INFINITE PROCESSES IN GREEK MATHEMATICS

by

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PREFACE

The thesis is a historico-philosophic survey of infinite processes in Greek mathematics. Beginning with the first emergence, in Ionian speculative cosmogony, of the underlying concepts of infinity, continuity, infinitesimal and limit, the survey follows the development of infinite processes through the ages as these concepts gain in clarity, it notes the positive contributions of various schools of thought, the negative and corrective influences of others and, in a broad way, establishes the causal chain that finally led to the more abstract processes of the mathematical schools of Cnidus (Eudoxus) and Syracuse (Archimedes).

At various stages of the survey important mathematical principles and axioms are singled out for fuller discussion. Where these are detected for the first time in this thesis a full analysis is given; obscurities are cleared up, appraisals offered and comparisons with modern equivalents made. Where the discovery is not new, current opinions are reviewed and weighed and, where it has been thought necessary, counter opinions are offered.

The central figures in the evolution surveyed in these pages turn out to be Thales, Pythagoras, Zeno, Aristotle, Eudoxus and Archimedes; they are not all of equal importance but each is — some as individuals, others only as typifying leading schools of thought — an indispensable link in the chain established.

What the author considers original and, mathematically, most important in this thesis is the full analysis of the Method of Exhaustion he presents. For the first time, to the best of his knowledge.

a) the Eudoxian number system in which this method operates is fully analysed,

b) the dependence of the method on this number system is established,

c) its essential difference from the modern integral calculus made clear, and

d) its two-fold aspect (as method of proof and method of discovery) revealed.

Other original, but minor, contributions, scattered through the various parts of the survey are new interpretations, explanations, derivations, proofs and comments. Wherever these occur throughout the text, if not explicitly attributed to others, they are held to be original.

Finally, the author acknowledges, with thanks, his indebtedness to Dr. Tobias Dantzig, under whose direction this survey was undertaken, for help received in the selection of titles and the organization of the material. Subjects which, though relevant to the survey, were thought too tedious to present in detail in the body of the thesis, and subjects which, though interesting, were not considered necessary to the main purpose of the survey have been consigned to the end, as appendixes I - IV.

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CHAPTER I

THE METHOD OF LIMITS

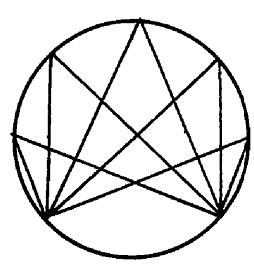
This survey begins, properly, with the Ionian School of philosophy. For in this school, especially in the cosmologies developed by its various members, we first find, with great lack of explicitness, it is true, and always vaguely and uncertainly, the basic concepts of infinity, the infinitesimal, limit and continuity emerging.

Thales of Miletus (640-546 B.C.). From information gathered from Proclus, Diogenes Laertius and other early historiographers and commentators, G.J. Allman[1] deduces that Thales, among his other achievements, had also formed the important mathematical conception of the geometric locus. He shows that Thales knew that:

1) A triangle is determined if its base and base angles are given,

2) If the base and the sum of the base angles of a triangle are given then an unlimited number of triangles exist satisfying the given conditions,

3) The vertices of all such triangles lie on the circumference of a circle.



This constitutes one of the earliest examples of an

[1] G.J. Allman, Greek Geometry from Thales to Euclid, Dublin, 1889, p. 13.

infinity of entities being looked upon as one distinguishable whole; it is also one of the earliest examples of a judgment being made about every member of an infinite set. We shall soon find the validity of such judgments being challenged.

<u>Anaximander of Miletus</u> (611-547). Paul Tannery[2] reconstructs Anaximander's cosmogony from fragments preserved by various early writers and finds that it is based on the following three assumptions:

1) The sky rotates daily around the earth,

2) In a circular movement the heavy tends toward the center, the light away from the center,

3) Heat is connected with motion, cold with immobility.

In this theory, from an endless, unlimited and shapeless mass, the ἄπειρον, which is subject to a circular movement, a flat disc, the earth, is formed at the center; then rings of water, air and fire (the other three elements) are thrown out, spreading away from the center in ever thinning layers but not without limit, for an infinite mass cannot rotate. The universe thus formed, however, is not stable; the celestial fire devours and dissipates the center and the outflung layers and thus, in the course of time, everything returns to the original state. But there is an end to every period of dissipation too, and the same reason that formed the universe once, reforms it. There is thus an endless succession of worlds in time and the only thing that remains immortal and imperishable is the circular movement.

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[2] Paul Tannery, <u>Pour l'histoire de la science hellène</u>, chapter on Anaximandre, Paris, 1930.

Tannery thinks that the word ἄπειρον as used in this context must not be taken in its customary sense (α-privative + πέρας, end) as this is inconsistent with Anaximander's view concerning rotation of an infinite mass. The word, he thinks, may very well be derived from πεῖρα (experience), and the privative α, in which case the meaning of the neuter noun, ἄπειρον, would be "that which is not experienced" both the active and the passive senses being permissible, that is, the word may mean not only "that which is not sensed" but also "that which does not sense". Hence, since the meaning "infinite" conflicts with the theory, the ἄπειρον of Anaximander may well be nothing else but the air, immobile, invisible, odorless, and so on, and thus unperceived by any of the senses. It is then a plenum devoid of all qualities or attributes perceptible by the senses, and thus "an indeterminate plenum".

Plausible though Tannery's interpretation is it must be rejected; for we have the following statement, attributed by Simplicius[3] to Theophrastus.

> Among those who admit only one principle, mobile but infinite, Anaximander, son of Praxiades of Miletus, who was a disciple and successor of Thales, says that the infinite is the principle and element of beings; besides, he is the one who first introduced the word principle (ἀρχή), meaning by this word not water or some other of the elements that we know of, but ....

in which it is clearly stated that, whatever this ἄπειρον may be, it is not any of the four elements (earth, water, air, fire). Further, we shall see, when we take up Aristotle's

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[3] Simplicius, <u>Physics</u>, 6 a, Vors. 15:21-34

study of the infinite, that he examined this ἄπειρον under the guise of ἀρχή and rejected it as impossible.

Anaximenes of Miletus[4], a younger contemporary of Anaximander and, by report, a pupil and friend of his, added the elaboration that the boundless air, subject to an eternal movement, is the source of everything. Expanding under the influence of heat, or contracting under that of cold, it has formed all the phases of existence.

Anaxagoras of Clazomenae[5] (500-428 B.C.) seems to have considered a chaotic mass, existing in some way from the beginning, and containing within it, in infinitesimally small fragments, endless in number, the seeds of things. These parts, of like nature with their wholes (the ὁμοιομερῆ of Aristotle which, as we shall see, were the basis of his conception of the continuous) were arranged, the like being segregated from the unlike, and summend into totals of like nature by the action of Mind (νοῦς), whose first manifestation in the universe was Motion. Anaxagoras held the evidence of the senses in slight esteem; for, though we seem to see things come into being and pass away, reflection (the νοῦς) tells us that death and growth are but new aggregations of the minute particles. It is easy to see in this philosophy the beginning of Atomism, the theory propounded by

[4] F.W.A. Mullach, Fragmenta Phil. Grae., i, 237-252

[5] Ibid

Leucippus (circ. 480 B.C.), pupil of Zeno. In this theory, an infinite empty space, and an infinity of atoms (ἄτομα = indivisibles) are the ultimate constituents of all things. His pupil and friend,

Democritus of Abdera (circ. 460 B.C.), developed further the philosophy of his teacher and, apparently, applied it to geometry. For, in the letter to Eratosthenes prefixed to "The Method" of Archimedes[6] we find the following statement.

> This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely, that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it.

Again, Plutarch[7] presents Democritus as saying:

> If a cone were cut by planes parallel to its base, what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will show the cone to be irregular, as having many indentations like steps, and unevennesses; and if they are equal, the sections will be equal and the cone will appear to have the property of a cylinder, namely, to be composed of equal and not unequal circles, which is very absurd.

Whether this is an argument in support of Atomism — for it seems to say that the indentations must be taken as existing (but small enough to be invisible) for otherwise there would be an absurd result — or, as Cajori[8] thinks, that it advances the view that matter is divisible to only a finite number of parts, one thing stands out clearly, it is

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6 T.L. Heath, The Method of Archimedes, Cambridge, 1912

7 Plutarch, De Comm. Not., Vol. IV, ed. Didot, p. 1321

8 Florian Cajori, American Mathematical Monthly, Vol. 22.

a distinct foreshadowing of the "Method of Indivisibles". Heath[9] is so much of this opinion that he even suggests a method of proof as probably the one used by Democritus.

Two triangular pyramids with equal bases and equal heights, cut by a plane parallel to the bases and dividing the heights in some ratio give equal sections; hence the pyramids, thought of as consisting of thin laminae, are equal. This would be an anticipation of Cavalieri's theorem. Democritus would then see that the pyramids into which a prism of the same base and height as the original pyramid can be divided satisfy this test of equality, so that the pyramid would be one third part of the prism. The extension to pyramids with polygonal bases would then be made by dividing the polygonal base into triangles. Finally, the extension to the cone might have been made, of course, without rigorous proof, by increasing indefinitely the number of sides of a regular polygonal base.

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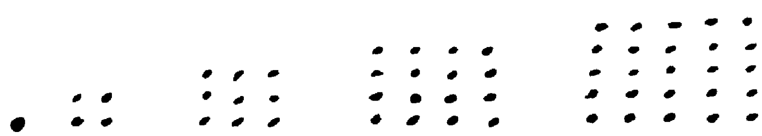
[9] T.L. Heath, History of Greek Mathematics, Vol. I.

CHAPTER II

PYTHAGOREAN ALGORITHMS

One of the well-known practices of Pythagoras and his school was that of σχηματογραφεῖν, or "representing numbers by figures". This practice was, of course, a consequence of the Pythagorean doctrine. For, in this doctrine, the geometric point was defined as "unity in position" (μονὰς θέσιν ἔχουσα) and, conversely, the unit of number as "a point without position" (στιγμὴ ἄθετος)[10]. Hence the natural tendency of representing numbers (pluralities of units) by aggregates of points, arranged in such shapes as the nature of the numbers might suggest.

In this practice a method was developed for the successive generation of numbers of a given type; this was the use of gnomons[11]. Thus, Proclus, Diogenes Laertius and Plutarch attribute[12] to Pythagoras the method of forming successive squares by the addition of equilateral gnomons

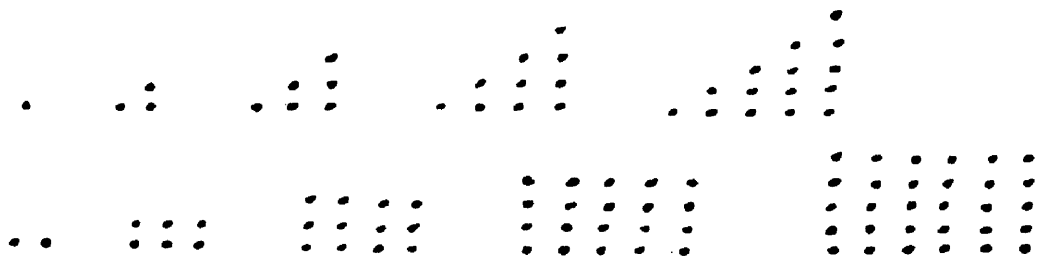


10 Proclus, Comment. on Euclid I, ed. Friedlein, 1873, p.95

11 γνώμων (γνῶναι, to know) = one who knows, that by which something is known, a criterion. In geometry: the figure (like a carpenter's square) which, added to any parallelogram, preserves the original shape (Euc. II, Def.2). More generally, the figure which, added to any figure, preserves the original shape (Heron of Alex., Def. 59).

12 Proclus, op. cit., Diog. Laert., ed. Hubner, Leipsic, VIII, 11.; Plutarch, Symposium, 6

while Lucian[13] and Aristotle mention the formation, by the Pythagoreans, of the triangular and the oblong numbers (ἑτερομήκεις ἀριθμοί) by the addition of the corresponding gnomons.



Another Pythagorean discovery of interest in this survey is that of what are now known as Pythagorean numbers, that is, numbers which satisfy the equation $x^2 + y^2 = z^2$. Proclus[14] says:

> But there are delivered certain methods of finding triangles of this kind (sc. right-angled triangles whose sides can be expressed by whole numbers) one of which they refer to Plato, but the other to Pythagoras, as originating from odd numbers. For Pythagoras places a given odd number as the lesser of the sides about the right angle, and when he has taken the square erected on it, and diminished it by unity, he places half the remainder as the greater of the sides about the right angle; and when he has added unity to this he gets the hypotenuse. Thus, for example, ... But the Platonic method originates from even numbers. For when he has taken a given even number he places it as one of the sides about the right angle, and when he has divided this into half, and squared the half, by adding unity to this square he gets the hypotenuse, but by subtracting unity from the square he forms the remaining side about the right angle.

The exact manner in which Pythagoras made this discovery is not known; Allman, however, makes the following plausible

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13 Lucian, βίων πρᾶσις, 4 vol. i, p. 317, ed. Jacobitz

14 Proclus, op. cit., p.428

suggestion[15]. The recurrence formula in the formation of the successive squares is $n_1^2 + (2n_1 + 1) = n_2^2$, in which $2n_1 + 1$ is the number of units in the gnomon added to the square n_1^2. It is easily seen that if $2n_1 + 1 = k^2$, where k is any odd number (not 1), then the recurrence formula becomes $n_1^2 + k^2 = n_2^2$ and thus the numbers n_1, k, n_2 are a solution of the Pythagorean triangle. The numbers n_1 and n_2 are, respectively, $\frac{k^2 - 1}{2}$, $\frac{k^2 + 1}{2}$. It is possible, of course, that Plato's rule also may have had such an origin. For two successive gnomons may have a sum which is a perfect square, that is $(2n - 1) + (2n + 1) = m^2$, m even, in which case $n = (m/2)^2$, which gives Plato's rule.

The higher polygonal numbers, though not definitely attributed to Pythagoras, were yet studied with a good deal of thoroughness by his later followers. A full account of these is given by Nicomachus of Gerasa[16] and by Theon of Smyrna[17]. The latter of these two later Pythagoreans (early part of second century A.D.) is important in this survey because of his famous algorithm on the "side and diagonal" numbers (thus named by Heath). It is enough to state here — the matter will be discussed fully in a subsequent chapter — that in his book "On the mathematics useful in the reading of Plato", in order to support his typically Pythagorean contention that

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[15] G.J. Allman, Enc. Brit., Werner Ed., New York, 1900

[16] Nicomachus of Gerasa, Introduction to Arithmetic, Tr. M.L. D'Ooge, Ann Arbor, 1938

[17] Theon de Smyrne, ed. Jaques Dupuis, Paris, 1892, p.72.

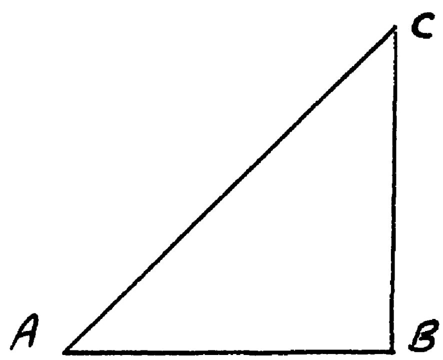
> as unity is the principle of all figures,
> according to the highest and generating ratio,
> so also is the ratio of the diagonal to the side
> found in the unit

he created an algorithm for the construction of a succession of ratios, $\frac{d_n}{s_n}$, which approach the limit $\frac{\sqrt{2}}{1}$.

CHAPTER III

THE INCOMMENSURABLES

An infinite process of a different kind is involved in the solution of another problem attributed to Pythagoras. Proclus[18] ascribes the discovery of the incommensurables to him but, unfortunately, does not give the method of discovery. It is commonly assumed now that the discovery was effected through the isosceles right-angled triangle. But the nature of this old proof (twice mentioned by Aristotle[19], and presented in some editions of Euclid as Prop. 117, Book X[20]), makes it more probable. Allman thinks[21], that this was accomplished by Pythagoras' successors. This proof is given by Heath in the following form.



Let $\frac{AC}{AB} = \frac{\alpha}{\beta}$, α, β relatively prime.

Then, $\frac{AC^2}{AB^2} = \frac{\alpha^2}{\beta^2}$

But since $AC^2 = 2\,AB^2$ we have $\alpha^2 = 2\beta^2$

Hence α^2 is even, whence α also is even; now, β is prime to α ; hence β must be odd.

Again, since α is even, let $\alpha = 2\gamma$. Then,

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18 Proclus, <u>op. cit.</u>, p. 65

19 Prior Analytics, I, C, XXIII and I, C, XLIV. ed. Bekker

20 Rejected by both Heath and Heiberg as an interpolation.

21 Allman, <u>loc. cit</u>.

$$4\gamma^2 = 2\beta^2 \text{ whence}$$
$$2\gamma^2 = \beta^2$$

and thus β^2, whence β also, is even. But β was shown to be odd.

Allman considers it more likely that the discovery was made by Pythagoras through the problem of cutting a line in extreme and mean ratio. This problem was solved by Pythagoras by the method known as "application of areas", a method presented in Euclid (Book VI) and definitely ascribed by Eudemus[22] to Pythagoras.

Now, from the solution of this problem it follows that if on the greater segment of a line so cut a part be taken equal to the less, the greater segment, regarded as a new line, will be cut in a similar manner; thus the process can be continued without end. On the other hand if a similar method were adopted in the case of any two lines which were capable of numerical representation (the less always being subtracted from the greater) the process would end. Hence would arise the distinction between <u>commensurable</u> and <u>incommensurable</u> lines.

Allman's theory is plausible; but there is evidence in Euclid to establish its soundness. We are told by Heath[23] that "the substance of Books VII-IX goes back at least to the Pythagoreans"; we can take the following definition and theorem then as being due at least to the Pythagoreans if not to Pythagoras.

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[22] Proclus, <u>op.cit.</u>, p. 65

[23] T.L. Heath, <u>History of Greek Mathematics</u>, 2:294

(A) Bk. VII, Def. 4. "Numbers composite to one another are those which are measured by some number as a common measure".

(B) Bk. VII, Prop. 1. "Two unequal numbers being set out, and the less being continually subtracted from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another".

Let the numbers be AB and CD, with AB greater than CD. Subtract CD repeatedly from AB until EB, less than CD, is left.

Subtract EB repeatedly from CD until FD, less than EB, is left; continue this process until a unit, GB, is left. I say AB, CD are prime to each other.

For, if not, let E, greater than a unit, measure them both. Then, since E measures CD, it measures AE; but it also measures AB; hence it measures EB.

Again, since E measures EB, it measures CF; but it also measures CD; hence it measures FD.

Again, since E measures FD, it measures EG; but it also measures EB; hence it measures the unit GB, the greater the less, which is impossible. Therefore, etc.

We note that in this proof lines are used to represent numbers, and hence the unit also. But how long a line must be taken to represent the unit? This question must have been faced and the conclusion easily arrived at that, even in the case where the two lines were not representations of whole numbers, but "part, or parts" the one of the other, or, as we would say, the one a rational fraction of the other, the length of the unit that measured them both did not matter; so long as a length was found, no matter how small, that made the process end then the lines were commensurable. Hence would arise the questions: but what if, no matter how small a line were found, at any given step, it never measured the one before it? Are there lines for which this process would never come to an end? And if there are such lines, how would one ever put them to the test since, obviously, no one can perform an infinity of operations?

At this point the known property of the "golden section" would come to mind and supply the answers to these questions: Apply this same process to the segments AB, BC of the line AC, divided in extreme and mean ratio at B,

and you can be sure, <u>beforehand</u>, that the process will never come to an end. Hence, there exist incommensurable lines.

I offer now, for purposes of comparison, the definition (C) and the theorem (D) below; their similarity to the set (A) and (B) above, in ideas involved, wording and method of proof are too striking to miss.

(C) Bk. X, Def. 1. "Those magnitudes are said to be commensurable which are measured by the same measure.

(D) Bk. X, Prop. 2. "If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable".

Let AB, CD be the given magnitudes, with AB less than CD, and let AB be repeatedly subtracted from CD until FD, less than itself, is left.

Let FD be repeatedly subtracted from AB until GB, less than itself, is left, and let this be continued indefinitely, with the remainder never measuring the one before it. I say AB, CD are incommensurable.

For, suppose some magnitude, E, measures them both. Then, since E measures AB it will measure CF, and since it also measures CD it will measure FD.

Again, since it measures FD it will measure AG and, since it also measures AB, it will measure GB.

Let this be continued until a magnitude HD is reached........ Bk X, 1 less than E. Since E measures CH and CD it also measures HD, the greater the less, which is impossible. Therefore, etc.

The strict correspondence of (C), (D) to (A), (B) is apparent; it points unmistakably to a derivation of the incommensurables such as the one suggested by Allman.

CHAPTER IV

MATHEMATICAL INDUCTION

We note in the derivation, and proof of the existence of the incommensurables just given, the first appearance of an important mathematical principle, intimately connected with the concept of infinity, namely, the principle of mathematical induction.

It was stated above that if Euclid's process were applied to the two segments of the "golden section" the process would not come to an end. It was pointed out that this was due to the fact that these two segments are such that there is assurance, beforehand, that after any step of the process, another was possible. For, let a, b, be the large and small segments of this section, respectively; then

$$\frac{a+b}{a} = \frac{a}{b}\text{, whence}$$

$$ab + b^2 = a^2\text{, or}$$

$$b^2 = a^2 - ab\text{, and thus}$$

$$\frac{b}{a-b} = \frac{a}{b}$$

which shows that b and $a - b$ are themselves the large and small segments, respectively, of a "golden section". But b and $a - b$ are also the next two segments in Euclid's algorithm. Hence, since b does not measure a neither does $a - b$ measure b and thus the process is endless.

There are two more matters in the above theory to be pointed out and appraised in which the principle of mathematical induction is — as in so many things in mathematics — involved; hence, for the clarification of what is to follow a more detailed discussion of this principle is advisable.

This principle may be stated, in the abstract, in the following way:

(I) Let S be a discrete series having the first term s_0; let P be a property such that if any term, s, of S has the property P then the successor of s (if s has a successor) also has it. Then if s_0 has the property P every term of S has it.

Consider now the discrete series whose terms are the successive pairs of line segments produced from an initial pair, (a_0, b_0), by Euclid's algorithm. This algorithm yields, from any term (a,b), its successor, (b, a − mb), provided b is not a measure of a; otherwise there is no successor and the series ends with (a,b).

Now let the first term, (a_0, b_0), consist of the pair of segments a_0, b_0, of the "golden section" and let (a,b) be any other term of the series. If a,b are the segments of a "golden section" we know, from the nature of this section, that b does not measure a, that Euclid's algorithm (in which m is 1 now) yields the successor (b, a − b) and that this successor is itself a "golden section". But the first term, (a_0, b_0), possesses this property (i.e., is a "golden section"); hence, every term of the series possesses this property.

It is seen that the principle, as stated, is applicable to all series in which every term (with the exception of the last, if there is a last) has a successor. Its great usefulness, however, occurs in series in which every term does have a successor; in such a case the principle makes it possible to make assertions concerning every term of the series, without enumeration of the individual terms (an operation which is, of course, impossible); such a case was the one met with in the derivation of the incommensurables given above.

A similar use of this principle, though somewhat more difficult to see, is involved in the definition of the finite cardinal numbers and hence, indirectly, of the infinite ones also, as conceived by G. Cantor[24], G. Peano[25] and B. Russell[26]. If we seek some property, present in every term of the series

(S) 1, 2, 3, ... n, ...

and in no other number, to use as a defining property of these numbers, we are tempted to seize on the fact that these numbers are all derivable from 1 by successive additions of one. But then, how many successive additions are to be permitted? A finite or an infinite number? If the former, then some finite numbers will be left out by the definition, which is not what we want; if the latter, then some number of (S) will be infinite, which is not what we want either.

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[24] G. Cantor, Math. Annal., 1885, Vol:XLVI, or transl. by P.E.B. Jourdain, The Theory of Transfinite Numbers, Chicago, 1915.

[25] G. Peano, Formulaire de Mathematiques, Turin, 1895

[26] B. Russell, Principles of Mathematics, New York, 1938, or Introduction to Mathematical Philosophy, New York, 1925

Besides, we must not use the notions of finite and infinite anyway, for otherwise our definition becomes circular.

We try to sidestep this difficulty by the use of another word and define the finite numbers, as some actually do in popular language, as those that can be reached, from 1, by successive additions of one, But then, what does the word "reached" mean in mathematics? If it means "actually arrived at" by counting, or some other such process, thenmost of the finite numbers will have to be left out, for a very small part of them can be so arrived at. If, on the other hand, only "conceptually arrived at" is meant, then the difficulties are increased, not lessened. For, what is the criterion of conceptual accessibility? Some people have no difficulty in arriving, in some way, at the infinite itself, conceptually. As a last resort we try the statement: The finite numbers are the numbers 1, 2, 3, 4, and so on, without end. But we soon realize that this only begs the question; for we have started out to define the series (S), and that is precisely what this series says, namely, 1, 2, 3, 4, and so on, without end.

The feeling persists however (as it should, of course) that the finite numbers are a perfectly definite class and that therefore some satisfactory way of defining them exists. A way does exist.

We recall the definition of the principle of mathematical induction (given as (I) above) and in it we put (S) for S, 1 for s_ω, n for s and n + 1 for "successor of s"; we obtain:

> Let (S) be a discrete series having the first term 1; let P be a property such that if any term, n, of (S) has the property P then n + 1 also has it. Then, if 1 has the property P every term of (S) has it.

We observe now that if P is the property of being "finite", as the word is intuitively understood, then the principle of mathematical induction asserts that all the terms of (S) are finite; for, clearly, 1 is finite, and n + 1 is finite if n is. Hence, we may define the finite numbers thus:

> The finite numbers are all those numbers which possess every property possessed by 1 which is such that n + 1 possesses it if n does.

This definition will be recognized, of course, as essentially that of B. Russell's[27]; it merely states, in precise language, what is only vaguely expressed by the usual popular phrases. Thus, by the use of the principle of mathematical induction, the difficulty of comprehending an infinity of objects within the scope of a single judgment has been overcome. This question will reappear in this survey.

The other matter that was to be pointed out is the appearance, for the first time, of still another principle of importance in mathematics, to wit, the principle of "infinite descent"[28]. This principle may be stated, in the abstract,

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27 Principles of Mathematics, Chap. XIV, or Introduction to Mathematical Philosophy, Chap. III.

28 T. Dantzig, Lectures on Selected Topics in the History and Philosophy of Mathematics, Dept. of Agriculture, Washington, 1940.

as follows:

> When in a repeated process which begins with a finite number, each step of the process yields a number which is smaller than the one obtained in the previous step, the process comes to an end.

An examination of Euclid's algorithm (given as prop. 1, Book VII above) reveals that this principle is tacitly invoked in it; for it is assumed that the process will, some time, lead to the unit, at which it stops. Tacitly again this principle was used later by Giovanni Campano[29] (c.1260) in proving the irrationality of the "golden section". Fermat[30] (1601-1665) made the first explicit use of this principle; by combining it with the law of contradiction he converted it into a powerful method for the proof of theorems in the theory of numbers. Thus, by its use, he proved that the area of a Pythagorean triangle is never a square; the assumption that such an area is a square leads to the result that there exists another Pythagorean triangle, smaller than the first, whose area is also a square, whence an infinite sequence of such triangles, each smaller than the one before it, which contradicts the principle in question. Fermat called his method of proof "la descente infinie ou indefinie"[31]. This method was used by Legendre (with acknowledgment of indebtedness to Fermat) in his "Essai sur la théorie des nombres" in the proof of a great number of theorems.

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[29] Cajori, <u>op.cit.</u>, p. 142

[30] A.M. Legendre, <u>Essai sur la théorie des nombres</u>, Paris, 1808

[31] Cajori, <u>op.cit.</u>, p. 169

It will be seen, of course, that this method of proof is not a principle in the same sense in which mathematical induction is a principle, that is, it is not an independent, first truth, accepted without proof (no matter how it may have been looked upon by its earlier users). It can be easily proved by mathematical induction; in fact, it may be looked upon as mathematical induction <u>in reverse</u> for, clearly, if any finite number can be reached from 1 then, reversing the process, from any finite number 1 can be reached.

It has been noticed, perhaps, that the infinite, as such, has not appeared once as an object of study among the Pythagoreans; instances of the concept have arisen many times in the mathematical processes and principles used by them, but they were only incidental. However, there is one exception to this remark; we have a comment, by Philoponous[32], according to which "In dealing with the infinite, to the Pythagoreans and Plato the relation of whole to part is that of infinite to infinite". The meaning of this is obscure, but it may well be a reference to the property of the infinite which was to cause its rejection by so many thinkers in the past and which was to become its defining property to Dedekind in more recent times. We do seem to have, however, in this statement of Philoponous', at least a recognition of the property of the infinite of having parts which are themselves infinite. But did the Pythagoreans, and Plato, go so far as to think of the infinite as having parts equal to itself?

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[32] Philoponous, <u>Commentaria in Aristotelem, Graeca</u>, Vol. 16, Berlin.

Did they see in this property a denial of the axiom "The whole is greater than its parts"? It is unfortunate that the comment does not make this clear.

CHAPTER V

THE PYTHAGOREAN AXIOM

We have seen that Pythagoras had identified the geometric point (a unit having position or, as Tannery put it, "the number one in space") and the unit of number (the point without position). It is necessary now, in order to provide the background requisite to an understanding of post-Pythagorean developments in infinite processes, to review briefly some of the other tenets of the Pythagorean doctrine. We are told by Diogenes Laertius[33] that

> Pythagoras taught that the principle of all things is the monad, or unit; arising from this monad is the infinite dyad.... from the monad and the infinite dyad arise numbers; from numbers points; from points lines, from lines planes, from these solid figures and from these sensible bodies....

and, by Aristotle[34], that

> The Pythagoreans seem to have looked upon number as the principle and, so to speak, the matter of which beings consist

and

> They supposed the elements of number to be the elements of being, and pronounced the whole heaven to be harmony and number

while from Philolaus of Thebes[35] (c.430 B.C.), the Pythagorean who gave the first written exposition of the doctrine, we have the statement

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[33] Laertius, op.cit., chapter on "Pythagoras"

[34] A. Seth, Enc. Brit., Werner ed., N.Y., 1900, XX:138

[35] Ibid

> Number is great and perfect and omnipotent, and the principle and guide of divine and human life.

It seems fairly certain then that the Pythagoreans had not only identified the unit of number with the geometric point, but that they had, further, identified, or rather confused, the geometric point with the physical point, or particle. From this it followed that not only were geometric loci pluralities of units (numbers) in position but that physical bodies also could be regarded as such. Hence the Pythagorean formula: Being is Number (τὰ ὄντα πολλά ἐστι); hence also much of what is otherwise occult or mystical in their doctrine (the belief, for instance, that physical bodies possess properties that are governed and revealed by number). We shall also see, in a subsequent chapter, that this doctrine had much to do with the paradoxes that Zeno propounded.

It has been shown above that the identification of the unit and the geometric point had led the Pythagoreans to represent certain classes of numbers by geometric figures which were made up of discrete points; but there is reason to believe that in their School the identification of arithmetic and geometry had gone much further. It is known[36] that Thales introduced the notions of the "equation" and the "proportion" into Greek mathematics; for among the theorems whose discovery is attributed to him are:

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[36] Allman, _op.cit._, p. 10

a) The three angles of a triangle are equal to two right angles,
b) The sides of equiangular triangles are proportional,

which, it is seen, explicitly introduce these concepts. From this modest beginning Pythagoras and his successors (notably Archytas[37] of Tarentum) created the elaborate Theory of Proportions[38], substantially as given in Euclid, Bk. VII. In this theory we find the first clear-cut instance of an attempt deliberately made to bring together two hitherto distinct branches of mathematics. Commenting on this theory Allman[39] says:

> Pythagoras elaborated the notion of proportion into a theory which reached the rank of a mathematical method, applicable to both arithmetic and geometry and in this respect he is comparable to Descartes, to whom is due the combination of Algebra and Geometry.

It is necessary now to search out the fundamental axiom by which the junction of arithmetic and geometry was effected in this theory. We have seen that the Pythagorean conception of the straight line was that of a sequence of juxtaposed points; this led, naturally, to their well-known assumption that any two line segments are commensurable. Now, from this it follows that, in this theory, to any line-segment, ρ , there corresponded a rational number, m/n, as its length.

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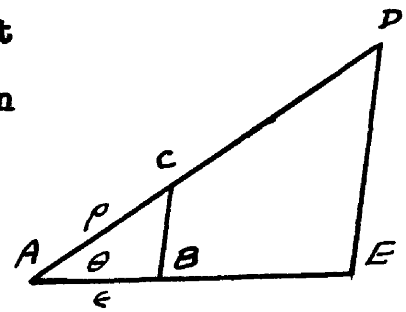
[37] Cajori, op.cit., p. 20

[38] Heath, History of Greek Mathematics, I:294

[39] Allman, loc.cit.



For, let the line-segment, ϵ, be taken as the unit of length, and lay out ρ and ϵ at any angle, θ, as shown in the figure. Join C and B and extend AC to any point D. Draw DE parallel to CB and cutting AB, extended, at E. Let μ denote the segment AD and ν the segment AE; then

$\mu/\nu = m/n$, since any two segments are commensurable.

But $\rho/\epsilon = \mu/\nu$, since the triangles ACB, ADE are equiangular. Hence, if r denote the length of ρ,

$r/1 = m/n$, that is, $r = m/n$, as was to be shown.

Again, we have seen (page 13 above) that it is assumed in this theory that to any two numbers, m, n, there correspond two line-segments μ, ν, whose ratio is m/n. It follows from this (by a proof similar to the one given above) that to any rational number, m/n, there corresponded a line-segment of length m/n. Putting these two results together we obtain the following fundamental axiom:[40]

> To every point on the line corresponds a rational number and, conversely, to every rational number corresponds a point on the line.

This may be looked upon as the Pythagorean analogue of the Cantor-Dedekind axiom.

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[40] It is well-known that the discovery of the existence of incommensurable lines was long kept a secret within the inner school; the final divulgence of this secret contributed of course, to the collapse of this theory and the entire Pythagorean doctrine.

If we state this axiom in the equivalent form

$$\frac{a}{b} = \frac{m}{n}$$

where a and b are line-segments and m, n are whole numbers, then the seemingly extreme views expressed by some later Pythagoreans acquire meaning and credibility. For we are told by Proclus[41] that "Eratosthenes looked on proportion as the bond (σύνδεσμος) of mathematics", and by Theon[42] that "Eratosthenes showed that all figures also result from the proportion", while, expressing his own views now, he adds: "Everything in mathematics is composed of proportions of some quantities, and the principle and (constituent) element of mathematics is the nature of the proportion".

That these views are not as extreme as they sound becomes clear if it is observed that much of the mathematics of that period can be — and was then — reduced to the simple notions of ratio and equality, that is, the notion of the proportion. Indeed, one of the most striking characteristics of the mathematics of that period is the omnipresence of the proportion.

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[41] Proclus, op.cit., p. 43

[42] Theon, op.cit., p. 182

CHAPTER VI

SUCCESSIVE RATIONAL APPROXIMATIONS

Perhaps the earliest and best known example of successive rational approximations to irrational numbers is that of Theon's[43] "side and diagonal" numbers. Theon stated his algorithm thus:

> Let two units be laid out, of which we take one as the side and the other as the diameter... add to the side the diameter and to the diameter two sides... the diameter is now 3 and the side 2. Again to the side add the diameter and to the diameter twice the side... the diameter is now 7 and the side 5... and the addition being thus continuously made the ratio alternates, the square on the diameter being now one more now one less than twice the square on the side... therefore the squares of all the diameters are the double of the squares of all the sides, alternately exceeding or falling short by one...

Symbolically, if s_n, d_n denote the "side and diagonal" numbers obtained at the n^{th} step then Theon's rule gives the formulas

$$\begin{aligned} s_{n+1} &= s_n + d_n \\ d_{n+1} &= 2s_n + d_n \end{aligned}$$

which yield the sequence of ratios $\dfrac{d_n}{s_n}$

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \ldots\ldots$$

beginning with $d_1 = 1$, $s_1 = 1$. In this sequence, Theon seems to say, the ratio $\dfrac{d_n}{s_n}$ alternates so that d_n^2 is now $2s_n^2 + 1$ now $2s_n^2 - 1$.

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[43] See p. above

J. Dupuis[44] sees in this the solution, in integers, of the indeterminate equation $y^2 = 2x^2 \pm 1$; for, suppose $y = a$, $x = b$ is such a solution. Then $a^2 - 2b^2 = \pm 1$. Form, by Theon's rule, the values $a' = a + 2b$, $b' = a + b$. We have:
$a'^2 - 2b'^2 = (a + 2b)^2 - 2(a + b)^2 = -a^2 + 2b^2 = \pm 1$, that is, a', b' are a solution also. Now, $a = 1$, $b = 1$ is a solution; hence Theon's rule gives an infinity of solutions.

However, Theon's construction has also been looked upon, notably by Heath[45], as an algorithm for successive approximations to the square root of 2. For, from

$$y^2 - 2x^2 = \pm 1$$

we get

$$\frac{y}{x} = \sqrt{2 \pm \frac{1}{x^2}} \qquad L\,\frac{y}{x} = \sqrt{2}$$

which show that the term $\frac{d_n}{s_n}$ of the above sequence converges to $\sqrt{2}$, being alternately less and greater.

Theon gave neither a proof nor a derivation of his algorithm. But Proclus[46] asserts that the Pythagoreans discovered the "side and diagonal" numbers, and that a proof of their characteristic property was given by "him" in the second book of the Elements. The "him" in question is, undoubtedly, Euclid; for, in Book II, prop. 10, we find the theorem:

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[44] Dupuis, <u>op.cit</u>. p. 72

[45] Heath, History of Greek Mathematics, Vol. 1; also Allman <u>op.cit.</u>

[46] Procli Diadochi in Platonis rempublicam commentarii, ed. Kroll, Vol. II, Teubner, 1901.

> "If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line",

which means that if AB is bisected, at C, and the whole line is extended to D

then $\overline{AD}^2 + \overline{BD}^2 = 2(\overline{AC}^2 + \overline{CD}^2)$

or, in modern notation

$$(2x + y)^2 + y^2 = 2\{x^2 + (x + y)^2\} \text{ , whence}$$
$$(2x + y)^2 - 2(x + y)^2 = 2x^2 - y^2, \text{ an identity.}$$

From this it is clear, says Heath, that if x_1, y_1 satisfy either of

$$2x^2 - y^2 = 1, \qquad 2x^2 - y^2 = -1$$

then $(x_1 + y_1)$, $(2x_1 + y_1)$ satisfy the other.

However, neither of these two views of the meaning of Theon's algorithm is satisfactory. There is no convincing evidence either that his purpose was to solve the indeterminate equations $y^2 - 2x^2 = \pm 1$ or that he wanted to find rational approximations to the square root of 2. On the other hand, his opening statement, "As unity is the principle of all figures, according to the highest and generating ratio, so also is the ratio of the diagonal to the side found in the unit", and Proclus' definite assignment of the "side and diagonal" numbers to the Pythagoreans, strongly suggest that

Theon's purpose was to defend the Master's doctrine that the unit is the constituent element of all numbers and the point that of all figures.

And it must be observed, in the proof given by Euclid, that <u>any whole number K</u> may be substituted for the unit, that is, if x_1, y_1, satisfy either of

$$2x^2 - y^2 = k, \quad 2x^2 - y^2 = -k$$

then $(x_1 + y_1)$, $(2x_1 + y_1)$ satisfy the other. Thus, for $K = 2$ say, and for $x_1 = 1$, $y_1 = 2$ we easily get the sequence

$$\frac{2}{1}, \frac{4}{3}, \frac{10}{7}, \frac{24}{17}, \ldots\ldots . \quad .$$

of successive rational approximations to the square root of 2. But such a choice of K would not do for the purpose of the algorithm; for now the ratio $\frac{d_n}{s_n}$ alternates so that the square on the diameter is now 2 more now 2 less than twice the square on the side. The unit does not play the all-important role of "generating principle" though the square root of 2 is approximated just as effectively as before.

A curious derivation of Theon's formulas, due to P. Bergh, is given by M. Cantor[47].

> "If we start with an isosceles right triangle, ABC, of side s_n and diagonal d_n, and extend each of the legs a distance d_n, to the points E and D, and complete the triangle EBD, then the new diagonal will be $2s_n + d_n$, and the new side $s_n + d_n$. If the perpendiculars AF, CG are drawn the proof is obvious".

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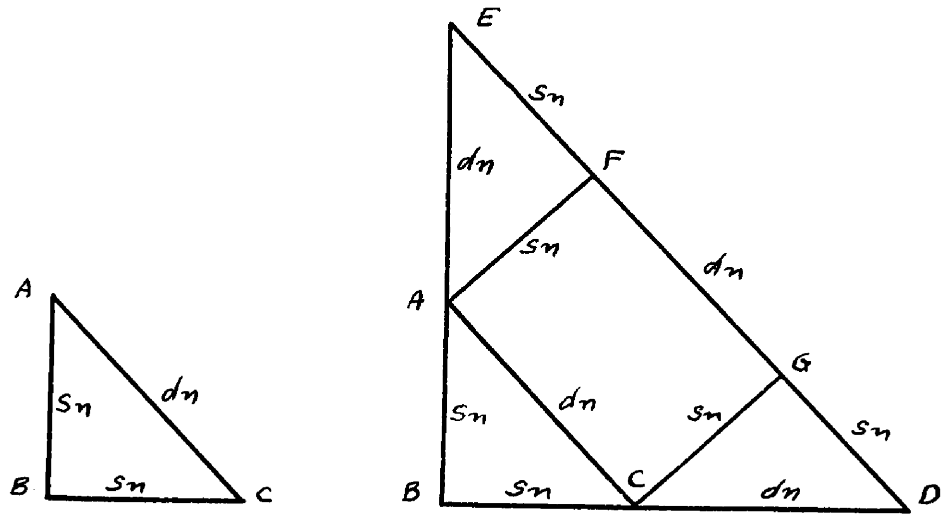
[47] Verlesungen uber Geshichte der Mathematik, I, p. 437; also Heath, <u>The Elements of Euclid</u>, 1:401



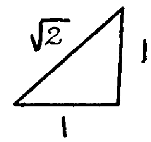
It is clearly implied in this construction that the sides and diagonals obtained in the way described are Theon's "side and diagonal" numbers. That they cannot be, however, becomes immediately evident when it is noticed that a pair of numbers, d_n , s_n , obtained by Bergh's method are never commensurable, whereas Theon's numbers always are. Again for any two pairs, d_n , s_n , and d_m , s_m , of Bergh's numbers, we always have

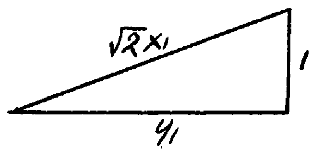
$$\frac{d_n}{s_n} = \frac{d_m}{s_m}$$

an equality which is never true of Theon's numbers. If, however, Bergh did not offer them as Theon's numbers (and it is hard to see what else he may have meant by them), then no derivation of Theon's numbers has been given.

But a satisfactory derivation of Theon's algorithm, based on the interpretation of its character advanced here (a defense of the Pythagorean doctrine) and proceeding entirely by Pythagorean considerations is easy to give.

The discovery of incommensurables was a serious blow to this doctrine; hence, that efforts were made to circumvent the difficulty may be safely assumed. Again, that the diagonal of the unit square is incommensurable with the side was known. It was natural then, and quite likely, that attention was given to the more general case

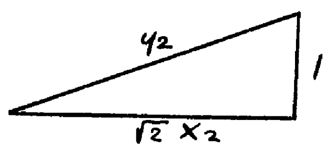




where one side is still preserved as a "unit", the important constituent element of the doctrine. From this, by the Pythagorean theorem

(1) $$y_1^2 = 2x_1^2 - 1$$

Similarly, for the case



which leads to

(2) $$y_2^2 = 2x_2^2 + 1.$$

From (1) and (2) it follows that

(3) $$2x_1^2 - y_1^2 = y_2^2 - 2x_2^2 = 1$$

Hence, by use of the Euclidean theorem given above,

(4) $$(2x + y)^2 - 2(x + y)^2 \equiv 2x^2 - y^2$$

Theon's formulas

$$\begin{aligned} x_i &= x_j + y_j \;; \qquad i, j = 1 \text{ or } 2 \\ y_i &= 2x_j + y_j \qquad\quad i \neq j \end{aligned}$$

follow from (4).

In the derivation of the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots\ldots$$

it is of interest to note that the principle of mathematical induction has been tacitly invoked. For, from the fact that $x_1 = 1$, $y_1 = 1$ satisfy the conditions $y^2 - 2x^2 = \pm 1$, and the fact that if x_1, y_1 satisfy it so do $x_2 = x_1 + y_1$, $y_2 = 2x_1 + y_1$, the infinite sequence follows.

Another interesting matter that may be pointed out in the derivation of this sequence is the example it affords of the failure of the ancients to make (at times obvious) generalizations. That any whole number, k, could be substituted for 1, in the equations $y^2 - 2x^2 = \pm 1$, has been pointed out above, when 2 was used to find another sequence of approximations to the square root of the same number, 2. But still another generalization, in another direction, is quite easy to make. K may be substituted for the number 2 of the given equations, to derive a sequence of approximations to the square root of k. Thus, taking $k = 3$ that is, starting with the equation

(1) $$y^2 - 3x^2 = 1,$$

we note that $x = 1$, $y = 2$ is a solution. Assuming that x_1, y_1 is a solution of (1) we seek numbers

(2) $$\begin{aligned} x_2 &= px_1 + qy_1 \\ y_2 &= mx_1 + ny_1 \end{aligned}$$

which are also a solution of (1), that is, such that

(3) $$\begin{aligned} y_2^2 - 3x_2^2 &= 1 \text{ , or} \\ (mx_1 + ny_1)^2 - 3(px_1 + qy_1)^2 &= 1. \end{aligned}$$

Rewriting (3) in the form

(4) $$(n^2 - 3q^2)y_1^2 + 2(mn - 3pq)x_1y_1 + (m^2 - 3p^2)x_1^2 = 1$$

we see that if

$$(5) \qquad \begin{aligned} n^2 - 3q^2 &= 1 \\ mn - 3pq &= 0 \\ m^2 - 3p^2 &= -3 \end{aligned}$$

this equation becomes

$$y_1^2 - 3x_1^2 = 1$$

true by assumption. Hence, if numbers p, q, m, n can be found to satisfy (5) the numbers x_2 , y_2 of (2) are a solution of (1). Now such numbers are: $m = 3$, $n = 2$, $p = 2$, $q = 1$. We have then the formulas:

$$\begin{aligned} x_2 &= 2x_1 + y_1 \\ y_2 &= 3x_1 + 2y_1 \end{aligned}$$

which yield, beginning with $x_1 = 1$, $y_1 = 2$, the sequence

$$\frac{2}{1}, \frac{7}{4}, \frac{26}{15}, \frac{97}{56}, \ldots \ldots$$

of rational approximations to the square root of 3.

The terms of the sequence thus found approach $\sqrt{3}$ from above. If approximations from below are desired use should be made of the equation $y^2 - 3x^2 = -k$, where k is any suitable constant. If, as in Theon's sequence, a sequence approaching $\sqrt{3}$ alternately from above and from below is desired, then constants c, k must be found such that if x_1 , y_1 is a solution of either of $y^2 - 3x^2 = c$, $y^2 - 3x^2 = -k$, then

$$\begin{aligned} x_2 &= px_1 + qy_1 \\ y_2 &= mx_1 + ny_1 \end{aligned}$$

is a solution of the other.

The next example of successive rational approximations to irrational numbers is furnished by the Alexandrian School. In the "Prolegomena to the Syntaxis of Ptolemy" — an anonymous manuscript, variously attributed to Diophantus, Pappus, or some later writer, living in Alexandria — is found the following passage.

> We shall show then how the square roots of given numbers are found... In finding the area of a triangle, by the general formula, Heron is led to the number 720, whose root he is to find. This is what he says .... "

This is followed by a set of instructions on how to proceed in order to find the desired root. As if written by Heron himself (who is noted for this fault) there is no hint of proof or derivation of the method used.

Reduced to modern symbolism the method may be presented thus. If A is a number, not a perfect square, and if a_0 is the nearest integral square root, then a closer approximation is given by

$$a_1 = 1/2 \left(a_0 + \frac{A}{a_0}\right)$$

and closer ones, in succession, according to the law

$$a_{n+1} = 1/2 \left(a_n + \frac{A}{a_n}\right)$$

Paul Tannery[48] suggests the following derivation of this formula. If A is the number whose root is to be found, and if a^2 is the nearest rational square <u>contained in A</u>, we may set

$$A = (a+b)^2 = a^2 + 2ab + b^2 = a^2 + r$$

and seek to find b. It is clear that

$$\frac{r}{2a} = \frac{2ab + b^2}{2a} = b + \frac{b^2}{2a} > b. \text{ Hence,}$$

$$a_1 = a + \frac{r}{2a} = a + \frac{A - a^2}{2a} = 1/2 \left(a + \frac{A}{a}\right)$$

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[48] P. Tannery, <u>Mémoires Scientifiques</u>, tome 1, No.53

(which is Heron's formula) is an approximate value, from above, of $\sqrt{A}$.

Again, if a^2 is the smallest square containing A we may set

$$A = (a-b)^2 = a^2 - (2ab - b^2) = a^2 - r.$$

It is clear that

$$\frac{r}{2a} = \frac{2ab - b^2}{2a} = b - \frac{b^2}{2a} < b. \quad \text{Hence}$$

$$a_1 = a - \frac{r}{2a} = a - \frac{a^2 - A}{2a} = 1/2\,(a + \frac{A}{a})$$

is an approximate value, again from above, of $\sqrt{A}$.

This method is described again by Theon of Alexandria and reappears, much later, in the Byzantine School, in the writings of Maximus Planudes (13th cent.) and Nicholas Rhabdas (14th cent.).

The Delian problem had been shown by Hippocrates of Chios to be reducible to the discovery of two geometric means between a and 2a; for from

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$$

it follows that $x^3 = 2a^3$. Of the many ways* invented by the mathematicians of this period for the solution of this problem one led to a geometric algorithm for the construction of successive approximations to the cube root of a number.

At the beginning of his third book Pappus[49] discusses

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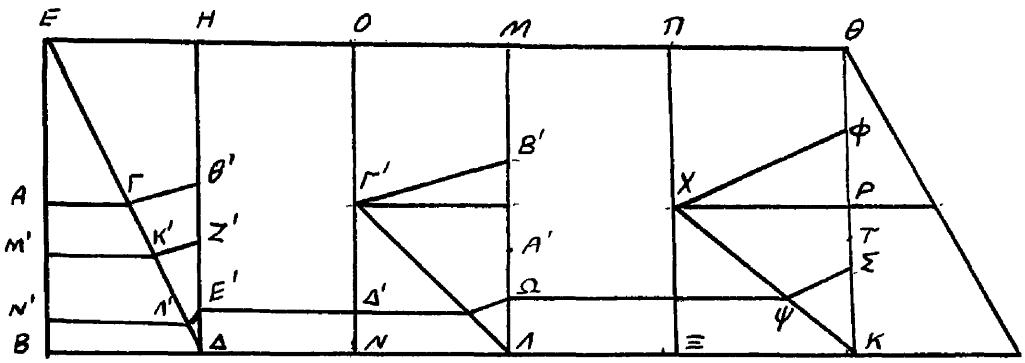
* See Appendix I for a "mechanical" solution.

49 Collection Mathematique, tr. Paul Ver Eecke, Paris, 1933, 2 vols.

at great length the problem of duplicating the cube and criticizes severely a solution submitted to him by a contemporary mathematician (whom he does not name) on the ground that it attempts to solve, by ruler and compass, a problem which is at once linear and solid. The solution in question is as follows:



Let AB, AΓ be two straights at right angles; draw through B the straight BΔ equal to AB; draw the join ΔΓ meeting BA at E; through E draw the straight EΘ parallel to AΓ ; extend BΔ ; draw through Δ the straight ΔH parallel to BE and put the straights ΔN, NΛ , ΛΞ , ΞK equal to BΔ.

Draw through the points N, Λ , Ξ , K, the straights NO, ΛM, ΞΠ , KΘ parallel to BE; put KP equal to BA and cut KP into two equal parts at the point Σ. Let ΣΘ be to a straight ΘT as the straight KΘ is to the straight ΘΣ and let ΘT be to a straight ΘΦ as the straight ΣΘ is to the straight ΘT. On ΞΠ cut off XΞ equal to AB and draw the join XK and the join XΦ.

Draw through Σ the straight ΣΨ parallel to the straight XΦ and, from Ψ, the straight ΨΩ parallel to the straight KΞ. Let ΩM be to a straight MA' as the straight

ΛM is to the straight MΩ, and the straight A'M be to a straight MB' as the straight ΩM is to the straight MA'. Cut off on the straight ON the straight NΓ' equal to the straight Ab and draw the join $\Gamma'\Lambda$ and the join $\Gamma'B'$.

Through Ω draw the straight $\Omega\Delta'$ parallel to the straight $\Gamma'B'$, and through Δ' the straight Δ'E' parallel to ΛN. Let the straight HE' be to a straight HZ' as the straight ΔH is to the straight HE' and let Z'H be to a straight HΘ' as the straight E'H is to the straight HZ'.

Draw the join $\Theta'\Gamma$, the straights Z'K', E'Λ', parallel to the straight $\Theta'\Gamma$ and, through K', Λ', the straights K'M', Λ'N' parallel to the straights AΓ, BΔ. It is to be shown that the straights M'K', N'Λ' are the mean proportionals of the straights AΓ, BΔ.

Pappus criticized the exactness of this solution but it appears, as Tannery has pointed out, that the criticism is wholly unmerited, for the inventor of this construction could not have meant it for anything but a means of indefinite approximation to the two means sought.

> "Let it be required to calculate $x = \sqrt[3]{a^2 b}$, $y = \sqrt[3]{ab^2}$", says Tannery[50], "and, to fix the ideas, let $b < a$. Let x_0 be an approximate value of x; the indicated construction amounts to taking for a closer approximation,

$$x_1 = a\left\{1 - \frac{a\,(a-b)}{a^2 + ax_0 + x_0^2}\right\},\quad y = b + \frac{(a-b)\cdot x_0^2}{a^2 + ax_0 + x_0^2}$$

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50 P. Tannery, <u>Mémoires Scientifiques</u>, tome 1, No. 8

If one puts $x_0 = x + \delta_0$, $x_1 = x + \delta_1$
one obtains easily

$$\frac{\delta_1}{\delta_0} = \frac{1}{1 + \dfrac{3x^2 + 3\delta_0 + \delta_0^2}{(a - x)(a + 2x + \delta_0)}}$$

One concludes from this that δ_1 is of the same sign as δ_0 and smaller in absolute value, and that δ_1/δ_0 decreases as x/a increases .....

Let it be required, for example, to find $\sqrt[3]{2}$, whose value is 1.2589 ... Make a = 1, b = 1/2; then $1/x_0 = \sqrt[3]{2}$; one has, in succession

$$\begin{aligned} 1/x_0 &= 4/3 &&= 1.3333 \ldots \\ 1/x_1 &= 37/29 &&= 1.2758 \ldots \\ 1/x_2 &= 6566/5197 &&= 1.2635 \ldots \\ 1/x_3 &= 104244667/82688489 &&= 1.2607 \ldots \end{aligned}$$

Or, by the second formula, giving the approximate values of y, making b = 1, a = 2, whence $x_0 = 3/2$,

$$\begin{aligned} y_0 &= 9/8 &&= 1.1250 \ldots \\ y_1 &= 46/37 &&= 1.2432 \ldots \end{aligned}$$

By taking the arithmetic means of the preceding values a much more rapid approximation is obtained

$$1/2 \left[4/3 + 9/8\right] = 59/48 = 1.2290 \ldots$$

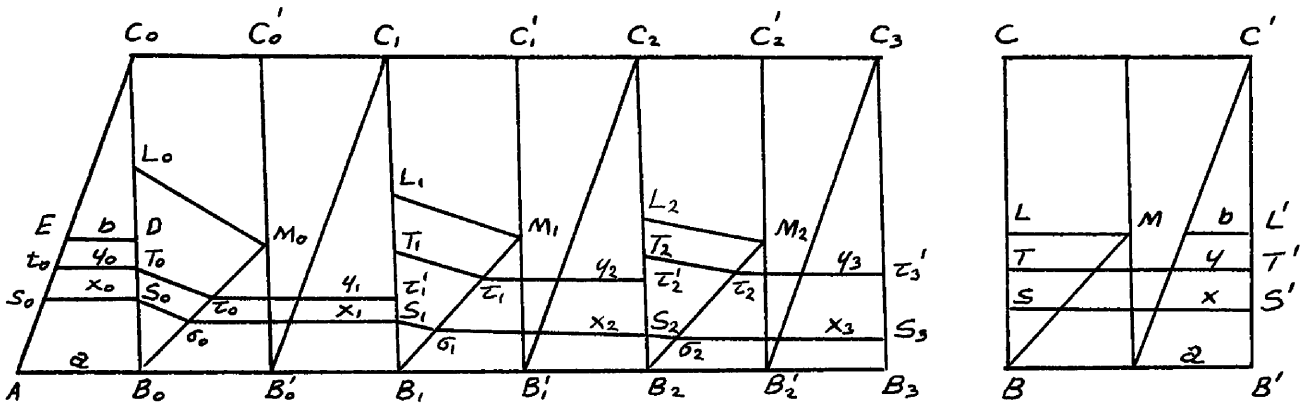
$$1/2 \left[37/29 + 46/37\right] = 2703/2146 = 1.2595 \ldots$$

It is doubtful if the ancients ever made use of this method; at any rate no known record of its use exists".

Thus ends Tannery's discussion of this method. But though Tannery has preferred to give an arithmetic explanation, the process, as given in Pappus, is geometric; that is, it does not give numerical values of the geometric means found. Of course, these values are rational if a and be are rational.

But it is of interest to explain the geometric construction itself, especially since Pappus has omitted the proof.

It is required to find successive approximations, x_i, y_i, to the two geometric means, x,y, between two given lines, a,b, where a > b.



Draw AB_0, B_0D, each equal to a, at right angles; draw DE, equal to b, at right angles to B_0D and complete the triangle AB_0C_0. Extend AB_0 to the right and draw through C_0 a parallel to it. Draw the equal rectangles $B_0C_0B_0'C_0'$, $B_0'C_0'B_1C_1$, $B_1C_1B_1'C_1'$, and so on. Let S_0 be the mid-point of B_0D.

On B_0C_0 find point T_0 such that $\frac{B_0C_0}{S_0C_0} = \frac{S_0C_0}{T_0C_0}$ and point L_0 such that $\frac{S_0C_0}{T_0C_0} = \frac{T_0C_0}{L_0C_0}$. Draw S_0s_0, T_0t_0 parallel to AB_0; these lines, x_0, y_0, are first approximations to the means sought.

Lay off $B_0'M_0$ equal to a and join L_0M_0, B_0M_0. Draw $T_0\tau_0$, $S_0\sigma_0$ parallel to L_0M_0 and $\tau_0\tau_1'$, $\sigma_0 S_1$ parallel to AB_1; draw $B_0'C_1$ cutting off x_1, y_1 on $\sigma_0 S_1$, $\tau_0\tau_1'$ respectively; x_1, y_1 are the next approximations to x, y.

On B_1C_1 find points T_1, L_1 making

$$\frac{B_1C_1}{S_1C_1} = \frac{S_1C_1}{T_1C_1} = \frac{T_1C_1}{L_1C_1}$$

On $B_1'C_1'$ lay off $B_1'M_1$ equal to a; join L_1M_1, B_1M_1 and draw $T_1\tau_1$, $S_1\sigma_1$ parallel to L_1M_1. Draw $\tau_1\tau_2'$, σ_1S_2 parallel to B_1B_2, and $B_1'C_2$ cutting off x_2, y_2 on σ_1S_2, $\tau_1\tau_2'$ respectively; x_2, y_2 are the next approximations to x , y.

On B_2C_2 find points T_2, L_2 making

$$\frac{B_2C_2}{S_2C_2} = \frac{S_2C_2}{T_2C_2} = \frac{T_2C_2}{L_2C_2}$$

On $B_2'C_2'$ lay off $B_2'M_2$ equal to a; join L_2M_2, B_2M_2 and draw $T_2\tau_2$, $S_2\sigma_2$ parallel to L_2M_2. Draw $\tau_2\tau_3$, σ_2S_3 parallel to B_2B_3, and $B_2'C_3$ cutting x_3 , y_3 on σ_2S_3, $\tau_2\tau_3$ respectively; x_3, y_3 are the next approximations to x, y.

Continue this process indefinitely; notice that the triangle $B_nL_nM_n$ approaches the limiting form BLM, the proportion

$$\frac{BC}{SC} = \frac{SC}{TC} = \frac{TC}{LC}$$

always holding. From this we have

$$\frac{B'C'}{S'C'} = \frac{S'C'}{T'C'} = \frac{T'C'}{L'C'}$$

from which $a/x = x/y = y/b$, a result which shows that the values x_n , y_n tend to are the geometric means between a and b.

CHAPTER VII

THE PARADOXES OF ZENO

Zeno of Elea (b. 500 B.C.), a pupil and friend of Parmenides, is said, by Plato[51], to have written certain "discourses" in which he defended the views of his teacher. In answer to a question put to him by Socrates, Zeno says:

> "You do not quite apprehend the true motive of this performance (the writing of the "discourses"), which is not really such an artificial piece of work as you imagine... For the truth is that these writings of mine were meant to protect the arguments of Parmenides against those who ridicule him, and urge the many ridiculous and contradictory results which were supposed to follow from the assertion of the One. My answer is addressed to the partisans of the Many, and intended to show that greater and more ridiculous consequences follow from their hypothesis of the existence of the Many, if carried out, than from the hypothesis of the existence of the One. A love of controversy led me to write the book in the days of my youth, and someone stole the writings, and I had therefore no choice about the publication of them ..."

Zeno's discourses unfortunately have not come down to our times and the only knowledge we have of his arguments against "the partisans of the Many" are — excluding the views expressed in the *Parmenides* as irrelevant here — the paradoxes discussed by Aristotle[52] in his *Physics* and attributed by him to Zeno, and repeated later, with some additional ones, in the *Commentaries* of Simplicius[53]. These

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51 *The Works of Plato*, "The Parmenides", tr. B.Jowett, N.Y.

52 Aristotle, *op.cit*. Physics VI

53 *Simplicii in Aristotelis Physicorum libros quattuor priores commentaria*, ed. H. Diels, Berlin, 1882.

paradoxes have, from the earliest times, been treated, by philosophers and commentators alike, as mere sophistic fallacies and due to the ignorance of one uninstructed in the subleties of the infinite. Much time and effort was expended, but with little success, to the detection and exposure of the errors involved in them. The failures of these attempts were due, in part, to the faulty theories of the infinite and continuity that were brought to bear on them, but to a larger extent to the fact that in none of these attempts had the true character of these paradoxes, as arguments directed to some specific purpose, the disproof of some tenet of ancient philosophy, been recognized.

Many, however, have accepted Zeno's paradoxes as "not really such an artificial piece of work" as could be expected from a young, pugnacious sophist defending the views of his master. Thus, Grote[54] sees in them an instance of the negative side of Grecian speculation, the probing, testing and scrutinizing, beginning then to come out more and more in philosophical inquiries and occupying as large a measure of the intellectual in philosophy as the positive side. "We shall find the two going hand in hand, and the negative vein the more impressive and characteristic of the two from Zeno downward in our history".

It was left for Paul Tannery[55] however to detect the specific aim of Zeno's paradoxes, rescue them from the scanty

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[54] History of Greece, VI:48

[55] Pour l'histoire de la science hellene, Zenon d'Elee, Paris, 1930.

esteem in which they were held, and re-present them to the world as a single, unanswerable polemic against the Pythagorean Doctrine.

> "Such a proposition is absolutely false", says Tannery; "a body, a surface, or a line is not a totality of juxtaposed points. The mathematical point is not a unit; it is a pure zero, a nothing of quantity".

Drawing first on Simplicius, whom he quotes as saying:

> "After having shown that if being has no magnitude it does not exist Zeno adds: 'If it exists it is necessary that every being have some magnitude, a certain thickness, and that there be intervals between its parts. And the same may be said of smaller parts; they too must have magnitudes and intervals between them. And what has been said once can always be repeated; there will thus be no last term where there are no parts different from one another! Thus, if there is a plurality then things must be at once large and small; so small that they have no magnitude, and so large that they are infinite' "

Tannery interprets as follows:

> If it be claimed that a body is made up of points then the principle of infinite divisibility leads to parts so small that they have no magnitude, which would also make their total, the body, of zero magnitude. But if it is assumed that these ultimate parts do have magnitude (the other horn of the dilemma), since there is an infinity of them the body must be infinitely large"

This argument, Tannery thinks, amounts to a rigorous proof that the infinitely divisible cannot be conceived of as a sum of indivisibles. He also points out that Simplicius errs in asserting that Zeno's paradox leads to the result that things are at once large and small, a result unacceptable because of the contradiction. It is rather a dilemma, each horn of which is individually unacceptable.

The next paradox examined by Tannery is the one in which, according to Simplicius, Zeno argues thus: "If there is a plurality, it is necessary that they be as many as they are, neither more nor fewer. Being as many as they are they are limited. But if there is a plurality, they are unlimited, for there are always others between units, and still others among these, and thus things would be unlimited". This he interprets as follows: to say that bodies are a sum of points is to say, implicitly, that the number of points is unlimited. On the other hand, the principle of divisibility requires that we admit the existence of points between any two distinct points; hence there must be an unlimited number of points. Zeno, by this argument, forces the adversary into a contradiction.

The other paradoxes examined by Tannery are the so-called 'arguments against motion' preserved, in very compressed form by Aristotle and variously presented in expanded form by later writers. As presented by Burnet[56] they are:

> 1:- You cannot traverse an infinite number of points in finite time. You must traverse the half of any given distance before you traverse the whole, and the half of that again before you traverse the whole, and the half of that again before you can traverse it. This goes on ad infinitum so that there are an infinite number in any given space, and thus it cannot be traversed in finite time.
>
> 2:- The second argument is the famous puzzle of Achilles and the tortoise. Achilles must first reach the place from which the tortoise started. By that time the tortoise will have got on a little way. Achilles must then traverse that, and still the tortoise will be ahead. He is always nearer, but he never makes up to it.

[56] Early Greek Philosophy, p. 322

3:- The third argument against the possibility of motion through space made up of points is that, on this hypothesis, an arrow in any moment of its flight must be at rest in some particular point, and thus at rest all the time.

4:- Suppose three rows of points in juxtaposition, as shown in fig. 1

```
      A  •  •  •  •  •  •          A •  •  •  •  •  •
fig.1 B  .  .  .  .  .  .   fig.2 B     .  .  .  .  .  .
      C  .  .  .  .  .  .          C        .  .  .  .  .  .
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One of these, B, is immovable, while A and C move in opposite directions with equal velocities so as to come into the positions shown in fig.2. The movement of C relatively to A will be double its movement relatively to B or, in other words, any given point in C has passed twice as many points of A as it has of B. It cannot, therefore, be the case that an instant of time corresponds to the passage from one point to another.

Tannery thinks that these arguments were given in Zeno's 'discourses' in the form of a dialogue. A Pythagorean adversary is to be convinced that space is not a sum of indivisible parts. The first argument shows that if space were really so constituted then it could not be traversed in finite time.

Against this the adversary objects (as Aristotle does) and says that the dichotomy of the space is not an actual one but only potential and thus it can still be traversed in finite time.

Zeno answers by the second argument in which this objection cannot be made.

The Pythagorean now pleads that he has conceded too much. Finite time itself, he argues, may be capable of infinite dichotomy, for isn't it also a sum of instants? And what prevents our making the successive positions assumed correspond to the successive instants of time?

Zeno now counters by his third argument. To every instant of time corresponds a definite position of the arrow. But to occupy a definite position is to be at rest. Hence the arrow is at rest every instant, and therefore the entire finite time.

The adversary now says that this is not the only correspondence that can be conceived. May not every instant of time correspond, not to a position, but to a change of position, a passage from point to point?

Zeno answers this with his fourth argument. If such a correspondence were possible then, since a point of C passes through two points of A whenever it passes through one point of B we have the absurd result that two instants are equal to one instant.

Tannery points out that Zeno is not (as is often overlooked) at all arguing against the possibility of motion. Indeed it is quite clear, after this re-constitution of Tannery's, that Zeno has only sought to show that, as he himself said, "greater and more ridiculous consequences (the impossibility of motion) follow from their hypothesis of the existence of the Many".

There seems to be no way open to doubt the correctness of Tannery's interpretation of the meaning and real aim of Zeno's paradoxes. The error they attacked was present in the

Pythagorean doctrine, they did reveal, successfully and devastatingly, this error and, as reconstituted by Tannery, they fit together too well and too logically for just this purpose, to have been meant for any other. Further, this interpretation of Tannery's has raised Zeno from a rather dubious former position to that of a "logician of the first rank".[57] We shall see that his influence on the trend mathematics was to take was great.

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[57] Cajori, American Mathematical Monthly, Vol. 22

CHAPTER VIII

ARISTOTLE'S CONCEPTION OF THE INFINITE

Aristotle[58] (b.384 B.C.) gave the best discussion in antiquity of the concepts of infinity and continuity. In various parts of his Physics, De Caelo, and Metaphysics we find various aspects of these concepts dealt with, in an apparently disconnected manner; yet careful study and piecing together reveal a sound and consistent investigation and analysis of these concepts.

Inquiring into the causes that give rise to the notion of the infinite Aristotle finds that they are:[59]

1: The nature of time — it has neither beginning nor end.
2: The division of magnitude — it is, potentially, an endless process.
3: The "coming to be" and the "passing away" — for that from which things continually come to be, and pass away, must be infinite.
4: The concept of "limitation" — This concept leads inevitably to the notion of the infinite. For, a thing is either limited or unlimited; if it is unlimited it is infinite. If it is limited, it has its limit in something which is either limited or unlimited. If this something is unlimited then it is infinite; if it is limited then it has its limit in something else which is either limited or unlimited, and so on. Thus, an infinity is arrived at again.
5: Magnitudes and things outside the heaven never give out in our thought — for if we think of anything outside the heaven we cannot refrain of thinking of something beyond that, and of something else beyond that, and so on.

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58 The Works of Aristotle, tr. and ed. J.A. Smith and W.D. Ross, Oxford, 1908-1930

59 Ibid, Physics, VI

Examining next the various senses in which the word "infinite" isused he finds that these differ according as the word is used of:

1: That which is incapable of being gone through, or traversed,
2: That which admits of being traversed the process, however, having no end,
3: That which with difficulty admits of being traversed,
4: That which naturally admits of being traversed but which is not actually traversed, or does not actually come to an end.

The first use of the word given above is the trivial one in which the word is applied to things which, by nature, are neither finite nor infinite; it is the way in which the voice might be said to be "invisible". The second is the strict use, the one to be examined. The third is the approximate, exaggerated use of the word, as in "the boundless, infinite sea". The fourth is that in which one speaks of "the endlessness of a ring or circle; there is no definite end.

Examining next the views that had been held by previous thinkers about the infinite he finds that it had been held to be:

1: An attribute of the elements (earth, water, air, fire) as subjects,
2: A subject itself (as by the cosmogonists)
3: An attribute of an abstract subject.

Finally, as to the processes that lead to the infinite he finds that they reduce to two. "What is infinite is so in respect of addition, or division, or both".

He finds that the first two of the views listed above involve an actual infinite; the last a potential one. He rejects the former as impossible, whether as a subject or as an attribute, and accepts the latter as possible under certain restricted conditions.

It is neither possible nor desirable to give all of Aristotle's arguments and proofs against the actual infinite in this survey. Some however will be given for the sake of the historic interest attached to them. It must first be explained that by an "actual" infinite Aristotle means, in some cases, a "physical" infinite, whether of one, two or three dimensions and that, apparently, to him the acceptance of such an infinite requires the acceptance of an infinite universe to contain it. "Not every definite magnitude can be exceeded for there would be something, in that case, bigger than the heaven".[60] But, an infinite heaven is impossible, he says. For, "if it is infinite then any two radii are infinite; hence the space between them is infinite. But an infinite space cannot be traversed. And yet, we see the heaven, <u>in its daily revolution</u>, traversing all space".[61] Hence, the heaven is not infinite and thus actual physical infinities do not exist.

Again, arguing now against the possibility of infinite bodies, he starts with the assumption that <u>the velocities of falling</u> bodies are proportional to their weights, an assumption

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[60] <u>Physics</u>, III, 207 b, 19-21

[61] De Caelo

which makes their times of motion, for a fixed distance, inversely proportional to their weights. "Hence", he continues, "for an infinite body the time of motion is infinitely small, no matter what the distance traversed, which is absurd. If you say it (the time of motion) is zero, then the infinite body does not move (contrary to the hypothesis); if you say it is a finite time, then a finite body can be found to perform this motion in this finite time and (thus) the finite and the infinite (bodies) have equal motions, which is absurd (since, in view of the assumption, they would then have equal weights).

Noticeable in these arguments of Aristotle's are the errors in his physical assumptions. Many centuries later, Galileo challenged them both, and of one of them he said:[62] "It is clear that Aristotle could not have made the experiment". But equally noticeable are some true notions Aristotle had concerning the finite and the infinite. For it is clearly implied, in his last argument, that a finite quantity, no matter how small, can, by multiplication, be made to exceed any other finite quantity, no matter how great, and that a finite quantity, no matter how great, cannot be made infinite by multiplication.

To explain now what Aristotle means by the "potential infinite" it is best to proceed by way of the few examples that he has himself used. We have his statement that "What is infinite is so in respect of addition, or division, or

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[62] Two New Sciences, tr. H.Crew and A.DeSalvio, New York, 1933, p. 65

both". Here, of course, infinite is not to be understood, naively, as meaning "of infinite magnitude" for, clearly, magnitudes do not become great as a result of division. The meaning of the statement then must be, simply, that the infinite arises, somehow, as a result of either of these operations, or both. Again, he says: "The infinite is so in virtue of its endlessly changing into something else ...." in which the underscored word is significant; and again, "The infinite must not be regarded as a particular thing ... but... though finite at any moment, always different from moment to moment", where again the underscored word is significant.

Consider now the operation of division as applied to an initial finite magnitude, a straight line say. The principle of "infinite divisibility", which Aristotle accepts, ensures an endless repetition of the process; the result is an endless sequence of terms. This is the "potential infinite" whose existence he asserts. "But", he explains[63], "not in the sense that it might ever be placed before you quite complete; rather you have to gain a knowledge of its existence. And you will have to understand that the fact that the division does not give out ensures a continued potentiality for the actuality, but not a completed existence". It is to be noticed that, with the "potential infinite" thus understood (to wit, as an infinite sequence, in modern terminology), all the passages quoted make good sense. But more evidence in support of this interpretation is available.

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[63] Metaphysics, Θ - 6, 1048 b, 13-18

Consider now the operation of addition and the following example of a potential infinite given by Aristotle. "In the case of a finite magnitude", he says, " you may take a definite fraction of it and add to it in the same ratio; if now the successive added terms do not include one and the same magnitude, whatever it may be, we shall find that addition gives a sum tending to a definite limit". It is not hard to see that Aristotle is describing here, no matter how awkwardly, a geometric series with common ratio less than one. But what is infinite about it? Certainly not the sum, as he himself, evidently, knows. The potential infinite lies in the endless succession of terms. And it is apparently a characteristic property of his "potential infinite" that, no matter how many terms of it be taken, there are always some more left outside of those taken. Hence, his next characterization:[64] "Not that outside of which there is nothing, but that outside of which there is always something, that is the infinite".

On the other hand, if the process is that of adding a finite magnitude, no matter how small, repeatedly to itself then (as was pointed out above) he knows that any finite magnitude will, some time, be exceeded. But, "Not every definite magnitude can be exceeded, for there would be something, in that case, bigger than the heaven". Hencehis conclusion that "The infinite therefore cannot exist, even potentially, in the sense of exceeding every finite magnitude as the result of successive addition".

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[64] Physics, III, 207 a 1-2

There is some confusion here, some vestige of Pythagoreanism perhaps, which somehow attaches physical mass to his conception of magnitude and thus prevents him from accepting even the possibility (potentiality) of "infinite magnitude". That this lack of abstraction may be the true cause of the difficulty here is supported by the fact that it would also explain why Aristotle is, none the less, ready to accept the sequence of the natural numbers as a "potential infinity". His acceptance of this infinity does not require him to accept an "infinite magnitude" also.

Heath[65] has discussed this question and found that Aristotle's "potential infinite" is a convergent infinite series. The view advanced here is that the "potential infinite" is an "infinite sequence", provided the law of formation of its successive terms does not ultimately lead to a contradiction of (rightly or wrongly) accepted truths. The proviso is legitimate, of course, though some of Aristotle's accepted truths could stand revision.

Mr. Abraham Edel in his very interesting and quite plausible "Aristotle's Theory of the Infinite"[66], goes a good deal beyond the view presented here. His thesis is (to the best of my understanding) that Aristotle's "potential infinite" is a large generalization of the interpretation given here, one that comes close to being, if stated

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[65] Elements of Euclid, Vol. 1, Cambridge, 1924

[66] Ph.D. Thesis, Faculty of Philosophy, Columbia University, New York, 1934

abstractly, an embodiment of the Principle of Mathematical Induction. Because of its great inherent interest I try to render Mr. Edel's thought more fully in Appendix II.

Aristotle's treatment of continuity is not as fortunate. His is not the conception of mathematical continuity, but rather an intuitive one, with little abstraction, of the continuity of sensible magnitude. It is explained, by Aristotle, as a type of contiguity, which is itself a type of succession.[67] "... things are called continuous when the touching limits of each become one and the same and are, as the word (συνεχές) implies, contained in each other; continuity is impossible if these extremities are two". His treatment, condensed and reordered, may be given as follows:

1: Things are said to be <u>together</u>, if they are in one place.
2: Things are <u>in contact</u> if their extremities are together.
3: That is <u>between</u> which is reached by a thing before it reaches another.
4: A thing is <u>in succession</u> to another if between it and the other there is nothing of the same kind.
5: A thing is <u>contiguous</u> to another if it is in contact with it and in succession to it.
6: Things are continuous if they are contiguous and if the extremities they have in contact are one and the same.

The definition is very unsatisfactory, of course. It makes use of many irrelevant notions, and it is hard to see, in (5) above, how a thing can be in contact with another and yet fail to be in succession to it. But, worst of all, it is impossible to form an adequate notion of the "things" employed in the definition. For since they have extremities,

[67] <u>Physics</u>, Vol. 3, 277 a, 11-13

they must have parts which are not extremities; now, if these parts are continuous, the whole definition becomes circular, and if they are discontinuous, the definition breaks down. However, Aristotle's definition has the great merit of recognizing, explicitly, that the notion of order is fundamental in, and prior to the notion of continuity. His use of the phrases "between", "in succession to", are ample evidence of this recognition.

CHAPTER IX

THE EUDOXIAN AXIOM

An anonymous scholiast on the fifth book of Euclid's Elements (believed by Heath to be Proclus) tells us that "some say that this book, containing the general theory of proportion, which is common to arithmetic, music, geometry, in a word to all mathematical science, is the discovery of Eudoxus, the teacher of Plato".[68]

It thus appears that this important branch of Greek mathematics is, at least by this scholiast, definitely attributed to Eudoxus. But additional evidence, due to Archimedes and, though indirect yet very convincing, may be adduced.

In the letter to Dositheus prefixed to the treatise "On the Sphere and Cylinder", Archimedes says: "I cannot feel any hesitation in setting them (i.e., these theorems) side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established, namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylinder which has the same base with the cone and equal height. For, though these properties were naturally inherent in the figures all along, yet they were in fact unknown to all the able geometers who lived before Eudoxus and had not been

[68] Taken by Heath (History of Greek Mathematics) from Euclid, Vol. V, p. 280, ed. Heiberg

observed by any one".

Again, in the letter to Eratosthenes, prefixed to "The Method", he says: "This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it".

But, in the letter to Dositheus prefixed to "The Quadrature of the Parabola" Archimedes says: "... and for the demonstration of this property the following lemma is assumed, that '<u>the excess by which the greater of two unequal areas exceeds the less can, by being added to itself, be made to exceed any finite area</u>'. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also that every cone is one third part of the cylinder having the same base as the cone and equal height <u>they proved by assuming a certain lemma similar to that aforesaid</u> ..."

The lemma of the last quotation is given once again by Archimedes, as one of the assumptions used in the book on "The Sphere and Cylinder", in the following form: "Of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with it and with one another".

It is established then, on the strength of this evidence, that Eudoxus gave the first irrefragable proof of certain theorems on the pyramid and the cone by means of a certain lemma similar to Archimedes' assumption. Now, an examination of these theorems, as preserved by Euclid in XII, 7 and XII, 10, reveals that the proof is by the "Method of Exhaustion" and that this method involves, explicitly, the lemma (Euclid, X, 1).

> "If there are two unequal magnitudes and from the greater is subtracted more than its half (or its half), and from the remainder is again subtracted more than its half (or its half), and if this is kept up continually there will be left some magnitude which will be less than the lesser of the two given magnitudes",

a lemma which Euclid proves by the assumption that:

> "By taking the lesser of two magnitudes it is possible, by multiplying it, to make it some time exceed the greater",

an assumption which he seems to base on two definitions of Book V, which state that

Def. 3. A ratio is a sort of relation in respect of size between magnitudes of the same kind (ὁμογενῶν).

Def. 4. Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

How similar Euclid's assumption is to Archimedes' may be seen from the following considerations.

Let a and b be any two comparable magnitudes, with $a < b$. Then, there exists an integer n such that

(1) $n \cdot a > b$, according to Euclid, while

(2) $n \cdot (b - a) > b$, according to Archimedes.

Now, since $a > o$ then $b - a < b$ whence, by (1), $n \cdot (b - a) > b$, which is (2). Again, since $a > o$ then $b + a > b$ whence, by (2), $n \cdot [(b + a) - b] > b$, that is, $n \cdot a > b$, which is (1).

And now, we may summarize the additional evidence that was to be adduced in support of Proclus' assertion concerning the authorship of Book V as follows. Archimedes asserted that Eudoxus gave the first irrefragable proof of the theorems concerning the volumes of the pyramid and the cone, by means of a lemma similar to his own. These proofs, as preserved by Euclid, involve an assumption which is similar (in fact, equivalent) to that of Archimedes, and this assumption makes its first appearance in Greek mathematics in Book V. Hence, Eudoxus' connection with this book is established.

It will be noticed that this brief investigation also establishes the fact that the axiom so often described as the "Axiom of Archimedes" in mathematical literature is due to Eudoxus; it is therefore more properly described as "The Axiom of Eudoxus".

CHAPTER X

THE EUDOXIAN THEORY OF PROPORTIONS

This theory cannot be described here in its entirety, admirable though it is and "in every respect superior to the algebraical method by which it is now generally replaced"[69]; but such parts of the theory as are at all involved in the object of this survey will be carefully examined.

The theory is founded on a number of definitions of which the most important are:[70]

(1) A magnitude is a part of a magnitude, the less of the greater, if it measures the greater.

(2) The greater is a multiple of the less when it is measured by the less.

(3) A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

(4) Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

(5) Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of the latter equimultiples respectively taken in corresponding order.

(6) Let magnitudes which have the same ratio be called proportional.

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[69] O. Henrici, F.R.S., Geometry, Enc. Brit. Vol.X:382 Werner Ed., New York, 1901

[70] T.L. Heath, Elements of Euclid, Vol. 2

(7) When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a <u>greater ratio</u> to the second than the third has to the fourth.

Of these definitions (1) and (2) serve the purpose of defining the word "multiple", used in (5) and (7); (6) is only nominal and thus unnecessary. Definition (3) however is worse than unnecessary; it is useless. For though it professes to be a definition of "ratio", it not only does not define this notion mathematically (even if it does classify the notion philosophically), but it also fails to impart to it any properties whatever that could make it usable in a mathematical system. However, no use is made of it in the theory, and it is believed by many to be a later addition, probably by Theon of Alexandria. Heath says: "It now appears certain this definition is an interpolation".[71]

The list narrows down now to (4), (5) and (7), defintions which have been copiously commented upon and variously interpreted through the ages. Thus, Pascal[72] thought that by "magnitudes of the same kind" Euclid meant magnitudes possessing the Eudoxian property. But this is not acceptable since there is evidence to support the view that Euclid accepted certain magnitudes as being "of the same kind (ὁμογενῆ)" even though they did not possess this property. For, in Book I he gives:

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[71] Henrici, <u>Loc. cit.</u>

[72] Blaise Pascal, <u>Pensées</u>, ed. W.F. Trotter, N.Y., 1910

Def. 8 A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Def. 9 And when the lines containing the angle are straight, the angle is called rectilinear,

in which, it is seen, the first definition puts all plane angles in the same genus and the second distinguishes rectilinear ones as a separate species. Then, in Book III, 16, he proves, after a fashion, that the angle a tangent to a circle makes with the circumference is less than any rectilinear angle, a fact which shows that though he thought of these angles (κερατοειδεῖς , horn-shaped) as being of the same kind (at least in the matter of magnitude, since he compared them), he yet denied that they possessed the Eudoxian property. For, if they did possess this property then, for any horn-angle H, and any rectilinear angle R, we would have, by the Eudoxian property, $n \cdot H > R$ whence $H > \frac{R}{n}$, which contradicts the theorem Euclid proved.

To allow therefore for this broader conception of "magnitudes of the same kind", which the above dicussion clearly shows Euclid had, the two definitions, (3) and (4), may now (as is, besides, quite permissible linguistically) be retranslated as follows:

Def. 3 A ratio is a sort of relation in respect of size between certain magnitudes of the same kind.

Def. 4 Those magnitudes of the same kind are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

We have then, in the acceptance of these angles, a clear instance of what Russell[73] has called since then the "relative infinitesimal", the correlative of the "relative infinite", dealt with, in recent years, by DuBois Reymond, [74] Borel[75], and briefly by Klein, Kasner and others.

However, the concept of non-Eudoxian magnitudes apparently did not lend itself to easy, intuitive understanding. For Proclus[76], in commenting on the many classifications of the notion of "angle" made by previous philosophers — for these had variously placed it in the categories of quantity, quality and relation — rejects the opinion that angle is a magnitude, saying: "If it is a magnitude, and all finite magnitudes of the same kind have a mutual ratio, all angles of the same kind, that is, which exist in surfaces, will have a mutual ratio. And hence the horn-shaped will have a ratio to the rectilinear. But things which have a mutual ratio may, by multiplication, exceed each other; and therefore it may be possible for the horn-shaped to exceed the rectilinear angle which, it is well-known, is impossible, since it is shown to be less than any rectilinear angle".

There is no explicit indication that Eudoxus himself had such a braod conception of magnitude; but the mere fact that he took the trouble to state his assumption concerning certain

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[73] Russell, Principles of Mathematics, p. 332

[74] P. DuBois Reymond, Infinitärcalcul, Hardy, Cambridge, tract #12.

[75] E. Borel, La Théorie de la Croissance, Paris, 1910

[76] Commentaries on Euclid's Book I, p. 141, Tr. T.Taylor, London, 1792

magnitudes proves that he at least considered the broader conception, though he decided to have nothing to do with it. It is not unfair, I think, to see in this a retreat, due in part to Zeno's criticisms. There is no explicit mention, as there is in the case of Archimedes, of the magnitudes of which Eudoxus made his assumption; but these turn out to be the same as those of Archimedes. For the assumption is applied to lines, in Book V, and to surfaces and solids in Book XII.

On the other hand, that the difficulties raised by Zeno were very skillfully circumvented by means of this assumption cannot be denied. Within the theory built up by Eudoxus, any magnitude (length, area, volume), no matter how small, can be surpassed in smallness, and any magnitude, no matter how large, can be surpassed in largeness. There is now no ultimate small and no ultimate large. The inconsistencies attacked by Zeno cannot now arise. Thus were banished the "points" of the Pythagoreans, as constituent elements of magnitude, the "indivisibles" of the Atomists, and all infinite magnitudes from Greek mathematics.

This does not mean, of course, that all conceptions of infinity were rendered impossible by the adoption of the Eudoxian axiom, for, obviously, one could still arrive at it by induction (we have seen that Aristotle did), and without violence to this axiom, provided one did not include it in the same class with the finite. One could, for instance, perceive by means of some insight or principle that there are more than n magnitudes in a class, no matter what natural number n might be, and assert an infinity of magnitudes in that class, provided this infinity itself was not thought of as a member of the same class. In fact, something entirely analogous to this (except for the assertion of the infinity) was done by Euclid himself when, in regard to the totality of primes he said (Book IX, 20), "Prime numbers are more than any assigned multitude of primes". There is no violence here to the Eudoxian axiom for though Euclid did state that (for convenience we denote the number of primes by N) $N > n$, for any natural number n, while the Eudoxian axiom asserts that for any two natural numbers, M', N', where $M' < N'$, an n exists such that $n \cdot M' > N'$, there is no contradiction, for Euclid did not put N in the class of natural numbers; the N of Euclid's is not the N' of the axiom.

Euclid's proof of this theorem was, of course, a creditable performance. Not so however his extreme cautiousness concerning the use of the word "infinite". For consider his evasiveness in the very enunciation of this theorem: "The prime numbers are more than any assigned number of primes". Not only does he apply the adjective "infinite" to this totality, but he even avoids reference to it as a "number". Zeno's criticisms were, indeed, effective.

That there was here an opportunity for the drawing of a valid definition of a certain type of infinity can be made clear by a brief examination of Euclid's proof of the theorem in question. This is, in substance, as follows:

Suppose the number of primes is finite. Let these be a, b . . . . k. Form now the number (a b ....k) + 1. Then

Case I — If this number is a prime, a new prime has been found, for it is, obviously, not any of a, b, ... k.

Case II — If it is not prime then it is divisible by some prime, p say, which cannot be identical with any of a, b, ... k; for if it were then it would divide (a b ...k), and since it also divides (a b ...k) + 1 by hypothesis, it would divide the difference, or unity, which is impossible.

Hence, in either case a new prime has been found. This contradiction proves that the number of primes is not finite.

It is obvious that the principle of mathematical induction is involved in this proof. For a discrete series is formed, whose terms are the successive primes, p_n , produced by Euclid's process from an initial prime, 1 say. This process yields, after any term p_n its successor p_{n+1} , which is either $(1 \cdot p_1 \cdot p_2 \ldots p_n) + 1$, if this is a prime, or some prime divisor of it if it is composite; in either case, Euclid's proof assures us that p_{n+1} is a prime different from any of its predecessors. Now, the first term of the series is a prime; also, if the n^{th} is a prime, so is the $(n + 1)^{th}$. Hence, every term of the series is a prime. But every term of the series has a successor; hence the series is infinite and thus the number of primes is infinite.

We note, incidentally, that this process cannot be used as an algorithm to obtain an endless succession of primes for no general way is known by which it can be decided, in every case, whether $(p_1 \cdot p_2 \ldots p_n) + 1$ is itself a prime or a product of new primes.

It is now time to examine definitions (5) and (7) and, to this end, I consider next the sense in which the word "magnitude" is used by Euclid for, as can be seen, this concept is fundamental in these definitions.

From axioms stated and others actually used in the <u>Elements</u>, Euclid seems to assign, in general, the following properties to any set of magnitudes, a, b, c, ... etc., of the same kind.

1) If $a = b$, $b = c$ then $a = c$
2) If $a < b$, $b < c$ then $a < c$
3) Of the relations $a < b$, $a = b$, $a > b$ one and only one holds.
4) $a + b$ is a magnitude of the same kind as a and b
5) If $a < b$ then an n exists such that $n \cdot a > b$ (for lengths, areas, volumes)
6) If a and b satisfy (5) then a/b, b/a exist.

Properties (5) and (6) are introduced, as has been noted, in Book V. And now, definitions (5) and (7) of Book V may be stated as follows. Let a and b be magnitudes of the same kind, c and d magnitudes of the same kind (the second kind not necessarily the same as the first); then

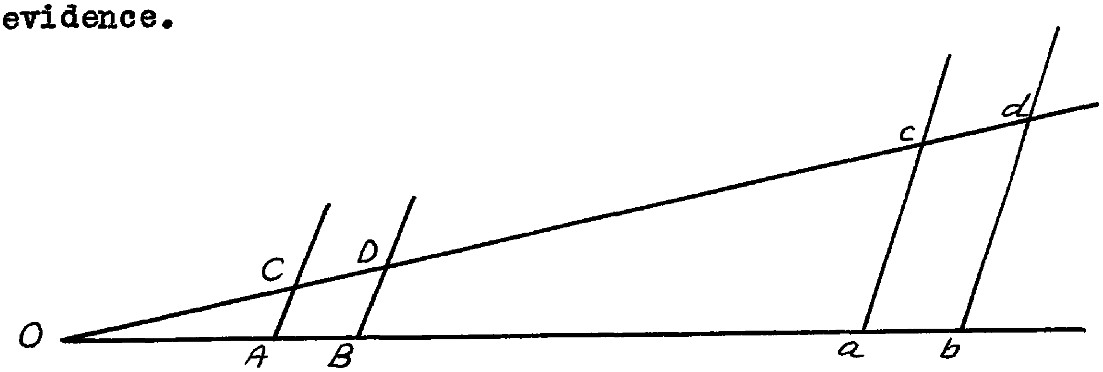
Def. 5 If, for every pair m, n of ~~the~~ whole numbers we have $m \cdot a \lesseqqgtr n \cdot b$ according as $m \cdot c \lesseqqgtr n\ d$ then $a/b = c/d$.

Def. 7 If, for some pair m, n of whole numbers we have $m \cdot a > n \cdot b$ but $m\ c \leqq n\ d$ then $a/b > c/d$.

We note that properties (3) and (6) of the previous paragraph are assumed in these definitions. We note also the high degree of abstraction imparted to the notion of ratio by these definitions; a and b may be lengths, c and d areas, yet $a/b = c/d$ if Def. 5 is satisfied. The Eudoxian ratio was then entirely independent of the nature of the magnitude in terms of which it was expressed.

These definitions of "equality" and "inequality" of ratios were criticized in the past on the ground that they do not offer criteria which can be applied, in every given case, to decide whether a certain ratio is or is not equal to another. For definition 5 requires one to examine an infinity of pairs m, n, which is actually an impossible task, and definition 7 requires one to find some pair m, n which, though it may exist, one may conceivably fail to find in a year's search. These objections will be recognized, of course, as the older counterpart of the questions agitated more recently by Kronecker and still more recently by the "intuitionist school" under the leadership of Brouwer and Weyl.

A. De Morgan[77] answers this criticism with the statement that "nevertheless, certain mathematical methods enable us to avoid this difficulty", and gives the following example in evidence.



> "If a series of parallels cut off consecutive equal parts from any one line which they cut, they do the same from any other. This premised, suppose any two lines, OA, OB, and take a succession of lines equal to OA, OB, drawing through every point a parallel to a given line. Draw any other line,

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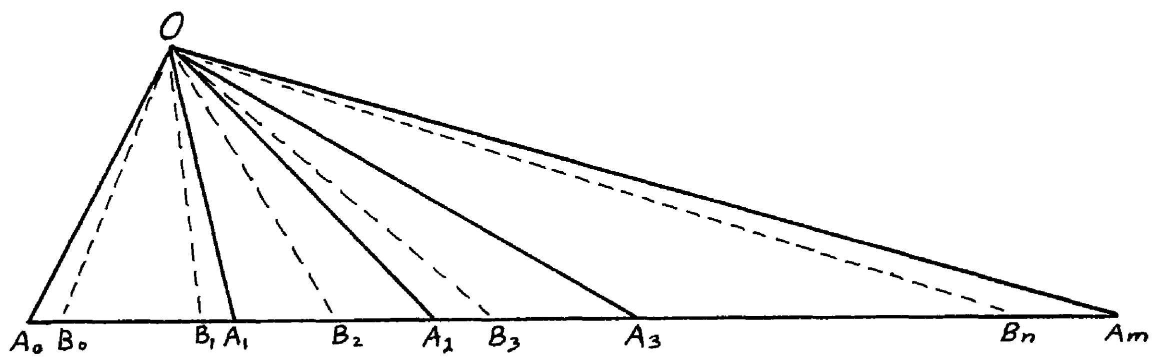
[77] Number and Magnitude, an Attempt to explain the 5th Book of Euclid, London, 1836, p. 72

OCD, intersecting all the parallels, from which the preliminary proposition shows that whatever multiple Oa is of OA, the same is Oc of OC; and whatever Ob is of OB the same is Od of OD. And if Oa be greater than, equal to, or less than Ob, Oc is greater than, equal to, or less than Od. Hence the definition of equal ratios applies precisely to the lines OA, OB, OC and OD which are, therefore, proportional. This gives the construction of Book VI, 12, or one analogous to it."

Writing to the same end, O. Henrici[78] shows, by the following example, how Euclid's definition of equality may yet be used. For greater clarity I underscore the theorems he uses as premises, and adjoin a figure, with references to it in parentheses, to illustrate his proof.



"Triangles of the same altitude are to one another as their bases, or if a and b are the bases, and α and β the areas of two triangles which have the same altitude, then $a/b = \alpha/\beta$.

To prove this we have, according to def. 5, to show that:

if $ma > nb$, then $m\alpha > n\beta$,
if $ma = nb$, then $m\alpha = n\beta$,
if $ma < nb$, then $m\alpha < n\beta$,

That this is true is, in our case, easily seen. We may suppose that the triangles (OA_0A_1 and OB_0B_1) have a common vertex (O), and their

78 Henrici, loc. cit.

> bases in the same line. We set off the base a along the line containing the base m times; we then join the different parts of division (A_2 , A_3 , ... A_m) to the vertex and get m triangles all equal to α . The triangle ($OA_0 A_m$) on ma as base equals, therefore, $m\alpha$. If we proceed in the same manner with the base b, setting it off n times, we find that the area of the triangle ($OB_0 B_n$) on the base nb equals $n\beta$, the vertex of all triangles being the same. But if two triangles have the same altitude, then their areas are equal if their bases are equal; hence $m\alpha = n\beta$ if ma = nb, and if their bases are unequal, then that has the greater area which is on the greater base; in other words, $m\alpha$ is greater than, equal to, or less than $n\beta$ according as ma is greater than, equal to or less than nb, which was to be proved".

It is interesting to observe how again the difficulty of including an infinity within the scope of an assertion has been overcome by the intervention of a principle. We have met examples in which the principle used for this purpose was mathematical induction, but in the two instances just given the empowering principle is not mathematical induction. Consider either of these examples, Henrici's say, and examine the premises used. Take the first, for example, and restate it thus: "If, for any two triangles having equal sides, etc."; it is at once evident that the word any is significant; it is the clue to the principle sought.

Consider the class, X, of elements x. The phrase 'any x' means 'an element of X, no matter which'. Hence the statement 'Any x has the property P' means

(A) If an element of X be taken, no matter which, it will be found to have the property P.

From this it is possible to infer, on a priori grounds that

(B) Every element of X has the property P.

In the case where X is a finite class the truth of (B) can be shown to follow from that of (A) by the argument that the elements of X can be individually examined, and in such an examination no x can be found which does not have the property P for the finding of such an x would contradict (A), which is supposed to be true.

In the case where X is infinite such an argument cannot be used to deduce (B) from (A) for the elements of X cannot be examined individually. But if X is an infinite class of the type known as a 'progression', the truth of (B) can be deduced from that of (A) by the principle of mathematical induction. For:

1: If the first element of X is taken it will be found to have the property P, by virtue of (A).

2: If any x is taken, it has the property P, by virtue of (A) again.

3: Any such x has a successor, since X is a progression.

4: If this successor is taken, it has the property P, by virtue of (A)

5: Every element of X has the property P, by the principle of mathematical induction.

But in the case of other types of infinity (to which mathematical induction does not apply) no means remains of asserting the property P for every member of the class except that which proceeds through the 'concept of X', the concept of which X itself is the 'extension'. This is the principle sought; it may be stated thus:

(C) If the concept, C, whose class is X, implies the property P, then every member of X has the property P.

The validity of this principle has been disputed, as is well known, by the intuitionists and, especially, by the finitists; and various contradictions now existing in the theory of infinite numbers have been attributed to it. Thus, Brouwer[79] insists that "The Komprehensions-axiom, by virtue of which all things which have a certain property are united to form an aggregate (Menge), even in the limited form used by Zermelo, is inadmissible in founding the aggregate theory, or is at least unserviceable. Only in the constructive definition of aggregates is a trustworthy foundation of mathematics to be found". But the rejection of principle (C), even for classes which can be ordered by Zermelo's axiom, renders the principle of the excluded middle invalid for all infinite classes except progressions. Brouwer does not hesitate; he rejects even this. "On this foundation"[80], he says, "particularly in the last half century, extensive false theories have been erected". He (and Weyl) abandons traditional logic and holds that a proposition of any theory is false if it contradicts an axiom of the theory, true if what it asserts can be constructed, and indeterminate if it contradicts no axiom but has not been constructed yet. An interesting

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[79] J. Pierpont, Bulletin American Mathematical Society, XXXIV:51

[80] Ibid, p. 52

peculiarity of this logic is that a proposition cannot be proved to belong to the third class. For, obviously, the assertion that the proposition will never be constructed cannot be proved directly; on the other hand, the indirect way, of showing that the hypothesis that it can be constructed leads to a contradiction, proves that the proposition is false, not indeterminate.

Whether the imputations made by these schools against the established theories of the infinite are just or not is not definitely known; for all that these schools have succeeded in demonstrating is that the contradictions do not arise in these infinities <u>if these infinities are left strictly alone.</u> And it must not be overlooked that new principles[81] have been invoked by the opposing school of thought also, by the supporters of traditional mathematics, which eliminate the paradoxes and contradictions from the theory of aggregates, <u>without necessitating the abandonment</u> of fully one half of the mathematics in existence. Russell[82], for instance, has shown that the principle "Whatever involves all of a collection cannot be one of the collection", the principle on which his Theory of Types is based, removes the paradoxes and contradictions from these theories without destroying them.

There is no a priori reason, as yet, for accepting either of these two solutions of the difficulty in preference to the

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[81] e.g., The Hilbert School and the Axiomatic Method, See J. Pierpont, <u>op. cit.</u>, p. 46-50

[82] <u>Principles of Mathematics</u>, p. 523-528.

other, and there cannot be until a complete theory of knowledge becomes available. But pragmatic considerations urge that, until then, those solutions be preferred which remove the difficulty without destroying a large part of what is left. There must not be another retreat before the infinite.

An interesting comparison may be made between the two principles, i.e., (C) above, and

(D) If in the progression X the first term has the property P, and if P is such that when any term, x, has the property its successor also has it, then every term of X has the property P.

As Poincaré[83] pointed out, the former is not an instrument of discovery while the latter is. For, if knowledge is only of the 'universal' then in asserting the property P of every member of X the former principle makes no discovery of new knowledge since the property P was already contained in the universal C. The latter however, proceeding from the observation of the property P in the particular, x (which is not knowledge), asserts it of the universal X (which is).

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[83] H. Poincaré, The Foundations of Science, tr. G.B. Halsted, New York, 1929. "Without doubt recurrent reasoning in mathematics and inductive reasoning in physics rest on different foundations, but their march is parallel, they advance in the same sense, that is to say, from the particular to the general", page 40.
"We can ascend only by mathematical induction, which alone can teach us something new", page 42.

CHAPTER XI

THE EUDOXIAN NUMBER SYSTEM

It is time now to examine the number system that Eudoxus constructed by means of the assumptions and definitions we have finished considering. Concerning this number system Heath says:[84] "Max Simon remarks, after Zeuthen, that Euclid's definition of equal ratios is, word for word, the same as Weierstrass' definition of equal numbers. So far from agreeing in the usual view that the Greeks saw in the irrational no number, Simon thinks it is clear from Euclid V that they possessed a notion of number in all its generality as clearly defined as, nay almost identical with, Weierstrass' conception of it". As if skeptical about this high opinion Heath continues thus: "Certain it is that there is an exact correspondence, almost coincidence, between Euclid's definition of equal ratios and the modern theory of irrationals due to Dedekind", and offers, in support of this view, the following demonstration.

> "Dedekind arrives at the following definition of an irrational number: 'An irrational number, α, is defined whenever a law is stated which will assign every given rational number to one and only one of two classes A and B such that
>
> 1: every number in A precedes any number in B,
>
> 2: there is no last number in A and no first in B;
>
> the definition of α being that it is the one number which lies between all numbers in A and

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[84] The Elements of Euclid, 2:125

all numbers in B'. [85]

Now let x/y and x'/y' be equal ratios in Euclid's sense. Then x/y will divide all rational numbers into two groups A and B; x'/y' will divide all rational numbers into two groups A' and B'.

Let a/b be any rational number in A, so that $a/b < x/y$; then $ay < bx$. But Euclid's definition asserts that in that case $ay' < bx'$ also. Hence also $a/b < x'/y'$; therefore any number of group A is also a number of group A'.

Similarly any number of group B is also a number of group B'. For if a/b belong to B, then $a/b > x/y$ and thus $ay > bx$. But in that case, by Euclid's definition, $ay' > bx'$; therefore also $a/b > x'/y'$. Thus, in other words, A and B are co-extensive with A' and B' respectively; therefore $x/y = x'/y'$ according to Dedekind as well as according to Euclid.

If x/y, x'/y', happen to be rational, then one of the groups, say A, includes x/y, and one of the groups, say A', includes x'/y'. In this case a/b might coincide with x/y; that is, $a/b = x/y$ which means that $ay = bx$. Therefore, by Euclid's definition, $ay' = bx'$; so that $a/b = x'/y'$. Thus the groups are again co-extensive. In a word then, Euclid's definition divides all rational numbers into two co-extensive classes, and therefore defines equal ratios in a manner exactly corresponding to Dedekind's theory".

It will be observed, of course, that this demonstration of Heath's does not establish a correspondence between Euclid's definition of equal ratios and the modern theory of irrationals due to Dedekind but rather (and perhaps this is Heath's meaning) between the definitions of equality in the two systems. But even thus, the correspondence exists in one way only, that is, if two ratios are equal in Euclid's sense then the real numbers determined by them are equal in Dedekind's

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[85] The postulation of a real number corresponding to every such "Cut" has been criticized by H. Weber and B. Russell; the "Cut" itself is now taken, by definition, as the real number.

sense, but not conversely. How far the two systems are from being in exact correspondence and how far Simon's estimate is from being a true one will be made clear in the analysis of the Eudoxian system that follows.

We recall the definition of a real number:

> If the class, C, of the rational numbers is separated, in any manner, into two non-empty classes, A and B, such that
> 1: any element of A is less than any element of B,
> 2: A has no last element,
>
> then the separation determined by the classes A and B is a real number, rational if B has a first element, irrational if it has not.

We denote the number thus defined by the symbol (A, B) and, for greater convenience in the comparison to be made, we exclude from the class C the number zero and all negative numbers.

We show now that a Eudoxian ratio, a/b, effects a separation of the rational numbers which is strictly analogous to the real number of the above definition. For, consider the class A of the rational numbers consisting of those for which $ma > nb$, and the class B of those for which $ma \leq nb$. Let such a separation be denoted by the symbol A/B. It is to be shown that

M. Every A/B is also an (A,B)

1: Let n/m be any rational number; then either $ma > nb$ or $ma \leq nb$; hence, n/m is either in A or else in B.

2: Let n/m be any element of A and r/s any one of B; then $ma > nb$ and $sa \leq rb$. Hence, $n/m < a/b \leq r/s$, that is, $n/m < r/s$. Hence, any element of A is less than any element of B.

3: Let a, b, be any two Eudoxian magnitudes of the same kind. Then an integer n exists such that $na > b$; hence $1/n < a/b$ and thus A has at least the rational number $1/n$ in it. Similarly, an m exists such that $mb > a$, whence $m/1 > a/b$. Thus B has at least the element $m/1$ in it. Therefore A and B are non-empty classes.

4: A has no last element. For, let n/m be any element of A; we show that an element of A, greater than n/m, exists.

Since n/m is in A, $n/m < a/b$ and thus $ma > nb$; therefore $ma - nb$ is a magnitude of the same kind as a and b and thus Eudoxian. Of course, so is mb. Hence, an integer k exists such that $k(ma - nb) > mb$. From this it follows that

a) $$1/k < \frac{am - nb}{mb}$$

And now, it is easily seen, the rational number n/m 1/k, greater than n/m, is in A. For, from a) above

$$1/k < (ma - nb)/mb,$$

$$< \frac{a}{b} - \frac{n}{m} \text{, whence}$$

$$\frac{n}{m} + \frac{1}{k} < \frac{a}{b} \text{, which was to be shown.}$$

And now, from 1:, 2:, 3:, 4: above it follows that the definition of (A, B) is satisfied by A/B. This concludes the proof.

We consider now the converse question, that is whether

N. Every (A, B) is also an A/B.

We paraphrase this question thus: Is there, for every separation of the rational numbers in the Dedekindian sense, a Eudoxian ratio , a/b, which effects this separation?

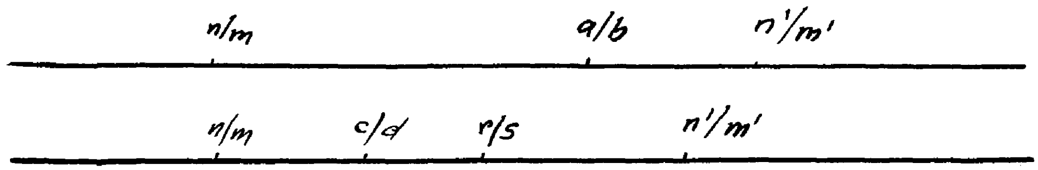
The answer to this question depends on the nature of the magnitudes a, b, c, etc. But whether these possess the property needed to assure the existence of the ratio in question, or not, must ultimately be a matter for postulation rather than proof. Hence the question to be answered reduces now to whether this postulation was made or not. It may be said that there is no mention of such a postulate in Greek mathematics, nor is there any indication of its use. On the other hand, it is fair, as will be shown, to think that an opportunity to formulate such a postulate did arise, but was not taken.

If we recall the statements of Eratosthenes, Theon of Smyrna and Proclus (p. 216 above) concerning proportion as the bond and common element of arithmetic and geometry, if we recall that incommensurable lines had already been discovered, that, in fact, the theory of proportion of Book V was created for the express purpose of coping with incommensurables, finally, if we note, in the following very natural derivation of the definitions of "equality" and "inequality" used in this book, a derivation, moreover, which was most probably the one employed in arriving at the said definition, if we note that all that was needed was the assertion of the converse of an observed fact, it will be seen that the opportunity was there.

For suppose that two pairs of magnitudes, a,b, and c,d, which may be incommensurable, are given,,and the ratios a/b, c/d are formed. If these are to be equal, that is, if they are to be the same ratio (see Def. 5 above) then, as is intuitively demanded and as the analogy from the case of commensurable ratios suggests, if a/b is less than, equal to or greater than ratio, n/m, then so must the other ratio, c/d be less than, equal to, or less than n/m. In other words,

$$\begin{array}{ll} \text{if } a/b < n/m & \text{then } c/d < n/m, \\ \text{if } a/b = n/m & \text{then } c/d = n/m, \\ \text{if } a/b > n/m & \text{then } c/d > n/m, \end{array}$$

from which the definition that was adopted, "$a/b = c/d$ if $ma \lesseqgtr nb$ according as $mc \lesseqgtr nd$" naturally arises. And it is obvious — too obvious to miss — that each of these ratios, a/b and c/d, leads easily to a separation of the numbers n/m into two groups, to wit, the group of those that are greater than it and that of those that are not. Hence, for two such ratios the comparison between them might have been shown, graphically, by means of two straight lines, thus:



From this would arise also, quite easily, the definition of "greater than". For, if a single ratio, r/s, lay between a/b and c/d, as shown in the figure, that is, if some r/s existed such that $r/s < a/b$ but $r/s \geq c/d$ then, obviously $a/b > c/d$. This, of course, would then be restated in the form given in Def. 7.

Indeed, it is difficult to imagine an easier or more natural approach to definitions 5 and 7 of Book V than the one outlined here. Therefore, it may be assumed that Eudoxus knew that any ratio separates the rational numbers into two mutually exclusive, and exhaustive groups. Hence, an easy opportunity was present to consider the possibility of the converse also and thus arrive at the postulate in question. As stated above, this opportunity was not seized by Eudoxus. On the other hand, he could not have been expected to either; for such a procedure would have been entirely foreign to the Greek habit of mind. As is, besides, illustrated in the number system Eudoxus constructed, the Greeks saw the magnitude first and then assigned a measure to it; they did not assert the existence of a magnitude to correspond to every measure.

It is important, however, to examine the consequences of the postulate in question. Suppose then that we have not only

M Every A/B is an (A, B), as was proved above, but also
N Every (A,B) is an A/B, by hypothesis.

It is easy to show that, in such a case, there is a one-to-one correspondence between the class of Dedekind Cuts and the class of Eudoxian ratios. For, let A_1/B_1 correspond to (A'_1, B'_1) and let (A'_2, B'_2) be any <u>other</u> Dedekind Cut. Then, we show, A_1/B_1 does not correspond to it also. For, if it did, we would have:

a) every element of A'_2 is an element of A_1, by N
b) every element of A_1 is an element of A'_1, by M.
c) every element of A'_2 is an element of A'_1, by a) and b); again
d) every element of B'_2 is an element of B_1, by N,
e) every element of B_1 is an element of B'_1, by M,
f) every element of B'_2 is an element of B'_1, by d) and e).

Now, from c) and f) $(A'_1, B'_1) = (A'_2, B'_2)$, by the definition [86] of equality of Dedekind Cuts. But this is impossible, since, by hypothesis, (A'_2, B'_2) is other than (A'_1, B'_1). Hence, A_1/B_1 cannot correspond to two distinct Dedekind Cuts. In a similar manner it can be shown that (A ,B)

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[86] W.F. Osgood, <u>Function of a Real Variable,</u> Peking, 1936 p. 47

cannot correspond to two distinct Eudoxian ratios. Thus, the correspondence is one-to-one.

Further, in this correspondence, the rank of corresponding elements is preserved; that is, if $A_1/B_1 < A_2/B_2$ then $(A'_1, B'_1) < (A'_2, B'_2)$. For, suppose that $A_1/B_1 < A_2/B_2$; then, for some m, n, $ma_1 < nb_1$ and $ma_2 \geqq nb_2$, whence $a_1/b_1 < n/m$ but $a_2/b_2 \geqq n/m$. From this it follows that B'_1 contains a rational number, n/m, not in B'_2; hence, by the definition of "less than" for Dedekind Cuts, $(A'_1, B'_1) < (A'_2, B'_2)$.

It follows from the last two results obtained that if the assumption N. is made the system of Eudoxian ratios becomes a linear continuum;[87] for it has been shown to be a series whose class is in one-to-one correspondence with the class of real numbers and ordinally[88] similar to it.

Yet, the Eudoxian continuum created with the help of this assumption is, in an essential respect, different from the Dedekindian continuum. For, if, according to the assumption, to every real number α there corresponds a ratio a/b such that $\alpha = a/b$ then, letting b be the unit of magnitude, $\alpha \cdot b = a$, that is, to every real number α there corresponds a magnitude of measure α. Hence, the assumption imparts to the geometric magnitudes a, b, . . . themselves the continuity of the Dedekindian system. If now, in this last result, the magnitudes a, b, . . . be regarded as geometric lengths, the assumption

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[87] E.V. Huntington, The Continuum as a Type of Order, Harvard Press, 1921

[88] Cantor, op. cit., p. 112.

reduces to the Dedekind-Cantor axiom concerning the geometric line.

On the other hand, if the assumption is not made the Eudoxian system is not a continuum (it is shown in Appendix III that it is a dense series). Further, this difference exists between the two systems: the Eudoxian is geometric in character, and dependent on the postulate N. if it is to be continuous. The Dedekindian is purely arithmetic, and independent of this postulate for its continuity.

As is wellknown, Dedekind's purpose[89] in the creation of the arithmetic continuum was to free analysis from all dependence on the notion of "measurable magnitude", and other notions foreign to arithmetic. For, as Russell[90] puts it, "Weierstrass, Dedekind, Cantor and their followers pointed out that if irrational numbers are to be significantly employed as measures of quantitative fractions (measurable magnitudes) they must be defined without reference to quantity (magnitude)". To the same effect, but more specifically, Dedekind[91] writes "All the more beautiful it appears to me that without any notion of measurable magnitudes and simply by a finite system of simple thought-steps man can advance to the creation of the pure continuous number-domain; and only by this means in my view is it possible for him to render the notion of continuous space clear and definite".

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89 R. Dedekind, Essays on Number, page 2, Chicago, 1924

90 Principles of Mathematics, p. 157

91 Dedekind, op. cit. p. 38

Thus did Eudoxus come within an ace of creating a continuum of ratios, and thus did Dedekind, twenty-two centuries later, brilliantly achieve his superior, purely arithmetic continuum, by taking up the problem where Eudoxus had left it. The discussion of the preceding pages has, undoubtedly, revealed the close connection between the two systems; to this can be added the following frank admission,[92] by Dedekind, of his indebtedness to Eudoxus.

> " . . . an irrational number is defined by the specification of all rational numbers that are less and all those that are greater than the number to be defined . . . this manner of determining it is already set forth in the clearest possible way in the celebrated definition of the equality of two ratios (Elements, V, 5). This same most ancient conviction has been the source of my theory as well as that of Bertrand and many other more or less complete attempts to lay the foundations for the introduction of irrational numbers into arithmetic".

The comparison of the two number systems given above was not meant, of course, to place them on the same level of excellence, nor should it be allowed to obscure the fact that the Eudoxian system was woefully incomplete.[93] This system contained all the rational numbers, all the algebraic irrationals of the quadratic and the biquadratic (the twenty-five species of the tenth book of Euclid), an infinity of algebraic irrationals of order 2^n (the infinite sequence of medials of Book X, 115), and the transcendental π (though its transcendentality, or even incommensurability was not definitely known); but it still lacked most of the algebraic irrationals

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[92] Ibid

[93] See also H.B. Fine, "Ratio, Proportion and Measurement in Euclid", Annals of Mathematics, XIX:1917.

and all (with the sole exception noted) the transcendentals which are, of course, by far the biggest part of the real number system.

As a consequence of this incompleteness not all the mathematical operations could be performed in this number system. In particular, an infinite sequence of ratios, a_1, a_2, . . . a_n, . . . rational or not, which satisfied Cauchy's criterion of convergence, $|a_{n+m} - a_n| < \epsilon$, $n \geqq N$, did not necessarily have a limit in this system. However, it is very unlikely that this defect inconvenienced Greek mathematicians at all. Greek mathematics had, by this time, taken a definite trend away from considerations of infinities and limits (the effect of Zeno's criticisms noted above) and had substituted for them considerations of simpler concepts, clearer and easier to grasp intuitively, and methods that could be applied with greater confidence. The Eudoxian number system was itself but a reflection of the mathematics of the time. Thus, though the question raised above could not arise, one that could, that was in fact substituted for it, was the converse question, that is, the question whether, for any ratio, a/b, of definite value and assumed (on intuitive or other grounds) to exist, approximations of definite value and arbitrarily close to the assumed one did or did not exist in their number system.

It can be shown that the Eudoxian number system, with the help of the Eudoxian axiom, was sufficient to guarantee the existence of such approximations. To this end we prove the following theorem.

"For any Dedekind Cut, (A, B), if it be assumed that there exists a Eudoxian ratio, a/b, determining the Eudoxian separation, A/B, then there exists a Eudoxian ratio, a'/b', such that $|a/b - a'/b'| < \delta$ where δ is any Eudoxian ratio".

Let the rational Eudoxian ratio ϵ be such that $\epsilon < \delta$; that there exists such an ϵ is assured by the fact that δ effects a separation of the rational numbers into two classes, and as was proved above, each of these has at least one element in it.

Let n_1/m_1 be an element of A and n_2/m_2 an element of B; then $n_2/m_2 > n_1/m_1$ and thus $n/m \equiv (n_2/m_2 - n_1/m_1) > 0$.

If $n/m < \epsilon$ we choose $a' = (n_1 + n_2)u$, $b' = (m_1 + m_2)u$, where u is the unit of magnitude, and form the ratio a'/b' $= \frac{(n_1 + n_2)u}{(m_1 + m_2)u}$, that is $a'/b' = \frac{n_1 + n_2}{m_1 + m_2}$, as the required ratio. As is easily verified $\frac{n_1}{m_1} < \frac{n_1 + n_2}{m_1 + m_2} < \frac{n_2}{m_2}$

Now, the ratio a'b' thus formed is either in A or in B. We consider these cases separately.

Case 1. Let a'/b' be in A; then $\frac{n_1}{m_1} < \frac{n_1 + n_2}{m_1 + m_2} < \frac{a}{b} \leqq \frac{n_2}{m_2}$, hence

$$\begin{aligned} \frac{a}{b} - \frac{a'}{b'} &= \frac{a}{b} - \frac{n_1 + n_2}{m_1 + m_2} \\ &\leqq \frac{n_2}{m_2} - \frac{n_1 + n_2}{m_1 + m_2} = \frac{m_1 n_2 - n_1 m_2}{m_2(m_1 + m_2)} \\ &< \frac{m_1 n_2 - n_1 m_2}{m_1 m_2} = \frac{n_2}{m_2} - \frac{n_1}{m_1} \\ &< \epsilon < \delta. \end{aligned}$$

Case 2. Let a'/b' be in B; then $\frac{n_1}{m_1} < \frac{a}{b} \leq \frac{n_1 + n_2}{m_1 + m_2} < \frac{n_2}{m_2}$; hence

$$\frac{a'}{b'} - \frac{a}{b} = \frac{n_1 + n_2}{m_1 + m_2} - \frac{a}{b} < \frac{n_1 + n_2}{m_1 + m_2} - \frac{n_1}{m_1}$$
$$< \frac{n_2 m_1 - n_1 m_2}{m_1 m_2} = \frac{n_2}{m_2} - \frac{n_1}{m_1}$$
$$< \epsilon < \delta.$$

Hence, in either case, $|a/b - a'/b'| < \delta$.

If, however, $n/m \geqq \epsilon$, we choose ν, μ such that $0 < \nu/\mu < \epsilon$ whence also $\nu/\mu < n/m$. By the Eudoxian axiom a k exists such that $k\ \nu/\mu > n/m$ whence $n_2/m_2 - n_1/m_1 \equiv n/m < k \cdot \nu/\mu$, or $n_2/m_2 < n_1/m_1 + k \cdot \nu/\mu$. We form the sequence of rational numbers:

(A) n_1/m_1, $n_1/m_1 + \nu/\mu$, $n_1/m_1 + 2\ \nu/\mu$, ... $n_1/m_1 + r\ \nu/\mu$, . . . $n_1/m_1 + k\ \nu/\mu$, and note that some of the elements of (A) are in A and some in B. Let $n_1/m_1 + r \cdot \nu/\mu$ be the last of those in A; then $n_1/m_1 + (r+1) \cdot \nu/\mu$ is in B. Let now $N_1/M_1 = n_1/m_1 + r\ \nu/\mu$, $N_2/M_2 = n_1/m_1 + (r+1)\ \nu/\mu$; we have $N_2/M_2 - N_1/M_1 = n_1/m_1 + (r+1) \cdot \nu/\mu - n_1/m_1 - r \cdot \nu/\mu = \nu/\mu < \epsilon < \delta.$ We choose $a' = (N_1 + N_2)u$, $b' = (M_1 + M_2)u$, we form $a'/b' = \frac{(N_1 + N_2)u}{(M_1 + M_2)u}$ and consider, precisely as before, the case where a'/b' is in A and that where it is in B. We find again, in either case, $|a/b - a'/b'| < \delta$ which completes the proof.

In what is to follow this theorem will be referred to, for convenience, as the Existence theorem; the name is justified by the fact that for any a/b and any Eudoxian ratio δ, the Eudoxian ratio a'/b' is shown by this theorem to <u>exist</u>.

CHAPTER XII

THE METHOD OF EXHAUSTION

It will be recalled that in tracing the authorship of the fifth book of Euclid to Eudoxus we discovered that the "first irrefragable proof" of the theorems that Archimedes assigned to Eudoxus was effected by the method of exhaustion; it may be taken then that, at least in the form found in Euclid, this method of proof is due to Eudoxus. However, an earlier and less perfect form of the method seems to have existed before Eudoxus; for, Simplicius, in his Commentary on Aristotle's Physics, preserves the following extract from Eudemus' History of Geometry.

> "Antiphon, having drawn a circle, inscribed in it one of those polygons that can be inscribed; let it be a square. Then he bisected each side of this square and through the points of section drew straight lines at right angles to them, producing them to meet the circumference; these lines evidently bisect the corresponding segments of the circle. He then joined the new points of section to the ends of the sides of the square, so that four triangles were formed, and the whole inscribed figure became an octagon. And again, in the same way, he bisected each of the sides of the octagon, and drew from the points of bisection perpendiculars; he then joined the points where these perpendiculars met the circumference with the extremities of the octagon, and thus formed an inscribed figure of sixteen sides. Again, in the same manner, bisecting the sides of the figure of sixteen sides and drawing straight lines, he formed a polygon of twice as many sides; and doing the same again and again until he had exhausted the surface, he concluded that in this manner a polygon would be inscribed in the circle, the sides of which, on account of their minuteness, would coincide with the circumference of the circle. But we can substitute for each polygon a square of equal surface; therefore we can, since the surface coincides with the circle, construct a square equal

to a circle".

Simplicius remarks, quoting Eudemus to the same effect, that the inscribed polygon would never coincide with the circumference of the circle, even if it were possible to carry the division of the area to infinity, since to suppose that it would set aside a geometrical principle which asserts that geometrical magnitudes are divisible ad infinitum.

A contemporary of Antiphon's, Bryson of Heraclea,[94] extended this method by circumscribing polygons about the circle at the same time that he inscribed others, thus obtaining an upper limit also to the area of the circle; he made the mistake however, of thinking that the arithmetic mean of the inscribed and circumscribed polygons was the area of the circle.

From these simple origins was evolved the justly famous method of Eudoxus, the method which, as improved and brilliantly used by Archimedes, could well be called the Greek Integral Calculus. How far beyond these origins this calculus went, what levels of logical rigor it attained and how close it came to the modern integral calculus will be made clear in the study that follows. But also in what respect these two calculi remained essentially distinct will be made equally clear; the views expressed by Heath, in the following quotation[95], concerning the Greek attitude toward the infinite and the infinitesimal should be observed.

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[94] Cajori, History of Mathematics, p. 23

[95] Works of Archimedes, Cambridge, 1897, p cxlii

"The time had not come for the acceptance of Antiphon's idea, and, perhaps as a result of the dialectic disputes to which the notion of the infinite gave rise, the Greek geometers shrank from the use of such expressions as 'infinitely great' and 'infinitely small' and substituted the idea of things 'greater or less than any assigned magnitude'. Thus, as Hankel says, they never said that a circle is a polygon with an infinite number of infinitely small sides; they always stood still before the abyss of the infinite and never ventured to overstep the bounds of clear conceptions. They never spoke of an infinitely close approximation or a limiting value of the sum of a series extending to an infinite number of terms."

That the notions of infinity, limit, infinitely close approximation and so forth had been met, and that some of these had been conceived with sufficient clarity to be used in mathematical computations or demonstrations has been shown in the preceding pages. But to those of the Greek mathematicians (as Eudoxus, Euclid, Archimedes, Apollonius) who paid attention to definiteness of result, clarity, and rigor of demonstration the use of these notions remained prohibited unless, as in the case of Archimedes, other, less objectionable means were available for the logical demonstration. Thus, quoting from Heath again,[96]

"Yet they must have arrived practically at such a conception, e.g., in the case of the proposition that circles are to one another as the squares of their diameters, they must have been in the first instance led to infer the truth of the proposition by the idea that the circle could be regarded as the limit of an inscribed regular polygon with an indefinitely increased number of correspondingly small sides. They did not, however, rest satisfied with such an inference; they strove after an irrefragable proof, and this, from the nature of the case, could only be an indirect one. Accordingly we always find, in proofs by the method of exhaustion, a demonstration that

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[96] Heath, loc. cit.

> an impossibility is involved by any other assumption than that which the proposition maintains. Moreover, this stingent verification, by means of adouble reductio ad absurdum, is repeated in every individual instance of the use of the method of exhaustion; there is no attempt to establish, in lieu of this part of the proof, any general propositions which could be simply quoted in any particular dase."

The method of exhaustion existed in two forms; as originated by Eudoxus (and as found exclusively in Euclid) it involved the construction of inscribed figures only in the surface or solid whose measure was to be found. But as improved by Archimedes (and as used by later writers) it involved the construction of circumscribed figures also in relation to the one whose measure was sought. The inscribed and the circumscribed figures are then allowed, by increasing the numbers of their sides, to approach coincidence with each other and with the figure whose measure is sought. But, it must be understood, Archimedes does not describe his method in this way; nor does he say at any time that the given figure is the limiting form of the inscribed or the circumscribed figure.

The resemblance of this procedure of Archimedes' to the modern conception of the integral is, of course, striking. Heath does not hesitate to say "he (Archimedes) performs genuine integrations". Paul Tannery also, after describing the first method, as found in Euclid, considers the extended method and says "The second, due to Archimedes, consists in summing values of a function corresponding to values of a variable in arithmetic progression; by multiplying by the

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[97] Notions Historiques, Paris, 1903, p. 340.

constant difference the successive values of the variable, and by supposing the constant difference to decrease indefinitely, the quadrature is thus obtained directly by passing to the limit. This second method corresponds to the Leibnizian conception of the primitive of the function f(x) as $\int f(x)dx$".

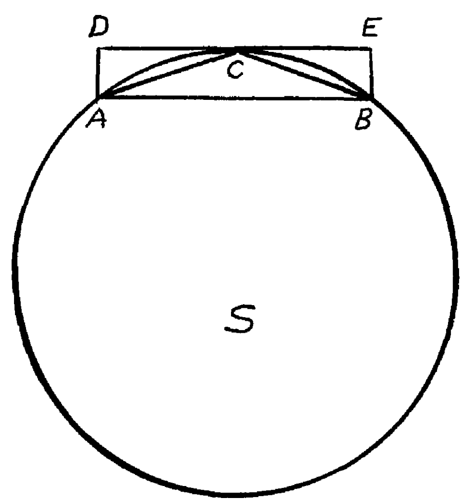
Such are Heath's and Tannery's views of this method; however, it will be maintained, and made clear, in the full description and analysis of the method to be given now that, because of the Greek attitude toward the infinite and the infinitesimal described above, Archimedes never even attempted to effect a quadrature by "passing to the limit", or "perform a genuine integration", if, by this phrase, Heath meant an integration in the modern sense. Further, it will be shown that, because of the defect in the Greek number system revealed in the study of the previous chapter, Archimedes could not have passed to a limit, even if he had attempted it, for the Eudoxian number system, in which this method operated, could not guarantee the existence of such a limit; nor was its existence ever asserted by postulation. In fact, to assume that Archimedes ever proceeded in this manner is to do injustice to his well-known sense of logical rigor. On the other hand, it will be shown that the form Eudoxus and Archimedes gave their method was largely determined by the nature of the medium in which it was to operate and that, for such a medium, no better method could be invented.

The following examples of the use of this method, selected from Euclid and Archimedes but presented here in more or less modernized notation will be found useful both for an understanding of the method itself and the analysis that is to follow.

The first one is the one mentioned by Heath in the first quotation above; it is Prop. 2, Book XII, Euclid. It is attributed by Heath to Eudoxus because the proof is exactly like those used in the theorems on the volumes of the pyramid and the cone, theorems known to be due to Eudoxus.

Proposition: Circles are to one another as the squares on the diameters.

Let S be a circle and AB the side of an inscribed regular polygon, P_1. Bisect arc AB at C, draw a tangent to S at C and raise the perpendiculars AD, BE to it, to form the rectangle ABED. Draw AC and BC.



The, we clearly have $\triangle$ ACB = 1/2 $\square$ ABED > 1/2 $\frown$ ACB. Hence, if P_2 is the inscribed regular polygon of side AC, $S - P_2 < 1/2(S - P_1)$; similarly $S - P_3 < 1/2(S - P_2)$, and so on. Hence, by Euc. X, 1 (see p. 62 above), a polygon, P_n, exists such that $S - P_n$ is less than any assigned area.[98]

Let now S and S' be the areas of two given circles and

[98] The Dependence of Euclid X, 1 on the Eudoxian axiom should be observed.

let d and d' be their respective diameters. It is to be proved that

(1) $$S/S' = d^2/d'^2$$

If (1) is not true let A be an area such that

(2) $$d^2/d'^2 = S/A,$$

where either $A < S'$ or $A > S'$.

Case 1. Suppose that $A < S'$.

Construct polygons, in the manner described above, in S' until a polygon, P'_n, is reached such that $S' - P'_n < S' - A$, whence also

(3) $$A < P'_n .$$

Inscribe in S a polygon, P_n, similar to P'_n; then

$$P_n/P'_n = d^2/d'^2$$
$$= S/A\text{, by (2) above, hence}$$

(4) $$P_n/S = P'_n/A.$$

Now, $P_n < S$, since P_n is inscribed in S; therefore, from (4), we must have $P'_n < A$; but this is impossible, by (3) above. Therefore case 1 is impossible.

Case 2. Suppose that $A > S'$. Then from (2) above

(5) $$d'^2/d^2 = A/S;$$

Let A' be such that $A/S = S'/A'$; then, from this and (5)

(6) $$d'^2/d^2 = S'/A'$$

where, since $S' < A$ we have

(7) $$A' < S.$$

Construct polygons in S until a P_n is reached such that $S - P_n < S - A'$,

(8) $$\therefore A' < P_n .$$

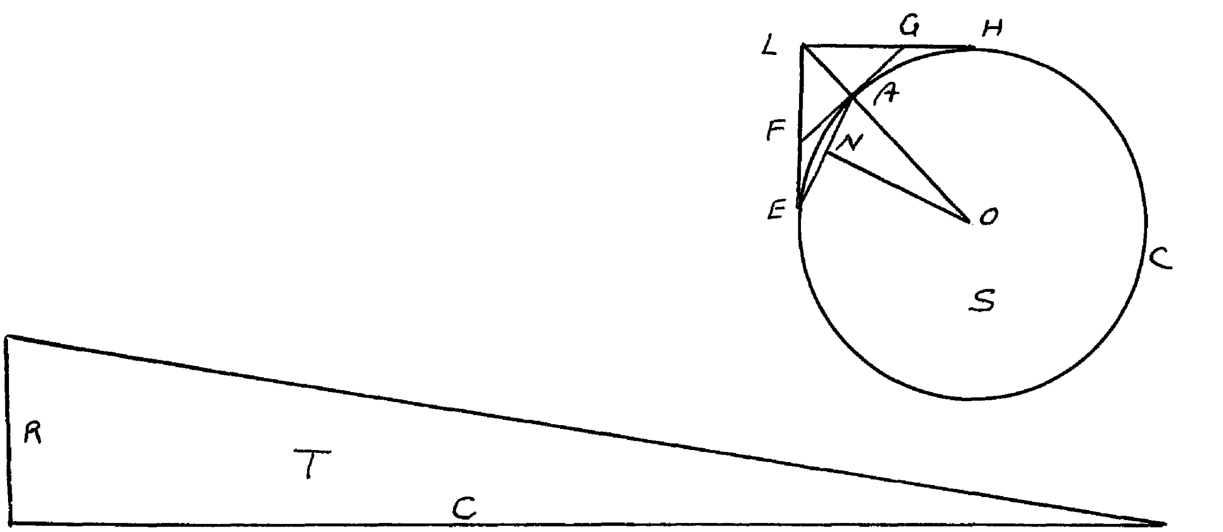
Inscribe in S' a polygon, P'_n, similar to P_n; then

$$P'_n/P_n = d'^2/d^2$$
$$= S'/A' \quad \text{, by (6) above, hence}$$
$$(9) \qquad P'_n/S' = P_n/A' .$$

Now, $P'_n < S'$, since P'_n is inscribed in S'; therefore, we must have $P_n < A'$, by (9) above; but this impossible by (8) above. Hence Case 2 is impossible. It follows then that $A = S'$ and thus, by means of (2), that (1) is true.

The next example is selected from Archimedes; it is Prop. 1, "On the measurement of the circle".

Proposition: The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle.



Let S be the area and C the circumference of the given circle, and T the area of the triangle described. Then, if $S \neq T$ either $S < T$ or $S > T$.

Case 1. Let $S > T$.

Inscribe polygons in S until a polygon, P_n , is reached[99]

[99] As proved in Euc. XII, 2 - the Eudoxian axiom again.

such that $S - P_n < S - T$, whence also

(1) $$T < P_n.$$

Let EA be the side of P_n, p its perimeter, ON the perpendicular on the former from the center. Then, ON is less than R, the radius, and p is less than C; hence $P_n < T$, which contradicts (1). Hence this case is impossible.

<u>Case 2</u>. Let $S < T$.

Circumscribe a polygon, P_1, about S and let two adjacent sides, touching the circle at E and H, meet in L. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection to form the new polygon P_2. Let A be the middle-point of arc EH, and FAG the tangent at A. Then, the angle LAG is a right angle, whence $LG > GA$ and thus $LG > GH$. It follows that $\triangle FLG > 1/2(\text{area } LEAH)$ and thus $P_2 - S < 1/2(P_1 - S)$; similarly, $P_3 - S < 1/2(P_2 - S)$, and so on. Hence, by Euc. X, 1,[100] there exists a P_n such that $P_n - S$ is less than any assigned area.

Let polygons then be circumscribed about S until one, P'_n, is reached such that $P'_n - S < T - S$, whence also

(2) $$P'_n < T$$

Let FG be the side of P'_n, p' its perimeter, and OA the perpendicular on FG from O. Then $OA = R$ and $p' > C$; hence $P'_n > T$, which contradicts (2). Therefore this case also is impossible. It follows then that $S = T$, which was to be proved.

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[100] See Footnote 98.

The next, and last, example chosen is Archimedes' quadrature of the area bounded by one turn of the spiral $r = a\theta$ and the line $\theta = 0$. The area is shown to be one third of the area of the circle $r = 2\pi a$ and the proof rests on the following two lemmas.

<u>Lemma 1</u>. (Prop. 10, On Spirals).

If $A_1, A_2, A_3, \ldots A_n$ be n lines forming an arithmetic progression in which the common difference is equal to A_1, the least term, then

$$n \cdot A_n^2 < 3 \cdot (A_1^2 + A_2^2 + \ldots + A_n^2), \text{ and}$$
$$n \cdot A_n^2 > 3 \, (A_1^2 + A_2^2 + \ldots + A_{n-1}^2).$$

Corollary. The results hold if similar plane figures are substituted for squares.

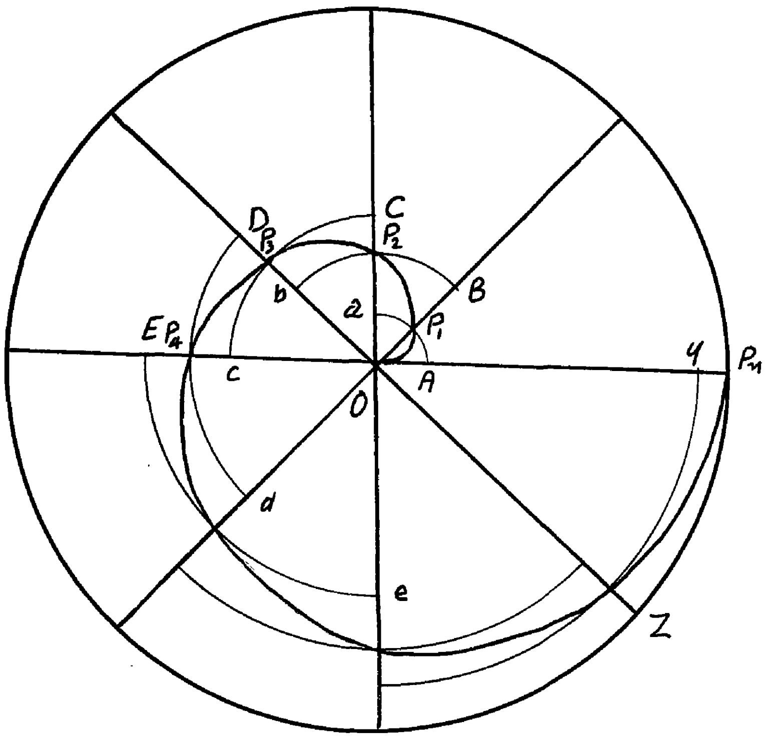
<u>Lemma 2</u>. (Cor. to prop. 21, 22, 23, On Spirals).

a) A figure can be circumscribed to the area of a spiral such that it exceeds the area by less than any assigned area.

b) A figure can be inscribed in it such that the area exceeds it by less than any assigned area.

The proof of the first lemms may be omitted as unessential to this study; it is enough to say that Archimedes proves it by means of n straight lines whose lengths are in arithmetic progression. The proof of the second one may be summarized as follows.

Given one complete turn of the spiral $O \, P_1 \, P_2 \ldots P_n$, draw a circle with center O and radius $O \, P_n$; bisect this circle and then bisect the halves, and then the quarters, and so on, until a sector ZOP_n is obtained, less than any assigned area (Archimedes here uses the Eudoxian axiom, $n \cdot a > b$, in the form $b/n < a$).



Then, by means of arcs of circles, $AP_1 a$, $BP_2 b$, $CP_3 c$, etc., construct the circumscribed figure $OAP_1 BP_2 C \ldots P_n$ and the inscribed figure $OP_1 aP_2 bP_3 c \ldots y$.

It is clear that the difference between the two figures is the sector ZOP_n, which is less than any assigned area. Hence, since the area of the spiral is intermediate in magnitude between these figures, each of these differs in magnitude from the area of the spiral, a fortiori, by less than any assigned area. It should be noticed that the radii OP_1, OP_2, ... OP_n are n lines in arithmetic progression, and that the common difference is OP_1, the least term.

Archimedes' proof of the theorem may be given now as follows. Let C be the area of the circle (see figure), and S the area of the spiral. It is to be shown that $S = 1/3\ C$.

If $S \neq 1/3\ C$ then either $S < 1/3\ C$ or $S > 1/3\ C$.

<u>Case 1</u>. Suppose that $S < 1/3\ C$.

Circumscribe the figure, F, about the spiral, making $F - S < 1/3\ C - S$ whence also

(1) $$F < 1/3\ C.$$

But $n \cdot OP_n^2 < 3(OP_1^2 + OP_2^2 + \ldots + OP_n^2)$ whence, substituting similar sectors for the squares, $C < 3F$, that is, $F > 1/3\ C$, which contradicts (1) above. Therefore this case is impossible.

<u>Case 2</u>. Suppose that $S > 1/3\ C$.

Inscribe the figure, f, in the spiral, making $S - f < S - 1/3\ C$, whence also,

(2) $$f > 1/3\ C.$$

But $n\ OP_n^2 > 3(OP_1^2 + OP_2^2 + \ldots + OP_{n-1}^2)$ whence, substituting similar sectors for the squares, $C > 3f$, that is, $f < 1/3\ C$, which contradicts (2) above. Therefore this case also is impossible.

Since S cannot be either greater or less than 1/3 C it must be equal to it.

It will be convenient to present, in the form of a lemma, a line of reasoning which is fundamental in the method of exhaustion.

<u>Lemma</u>. Let a ratio, a/b, be assumed to exist, and let c/d be a Eudoxian ratio. Thus, a/b is merely known to exist; c/d is known in value also.

Now, (see H, I, Appendix III)

a) One and only one of the following relations holds:

$$a/b > c/d, \quad a/b = c/d, \quad a/b < c/d,$$

and, by the existence theorem,

b) Given any ratio, δ , a ratio ρ exists such that $|a/b - \rho| < \delta$

If it be assumed now that $a/b > c/d$ then, letting $\delta_1 = a/b - c/d$ a ratio, r, exists, by (b), such that

c) $$a/b - r < \delta_1 .$$

If it be assumed that $a/b < c/d$ then, letting $\delta_2 = c/d - a/b$, a ratio, R, exists, by (b), such that

d) $$R - a/b < \delta_2$$

And now, if it can be shown that r satisfies the relation

e) $$r < c/d$$

and R the relation

f) $$R > c/d$$

then $$a/b = c/d.$$

For, from (c), $a/b - r < a/b - c/d$, whence $c/d < r$; but this is impossible, by (a), in view of (e).

Again, from (d), $R - a/b < c/d - a/b$, whence $R < c/d$; but this is impossible, by (a), in view of (f).

Therefore, the assumptions $a/b > c/d$ and $a/b < c/d$ are both false. Hence, by (a), $a/b = c/d$.

It will be convenient, also, to make some provision for the fact that in some of the applications made of this method the investigation is about ratios while in others it is about absolute magnitudes. Thus, in the first example presented above a ratio was to be found, while in the second and third, absolute magnitudes. This difference may be allowed for either by treating ratios of magnitudes as magnitudes (of a different kind), or by treating Archimedes' magnitudes as ratios, by considering in their place numerical ratios with denominator 1. The latter course will be adopted here, as permitting the continued use of the vocabulary employed thus far.

The analysis of the method, for the general case, may now be given as follows.

Let X be the geometric figure to be measured and a/b its measure, assumed to exist but unknow as yet.

Let c/d be a ratio, known in value, adopted as an estimate of a/b. Let f_n be the measure of a figure inscribed in X and F_n that of a figure circumscribed about X.

Then, the proof that $a/b = c/d$ can be analyzed into the following steps.

1) $f_n < a/b < F_n$, for any n, by assumptions 1 – 4, Sphere and Cyl., 1.
2) $f_n < c/d < F_n$, for any n, by individual proof in each case.
3) For any δ_1 an r exists such that $a/b - r < \delta_1$, by the existence theorem.
4) For any δ_2 an R exists such that $R - a/b < \delta_2$, by the same theorem.
5) If $a/b > c/d$ then $a/b - c/d$ is chosen for δ_1 and f_n for r; it follows from (3) that $c/d < f_n$, which contradicts (2).

6) If $a/b < c/d$ then $c/d - a/b$ is chosen for δ_2 and F_n for R; it follows from (4) that $F_n < c/d$, which contradicts (2).

7) From (5) and (6) it follows that $a/b = c/d$, by the lemma.

That the examples presented above fall into the general pattern of this analysis is easy to show.

Thus, in the first example, it was shown that $S'/S = d'^2/d^2$. Here S'/S is the ratio a/b and A/S, equal to d'^2/d^2, is the ratio c/d. Denoting by p'_n and P'_n the polygons inscribed and circumscribed, respectively, in S', then p'_n/S and P'_n/S are the ratios f_n and F_n respectively.

We have:

1) $p'_n/S < S'/S < P'_nS$, by assumptions 1—4, not stated in Euclid explicitly but used nevertheless.

2) $p'_n/S < A/S < P'_nS$, by proof, as follows:

Let p'_n be inscribed in S' (as shown by Euclid) and let p_n, similar to p'_n, be inscribed in S. Then, $p'_n/p_n = d'^2/d^2$ $(= A/S)$. But $p_n < S$; hence $p'_n/S < p'_n/p_n = A/S$.

Again, let P'_n be circumscribed about S' (as shown by Archimedes), and let P_n, similar to P'_n, be circumscribed about S. Then $P'_n/P_n = d'^2/d^2$ $(= A/S)$. But $P_n > S$; hence $P'_n/S > P'_n/P_n = A/S$/

3) $S'/S - r < \delta_1$, by the existence theorem.

4) $R - S'/S < \delta_2$, by the same theorem.

5) If $S'/S > A/S$, let $\delta_1 = S'/S - A/S$ and $r = p'_n/S$. Then, from (3), $S'/S - p'_n/S < S'/S - A/S$, whence $A/S < p'_n/S$, which contradicts (2).

6) If $S'/S < A/S$, let $\delta_2 = A/S - S'/S$ and $R = P'_n/S$. Then, from (4), $P'_n/S - S'/S < A/S - S'/S$, whence $P'_n/S < A/S$, which contradicts (2).

7) From (5) and (6) we have, by the lemma, $S'/S = A/S$ $(=d'^2/d^2)$, q.e.d.

It must be pointed out, however, that though, for the purpose of this analysis, the contradiction in step (6) was revealed by means of circumscribed polygons, the procedure followed by Eudoxus was entirely different. He used, instead, the following device.

> If $S'/S < A/S$ $(=d'^2/d^2)$, let $A'/S' = d^2/d'^2$, whence also $A/S = S'/A'$. Then, $S'/S < S'/A'$ $(=d'^2/d^2)$ and thus, by taking reciprocals, $S/S' > A'/S'$ where $A'/S' = d^2/d'^2$.

It is seen that this device converts the hypothesis of step (6), for the circles S', S, into the hypothesis of step (5) for the circles S, S'; since the latter was shown to involve a contradiction, the former does also. This device is characteristic of all of Eudoxus' proofs preserved in Euclid; he seems to have based his method on Antiphon's idea (of inscribing figures) exclusively. Bryson's addition to this idea does not appear once in Euclid.

In the second example it was to be shown that S, the area of the circle, known to exist but unknown in value, is equal to the area T, of the triangle described, known in value also. Hence, S is the ratio a/b, T the ratio c/d, and p_n, P_n, the areas of the inscribed and circumscribed polygons, respectively, are the ratios f_n and F_n.

We have:

1) $p_n < S < P_n$, by assumptions 1–4 Sphere and Cylinder, 1.

2) $p_n < T < P_n$, by proof, as given on page 100 above.

3) For any δ_1 a ratio r exists such that $S - r < \delta_1$, by the existence theorem.

4) For any δ_2 a ratio R exists such that $R - S < \delta_2$, by the existence theorem.

5) If $S > T$, let $\delta_1 = S - T$ and $r = p_n$; then, from (3), $S - p_n < S - T$, whence $T < p_n$, a contradiction of (2).

6) If $S < T$, let $\delta_2 = T - S$ and $R = P_n$; then, from (4), $P_n - S < T - S$, whence $P_n < T$, a contradiction of (2).

7) From (5) and (6) it follows, by the lemma, that $S = T$, q.e.d.

Finally, in the third example, it was to be shown that S, the area of the spiral, known to exist but unknown in value, is equal to 1/3 C, where C is the area of the circle, known in value also. Hence, S is the ratio a/b, 1/3 C is the ratio c/d, and the areas f and F of the inscribed and circumscribed figures respectively are the ratios f_n and F_n.

We have:

1) $f_n < S < F_n$, by assumptions 1 - 4, Sphere and Cylinder, 1.

2) $f_n < 1/3\ C < F_n$, by proof, as given on page 103 above.

3) For any δ_1 a ratio r exists such that $S - r < \delta_1$, by the existence theorem.

4) For any δ_2 a ratio R exists such that $R - S < \delta_2$, by the existence theorem.

5) If $S > 1/3\ C$ let $\delta_1 = S - 1/3\ C$ and $r = f_n$; then, from (3) $S - f_n < S - 1/3\ C$, whence $1/3\ C < f_n$, a contradiction of (2).

6) If $S < 1/3\ C$ let $\delta_2 = 1/3\ C - S$ and $R = F_n$; then, from (4), $F_n - S < 1/3\ C - S$, whence $F_n < 1/3\ C$, a contradiction of (2).

7) From (5) and (6) if follows, by the lemma, that $S = 1/3\ C$, q.e.d.

And now we may summarize as follows. The measure, a/b, of the figure X is, in the first instance, accepted as existing, by the simple intuition that to any magnitude corresponds a Eudoxian ratio as its measure. Then, as the next step, sequences of ratios,

$$f_1, f_2, \ldots f_n \text{ and } F_1, F_2, \ldots F_n$$

are constructed as measures of inscribed and circumscribed figures, respectively, and, by virtue of assumptions 1 – 4, Sphere and Cylinder, 1., (assumptions which, roughly speaking, merely state that a figure contained in another has a measure less than that of the other), it is asserted that, for any f_n, F_n, $f_n < a/b < F_n$.

Next, it is proved in each case (and is guaranteed for all cases by the existence theorem) that the approximations f_n, F_n, can be made as close as is desired, that is, for any δ_1, δ_2, f_n and F_n exist such that

a) $\quad a/b - f_n < \delta_1$

b) $\quad F_n - a/b < \delta_2$

Then the estimate c/d is produced — how it is found will be considered later — and it is shown, by special proof in each case (and in this one may see a first inadequacy in this calculus for, obviously, such a proof could not be of general scope) that, for any f_n, F_n,

c) $\quad f_n < c/d < F_n$.

The ratios a/b, c/d are then compared. It is first supposed that $c/d < a/b$ and then, by (a), it is deduced that for some f_n, the relation $a/b - f_n < a/b - c/d$ holds, whence $c/d < f_n$, which is false by (c). Then it is supposed that $a/b < c/d$ whence, by (b) it is deduced that for some F_n, $F_n - a/b < c/d - a/b$ which leads to $F_n < c/d$, which is false by (c). From these results, and the lemma, it is finally deduced that $a/b = c/d$.

It is clear then that, in this calculus, a/b is not arrived at as the limit of sequence converging to it either from above or from below; and it has been shown that such procedures could not have been carried out in the Eudoxian number system in which this calculus operated. On the other hand, it is equally clear that this calculus does make use of approximation sequences, either from below, or both from above and from below; and it has been shown that for such procedures the Eudoxian number system was amply adequate. In the difference between these two procedures lies the <u>essential difference</u> between the two calculi, the ancient and the modern, mentioned earlier in this chapter. The former, starting from a/b, merely constructs one or two sequences, always finite, but coming as close to a/b as is desired; the latter, starting from (one or) two sequences, follows them to their limits, if these exist and, provided they are equal, defines a/b as this common value.

It will be noticed, of course, that among the principles this calculus depended on were the logical principles of contradiction and of the excluded middle, the Eudoxian axiom, and the assumptions 1 – 4 of Archimedes. But what this calculus chiefly depended on, in fact, what it seems to have been perfectly adapted to, as a method of proof, is the number system in which it operated; indeed, for such a number system this calculus was a perfect tool. This is not surprising, of course, for the same man who created the medium also invented the tool to be used in it.

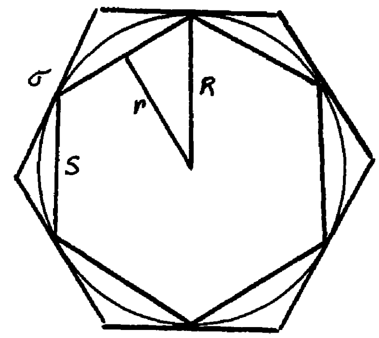
The analysis of the method of exhaustion undertaken here has, so far, sought to reveal its nature as a <u>method of proof</u>; it is time now to examine it as a <u>method of discovery</u>. How the estimate, c/d, <u>was proved</u> to be equal to a/b has been shown; it is time to consider the question of how c/d <u>was discovered</u>.

The answer to this question must be looked for, in the examples given, in the relations

$$f_n < S < F_n,$$
$$f_n < c/d < F_n,$$

which appeared as the first and second steps in the above analysis. It will be recalled that f_n and F_n also satisfy the conditions $S - f_n < \delta_1$ and $F_n - S < \delta_2$. We examine these relations then in the examples presented above and try to discover, from the evidence available, how the ratio c/d could be (and thus how it, <u>possibly</u>, was) arrived at in each case. We omit the first example in this examination since the above relations do not, strictly, apply to it — it will be recalled that Eudoxus did not use circumscribed figures, F_n, but employed a device instead.

In the second example an estimate, T, of the area, S, of the circle is to be found. Let C be the circumference of the circle and R its radius. Let p_n be the area of an inscribed regular polygon of n sides; let p be its perimeter, s its sides and r its apothem.



Let P_m be the area of a circumscribed polygon of m sides; let P be its perimeter, σ its side and R its apothem. Then, by assumptions 1 – 4,

(1) $$p_n < S < P_m \quad \text{, and}$$

(2) $$p < C < P \; .$$

From (2), by multiplying by 1/2 R

$$1/2\ R\ p < 1/2\ R\ C < 1/2\ R\ P \text{ , whence}$$

$$1/2\ r\ p < 1/2\ R\ C < 1/2\ R\ P \text{ , since } r < R. \text{ From this}$$

(3) $$p_n < 1/2\ RC < P_m \; .$$

In (3) we have a derivation of the estimate, 1/2 R C, of the area of the circle, with a proof that it is greater than p_n , for any n, and less than P_m, for any m. No actual summation of parts is necessary; for the inscribed isosceles triangles (with apothem r) <u>are seen</u> to form the polygon p_n , and the outer ones (with apothem R) the polygon P_m .

In the next example, from Archimedes again, an estimate of the area, S, of the spiral $r = a\,\theta$ is to be found. Let the figures of areas f_n and F_n be drawn in and about the spiral, as on page 102 above; then, by assumptions 1 – 4,

(1) $$f_n < S < F_n \; .$$

Now, obviously, F_n is the sum of the similar circular sectors with radii OP_1, OP_2, ... OP_n. Since all the angles about the center, O, are equal, these radii form an arithmetic progression with common difference OP_1. Hence, writing k for OP_1, we have

$$OP_1 = k,\ OP_2 = 2k,\ OP_3 = 3k,\ .\ .\ .\ OP_n = nk.$$

By squaring and adding now we get

(2) $OP_1^2 + OP_2^2 + OP_3^2 + \ , \ . \ . \ . \ + OP_n^2 = k^2(n^3/3 + n^2/2 + n/6) > k^2 n^3/3$

Again, it is obvious that f_n is the sum of the circular sectors with radii OP_1, OP_2, . . . OP_{n-1}. We subtract $OP_n^2 = k^2 n^2$ from each side of (2) and get

(3) $OP_1^2 + OP_2^2 + \ , \ . \ . \ + OP_{n-1}^2 = k^2(n^3/3 - n^2/2 + n/6) < k^2 n^3/3.$

The results (2) and (3) above are Archimedes' Lemma 1 (page 101 above); they are obtained by him in the manner characteristic of the time, that is, by means of geometric lines.

Substituting similar sectors now for the squared radii — which is equivalent to multiplying by $\theta_n = \pi/n$ — and observing that $k = r/n$, where r is the radius of the circle, we get, from (2)

4) $\quad F_n > k^2 n^3/3 \cdot \pi/n = \pi/3\ (r/n)^2 n^2 = 1/3\ C,$

where C is the area of the circle. In the same way we get from (3),

5) $$f_n < 1/3\ C.$$

(4) and (5) may now be combined into

6) $$f_n < 1/3\ C < F_n.$$

In (6) we have Archimedes' derivation of the estimate, 1/3 C, of the area of the spiral. In this case a summation of a finite series has actually been performed; the resemblance to the modern conception of the integral is a little closer.

The method used by Archimedes in this example is one that he has used in other examples also, though by no means in all. Its chief characteristic is the summation of a finite series; and it is probable that those commentators (Heath, Tannery, and others) who have maintained that Archimedes performs integrations, in the modern sense, or passes to the limits of sequences, were led into these erroneous beliefs by hasty consideration of this characteristic. It is worth-while then to contrast to the Archimedean method the following modern procedures for obtaining c/d.

<u>First method</u>. Proceeding intuitively, one may, given the bounded surface, S, assert that the area, a/b, unknown as yet, exists; then, letting f_n, and F_n have meanings as before, engage in the following steps of reasoning.

a) $f_n < a/b < F_n$, by the same assumption as that used by Archimedes.

b) $L\ f_n \leq a/b \leq L\ F_n$, from (a) and the concept of the "limit".

c) $L\ f_n = L\ F_n = c/d$ say, by actual evaluation,

d) $a/b = c/d$, from (a) and (c).

There is no conflict with Greek intuitionism here, but concepts are involved which were no part of Greek mathematics. The concept of the <u>limit</u> is used in step (b), which implies the consideration of an <u>infinite</u> sequence and presupposes a continuous number system. Applying this method to the last example considered above, and using the same notation, we have:

$$f_n < a/b < F_n \text{ , whence}$$

$$L\ f_n \leq a/b \leq L\ F_n \text{ , that is}$$

$$L\ \pi/n \sum_{i=1}^{n-1} OP_i^2 \leq a/b \leq L\ \pi/n \cdot \sum_{i=1}^{n} OP_i^2 \text{ ,}$$

where f_n and F_n have meanings as before. From relations (2) and (3), of page 113, we get

$$L\ \pi/n\ (r/n)^2 \cdot (n^3/3 - n^2/2 + n/6) \leq a/b \leq L\ \pi/n(r/n)^2\ (n^3/3 + n^2/2 + n/6)$$

from which, by evaluation of the limits,

$$1/3\ \pi r^2 \leq a/b \leq 1/3\ \pi r^2 \text{, that is, } a/b = 1/3\ C.$$

Second Method. Discarding the notions and assumptions of intuitive geometry employed in the first method (but, it may be observed, still following the path pointed out by intuition) one may, more strictly, and on a purely arithmetic basis,

a) Prove that $\quad L\ f_n$, $L\ F_n$ exist,

b) Prove that $\quad L\ f_n = L\ F_n$,

c) Evaluate this limit, to obtain c/d say,

d) Define the area as c/d.

In this procedure the same concepts, foreign to Greek thought, are involved; but in addition there is direct conflict with Greek intuitionism. To the Greek mind an area did not exist merely by virtue of a definition.

The point need not be labored further; it is established I think, that the calculus of Eudoxus and Archimedes, as formally presented at least, is strictly finite in character.

But, as will be shown presently, Archimedes did use, heuristically and quite apart from the Method of Exhaustion, concepts beyond the finite. In preliminary investigations of some theorems he made use, purely as means to discovery, of processes which were essentially summations of infinite sequences of infinitely small elements.

These processes are contained in the famous "Method" of Archimedes,[101] given to the world by Heiberg, following his discovery, in Constantinople, of the manuscript relating thereto, in 1906.

We find, in the letter to Eratosthenes, prefixed to this book, the following comment, by Archimedes, on this method.

> "I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry aterwards because their investigation by this method did not furnish an actual demonstration".

Archimedes applies this method to two problems in this book: the evaluation of the area of a segment of a parabola and the evaluation of the volume of a sphere. Since the treatment of these problems is the same, it will be enough to consider only one in this survey, say that on the volume of the sphere.

Having stated, in his customary careful way, the propositions which he assumes (these are, in this case, propositions from his book On the Equilibrium of Planes), and having first

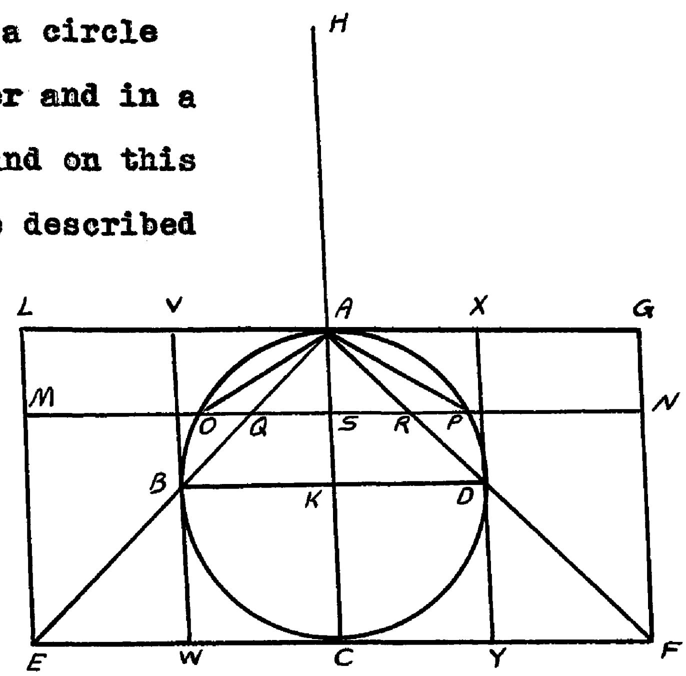
[101] Heath, The Method of Archimedes, treating of Mechanical problems, Cambridge, 1912.

dealt with the segment of the parabola, Archimedes then proceeds to "investigate", by this method, the proposition that

> Any sphere is, in respect of solid content, four times the cone with base equal to a great circle of the sphere and height equal to its radius.

This investigation (rendered here in Archimedes' own style, as translated by Heath) proceeds thus:

Let ABCD be a great circle of a sphere, and AC, BD diameters at right angles to one another. Let a circle be drawn about BD as diameter and in a plane perpendicular to AC, and on this circle as base let a cone be described with A as vertex. Let the surface of this cone be produced and then cut by a plane through C parallel to its base; the section will be a circle on EF as diameter. On this circle as base let a cylinder be erected with height and axis AC, and produce CA to H, making AH equal to AC.



Let CH be regarded as the bar of a balance, A being its middle point.

Draw any straight line, MN, in the plane of the circle ABCD, and parallel to BD. Let MN meet the circle in O, P, the diameter AC in S, and the straight lines AE, AF in Q,R, respectively. Join AO.

Through MN draw a plane at right angles to AC; this plane will cut the cylinder in a circle with diameter MN, the sphere in a circle with diameter OP, and the cone in a circle with diameter QR.

Now, since $MS = AC$, and $QS = AS$,

$$
\begin{aligned}
MS \cdot SQ &= CA \cdot AS \\
&= AO^2 \\
&= OS^2 + SQ^2
\end{aligned}
$$

and, since $HA = AC$

$$
\begin{aligned}
HA : AS &= CA : AS \\
&= MS : SQ \\
&= MS : MS \cdot SQ \\
&= MS : (OS^2 + SQ^2), \text{ from the above result} \\
&= MN : (OP^2 + QR^2); \text{ hence}
\end{aligned}
$$

HA : AS = (circle on diameter MN) : (circle on diameter OP + circle on diameter QR),

that is

HA : AS = (circle in cylinder) : (circle in sphere + circle in cone).

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about A, with the circle in the sphere together with the circle in the cone, if both the latter circles are placed with their centers of gravity at H.

Similarly for any three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LF parallel to EF.

If we deal in the same way with <u>all the sets of three circles</u> in which planes perpendicular to AC cut the cylinder, the sphere, and the cone, and <u>which make up those solids</u>

respectively, it follows that the cylinder, in the place where it is, will be in equilibrium about A with the sphere and the cone together, when both are placed with their centers of gravity at H.

Therefore, since K is the center of gravity of the cylinder,

$$HA : AK = (\text{cylinder}) : (\text{sphere} + \text{cone } AEF)$$

But $\qquad HA = 2AK$; therefore

$$\text{cylinder} = 2\ (\text{sphere} + \text{cone } AEF).$$

Now, $\qquad \text{cylinder} = 3\ (\text{cone } AEF)$, by Euclid XII, 10;

therefore, $\qquad \text{cone } AEF = 2\ (\text{sphere}).$

But, since $\qquad EF = 2\ BD,$

$$\text{cone } AEF = 8\ (\text{cone } ABD);$$ therefore

$$\text{sphere} = 4\ (\text{cone } ABD),$$ q.e.d.

In Proposition 34, On the Sphere and Cylinder, Archimedes proves, by the method of exhaustion and, apparently, quite independently of the above investigation, that the estimate, 4 (cone ABD), is the true volume of the sphere; but he says, elsewhere in the preface to "The Method", that,

> "It is of course easier, when we have previously acquired, by the Method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge".

It is clear that in this example of Archimedes' Method (see underscored lines) a solid is thought of as the sum of an infinity of infinitely small elements. Now, we have seen that Pythagoras had taught that "... from lines come planes, and from these solid figures, etc.", and that Democritus had thought of the cone and the cylinder as being composed of circles; we have also seen that Archimedes was not unacquainted with these views. It is probable then that in these views lies the source of this conception of the solid exhibited by Archimedes in his Method. For, not only does his mention of Democritus, and his theorem, in such close connection with his Method strongly suggest it, but we also have the following indirect evidence. His statement that Eudoxus gave the first "irrefragable" proof of the theorem implies that he knew of other, refragable, proofs. One such proof, that he knew of, was that of Democritus. It is true that he does not say that Democritus' proof was refragable, but only that "he did not prove it (the theorem)", but this simply means that he did not consider Democritus' proof good enough. For he also says, about his own method, which is similar to that of Democritus', that it does not supply a proof but rather "a sort of indication that the conclusion is true".

For still other methods employed by Archimedes for obtaining an estimate of a geometric magnitude which he wished to evaluate, we examine the following, very revealing statement, which forms the last paragraph of The Method.

> "From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base

equal to the surface of the sphere and height equal to the radius".

To make the meaning of this passage clearer, let r, C, A, be the radius, circumference and area, respectively, of a great circle of the sphere, and let S and V be the area and volume, respectively, of the sphere itself. We have seen that Archimedes proved (Prop. 1, Measurement of a Circle) that:

1) $$A = 1/2 \cdot r \cdot C$$

and (Prop. 34, On the Sphere and Cylinder) that

2) $$V = 4 \cdot 1/3 \cdot r \cdot A$$

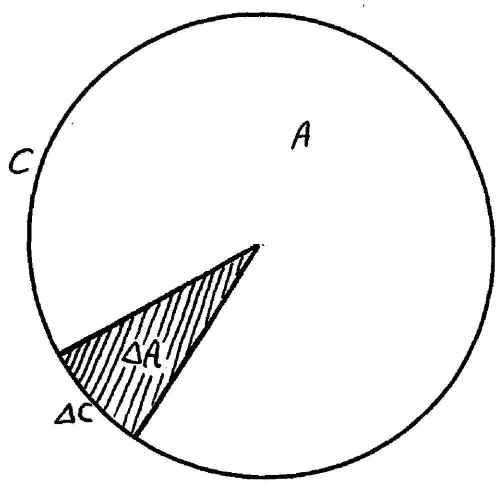
The passage quoted says now that "judging" from (1) Archimedes "apprehended" that, "in like manner",

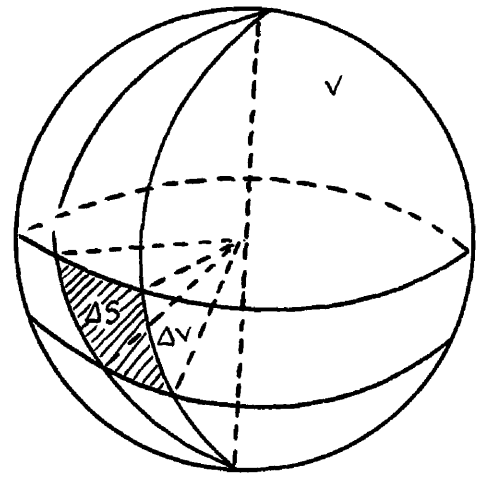
3) $$V = 1/3 \cdot r \cdot S$$

and, from (2) and (3), that he "conceived the notion that"

4) $$S = 4 \cdot A.$$

This estimate he proved, in the usual manner, in Prop. 33, On the Sphere and Cylinder, to be the true area of the sphere. But of greater interest than the derivation of the estimate $S = 4 \cdot A$ is the derivation of the estimate $V = 1/3 \cdot r \cdot S$ by analogy from $A = 1/2 \cdot r \cdot C$. The analogy involved is, very probably, the one exhibited in the figures below.





$$\Delta A = \tfrac{1}{2} r \Delta C$$
$$\therefore A = \Sigma \Delta A = \tfrac{1}{2} r \Sigma \Delta C = \tfrac{1}{2} r C.$$

$$\Delta V = \tfrac{1}{3} r \Delta S$$
$$\therefore V = \Sigma \Delta V = \tfrac{1}{3} r \Sigma \Delta S = \tfrac{1}{3} r S$$

In the derivation of this estimate also a solid is conceived as the sum of an infinity of infinitely small elements. Two differences from the previous example are to be noted: the infinitely small elements are not plane sections this time, further, the summation is more direct; the laws of equilibrium are not needed.

And now we can state, in summary, that the method of exhaustion, as a method of discovery, did not make use of a general method, but depended rather, according to the problem being considered at the time,

a) on direct intuition, as in the case of the circle ($A = 1/2 \cdot r \cdot C$),

b) or on the summation of a finite series, as in the case of the spiral ($S = 1/3 \cdot A$),

c) or on the "method of sections", assisted by the laws of equilibrium, as in the case of the sphere ($V = 4 \cdot 1/3 \cdot r \cdot A$),

d) or on an intuitive summation of infinitesimals, some times suggested by analogy, as in the case of the sphere (second estimate, $V = 1/3 \cdot r \cdot S$),

e) or on an algebraic deduction, as in the case of the area of the sphere, $(S = 4 \cdot A)$.

The lack of a general method of discovery limited the applicability of this method; its successful use in any given case was more a matter of ingenuity on the part of the investigator than an application of a standard method. Nevertheless, this method remained in existence, in the form in which Archimedes left it, for almost two thousand years, as the chief tool of mathematics within the field of its applicability. It was used again, by Pappus (c.380 A.D.), was preserved in the East in the works of Ibn Al-Haitam (died 1039), it spread in Europe, principally through the translations of Archimedes that began to appear at about the middle of the fifteenth century, was used by Galileo (c. 1630), a little later by Gregory St. Vincent (c. 1650), and was finally superseded by the methods of Kepler, Guldin and Cavalieri.

It is to be noted, however, that these new methods only abandoned the method of exhaustion as a method of proof, thus losing much in the matter of rigor; they retained its concepts and processes as a method of discovery. Kepler made wide use of the ancient Greek conception of the solid as composed of an infinity of planes, and the plane of an infinity of lines; in particular, "he conceived the circle to be composed of an infinite number of triangles having their common vertices at the center and their bases in the circumference; and the sphere to consist of an infinite number of pyramids",[102] which, as has been seen, was the Archimedean conception also. Again, it is well known that Cavalieri's method, which made use of similar concepts, was prompted by that of Kepler's, and as to that of Guldin's[103], which obtained volumes by the use of these concepts and that of the center of gravity, we have seen it already forecast in The Method of Archimedes.

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[102] Cajori, History of Mathematics, p. 160

[103] See Appendix IV.

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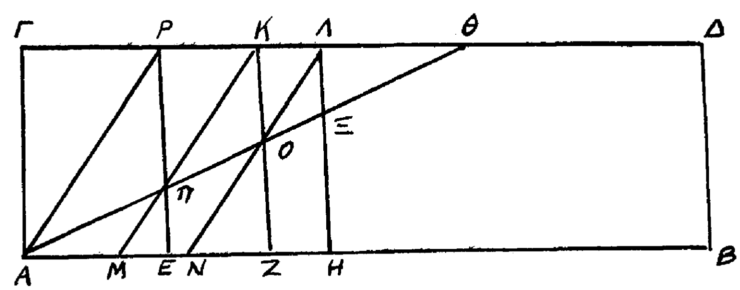
APPENDIX I

THE "MESOLABE" OF ERATOSTHENES

Eutocius[1] (in his commentary on Archimedes' On the Sphere and Cylinder) mentions a "votive monument" erected by Eratosthenes of Cyrene (b. 276 B.C.) to Ptolemy[2] on which was fixed a representation in bronze of his mechanical contrivance for solving the problem of the two geometric means, and on which appeared also a short proof and an epigram, the epigram stating, among other things, that the same contrivance would equally well serve to interpolate any desired number of means between the two lines.

In Book III, Prop. 5, Pappus[3] gives the following description of this contrivance.



> "Let ABΓΔ be a rigid rectangle and in it three equal triangles, AEP, MZK, NHΛ, the one, AEP, fixed in position, the other two, MZK, NHΛ, free to slide along AB.

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[1] Heath, History of Greek Mathematics, vol. 2

[2] Ptolemy Euergetes who, about 246 B.C. had called Eratosthenes to Alexandria to be a tutor to his son, and superintendent of the library.

[3] Ver. Eecke, loc. cit.

> Cut off $\Lambda\Xi$ equal to one half of $A\Gamma$ and slide the triangles MZK, NHΛ until the points A, Π, O, Θ are on the same straight line. Then ΠP, OK are the means sought".

It willbe seen that the contrivance solves the problem of the duplication of the cube; but Pappus observes that "If the ratio of cube to cube is some other, then the ratio of $A\Gamma$ to $\Lambda\Xi$ must be made equal to this other". From this remark and the epigram stated by Eutocius it is clear that the "Mesolabe" was capable of solving the more general problem

$$x^n = k\,a^n .$$

APPENDIX II

MR. ABR. EDEL'S THESIS

Aristotle recognizes, explicitly, two kinds of potentiality. There is first the potentiality of activity, as in seeing. One sees potentially (i.e., one is able to see) and one sees actually. And after one has seen, actually, nothing has changed in him in respect of this potentiality. Then there is the potentiality corresponding to change. For example, the stone is potentially a statue; but after the stone has once become a statue it does not revert to its former state in respect to that potentiality.

However, says Mr. Edel, there is still another kind of potentiality, about which Aristotle is not explicit but which there is ample evidence to believe he did consider, and that is the potentiality that corresponds to repeated change. Aristotle considers that a thing is potentially some thing else when, under certain circumstances, the latter will result through some single process or activity in which the former is engaged. But if several processes or activities are necessary (as earth becoming bronze, bronze a statue), one cannot speak of the one as being, potentially, the other. Thus, this potentiality is applicable to cases in which the actuality is obtainable in one step. Yet, if in a series of events involving a repeated change (and where potentiality cannot be said to reside in the individual terms of the series), if in such a series there exists a subject involved in the change, or several interrelated subjects, which constitute a condition, or set of conditions, for the repetition of the process (and therefore the successive terms of the series) then there can be said (Mr. Edel said it, so did Kant, but did Aristotle?) to be a potentiality, residing not in the individual terms, but in the subject which serves as a condition for any term, after the previous one has been actualized. This, Mr. Edel maintains, is the potentiality that characterizes Aristotle's "potential" infinite.

The strength of Mr. Edel's argument lies especially in the examples he has selected from Aristotle. These are: day, the Olympic Games, the generations of men, the division of magnitudes. For, any day follows inevitably, after an interval, from the very same system of conditions which caused the previous one; any celebration of the Olympic games from the same complex of Greek religious beliefs which produced the previous one, any generation of men follows the previous one by the action of the same faculties of human nature which produced the previous one, and any division of magnitude follows the previous one through the same conditions (the divisibility of magnitude, the will of the divider to divide, etc.) which caused the previous one.

Stated more abstractly, what Mr. Edel says is that the continued repetition of a process, occurring in a medium which can guarantee that after any step another can be taken, is an actualization which is always going on but is never complete, but that the corresponding potentiality is a complete whole and that this potentiality underlies Aristotle's infinite.

If this is Aristotle's conception of the infinite (and Mr. Edel admits that he has not found an explicit statement of it in Aristotle's works), then, clearly Aristotle is the fore-runner of the Peano, Frege, Cantor, Russell school which derives the infinite by the principle of mathematical induction. Cantor does not make explicit use of this principle, but he uses it nevertheless. Russell, with more philosophic penetration, has singled it out and made it the basis of his "inductive number". It is true that Dedekind defined the infinite by another of its properties, namely, that of having a part equal to itself. But this is an unordered infinite. To reduce it to the ordered kind, Zermelo's axiom is needed. This, of course, serves to introduce into it the necessary order.

APPENDIX III

THE SERIES OF EUDOXIAN RATIOS

We recall Euclid's definitions 5 and 7:

Def. 5. $a/b = c/d$ if, for every pair m, n, $ma \lesseqgtr nb$ according as $mc \lesseqgtr nd$.

Def. 7. $a/b > c/d$ if, for some pair, m, n, $ma > nb$ but $mc \leqq nd$, and prove, in succession,

A. If $a/b = c/d$ then $c/d = a/b$.

For, if $mc < nd$ then $ma < nb$,
if $mc = nd$ then $ma = nb$,
if $mc > nd$ then $ma > nb$,

since the assumption that any of these conditions is not satisfied leads to a contradiction of the hypothesis $a/b = c/d$; hence, by Def. 5, $c/d = a/b$.

B. If $a/b = c/d$ and $c/d = e/f$ then $a/b = e/f$.

For, if $ma \lesseqgtr nb$ according as $mc \lesseqgtr nd$, and $mc \lesseqgtr nd$ according as $me \lesseqgtr nf$ then $ma \lesseqgtr nb$ according as $me \lesseqgtr nf$; hence, by Def. 5, $a/b = e/f$.

C. If $a/b > c/d$ then $a/b \neq c/d$.

This follows from the incompatibility of definitions 5 and 7.

D. If $a/b > c/d$ then $c/d \neq a/b$,

For the same reason as in C. above.

E. If $a/b > c/d$ then $c/d \ngtr a/b$.

For, since $a/b > c/d$ then, for some m, n,

1: $ma > nb$ but $mc \gtreqless nd$, by Def. 7.

If possible, let $c/d > a/b$, then, for some μ, ν,

2: $\mu c > \nu d$ but $\mu a \gtreqless \nu b$, by Def. 7.

We prove 1: and 2: incompatible. Let $\mu = (r/s)m$ and $\nu = (p/q)n$; then, by substituting in 2:

3: $qrmc > spnd$ but $qrma \gtreqless spnb$.

Also, from 1: by multiplication by qr,

4: $qrma > qrnb$ but $qrmc \gtreqless qrnd$.

Now from the first of 3: and the second of 4:

5: $qrnd > spnd$ whence $qr > sp$.

Also, from the second of 3: and the first of 4:

6: $spnb > qrnb$ whence $sp > qr$.

But the results 5: and 6: are contradictory. Hence the theorem is true.

F. Definition. c/d is said to be "less than" a/b (a relation here indicated thus: $c/d < a/b$), if $a/b > c/d$.

This definition is justified by the results C., D., E., above. This definition is not given explicitly in Book V, but the relation "less than" is used there precisely in the sense of this definition.

G. If, for some m, n, $ma < nb$ but $mc \gtreqless nd$ then $a/b < c/d$.

This follows from Def. 7. and F above.

H. Of the relations $a/b < c/d$, $a/b = c/d$, $a/b > c/d$ only one can hold.

For, from C., D., E., F., it follows that these relations are mutually incompatible.

I. Of the relations $a/b < c/d$, $a/b = c/d$, $a/b > c/d$ one must hold.

For, given any two ratios, a/b and c/d, the criterion of equality, $ma \lesseqgtr nb$ according as $mc \lesseqgtr nd$, for any m, n, either is or is not satisfied. If it is, then $a/b = c/d$ and the theorem holds. If it is not, then it fails for one of the following three reasons:

1: for some m, n, $ma < nb$ but $mc \geqq nd$, or
2: for some m, n, $ma = nb$ but $mc \lessgtr nd$, or
3: for some m, n, $ma > nb$ but $mc \leqq nd$,

since, from (3) of p. 71 above, we have that any two magnitudes of the same kind, ma and nb, must satisfy one and only one of the relations $ma < nb$, $ma = nb$, $ma > nb$, and the three reasons given above contain all the logical possibilities.

If case 1: occurs then $a/b < c/d$, by G. above.

If case 2: occurs in the form $ma = nb$ but $mc > nd$ then $c/d > a/b$, by Def. 7. But if it occurs in the form $ma = nb$ but $mc < nd$ then $c/d < a/b$, by G.

If case 3: occurs then $a/b > c/d$, by Def. 7.

Hence, under any circumstances, the theorem holds.

J. If $a/b > c/d$ and $c/d > e/f$ then $a/b > e/f$.

From the hypothesis and Def. 7, we have:

a) for some m, n, $ma > nb$ but $mc \leqq nd$, and

b) for some μ, ν, $\mu c > \nu d$ but $\mu e \leqq \nu f$.

Let $\mu = (r/s)m$ and $\nu = (p/q)n$ and substitute in b); we get:

c) $qrmc > spnd$ but $qrme \leqq spnf$.

Also, from a), by multiplication by qr,

d) $qrma > qrnb$ but $qrmc \gtreqless qrnd$.

Now, from the first of c) and the second of d),

e) $spnd < qrnd$, whence $sp < qr$.

Substituting from e) in the second of c),

f) $qrme \gtreqless spnf < qrnf$, whence $me < nf$.

We have then, from a), that $ma > nb$, and from f), that $me < nf$; hence, by Def. 7., $a/b > e/f$, which was to be shown.

K. The class of Eudoxian ratios, ordered by the relation "less than", as defined, is a series.

For

a) If a/b, c/d are any two distinct ratios, then either $a/b < c/d$ or $c/d < a/b$, by H and I above.

b) If $a/b < c/d$ then $a/b \neq c/d$, by D. above.

c) If $a/b < c/d$ and $c/d < e/f$ then $a/b < e/f$, by J. above.

L. The series of Eudoxian ratios is dense.

Let $a/b < c/d$, where a, b, c, d are magnitudes of the same kind, and let u be the unit of magnitude. We show that there exists a ratio, f/g, such that $a/b < f/g < c/d$.

Since $a/b < c/d$ then, by Def. 7., for some m, n, $cm > nd$ but $ma \gtreqless nb$. Now since $mc > nd$, $mc - nd$ is a Eudoxian magnitude of the same kind as c and d; of course, so is md. Therefore, by the Eudoxian axiom, an integer k exists such that $k(mc - nd) > md$; from the last result we deduce

$$\frac{1}{k} < \frac{mc - nd}{md}$$

$$< \frac{c}{d} - \frac{n}{m}\text{, whence}$$

$n/m + 1/k < c/d$. We have then,

$$\frac{a}{b} \leq \frac{nu}{mu} < \frac{(kn + m)u}{kmu} < \frac{c}{d}$$

We take $f = (kn + m)u$, $g = kmu$. For such an f and g, $a/b < f/g < c/d$, which was to be shown.

APPENDIX IV

The problem of the Pappus-Guldin theorem has long been a source of vexation. In the advance notice of Book 7 of Pappus' Mathematical Collection one finds the statement that

> The ratio of perfect solids created by the revolution of plane areas about an axis in their plane consists of the ratio of the revolved areas and that of the straights similarly drawn from the barycentric points situated in these planes to the axis; and the ratio of imperfect solids created by the revolution of plane areas about an external axis in their planes is composed of the ratio of the plane areas and that of the arcs described by the barycentric points situated in these planes.

with the additional information that "a great many other theorems of this kind had been proved". One is naturally led to believe that somewhere in Book 7 one will find this theorem, or some problem bearing on it, dealt with in some way. For it was a practice of Pappus, consistently observed throughout the Mathematical Collection, to introduce in the first few pages of each book the subjects to be treated therein. One is puzzled therefore to find that the theorem, or problem bearing on it, is not even mentioned in Book 7, or anywhere else in the entire Mathematical Collection.

Ver Eecke[1] thinks (and states that F. Hultsch[2] is of the same opinion) that the passage in question is an interpolation. He bases his belief not only on the peculiar, isolated

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[1] Ver Eecke, op. cit

[2] Fr. Hultsch, Pappi Alexandrini Collectionis ...., Berlin, 1876-78.

appearance of the passage in the Mathematical Collection but also on the following considerations.

1:- The language in which the passage is couched is less pure, and less clear, than that of the rest of the book and contains words nowhere else used in it;

2:- The first editions of the Mathematical Collection in Europe (Commandinus, Pisa, 1588; reprinted, Venice, 1589; Pisa, 1602) do not contain this passage, but the next one (a revised edition of Commandinus', put out by C. Manolessius, 1660 Bologna) does.

It is clear that this evidence effectively disposes of Pappus as the author of the theorem in question; and it is equally clear that it also settles the question as to whether Guldin borrowed his theorem from the Mathematical Collection or not, for the date of his death (1643) precludes the possibility of his having even seen the Manolessius edition.

But Ver Eecke rather spoils this otherwise sound, though partial solution of the Pappus-Guldin problem by an apparent willingness to believe in "the remarkable fact that some quite eminent geometer lived and wrote, after Pappus, in the period of decadence", and by stating that Guldin must be given credit for a genuine "rediscovery" (sic). Having shown that Guldin could not have obtained his theorem from the Mathematical Collection he rejects, or does not consider, the converse possibility of Guldin's theorem finding its way into the Mathematical Collection. And yet much can be said in support of such a view.

Guldin's Centrobaryca, in which the theorem in question is first stated, and used to evaluate a good many volumes of revolution, appeared many years before the Manolessius edition. Kepler's Nova Stereometria Doliorum; accessit stereometriae Archimideae supplementum (1615), in which similar problems are solved (though by a different method), and Cavalieri's Geometria indivisibilibus .... (1635), in which like problems again are dealt with, had already made their appearance. Above all, the long and bitter dispute between Guldin and Cavalieri, in which each attacked the method of the other and neither could adequately defend his own had, long since, flared up, raged on for years, and ended after the death of both opponents (Cavalieri's in 1647). Manolessius certainly had had time, and opportunity, to learn not only of Guldin's theorem but also of the "great many other theorems of this kind" that had been proved. That he was also not incapable of making illicit entries in his revision of Commandinus' edition is attested by Ver Eecke, by Hultsch and by Heath[3]. It follows then that the hypothetical "eminent geometer, after Pappus, in the period of decadence" must be rejected. For not only is there no record, or even hint, of the existence of such a person in all history but, since the explanation suggested above is so much more probable, there is even no necessity for the hypothesis of his existence.

It is almost certain then that the theorem in question, in the form currently met with, is due to Guldin; yet the discovery is not entirely his.

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[3] Heath, "Pappus", Encyclopoedia Britannica, Werner Ed. New York, 1900, Vol. XVIII, p. 231.

It is well known that Guldin had been, for a considerable time, in close association with Kepler, as an understudy, before succeeding him to the chair of Astronomy at the gymnasium of Gratz; and, as is also well known, Kepler, the more eminent of the two mathematicians, had also occupied himself with similar problems. It is not suggested here that Guldin obtained his discovery from Kepler — there is no record of such a charge against Guldin — but it is suggested that both Kepler and Guldin, probably at this time, had had access to some copy of Archimedes' Method. For it cannot be by mere coincidence that the long-forgotten and until then unused ideas of Archimedes all of a sudden reappear in mathematical investigations, at about the same time, in the works of Kepler and Guldin. It has been shown (Chap. XII supra) that Kepler's division of the areas of the circle and the sphere into infinitesimal elements is, word-for-word, that of Archimedes; and it is generally known, of course, that Archimedes was the first to use the center of gravity in the evaluation of volumes and areas.