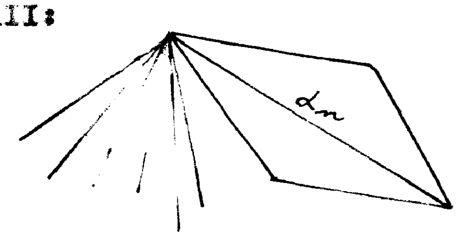


Figure 10

Since β_n can meet any other $\beta_i, i = 1,2,\ldots,n-1$ a simple addition to the solution of Case VII will suffice here.

e) if $n < 5$, the result follows easily from parts a,b,c. The case $n = 5$ is tabulated in Table I Column 3. Columns 1 and 2 give the basic three and four arc configurations which lead to legitimate representations of the five arc cases. If all five arcs meet in a common end point we represent the configuration as in Row 1. If four arcs meet in a common end point, and no five do, we have two distinct possibilities. The fifth arc may have a free end, or it may not (see Rows 2 and 3). If three arcs meet in a common end point, and no four arcs do, the fourth may or may not have a free end (see Rows 4 and 10, Column 2). The possible positions for the fifth arc are limited only by the condition that no four arcs may meet in a common end point. Duplications have been omitted in Table I.

If no more than two arcs meet in a common end point we have exactly two possibilities (see Rows 11 and 12).

The solutions for all these cases except that in Row 12 have been indicated in the preceding proofs.

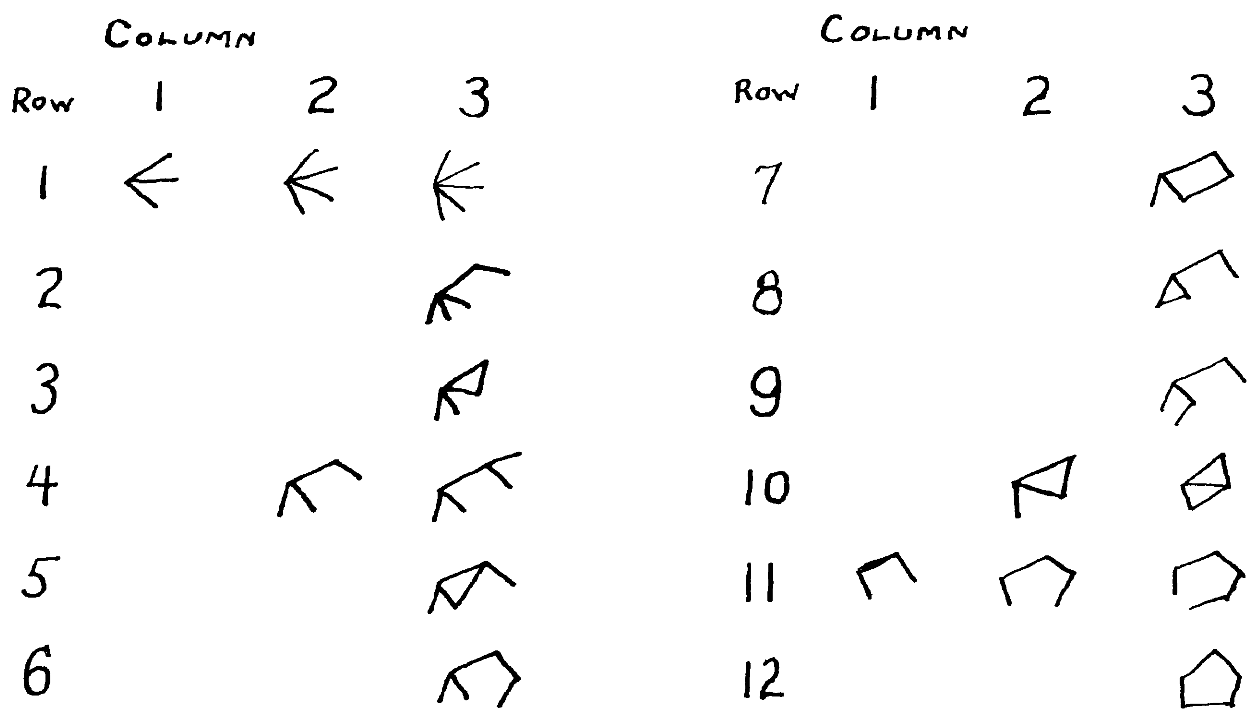


TABLE I


Column
1 2 3 4
Column
Row 1 2 3 4
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30

TABLE II

The case n = 6 is tabulated in Table II in a similar manner. The solutions for the cases in Rows 1,2,3,4,5,6,7,8,9,10,11,12,15,18,20,~~21~~,22, 29,30 have been indicated in the proof of parts a, b, c, d of this theorem. The cases in Rows 13,14,23, ,26 can be solved by a modification of solutions given in the preceding work. The cases represented in Rows 16,19,24, 21, 25, and 28 are as yet unsolved. The solutions for the cases ~~r~~epresented in Rows 17 and 27 are indicated below.

Table II, Row 17:

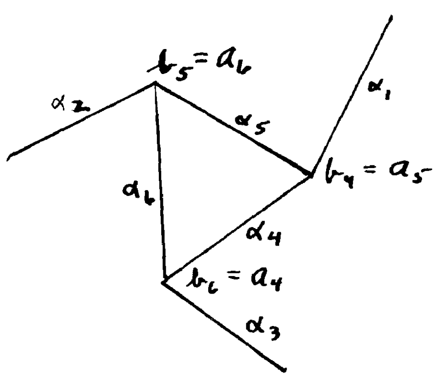


Figure 11

Define $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3$. Let $c_1 = (a_4b_4 : \beta_1)$, and $c_2 = (c_1a_4 \text{ of } \alpha_4 : \beta_3)$. Define $\beta_4 = (\widehat{a_4c_2} \text{ of } \beta_3) \cup (\widehat{c_2c_1} \text{ of } \alpha_4) \cup (\widehat{c_1b_4} \text{ of } \alpha_1)$. Let $c_3 = (b_5a_5 : \bigcup_{i=1}^{3} \beta_i)$. Note that c_3 is not on β_3. Let $c_4 = (c_3b_5 \text{ of } \alpha_5 : \beta_2)$. Define

$\beta_5 = (\widehat{b_5c_4} \text{ of } \beta_2) \cup (\widehat{c_4c_3} \text{ of } \alpha_5) \cup (\widehat{c_3a_5} \text{ of } \beta_1 \cup \beta_4)$. Let $c_5 = (b_6a_6 : \beta_2 \cup \beta_5)$, and $c_6 = (c_5b_6 \text{ of } \alpha_6 : \beta_3 \cup \beta_4)$. Define $\beta_6 = (\widehat{a_6c_5} \text{ of } \beta_2 \cup \beta_5) \cup (\widehat{c_5c_6} \text{ of } \alpha_6) \cup (\widehat{c_6b_6} \text{ of } \beta_3 \cup \beta_4)$.

Table II, Row 27:

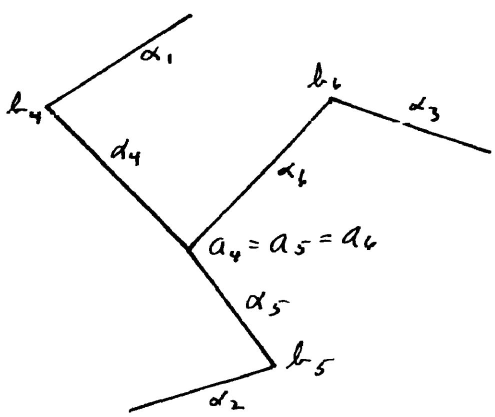


Figure 12

Define $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3$. Let $c_1 = (a_4 b_4 : \beta_1)$. Define $\beta_4 = (\widehat{b_4 c_1} \text{ of } \beta_1) \cup (\widehat{c_1 a_4} \text{ of } \alpha_4)$. Let $c_2 = (a_5 b_5 : \beta_2)$, and $c_3 = (c_2 a_5 \text{ of } \alpha_5 : \beta_4)$. Define $\beta_5 = (\widehat{a_5 c_3} \text{ of } \beta_4) \cup (\widehat{c_3 c_2} \text{ of } \alpha_5) \cup (\widehat{c_2 b_5} \text{ of } \beta_2)$. Let $c_4 = (a_6 b_6 : \beta_3)$, and $c_5 = (c_4 a_6 \text{ of } \alpha_6 : \beta_4 \cup \beta_5)$. Define $\beta_6 = (\widehat{b_6 c_4} \text{ of } \beta_3) \cup (\widehat{c_4 c_5} \text{ of } \alpha_6) \cup (\widehat{c_5 a_6} \text{ of } \beta_4 \cup \beta_5)$.

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