

ALMOST SYMMETRIC SPACES
AND GRAVITATIONAL RADIATION

by

Richard Alfred Matzner

Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

1967

copy 1

APPROVAL SHEET

Title of Thesis: Almost Symmetric Spaces and Gravitational Radiation

Name of Candidate: Richard Alfred Matzner
Doctor of Philosophy, 1967

Thesis and Abstract Approved:

Charles W. Misner

Charles W. Misner
Professor of Physics
Department of Physics and Astronomy

Date Approved: July 14, 1967

App. only

68-65-29

R. A. MATZNER
Physics, General
U of Maryland

ABSTRACT

Title of Thesis: Almost Symmetric Spaces and Gravitational Radiation.

Richard Alfred Matzner, Doctor of Philosophy, 1967.

Thesis directed by: Professor Charles W. Misner.

Supervisor:

Techniques are given which can lead to invariant approximation techniques. The basis of these techniques is a definition for "almost" Killing fields, which are the analogue of the Killing vector fields present when the space has a symmetry. The integral ~~curves~~ of these fields can be taken as coordinate lines, and then the variation of the metric tensor along these lines is the slowest possible, in a global sense.

While this method can be applied to practically any situation, particularly when the deviation from symmetry is small, and might be suited especially to problems of slow motion and equations of motion, this work concentrates on applications to gravitational radiation.

Some examples are given. After initial computations for simple situations; an exact calculation on a perturbed flat 2-dimensional torus, and a calculation for a linearized weak gravitational wave pulse, we turn to the type of space considered by Isaacson.

Associated with every "almost" Killing field is a real scalar functional $\lambda[\xi]$. For positive definite spaces $\lambda \geq 0$ and $\lambda = 0$ only when ξ is Killing. In application to spaces of the Isaacson type, we show that λ contains an (additive) term which measures the average value of the quantity $T^{(av)}_{\alpha\beta} \xi^\alpha \xi^\beta$ over the whole space. Here $T^{(av)}_{\alpha\beta}$ is the average effective stress tensor defined by Isaacson. If ξ is Killing in the background, then λ consists only of this $T^{(av)}_{\alpha\beta} \xi^\alpha \xi^\beta$ term. We give applications of this fact to the Robertson-Walker metric constant time slices.

The cosmological solution due to Taub is also investigated by these methods (Appendix D). This solution has homogeneous but not isotropic spacelike sections. The complete spectrum of the operator $-\mathcal{D}$, of which the "almost" Killing field is the ground state, is found in the spacelike sections of this solution.

An averaging scheme is given to separate the background from the waves in a situation with gravitational radiation in a slowly curving background, by means of averaging along the "almost" Killing fields and the few lowest eigen solutions of $\Delta \xi + \lambda \xi = 0$. The Taub solution is used as an example for this also. We show an iterative averaging scheme which finds a background 3-sphere to the Taub closed 3-space slices. This averaging scheme yields an intuitively reasonable background. However, it can be compared with another intuitively appealing averaging scheme: defining the average as the 3-sphere with the same volume. The two methods give different 3-spheres as the background; hence different energy density in the remainder, the gravitational radiation. Criteria have not yet been found to specify which background may be the optimum one for the Taub space; in particular neither of these backgrounds evolve like a radiation dominated Robertson-Walker solution. However, reasons are given which make the uniqueness plausible for situations with high frequency radiation.

Some applications and examples are given in the Appendices, and some of the remaining problems are outlined in the concluding chapter.

FOREWORD

Many of the principal results in this thesis will be found in the Appendices. This arrangement was chosen so that these topics could be written in a form directly suitable for journal publication. Thus Appendices C and D are each self contained units in the form of journal papers, while the main body of the thesis discusses the general setting and motivation for the problems treated in these Appendices, and states further, as yet unsolved, problems to which they lead, indicating some possible approaches to these further problems. Appendix C is a reproduction of a paper which has already been published, which was co-authored with C.W. Misner.

When making a reference to an equation which occurs in another part of the thesis we shall refer to it by the notation of the table of contents. For example eq. (2.5) means eq. (5) of Chapter II, eq. (A-7) means eq. (7) of appendix A. References of the form (Synge, 19--) are to be found in the Bibliography at the end of the thesis. Footnotes for the main body of the thesis are found at the end of the main section (before the Appendices), while within some of the Appendices (conforming to the requirements of journals for which they are intended) references are given as footnotes to be found at the end of that same Appendix.

I gratefully acknowledge the financial support obtained from a National Science Foundation grant and a grant held by Professor Joseph Weber from the National Aeronautics and Space Administration.

I would like to express my deepest gratitude to my advisor Professor Charles Misner for the close support and constant encouragement he has given during the course of this work.

Part of this work was carried out at the Department of Applied Mathematics and Theoretical Physics, the University of Cambridge. I gratefully acknowledge the hospitality of the Department of Applied Mathematics, and many

stimulating and fruitful discussions with members of the University of
Cambridge during this past year.

TABLE OF CONTENTS

Chapter	Page
FOREWORD.....	ii
INTRODUCTION AND OUTLINE.....	1
I. REVIEW: SLOW MOTION AND EQUATIONS OF MOTION, LINEARIZED GRAVITATIONAL RADIATION.....	5
II. REVIEW: GRAVITATIONAL RADIATION.....	13
III. WAVE-NEWTONIAN FIELD SEPARATION AND INTERACTION.....	24
IV. SYMMETRY AND KILLING VECTORS.....	28
V. ALMOST SYMMETRIC SPACES.....	33
VI. EXAMPLES.....	53
VII. APPLICATIONS TO SHORT WAVELENGTH GRAVITATIONAL RADIATION.....	58
VIII. SPECIFICATION OF A "BEST" BACKGROUND.....	64
IX. CONCLUSION; OUTLOOK.....	69
APPENDIX A. APPROXIMATE CALCULATION OF GRAVITATIONAL RADIATION FROM COLLAPSING AXI-SYMMETRIC SYSTEMS.....	75
APPENDIX B. ELECTRONIC COMPUTATIONAL METHODS FOR GENERAL RELATIVITY.....	96
APPENDIX C. GRAVITATIONAL FIELD EQUATIONS FOR SOURCES WITH AXIAL SYMMETRY AND ANGULAR MOMENTUM.....	102
APPENDIX D. THE SYMMETRY OF THE SPACE SECTIONS OF THE TAUB COSMOLOGICAL SOLUTION.....	120
WORKS CITED.....	142

FIGURES AND TABLES

Figure	A1.	92
	A2.	94-95
	C1.	118
Table	A1.	91

Introduction and Outline

The theory of Relativity is based on an explicitly covariant set of equations: $R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)$. However, exact solutions to this set of equations are available only for very special situations, such as those with a high degree of symmetry. The solution to a physical problem with source and asymptotically flat space which is not symmetric in any sense involves a complexity that makes it impossible to obtain at the current stage in the Mathematical theory of nonlinear differential equations.

The approach has thus often been to use approximation methods to obtain solutions when they cannot be obtained from symmetry, say. The approximation methods usually destroy explicit covariance of the problem. The result is derived in one particular coordinate frame and extreme care must be taken to ensure that the coordinate conditions are all explicitly stated and understood. Invariant methods have so far been used only when the space under consideration was symmetric, and so contained a Killing field. But when there is no symmetry, there has heretofore not even been a way of specifying the quantitative lack of symmetry in the space. This paper presents such a quantitative measure, and gives a definition of "almost" Killing fields which generalize the Killing fields of symmetric manifolds. Although these ideas should be very helpful in the study of slow motion -- both of the type considered by Einstein, Infeld, and Hoffmann, and of the type exemplified by a slowly contracting non-spherical collection of matter with pressure -- we have found it easier to apply them to some instances of gravitational radiation. The outline of this thesis is as follows.

Chapters I and II are reviews of slow motion and of gravitational radiation. In Chapter II we include a discussion of the results of Isaacson (1967) for high frequency gravitational radiation. We will use Isaacson's

results frequently in the discussion of applications. Chapter III is a brief discussion of the equivalence principle and its relation to the problem of separating waves from background in situations with gravitational radiation.

Chapter IV is a short account of Killing vectors, and their utility in practically every calculation that one might consider doing, and for defining conserved energy and momentum quantities.

Chapter V is the central chapter of this work. It contains the definitions and a discussion of some of the properties of the fields we call almost Killing. These fields generalize the idea of Killing fields to spaces which are not symmetric. The field so defined specifies coordinate lines along which the variation of the metric tensor is the slowest possible in a global sense. Thus it generalizes the Killing fields in spaces with a symmetry where the metric tensor does not change along Killing trajectories. For closed spaces with positive definite metrics, existence and differentiability of solutions to the defining equations (5.1), (5.2) can be proved, and we quote the relevant theorem. The method postulated here has strong analogies to elasticity theory, and we show that in fact for 2-dimensional surfaces which can be imagined embedded in flat 3-space, there is an operational method of determining the symmetry parameter (λ). It is just the square of the lowest vibrational frequency-- assuming a certain type of stress strain relation--when the surface is a physical shell constrained by rigid sliding contact surfaces (no normal motion and no transverse stress at the surfaces). This analogy to elasticity permits a simple heuristic proof of the existence theorem for solutions to (5.2).

In Chapter V we also give a generalization of a theorem by Yano and Bochner (1952), and a brief discussion of some spaces of Minkowski signature, where the quantity $\sum^{\alpha;\beta} S(\alpha;\beta)$ may equal zero even though

$$\sum S(\alpha;\beta) \neq 0.$$

Chapter VI presents applications of these ideas to two examples.

In the first, we find exactly the vector field which is our almost Killing field under certain restrictions, for a wavelike perturbation on a flat 2-dimensional torus. The second is an application of these ideas to a Minkowski 4-space containing a linearized plane gravitational wave pulse.

Chapter VII contains the application of the techniques of almost Killing fields to high frequency gravitational radiation, of the type considered by Isaacson. It is shown that the real functional $\lambda[\xi]$ associated with every vector field ξ measures some parameters of the radiation. In the simplest case this parameter is the "energy density" of the radiation, but if a sufficient number of vector fields can be invariantly defined in the background the average gravitational "stress" associated with the wave may also be measured.

Chapter VIII is a discussion of the problem taken up in Chapter III, but with a slightly more precise objective in mind. We postulate that the background for an Isaacson type space can be found by the method of averaging along the invariantly defined vector fields which are solutions of (5.1) and (5.2). This may be essential if Isaacson's scheme is to have any computational advantages, since his program always requires such a splitting. For radiation of short enough wavelength, one would expect any averaging to be effective. We attempt to give a method which is powerful enough to work even when the radiation is definitely not short wavelength. For instance, Appendix D discusses this scheme for the Taub cosmological solution which is apparently a R.W. type of space but with the longest wavelength gravitational radiation that will fit into it giving the energy density to curve up the space. These studies of averaging procedures to define the background-wave decomposition of a metric are so far only exploratory and give no definitive results.

Chapter IX discusses further questions that these ideas may be applied to, and directions for further research.

The appendices A,B,C are examples of methods mentioned in the text. Appendix D, as we mentioned above, is a calculation applying the ideas here to the Taub comological solution.

Chapter I. Review: Weak Fields, Slow Motion and Equations of Motion,
Linearized Gravitational Radiation

The term "slow motion" immediately brings to mind the Einstein-Infeld-Hoffmann (EIH) method, one way of treating the equations of motion in General Relativity. Because of this historical fact and because the question of gravitational radiation has often been considered via the equations of motion, we devote this introductory chapter to a review of this and similar methods (including the "Fast" approximation which does not suppose slow motion).

The first approximation level to be considered in Relativity is the linear approximation. This means that one assumes a small deviation from flat space and writes $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ where $\eta_{\mu\nu} = \text{diag}(+---)$ is the Minkowski metric and $\epsilon h_{\mu\nu}$ is the difference between the actual metric and the assumed flat background. The procedure is to then linearize the Einstein equations $G_{\mu}^{\alpha} = 8\pi T_{\mu}^{\alpha}$ in ϵh ; by a suitable choice of coordinate conditions these equations may be put in the form

$$\frac{1}{2} \epsilon \square \psi^{\alpha\beta} = 8\pi T^{\alpha\beta} \quad \text{where} \quad \psi^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2} \eta^{\alpha\beta} h^{\sigma}_{\sigma} \quad (1.1)$$

(Landau and Lifshitz, 1962, Ch. 11).

These equations contain Newtonian gravitation as is well known. For now assume slow motions as well as weak fields. They can then be reduced to the form

$$-\epsilon/2 \nabla^2 \psi^{00} = 8\pi \rho$$

which gives $\epsilon \psi^{00} = 4\phi$ with ϕ = the Newtonian potential

$\sim -m/r$. The equations of motion (which are the geodesic equations in the space described by this metric) then become:

$$\frac{d}{dt} m \frac{dx^i}{dt} = \phi_{,i} \quad (1.2)$$

the Newtonian equations of motion. These are the original slow motions which several centuries of extremely precise astronomical observations have verified, since their formulation by Newton. The Newtonian equations of motion are necessarily the starting point for all slow motion approximations since Relativistic results must reproduce Newton's, where the Newtonian theory applies. The accuracy of the Newtonian predictions and the magnitude of the Relativistic corrections in (say) planetary motion problems can be appreciated by noting that the Newtonian potential of the sun at the Earth's orbit is about 10^{-8} (the Earth's potential at its surface is about 10^{-8} (Synge, 1960). Thus the second order correction to the motion will be one part in 10^8 . [If one thinks in an invariant manner, and calculates the curvature tensor, the familiar result from the Schwarzschild solution $R_{\dots} \sim m/r^3$ shows that it is given by the density. The density of the sun is ~ 6 gm/cc; the average density of the sun is ~ 1 gm/cc. Thus the curvature tensor of the sun at the Earth's orbit is $\sim (R_{\odot}/R_E)^3 \sim 10^{-7}$ gm/cc. The curvature tensor invariantly described gravitational quantity for the sun is about 100 times weaker at the surface of the Earth than is the Earth's. To reconcile this fact with the usual methods, and especially with Newtonian results, one must realize, as Dixon (1967) has emphasized, that the relevant feature is the linearity of the equations for small fields. The orbit of the Earth is very similar to that of an infinitesimal test particle with the same initial conditions, because of this additive linearity.]

Of course, one can also consider gravitational radiation as well as slow motions like planetary motions which are so slow that radiation is negligible. The theory of General Relativity permits energy transport by gravitational means, and this can be seen even in the weak field approximation. The time dependent form of eq (1.1) contains the D'Alembertian operator

has wave-like solutions. There are problems with making such a straightforward statement, the problems of coordinate conditions. The specification of the coordinate gage used is a matter of some taste; it can, however, affect the results in naive applications of pseudotensor calculations. Briefly bypassing these problems-- although they are really the principal questions we intend to come to grips with in this work-- calculations can be carried out which are analogous to the calculation of radiation in classical electrodynamics. They give the familiar result (Landau and Lifshitz, 1962) showing that there is no dipole radiation, but that the first nonvanishing radiation terms are the quadrupole terms. In fact, one finds

$$\frac{dE}{dt} = -\frac{G}{45c^3} \sum_{i,j} \frac{d^3}{dt^3} D_{ij}$$

to this order for the energy loss from the system by gravitational radiation as calculated by "common-sense" applications of pseudotensor methods. Here D_{ij} is the quadrupole moment tensor of the source. (An example of the utility of this expression is given in Appendix A, where the linearized theory radiation is calculated for several cases of collapsing objects undergoing Newtonian motion.) The calculation of gravitational radiation by linearized theory obviously requires weak fields, and thus Newtonian situations such as collapse from non-Relativistic initial conditions. Planetary motions are well suited to this method. Although strong fields are excluded, high velocities are apparently not excluded. Thus the radiation from near-miss hyperbolic orbits can be calculated by this method, although high velocity bound orbits cannot be so treated. We shall encounter this distinction again below.

Several methods have been invented to carry the possibilities inherent

in the weak field approximation to higher orders. They divide-- very roughly-- into the methods associated with the names Einstein, Infeld, and Hoffmann, and the methods like the "Fast" approximation.

The most famous treatment of the problem of motion in General Relativity is that initiated by Einstein, and carried on by Einstein, Infeld, and Hoffmann. The first paper on this topic was published by Einstein and Grommer (1927). They showed that the geodesic motion of a test particle follows from the field equations. The paper of Einstein, Infeld, and Hoffmann (1938) finally went beyond the simple motion of test particles on geodesics in an external field, and formulated an approximation method which would permit the simultaneous calculation of the gravitational field and the motion of its sources. A parallel, independent development along these lines was carried out by Fock and his collaborators, beginning with the papers by Fock on the equations of motion, published in 1939.

The methods of Fock differ from those of EIH in that Fock fixes the coordinate conditions to be harmonic conditions: $(-g)^{-\frac{1}{2}} \{(-g)^{\frac{1}{2}} g^{\alpha\beta}\}_{,\beta} = 0$ at the outset and remains in this gage throughout his derivation, and makes very extensive use of the gage in simplification of the expressions which appear. (See, e.g., Fock, 1964.) Also, Fock (and Papapetrou, 1951) makes specific assumptions about the matter tensor; in particular that it is nonsingular.

The EIH approach, on the other hand, assumes particles will be given by singularities in the fields, and so looks for solutions to either $G^{\alpha\beta} = 0$, or to $G^{\alpha}_{\beta} = 8\pi T^{\alpha}_{\beta}$, where T^{α}_{β} has some (modified) delta function singularity. The difference we are most interested in, however, is the coordinate condition applied in the EIH method. One possible coordinate condition is $\gamma^{\nu\rho}_{,\rho} = 0$, $\gamma^{mm}_{,m} = 0$, where $\gamma^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$ (Goldberg, 1962).

These are not harmonic conditions, since they are not the full set $\gamma^{\mu\nu}, \nu = 0$. They differ very little from harmonic coordinates when $\gamma^{\mu\nu}$ is small, however, and the EIH method adjusts them at each step of the approximation, and iteration schemes allow the method to be carried out step by step, with the correct coordinate condition applied at each order.

The recent work of Chandrasekhar (1965) is similar to the programs of EIH and Fock. Chandrasekhar has expanded the coupled Einstein and hydrodynamics equations in powers of v/c , keeping terms to the first post-Newtonian order. As in all such work, a coordinate condition is necessary; the condition used by Chandrasekhar is

$$\frac{1}{2} \frac{\partial h^i_i}{\partial x^0} - \frac{\partial h^i_0}{\partial x^i} = 0$$

(where $g_{\mu\nu} = \gamma_{\mu\nu} + \epsilon h_{\mu\nu}$),

a condition which he is careful to check at the end of his calculations.

The Chandrasekhar development presents the quantities that appear in physically intuitive terms; for instance

$$\epsilon h_{00} = -2U + (2U^2 + 4\Phi^2) + O(v^6)$$

where U is the Newtonian gravitational potential (in this coordinate frame) and $\nabla^2 \Phi = -4\pi \rho \phi$ where

$$\phi = v^2 + U + \frac{1}{2}(\text{internal energy}) + \frac{3p}{2\rho},$$

showing the effect of the energy density of the gravitational field acting itself to produce more gravitational field, giving the post-Newtonian correction to the metric.

It might seem that similar methods could be used to give the next approximation after linear theory for waves. However, this is not so. Trautman (1965) has shown that if one starts with a linearized solution with

outgoing radiation and imposes a gage which is the harmonic gage condition to first order, one finds that the potentials $h_{\alpha\beta}^{(1)}$ satisfy

$$\square h_{\alpha\beta}^{(1)} \equiv \eta^{\mu\nu} h_{\alpha\beta,\mu\nu}^{(1)} = 0 .$$

If one then assumes the same coordinate condition for $h_{\alpha\beta}^{(2)}$ and writes the source due to the effective stress tensor of the radiation which generates

$$h_{\alpha\beta}^{(2)} : \square h_{\alpha\beta}^{(2)} = Q(h^{(1)}) , \text{ one finds that } h^{(2)} \text{ behaves as } (\log r)/r .$$

Thus it does not vanish fast enough at infinity, contradicts the Sommerfeld condition, and gives infinite energy. Thus the approximation scheme has broken down. We will mention the explanation of this failure after the discussion of the Asymptotic methods and Isaacson's results. It can be avoided by solving the wave and the background consistently as a first step, and then using this substitution method for higher terms. (Isaacson 1967).

Further work on the equation of motion has been recently done by Dixon (1967). He has considered the full solutions and has not separated them by any approximation methods. He has given a multipole expansion which determines the motion of the body in terms of its moments in an asymptotic series-- asymptotic because it must be stopped before the m 'th term, where m is a number such that $\frac{\partial^m g_{ij}}{\partial x^m}$ varies appreciably over the size of the particle. Thus it is implicitly assumed as in classical motion of "small" extended masses that the first few multipole moments give a sufficiently accurate description of the motion of the particle. Dixon's equations are exact, they give the motion of the particle once the total field is known. **T**he total field, however, includes the field of the particle under consideration, as well as the external sources. There will be no infinite field problems because Dixon always assumes his particles are extended bodies. However, the problem of motion has not yet been fully solved in this way since a scheme for satisfying the motion equations without knowing the fields

a priori has not yet been given. The multipole approach is interesting because of its invariance and because of the ability to sidestep the question of coordinate conditions.

Although the EIH method is excellent for the type of problem found in planetary orbit calculations, for instance, it is not at all suited to calculations of intense radiation. This is because of its assumption of slow motion which implies weak fields, since strong fields accelerate bodies to high velocities. But the weak field assumption means the observer is always in the near field region, where it is extremely difficult to distinguish radiation from induction phenomena. Another type of approximation has been made, the "Fast" approximation. (See for instance, Goldberg, 1962) This method extends the weak field results by assuming only that the potential m/r is small, but that v/c is not necessarily small. This scheme is still unsuitable for calculating intense radiation from bound planetary orbits, but apparently can give correct answers in the case of near miss hyperbolic orbits, where the radiation may be fairly intense even though the fields are not strong.

Recently, Carmeli (1965) has given methods of finding the equations of motion by assuming that the metric can be split into a part associated with the particle and an external field. He obtains the solution to the motion as a sum of powers of the mass of the body under consideration. By making suitable assumptions, both the EIH and the Fast approximation can be obtained from Carmeli's results. Furthermore he has been able to show that some strange, apparently antidamping, terms found by Havas and Goldberg (1962) are exactly cancelled by some more obscure terms which had previously been neglected. However, as Carmeli has pointed out, the physical interpretation of the terms that appear in these equations is still confused and more work

remains to be done from this approach. (The problem of equations of motion is still lively even in classical electrodynamics. See Kaup, 1966, 1967)

We have given this sketch to show how the problems of coordinate conditions and equations of motion are interrelated, and to show how the treatments of equations of motion make certain assumptions about gravitational radiation. Of course, assuming that there is an EIH solution which to first order is Newtonian motion postulates that there is no strong radiation likely to disrupt the system while it is being investigated. The separation into slow motion and waves so that the waves can be excluded in these approximate schemes is very similar to the separation we will discuss in Chapters III and VII below. First, however, we will give an account of the other viewpoints on gravitational radiation; viewpoints which are far removed from the questions of equations of motion.

Chapter II. Review: Gravitational Radiation

a) Exact Gravitational Radiation

There is as yet no example of a vacuum solution to Einstein's equations which represents a spatially bounded source emitting gravitational radiation. There are several reasons for this. The nonlinearity of the gravitational equations is the notorious complicating factor. Expansions of the Fourier type are excluded (plane waves of finite amplitude have infinite effective energy) and the apparatus of the classical electrodynamics approach, which has after all only yielded precise answers to this type of question recently (Bondi, 1960, Misner and Zepolsky, 1966) in the simpler electromagnetic case cannot at all be applied to the nonlinear gravitational problem. The nonlinearity exhibits itself in a characteristic way, in infinite energy densities. As pointed out previously, in trying to find the corrections to a linearized solution with waves if one attempts to apply a straight forward Fourier expansion in terms of the plane wave solution which can be obtained from the linearized approximation, one discovers that the energy density in the wave causes a curving of the space near it so that a correction is needed which is large, in the sense that it has a term of order $(\log r)/r$ far from the source. These questions have been considered by Bondi, Van der Burg, and Metzner (1962) and also by Isaacson (1967).

The situation is much simpler when the problem is highly symmetric. Cylindrically symmetric waves have been known for a long while [see, e.g. Einstein and Rosen (1937), Rosen (1937), and Wheeler and Weber (1957)]. Recently there has been some success in finding other idealized exact solutions which contain waves (but not from a bounded source.) Brill (1959) has shown how to prescribe time symmetric axially symmetric waves, which describe an imploding- exploding (source-free) wave. He takes a metric form:

$$ds_3^2 = \Psi^4(\rho, z) [e^{2\lambda g_1(\rho, z)} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] \quad (2.1)$$

in the instant of time symmetry $t=0$. If q_1 is a function of bounded support then Ψ is given by the initial value problem, ${}^{(3)}R(t=0) = 0$

which is ${}^{(3)}\nabla^2 \Psi = -\frac{\lambda}{2} ({}^{(2)}\nabla q_1)^2 \Psi$ (λ is a parameter, $\lambda > 0$). (2.2)

Brill shows that there are everywhere regular solutions (in the $t=0$ 3-space) only with an asymptotic form $\Psi \sim A(\lambda) \left[1 + \frac{m(\lambda)}{2r} \right]$, $m > 0$; and only when $\lambda q_1 \leq \lambda$, everywhere for a definite limiting λ . For every positive value of λ less than this limit there is one and only one regular solution which is asymptotically flat; this solution describes a localized gravitational wave whose energy (measured at infinity) is m . Araki (1959) has discussed this problem further.

Work has been done also by Robinson and Trautman (1960) who have investigated all vacuum fields ^{which admit a} hypersurface orthogonal non-shearing geodesic ray congruence. These contain (all) pure radiation fields with rotation free rays. These do however contain also the Schwarzschild solution so it is not safe to characterize them as entirely wavelike solutions. Robinson has pointed out that the Schwarzschild solution is a ($\frac{1}{2}$ advanced) + ($\frac{1}{2}$ retarded) solution of a wave type. Other exact gravitational wave solutions are given by Hély (1959), Peres (1959), Bondi (1957), Takeno (1957), Kundt (1958), and Jordan, Ehlers and Kundt (1960). These waves, though interesting, are still far from the solution which shows a finite source emitting a finite amount of radiation. All of these solutions mentioned here have algebraically special Petrov type Riemann tensors. (See Petrov, 1954; see also the next section.)

b) Characterization of the Riemann Tensor

Characterization of gravitational fields in terms of distinguished (null) rays has been given by Debever (1959) and Sachs (1961). They have shown that the Riemann tensor can be classified by the number and kind of null vector rays it admits [in the sense of eq. (2.3) below] at each point. The basis of their classification is a theorem (Sachs, 1961) that in every empty space-time there exists at least one and at most four directions $k^\alpha \neq 0$ such that

$$k_{[\alpha} R_{\beta]\epsilon\lambda\delta} k^\epsilon k^\lambda = 0, \quad k^\sigma k_\sigma = 0. \quad (2.3)$$

The classification then proceeds by telling how many such vectors there are and whether any are coincident. This tensor method gives the same classification as the earlier matrix method of Petrov (1954).

We give the correspondence of types here since we shall later use the Petrov notation:

- Type I, completely general, four distinct null vectors;
- Type D, three null directions (one doubly degenerate);
- Type II, two null directions (each doubly degenerate);
- Type III, two null directions (one triply degenerate);
- Type N, one quadruply degenerate null direction.

The interest in classification of the Riemann tensor is that the field at large distances from a bounded source tends to a type N Riemann tensor, as we mention in the next section, on asymptotic methods.

c) Asymptotic Methods

The most successful discussion of gravitational radiation from a finite source is that which utilizes the ideas first set out by Bondi, Van der Burg and Metzner (1962). These workers (considering only the axisymmetric case) developed the Einstein equations in a null coordinate system so that outgoing light rays had a constant retarded time. They fixed the coordinate system in an ⁱⁿvariant way, and could then write the metric as

$$ds^2 = (Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma})du^2 + 2e^{2\beta} du dr + 2Ur^2e^{2\gamma} du d\theta - r^2(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2\theta d\phi^2). \quad ($$

Here V , β , U , and γ are functions of u , r , and θ . By assuming flatness at null infinity, Bondi can choose coordinates u and θ so that g_{uu} is positive as $r \rightarrow \infty$. There is then a coordinate patch near infinity where $g_{uu} > 0$, and this is assumed to be the region at infinity surrounding a bounded source. As null rays are followed in to smaller values of r , a point is found where the neighboring rays intersect. An envelope is then drawn outside all the points at which such crossings occur, and the coordinate patch described here is then the whole region outside that envelope.

The seven non-trivial Einstein equations can be broken into two groups. The first set (the main equations) consists of $R_{rr} = R_{re} = R_{ee} = R_{\phi\phi} = 0$. Of the other equations, $R_{ur} = 0$ follows from the Bianchi identities. Under these circumstances, both $R_{ue} = 0$ and $R_{uu} = 0$ are satisfied everywhere because of the Bianchi identities, if they hold at some r for all values of u and θ . The envelope where these supplementary conditions hold is taken to be the envelope surrounding the source, as described above.

The main equations can be split further into two groups. The equations $R_{rr} = 0$, $R_{re} = 0$, and $e^{2(\beta-\gamma)}R_{ee} - r^2R_{\phi\phi}e^{2\beta} = 0$ contain only differentiation in the hypersurface $u = \text{constant}$ (hypersurface equations). The remaining equation $R_{\phi\phi} = 0$ contains one derivative with respect to

u , of the form $\frac{\partial^2}{\partial u \partial r} (r\gamma)$. Thus these equations can be solved if for some u , γ is given. The hypersurface equations then give β , U , and V . When these are known, the equation $R^\phi_\phi = 0$ gives the value of γ for a later time step, and the entire solution is known.

Bondi proceeded by writing asymptotic expansions for the variables U , V , β , and γ . The requirement of outgoing radiation limits the asymptotic forms of these quantities. A solution can be found by substituting the asymptotic expansions into the main equations.

The leading terms in the asymptotic expansion can be identified with the physical properties of the source. For instance, Bondi, et. al. write

$$\begin{aligned}\gamma &= c(u, \theta)r^{-1} + \dots, \\ V &= r - 2M(u, \theta) + \dots.\end{aligned}$$

The entire solution (for outgoing asymptotically flat solutions) depends at infinity only on $c(u, \theta)$.

The leading term of one of the supplementary conditions is

$$M_{,u} = -c_{,u}^2 + \frac{1}{2}(c_{,\theta\theta} + 3c_{,\theta} \cot \theta - 2c)_{,u}.$$

The quantity

$$m(u) = \frac{1}{2} \int_0^\pi M(u, \theta) \sin \theta \, d\theta$$

is the mass in the static case. Furthermore (by the supplementary condition),

$$\frac{dm}{du} = -\frac{1}{2} \int_0^\pi (c_{,u})^2 \sin \theta \, d\theta.$$

Thus we find that the mass of the system can only decrease and it depends only on the news function; an arbitrary function of two variables, which describes the modulation of the radiation sent out by the source. [If the

situation were not axi-symmetric there would be two news functions, completely describing the two degrees of freedom in gravitational radiation. This has been discussed by Sachs (1962).]

The methods utilized in the Bondi analysis have been used by other workers to investigate asymptotically flat solutions. The coordinate r is (within allowed gauge transformations) a luminosity (L) distance; i.e. $L=L_0(r_0/r)^2$. The use of such a coordinate— which is invariantly defined in most physically defined situations—leads to some close analogies with electromagnetic radiation away from the source. One example is the peeling-off theorem, which shows that an asymptotically flat space with unmixed radiation must have a Riemann tensor that behaves asymptotically like (indices suppressed).

$$R \sim {}_0N/r + {}_0III/r^2 + {}_0II/r^3 + {}_0I/r^4 + {}_0I'/r^5 + O(r^{-6}) ,$$

where the N , II , III , etc. refer to Petrov types of the Riemann tensor (Petrov 1954); the left subscript zero means that these terms are covariantly constant along the outgoing rays. (Sachs 1961; Goldgerg and Sachs, 1962).

Other investigations of this type have been carried out by Newman and Penrose (1962) and by Newman and Unti (1962) and by Janis and Newman (1965). Here the emphasis is on the asymptotic description of the fields in terms of the curvature tensor, instead of concentrating on the metric as Bondi did. The results are similar to those obtained by Bondi and his coworkers. Recent work from this viewpoint has been done by Couch, Torrence, Janis, and Newman (1967). We will discuss this paper below in Chapter III when we consider the interaction of wave and "Newtonian" fields.

d) Canonical Formulations: ADM

To treat gravitational radiation correctly one apparently needs to specify coordinate conditions or to consider asymptotic situations (as was done by Bondi et.al.) where the coordinates were more easily pinned down by physical requirements, because one supposes the space is almost flat near infinity.

A formulation of the problem of solutions to Einstein's equations which includes a discussion of radiation and which specifies the coordinate conditions in an unusual way has been given by Arnowitt, Deser, and Misner (ADM; 1962). Their interest was directed toward casting the Einstein equations into a form which makes the necessary field quantities explicit, and suppresses the redundant field variables which appear because of the covariance of the theory. They consider a 3+1 separation of the 4-space and work in a 3-dimensional spacelike surface. They separate the "Coulomb" terms which are due to massive sources, from the transverse-traceless parts of the metric which are the wave parts. The "natural" coordinate conditions are non-local ones, involving integrals over the whole 3-surface. The reason such conditions are natural can be seen by considering the vector (electromagnetic) case. There one writes ⁽¹⁾

$$\underline{A} = \underline{A}^T + \underline{A}^L, \quad \text{where}$$

$$A^T_{|n} = 0, \quad \epsilon_{ijk} \partial_i A^L_k = 0.$$

Thus $A^L_k = \phi_{,k}$ and one can find the longitudinal (Coulomb) part A^L_k by noting that

$$A^k_{|k} = A^L k_{|k} = \phi^{,k}_{|k}.$$

One had then to solve this elliptic equation, which in flat space has the

solution

$$\phi(x') = \frac{1}{4\pi} \int \frac{A(x)^k{}_{ik} d^3V}{|x-x'|}$$

The gradient of ϕ yields $\underline{\underline{A}}^L$ and by subtraction one obtains $\underline{\underline{A}}^T$, the sourcefree part of $\underline{\underline{A}}$.

The ADM coordinate conditions are very different from the local differential coordinate conditions usually assumed in the theory. [In the limit of short wavelength, these ADM coordinate conditions become local (Misner, 1967, private communication).]

The ADM approach has given useful expressions for energy and momentum quantities when gravitational radiation is present. Asymptotically defined quantities can be constructed when radiation is present, and by investigating the behavior of the canonical variables, one finds that a Poynting vector can be defined, for instance, which has the expected vectorial transformation properties under coordinate changes that are asymptotically Lorentz transformations.

The canonical formulation is of interest for the problem of quantization of the gravitational field. To this end ADM have ^{written} the canonical Poisson brackets for the motion of the field. Other (earlier) work on this subject was done by Dirac (1950, 1958, 1959). Pirani and Schild applied some of the earliest of Dirac's (1950) canonical quantization procedures to the gravitational field. Some recent work comparing the Dirac and ADM approaches has been done by Anderson (1966).

e) High Frequency Gravitational Radiation

Brill (1964), Brill and Hartle (1964), and Isaacson (1967) have recently done investigations which give a better understanding of the behavior of short wavelength gravitational radiation. In particular, a stress-energy for the radiation can be averaged in a suitable way so that the resultant energy density is a positive definite quantity. The radiation's energy density arises as discussed in Chapter III below because of the nonlinear nature of the Einstein equations. When coordinate conditions are picked in which they can be compared, the averaged stress tensor agrees with the pseudotensor (for instance the pseudotensor given by Landau and Lifshitz, 1962, p341). The averaged energy tensor is, however, invariant over a much wider range of gage transformations than the usual pseudotensor treatment allows.

Isaacson has considered the following situation. Suppose $g_{\alpha\beta}$ is a vacuum metric which admits a coordinate system (Isaacson : Steady Coordinates) such that the metric can be written

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + \epsilon h_{\alpha\beta} ,$$

where the metric $\gamma_{\alpha\beta}$ is a slowly varying function of position and $h_{\alpha\beta}$ satisfies a certain generalized wave equation $\square' h_{\alpha\beta} = 0$ in the space given by $\gamma_{\alpha\beta}$. Further, we demand that the averaged stress tensor (defined by Isaacson) from $h_{\alpha\beta}$ should give the background $\gamma_{\alpha\beta}$ when inserted as a source into the field equations for $\gamma_{\alpha\beta}$. We symbolically write

$$R_{\alpha\beta}^{(0)}(\gamma) = -\epsilon^2 \langle R_{\alpha\beta}^{(2)}(h, \gamma) \rangle , \quad (2.5)$$

$$\epsilon R_{\alpha\beta}^{(1)}(h, \gamma) = \epsilon \square' h_{\alpha\beta} = 0 , \quad (2.6)$$

where the numbers $0, 1, 2, \dots$ refer to the powers of ϵ appearing in the expansion of the equation $R_{\alpha\beta}(\gamma + \epsilon h) = 0$. Since we are interested in high frequency radiation, we assume that $h_{\alpha\beta}$ is a rapidly varying function of position.

The equations (2.5) and (2.6) must be solved simultaneously in a consistent manner, since the equation for $h_{\alpha\beta}$, (2.6), involves $\gamma_{\alpha\beta}$. But $\gamma_{\alpha\beta}$ is given by equation (2.5) which involves the averaged stress tensor of $h_{\alpha\beta}$ as a source. The radiation is causing a "Newtonian" field because it has energy. The "Newtonian" field simultaneously affects the motion of the radiation.

The self-consistent requirement imposed by equations (2.5) and (2.6) means that a derivative of h must be of order ϵ^{-1} , i.e. $\partial^m h = O(\epsilon^{-m})$. This can easily be seen by the following argument due to Isaacson. Derivatives of the background are $\partial\gamma \sim \gamma L^{-1}$, the derivatives of h are $\partial h \sim h \lambda^{-1}$, where L is a typical length in the background and λ is the wavelength of the high frequency radiation; $L \gg \lambda$. The "energy density" in the wave is then $\rho \sim c^4 G^{-1} \epsilon^2 \lambda^{-2}$ and the curvature of the background is $\sim L^{-2}$. Then we have, by the Einstein equations,

$$R_{\alpha\beta} \sim L^{-2} \gtrsim (Gc^{-4})(c^4 G^{-1})(\epsilon/\lambda)^2 \Rightarrow \epsilon \lesssim \lambda/L.$$

If there is matter present which is also curving up the space the inequality holds; if the curvature is due totally to the waves, we have approximate equality, $\epsilon \sim \lambda$ (we take $L \approx 1$). This means that $\lambda \rightarrow 0$ and $\epsilon \rightarrow 0$ are the same limit for a fixed background, and in further discussion of the Isaacson method we will write $O(\lambda)$ instead of $O(\epsilon)$ to emphasise this.

Isaacson has also given a method of solving the equations which is similar to the W.K.B. method (the solution is the first term in an asymptotic expansion in λ). Isaacson has in particular shown that when the equations are solved in this self consistent manner, the problems of logarithmic terms, mentioned by Trautman (1965) and discussed in Chapter I, in radiative solutions are avoided. Apparently the logarithmic terms are caused because in the linearized theory the waves move along flat-space null cones, which are different by large amounts from the physical null cones in the space which is curved by the energy density of the radiation. [The null cones in flat space are $\text{const}=t-r$; those in the Schwarzschild solution are $\text{const}=t-r-2m \ln(r-2m)$. This is suggestive of the source of the logarithmic terms that Trautman finds. Clearly a very large shift must take place to move the null cones to the correct position (at least as big as the first approximation itself) .]

[The asymptotic methods of Bondi, et.al. (1962) avoid the logarithmic terms because they especially center their attention on the physical null cones of the full solution.]

Chapter III. Wave-Newtonian Field Separation and Interaction

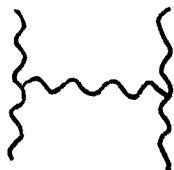
One of the outstanding problems remaining in the Isaacson approach to high frequency gravitational radiation is the necessity of splitting the metric into a slowly varying background with an easily identifiable wave in it. One must first find a "steady" coordinate system before Isaacson's results can be applied.

This difficulty and the related one of the equivalence principle; that the waves have mass and so curve up the space you wish to investigate them in, are questions that have somehow to be answered in discussion of gravitational radiation. Because of their central importance, we give a brief survey and review of this topic.

The equivalence principle is actually the basic difficulty in finding a background space-- i.e. in finding available coordinate conditions. We are well aware that any gravitational field can be transformed away in an infinitesimal volume, simply by going to a free fall system. Thus when one of the pseudotensors is used to calculate the flux of gravitational momentum or energy at infinity, one must be careful to take appropriate coordinate conditions since the pseudotensor can always be annulled at a point by coordinate transformations.

One should point out, however, that by considering observers at infinity in their asymptotic characterization of gravitational radiation, Bondi, et.al. (1962) have been able to discuss the mass and momentum carried out to infinity. And the work of Newman and Penrose (1962) who discuss the Riemann tensor in tetrad frames clearly avoids such difficulties since they deal directly with invariant quantities. Also, the canonical formalism of A.D.M. fix the coordinate conditions so that energy and momentum can be defined in asymptotically flat space.

The equivalence principle can be stated in another way: that every energy disturbance has a mass and thus generates a Newtonian gravitational field. This has important consequences in the quantized theory of gravitation. Recall that the cross section for the deflection of light (massless radiation) by a massive body, like the sun, is $\frac{d\sigma}{d\Omega} \sim m_0^2$ (for a discussion, see Matzner, 1967). Then one can immediately estimate the differential cross section for graviton-graviton scattering. In the c.m. frame each graviton has energy (and momentum) E . One of them (a massless particle) sees a Newtonian field due to the mass of the other, E . The differential cross section is then $\frac{d\sigma}{d\Omega} \sim E^2$. One can even correctly predict the strong forward peak in the cross section due to the long range Newtonian field. Note that the quantum of action, \hbar , does not enter. The detailed results for quantized gravity show that the term due to the equivalence principle,



actually does dominate for graviton-graviton scattering (DeWitt, 1967).

This creation of a "Newtonian" field is of course the self-consistency aspect of radiation that Isaacson has pointed out. We have just noted that gravitational radiation will interact with a Newtonian field, no matter what energy is the source of the field. It will interact just as well with its own Newtonian field. This phenomenon is responsible for "tails". An example of such a tail has been given by Couch, Torrence, Janis, and Newman (1967) who considered an approximation scheme for the tetrad components of the Riemann tensor, starting from flat space. They construct a field which is to first order Schwarzschild with an outgoing wave pulse. To second order they can construct a solution which has a spherical outward moving

wavefront; ahead of the wavefront is quiet (but there is of course a long range Newtonian field due to the central mass). However, behind the wavefront is a combination of ingoing and outgoing radiation. A solution with only outgoing radiation is not possible to this order. The explanation is the backscatter of radiation off the static Schwarzschild field, which can be entirely due to the energy in the radiation. The earlier work in this type of approximation scheme has been given by Janis and Newman (1965), and Torrence and Janis (1967).

Isaacson (1967) has also considered this problem, starting from the Vaidya metric. (Vaidya, 1951, 1953; see also Lindquist, Schwartz, and Misner, 1965). This is a spherically symmetric solution which has a stress tensor $T_{\mu\nu} = \sigma k_\mu k_\nu$, where k_μ null vector in the outward radial direction. Thus this metric describes the flow of disordered radiation in an outward, spherically symmetric manner. Isaacson takes the Vaidya solution as his background $\gamma_{\mu\nu}$; he then postulates weak outgoing waves $\epsilon h_{\mu\nu}$ which are transverse traceless. He shows that these waves give a stress tensor with the correct form (after averaging and using the WKB approximation) required by the Vaidya metric. Since the WKB approximation is a short wavelength approach, Isaacson does find he can obtain this form for the Vaidya metric, which requires that all quantities depend on the retarded time, u . As Isaacson has pointed out, this means that there is no tail or backscatter to this order. However, a detailed investigation of the (linear) equation (2.6) for the $h_{\alpha\beta}$ shows that for finite frequency waves, backscatter must occur. [Investigation of the propagation of tensor waves in the Schwarzschild metric via an equation of the type of (2.6) has been carried out by Edelman (1967) and by Vishveshwara (1967). A simpler situation which contains all the interesting features of gravitational radiation is

the behavior of scalar waves in the Schwarzschild field; see Matzner , (1967). All of these investigations show that backscatter will occur for finite frequency waves.] Because backscatter will occur for radiation of finite frequency, one must conclude that the Vaidya metric is only a "geometrical optics" approximation to a real situation. The detailed structure of the solution for lower frequencies will of course depend on the detailed structure of the radiation considered.

After this brief discussion of Isaacson's methods and some of their properties and the difficulty of a wave background separation, we suggest a few problems which may lead to an extension of his type of treatment. Since the "Steady" coordinate system is essential in his development, three questions are suggested. Given a candidate space, is there a "Steady" coordinate system; if there is, how does one find it; and can one define a "best" background against which the wave separation is optimal? We give some tentative answers to these questions below. The verification that a background does exist is fairly straight forward, and in Appendix D we give a calculation which is an example of how to find a background metric when there is a wave present. Also in that Appendix, we give some tentative comments about the question of optimum background. The results of these topics are as yet inconclusive, however.

This section has hopefully given an impression of the importance of the wave background separation. The recurrent feature of these investigations is that in the slow/fast or wave/background separation, both aspects must be investigated. The problem must be treated as a coherent whole. The waves make a background which interacts with the waves. There is so much interdependence that these two cannot be separated.

Chapter IV. Symmetry and Killing Fields

In the discussion in the previous paragraphs, we have emphasized that the problem of slow/fast separation must be treated as a coherent single entity. Nevertheless, the initial steps of such an investigation usually concentrate on one or the other aspect of the separation.

This work originally began with the consideration of collapsing axisymmetric systems. From a Newtonian viewpoint, Lin, Mestel, and Shu (1965) have given solutions for the motion of pressureless nonspherical dust clouds undergoing homologous collapse. Appendix A gives the calculation of the gravitational radiation according to linearized theory for these Newtonian motions (and also some linearized calculations done in strong field situations to find order of magnitude radiation intensities). The results are much as might have been expected; non-relativistic initial conditions give little gravitational radiation. On the other hand, the radiation is precisely what one gets by order of magnitude estimates. There are no selection rules forbidding the radiation.

However Birkhoff's theorem guarantees that there are radiationless collapses to strong field configurations, the spherically symmetric ones. We may thus expect that by taking only slightly aspherical collapse, only small amounts of radiation will be released.

There has been some interest in a closely related problem lately. In particular, a large amount of work on perturbations of spherically symmetric static situations has been done by Compolattaro and Thorne (1967), and suggestions for similar investigations have been made by Misner and Zepolsky (1966). The Compolattaro-Thorne method begins by assuming one is given a spherical solution representing a star at rest, say. One then considers the effect of small perturbations from the spherical state. Compolattaro and

Thorne then sketch the derivation of the equations for both the perturbations in the vacuum and in the matter part of the solution. Even though spherically symmetric metrics are simple and even though one keeps only linear terms, the calculational task for the Compolattaro-Thorne scheme is formidable. For this reason, several investigators (Fletcher, Clemens, Matzner, Thorne, and Zimmerman, 1967) have turned to computer calculation to eliminate the drudgery of calculation of the Riemann tensor and the equations of motion. The programs developed at the University of Maryland by the present author and R. W. Clemens are discussed in Appendix B. We also discuss the range of application and include some samples of the type of results that can be obtained.

The alternative way to obtain small radiation in strong field situations is to suppose that there is a pressure field which keeps the situation only slowly changing, even through the gravitational fields are strong. By keeping the motion slow enough, the radiation can be made as small as desired. The problem has been investigated by Levi (1965). He has found that one can start with an axisymmetric metric which is static and of the Weyl form (see, e.g. Appendix C for a discussion of these Weyl types of metrics). If one allows the situation to be time varying, then to first order in velocity, the diagonal terms are the same instantaneous functions of the source as they are in the static case, but the off diagonal terms are no longer zero; the equations

$$R^{oi} = 8\pi T^{oi}$$

give linear equations for them to this order. Thus, by taking this situation slowly enough changing, the motion can be completely described. One thing that Levy has found (following an idea of Bondi, 1964) is the expression for a "Newtonian Poynting Vector", which gives the momentum and energy which is transported even in this completely nonradiative situation.

The difficulty with the Levy approach, or with the methods discussed in Chapter I is that they become prohibitively complicated. Invariant methods, if they were available, would certainly be more useful because one expects they would remove the clutter of approximation steps and coordinate conditions completely. They would probably either be completely inapplicable, or would yield a result by straight forward calculation.

The best - and so far the only - invariant procedures are applicable when the space under consideration has a symmetry. Riemannian spaces which possess a symmetry are those in which a coordinate system may be found with the metric tensor independent of one of the coordinates. They are equivalently characterized by the fact that they admit a solution ξ^a to Killing's equation ^{(1), (2)}

$$\mathcal{L}_{\xi} g_{AB} \equiv 2 \xi^C (A \parallel B) = 0. \quad (4.1)$$

The preferred coordinate system mentioned above is obtained by picking coordinates such that $\xi^A \equiv \left(\frac{\partial}{\partial x^0} \right)^A = \delta_{(0)}^A$. The metric is then clearly independent ⁽³⁾ of x^0 .

Practically every calculation is simplified when the space admits a Killing field, and correspondingly, when there is no Killing field, the sheer calculational difficulties multiply. For instance, calculation of the effects of small deviations from exact symmetry in cosmological solutions must often be treated in an approximate manner.

We have included as Appendix C a paper (coauthored with C. W. Misner) on the field equations for vacuum spaces with stationary axial symmetry. (These spaces have two Killing vectors describing their axial symmetry and time independence.) The simplifications because of the Killing fields are tremendous. As an example of the simplification available when effective use is made of the existence of Killing vectors, one should note the simplicity of the derivation of the field equations given in Appendix C. This

can be compared with the very tedious algebra of calculating the field equations directly from the classical formulae, starting from the metric form of equation (C.11). The classical methods, although greatly simplified because neither the coordinate ϕ nor t appears, cannot take full advantage of the simplifications available when there is a Killing field. When there is no Killing field, and the classical methods or equivalent techniques are the only ones applicable, the difficulties are truly formidable. (This is another place where the electronic computational methods described in Appendix B can be useful).

Conserved quantities arise from Killing vectors and the vacuum field equations in the way sketched in Appendix C. Komar in particular (1962) has discussed methods of weakening the Killing equations to admit spaces which are not symmetric, in order to take advantage of techniques analogous to those available when there is a Killing field. Although there are clearly some characteristics of Killing fields which depend on the Killing property, some of the uses of Killing fields, in particular their use to define "conserved" quantities, depends only on the fact that they are invariantly defined vector fields. (The "conserved" quantities defined by plugging an arbitrary vector field into Komar's formulae may be formally conserved, but may yield no useful conserved quantities.) Komar has suggested the following weakening of the Killings equations (to "semi-Killing" fields)

$$\xi^\alpha \xi_{(\alpha} \xi_{\beta)} = 0 \quad , \quad (4.2)$$

$$\xi^\alpha_{;\alpha} = 0 \quad . \quad (4.3)$$

He always demands that the field tend to Killing at infinity.

Another suggestion by Komar is that the vector field ξ^α is proportional to $t_{,\alpha}$ where $t = \text{constant}$ is a spacelike surface and t satisfies the equation

$$t^{;\mu} t_{;\nu} t_{;\mu}{}^{;\nu} = t^{;\mu} t_{;\mu} t_{;\nu}{}^{;\nu} \quad , \quad (4.4)$$

i.e., the constant t hypersurfaces are minimal. This has the advantage that it may be possible to discuss the existence of global solutions to eq. (4.4), since it is an elliptic equation. (Misner, 1967, private communication). However, in some situations, e. g. spherically symmetric collapse, Misner⁽¹⁹⁶³⁾ has shown that the Komar object defined by using this minimal field (as a substitute for the energy which is defined for stationary situations) results in a quantity which is not conserved, and corresponds to the Schwarzschild mass only when the constituent matter of the collapsing object is completely dispersed. (Note that the Schwarzschild mass is well defined in the spherical case.)

While the other suggestion of Komar-- that the field be semi-Killing-- seems to be a fairly weak requirement, there is no geometrical justification for the equations, in contradiction to the situation for the minimal fields, and for the vector fields defined in this work in the next chapter.

Chapter V. Almost Symmetric Spaces

a) Definitions and Interpretation

Our discussion will initially be in terms of positive definite manifolds, but we indicate the generalization to spaces of Minkowski signature. The treatment will be general in the sense that we will not have to assume the deviation from symmetry is small, although we may do so at times to make

interpretation easier.

Our definition for spaces which are not symmetric and hence have no non-trivial solution to the equation $\sum (A|B) = 0$ is the following. We characterize the amount of symmetry in a (positive definite) Riemannian space M by considering the minimum possible value of the expression

$$0 \leq \lambda \left[\sum \xi \right] = \frac{\int \sum (A|B) \sum \xi}{\int \sum A \sum A} dV \quad ; \quad dV \equiv \sqrt{g} d^m x. \quad (5.1)$$

Here $\sum \xi$ is an arbitrary vector field, and the quantity λ is the ratio of integrals of scalar fields over the space. Since the metric is positive definite, $\lambda \left[\sum \xi \right]$ is zero iff $\sum \xi$ is Killing. We have imposed the normalization condition in (5.1), dividing by the integral of the squared length of the vector, to exclude zero fields which are always solutions of Killing's equation. We shall take as our criterion for the "almost Killing" field that it minimize λ compared to all other choices of the vector field. Objects like the right side of (5.1) may have more than one stationary point, so we emphasize that the most interesting value of λ and the correspondingly most interesting vector field associated with it are λ_0 , the smallest stationary value, and $\sum \xi_0$, the "ground state" vector field.

By standard arguments, assuming the compactness of M , or restricting the class of vector fields to allow neglect of surface terms at infinity, the variational problem defined in equation (5.1) is the same as the problem of finding the eigenvalues λ in the equation

$$\sum (A|B)_{|B} + \lambda \sum A = 0, \quad (5.2)$$

and λ_0 corresponds to the smallest (for positive definite spaces) of these

The ground state vector field $\frac{\partial}{\partial x^0}$ may be characterized in the following way. In a coordinate system in which $\frac{\partial}{\partial x^0} = \delta_{(0)}^A$, then

$$g_{00} = \frac{\partial}{\partial x^0} \cdot \frac{\partial}{\partial x^0}, \text{ and}$$

$$\frac{dg_{AC}}{\sqrt{g_{00}} dx^0} \equiv \frac{dg_{AC}}{dS} = \frac{\partial}{\partial x^0} \frac{(A|1|C)}{\sqrt{g_{00}}} \quad (5.3)$$

Here dS is an element of proper length along $\frac{\partial}{\partial x^0}$. It is apparant that (5.1) is an integral "average square" of this quantity, but the "average" of the ratio is given by the ratio of the averages of the numerator and denominator. Even though we have a small eigenvalue λ_0 , it is difficult to use λ_0 to put bounds on the quantity (5.3), since $\frac{\partial}{\partial x^0}$ may vanish at some points for global topological reasons (for instance nonsingular vector fields on a sphere must vanish somewhere).⁴ The integral average (5.1) thus forces us to accept behaviour which is locally rapid (e.g. schematically g_{AB} a smooth curve with a few kinks) as being smooth in a global sense. On the other hand, the vector field defined by equation (5.1) obviously chooses the coordinate lines for x^0 which give the slowest dependence of g_{AB} on x^0 , in some global sense.

For the moment assuming the existence of solutions, we can get an upper bound for the quantity λ_0 . By definition λ_0 is the minimum value of the integral (5.1) so any test function gives a bound. By considering a geodesic patch of radius L at some point in \mathcal{M} , and by taking a test field:

$$\Psi_A = \delta_A^{(0)} [1 - 2L^{-1}(\sum x^A x^A)^{1/2}] \text{ for } \sum x^A x^A \leq \frac{1}{4} L^2,$$

$$\Psi_A = 0 \text{ otherwise,}$$

it is easy to estimate that $0 \leq \lambda[\Psi] \leq 2L^{-2}(m+1)(m+2)$, where m is the dimension of the space. [Thus, on the surface of a cube of edge length l

for instance, the maximum possible L is $2/\sqrt{2}$ and $\lambda_0(\text{cube}) \leq 2 \cdot 2 \cdot 2^{-2} \cdot 3 \cdot 4 = 48 \cdot 2^{-2}$]. The size of a geodesic coordinate patch is roughly given by $L^{-2} \sim R \dots$ (the Riemann tensor), so we have a rough bound for λ_0 in terms of the curvature.

It is important to note that estimates of this type hold for the eigenvalue λ_0 in any space. The idea of "almost symmetric" enters when it turns out that $\lambda_0 \ll L^{-2}$, where L is a typical curvature length of the problem. We will present some examples of this type of behaviour in Chapters VI and VII below, but a simple example is the unit 2-sphere, where $\lambda_0 = 0$, while the only available length is the radius (=1).

b) Existence and Differentiability for Closed Positive Definite Metric Spaces.

For the rest of this paper, except where noted otherwise we will assume that we are working with a positive definite Riemannian C^∞ manifold \mathcal{M} , and we assume that \mathcal{M} is compact or that boundary conditions are chosen so that integration by parts is possible with neglect of surface terms. We denote the Hilbert space of all square integrable vector fields $\underline{\xi}$ on \mathcal{M} by $\mathbb{L}(\mathcal{M})$. The norm is

$$\|\underline{\xi}\|_{\mathcal{M}} = \left(\int dV \underline{\xi} \cdot \underline{\xi} \right)^{1/2} < \infty.$$

The demand that $\lambda[\underline{\xi}]$ be stationary yields, as usual, a second order equation:

$$\begin{aligned} \delta \left[\lambda \int \xi^A \xi_A dV \right] &= \delta \left[\int \xi^{(A|B)} \xi_{(A|B)} dV \right], \\ 2\lambda \int \xi^A \delta \xi_A dV &= 2 \int \xi^{(A|B)} \delta \xi_{A|B} dV \\ &= 2 \int \left(\left[\xi^{(A|B)} \delta \xi_{A|B} \right]_{|B} - \xi^{(A|B)}_{|B} \delta \xi_A \right) dV. \end{aligned}$$

The compactness of \mathcal{M} (or the boundary condition at infinity) makes the first term on the right vanish. Then, since $\delta \xi_{A|B}$ is an arbitrary variation, we find

$$\xi^{(A|B)}_{|B} + \lambda \xi^A = 0. \quad (5.2)$$

We will take the definitions (5.1) and (5.2) to be the defining equations for the preferred vector fields in \mathcal{M} . Equation (5.2) is the generalization (because of the term λ) of the second order equation equivalent to Killing's equation given by Yano and Bochner (Ref. 2, p.57). It is clear that a solution to (5.2) with $\lambda = 0$ is Killing and vice versa. In Minkowskian signature

metrics, the stationarity of (5.1) still implies (5.2) but the equivalence of (5.2) for $\lambda=0$ to Killing's equation no longer holds.

The derivation of equation (5.2) shows that the operator

$$-\frac{D}{Dx^B} g^{F(BA)} \delta_c \frac{D}{Dx^F} = -\mathcal{D} \zeta^A \quad (5.4)$$

is positive. (The notation is $\frac{D}{Dx^B} \xi^C \equiv \xi^C \parallel_B$.) It is positive definite if there are no Killing vectors or if we exclude them. Also, because of the compactness of \mathcal{M} , or the boundary conditions at infinity, \mathcal{D} is self adjoint on $\mathcal{L}(\mathcal{M})$.

It is clear that there are at least as many solutions to (5.2) as there are Killing vectors. We are of course interested in the case where there are solutions which are not Killing vectors. Consider only the subspace $\mathcal{L}'(\mathcal{M}) \subset \mathcal{L}(\mathcal{M})$ which is orthogonal to the finite number of Killing vectors in \mathcal{M} . The operator $-\mathcal{D}$ is then positive definite and in fact is strongly elliptic.⁶ We may then apply the theorem quoted by Kodaira and Spencer (Ref. 6, Theorem I) for compact \mathcal{M} to find that $-\mathcal{D}$ has a complete countable set of differentiable eigenfunctions e_n with real eigenvalues whose only accumulation point is $+\infty$. (The completeness means, if Ψ differentiable, $\Psi \in \mathcal{L}'(\mathcal{M})$ then $\Psi = \sum_{n=1}^{\infty} a_n e_n$ where $a_n = \int dV \Psi \cdot e_n$, and the series converges in $\mathcal{L}'(\mathcal{M})$.)

Thus we have all the expected "nice" properties of the operator $-\mathcal{D}$ on the compact manifold \mathcal{M} . In particular, we know that a differentiable ground state solution ψ_0 exists. On compact manifolds, then, there will be uniform bounds for the quantities $\int_{\mathcal{M}} \psi_0 \cdot \psi_0$ and $\int_{\mathcal{M}} \psi_0 \psi_0^{(A||B)}$ and for all the other derivatives of ψ_0 .

The quadratic form in (5.1) may be written

$$\int_{\mathcal{M}} \psi_0^{(A||B)} \int_{\mathcal{M}} \psi_0^{(A||B)} = \int_{\mathcal{M}} \psi_0^{(A||B)} \int_{\mathcal{M}} \psi_0^{(A||B)} \int_{\mathcal{M}} \psi_0^{(A||B)} \int_{\mathcal{M}} \psi_0^{(A||B)}$$

$$\int_{\mathcal{M}} \psi_0^{(A||B)} \int_{\mathcal{M}} \psi_0^{(A||B)} = \frac{1}{2} \left(g^{AM} g^{BN} + g^{AN} g^{BM} \right) \int_{\mathcal{M}} \psi_0^{(A||B)} \int_{\mathcal{M}} \psi_0^{(A||B)} \quad (5.5)$$

where

The most general positive definite form with these symmetries which depends only on the metric is

$$C^{ABMN} + \mu g^{AB} g^{MN}, \quad \mu > -\frac{1}{2}.$$

Although the μ -addition is non-negative, even in spaces of Minkowski signature \uparrow for $\mu > 0$, the equation analogous to (5.2)

$$\sum_{\parallel B} A^{\parallel B} + \sum_{\parallel B} B^{\parallel A} (1 + \mu) + (R^A_c + \lambda \delta^A_c) \xi^c = 0, \quad (5.6)$$

is not qualitatively different from (2). Dealing only with the μ term gives conditions only on the divergence of $\sum_{\parallel B} A^{\parallel B}$, and allows too many solutions. If we have any C^{ABMN} contribution, then the equation is qualitatively like (5.2) and the only criterion for the choice of μ seems to be aesthetics, which suggests $\mu = 0$, as we take here.

c) An Analogue to Elasticity Theory.

The tensor C^{ABMN} defined in (5.5) is formally similar to the elasticity strain coefficients given by Green and Zerna⁷ for isotropic elasticity in a uniform medium (with Poisson ratio identically zero because we set $\mu = 0$). The similarity of the equations to an elasticity theory is no accident. In elasticity, the strain components $u_{(i|k)}$ measure the Lie derivature of the metric along the displacement field \underline{u} . This can be seen physically in a coordinate system such that there is no relative coordinate velocity between particles. (This means that the field u^i must have constant components in these coordinates.) Then the metric gives the distance between particles, and the strain tensor is $g_{ij, \alpha} u^\alpha$, which is $L_u g_{ij}$ in this coordinate system.

The minimization problem set here is in fact completely analogous to the eigenvalue problem for vibrations of closed elastic shells, under the boundary conditions of sliding rigid contact (the type of boundary condition at the interface

between a turning shaft and an immobile bearing).

To see this, consider a thin shell of uniform density, and which is described by a middle surface $\mathcal{R}(\theta^1, \theta^2)$, and at each point a thickness $2h(\theta^1, \theta^2)$ measured along the normal $\underline{m}(\theta^1, \theta^2)$ at each point. The surfaces of the shell are at $\pm h$. We assume linear isotropic elasticity and define an ordinary Cartesian frame X^a (labelled by latin indices from the beginning of the alphabet), and an intrinsic coordinate system adjusted to fit the shell. The intrinsic system will use as coordinates the θ^1, θ^2 parameters giving the middle surface and θ^3 , defined as the distance along the normal to the middle surface. Every point on the same normal $\underline{m}(\theta^1, \theta^2)$ will have the same coordinates θ^α ; its coordinate θ^3 will be the distance along the normal to the middle surface. (We will have Greek indices running and summing over 1 and 2.)

The middle surface will have some metric form $\bar{g}_{\alpha\beta}$ which gives the formula for length $d\ell$ in terms of the coordinate differentials: ⁽¹⁾

$$d\ell^2 = \bar{g}_{\alpha\beta} d\theta^\alpha d\theta^\beta \quad (\text{in the middle surface}).$$

Let

$$K_{\alpha\beta} = -(\underline{m})_{\alpha;\beta}$$

be the second fundamental form of the middle surface. (The covariant derivatives denoted by a colon are in the 3-space endowed with the metric $\bar{g}_{\alpha\beta}$, $\bar{g}_{\alpha 3} = 0$, $\bar{g}_{33} = 1$ and connection $\bar{\Gamma}_{\kappa\lambda}^j = \frac{1}{2} \bar{g}^{ij} (\bar{g}_{ik,\lambda} + \bar{g}_{i\lambda,k} - \bar{g}_{\lambda k,i})$; $K_{\alpha\beta}$ so defined is a 2-tensor defined on the middle surface.) The metric throughout the entire finite thickness can now be written (exactly): ⁽⁶⁾

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} - 2\theta^3 K_{\alpha\beta} + (\theta^3)^2 K_\alpha{}^\lambda K_{\lambda\beta},$$

$$g_{\alpha 3} = 0, \quad g_{33} = 1.$$

Here $K_\alpha{}^\lambda = \bar{g}^{\lambda\sigma} K_{\alpha\sigma}$. We will denote the covariant derivatives with respect to this flat space metric also by a colon, since $g_{ij} = \bar{g}_{ij}$ on the middle surfaces.

i) The Natural Frequencies

We now turn to the eigenvalue problem:

$$\tau^{ab}_{,b} + \omega^2 \xi^b = 0, \quad (5.7)$$

for a finite thickness shell, which is equivalent to finding the stationary points of the quantity

$$\lambda[\xi] = \frac{\int \tau^{ab}(\xi) \xi_{a,b} dV}{\int \xi^a \xi_a dV}. \quad (5.8)$$

Here $\tau^{ab}(\xi)$ is the three dimensional stress tensor, and the comma denotes partial derivative.

In intrinsic coordinates, the differential equation (5.7) is

$$\tau^{ij}_{;j} + \omega^2 \xi^i = 0. \quad (5.9)$$

Consider only the shell component of this: $i = \alpha$. Our assumption of sliding contact makes the $i = 3$ component of (5.9) a constraint equation which gives the normal forces in the motion. We will assume it is satisfied; the existence of a solution is necessary for what follows, but its explicit form is not.

Writing equation (5.9) out in full we find

$$\begin{aligned} \tau^{\alpha\beta}_{, \beta} + \Gamma^{\alpha}_{\beta\sigma} \tau^{\beta\sigma} + \Gamma^{\beta}_{\sigma\beta} \tau^{\alpha\sigma} + \omega^2 \xi^{\alpha} \\ + \tau^{\alpha 3}_{, 3} + 2 \Gamma^{\alpha}_{\beta 3} \tau^{\beta 3} + \Gamma^{\alpha}_{3 3} \tau^{33} + \Gamma^3_{\alpha 3} \tau^{\alpha k} = 0. \end{aligned} \quad (5.10)$$

The Christoffel symbols Γ^i_{jk} are defined as

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{kl,j})$$

In the intrinsic coordinate system, $\Gamma^{\alpha}_{33} = \Gamma^3_{\alpha 3} = 0$. Of the

remaining quantities in (5.10), $\tau^{\alpha 3}$ must tend to zero with h , since otherwise our boundary condition that $\tau^{\alpha 3} = 0$ at $\theta^3 = \pm h$ would mean the term $\tau^{\alpha 3},_3$ would become infinite as $h \rightarrow 0$. The other quantities in (5.10) are finite, and so we conclude that any finite frequency solution will have $\tau^{\alpha 3} \rightarrow 0$ as $h \rightarrow 0$.

At this point we must explicitly consider the form of the stress tensor for isotropic elasticity. It is $\tau^{ij} = \mu'(g^{ir}g^{js} + g^{is}g^{jr} + \frac{2\eta}{1-2\eta}g^{ij}g^{rs})\xi_{(r,s)}$, (5.11) where μ' is the shear modulus and η is Poisson's ratio.

Consider the 3-3 component of equation (5.11):

$$\tau^{33} = \mu' \left(2g^{33}g^{33} + \frac{2\eta}{1-2\eta}g^{33}g^{33} \right) \xi_{3,3} + \frac{2\eta\mu'}{1-2\eta}g^{33}g^{\alpha\beta}\xi_{\alpha,\beta}$$

Now $\xi_{3,3} = \xi_{3,3} - \Gamma^k_{33}\xi_k = \xi_{3,3}$. So we see that $\xi_{3,3}$ must vanish in the limit as $h \rightarrow 0$, or else the eigenvalue, λ [computed via the covariant form of (5.8), say], will become infinite, since the term $(\xi_{3,3})^2$ will occur, and we require that $\xi_3 = 0$ on the surfaces of the shell.

The result on ξ_3 and $\tau^{\alpha 3}$ was obtained by requiring that the corresponding frequency stay finite. We will give examples of vector fields in subsection (V.c.iii) below which do in fact have finite λ and thus bound finite frequency eigensolutions. Physically, we are excluding in this way shear and longitudinal waves which reflect back and forth between the two bounding surfaces of the shell. These waves have velocity $c_s \sim \sqrt{\mu'}$ and $c_l \sim [2\mu'(1+\eta)/(1-2\eta)]^{1/2}$ respectively. Thus the frequencies corresponding to such motions reflecting between the shell surfaces are of the order $\omega_e \sim c/h$. So our shell theory equations and conclusions only hold for $\omega^2 \ll \omega_e^2$. This is a restriction on either the highest frequency or on the largest thickness we can legitimately consider by shell methods. We shall see, however, that there are a countably infinite number of eigensolutions with frequencies independent of h for small h , so we can always carry the

discussion of the spectrum in terms of the shell quantities arbitrarily far if h is small enough.

ii) Reduction to Surface Tensors

To obtain surface equations we use the form (5.11) for $\tau^{\alpha\beta}$ and integrate (5.10) through the thickness of the shell. With the simplifications we have found so far, we obtain, for $h \rightarrow 0$

$$0 = h^{-1} \int_{-h}^h \sqrt{\theta/g} (\tau^{\alpha\beta}_{, \beta} + \Gamma^{\alpha}_{\beta\sigma} \tau^{\beta\sigma} + \Gamma^{\beta}_{\sigma\beta} \tau^{\alpha\sigma} + \omega^2 \xi^{\alpha}) d\theta^3. \quad (5.12)$$

Here g is $\det g_{ij}$, $\bar{g} \equiv \det \bar{g}_{\alpha\beta} = \det \bar{g}_{ij}$. The components of $\tau^{\alpha\beta}$ contain terms [see eqn. (5.11)]

$$\frac{2\eta\mu'}{1-2\eta} g^{\alpha\beta} g^{33} \xi_{3,3};$$

the integral of the term vanishes as $h \rightarrow 0$, for

$$\int_{-h}^h \sqrt{g} g^{\alpha\beta} \xi_{3,3} d\theta^3 = \int_{-h}^h (\sqrt{g} g^{\alpha\beta} \xi_{3,3})_{,3} d\theta^3 - \int (\sqrt{g} g^{\alpha\beta})_{,3} \xi_3 d\theta^3 \rightarrow 0 \text{ since } \xi_3 \rightarrow 0.$$

[The term $\int_{-h}^h \sqrt{g} \tau^{\alpha 3}_{,3} d\theta^3$, which was omitted in (5.12), vanishes by a similar argument.] We will denote the quantity which is $\tau^{\alpha\beta}$ with the $\xi_{3,3}$ term deleted by $\tau'_{\alpha\beta}$, and define $C^{(\alpha\beta)(\lambda\sigma)}$ by $\tau'^{\alpha\beta} = C^{(\alpha\beta)(\lambda\sigma)} \xi_{(\lambda;\sigma)}$, and write $\bar{C}^{\alpha\beta\lambda\sigma} \equiv C^{\alpha\beta\lambda\sigma}|_M$, (the restriction of the middle surface).

Define a surface vector $\bar{\xi}^{\alpha}$:

$$\bar{\xi}^{\alpha} = h^{-1} \int_{-h}^h \xi^{\alpha} \sqrt{\theta/g} d\theta^3 \quad (\text{with } \bar{\xi}^{\alpha} \equiv \lim_{h \rightarrow 0} h^{-1} \int_{-h}^h \xi^{\alpha} \sqrt{\theta/g} d\theta^3)$$

and a surface tensor

$$h \bar{\tau}^{\alpha\beta} = h^{-1} \int_{-h}^h \tau'^{\alpha\beta} \sqrt{\theta/g} d\theta^3.$$

Now (using the continuity of g_{ij} and its derivatives)

$$\begin{aligned} \bar{\tau}^{\alpha\beta} &\equiv \lim_{h \rightarrow 0} h \bar{\tau}^{\alpha\beta} \\ &= \bar{C}^{\alpha\beta\lambda\sigma} \cdot \lim_{h \rightarrow 0} h^{-1} \int_{-h}^h \sqrt{\theta/g} (\xi_{\lambda;\sigma} - \Gamma^{\alpha}_{\lambda\sigma} \xi_{\alpha} - \Gamma^{\beta}_{\lambda\sigma} \xi_{\beta}) d\theta^3, \end{aligned}$$

and

$$\begin{aligned} \lim_{h \rightarrow 0} h^{-1} \int_{-h}^h \sqrt{g/g} (\bar{\Sigma}_{\lambda, \sigma} - \Gamma_{\lambda \sigma}^{\alpha} \bar{\Sigma}_{\alpha} - \Gamma_{\lambda \sigma}^3 \bar{\Sigma}_3) d\theta^3 \\ = \bar{\Sigma}_{\lambda, \sigma} - \bar{\Gamma}_{\lambda \sigma}^{\alpha} \bar{\Sigma}_{\alpha} \end{aligned}$$

where we denote by $\bar{\Gamma}_{\lambda \sigma}^{\alpha}$ the Christoffel symbol in the middle surface, we have used the fact that $\bar{\Sigma}_3 \rightarrow 0$ as $h \rightarrow 0$, and we have assumed that the derivatives of h become negligible as the thickness vanishes.⁹

Thus

$$\begin{aligned} \bar{\Sigma}^{\alpha \beta} &= \mu' (\bar{g}^{\alpha \sigma} \bar{g}^{\beta \lambda} + \bar{g}^{\beta \sigma} \bar{g}^{\alpha \lambda} + \frac{2\eta}{1-2\eta} \bar{g}^{\alpha \beta} \bar{g}^{\lambda \sigma}) \bar{\Sigma}(\lambda, \sigma), \\ &= \bar{C}^{\alpha \beta \lambda \sigma} \bar{\Sigma}(\lambda, \sigma), \end{aligned} \quad (5.13)$$

where the slash denotes the covariant derivative in the middle surface.

The integrated equation (5.12) then becomes, as $h \rightarrow 0$,

$$\begin{aligned} 0 &= \omega^2 \bar{\Sigma}^{\alpha} + \bar{\Sigma}^{\alpha \beta}{}_{, \beta} + \lim_{h \rightarrow 0} h^{-1} \int_{-h}^h (\Gamma_{\beta \sigma}^{\alpha} \bar{\Sigma}^{\beta \sigma} + \Gamma_{\sigma \beta}^{\beta} \bar{\Sigma}^{\alpha \sigma}) \sqrt{g/g} d\theta^3, \\ &= \omega^2 \bar{\Sigma}^{\alpha} + \bar{\Sigma}^{\alpha \beta}{}_{, \beta} \end{aligned} \quad (5.14)$$

with a surface stress tensor $\bar{\Sigma}^{\alpha \beta}$ given by (5.13). Clearly equation (5.14) is equivalent to a two dimensional integral problem: the stationary points of

$$\lambda = \frac{\int \bar{\Sigma}^{\alpha \beta} \bar{\Sigma}(\alpha, \beta) \sqrt{g} d\theta^1 d\theta^2}{\int \bar{\Sigma}^{\alpha} \bar{\Sigma}_{\alpha} \sqrt{g} d\theta^1 d\theta^2}. \quad (5.15)$$

We note that if we substitute the 2-dimensional version of eq. (5.5) into eq (5.13) for $\bar{\Sigma}^{\alpha \beta}$, we get precisely the 2-dimensional symmetry problem equations from (5.14) and (5.15).

iii) Finite Frequency Vector Fields

To complete this discussion we show that there are some vector fields which have finite frequency in the limit $h \rightarrow 0$. For take $\bar{\Sigma}_3 = 0$, and

at each point of the middle surface define ξ_α arbitrarily, but define ξ_α away from the middle surface by

$$\xi_{\alpha,3} = \frac{1}{2} \Gamma^\sigma_{\alpha 3} \xi_\sigma$$

Then $\xi_{(3;3)} = \xi_{(\alpha;3)} = 0$, and the minimizing integral [the covariant form of equation (5.8)] is

$$\lambda_h = \frac{\int \bar{c}^{\alpha\beta\lambda\sigma} \xi_{(\alpha;\beta)} \xi_{(\lambda;\sigma)} d\theta^3 d\theta^1 d\theta^2 \sqrt{\bar{g}}}{\int \xi_\beta \xi_\alpha \bar{g}^{\alpha\beta} d\theta^3 d\theta^1 d\theta^2 \sqrt{\bar{g}}}$$

which clearly has a finite limit independent of h (since only the metric terms involve θ) as $h \rightarrow 0$:

$$\lim_{h \rightarrow 0} \lambda_h = \frac{\int \bar{c}^{\alpha\beta\lambda\sigma} \bar{\xi}_{(\alpha;\beta)} \bar{\xi}_{(\lambda;\sigma)} d\theta^3 d\theta^1 d\theta^2 \sqrt{\bar{g}}}{\int \bar{\xi}_\beta \bar{\xi}_\alpha \bar{g}^{\alpha\beta} d\theta^3 d\theta^1 d\theta^2 \sqrt{\bar{g}}} < \infty \quad (5.16)$$

This limit is the same as the quantity given by equation (5.15).

iv) Completeness as $h \rightarrow 0$

We have quoted in Section V.b. a theorem which proves the existence and completeness of solutions to equation (5.2). We give here a heuristic verification of the completeness of the shell solutions eigensolutions of (5.14) and a discussion of the approach to this completeness as the shell thickness vanishes. Suppose $\bar{\xi}_\alpha$ is defined on the middle surface. We can

define an associated vector field ζ_i defined in the finite thickness shell by: $\zeta_3 = 0$, $\zeta_\alpha|_h = \bar{\xi}_\alpha$

and $2\zeta_{\alpha,3} = \Gamma^\sigma_{3\alpha} \zeta^\sigma$. This vector field then has $\zeta_{(3;3)} = \zeta_{(3;\alpha)} = 0$.

We can expand ζ in terms of the complete set of eigensolutions of the finite thickness shell. We have shown [equation (5.16)] that the quantity $\lambda_h[\zeta]$ associated with this vector field has a finite limit, $\lambda[\bar{\xi}]$ as $h \rightarrow 0$.

Consequently, when ζ is expressed in terms of the normalized complete set $\{E_{(m,h)}\}$ for the finite thickness shell,

$$\zeta = \sum_m a_{(m,h)} E_{(m,h)} \quad (5.17)$$

we have the energy

$$\lambda_h [\xi] = \sum_m a_{(m,h)}^2 \lambda_{(m)h} < \infty. \quad (5.18)$$

We have seen above that vector fields \sum_m which have a non-zero third component \sum^3 or a non-zero stress component $\tau^{\alpha 3}$ have corresponding frequencies which diverge at least like h^{-1} as $h \rightarrow 0$. Thus, the coefficients which multiply such fields in equation (5.18) must vanish at least as $h^{1/2}$. Thus, as $h \rightarrow 0$ the expansion is entirely in terms of eigen-solutions with \sum^3 and $\tau^{\alpha 3}$ vanishing. In this limit, we find, integrating equation (5.17) through the shell thickness,

$$\bar{\xi} = \sum_m a_m \bar{E}_{(m)}$$

the completeness relation in terms of the shell eigenfunction. The physically interesting point is that finite frequency motions become motions only in surfaces which are parallel to the middle surface, and with no shear between such surfaces. This means that in the limit of thin shells with these boundary conditions, every point in the shell describes a motion given by the surface equation (5.14) and in addition, all finite λ motions are described by expansions in the eigenfunctions of equation (5.14).

An example of the situation we visualize is a closed 2-surface whose symmetry we wish to measure. We form a frictionless elastic shell over the surface (with 2-dimensional Poisson ratio $\equiv 0$ since $\mu=0$), with the shell initially unstrained so that it resists both compression and expansion. Then the asymmetry of the object is measured by the square of the fundamental oscillation frequency if we perturb the shell. If it has a neutral mode, the surface has a Killing vector.

That shell completeness follows from 3-space completeness has just been shown and we should note that all these results on the solutions for 2-dimensional shells can be generalized for any finite number of dimensions. Furthermore any Riemannian manifold can be embedded in a Euclidean space of

sufficiently high dimension (Nash 1956). And the existence and differentiability and completeness of solutions for elasticity has been carried out for arbitrary number of dimensions. This is a sketch for an alternate heuristic proof of the theorem of Kodaira and Spencer quoted in Section (V.b), for the existence and completeness of solutions to equations (5.1) or (5.2).

d) A Theorem of Yano

After the qualitative discussion of the preceding section, we give a precise result. In positive definite metric spaces the eigenvalue λ_0 is clearly non-negative. But it is possible to obtain a better lower bound in some cases by noting the following.

We have

$$\int \nabla_{(A|B)} \nabla_{(A|B)} dV = - \int \nabla_{(A|B)} \nabla_{(A|B)} dV + \int (\nabla_{(A|B)} \nabla_{(A|B)})_{|B} dV.$$

The second term on the right vanishes by the compactness of the space or by the boundary conditions at infinity. Further, the integrand of the first term on the right is

$$\nabla_{|B}^{A|B} + \nabla_{|B}^{B|A} = \nabla_{|B}^{A|B} + \nabla_{|B}^{B|A} + R^A{}_C \nabla^C.$$

Thus, if the space is compact or if we impose stronger than usual conditions on the vanishing of $\nabla^A_{|B}$ at infinity (note this is an unsymmetrized derivative):

$$\int \nabla_{(A|B)} \nabla_{(A|B)} dV = \int \nabla_{(A|B)} \nabla_{(A|B)} dV + \int \nabla^A_{|A} \nabla^B_{|B} dV - \int R^A{}_C \nabla^C \nabla^B_{|B} dV.$$

Consequently, in positive definite metric spaces,

$$\int (\lambda \delta^A{}_C + R^A{}_C) \nabla^C \nabla^B_{|B} dV \geq 0.$$

This holds for any vector $\nabla_{|B}$ and the associated $\lambda[\nabla_{|B}]$. In particular,

$$\lambda_0 \int \nabla_{|B} \nabla^B_{|B} dV \geq \int R^A{}_C \nabla^C \nabla^B_{|B} dV.$$

This is an improved bound in those cases where $R^A{}_C$ is a negative definite quadratic form on the manifold.

$$\nabla^A R^C{}_A \nabla_C \leq -\lambda_{\text{Ricci}} \nabla^A \nabla_A$$

for some positive number λ_{Ricci} and for all vectors $\nabla_{|B}$ and all points of the manifold. (With this sign convention a hyperboloid has constant

negative definite R_A^c .)

Thus we have the lower bound for λ :

$$\lambda \geq \lambda_{\text{Ricci}} . \quad (5.19)$$

This derivation is a generalization of that of Yano (Ref 2, p.39) to prove that there are no Killing vectors on compact manifolds if R_A^A is a negative definite quadratic form. The advantage of the present formulation is that it gives a criterion, λ , of the deviation from symmetry.

e) Null Killing Tensors

In spaces of Minkowski signature, the quantity $\xi^{(\alpha;\beta)}_{\xi(\alpha;\beta)}$ may become negative, or may be zero even when $\xi^{(\alpha;\beta)}$ is not zero. This complicates the application of the methods described here to simple exact solutions to Einstein's equations, such as those solutions due to Schwarzschild, Kerr (1963), and Vaidya.^{12,13} In each of these solutions, there are vectors which are not null but whose symmetrized derivative is a null tensor. They are the Killing vectors in the flat space $b_{\alpha\beta}$ which is a base metric for each of these solutions, in the sense of equation (5.20) below.

In fact, each of these solutions is a member of a general class of metrics which can be written

$$ds^2 = (b_{\alpha\beta} + k_\alpha k_\beta) dx^\alpha dx^\beta \equiv g_{\alpha\beta} dx^\alpha dx^\beta, \quad (5.20)$$

where $b_{\alpha\beta}$ is some background and k_α is null in the background: $b^{\alpha\beta} k_\alpha k_\beta = 0$, and hence is also null in the full metric. Suppose $\xi_{\underline{\mu}}$ is Killing in the background. Then there is a coordinate frame such that $\xi_{\underline{\mu}}$ has constant (contravariant) components and $b_{\alpha\beta, \rho} \xi^{\rho} = 0$. Then in this coordinate system,

$$2 \xi_{(\alpha;\beta)} = \mathcal{L}_{\xi_{\underline{\mu}}} k_\alpha k_\beta = (k_\alpha k_\beta)_{, \rho} \xi^{\rho},$$

and, since

$$g^{\alpha\gamma} = b^{\alpha\gamma} - k^\alpha k^\gamma$$

and

$$\begin{aligned} k^{\alpha, \rho} k_\alpha \xi^{\rho} &= (b^{\alpha\sigma} k_{\sigma, \rho} k_\alpha) \xi^{\rho} \\ &= (k_{\sigma, \rho} k^\sigma) \xi^{\rho} \\ &= \frac{1}{2} (k^\alpha k_\alpha)_{, \rho} \xi^{\rho}, \end{aligned}$$

the square of $2 \zeta_{(\alpha; \beta)}$ is

$$b^{\alpha\delta} b^{\beta\lambda} (k_\alpha k_\beta)_{,p} \zeta^p (k_\gamma k_\lambda)_{,\sigma} \zeta^\sigma = 0.$$

It is easy to see that $\zeta_{(\alpha; \beta)}$ has vanishing trace also, so the possibility suggested in Section V a) that adding a μ -term would lead to non-negative results is seen to be inapplicable here. Metrics like (5.20) but with $b_{\alpha\beta}$ flat have been studied by Kerr and Schild.⁽¹⁴⁾

In the case where $\zeta_{(\alpha; \beta)}$ becomes a null tensor for some null vector ζ^α , we may define a limiting process. Let γ^M be a (so far arbitrary) timelike vector. Then define $\eta^M = \zeta^M + \epsilon \gamma^M$, $\epsilon > 0$;

$$\eta_{(\alpha; \beta)} \eta^{(\alpha; \beta)} = \zeta_{(\alpha; \beta)} \zeta^{(\alpha; \beta)} + 2\epsilon \gamma_{(\alpha; \beta)} \zeta^{(\alpha; \beta)} + \epsilon^2 \gamma_{(\alpha; \beta)} \gamma^{(\alpha; \beta)}$$

The first term is zero by hypothesis, as is $\zeta_\alpha \zeta^\alpha$. Thus

$$\lim_{\epsilon \rightarrow 0} \frac{\int \eta_{(\alpha; \beta)} \eta^{(\alpha; \beta)} dV}{\int \eta^\alpha \eta_\alpha dV} = \frac{\int \gamma_{(\alpha; \beta)} \zeta^{(\alpha; \beta)} dV}{\int \gamma_\alpha \zeta^\alpha dV}.$$

Since a timelike vector is never orthogonal to a null vector, the denominator is always positive.

It is not in general clear whether the limit is unique. However, if ζ^α tends to zero sufficiently fast at infinity,

$$\frac{\int \gamma_{(\alpha; \beta)} \zeta^{(\alpha; \beta)} dV}{\int \gamma_\alpha \zeta^\alpha dV} = - \frac{\int \gamma_\alpha \zeta^{(\alpha; \beta)}_{, \beta} dV}{\int \gamma^\alpha \zeta_\alpha dV} \quad (5.21)$$

If ζ^α is a solution to $\zeta^{(\alpha; \beta)}_{, \beta} + \lambda \zeta^\alpha = 0$, for some eigenvalue λ , then even if ζ^α is null, we can define the characteristic integrals by $\lambda = \text{L.H.S. of (5.21)}$ for arbitrary timelike vector γ^M which goes to zero

sufficiently fast at infinity. In this case the limit in the integral definition is independent of the timelike vector used in the limiting process, and the integral definition agrees with the eigenvalue given by the differential equation.

On the other hand, if the limit is λ , independent of the timelike vector γ^α (so long as γ^α vanishes sufficiently fast at infinity), then λ is clearly a solution of the differential equation, by (5.21). So, for even null vectors, we can define a modified integral which is equivalent to the differential equation.

Chapter VI. Examples

a) Exact Calculation on Torus for λ , $\frac{2\pi i}{\sigma}$.

The metric for a flat torus is $ds^2 = dx^2 + dy^2$, where the points $(x+1, y)$, (x, y) , and $(x, y+1)$ are identified. Differentiable functions on the torus must be doubly periodic with unit period.

We consider a curved torus with metric

$$ds^2 = e^{4\sigma(y)} dx^2 + dy^2$$

Here points are again identified as above by their coordinates, and $\sigma(y)$ is periodic in y with unit period.

This space still has a Killing vector giving translation in the x-direction. We will consider vectors which are orthogonal to the x-Killing ground state, and to simplify algebra we will in fact consider only vectors pointwise orthogonal. Then it is clear the ground state vector itself will not depend on x and we write:

$$\left(\frac{\xi}{\eta}\right)^i = \left(\frac{\xi^x}{\xi^y}, \frac{\xi^y}{\xi^y}\right) = [0, e^{-\sigma} u(y)]$$

The eigenvalue equation $\mathcal{D}^M \xi^M + \lambda \xi^M = 0$ becomes

$$f^{-1/5} \left[\frac{d^2 u}{dy^2} + \left(\lambda - \frac{1}{5f} \frac{d^2 f}{dy^2} \right) u \right] = 0, \tag{6.1}$$

with $f = e^{5\sigma}$.

We desire a "wavelike" perturbation which makes (6.1) explicitly manageable. We thus pick f so that (6.1) is a Mathieu equation. This requires that f itself be a Mathieu function; since $f^{4/5}$ is g_{xx} , we must take f to be the only nowhere zero periodic solution to Mathieu's equation, namely: ¹⁵

$$f = ce_0(ky; 5\epsilon/2) = 1 - \frac{5\epsilon}{4} \cos 2ky + \dots$$

Here k is $m\pi$, since the basic period is unity, and ϵ , which is assumed small for this expansion, is the amplitude of the small waves in the metric component.

Now f is inserted into (6.1) which itself becomes Mathieu equation. By a general theorem on the Sturm-Liouville equation¹⁶ we know that if there is a solution with no zeros, it must be the ground state, which we seek to find λ_0 . We are again forced to take a solution ce_0 , and obtain

$$u = ce_0(ky; \epsilon/40), \text{ so that}$$

$$\frac{f''}{f} = f^{-1/5} u \approx 1 + \epsilon \cdot \frac{19}{80} \cos 2ky + \dots$$

and

$$\lambda_0 \approx \frac{\epsilon^2 k^2}{2} + \dots$$

are the solutions found from the metric perturbation

$$g_{xx} \approx 1 - \epsilon \cos 2ky + \dots$$

From our arguments in Section Va, we expect $\sqrt{\lambda_0}$ to be some sort of average derivative of the metric. We see that this is the case, since

$$g_{xx,y} = -2k\epsilon \cos 2ky, \text{ so that } \langle (g_{xx,y})^2 \rangle = 2k^2\epsilon^2$$

The ϵ^2 factor in λ_0 is to be expected, because the flat torus is a space which minimizes λ , so any deviations would be like ϵ^2 . The term k^2 suggests that it is the "energy content" of the waves which determines the size of λ . This ties in with our estimates of λ made above in terms of the Riemann tensor. It is interesting to calculate the scalar R , which completely characterizes the curvature:

$$R = -4 \left(\frac{d^2\sigma}{dy^2} + 2 \left(\frac{d\sigma}{dy} \right)^2 \right)$$

For small amplitude waves,

$$\sigma \approx -\frac{\epsilon}{4} \cos 2ky \quad \text{and}$$

$$R \approx -4 \left(\epsilon k^2 \cos 2ky + \frac{1}{2} \epsilon^2 k^2 \sin^2 2ky \right).$$

The ground state thus gives a much smaller value of λ_0 (by a factor ϵ)

than our previous crude estimates yielded. On the other hand, if we average R , we have

$$|\langle R \rangle| = e^2 k^2,$$

which is twice the eigenvalue

λ . In this case, at least, the ground state eigenfunction is sampling the average (over many wavelengths) of the disturbance in the space. We shall see this is a general phenomenon, and shall meet it again in the application of these ideas to spaces containing gravitational radiation, which we take up in Chapter VII.

b) Linearized Gravitational Waves.

One example of a space with a null Killing vector is flat space with a plane wave (linearized) gravitational wave on it. We shall give a schematic derivation here which shows what happens to the Killing vector as we go from an idealized plane pulse to a situation where there is a spread of directions in a wave packet.

The metric in this situation can be gaged¹⁷ so that

$$ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta}) dx^\alpha dx^\beta$$

where

$$h_{0i} = 0, \quad h_{ab} = \int h_{ab}(\underline{k}) e^{i\underline{k} \cdot \underline{x} - i|k|t} d^3k.$$

We suppose this wave packet is fairly well localized at $t=0$, and that it has center momentum $\underline{k}_0 = |k_0| \hat{z}$.

The coefficient of t in the exponent can be expanded about \underline{k}_0 :

$$|k| = |k_0| + \underline{\Delta k} \cdot \frac{\underline{k}_0}{|k_0|} + \frac{\underline{\Delta k} \cdot \underline{\Delta k}}{|k_0|} \left(1 - \frac{k_0 k_0}{|k_0|^2} \right) + \dots \quad (6.2)$$

The coefficient of \hat{z} in this expansion stops at the linear term Δk_z , because $|k_0|$ is linear in $\underline{\Delta k}$ if $\underline{\Delta k}$ is parallel to \underline{k}_0 ; the wave packet will thus not spread front-to-back.

Since $\Delta k_x = k_x$, $\Delta k_y = k_y$ and $d \Delta k_z = dk_z$ (because

$\underline{k}_0 = |\underline{k}_0| \hat{z}$) we find to the order written out in (6.2):

$$h(x,t) = \int dk_x dk_y e^{i(k_x x + k_y y)} e^{-i(k_x^2 + k_y^2)t/|\underline{k}_0|} \tilde{h}(k_x, k_y, z-t). \quad (6.3)$$

Here h is a typical h_{ab} and $\tilde{h}(k_x, k_y, z)$ is the two-dimensional Fourier transform of $h(\underline{x}, 0)$. The integration can be carried further to show that the lateral growth is via diffraction Green's functions.

However, the form (6.3) is more physically transparent. We find:

$$\begin{aligned} \frac{\partial h}{\partial z} &= \int dk_x dk_y e^{i k_x x + i k_y y} e^{-i(k_x^2 + k_y^2)t/|\underline{k}_0|} \frac{\partial}{\partial z} \tilde{h}(k_x, k_y, z-t) \\ &= i k_z h, \end{aligned}$$

$$\begin{aligned} \frac{\partial h}{\partial t} &= -\frac{\partial h}{\partial z} - i \int \frac{(k_x^2 + k_y^2)}{|\underline{k}_0|} dk_x dk_y e^{i k_x x + i k_y y} e^{-i(k_x^2 + k_y^2)t/|\underline{k}_0|} \tilde{h}(k_x, k_y, z-t) \\ &= -\frac{\partial h}{\partial z} - i \frac{(\Delta k_\perp)^2}{|\underline{k}_0|} h. \end{aligned}$$

(defining k_z and Δk_\perp).

Although k_z ($\approx |\underline{k}_0|$) and Δk_\perp are functions of position and time, we will assume they are some representative constant values, and we will take a test vector with constant components: $\xi^t = a$, $\xi^z = b$. We then have

$$\mathcal{L}_\xi h = a i [-k_z + (\Delta k_\perp)^2/|\underline{k}_0|] h + i b k_z h.$$

And

$$(\mathcal{L}_\xi h)^2 = k_z^2 \left[(b-a)^2 + a^2 \left(\frac{(\Delta k_\perp)^2}{k_z k_0} \right)^2 - 2a(b-a) \frac{(\Delta k_\perp)^2}{k_z k_0} \right] |h|^2$$

We consider only the ratio

$$"d\lambda" = \frac{(\mathcal{L}_\xi h)^2}{\xi^t \xi^z}$$

without integrating. We take $a = 1$, $b = 1 + \epsilon$. The minimum of the ratio " $d\lambda$ " is then given [for $(\Delta k_{\perp})^2 (k, k_0)^{-1}$ small] when

$$\epsilon = - \frac{(\Delta k_{\perp})^2}{k, k_0} < 0 .$$

Thus the minimizing vector is timelike and tends to the null vector as $(\Delta k_{\perp})^2 \rightarrow 0$.

The value of the ratio is, in this limit

$$"d\lambda" \simeq 2 \left(\frac{(\Delta k_{\perp})^2}{k_0} \right)^2 |h|^2 .$$

The eigenvalue tends to zero if the wave vanishes, or if it becomes more nearly plane fronted. The deviation from symmetry goes quadratic in both the amplitude of the wave and in the quantity $\Delta k_{\perp} (\Delta k_{\perp} / k_0)$.

Chapter VII. Applications to Short Wavelength Gravitational Radiation

Let us consider the values of the functional $\lambda[\xi]$ in one of the Isaacson (1967) metrics which have the form (see Section II.e)

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + \epsilon h_{\alpha\beta}$$

where

$$\gamma = \Theta(1), \quad \partial\gamma = \Theta(1), \quad h = \Theta(1),$$

and

$$\epsilon \partial h = \Theta(1)$$

Recall we write $\Theta(\star)$ instead of $\Theta(\epsilon)$.

Write $\mathcal{L}_{\xi} \gamma_{\alpha\beta} = \dot{\gamma}_{\alpha\beta}$ and assume ξ is only slowly varying, so $\dot{\xi} = \Theta(1)$, $\partial \dot{\xi} = \Theta(1)$. Then we find

$$\Theta(1) = 2 \int \xi_{(\alpha;\beta)} = \dot{\gamma}_{\alpha\beta} + \epsilon h_{\alpha\beta,\sigma} \xi^\sigma + \Theta(\lambda).$$

And since $g^{\alpha\beta} = \gamma^{\alpha\beta} + \Theta(\lambda)$ we find

$$4 \int \xi_{(\alpha;\beta)} \xi^{(\alpha;\beta)} = \dot{\gamma}_{\alpha\beta} \dot{\gamma}_{\mu\nu} \gamma^{\alpha\nu} \gamma^{\mu\beta} + 2\epsilon h_{\alpha\beta,\sigma} \xi^\sigma \dot{\gamma}_{\mu\nu} \gamma^{\mu\alpha} \gamma^{\nu\beta} + \epsilon^2 (h_{\alpha\beta,\sigma} h_{\mu\nu,\rho}) \xi^\sigma \xi^\rho \gamma^{\alpha\mu} \gamma^{\beta\nu} + \Theta(\lambda). \quad (7.1)$$

To this order the denominator in (5.1) is just $\int \sqrt{-\gamma} \xi^\alpha \xi^\beta \gamma_{\alpha\beta} d^4x$, and upon integration the first term in (7.1) yields a number depending only on ξ^α and $\gamma^{\alpha\beta}$, which we denote by $4\lambda_\gamma[\xi]$. Since we have assumed $\xi = \Theta(1)$, $\delta\xi = \Theta(1)$, $\delta\gamma^{\alpha\beta} = \Theta(1)$, the integral of the second term in (7.1), although a priori of order unity, is actually much smaller, since it is the integral of a rapidly oscillating quantity $h_{\alpha\beta,\sigma}$ times slowly varying factors. Thus this term is at largest $\Theta(\lambda)$ and we need consider only the last term, which, (again noting the product of rapidly and slowly varying terms) we write:

$$\int \epsilon^2 \langle \gamma^{\mu\alpha} h_{\alpha\beta,\sigma} \gamma^{\beta\nu} h_{\nu\mu,\rho} \rangle \xi^\sigma \xi^\rho \sqrt{-\gamma} d^4x + \Theta(\lambda).$$

The average is over many wavelengths but over a region which is much smaller than the scale of the slowly changing background. Let the colon denote covariant derivative in the background, then noting that $\Theta(\lambda^{-1}) = h_{\alpha\beta,\sigma} = h_{\alpha\beta;\sigma} + \Theta(1)$, we write this term as

$$\int \epsilon^2 \langle \gamma^{\mu\alpha} h_{\alpha\beta;\sigma} \gamma^{\beta\nu} h_{\nu\mu;\rho} \rangle \xi^\sigma \xi^\rho \sqrt{-\gamma} d^4x + \Theta(\lambda) = 32\pi \int T_{\sigma\rho}^{(\text{av})} \xi^\sigma \xi^\rho \sqrt{-\gamma} d^4x + \Theta(\lambda).$$

Here $T_{\sigma\rho}^{(\text{av})} = \frac{\epsilon^2}{32\pi} \langle \gamma^{\mu\alpha} h_{\alpha\beta;\sigma} \gamma^{\beta\nu} h_{\nu\mu;\rho} \rangle$

is the average stress tensor of gravitational waves as defined by Isaacson. It is this average stress tensor which determines the background γ according to

$$R_{\alpha\beta}(\gamma) = 8\pi \left(T_{\alpha\beta}^{(\text{av})} - \frac{1}{2} \gamma_{\alpha\beta} T^{(\text{av})} \right). \quad \text{This term is clearly also independent of } \lambda \text{ as } \lambda \rightarrow 0, \text{ and we find}$$

$$\lambda[\xi] = \lambda_\gamma[\xi] + \lambda_{\text{rad}}[\xi] \quad (\text{for high frequency radiation}), (7.2)$$

where

$$\lambda_{\text{rad}} \left[\frac{\xi}{\xi} \right] = \frac{\int 8\pi T_{\alpha\beta}^{(a\nu)} \xi^\alpha \xi^\beta \sqrt{-\gamma} d^4x}{\int \xi^\alpha \xi^\beta \gamma_{\alpha\beta} \sqrt{-\gamma} d^4x}$$

Both terms are independent of λ in the short wavelength limit. If ξ^α is timelike, then $T_{\alpha\beta}^{(a\nu)} \xi^\alpha \xi^\beta > 0$

We note, moreover, that the additional term λ_{rad} allows in some sense a distinction to be made between dust and gravitational radiation filled universes; the eigenvalue λ is lower for dust universes with the same large scale "shape". Since λ_γ can be bounded by a curvature (in the background $\gamma_{\alpha\beta}$), and since $T_{\alpha\beta}^{(a\nu)}$ is clearly also a curvature in the background metric, we see that $\lambda \left[\frac{\xi}{\xi} \right]$ is indeed sampling only the large scale curvature of the space, and "smoothing over" the ripples, as might be expected from an integral estimate.¹⁹ This derivation makes explicit the result suggested by the torus calculation in Section V a).

To carry the discussion of the energy density in gravitational radiation a bit further, let us consider the Robertson-Walker (R-W) metrics. These have a metric form

$$ds^2 = -d\tau^2 + \Omega^2(\tau) d\sigma^2 \quad (73)$$

where $d\sigma^2$ is the line element of the homogeneous and isotropic space sections; these sections are flat or have unit (by choice of length scale) positive or negative curvature. Because of the high symmetry, these sections each have six Killing vectors; translation along and rotation around each of the axes for the flat sections, two sets of "rotations" for the curved ones. (For a discussion of the metrics of this type, see, e.g. Hawking²⁰).

Because the τ constant sections in these metrics have Killing vectors, if we form the ratio (for F_i a vector in the 3-space, described by the 3-metric

$${}^3g_{ij} = g_{ij}, \quad {}^3g_{i2} {}^3g^{ik} = \delta^k_i)$$

$${}^{(3)}\lambda \left[\frac{\xi}{\xi} \right] = \frac{\int \xi^{(i,j)} \xi^{(i,j)} \sqrt{{}^3g} d^3x}{\int \xi_i \xi_j g^{ij} \sqrt{{}^3g} d^3x}$$

${}^{(3)}\lambda_0 = 0$ since we can pick ξ Killing. In this case, because of the symmetry we have ${}^{(3)}\lambda_0 = 0$ even though there is a length scale introduced by $\Omega(\tau)$.

For an example of a space which is on the large scale identical to the R-W types, and in spirit of recent observational discoveries, we suppose that the universe is at the present time is given by a R-W form and that its behaviour is dominated by the matter in it, but that it contains 3^0K black-body gravitational radiation distributed in a uniform and isotropic way through the universe.²¹ In the "now" constant time slice, we compute ${}^{(3)}\lambda \left[\frac{\xi}{\xi} \right]$ for ξ Killing in the background, and by arguments like those leading to equation (7.2), we find

$${}^{(3)}\lambda \left[\frac{\xi}{\xi} \right] = {}^3\lambda_{\text{rad}} \left[\frac{\xi}{\xi} \right] = \frac{8\pi \int T^{(av)}_{ij} \xi^i \xi^j \sqrt{{}^3\gamma} d^3x}{\int {}^3\gamma_{ij} \xi^i \xi^j \sqrt{{}^3\gamma} d^3x},$$

where here γ_{ij} is the background R-W metric. Because of the assumed isotropy and homogeneity of both the background and the radiation, $T^{(av)}_{ij}$ must be proportional to γ_{ij} where (since this is a massless radiation field) the proportionality factor is $\rho/3$. Since $\rho = \rho(\tau)$ is constant on space slices, we have ${}^{(3)}\lambda \left[\frac{\xi}{\xi} \right] = {}^3\lambda_{\text{rad}} \left[\frac{\xi}{\xi} \right] = \frac{8\pi}{3} \rho(3^0\text{K black body})$. [In this simple situation it is clear that a minimum with respect to the background is a minimum in the full metric since $\lambda = \lambda_\gamma + \text{const}$. In fact, by an argument similar to that for first order perturbation theory in quantum mechanics, it is easy to see that to first order in $T^{(av)}_{\mu\nu}$, the minimum is given by the same vector field in the full metric as in the background. One can also calculate the first order correction to the vector fields exactly as is done in

nonrelativistic quantum theory:

$$\zeta_{(m)} = \zeta_{(m)} + 8\pi \sum_{k \neq m} \frac{\int \zeta_{(m)} \cdot \overleftrightarrow{T}_{(av)} \cdot \zeta_{(k)} dV}{\lambda_{(m)} - \lambda_{(k)}} ,$$

where $\zeta_{(i)}$ is the i^{th} eigenvector of $-\mathcal{D}$ in the background and $\lambda_{(i)}$ is the corresponding eigenvalue, and $\zeta_{(i)}$ is the i^{th} eigenvector of the perturbed $-\mathcal{D}$. In this formula, the vectors $\zeta_{(i)}$ and $\zeta_{(i)}$ are considered referred to the background space $\delta_{\alpha\beta}$.

Since 3°K black body radiation has an energy density of $\sim 10^{-34}$ gm/cc, we

see that this yields a much longer characteristic length than that provided by the large scale background. (The background scale is necessarily shorter than that given by the amount of matter observed in the galaxies $\sim 10^{-30}$ gm/cm³, and even shorter - **magic** density $\sim 10^{-29}$ gm/cm³ - if we assume that there is sufficient deceleration to close the universe with the observed Hubble velocity).

Actually, the energy density of the radiation contributes to the curving of the background. In the previous example the radiation was as symmetric in the large as the background.²² However, if the background is determined by some other factor, say a distribution of dust or electromagnetic radiation, then the gravitational radiation, if weak enough, will not change the background significantly and such an integral over the different Killing vectors in the background will give six different numbers (for R-W background) characterizing the "stress" in the gravitational radiation.

This result is perhaps the most interesting of this work. We have here apparently an invariant method for specifying some parameters of gravitational radiation.

Chapter VIII. Specification of a "Best" Background

A metric which appears to contain gravitational radiation can always be analysed by computing the Riemann tensor components in an orthonormal frame. If there is radiation present which is curving up the space, we have seen that its characteristic Riemann tensor will be

$$R_{\dots} \sim \epsilon \partial^2 h = \mathcal{O}(\epsilon^{-1}) = \mathcal{O}(\lambda^{-1}),$$

and so will be overwhelming in the short wavelength limit. The integral method given here, when applied to a space of the Isaacson type, gives a finite result for the integrals involved, even when $\lambda \rightarrow 0$, and in fact the limit is of the order of the large scale background curvature, and hence gives the average stress in the gravitational radiation. Thus we already have a beginning of a specification program. The Isaacson metrics can be singled out from among the (perhaps) wider class of metrics which have Riemann tensor variations (in a tetrad frame) of order λ^{-1} on a length scale λ .

For applications to radiation, another very useful tool would be the ability to find the optimum background for splitting the metric in "Steady" coordinates. Here we have a less clear criterion, but we suggest that the background could be found by averaging along the first few eigensolution of the equation $(\Delta + \lambda)\xi = 0$. (If we imagined the space were continuously distorted from a symmetric one, the Killing fields would change into other fields which would be the lowest eigenvalue fields, at least for small distortions. It is these that we have in mind.) We present, in Appendix D a calculation of the complete spectrum of $-\Delta$ for the constant time slices in a cosmological solution due to Taub. We also give, and discuss, a way of averaging this metric along non-Killing vector fields (fields which would be Killing if the space were completely symmetric) to find a background space. We briefly discuss the question whether this is the "best" background which can be found.

The last problem is still an open one. The calculations in Appendix D

rely very heavily on vector fields which can be defined in the Taub slices with the aid of the group structure implied by the exact Killing vectors which remain. Even here, we find that technical problems hinder the completion of the discussion. In particular, we had to leave incomplete the discussion of a smoothed averaging due to the algebraic complexity involved. We hope to remedy this gap in the near future.

A more fundamental difficulty is that we have found that two reasonable suggestions for averaging: a) the averaged metric obtained by Lie transport along the eigenvector fields of $-\mathcal{D}$, and b) the average defined as the sphere with the same volume as the Taub space slice, lead to two different backgrounds. Since these backgrounds have different time dependence, we find two different numbers for the energy density of the "radiation", the anisotropy in the Taub slices.

One might hope that the wave-background separation found in Appendix D would --for one of the averagings--lead to a R-W background whose motion was determined by massless radiation, since we like to think of the anisotropy in the Taub slices as gravitational radiation.

We can investigate this by writing the averaged R-W space found in Appendix D in the full 4-dimensional form which makes its (3-sphere) x (time) structure apparent, as

$$ds^2 = -e^{g(t)} (2\ell)^2 \left[d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) \right] + \left[-1 + \frac{2(mt + \ell^2)}{t^2 + \ell^2} \right]^{-1} dt^2 . \quad (8.1)$$

Here χ, θ, φ are angle coordinates on the 3-sphere. m, ℓ are constant real lengths. This is not quite the standard R.W. form; we need only make the transformation

$$d\tilde{t} = \left[-1 + \frac{2(mt + \ell^2)}{t^2 + \ell^2} \right]^{-1/2} dt ,$$

to put in the R.W. form (see, e.g., Hawking, 1966).

To see that neither of the averaging schemes defined in Appendix D gives rise to a traceless effective stress tensor, we need only note that the unique solution with ${}^4R=0$ is the radiation dominated model, which has a time dependent radius given by

$$g_{ii} = -R_0^2 e^{g(t)} = -R_0^2 \left(1 - \frac{\hat{r}^2}{R_0^2}\right) \quad (\text{no sum on } i), \quad (8.2)$$

where $2R_0 = \mathcal{Z}(t_+) - \mathcal{Z}(t_-)$,
with $t_{\pm} = m \pm (m^2 + l^2)^{1/2}$.

We set $\mathcal{Z}(t_-) = 0$, and $\hat{r} = \mathcal{Z} - R_0$.

Near the singular point, $\mathcal{Z} = 0$, this has the behavior

$$e^{g(t)} \sim \mathcal{Z}. \quad (8.3)$$

Near the singular point the relation between t and \mathcal{Z} in our averaged background eq. (8.1) is

$$d\mathcal{Z} \sim dt (t - t_-)^{-1/2} \Rightarrow \mathcal{Z} \sim (t - t_-)^{1/2}$$

For the equal volume averaging we have from Appendix D

$$e^{g(t)} = \left[\frac{(t^2 + l^2)}{(2l)^2} \frac{(t - t_-)}{2l} \frac{(t - t_+)}{2l} \right]^{1/3} \sim (t - t_-)^{1/3}$$

so

$$e^{g(t)} \sim \mathcal{Z}^{2/3} \quad (\text{equal volumes}). \quad (8.4)$$

This result has been noted in general for anisotropic homogeneous universes near the singularity. (Misner, 1967).

For the metric obtained by averaging along the invariant vector fields we have

$$e^{g(t)} = \left(\frac{2}{3} \frac{(2R)^2}{t^2 + l^2} + \frac{1}{3} \frac{t^2 + l^2}{(t - t_-)(t - t_+)} \right)^{-1} \sim (t - t_-),$$

so

$$e^{g(t)} \sim \mathcal{Z}^2 \quad (\text{Lie transport average}). \quad (8.5)$$

Since neither of these have the required time behavior of the radiation dominated R.W. solutions, the effective stress tensor cannot be traceless in

either case. [This result of non-vanishing trace does not contradict the result of Brill (1964) that the effective averaged stress tensor for small amplitude short wavelength gravitational waves is traceless to the lowest order in the deviation from the background. The waves in Taub space are neither short wavelength nor weak.]

It might be hoped that the situation would be somewhat better near the largest expansion instant of the space where the radiation is less intense- the space is more symmetric- but the trace can clearly vanish only at discrete instants of time because of the analyticity of the Taub solution.

One must stress again that the unique background which is R.W. radiation dominated and has the correct singular instants t_{\pm} is that given by eq.(8.2). This space bears no obvious relation to the two previously defined averages, but can be considered as a third possible candidate for a background space.

Some of this ambiguity in defining a background is due to the fact that Taub space contains only a few wavelengths of radiation. The Isaacson scheme, on the other hand, assumes short wavelength radiation. We would expect that in such cases averaging over only a few vector fields -instead of the infinite number we needed for Taub space- would lead to unique results. In particular, for the high frequency radiation problem of Isaacson, one would want to use only a few averagings with a smoothing function that averaged over many wavelengths, but did nothing to change a scale the size of the background. (As mentioned previously, the problem in this respect with the Taub slices is that they are not much bigger than one wavelength of radiation.) The "few" vector fields are necessary because one wants to average over a volume, not just along a line; sufficient averagings are necessary to span such a small volume. The averaging Isaacson requires is some unspecified averaging process over such a volume. For the Taub slices, as we showed above, two averaging ideas lead to two different definitions of the energy density,

so we are unable to decide which is "best". We hope that further investigation of this question for short wavelength situations may lead to a justification of the averaging scheme given here. Certainly any scheme which requires averaging along invariantly defined vector fields will be best suited by the fields defined here since they are specified by the metric itself. Other schemes, like the constant volume one defined for the Taub space, do not seem to be applicable at all to local averaging.

Chapter IX. Conclusion; Outlook

Besides the problem of finding whether a background exists and what it is, several questions remain. The first is to better understand why the method here results in finding out anything about the gravitational radiation. The answer lies in the Ricci identity. Let u^α be a general vector field and define $\dot{u}^\alpha \equiv u^\alpha_{;\beta} u^\beta$. Then the Ricci identity, contracted by u^δ , is

$$u^\delta [u_{\alpha;\delta;\gamma} - u_{\alpha;\gamma;\delta}] = R_{\alpha\beta\gamma\delta} u^\beta u^\delta.$$

Thus

$$(u_{\alpha;\gamma})^\cdot - \dot{u}_{\alpha;\gamma} + u_{\alpha;\delta} u^\delta_{;\gamma} + R_{\alpha\beta\gamma\delta} u^\beta u^\delta = 0.$$

The symmetric trace free part of the equation gives the propagation of the trace free part of $u_{(\alpha;\beta)}$ along the rays of u^α , in terms of quantities defined by the vector field u^α and the geometrical object $C_{\alpha\beta\gamma\delta} u^\beta u^\delta$. The Weyl tensor contracted into $u^\beta u^\delta$.

The Weyl tensor (= the Riemann tensor in vacuum) is a "square of derivatives" of the metric. Hence the shear (traceless part of $u_{(\alpha;\beta)}$) measures in some sense the energy density of the gravitational radiation. We note that due to the gage conditions which may be imposed on $h_{\alpha\beta}$ (Isaacson, 1967, p.35), the trace term h^α_α can be set equal to zero as the wavelength goes to zero. Thus the divergence of the vector field ξ does not contribute to λ_{rad} in equation (7.1), and we see we are really measuring the trace free part of $\xi_{(\alpha;\beta)}$. The integral performed to obtain λ_{rad} in (7.1) measures the integral total of the shear in the test vector field, and so measures the accumulated "energy density" in the wave.

The discussion of the energy density of gravitational radiation has been carried out in terms of the shear of null (light) rays by Penrose (1967) and in terms of the shear of timelike geodesics by Hawking (1966).

Because the trace h^α_α does not enter λ_{rad} , our choice of $\mu=0$ in

Section Vb. was not really relevant for gravitational radiation, since it does not affect the result for λ_{rad} . However, if the space under consideration is invariant under conformal transformations, there would be non-Killing solutions if we took $\mu = -m^{-1}$, which subtracts all of the trace out of the form C^{ABMN} . See (Yano and Bochner, 1953, p. 72). We tentatively remain with our original choice, $\mu = 0$, although the choice $\mu = -m^{-1}$ for a conformally covariant expression may prove useful in investigations which utilize the conformal invariance of the Weyl tensor in the study of gravitational radiation.

Further remaining problems, include those emphasized in the previous section; to better specify the determination of a background by these methods, and a criterion for determining the "best background".

As we have seen, the ground state eigenvalue measures, for different spaces with the same large scale shape, the contributions of the gravitational energy density in the space. This gives an additional length scale $\lambda_{\text{rad}}^{-1/2} \left[\frac{m}{\sigma} \right]$ in addition to the sizes obviously present in a situation with large scale size $\sim L$ and Riemann tensor variations on a length scale λ . With each invariantly defined vector in the background, there is an associated length scale which measures some component of the stress of the gravitational energy.

The question remains: can a complete specification of the space be done in this way? Restricting consideration to just the ground state $\frac{m}{\sigma}$, this does not seem possible because the eigenvalue λ contains parts due to the background as well as due to the radiation and there seems to be no way to separate them. However, it is plausible that the entire spectrum of the operator $-\mathcal{D}$, such as we calculated in Appendix D for the Taub constant time slices, may give a sufficiently powerful specification of the space it is expressed in, that the complete solution, background plus radiation, can be

expressed in terms of an expansion in the eigenvectors and eigenvalues of $-\mathcal{D}$. While this complete specification may be overambitious, it is hoped that a more clear-cut identification of spaces which satisfy Isaacson's requirement (that they admit a "Steady" coordinate system) will be possible by these methods. This is certainly an important application, if it can in fact be done. We have already mentioned (Chapter VIII) that the quantity $\lambda[\xi]$ gives a finite result even as $\lambda \rightarrow 0$ for spaces of the Isaacson type, and this may be the beginning of such a specification scheme.

The method presented here in equations (5.1) and (5.2) is a straightforward generalization of the idea of a Killing field. The differential equation (5.2), in spaces of Minkowski signature, can be considered a coordinate condition for the time, say. Detailed investigation of this idea may yield very useful results in the future.

We also have left for future investigation the question of using the vector fields defined by these recipes to give new candidates for conserved momentum or energy objects. This may also prove quite a fruitful field of investigation.

Perhaps the most significant and unexpected results of the ideas in the work described here are the applications to spaces which contain short wave gravitational radiation, and their use to specify some numerical parameters for the radiation.

Clearly there are a great many topics for investigation which are suggested by the results of this work. If the specification of the manifold in terms of the spectrum of its operator $-\mathcal{D}$ succeeds, it will probably provide, via an extremely circuitous route, the answer to the fundamental problem which prompted this investigation: to find an invariant way of doing problems of slow motion in Relativity.

Footnotes

1. Round brackets mean symmetric part, square brackets mean antisymmetric part. The double stroke (\parallel) means covariant derivative in a general Riemannian space, the semicolon ($;$) means covariant derivative in a 4-dimensional Minkowski signature space, V_4 , the slash ($|$) means covariant derivative in a 3-dimensional positive definite subset of V_4 . Ordinary derivative is indicated by a comma.
2. K. Yano and S. Bochner, Curvature and Betti Numbers, Princeton, (1953)
3. A. Trautman, "Foundations and Current Problems in General Relativity" in Lectures on General Relativity (Brandeis Summer Institute in Theoretical Physics), Prentice Hall, Englewood Cliffs, New Jersey, (1964)
4. We expect, however, that the ground state vector field will have few zeros. It is known that the nodes of this vector field cannot be separating hypersurfaces (that is, they cannot separate the domain into disjoint parts). See R. Courant and D. Hilbert, Methods of Theoretical Physics, Vol. 1, Interscience, New York, page 452 (1953). If we assumed that M and g_{AB} were analytic instead of merely C^∞ , then there could be no zeros which are $(n-1)$ -dimensional subsets of hypersurfaces. In any case, this theorem simplifies the construction of coordinate systems which utilize the ground state vector field as one congruence of coordinate lines. Such a coordinate system will fail at the (at most) $(n-1)$ -dimensional regions where the vector field vanishes, so "patching" over the nodes with geodesic coordinates should be simple.
5. K. Kodiyara and D. C. Spencer, Ann. Math., 71, 43, (1960)
6. A. E. Green and W. Zerna, Theoretical Elasticity, Oxford, (1954), pl62.
7. The discussion in this section is similar to that of A.E. Green and W. Zerna, Theoretical Elasticity, Oxford, (1960), Chapter 10, where the (equilibrium) theory of shells is discussed. The notation, however, follows T.J. Willmore, Differential Geometry, Oxford, (1958). The discussion given here completely in tensor formalism may be more accessible to readers familiar with such methods than is that of Green and Zerna, who use a combination of tensor and triad component objects. The reduction to shell theory surface equations given here is essentially the same as Green and Zerna's method.
8. See, e.g. Willmore, op. cit., Chapter III and page 230; also Green and Zerna, op. cit., pp 26-37. The two eigenvalues of the matrix K^α_β are the principle curvatures of the middle surface.
9. There are actually two different limiting ratios to be considered. The first is the ratio of the wavelength of the waves which might reflect between the two surfaces, $\sim h$, to the wavelength λ of the oscillations in the shell (parallel to it) which can be of order L , a radius of

curvature of the shell, but which may be smaller. This is our requirement of finite frequencies as $h \rightarrow 0$. The other limit is the requirement that the geometrical quantities not be significantly different from those on the middle surface, i.e., $L \gg h$, independent of the wavelength we consider. Both $L \gg h$ and $\Lambda \gg h$ must be satisfied for the discussion here to be valid.

10. S.G. Mikhlin, The Problem of the Minimum of a Quadratic Functional, Holden Day, San Francisco, (1965), and references there.
11. The boundary conditions demanded for non-compact spaces for this derivation are rather strict, so this does not constitute a proof of the nonexistence of Killing vectors on negative definite open surfaces. In particular, a spacelike 3-hyperboloid has six Killing vectors, each of which would give surface integral terms in equation (5.19).
12. R.W. Lindquist, C.W. Misner, and R.A. Schwartz, Phys. Rev. 137, B1364, (1965).
13. P.C. Vaidya, Proc. Indian Acad. Sci. A33, 264, (1951), and Curr. Science 21, 96, (1952)
14. R.P. Kerr and A. Schild, in Convegno Sulla Relativita Generale: Problemi Dell Energia e Onde Gravitazionali, (Proceedings of Conference in honor of the fourth centenary of the birth of Galileo) published by Comitato Nazionale per le Manifestazioni Celebrative, Rome, (1964)
15. N.W. McLachlan, Theory and Application of Mathieu Functions, Dover paperback, New York, (1964). We use his notation.
16. E.A. Coddington, and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York, (1955), page 212.
17. L.D. Landau and E.M. Lifshitz, Classical Theory of Fields, Pergamon Press, (second edition, 1962)
18. R.A. Isaacson, Ph.D. thesis, University of Maryland Department of Physics and Astronomy, (1967), p. 50.
19. We must still justify the assumption that $\delta \xi \sim 1$ for the ground state. We note that to change the integrals of the second and third terms in equation (7.1) would require $\delta \xi \sim \lambda^{-1}$, so that oscillations in ξ will be "in phase" with those in the radiation field. In this case the second integral would be $O(\lambda^{-1})$, while the third would still be $O(1)$. The first integral would then, however, be $\lambda \int \xi = O(\lambda^{-1})$. So we conclude that the ground state eigenvalue can certainly be made smaller by taking a slowly varying vector field, and the ground state eigenvalue is given by equation (7.2).
20. S.W. Hawking, Ap.J. 140, 1, (1964)

21. The observations of 3°K electromagnetic radiation show that it is in fact remarkably isotropic. See R.B. Partridge and D.T. Wilkenson, Phys. Rev. Lett 18, 557, (1967)
22. The gravitational radiation can supply the entire energy density curving up the space. The solution with the R-W symmetries was discussed by D.R. Brill, Nuovo Cimento Suppl. 2, no. 1, (1964). The value of $\lambda[\xi]$ for ξ Killing in the background of this space gives $\rho_{\xi} = -\dot{\Omega}(t)/\Omega$, the deceleration parameter for the large scale evolution.

Appendix A

Approximate Calculation of Gravitational Radiation
from Collapsing Axi-Symmetric Systems

I. Introduction

The problem of radiation in General Relativity can be easily treated only in the weak field limit. The unfortunate feature of this linearized theory approach is that it becomes invalid just when the radiation becomes large enough to be interesting. It does, one presumes, provide useful qualitative estimates even in the strong field region, and this paper provides some such estimates.

The paper deals principally with two problems: radiation from a uniform oblate spheroid of pressureless matter undergoing Newtonian collapse, (the prolate case is discussed qualitatively) and a calculation of the radiation from two equal mass objects coming straight on at each other. In the second case the calculation is with Newtonian interaction and the weak field theory, but with the answer carried into a strong field region.

II. Collapsing Axi-Symmetric Systems

The idea is the following. A massive system initially not at equilibrium, with an initially large quadrupole moment and with a radius only slightly larger than its Schwarzschild limit will, according to linear theory, radiate an amount of energy equal to a good fraction of its rest mass in times less than or comparable to one period of its oscillation or rotation frequency. Thus a deformed neutron star, for instance, if it had a radius $r \sim 2m$, would not execute any sustained oscillations. It would just "deflate" to its equilibrium configuration, radiating away its excess deformation energy. Thus in the strong field regime a single collapse of an object might radiate as much energy as a binary system in circular orbits would. (More precisely, when the binary system got down into strong fields, its final few cycles would not resemble a spiral at all, but would look like a collapse.) (Misner 1965)

For this reason, and because rotating systems are more difficult to treat, we will deal only with collapsing systems. The motions will be calculated from Newtonian theory and the radiation from linearized gravity theory.

III. Uniform Spheroids

In a recent paper (Lin 1965), C.C. Lin et.al. calculate the collapse of uniform pressureless spheroids (uniform means $\rho(r,t) = \rho(t)$). He obtained results for both prolate and oblate ellipsoids, with the following general features:

- a) the spheroids stay uniform;
- b) the eccentricity increases with time: an oblate spheroid becomes a disc, a prolate spheroid becomes a line;
- c) collapse to a disc or line occurred in all cases for a time $t_c \sim .5 \rho_0^{-1/2}$ (ρ_0 is the initial density).

The Lin results are given both as a power series expansion and as computer obtained result. The agreement between the two methods is excellent for the oblate case and reasonable in the prolate case.

We have taken Lin's oblate case power series solution and computed the radiation given by

$$L = \frac{1}{45} \sum_{\alpha\beta} \left[\ddot{Q}_{\alpha\beta} \right]^2$$

where $Q_{\alpha\beta}$ is the quadropole moment of the radiating system and $\dot{}$ means $\partial/\partial t$. (Landau 1951) Some plots of the dimensionless number

$$\left[\frac{5}{m} \frac{\ddot{Q}}{\omega_0^2} - \frac{1}{\rho_0} \frac{1}{3/2} \right]^2$$

are found in figure (1) for various values of Z_0/ω_0 . (We denote the initial semi-minor and semi-major axes by Z_0 , ω_0 and for ellipsoids $Q = Q_{xx} = Q_{yy} = -Q_{zz} (\frac{1}{2})$.) We then see that there is a strong peaking toward the end of the collapse, and the radiation in the initial stage is insignificant. The FORTRAN IV program and the output from which the graph was plotted are given in Section V.

The conclusion to be drawn from these calculations is that for non-relativistic initial conditions the radiation is negligible. For instance, inserting the sun's density, mass and radius into the normalizing factor:

$$\frac{5}{m} \frac{1}{\omega_0^2} \frac{1}{\rho_0^{3/2}}$$

since we have (Synge 1960)

$$m_\odot \approx .5 \times 10^{-6} \text{ sec}$$

$$\omega_0 \approx 2.3 \text{ sec}$$

$$\rho_0^{-1/2} \approx 3.9 \times 10^3 \text{ sec}$$

so

$$\left[\frac{5}{m_\odot} \frac{1}{\omega_0^2} \frac{1}{\rho_0^{3/2}} \right]^2 = \left[\frac{5}{.5 \text{ sec}} \times 10^6 \frac{1}{5.37 (\text{sec})^2} \right]^2$$

$$\left[(60) \times 10^{+9} (\text{sec})^3 \right]^2 \approx 10^{34}$$

For gravitational collapse with $Z_0/\omega_0 = .95$, we have (Figure 1)

$$\left[\frac{5}{m} \rho_0^2 \frac{\ddot{Q}}{\rho_0^{3/2}} \right]^2 \leq 10^{12}$$

so

$$[\ddot{Q}]^2 \leq 10^{12-34} = 10^{-22}$$

To put in all the factors, we note that

$$\sum [\ddot{Q}_{\alpha\beta}]^2 = 6[\ddot{Q}]^2$$

so

$$L = \frac{1}{45} \sum [\ddot{Q}_{\alpha\beta}]^2 \approx 10^{-23} \approx 10^{-17} M_{\odot} / \text{sec}.$$

Since the collapse time $\rho_0^{1/2}$ is $\rho_0^{1/2} = 3.9 \cdot 10^3$ sec and we find that the last small fraction, say the last thousandth, of the collapse is the time of significant radiation, the total radiation in this collapse is

$$\sim 10^{-17} - 10^{-16} M_{\odot}.$$

This result justifies the statement above about small radiation from non-relativistic initial conditions.

The prolate spheroid radiation was not calculated because of inaccuracies in the power series expansion given by Lin (as checked against his numerical integration results), but it would not be qualitatively different.

IV. Colliding Massive Particles

These computations show that to find a situation where the radiation will be large, we must follow the collapse beyond the first stages--we must follow the collapse of a disc or of a line.

We make the mathematics even simpler by taking a very simple physical situation. Two particles, each of mass m , are separated by a distance r and interact via Newton's Law. (We take this as a simple idealization of the case of a collapsing line.) Then the equation of motion is (fig. 2)

$$\mu \ddot{r} = -\frac{m^2}{r} \quad (\mu = \frac{m}{2}) \quad .$$

This has the solution given in parametric form:

$$r = r_0 \cos^2 \theta$$

$$t = \sqrt{\frac{r_0}{2m}} \left(\theta + \frac{1}{2} \sin 2\theta \right).$$

(r_0 = separation where $\dot{r} = 0$)

Since $Q_{zz} = -2Q_{xx} - 2Q_{yy} = mr^2 = Q$ we can compute the quantity of interest,

Q .

We have $(k = \sqrt{\frac{2m}{r_0^3}})$

$$d\theta/dt = k(2 \cos^2 \theta)$$

$$\dot{Q} = -m k r_0^2 \sin 2\theta$$

$$\ddot{Q} = -m k^2 r_0^2 \left(1 - \frac{\sin^2 \theta}{\cos^2 \theta} \right)$$

$$\ddot{\dot{Q}} = m k^3 r_0^2 \left(\frac{\sin \theta}{\cos^5 \theta} \right).$$

$$[\ddot{\dot{Q}}]^2 = m^2 \left(\frac{2m}{r_0^3} \right)^3 r_0^4 \left(\frac{1-r/r_0}{(r^5/r_0^5)} \right) = \frac{1}{4} \left(\frac{2m}{r} \right)^5 (1-r/r_0) \quad .$$

So (since the definition of Q in terms of m, r has changed from the ellipsoidal case) we get

$$L = \frac{1}{45} \frac{3}{2} [\ddot{Q}]^2 = \frac{1}{120} \left(\frac{2m}{r}\right)^5 \left(1 - \frac{r}{r_0}\right)$$

We note that if the two masses were in circular orbit around each other, the radiation when their separation is r is given by (Vishveshwara 1964)

$$L_0 = \frac{2}{5} \left(\frac{2m}{r}\right)^5$$

We see that although radiation in the two systems is comparable, it is larger by a factor of fifty or so in the binary orbit case.

We can also compute the total energy emitted by each of these systems.

i.e.:

$$\begin{aligned} E &= \int_{t_0}^t L(t) dt \\ &= \int_{r_0}^r L(r) \frac{dt}{dr} dr \end{aligned}$$

For the colliding particles, taking $r_0 = \infty$, we have

$$E = \frac{1}{120} \int_{+\infty}^r (2m)^5 \frac{1}{r^5} \left(\frac{dt}{dr}\right) dr$$

now

$$\frac{dt}{dr} = [r_0 \theta_t]^{-1} = \left[(1-r/r_0)^{1/2} \sqrt{\frac{2m}{r}}\right]^{-1} = \left[\sqrt{\frac{2m}{r}}\right]^{-1} \text{ for } r_0 = \infty$$

so

$$E = \frac{(2m)^{9/2}}{120} \int_{\infty}^r \frac{1}{r^{9/2}} dr = \frac{1}{420} \cdot 2m \left(\frac{2m}{r}\right)^{7/2} \quad (A)$$

This is the amount of energy radiated for a fall from infinity. We note that this equals the rest mass of the system ($2m$) only when

$\frac{1}{420} \left(\frac{2m}{r}\right)^{7/2} = 1$. Note that this happens only for some value of r less than $2m$. The total kinetic energy of the system is equal to the potential energy since they started at rest:

$$K.E. = \frac{m}{r}.$$

So the radiation begins to become appreciable when the energy radiated approaches the K.E.;

$$\frac{m}{2} \left(\frac{2m}{r}\right) \approx \frac{1}{105} \frac{m}{2} \left(\frac{2m}{r}\right)^{7/2}.$$

This happens only when

$$1 = \frac{1}{105} \left(\frac{2m}{r}\right)^{5/2}.$$

Again, this is for some r less than $2m$.

We may compare the result (A) with the total radiation from a binary system as it decays from an infinitely large orbit to an orbit of radius r . It is just the binding energy at that radius:

$$\Delta E = \frac{m}{2r}.$$

We see that the total energy radiated from the binary system is much greater than that from the "colliding" system, since the binary takes so much longer to collapse to any given radius, and the luminosities are roughly comparable at comparable radii. The same comparison as above shows that the total collapse energy radiated does not approach that radiated from the binary until $r < 2m$. Table 1 gives a comparison between the two systems of luminosity and total

energy radiated for several interesting radii.

We may take the comparison between binary and collapsing systems as an indication that even the linearized theory predicts a "tailing-off" in the radiation from a binary system. In particular, we expect that for some small radius r_c circular orbits will become unstable. (For a test particle around a central mass M the full theory gives this radius as $6M$. Not enough is known about the two-body problem in general to make any more accurate estimates of r_c in the equal mass case.) A binary then gets perturbed so much it becomes a colliding system--with a loss of about two orders in luminosity. The full theory would presumably reduce the luminosity even more since the system would be retarded by its own radiation reaction and red-shifts would reduce luminosity observed at infinity.

It is interesting that if we demand that $r > 2m$, then the binary system will have radiated only $1/4$ its energy by the final instant (assuming the orbits stayed circular). A head-on collapse would have radiated only $\frac{1m}{210}$. If a binary were perturbed into a head-on collapse at radius $6m$, the total radiation when it reached $2m$ would be

$$\frac{\Delta m}{m} = \frac{1}{12} + a$$

where the $1/12$ comes from the binary part of the motion and $a \approx 1/200$ comes from the collapse.

I thank Dr. C.W. Misner for suggesting this problem and for very helpful advice.

Computer usage was through the Computer Science Center, U. of Md., and NASA grant NsG-398 to the Computer Science Center, U. of Md.

V. COMPUTER PROGRAM AND RESULTS

The following pages give a listing of the FORTRAN IV program and a partial listing of the results. Input for the program is coefficients from C.C. Lin, table A1, (Lin, 1965). The collapse times are also from Lin.

In the output, the quantities listed for various values of Z_0/ω_0 are

$$A = \left[\frac{5}{m} \frac{Q}{\omega_0^2} \frac{1}{\rho_0^{3/2}} \right]^2, \quad t_{\rho_0}^{1/2}, \quad \text{and} \quad \ln_{10} A.$$

MATZNER
AMAIN

205/66/071
- EFN SOURCE STATEMENT - IFN(S) -

DATE 05/02/66

```

201 FORMAT(6F12.6)
202 FORMAI(4E24.8)
101 READ(5,201) ZPI,E2,E42,AZ,A2Z,DTH
WRITE (6,201)ZPI,E2,E42,AZ,A2Z,DTH
EPZSQ=1.-ZPI*ZPI
TH=0.
100 E4=E2*E42
SINTH=SIN(TH)
COSTH=COS(TH)
SIN2TH=SIN(2.*TH)
COS2TH=COS(2.*TH)
SIN4TH=SIN(4.*TH)
COS4TH=COS(4.*TH)
DEN=1.-A2Z+(1.+A2Z)*COS2TH
THT=AZ/DEN
THTT=2.*AZ*SIN2TH*(1.+A2Z)*THT/(DEN*DEN)
THTTT=4.*AZ*AZ*(1.+A2Z)*THT*((1.+A2Z)*3.*SIN2TH**2/DEN**4+COS2TH
1*DEN**(-3))
SQT=-4.*EPZSQ* COSTH**3*SINTH-(1.-EPZSQ)*(E2 *SIN4TH+4.*(2.*E4+E2*
1E2)*COSTH*SINTH**3+12.*E2*E4*(SINTH**5)/COSTH+(4.*E2*E4+8.*E4*E4)*
2SINTH**7/COSTH**3+4.*E4*E4*SINTH**9/COSTH**5)
SQTT=-EPZSQ*(-3.*SIN2TH**2+4.*COSTH**4)-(1.-EPZSQ)*(4.*E2*COS4TH+3
1.*(2.*E4+E2*E2)*SIN2TH**2+(60.*E2*E4-4.*(2.*E4+E2*E2))*SINTH**4+(4
20.*E2*E4+56.*E4*E4)*SINTH**6*COSTH**(-2)+(12.*E2*E4+60.*E4*E4)*
3SINTH**8*COSTH**(-4)+20.*E4*E4*SINTH**10*COSTH**(-6))
SQTTT=-EPZSQ*(-6.*SIN4TH-16.*SINTH*COSTH**3)-(1.-EPZSQ)*((-16.*E2+
1 6.*(2.*E4+E2*E2))*SIN4TH+(60.*E2*E4-4.*(2.*E4+E2*E2))*4.*SINTH**3
2*COSTH+(40.*E2*E4+56.*E4*E4)*6.*SINTH**5*COSTH**(-1)+(176.*E2*E4+
3592.*E4*E4)*SINTH**7*COSTH**(-3)+(48.*E2*E4+440.*E4*E4)*SINTH**9*
4COSTH**(-5)+120.*E4*E4*SINTH**11*COSTH**(-7))

A=SQTTT*THT**3+3.*SQTT*THTT*THT+SQT*THTTT
ASQ=A*A
ASLG=ALOG10(ASQ)
TIME=(1./AZ)*(TH+.5*SIN2TH-A2Z*(TH-.5*SIN2TH))
WRITE(6,202)A,ASQ,TIME ,ASLG
TH=TH+DTH
IF(TIME-.6)100,101,101
END

```

$$z_0/\bar{\omega}_0 = .95$$

A	$tp_0^{1/2}$	$\ln_{10} A$
0.	0.	-0.17014118E 39
0.30817884E-01	0.34935624E-01	-0.15111972E 01
0.12946945E 00	0.69695140E-01	-0.88783268E 00
0.31623791E 00	0.10410420E 00	-0.49998607E 00
0.63118071E 00	0.13799195E 00	-0.19984628E 00
0.11459794E 01	0.17119277E 00	0.59176825E-01
0.19867271E 01	0.20354787E 00	0.29813823E 00
0.33775930E 01	0.23490693E 00	0.52860732E-00
0.57261888E 01	0.26512957E 00	0.75786567E 00
0.97956091E 01	0.29408678E 00	0.99103145E 00
0.17063793E 02	0.32166220E 00	0.12320756E 01
0.30505379E 02	0.34775324E 00	0.14843764E 01
0.56372162E 02	0.37227218E 00	0.17510647E 01
0.10846065E 03	0.39514698E 00	0.20352722E-01
0.21895215E 03	0.41632205E 00	0.23403493E 01
0.46783019E 03	0.43575876E 00	0.26700883E 01
0.10690496E 04	0.45343588E 00	0.30289979E 01
0.26464912E 04	0.46934973E 00	0.34226705E 01
0.72162819E 04	0.48351427E 00	0.38583135E 01
0.22158714E 05	0.49596092E 00	0.43455446E 01
0.78994424E 05	0.50673828E 00	0.48975965E 01
0.34128829E 06	0.51591162E 00	0.55331214E 01
0.19000831E 07	0.52356225E 00	0.62787727E 01
0.14879731E 08	0.52978667E 00	0.71725951E 01
0.18539134E 09	0.53469566E 00	0.82680895E 01
0.43623360E 10	0.53841311E 00	0.96397191E 01
0.24881862E 12	0.54107484E 00	0.11395883E 02
0.52766826E 14	0.54282723E 00	0.13722361E 02

+collapse

$$z_0 / \bar{\omega}_0 = .9$$

0.	0.	-0.17014118E 39	
0.96045071E-01	0.35305694E-01	-0.10175249E 01	
0.40429641E 00	0.70431785E-01	-0.39330012E 00	
0.99082108E 00	0.10520046E 00	-0.40047630E-02	
0.19869888E 01	0.13943749E 00	0.29819543E 00	
0.36301542E 01	0.17297395E 00	0.55992508E 00	
0.63428862E 01	0.20564789E 00	0.80228693E 00	
0.10887160E 02	0.23730604E 00	0.10369146E 01	
0.18671295E 02	0.26780521E 00	0.12711744E 01	
0.32381443E 02	0.29701384E 00	0.15102962E 01	
0.57331995E 02	0.32481325E 00	0.17583970E 01	
0.10448353E 03	0.35109881E 00	0.20190478E 01	
0.19753020E 03	0.37578107E 00	0.22956335E 01	
0.39050224E 03	0.39878657E 00	0.25916236E 01	
0.81436424E 03	0.42005860E 00	0.29108187E 01	
0.18097385E 04	0.43955778E 00	0.32576158E 01	
0.43384399E 04	0.45726242E 00	0.36373336E 01	
0.11393253E 05	0.47316881E 00	0.40566477E 01	
0.33434570E 05	0.48729116E 00	0.45241957E 01	
0.11257274E 06	0.49966154E 00	0.50514332E 01	
0.45064880E 06	0.51032948E 00	0.56538382E 01	
0.22520185E 07	0.51936158E 00	0.63525719E 01	
0.15021865E 08	0.52684074E 00	0.71767239E 01	
0.14670204E 09	0.53286538E 00	0.81664362E 01	
0.23882802E 10	0.53754849E 00	0.93780854E 01	
0.78670103E 11	0.54101641E 00	0.10895810E 02	collapse

$$z_0 / \bar{\omega}_0 = .8$$

0.	0.	-0.17014118E 39	
0.37306343E 00	0.36200477E-01	-0.42821732E 00	
0.15751752E 01	0.72212893E-01	0.19732885E 00	
0.38802944E 01	0.10785106E 00	0.58886469E 00	
0.78394156E 01	0.14293254E 00	0.89428369E 00	
0.14464313E 02	0.17728046E 00	0.11602978E 01	
0.25593499E 02	0.21072525E 00	0.14081297E 01	
0.44624295E 02	0.24310640E 00	0.16495714E 01	
0.78018227E 02	0.27427398E 00	0.18921961E 01	
0.13851590E 03	0.30409026E 00	0.21414996E 01	
0.25231402E 03	0.33243094E 00	0.24019414E 01	
0.47591636E 03	0.35918649E 00	0.26775306E 01	
0.93802528E 03	0.38426322E 00	0.29722146E 01	
0.19506724E 04	0.40758422E 00	0.32901843E 01	
0.43267883E 04	0.42909011E 00	0.36361657E 01	
0.10369012E 05	0.44873966E 00	0.40157374E 01	
0.27272091E 05	0.46651017E 00	0.44357184E 01	
0.80290688E 05	0.48239773E 00	0.49046652E 01	
0.27134984E 06	0.49641723E 00	0.54335296E 01	
0.10878084E 07	0.50860225E 00	0.60365524E 01	
0.54015285E 07	0.51900467E 00	0.67325167E 01	
0.35207638E 08	0.52769419E 00	0.75466369E 01	
0.32642647E 09	0.53475763E 00	0.85137853E 01	collapse

$$z_0 / \tilde{\omega}_0 = .7$$

0.	0.	-0.17014118E 39
0.78124551E 00	0.37419950E-01	-0.10721246E 00
0.33096123E 01	0.74640905E-01	0.51977713E 00
0.81989450E 01	0.11146585E 00	0.91375798E 00
0.16698763E 02	0.14770175E 00	0.12226843E 01
0.31142733E 02	0.18316142E 00	0.14933567E 01
0.55862295E 02	0.21766546E 00	0.17471188E 01
0.99065219E 02	0.25104402E 00	0.19959212E 01
0.17682373E 03	0.28313846E 00	0.22475405E 01
0.32190746E 03	0.31380301E 00	0.25077311E 01
0.60433385E 03	0.34290617E 00	0.27812769E 01
0.11819385E 04	0.37033204E 00	0.30725949E 01
0.24329920E 04	0.39598148E 00	0.33861407E 01
0.53301352E 04	0.41977312E 00	0.37267383E 01
0.12586291E 05	0.44164410E 00	0.40998978E 01
0.32521308E 05	0.46155081E 00	0.45121681E 01
0.93662303E 05	0.47946922E 00	0.49715649E 01
0.30770272E 06	0.49539521E 00	0.54881314E 01
0.11877043E 07	0.50934453E 00	0.60747083E 01
0.55985297E 07	0.52135269E 00	0.67480740E 01

← collapse

$$z_0 / \tilde{\omega}_0 = .6$$

0.	0.	-0.17014118E 39
0.12621644E 01	0.38875066E-01	0.10111590E 00
0.53665859E 01	0.77537919E-01	0.72969809E 00
0.13377470E 02	0.11577847E 00	0.11263740E 01
0.27489139E 02	0.15339084E 00	0.14391612E 01
0.51874277E 02	0.19017544E 00	0.17149521E 01
0.94452729E 02	0.22594095E 00	0.19752145E 01
0.17063219E 03	0.26050622E 00	0.22320610E 01
0.31151139E 03	0.29370210E 00	0.24934739E 01
0.58272817E 03	0.32537313E 00	0.27654660E 01
0.11301688E 04	0.35537907E 00	0.30531433E 01
0.22978464E 04	0.38359633E 00	0.33613210E 01
0.49539037E 04	0.40991919E 00	0.36949476E 01
0.11467182E 05	0.43426085E 00	0.40594567E 01
0.28914565E 05	0.45655432E 00	0.44611167E 01
0.80803579E 05	0.47675305E 00	0.49074307E 01
0.25565015E 06	0.49483144E 00	0.54076460E 01
0.94072805E 06	0.51078509E 00	0.59734641E 01
0.41697583E 07	0.52463080E 00	0.66201109E 01

← collapse

$$z_0/\bar{\omega}_0 = .5$$

0.	0.	-0.17014118E 39
0.16626293E 01	0.40945169E-01	0.22079543E 00
0.70980641E 01	0.81660176E-01	0.85113992E 00
0.17815425E 02	0.12191716E 00	0.12507962E 01
0.36969860E 02	0.16149283E 00	0.15678478E 01
0.70677898E 02	0.20017072E 00	0.18492837E 01
0.13082810E 03	0.23774331E 00	0.21167010E 01
0.24120180E 03	0.27401415E 00	0.23823805E 01
0.45135459E 03	0.30879977E 00	0.26545178E 01
0.86974579E 03	0.34193156E 00	0.29393924E 01
0.17475993E 04	0.37325743E 00	0.32424419E 01
0.37059044E 04	0.40264332E 00	0.35688942E 01
0.83987142E 04	0.42997457E 00	0.39242128E 01
0.20629158E 05	0.45515705E 00	0.43144815E 01
0.55822108E 05	0.47811809E 00	0.47468063E 01 ← collapse
0.16974750E 06	0.49880722E 00	0.52298034E 01

$$z_0/\bar{\omega}_0 = .4$$

0.	0.	-0.17014118E 39
0.18238159E 01	0.44010153E-01	0.26098100E 00
0.78216524E 01	0.87764557E-01	0.89329852E 00
0.19782900E 02	0.13101002E 00	0.12962900E 01
0.41506016E 02	0.17349844E 00	0.16181110E 01
0.80510939E 02	0.21498927E 00	0.19058549E 01
0.15179586E 03	0.25525193E 00	0.21812600E 01
0.28627894E 03	0.29406814E 00	0.24567894E 01
0.55064465E 03	0.33123401E 00	0.27408715E 01
0.10966746E 04	0.36656221E 00	0.30400778E 01
0.22919984E 04	0.39988373E 00	0.33602143E 01
0.50929996E 04	0.43104962E 00	0.37069737E 01
0.12201560E 05	0.45993246E 00	0.40864154E 01 + collapse
0.32018754E 05	0.48642767E 00	0.45054045E 01

$$z_0/\bar{\omega}_0 = .3$$

0.	0.	-0.17014118E 39
0.16317852E 01	0.48494805E-01	0.21266300E 00
0.70344836E 01	0.96696896E-01	0.84723222E 00
0.17948492E 02	0.14431648E 00	0.12540280E 01
0.38131619E 02	0.19106960E 00	0.15812853E 01
0.75199096E 02	0.23668093E 00	0.18762126E 01
0.14477867E 03	0.28088658E 00	0.21607046E 01
0.28017451E 03	0.32343668E 00	0.24474286E 01
0.55600022E 03	0.36409792E 00	0.27450750E 01
0.11496110E 04	0.40265586E 00	0.30605509E 01 + collapse
0.25124252E 04	0.43891707E 00	0.34000932E 01

$$z_0/\bar{\omega}_0 = .2$$

0.	0.	-0.17014118E 39
0.10219806E 01	0.56437788E-01	0.94426512E-02
0.44316662E 01	0.11251996E 00	0.64656705E 00
0.11420348E 02	0.16789448E 00	0.10576794E 01
0.24609578E 02	0.22221636E 00	0.13911042E 01
0.49451669E 02	0.27515113E 00	0.16941809E 01
0.97495201E 02	0.32637817E 00	0.19889833E 01
0.19427429E 03	0.37559398E 00	0.22884153E 01
0.39946351E 03	0.42251506E 00	0.26014771E 01
0.86194712E 03	0.46688093E 00	0.29354806E 01

collapse

$$z_0/\bar{\omega}_0 = .1$$

0.	0.	-0.17014118E 39
0.27169368E 00	0.75573904E-01	-0.56592046E 00
0.11862155E 01	0.15084679E 00	0.74163596E-01
0.30922672E 01	0.22502264E 00	0.49027701E 00
0.67740870E 01	0.29771542E 00	0.83085077E 00
0.13911738E 02	0.36845388E 00	0.11433814E 01
0.28194206E 02	0.43678632E 00	0.14501599E 01

collapse

$$z_0/\bar{\omega}_0 = .05$$

0.	0.	-0.17014118E 39
0.50204316E-01	0.10372418E 00	-0.12992590E 01
0.22006238E 00	0.20674193E 00	-0.65745421E 00
0.57752028E 00	0.30835386E 00	-0.23843276E 00
0.12773183E 01	0.40787466E 00	0.10629914E 00
0.26566351E 01	0.50463987E 00	0.42433192E 00

collapse

a) Binary (circular orbits)

$$L = \frac{2}{5} \left(\frac{2m}{r}\right)^5 \quad \frac{\Delta m}{m} = \frac{m}{2r}$$

b) Head-on Collapse

$$L = \frac{1}{120} \left(\frac{2m}{r}\right)^5 \quad \frac{\Delta m}{m} = \frac{1}{210} \left(\frac{2m}{r}\right)^{7/2}$$

	L	$\frac{\Delta m}{m}$	L	$\frac{\Delta m}{m}$
3m	5.26×10^{-2}	$\frac{1}{6}$	1.09×10^{-3}	1.15×10^{-3}
6m	1.65×10^{-3}	$\frac{1}{12}$	3.43×10^{-5}	1.02×10^{-4}
12m	5.15×10^{-5}	$\frac{1}{24}$	1.07×10^{-4}	$.9 \times 10^{-5}$
20m	4.0×10^{-6}	.025	8.33×10^{-8}	1.51×10^{-6}

Table 1: Comparison of luminosity L and fractional change in energy $\frac{\Delta m}{m}$ for a two body system (each component of mass m) in two configurations

a) "Binary" circular orbits, and

b) Head-on collapse (no angular momentum). Units are relativistic:

$$G = c = 1$$

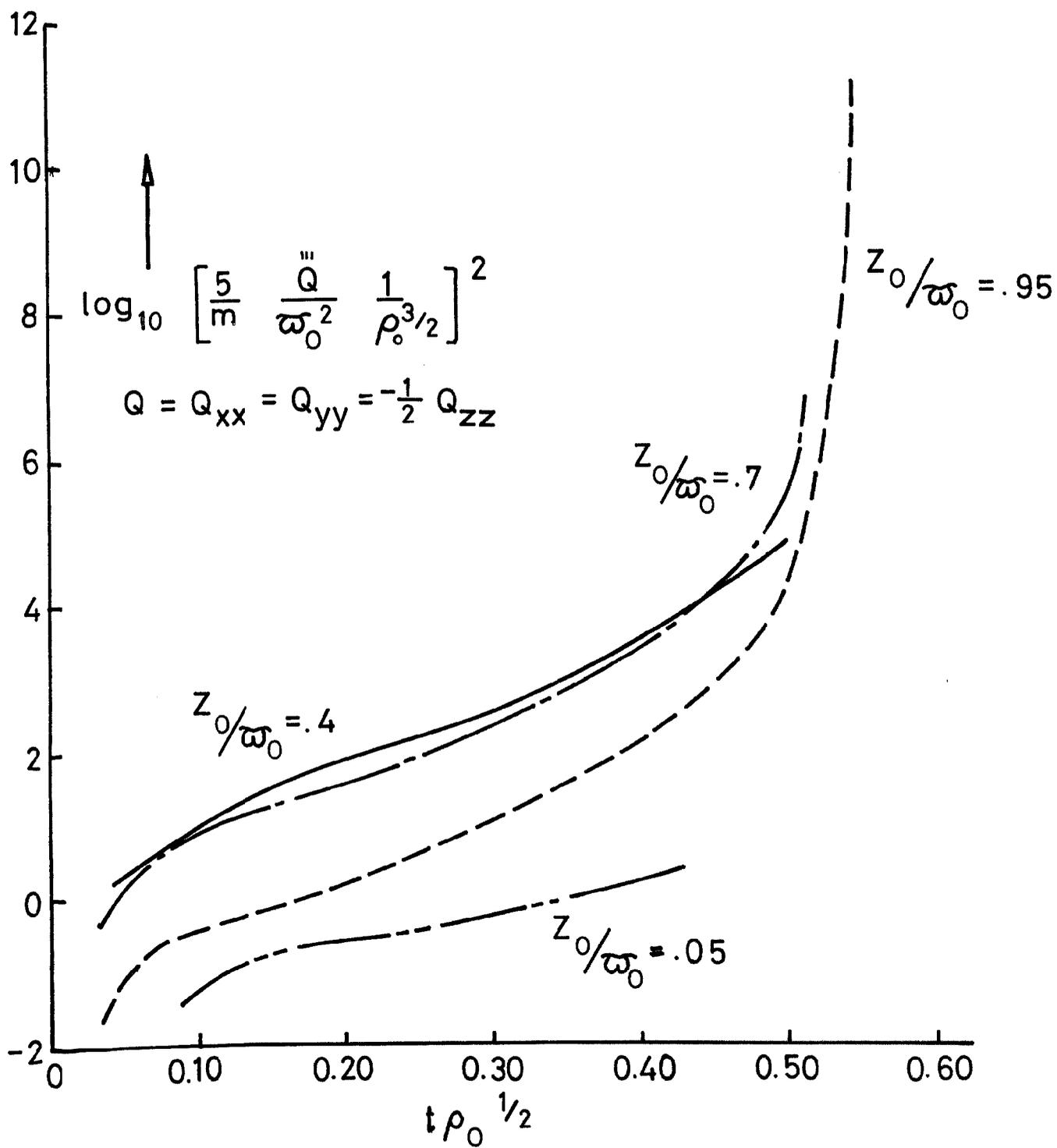


FIG. 1.

Caption for Figure 1:

Figure 1. Logarithm of the dimensionless luminosity

$[\frac{5}{m} \ddot{Q}/\bar{\omega}_0^2 1/\rho_0^{3/2}]^2$ for pressureless collapse of oblate dust ellipsoids.

The time is given in dimensionless units $\tau = t\rho_0^{1/2}$, with ρ_0 the initial dust density. The different curves are for several values of the initial parameter $Z_0/\bar{\omega}_0$ (initial ratio of semi-minor to semi-major radius).

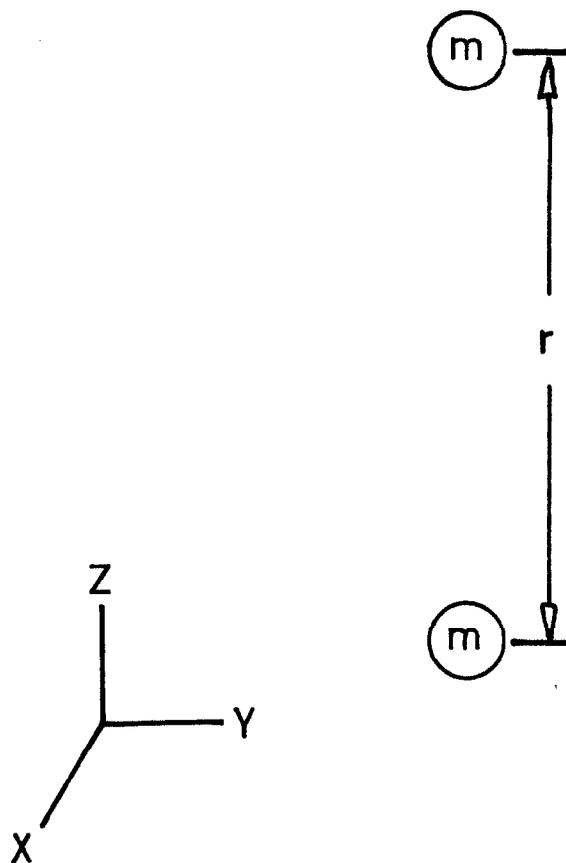


FIG 2 TWO PARTICLES SEPERATED BY DISTANCE r IN THE Z DIRECTION.

Appendix B

Electronic Computational Methods for General Relativity

The computational difficulties in finding the Einstein field equations once given the metric form can be overwhelming. In simple cases, direct hand calculations using the classical formulae (e.g. Landau and Lifschitz, 1962) or using the more modern techniques of differential geometry (e.g. Misner, 1963) are not too difficult. However, in really complicated situations, where the problem has little symmetry, the calculation can be a monumental task. In response to this, several investigators, including the author and R. W. Clemens at Maryland (Fletcher, Clemens, Matzner, Thorne, and Zimmerman, 1967; see also Fletcher, 1965, 1966, Clemens and Matzner, 1967, Thorne and Zimmerman, 1967) have turned to utilization of computer manipulation of algebraic structures. The computer techniques have in fact become quite sophisticated. Functional differentiation is a standard feature.

The University of Maryland system has provisions for calculation of most of the important geometric quantities, given $g_{\alpha\beta}$. It calculates, for instance, $R^{\alpha}_{\beta\gamma\delta}$, $R_{\alpha\beta}$, R and G^{α}_{β} . The main program has facilities for putting the calculated output on magnetic tape, to avoid the huge amounts of work which would be required to key-punch the material for re-input for later manipulation. Auxiliary programs have been devised to maintain and update a library of such output, and "restart" programs have been written which allow later manipulation of the results. For instance, programs have been developed which allow the substitution of quantities for others, and which allow termination at any desired order in an expansion of a small parameter.

The present state of computer technology limits the complexity of the jobs that can be run successfully. This is because of time limitations, and because of computer storage space limitations. Typical long runs on the 7094 run to (order of) one hour. (Although the exact Schwarzschild solution runs in a matter of seconds and a Bondi-Metzner metric like (2.4) takes about 45 minutes.) The other limitation is storage

size. Expressions being manipulated must be small enough so that if, for instance, $a + b$ is desired, both a and b can be simultaneously written into the core (high speed) storage area of the computer to be manipulated. Since the routines for algebraic manipulation are long and complicated, there is only a small amount of "free" core for this.

The complexity of the problem under consideration is the determining feature that separates the possible jobs from the impossible. The current generation of 7094 type computers, with the IBM FORMAC language, are really suitable for calculations such as perturbations on Schwarzschild. (See Campollatoro and Thorne, 1967). More difficult problems often exceed the computer's capabilities. It is hoped that the next generation of computers (of the high speed large store type) will effectively remove this limitation. It has been the author's experience that since only about 1/4 of the "core" of a 7094 is available for expression storage, even modest extra amounts of high speed storage area (in terms of the amount built into a 7094) would remove practically all limitations on the complexity of the intelligible problems that could be handled. (Intelligibility in this context can be quantitatively measured - although with only a rough cutoff - simply by placing the printed out put on a scale. More than 1 kg. \Rightarrow unintelligible, and this is a generous upper limit.)

We conclude with some sample calculations for the Schwarzschild spherically symmetric form:

$$ds^2 = e^{\lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - e^{\nu} dt^2$$

The results given on the following pages show the input metric, the determinant of the metric, and the Ricci tensor components. The quantity being computed is indicated by the heading at the top of the page; the relevant indices are printed just before the quantity corresponding to those indices.

The results are in standard FORTRAN notation, but for readers unfamiliar with computer output, we give the following explanations. Raising to a power is written

thus: $x^2 \equiv X^{**2}$, the functions $\sin x \equiv \text{FMCSIN}(X)$,
 $e^\lambda \equiv \text{FMCEXP}(\text{LAMBDA})$. Note that the Ricci tensor components are printed
twice (for irrelevant technical reasons); the dollar sign (\$) marks the end of an
expression so that the two copies can be distinguished. We have translated the
notation into more mathematical appearing formulae for the metric, the determinant
and some of the Ricci tensor components.

IT COVAR METRIC = $g_{\mu\nu}$

1
XP(PHI)§

← $g_{11} = e^{\psi}$

2

← $g_{12} = 0$

3

← $g_{13} = 0$

4

← $g_{14} = 0$

2
§

← $g_{22} = r^2$

3

← $g_{23} = 0$

4

← $g_{24} = 0$

3
*FMCSIN(THETA)**2§

← $g_{33} = r^2 \sin^2 \theta$

4

← $g_{34} = 0$

4
EXP(LAMBDA)§

← $g_{44} = -e^{\lambda}$

G(μ, ν)
14*FMCSIN(THETA)**2*FMCEXP(PHI+LAMBDA)§

$\det(g_{\mu\nu}) = -r^4 \sin^2 \theta e^{\psi + \lambda}$

PUT COVAR METRIC = $g_{\mu\nu}$

1 1
CEXP(PSI)\$

1 2

1 3

1 4

2 2
2\$

2 3

2 4

3 3
2*FMCSIN(THETA)**2\$

3 4

4 4
CEXP(LAMBDA)\$

1 G(μ, ν)
4*4*FMCSIN(THETA)**2*FMCEXP(PSI+LAMBDA)\$

$$\leftarrow g_{11} = e^{\psi}$$

$$\leftarrow g_{12} = 0$$

$$\leftarrow g_{13} = 0$$

$$\leftarrow g_{14} = 0$$

$$\leftarrow g_{22} = r^2$$

$$\leftarrow g_{23} = 0$$

$$\leftarrow g_{24} = 0$$

$$\leftarrow g_{33} = r^2 \sin^2 \theta$$

$$\leftarrow g_{34} = 0$$

$$\leftarrow g_{44} = -e^{\lambda}$$

$$\det(g_{\mu\nu}) = -r^4 \sin^2 \theta e^{\psi + \lambda}$$

3 3

R*FMCSIN(THETA)**2*FMCEXP(-PSI)*FMCCIF(PSI,(R,1))*2**(-1)+R*FMCSIN
 (THETA)**2*FMCEXP(-PSI)*FMCCIF(LAMBDA,(R,1))*(-2**(-1))+
 FMCSIN(THETA)**2-FMCSIN(THETA)**2*FMCEXP(-PSI)\$

R*FMCSIN(THETA)**2*FMCEXP(-PSI)*FMCCIF(PSI,(R,1))*2**(-1)+R*FMCSIN
 (THETA)**2*FMCEXP(-PSI)*FMCCIF(LAMBDA,(R,1))*(-2**(-1))+
 FMCSIN(THETA)**2-FMCSIN(THETA)**2*FMCEXP(-PSI)\$

**

3 4

0\$

0\$

**

4 4

R**(-1)*FMCEXP(-PSI+LAMBDA)*FMCCIF(LAMBDA,(R,1))+FMCEXP(-PSI+LAMBDA)
 A)*FMCDIF(PSI,(R,1))*FMCCIF(LAMBDA,(R,1))*(-4**(-1))+FMCEXP(-
 PSI+LAMBDA)*FMCDIF(LAMBDA,(R,1))*2*4**(-1)+FMCEXP(-PSI+LAMBDA)*F
 MCDIF(LAMBDA,(R,2))*2**(-1)+FMCCIF(PSI,(T,1))*FMCDIF(LAMBDA,
 (T,1))*4**(-1)+FMCDIF(PSI,(T,1))*2*(-4**(-1))+FMCCIF(PSI,(T,2))*(-
 2**(-1))\$

R**(-1)*FMCEXP(-PSI+LAMBDA)*FMCCIF(LAMBDA,(R,1))+FMCEXP(-PSI+LAMBDA)
 A)*FMCDIF(PSI,(R,1))*FMCCIF(LAMBDA,(R,1))*(-4**(-1))+FMCEXP(-
 PSI+LAMBDA)*FMCDIF(LAMBDA,(R,1))*2*4**(-1)+FMCEXP(-PSI+LAMBDA)*F
 MCDIF(LAMBDA,(R,2))*2**(-1)+FMCCIF(PSI,(T,1))*FMCDIF(LAMBDA,
 (T,1))*4**(-1)+FMCDIF(PSI,(T,1))*2*(-4**(-1))+FMCDIF(PSI,(T,2))*(-
 2**(-1))\$

**

Appendix C

Gravitational Field Equations for Sources with
Axial Symmetry and Angular Momentum

(published as a joint paper with C. W. Misner, which appeared in
Phys. Rev. 154, 1229, 1967)

Abstract

The investigation of stationary axially symmetric gravity fields leads to a reduced system involving two field variables which describe the "Newtonian" and the "rotation" part of the metric. This paper presents a parametrization of this reduced problem, which exhibits a previously unnoticed symmetry. Although the symmetry group (isomorphic to homogeneous Lorentz transformations on $2 + 1$ dimensional space) has a trivial action corresponding to unimodular linear transformations of the ϕt coordinate pair, its existence "explains" the existence of a very simple new Lagrangian for the reduced field equations, and the relatively simple form in which these equations (and the corresponding surface independent flux integrals for mass and angular momentum) can now be written.

Introduction and Summary

Previous studies¹⁾²⁾³⁾⁴⁾⁵⁾ of stationary vacuum solutions of Einstein's equations with axial symmetry have shown that the difficulties can be isolated in a reduced system involving only two independent coupled second order equations in the two basic unknown functions entering the metric. In this paper we point out a previously unnoticed symmetry group (isomorphic to the homogenous Lorentz transformations in 2 + 1 dimensional space) for this reduced problem. This symmetry governs the various ways in which the metric components can be expressed in terms of the two basic functions (field variables). In terms of two field variables α and β which we define, the reduced problem is summarized in a simple Lagrangian

$$\mathcal{L} = (\underline{\nabla}\beta)^2 - \cosh^2\beta (\underline{\nabla}\alpha)^2$$

involving only vector operations in flat Euclidean 3-space. For the corresponding field equations,

$$\underline{\nabla} \cdot \underline{M} = 0$$

where

$$\underline{M} \equiv e^{-i\alpha} (\underline{\nabla}\beta + \frac{1}{2} i \sinh 2\beta \underline{\nabla}\alpha) \quad ,$$

only those solutions with axial symmetry are accepted. For solutions satisfying appropriate conditions which guarantee that the corresponding metric is asymptotically flat and non-singular outside some bounded (source) region, the integral

$$\int_{\Sigma} (\underline{M} + \underline{\nabla} \cdot \underline{\rho}) \cdot d^2\underline{S} = 8\pi(m + i J)$$

(where $\rho^2 = x^2 + y^2$) has the same value on every closed 2-surface Σ surrounding the source, and gives the mass m and total angular momentum J of the system.

In the special case $\alpha = 0$ studied by Weyl and Levi-Civita⁵⁾, the metric is static and the function $\psi = \frac{1}{2}(\beta + \ell n\rho)$ satisfies the linear Laplace equation as in Newtonian theory. Angular momentum in the system will require a non-constant α so the field equations are no longer linear. The transformation group in $\alpha\beta$ space which characterizes the simplicity of the above Lagrangian does not act to produce usefully different solutions from any given one, for we find that these group transformations are equivalent to constant, unit determinant, linear transformations among the ϕt coordinates in the metric. Thus we have not been able to use the tantalizing simplicity of the Lagrangian to yield new solutions. In fact the only known metric (due to Kerr)¹⁵⁾ with both $J \neq 0$ and $m \neq 0$ in the class considered here gives prohibitively complicated forms for the fields α and β .

I. Metric Form and Symmetries

Einstein's equations for time independent metrics with axial symmetry have been discussed by several authors¹⁾²⁾³⁾⁴⁾⁵⁾. This paper will deal with vacuum solutions within a class in which the sources have an angular momentum distribution. In particular, stationary flow in the ϕ direction will be allowed in the source (although flow in other directions is excluded). These statements reflect the symmetries embodied in the metric form⁶⁾ which we assume:

$$ds^2 = g_{zz}(d\rho^2 + dz^2) + g_{XY} dx^X dx^Y \quad (1)$$

X and Y range and sum over ϕ and t .

The axially symmetric and stationary properties of the problem allow a metric form in which all the components of the metric are independent of ϕ and t . We have assumed that there is also a symmetry of the system:

$$(\phi, t) \rightarrow (-\phi, -t) \quad (2)$$

which leaves the metric invariant. It is this symmetry which eliminates the cross terms between the ρz and ϕt parts of the metric, since for instance $d\rho dt$ would change sign under (2). If there were matter flowing in the source in the ρ direction under (2), we would not expect the exterior metric to be invariant. We see that flow in the z direction is also excluded, but flow around the axis is allowed since $\frac{\partial \phi}{\partial t}$ does not change under (2). Since $d\phi dt$ is invariant under (2) we must in general write for the ϕt part ds_2^2 of the metric:

$$ds_2^2 \equiv g_{XY} dx^X dx^Y = g_{\phi\phi} d\phi^2 + 2g_{\phi t} d\phi dt + g_{tt} dt^2. \quad (3)$$

The form (1) is invariant under conformal transformations in the ρz plane. Since they do not involve ϕ or t , such transformations do not disturb the stationary axially symmetric character of the metric. The functional forms of g_{zz} and g_{XY} will change, and the form of the equation for the axis of rotation will be changed.⁷⁾

We shall demand that acceptable solutions be asymptotically flat. In this paper, asymptotically flat means

$$g_{\mu\nu} - \eta_{\mu\nu} = O\left(\frac{1}{r}\right) \quad (r \rightarrow \infty) \quad (4)$$

where $r^2 = x^2 + y^2 + z^2$ in rectangular coordinates given by the transformation

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \\ t &= t \end{aligned} \quad (5)$$

Demanding that (4) be satisfied gives the following asymptotic behavior for the components given by equation (1):

$$\begin{aligned} g_{tt} &= -1 + O\left(\frac{1}{r}\right) \\ g_{zz} &= 1 + O\left(\frac{1}{r}\right) \\ \frac{g_{\phi t}}{\rho} &= O\left(\frac{1}{r}\right) \\ f &\equiv \frac{g_{\phi\phi}}{2} - g_{zz} = O\left(\frac{1}{r}\right) \end{aligned} \quad (6)$$

It is a simple exercise in harmonic function theory to show that any metric with the symmetries assumed here which satisfies (4) can be put in the form (1) even with the further condition (12) below while maintaining evident asymptotic flatness in the sense that Eqs. (6) are satisfied. The condition that the metric be differentiable at the axis--assumed to have the equation $\rho = 0$ --gives for $\rho = 0$ that $g_{\phi t}/\rho^2$ be finite and

$$f = 0 = \frac{\partial f}{\partial \rho}$$

$$\frac{\partial g_{zz}}{\partial \rho} = 0 = \frac{\partial g_{tt}}{\partial \rho} \quad (7)$$

$$\frac{\partial}{\partial \rho} \left(\frac{g_{\phi t}}{\rho^2} \right) = 0 \quad .$$

II The Field Equations

It is well known¹⁾²⁾³⁾⁴⁾⁵⁾ that with the metric form (1) the field equations

$$R_{XY} = 0 \quad (8)$$

involve only g_{XY} and that the equations

$$R_{\rho\rho} = R_{zz} = R_{\rho z} = 0 \quad (9)$$

give g_{zz} by simple line integrals, once g_{XY} is known. Consequently, we deal with (8).

One of the equations in (8) is:

$$0 = g_{zz} (\det g_{XY})^{1/2} (R_{\phi}^{\phi} + R_t^t) = \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} \right) (\det g_{XY})^{1/2} . \quad (10)$$

Thus $\sqrt{\det g_{XY}}$ is a harmonic function of ρ and z and we may write

$$\det g_{XY} \equiv \bar{\rho}(\rho, z)^2 . \quad \text{By a conformal transformation}$$

we may make $\bar{\rho}$ one of the coordinates; and then with \bar{z} that harmonic function of ρ and z conjugate to $\bar{\rho}(\rho, z)$ we find⁸⁾ (after dropping the bars) in these new coordinates that

$$ds^2 = g_{zz} [d\rho^2 + dz^2] + g_{XY} dx^X dx^Y \quad (11)$$

with

$$\det g_{XY} = -\rho^2 . \quad (12)$$

To continue, we choose a parametrization of g_{XY}

$$[g_{XY}(\alpha, \beta)] = \begin{pmatrix} g_{tt} & g_{\phi t} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \cos \frac{\alpha}{2} & \sin \frac{\alpha}{2} \\ -\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} -\rho e^{\beta} & 0 \\ 0 & \rho e^{-\beta} \end{pmatrix} \begin{pmatrix} \cos \frac{\alpha}{2} & -\sin \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix}. \quad (13)$$

The resulting metric components are

$$\begin{aligned} g_{tt} &= -\rho (\cos\alpha \cosh\beta + \sinh\beta) \\ g_{\phi\phi} &= \rho (\cos\alpha \cosh\beta - \sinh\beta) \\ g_{\phi t} &= \rho \sin\alpha \cosh\beta. \end{aligned} \quad (14)$$

The field variables are β , the "Newtonian" field, and α , the "rotation" field. From (13) $\det g_{XY} = -\rho^2$. The conditions (7) for regularity on the axis $\rho = 0$ then require that $\alpha\rho^{-2}$ and $2\psi \equiv \beta + \ln\rho$ be differentiable there. (That is, be finite and have $\frac{\partial}{\partial\rho} = 0$ at $\rho = 0$). Similarly, conditions (6) for asymptotic flatness require that ψ , $\alpha\rho$ and $\alpha\rho^{-1}$ all be $O(\frac{1}{r})$.

We have satisfied Eq. (10) by the choice (14) for the metric, and the remaining field equations are now most easily found in the following way. Suppose ξ^μ is a Killing vector: $\xi_{(\mu;\nu)} = 0$. Then the quantity⁹⁾

$$E^\mu(\xi) = \frac{1}{2} (\xi^{\mu;\nu} - \xi^{\nu;\mu})_{;\nu}$$

becomes simply

$$E^\mu(\xi) = -R^\mu_{\nu} \xi^\nu. \quad (15)$$

In particular,

$$E^t(\xi) = -R^t_{\nu} \xi^{\nu} \quad (16)$$

In the problem under consideration, the ϕ and t Killing vectors (with components δ^{μ}_{ϕ} and δ^{μ}_t respectively) give (using metric form (13))

$$\begin{aligned} -R^t_t = E^t(t) &= \underline{\nabla} \cdot [\underline{\nabla}(\ln\rho) + \cos\alpha \underline{\nabla}\beta + \frac{1}{2} \sin\alpha \sinh 2\beta \underline{\nabla}\alpha] = 0 \\ -R^t_{\phi} = E^t(\phi) &= \underline{\nabla} \cdot [-\cosh^2\beta \underline{\nabla}\alpha - \sin\alpha \underline{\nabla}\beta + \frac{1}{2} \cos\alpha \sinh 2\beta \underline{\nabla}\alpha] = 0 \end{aligned} \quad (17)$$

The symbol $\underline{\nabla}$ has its flat 3-space meaning:

$$\underline{\nabla}^i \underline{\nabla}_i \alpha = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \alpha \quad \text{in coordinates } x, y, z \text{ given by}$$

transformation (5).

We drop the $\underline{\nabla}(\ln\rho)$ term since $\underline{\nabla}^2(\ln\rho)$ vanishes away from the axis $\rho = 0$, but we must then remember to check regularity at $\rho = 0$ ⁽¹⁰⁾ in any solution. These equations⁽¹¹⁾ are then equivalent to

$$\begin{aligned} \underline{\nabla} \cdot (\cosh^2\beta \underline{\nabla}\alpha) &= 0 \\ \underline{\nabla}^2\beta + \frac{1}{2} \sinh 2\beta (\underline{\nabla}\alpha)^2 &= 0 \end{aligned} \quad (18)$$

or equivalently

$$\begin{aligned} \underline{\nabla} \cdot \underline{M} &= 0 \\ \underline{M} &\equiv e^{-i\alpha} \left[\underline{\nabla}\beta + \frac{i}{2} \sinh 2\beta \underline{\nabla}\alpha \right] \end{aligned} \quad (19)$$

Since the quantities $E^{\nu}(\xi)$ are Komar's⁽⁹⁾ conserved quantities, we have an integral conservation law:

$$\int_{\Sigma} (\underline{M} + \underline{\nabla}(\ln\rho)) \cdot d\underline{s} = 8\pi [m + iJ] \quad (20)$$

where m and J are the total mass and angular momentum, respectively, of the source, and the 2-surface Σ is any 2-surface completely containing the source.

Equations (18) may be derived from the variation of an action integral

$$I = \int \mathcal{L} d\zeta \quad (21)$$

where $d\zeta = dx dy dz = 2\pi\rho d\rho dz$ is the flat 3-space volume element and

$$\mathcal{L} = (\nabla\beta)^2 - \cosh^2\beta (\nabla\alpha)^2. \quad (22)$$

The form

$$d\ell^2 = d\beta^2 - \cosh^2\beta d\alpha^2 \quad (23)$$

suggested by the Lagrangian is the metric for a hyperboloid given by

$$(\vec{\lambda})^2 = \xi^2 + \eta^2 - \tau^2 = 1.$$

This is the unit distance hyperboloid in a Lorentz 3-space with ξ, η space-like and τ time-like coordinates (Fig. 1). Realizing this, the Lagrangian may be written

$$\mathcal{L} = g^{ij} h_{AB} \frac{\partial y^A}{\partial x^i} \frac{\partial y^B}{\partial x^j} \quad (24)$$

where g^{ij}, x^i, x^j refer to the real Euclidean 3-space and h_{AB}, y^A, y^B are the metric tensor and coordinates on the hyperboloid in some coordinate system. Eq. (24) allows immediate changes of the field variables α, β to any other parameterization. If $\vec{\lambda}$ refers to a point on the hyperboloid, (24) can be written

$$\mathcal{L} = (\nabla\vec{\lambda})^2 \quad (25)$$

From this point of view the Lagrangian is just a kinetic term--the square of the gradient of this field quantity. Eq. (25) makes it fairly clear that no parameterization will significantly alter the form of the equations. Their form could be changed one step further back by giving up the specialization (12) with its coordinate condition $\sqrt{-\det g_{XY}} = \rho$.

Besides the substitutional transformations which amount to changing coordinates on the hyperboloid, the Lagrangian is obviously invariant under those transformations in the ξ, η, τ space which leave the hyperboloid invariant. These are just the Lorentz transformations in that space. These transformations are discussed in part III.

III Transformations in the Field Variables Leaving
the Lagrangian Invariant - Lorentz Transformations
in Minkowskian 3-Space

Since a hyperboloid in Minkowski space is a surface of constant curvature, all of its points are equivalent. The transformations which leave the hyperboloid invariant have generators (Fig. 1)⁽¹²⁾

$$\begin{aligned}
 L_{\tau} &= \frac{\partial}{\partial \alpha} \\
 L_{\xi} &= \cos \alpha \tanh \beta \frac{\partial}{\partial \alpha} + \sin \alpha \frac{\partial}{\partial \beta} \\
 L_{\eta} &= +\sin \alpha \tanh \beta \frac{\partial}{\partial \alpha} - \cos \alpha \frac{\partial}{\partial \beta} .
 \end{aligned}
 \tag{26}$$

Here the subscript on the generators L_i names the invariant axis under the rotation.

Direct calculation gives⁽¹³⁾

$$\begin{aligned}
 L_{\tau} M &= -iM \\
 L_{\xi} M &= \cosh^2 \beta \nabla \alpha \\
 L_{\eta} M &= -i \cosh^2 \beta \nabla \alpha .
 \end{aligned}
 \tag{27}$$

Using finite Lorentz rotations (or their infinitesimal counter-parts (26)) and (27), new solutions may be obtained from others by applying the group operations. From (26) and (27) it might seem that we would get a different solution with a different value of angular momentum just by increasing α slightly- using L_{τ} or its finite form, G_{τ} . But this is not the case. From (13) we see that the transformation $\alpha \rightarrow \alpha + \gamma$ (γ constant) is equivalent to the transformation

$$\begin{aligned}
 t &\rightarrow t \cos \gamma/2 - \phi \sin \gamma/2 \\
 \phi &\rightarrow t \sin \gamma/2 + \phi \cos \gamma/2 .
 \end{aligned}
 \tag{28}$$

Consider now a finite transformation G_η generated by $L_\eta^{(16)}$.

Then we have:

$$\begin{aligned}\tau' &= \sinh \beta' = \sinh \beta \cosh u - \cosh \beta \cos \alpha \sinh u \\ \xi' &= \cosh \beta' \cos \alpha' = \cosh \beta \cosh u \cos \alpha - \sinh \beta \sinh u \\ \eta' &= \eta = \cosh \beta \sin \alpha\end{aligned}\quad (29)$$

Then the new metric components g'_{XY} are given from (14):

$$\begin{aligned}g'_{tt} &= -\rho(\cos \alpha \cosh \beta (\cosh u - \sinh u) + \sinh \beta (\cosh u - \sinh u)) = g_{tt} e^{-u} \\ g'_{\phi\phi} &= \rho(\cos \alpha \cosh \beta (\cosh u + \sinh u) - \sinh \beta (\cosh u + \sinh u)) = g_{\phi\phi} e^u \\ g'_{\phi t} &= \rho \cosh \beta \sin \alpha = g_{\phi t}\end{aligned}\quad (30)$$

The field equations $R_{AB} = 0$, $A, B = \rho, z$, show that g_{zz} is unchanged under (29) ⁽¹⁴⁾. We thus have

$$ds'^2 = g_{zz} [d\rho^2 + dz^2] + g_{\phi\phi} e^u d\phi^2 + 2g_{\phi t} d\phi dt + g_{tt} e^{-u} dt^2.$$

Note that this form violates conditions (6) because of the factor

$e^u g_{\phi\phi}$. We can remove this undesirable behavior by a simple coordinate transformation:

$$\bar{\phi} = e^{u/2} \phi, \quad \bar{t} = e^{-u/2} t \quad (31)$$

Then

$$ds'^2 = g_{zz} [d\rho^2 + dz^2] + g_{\phi\phi} d\bar{\phi}^2 + 2g_{\phi t} d\bar{\phi} d\bar{t} + g_{tt} d\bar{t}^2.$$

The transformation G_η is thus equivalent to a coordinate transformation

in ϕ and t . Note that both G_τ and G_η give rise to linear coordinate

transformations of determinant unity, so the same will also be true of

$G_\xi = G_\tau^{-1} G_\eta G_\tau$. Thus our investigation of the generators is

complete.

Footnotes for Appendix C.

- 1) A. Papapetrou, Preprint (1965), to be pub. Ann. Inst. H. Poincaré, Paris.
- 2) A. Papapetrou, Ann. der Physik 12 (1953) 209
- 3) T. Lewis, Proc. Roy. Soc. A136 (1932) 176
- 4) W. J. Van Stockum, Proc. Roy. Soc. Edin. 57 (1937) 135
- 5) J. L. Synge "Relativity: The General Theory" (North-Holland, Amsterdam, 1960) Ch. 8, § 1, and earlier references mentioned there.
- 6) Papapetrou (footnote 1) has given an algebraic proof (exhibiting the coordinate transformations) to show that the form (1) may be imposed on all stationary/axi-symmetric fields satisfying the vacuum equations outside a world tube of finite space-like section.
- 7) An example is flat space in parabolic coordinates where the metric is $ds^2 = (\xi_1^2 + \xi_2^2)(d\xi_1^2 + d\xi_2^2) + \xi_1^2 \xi_2^2 d\phi^2 - dt^2$. The correspondence to cylindrical coordinates is

$$\begin{aligned} \rho &= \xi_1 \xi_2 \\ \phi &= \phi \\ z &= \frac{1}{2} (\xi_1^2 - \xi_2^2) \\ t &= t. \end{aligned}$$

z and ρ are conjugate harmonic functions of ξ_1 and ξ_2 (The real and imaginary parts of $1/2 (\xi_1 + i\xi_2)^2$ respectively). But the axis is given by $\xi_1 = 0$ for $z > 0$ or $\xi_2 = 0$ for $z < 0$.
- 8) This is the usual choice for ρ and z ; see 1)-5) above.
- 9) A. Komar, Phys. Rev. 127 (1962) 1411.
- 10) As stated following eqn. (14) above.

- 11) Papapetrou (footnote 1) has given a different form for the metric and has obtained slightly more complicated second order equations as well as the corresponding fourth order equations.
- 12) An excellent discussion of the unit distance hyperboloid and possible coordinates and curves on it can be found in E. Schroedinger, Expanding Universes (Cambridge Press, Cambridge, 1956) particularly Chapter 1.
- 13) We note that $[L_i, \nabla] = 0$, $i = \xi, \eta, \tau$.
- 14) One of the equations $R_{AB} = 0$, $A, B = \rho, z$ is $R_{\rho z} = 0$. It reads
- $$(\ln g_{zz})_{,z} = -\frac{1}{2\rho} (2g_{\phi t, \rho} g_{\phi t, z} - g_{tt, \rho} g_{\phi\phi, z} - g_{tt, z} g_{\phi\phi, \rho}).$$
- The RHS of this equation is invariant under (30) as are the other equations $R_{AB} = 0$.
- 15) R.P. Kerr, Phys. Rev. Lett. 11 522 (1963).
- 16) u is the hyperbolic angle of the "rotation" due to G_n .

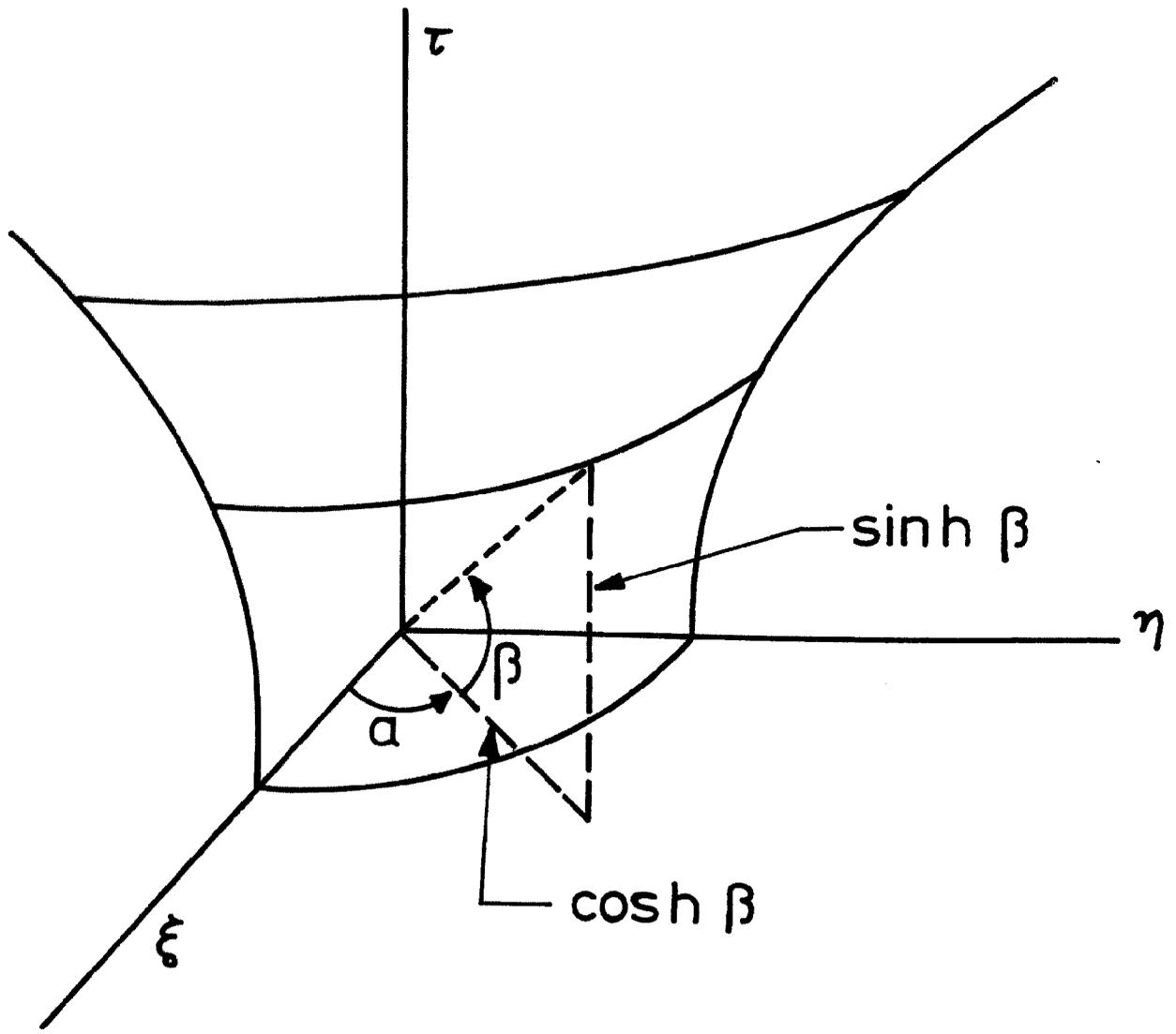


Fig 1

Fig. 1: The unit distance hyperboloid in the Minkowskian 3-space with ξ, η spacelike variables and τ timelike. The relations of α, β to ξ, η, τ are $\xi = \cos \alpha \cosh \beta$, etc., as indicated.

Appendix D

The Symmetry of the Space Sections of the Taub Cosmological Solution

Abstract

Calculations are given which are applications of the ideas of almost symmetric spaces previously developed by the author. The particular example chosen is a cosmological solution due to Taub which is a generalization of the closed Robertson-Walker metrics.

The complete eigenvalue spectrum of the differential operator $-\mathcal{D}\xi = -\sum_{ij} \xi^{(ij)}$ is given when the vector ξ lies in the 3-space $\mathcal{M}(t)$ in the Taub solution (which is characterized by being homogeneous for each t), and the covariant derivatives are taken in $\mathcal{M}(t)$. The method is to utilize the symmetries of $\mathcal{M}(t)$ (four Killing fields) and the topological equivalence of $\mathcal{M}(t)$ to the 3-sphere S^3 , to express the eigen values in terms of easily calculated quantum numbers and to express the eigen functions ξ as a "rigid rotator" eigen function [which is specified by its quantum numbers under transformations by the rotation group ($\cong S^3$)], times an invariantly defined vector field.

The equivalence of $\mathcal{M}(t)$ to S^3 is also utilized to construct iteratively a background space which is completely symmetric, i.e. is metrically S^3 . The method of accomplishing this is to average the metric tensor along the vector fields which can be invariantly defined on $\mathcal{M}(t)$, and which fields are also characterized by being the nowhere zero non-Killing eigenvectors (corresponding to the few lowest eigenvalues) of $-\mathcal{D}$. The purpose of these exercises is to characterize the behaviour of the eigenvalue spectrum, in the hopes of being able to reconstruct the entire metric from a knowledge of the spectrum of $-\mathcal{D}$. The averaging process given here is similarly a model of the idea of invariantly defining a background space by averaging along the eigenvectors corresponding to some of the lowest eigen values of $-\mathcal{D}$. Both these ideas may be important in the consideration of gravitational radiation.

I Introduction

In the main body of this work we have given a criterion for the numerical specification of the lack of symmetry in a Riemannian manifold. The method consists of minimizing (finding the stationary points for Minkowski signatures) quantity

$$\lambda = \frac{\int \mathbb{F}^{(\alpha;\beta)} \mathbb{F}_{(\alpha;\beta)} dV}{\int \mathbb{F}^\alpha \mathbb{F}_\alpha dV} \quad (1)$$

where \mathbb{F} is an arbitrary vector field (satisfying boundary conditions that \mathbb{F} vanish sufficiently fast at infinity in open spaces) and the integrand in the numerator is the square of the symmetrized covariant derivative.

It was also shown that the eigenvalues λ measure some parameters of short wave gravitational radiation. In particular, for spaces of the same large scale shape (given by a suitable background metric $\gamma_{\mu\nu}$) we have

$$\lambda = \lambda_\gamma + 8\pi \frac{\int T_{(\alpha\nu)} \mathbb{F}_\mu \mathbb{F}_\nu \sqrt{-\gamma} d^4x}{\int \mathbb{F}_\alpha \mathbb{F}^\alpha \sqrt{-\gamma} d^4x} + \mathcal{O}(\lambda) \quad (2)$$

where λ_γ is a function only of the background and the vector \mathbb{F} , and $T_{(\alpha\nu)}$ is an average stress tensor (as defined for instance by Isaacson⁽¹⁾) of the short wave gravitational radiation (of wavelength λ).

In considering the problem of gravitational radiation from this viewpoint several problems are suggested. One is to find a background if one exists in space presumably containing a background plus wave. Isaacson assumes that if separation is possible, it can be found by sufficiently intense inspection. But it would be much more satisfactory to have an invariant method of finding the background when one exists. We propose that the background can be found by averaging along the eigenvectors corresponding to the few lowest eigenvalues of λ . We will consider this question for the Taub space slices in Section IV but we shall need the lowest eigenvalues and eigenvectors, which we find in section

The eigenvalue spectrum is one of the requisites for still another problem suggested by Isaacson's investigation: to more completely specify the metric.

That is, we wish to find the background and the derivations completely and invariantly, without making any short wavelength assumptions as are done by Isaacson. (This would clearly include the specification of the background done in Section IV.) We expect that this project can be accomplished through use of the eigenvalue spectrum of (1), or alternately the second order operator defined by (1):

$$-\mathcal{D}^{\mu\nu} g_{\mu\nu} \equiv -\mathcal{M}^{(\mu;\alpha)}{}_{;\alpha} \quad (3)$$

(The problem of specifying the metric is then analogous to finding a potential in quantum mechanics from a knowledge of only the eigenfunctions and energy eigenvalues.) For these reasons we will consider the spectrum of $-\mathcal{D}$ in Section III.

The problems of Minkowski signature in considering equation (1) or (3) are manifest, so in investigating the ideas suggested here, we will confine ourselves to a positive definite manifold which has some intrinsic interest. We shall consider the symmetry of closed spacelike 3-manifolds $\mathcal{M}(t)$ which are the time slices of the Taub part of Taub-NUT space [given by the t-constant part of equation (4) below]. Since this manifold is compact and analytic, the eigenvalue spectrum, [the stationary values of (1)] will correspond to a countable set of analytic vector fields on the manifold $\mathcal{M}(t)$. The integrals in (1) are positive and bounded on these compact $\mathcal{M}(t)$

For each t the manifold $\mathcal{M}(t)$ is topologically a 3-sphere, S^3 . The spatial metric possesses four Killing fields; three of which describe the spatial homogeneity of the spaces, and the fourth giving the one axis of isotropy at each point (Section II below) $\mathcal{M}(t)$ can in fact be characterized by its homogeneity.

Two other invariant vector fields (the two other symmetries of S^3) can be defined on $\mathcal{M}(t)$, due to an especially simple topological equivalence of $\mathcal{M}(t)$ to S^3 .

The eigenspectrum of equation (1) or (3) clearly starts at $\lambda_0 = 0$, because there are Killing vectors. We shall be interested in the other eigenvalues and eigenvectors. We shall in fact find the complete spectrum of eigenvalues and the

complete set of corresponding eigenvectors.

II The Taub Solution

Taub⁽²⁾ obtained a cosmological solution which is in a sense a generalization of the Robertson-Walker⁽³⁾ (R-W) metrics. It has $R_{\alpha\beta} = 0$ and homogeneous but not isotropic $t = \text{constant}$ space sections $\mathcal{M}(t)$. In one particular coordinate system⁽⁴⁾

$$ds^2 = (t^2 + l^2) (d\theta^2 + \sin^2\theta d\phi^2) + U(t) (2l)^2 (d\psi + \cos\theta d\phi)^2 - 2(2l) (d\psi + \cos\theta d\phi) dt. \quad (4)$$

Here $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$ are coordinates on S^3 ; l is a constant length ($l > 0$) and $U(t)$ is

$$U(t) = -1 + \frac{2(mt + l^2)}{t^2 + l^2} = \frac{(t - t_-)(t - t_+)}{t^2 + l^2},$$

where $t_{\pm} = m \pm (m^2 + l^2)^{1/2}$; m is another positive constant length. The function $U(t)$ is positive only for $t_- < t < t_+$; for t in this range, $\mathcal{M}(t)$ is spacelike. The surfaces $t = t_{\pm}$ are null surfaces which

bound the solution from another region of 4-space described by the empty space Newman, Unti, and Tamborino (NUT) space⁽⁵⁾, which is given by the same metric, (4), in the region where $U(t) < 0$. In NUT space, t is a spacelike coordinate.

In the following we use $A^2 = t^2 + l^2$ and $B^2 = 4l^2 U$.

The homogeneity of the space-slices is demonstrated by the three Killing vectors (which are Killing in $\mathcal{M}(t)$ for each t (4)):

$$\xi_{\theta} = -\sin \phi \partial_{\theta} - \cos \phi (\cot \theta \partial_{\phi} - \csc \theta \partial_{\psi})$$

$$\xi_{\phi} = \cos \phi \partial_{\theta} - \sin \phi (\cot \theta \partial_{\phi} - \csc \theta \partial_{\psi})$$

$$\xi_{\psi} = \partial_{\psi}$$

Although these look like rotations they have no fixed points, since they have nonvanishing length; see eqn.(8).

There is also one more Killing vector in $\mathcal{M}(t)$:

$$\xi_{\psi} = -\partial_{\psi}$$

In addition we may define two more vectors which are invariantly distinguished on the 3-sphere, but are not Killing in $\mathcal{M}(t)$:

$$\xi_{\theta} = -\sin \psi \partial_{\theta} + \cos \psi (\csc \theta \partial_{\phi} - \cot \theta \partial_{\psi})$$

$$\xi_{\psi} = -\cos \psi \partial_{\theta} - \sin \psi (\csc \theta \partial_{\phi} - \cot \theta \partial_{\psi})$$

The topological equivalence mentioned in Section I is simply given by the coordinates

θ, ϕ, ψ , which would be the Euler angle coordinates on S^3 if

$$t^2 + \ell^2 = 4\ell^2 \alpha.$$

The Taub solution is a generalization of the closed R.W. model in the sense that $\mathcal{M}(t)$ for each $t \in \langle t_-, t_+ \rangle$ has only four Killing vectors. The corresponding slice in the R.W. metric is topologically S^3 also, but the metric there does not distinguish any direction on the 3-sphere, and the R.W. metrics have six Killing vectors in each time-slice; the full set of generators for the symmetries of the 3-sphere.

III The Eigenvalues of $-D$ on $\mathcal{M}(t)$

Because of the large number of symmetries still available in the Taub space slice

We are able to find the spectrum of the operator $-\mathcal{D}$ explicitly.

We have the following commutation relations for the invariant vector fields⁴ (where these are Lie brackets):

$$\begin{aligned} [\xi_i, \xi_j] &= -\epsilon_{ijk} \xi_k, \\ [\xi_i, \eta_j] &= 0, \\ [\eta_i, \eta_j] &= -\epsilon_{ijk} \eta_k. \end{aligned} \quad (5)$$

and $-\mathcal{D}$ which plays a role like the Hamiltonian in quantum mechanics, commutes with ξ_i and η_z . ($-\mathcal{D}$ commutes with each of these because they are Killing vectors and $-\mathcal{D}$ is comprised only of the metric and covariant derivatives.) The commutation equations(5) for the ξ_i show that they behave like an angular momentum, and we may pick one, say ξ_z , as well as $-\mathcal{D}$ and η_z and ξ^2 to be part of our set of commuting operators.

Also, the projection operator $P_z \equiv B^{-2} \eta_z \cdot$ commutes with \mathcal{D} and with the rest of the operators, because it consists only of the Killing vector η_z , the metric (\cdot), and the constant [in $\mathcal{M}(t)$] B^2 . Thus the orthogonal projection $P_1 \equiv 1 - P_z$ commutes with the rest of the operators. P_1 projects into the direction orthogonal to η_z . Because P_z commutes, we see that if our eigenvector is parallel to η_z , it will not be mixed into the $\eta_x - \eta_y$ directions, and vice-versa. Since the only quantity in the metric which picks out a direction in the $\eta_x - \eta_y$ plane is η_z we may take as a basis in this plane the quantities which are eigenvectors of $-i\eta_z$, i.e. $\eta_{\pm} = \eta_x \pm i\eta_y$. We will label all our functions with the quantum number s , the "z component of internal spin"; $s=0$ for η_z , $s=\pm 1$ for η_{\pm} . Each of our functions ψ_s will then be $\psi_s = f \eta_s$ where f is a scalar function and $\eta_{s=1} = \eta_{\pm}$, etc. Clearly an algebraic

projection operator can be defined to give s :

$$S_{op} = DP_L ; D = \frac{1}{2} A^{-2} \{ (\underline{\eta}_x + i \underline{\eta}_y)(\underline{\eta}_x - i \underline{\eta}_y) - (\underline{\eta}_x - i \underline{\eta}_y)(\underline{\eta}_x + i \underline{\eta}_y) \}$$

Since $\underline{\eta}_z$ distinguishes $s=+1$ from $s=-1$, we expect the eigenvalues of $-D$ to depend on s .

The vectors $\underline{\xi}_i, \underline{\eta}_i$ are the generators of motions on S^3 . Coordinates on S^3 are Θ, Φ, Ψ ; Euler angles. We thus see that a specification of the quantum numbers $-i \underline{\xi}_z^2 = j_z, -\underline{\xi}^2 = j(j+1)$, and $-i \underline{\eta}_z = k$ will completely determine the functional form of scalar eigen functions on $\mathcal{M}(t)$ to be, in fact, a "rigid rotator eigen function" (see Ref. 6). (Since $\underline{\xi}_x + i \underline{\xi}_y$ commutes with $-D$, j_z will not actually enter the formula for the eigenvalue.) We have just given the vector part of the eigenfunction specified by s . Thus λ , the eigenvalue of $-D$, is $\lambda = \lambda_{jks}$. (Even if $\underline{\eta}_z$ were not a symmetry, we would expect to be able to express the eigenfunctions as a sum over different k -values. Thus even the case of completely anisotropic space slices $\mathcal{M}(t)$ [only $\underline{\xi}_i$: Killing] should also be possible by this method.)

The commutation relations allow both j and k to be half integral, but they are in fact integral, as we can see by the following argument. Suppose $\underline{\zeta}$ is an eigen vector of $\underline{\xi}^2$:

$$\begin{aligned} \underline{\xi}^2 [\underline{\zeta}] &= [\underline{\xi}_i, [\underline{\xi}_i, f \underline{\eta}_z]] = \underline{\eta}_s \underline{\xi}_i [\underline{\xi}_i [f]] \\ &= -\underline{\eta}_s f j(j+1) \\ &= -j(j+1) \underline{\zeta} . \end{aligned} \quad [\text{We used eqns. (5) .}]$$

But since $j(j+1)$ was obtained by the differentiation of a single valued scalar function, it must be integral. Similarly, if $\underline{\zeta}$ is an eigen function of $\underline{\eta}_z$,

$$\begin{aligned} -i \underline{\eta}_z [\underline{\zeta}] &= -i [\underline{\eta}_z, f \underline{\eta}_s] \\ &= f [-i \underline{\eta}_z, \underline{\eta}_s] + \underline{\eta}_z [f] \underline{\eta}_s \\ &= s f \underline{\eta}_s + (k-s) f \underline{\eta}_s = k \underline{\zeta} , \end{aligned}$$

But S is integer and $(k \cdot s)$ is obtained by differentiation of a single valued scalar function, so k is an integer.

In the following we will take advantage of the formula

$$\mathcal{L}_{\xi} g^{ab} = g^{ab}{}_{;c} \xi^c - \xi^a{}_{;c} g^{cb} - \xi^b{}_{;c} g^{ca} \equiv -2 \xi^{(a;b)},$$

and also⁽⁷⁾

$$\oint \mathcal{L}_{\xi} (\text{scalar density}) d^4x = 0, \quad (6)$$

where this integration is over a compact orientable surface, and the d^4x is coordinate volume; \underline{P} is any vector field.

We thus consider the eigenvalue ratio eqn (1):

$$\lambda = \frac{\int \mathcal{L}_{\xi} g'' : \mathcal{L}_{\xi} g'' \sqrt{|g|} d^3x}{\int \xi \cdot \xi \sqrt{|g|} d^3x},$$

where we indicate by g'' the contravariant metric in $\mathcal{M}(t)$ (as distinct from its components g^{ab}). The two dots between the terms in the numerator indicate that we take the double contraction between the two second rank 3-tensors $\mathcal{L}_{\xi} g''$. While we should a priori take the variation of this ratio, we know that λ is a function of only j, k, s , so we construct vectors $\underline{\xi}$ which have these quantum numbers. These vectors are then automatically eigenvectors of and the integral ratio gives the eigenvalue directly.

We use the following formula due to Misner⁽⁸⁾:

$$g'' = A^{-2} \underline{\xi}_i \otimes \underline{\xi}_i + (B^{-2} - A^{-2}) \underline{\eta}_z \otimes \underline{\eta}_z \quad (7)$$

where we have abbreviated

$$B^2 = B^2(t) \equiv 4\ell^2 U(t) \quad \text{and} \quad A^2 = A^2(t) \equiv t^2 + \ell^2.$$

Then,

$$\begin{aligned}
-2 \zeta^{(\cdot\cdot\cdot)} &= \mathcal{L}_{\zeta} g^{\cdot\cdot} \\
&= A^{-2} ([\zeta, \xi_i] \otimes \xi_i + \xi_i \otimes [\zeta, \xi_i]) \\
&\quad + (B^{-2} - A^{-2}) ([\zeta, \eta_z] \otimes \eta_z + \eta_z \otimes [\zeta, \eta_z]) .
\end{aligned}$$

Since $[\zeta, \rho] = -[\rho, \zeta]$ for any vectors ζ, ρ , we write

$$\begin{aligned}
(\mathcal{L}_{\zeta} g^{\cdot\cdot}) : (\mathcal{L}_{\zeta} g^{\cdot\cdot}) &= \\
&= [\mathcal{L}_j A^{-2} (\bar{\zeta} \otimes \xi_j + \xi_j \otimes \bar{\zeta}) + \mathcal{L}_z (B^{-2} - A^{-2}) (\bar{\zeta} \otimes \eta_z + \eta_z \otimes \bar{\zeta})] : \\
&= [\mathcal{L}_i A^{-2} (\zeta \otimes \xi_i + \xi_i \otimes \zeta) + \mathcal{L}_z (B^{-2} - A^{-2}) (\zeta \otimes \eta_z + \eta_z \otimes \zeta)] .
\end{aligned}$$

We write $\mathcal{L}_i = \mathcal{L}_{\xi_i}$ and $\mathcal{L}_z = \mathcal{L}_{\eta_z}$. This should cause no confusion since we will never explicitly consider the operator \mathcal{L}_{ξ_z} . We have written complex conjugates here because we want to introduce the complex vectors η_s . We multiply out the square and use equation (6) to integrate by parts so that each term becomes

$$\frac{1}{2} (\bar{\zeta} \cdot [] + \zeta \cdot [])$$

We give the A^{-4} term as an example:

$$\begin{aligned}
&\int A^{-4} \mathcal{L}_j (\bar{\zeta} \otimes \xi_j + \xi_j \otimes \bar{\zeta}) : \mathcal{L}_i (\zeta \otimes \xi_i + \xi_i \otimes \zeta) \sqrt{g} d^3x \\
&= -\frac{1}{2} \int \sqrt{g} d^3x A^{-4} [\mathcal{L}_i \mathcal{L}_j (\bar{\zeta} \otimes \xi_j + \xi_j \otimes \bar{\zeta}) : (\zeta \otimes \xi_i + \xi_i \otimes \zeta) \\
&\quad + (\bar{\zeta} \otimes \xi_j + \xi_j \otimes \bar{\zeta}) : \mathcal{L}_j \mathcal{L}_i (\zeta \otimes \xi_i + \xi_i \otimes \zeta)]
\end{aligned}$$

(We used $\mathcal{L}_i \sqrt{g} = 0$ since ξ_i is Killing.)

The coefficient of $A^{-4} \sqrt{g} d^3x$ in the integrand may be written, because of the symmetry, and because $[\xi_i, \xi_j] = 0$ as

$$I = \mathcal{L}_i ([\xi_j, \bar{\xi}] \otimes \xi_i) : (\xi_j \otimes \xi_i + \xi_i \otimes \xi_j) \quad + \text{complex conjugate}$$

Furthermore, $[\xi_i, \xi_j] \cdot \xi_k = -\epsilon_{ijk} \xi_l \cdot \xi_l = 0$ and we can write out this expression, using

$$\xi = f \underline{\eta}_s :$$

$$\begin{aligned} I = & \sum_i [\xi_j, [\bar{f}]] (\underline{\eta}_s \cdot f \underline{\eta}_s) (\xi_j \cdot \xi_i) \\ & + \sum_i [\xi_j, [\bar{f}]] (\underline{\eta}_s \cdot \xi_i) (\xi_j \cdot f \underline{\eta}_s) \\ & + (\xi_i \cdot \underline{\eta}_s) \xi_j [\bar{f}] ([\xi_i, \xi_j] \cdot f \underline{\eta}_s) + \text{C.C.} \end{aligned}$$

In the following we need to know the dot products of the vectors $\xi_i \cdot \xi_j, \underline{\eta}_s$. In fact we have (as can be verified from the definition of $\xi_i, \underline{\eta}_j$ given in the text; see also Ref. 4):

$$\xi_i \cdot \xi_j = A^2 \delta_{ij} + (B^2 - A^2) \eta_i \eta_j \quad \text{and} \quad (8)$$

$$\underline{\eta}_z = -m_i \xi_i, \quad \text{where} \quad (9)$$

$$(m_x, m_y, m_z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (10)$$

$$\text{Furthermore, } \eta_{\pm} = m_{\pm a} \eta_a \quad \text{and} \quad (11)$$

$$(m_{+x}, m_{+y}, m_{+z}) = e^{i\psi} (-\cos \theta \cos \phi + i \sin \phi, -\cos \theta \sin \phi - i \cos \phi, \sin \theta) \quad (12)$$

$$\begin{aligned} \text{and} \quad \overline{m_{+a}} &= m_{-a}, \\ m_{+a} m_{+a} &= m_{+a} m_a = 0 \\ \overline{m_{+a}} m_{+a} &= m_a m_a = 1 \end{aligned} \quad (13)$$

Using these relations we see that the top line in I is

$$= (\underline{\xi} \cdot \underline{\xi}) \{ -A^2 (j(j+1)) - (B^2 - A^2) (k-s)^2 \} + \text{C.C.}$$

The last two lines in I may be combined into

$$\begin{aligned} L.L. &= f(\bar{\eta}_s \cdot \xi_i) \left\{ \xi_i [\xi_j [\bar{F}]] \xi_j \cdot \eta_s + \xi_i [\bar{F}] ([\xi_i, \xi_j] \cdot \eta_s) \right\} + c.c. \\ &= f(\bar{\eta}_s \cdot \xi_i) \left\{ \mathcal{L}_i (\xi_j [\bar{F}] \xi_j \cdot \eta_s) \right\} + c.c. \end{aligned}$$

[Since ξ_i commutes with η_s and with the metric (\cdot) .]

So

$$\begin{aligned} L.L. &= \delta_{s_0} f(-m_i B^4) \mathcal{L}_i (\xi_j [\bar{F}] (-m_j)) \\ &\quad + (1 - \delta_{s_0}) f(\bar{m}_{s_i} A^4) \mathcal{L}_i (\xi_j [\bar{F}] m_{s_j}) + c.c. \\ &= \delta_{s_0} B^2 (\bar{\xi} \cdot \xi) \bar{F}^{-1} \eta_z [\eta_z [\bar{F}]] \\ &\quad + A^2 (1 - \delta_{s_0}) (\bar{\xi} \cdot \xi) \bar{F}^{-1} \eta_s [\eta_s [\bar{F}]] + c.c. \\ &= -\delta_{s_0} B^2 (\bar{\xi} \cdot \xi) (k-s)^2 \\ &\quad - (1 - \delta_{s_0}) (\bar{\xi} \cdot \xi) A^2 (j(j+1) - k(k-s)) \end{aligned}$$

In the second line of the last equality we used the general formula

$$L_{\bar{z}} L_{\pm} = l(l+1) - l_z(l_z+1) \quad \text{plus the fact that in this case } l_z \text{ is } (k-s) \text{ and}$$

$\underline{\eta}_i \cdot \underline{\eta}_i = \sum_i \xi_i \cdot \sum_i \xi_i$ and that the order of the operators depends on s .

Combining these quantities we get from $\underline{T} + \underline{L} + \underline{L}$:

$$(\underline{\xi} \cdot \underline{\xi}) \left\{ -A^2 (j(j+1) - (k-s)^2) - B^2 (k-s)^2 (1 + \delta_{s0}) - (1 - \delta_{s0}) (j(j+1) + k(k-s)) \right\}.$$

The other terms work out in similar but more straightforward fashion, and the result is

$$\lambda_{jks} = \frac{1}{2} \left\{ B^2 (1 + \delta_{s0}) \left[\frac{k-s}{A^2} + k(B^{-2} - A^{-2}) \right]^2 + A^{-2} [j(j+1) - (k-s)^2] + (1 - \delta_{s0}) A^{-2} [j(j+1) - k(k+s)] \right\}, \quad (14)$$

Here $s = \pm 1, 0$; j and k are integer, and j is required to satisfy

$$j(j+1) \geq (k-s)^2 \quad \text{and} \quad j(j+1) \geq k(k-s)$$

We finally note that a solution $\underline{\xi}_+ = \underline{\eta}_+ = \underline{\eta}_x + i \underline{\eta}_y$ has $k = s = 1$, and has $j = 0$ since $[\underline{\eta}_i, \underline{\xi}_j] = 0$. Its eigenvalue is

$$\lambda_{011} = \frac{1}{2} B^2 (B^{-2} - A^{-2})^2.$$

$\underline{\xi}_- = \underline{\eta}_- = \underline{\eta}_x - i \underline{\eta}_y$ has the same eigenvalue. The eigenvalue vanishes when

$A^2 = B^2$ in which case the 3-space $\mathcal{M}(t)$ is instantaneously isotropic as well as homogeneous. Thus at this instant $\underline{\eta}_x$ and $\underline{\eta}_y$ are Killing and we identify them with the "missing" Killing vectors in general $\mathcal{M}(t)$.

IV The Isotropic Background in $\mathcal{M}(t)$

It has been suggested^(9,10) that Taub Space is a R.W. type solution, similar to Brill's radiation filled one,⁽⁹⁾ but with the longest wavelength gravitational wave that will fit into it giving the energy density to close it, in place of the matter or radiation in the usual R.W. forms. We show here a way of finding the underlying space

slices - the closed constant curvature spaces in the R.W. Metric - by a procedure of iteratively averaging the metric for the Taub slice $\mathcal{M}(t)$ along the few lowest-eigenvalue eigensolutions of (3).

In the 3-space, six averaging vector fields are necessary to obtain a completely homogeneous and isotropic background space. For one can, by averaging over three fields which describe homogeneity in a 3-space, obtain ^{an} isotropic but completely homogeneous space. Further averaging along these vector fields, which are then Killing, will have no effect. **More** averaging fields are needed.

In this section we shall make extensive use of the invariant definition of the vectors $\underline{\eta}_i$ and $\underline{\xi}_i$. We shall average along these fields, and we note that $\underline{\eta}_i$ and $\underline{\xi}_i$ are well suited to this averaging since of all the eigenvectors of $-D$ in $\mathcal{M}(t)$, only $\underline{\eta}_i$ and $\underline{\xi}_i$ are everywhere non-zero. This is easily seen because eigenvectors are of the form $f(\theta, \phi, \psi) \underline{\eta}_s$ and of the eigenfunctions f of the rotation group, none except the constant is everywhere nonzero.⁽⁶⁾ [Formulae (8) and (9) show that $\underline{\xi}_i \cdot \underline{\xi}_i \neq 0$ and $\underline{\eta}_i \cdot \underline{\eta}_i \neq 0$ for each i .] Our criterion for an invariantly defined field along which we will average will then be that they are the nowhere zero eigensolutions of equation (3). It clearly will be pointless to average along a Killing vector by a method of Lie transport as we will do here, so in this case we have initially only two vectors to consider averaging along, $\underline{\eta}_x$ and $\underline{\eta}_y$. It seems plausible that in at least some completely non-symmetric situations - such as those obtained by small general perturbations from a R.W. form - we can have up to six such candidates for the vector fields in the three space. It is, of course possible that by averaging along one vector field we destroy the Killing nature of another. We shall see that after averaging along $\underline{\eta}_x$, say, in the space $\mathcal{M}(t)$, $\underline{\eta}_z$ is no longer Killing. (If, as we do here, we average along a complete trajectory of $\underline{\eta}_x$, then afterward $\underline{\eta}_x$ is clearly Killing.) Because of the commutation relations eq. (5), the $\underline{\xi}_i$ stay Killing under $\underline{\eta}_i$ averaging.

Now consider averaging the contravariant metric tensor g^{ab} of $\mathcal{M}(t)$. We do this by a process of Lie transport. Pick one of the averaging vector fields, say $\underline{\eta}_x$. We then average $g^{ab}(p)$ along $\underline{\eta}_x$ by carrying the space back to the point p along the trajectories of $\underline{\eta}_x$ and averaging the values of g^{ab} there for the whole trajectory of $\underline{\eta}_x$. We use the form eq.(7) for the contravariant metric. After the averaging $\underline{\eta}_x$ will clearly be a Killing vector of the resultant space. We denote the back translated metric (a parameter length r along the curves) as $g^{ab}(p; rx)$. Formulae in Schouten⁽⁷⁾ then give for this finite translation:

$$\begin{aligned} g^{ab}(p; rx) &= e^{r\mathcal{L}_x} g^{ab}(p) \\ &\equiv g^{ab}(p) + r(\mathcal{L}_x g^{ab})(p) + \frac{r^2}{2}(\mathcal{L}_x^2 g^{ab})(p) + \dots \end{aligned}$$

The symbol \mathcal{L}_x denotes Lie differentiation along $\underline{\eta}_x$. Since $[\underline{\eta}_i, \underline{\xi}_i] = 0$ and by eqns.(5) we see that

$$\begin{aligned} \mathcal{L}_x(\underline{\eta}_z \otimes \underline{\eta}_z) &= \underline{\eta}_y \otimes \underline{\eta}_z + \underline{\eta}_z \otimes \underline{\eta}_y, \\ \mathcal{L}_x^2(\underline{\eta}_z \otimes \underline{\eta}_z) &= -2\underline{\eta}_z \otimes \underline{\eta}_z + 2\underline{\eta}_y \otimes \underline{\eta}_y, \text{ and} \quad (15) \\ \mathcal{L}_x^3(\underline{\eta}_z \otimes \underline{\eta}_z) &= -4 \mathcal{L}_x(\underline{\eta}_z \otimes \underline{\eta}_z), \end{aligned}$$

we obtain [remember the analyticity of $\mathcal{M}(t)$]

$$g^{ab}(p; rx) = g^{ab}(p) + \frac{1}{2}(\mathcal{L}_x g^{ab})(p) \sin 2r + \frac{1}{2}(\mathcal{L}_x^2 g^{ab})(p) \frac{1}{2}(1 - \cos 2r). \quad (16)$$

The range of the path parameter r is 0 to 4π , as can be seen by considering the equivalent quaternion translation on the 3-space. (See Ref. 8,

appendix B; this whole discussion of translations could be done in terms of quaternions on S^3 .) We will average over the whole range of V and use (8) $\underline{\eta}_i \otimes \underline{\eta}_i = \sum_{\underline{\xi}_i} \underline{\xi}_i \otimes \underline{\xi}_i$ to obtain

$$g''(p; \overline{r}_x) = A^{-2} (\underline{\xi}_i \otimes \underline{\xi}_i) + \frac{1}{2} (B^{-2} - A^{-2}) (\underline{\xi}_i \otimes \underline{\xi}_i - \underline{\eta}_x \otimes \underline{\eta}_x). \quad (17)$$

This can be put again into the form of a Taub slice, $\mathcal{M}(t)$, but now with a smaller asymmetry, and now $\underline{\eta}_x$ is Killing, but $\underline{\eta}_z$ is not.

If we now average with a nowhere zero non-Killing vector in this space, say

$\underline{\eta}_y$, we will get by cyclic symmetry:

$$g''(p; \overline{r}_x, \overline{s}_y) = A^{-2} (\underline{\xi}_i \otimes \underline{\xi}_i) + \frac{1}{2} (B^{-2} - A^{-2}) (\underline{\xi}_i \otimes \underline{\xi}_i - \frac{1}{2} [\underline{\xi}_i \otimes \underline{\xi}_i - \underline{\eta}_y \otimes \underline{\eta}_y]). \quad (18)$$

We see that if this cyclic averaging is continued, the anisotropic terms $\underline{\eta} \otimes \underline{\eta}$ have a coefficient which vanishes as 2^{-m} in the n^{th} term. The coefficient of $\underline{\xi}_i \otimes \underline{\xi}_i$ approaches $A^{-2} + \frac{1}{2} (B^{-2} - A^{-2}) \sum_0^\infty (-2)^{-m} = \frac{2}{3} A^{-2} + \frac{1}{3} B^{-2}$. This metric $g'' = \text{const.} \times \underline{\xi}_i \otimes \underline{\xi}_i$ is metrically S^3 , and we have succeeded in finding a symmetric background.

We pointed out above that the $\underline{\xi}_i$ vector fields are completely invariant under this averaging, since the $\underline{\eta}_i$ fields all commute with the $\underline{\xi}_i$. At each step in this iterative process, there are thus only two fields along which the averaging has any effect; the two $\underline{\eta}$ fields which do not appear in the metric at that step. (This is because we have averaged along the entire trajectory - all around the space - at each step.) A calculation for $g''(p; r_y)$ (i.e. for the metric dragged back along the vector field $\underline{\eta}_y$ instead of along the field $\underline{\eta}_x$) analogous to that leading equation (16) gives

$$g''(p; r_y) = g''(p) + \frac{1}{2} (\mathcal{L}_y g'')(p) \sin 2r + \frac{1}{2} (\mathcal{L}_y^2 g'')(p) \frac{1}{2} (1 - \cos 2r). \quad (19)$$

This is the same as (16) but with \underline{L}_y in place of \underline{L}_x (\underline{L}_y means $\underline{L}_{\underline{n}_y}$). Similarly, the equation analogous to (17) becomes

$$g''(p; \bar{r}_y) = A^{-2}(\underline{\xi}_i \otimes \underline{\xi}_i) + \frac{1}{2}(B^{-2} - A^{-2})(\underline{\xi}_i \otimes \underline{\xi}_i - \underline{\eta}_y \otimes \underline{\eta}_y) \quad (2)$$

The only difference is the appearance of $\underline{\eta}_y$ instead of $\underline{\eta}_x$. Recalling again that the $\underline{\eta}$ fields can only act among themselves, we see that if we carry out the next step, i.e. obtain the equation analogous to (18), for one of the choices of non-Killing field, we obtain an equation exactly like (18), except possibly with a different subscript on the $\underline{\eta}$ field. It is easy to see that this is so at each step of the iteration. The point of this is that the limit for the metric is independent of the order of averaging in this averaging scheme, where we average along the whole trajectory of the vector field.

We need not, however average the metric all around the space. Suppose we have a C^∞ weighting function $w(r)$ with the property that $\int_{-\epsilon}^{2\pi-\epsilon} w(r) dr = 1$, and that there is an open interval $I_w \subset (-2\pi + \epsilon, 2\pi - \epsilon)$; $\epsilon, \epsilon' > 0$, such that the support of w is contained in I_w . We also assume that $\frac{dw(r)}{dr} \leq 0$

Then, since the weight function is symmetric, if we average equation (16), the term proportional to $\sin 2r$ vanishes and contributes nothing in the average. However, in averaging the term $(1 - \cos 2r)$ we will obtain not 1, but $1 - \delta$, where δ ($1 > \delta > 0$) depends on the width of the weight function.

To investigate the effect of a finite width for the averaging function, let us write the contravariant metric as

$$g'' = \left\{ A^{-2} + \frac{1}{3}(B^{-2} - A^{-2}) \right\} (\underline{\xi}_i \otimes \underline{\xi}_i) + (B^{-2} - A^{-2}) \left[\frac{1}{3}(\underline{\eta}_z \otimes \underline{\eta}_z - \underline{\eta}_x \otimes \underline{\eta}_x) + \frac{1}{3}(\underline{\eta}_z \otimes \underline{\eta}_z - \underline{\eta}_y \otimes \underline{\eta}_y) \right]$$

where we used $\underline{\xi}_i \otimes \underline{\xi}_i = \underline{\eta}_k \otimes \underline{\eta}_k$.

Suppose we consider averaging this quantity with the weight function w , along the vectors $\underline{\eta}_i$, taken in any order, but each taken an infinite number of times. Note that the first term in g'' is invariant under this averaging; if

the second term tends to zero under this averaging, we have shown that the averaging process is independent of the order of averaging along the vector fields, and of the (non-zero) width of the smoothing function w .

In fact, for averaging over the whole trajectories of the vector fields, we have just seen that this term does vanish. Straightforward brute force calculations indicate that this is ~~true~~ when there is a weighting function w (so long as w is not a delta function). However, a straightforward intuitively clear proof has not yet been found, and we leave this as a true but ~~un~~unproven statement.

V Remaining Problems

We have been able to perform this averaging because of the invariant description of the vectors η_i and ξ_i . One invariant characterization of these vectors is that they are the nowhere zero eigensolutions to $(D+1)\xi = 0$. They are also the fields that would be obtained from the Killing fields on a spherical space which is continuously deformed into the Taub shape. (These fields, if the deformation from symmetry is big enough, need not be the lowest eigenvalue vector fields; see section D.III.) We suspect that this latter characterization is the relevant one; in particular, we expect that averaging on a 2-sphere can be carried out by averaging along continuous vector fields in the way we did in Section D.IV, even though each such fields has at least two zeros. Further investigation of this question remains to be done, and the specification of just what fields averaging can effectively be done along. In particular, since the spectrum of $-D$ specifies a countable number of vector fields on the space, we apparently have all these as candidates for averaging fields.

Another question remaining to be settled is that of the smoothing function. We have not given a completely satisfactory discussion of the "damping" method of averaging. However, this is only a formal problem, we have given the qualitative features. The result we obtain under averaging along these fields is unique, with or

without the smoothing. This essentially is because an infinite number of averagings will bring adjacent values of a function into equality, even if there is a smoothing function. Note that there are only four wavelengths of "radiation" in the space $\mathcal{M}(t)$, since $\cos 2r$, $r \in [0, 4\pi]$, appears in Eq. (16). Presumably, when the averaging can be done over many wavelengths, in the situation Isaacson⁽¹⁾ discusses, the smoothing functions will not be critical.

However, these problems are still to be considered in a general framework. Thus, in general we need to define the suitable averaging vector fields, and to investigate the uniqueness and dependence on smoothing functions of such averaged results.

One might wonder what the meaning of the averaged space is. A completely symmetric space which is a background for a space which is topologically S^3 is S^3 itself. The only unknown quantity remaining is the radius of the sphere. It might be expected that a sphere of the same volume as $\mathcal{M}(t)$ would be the simplest such average. However, the average defined here does not lead to the sphere of this radius.

Since the $\underline{\eta}_i$ are orthogonal, we can write the covariant Taub metric in terms of σ^i , the 1-forms dual to the $\underline{\eta}_i$ as

$$ds^2 = A^2(\sigma^x \sigma^x + \sigma^y \sigma^y) + B^2 \sigma^z \sigma^z.$$

To obtain this form we used (again) $\underline{\eta}_i \otimes \underline{\eta}_i = \sum_i \underline{\xi}_i \otimes \underline{\xi}_i$. The integral defining the volume of the space is $\int \sqrt{g} \sigma_i \wedge \sigma_j \wedge \sigma_k$.

The covariant form of the averaged metric which has the same volume as $\mathcal{M}(t)$ is thus given by

$$ds^2 = (A^4 B^2)^{1/3} (\sigma^x \sigma^x + \sigma^y \sigma^y + \sigma^z \sigma^z),$$

where we used the fact that the metric coefficients are constant in $\mathcal{M}(t)$.

The averaged metric we obtained Section D.IV, by averaging along vector fields is

$$\left(\frac{2}{3} A^{-2} + \frac{1}{3} B^{-2}\right)^{-1} (\sigma^x \sigma^x + \sigma^y \sigma^y + \sigma^z \sigma^z);$$

the ratio of the volume of this space to the volume of $\mathcal{M}(t)$ is thus

$$\left(\frac{2}{3} A^{-2} + \frac{1}{3} B^{-2}\right)^{-3/2} A^2 B$$

The ratio is equal to unity only when $A = B$; then the space is spherical.

It would be encouraging if the averaged sphere we obtain by the invariant vector fields were to have some desirable property as a background metric; for instance in a dynamical sense. Thus the question is how much the time dependence of A^2 and B^2 differentiates the equal-volume sphere from the invariantly averaged one. Does the motion of this sphere have more desirable characteristics as regards to the effective energy density needed to cause the motion? We here have two candidates for averaged backgrounds, each of which defines a (different) energy density in the gravitational radiation (the anisotropy). We hope that later investigations will show whether there is any reasonable criterion for picking one or the other. We do feel that the consideration of shorter wavelength radiation situation will lead to an unambiguous background of the Isaacson⁽¹⁾ type.

Finally, we point out that entirely naive considerations can lead to the averaging result we have found here. For

$$\sum_i \otimes \sum_i = \underline{n}_x \otimes \underline{n}_x + \underline{n}_y \otimes \underline{n}_y + \underline{n}_z \otimes \underline{n}_z$$

One would expect that in a sum of squares, where nothing picks out a direction (and nothing does, on the sphere we average over), we would have

$$\frac{1}{3} \sum_i \otimes \sum_i = \langle \underline{n}_k \otimes \underline{n}_k \rangle,$$

where we sum on i but not on k , and the brackets indicate the average. This is the result we obtained in the previous section, and in fact is so simple that one is inclined to demand it of all candidates for averaging schemes. The constant volume scheme clearly does not have this property.

Footnotes for Appendix D.

- (1) R. A. Isaacson, Ph.D. dissertation, University of Maryland, Department of Physics and Astronomy, (1967).
- (2) A. H. Taub, Ann. Math. 53, 472, (1951).
- (3) For a modern exposition of these metrics see S. W. Hawking, Ap.J. 145, 544, (1966).
- (4) C. W. Misner and A. H. Taub, paper submitted to J.E.T.P. (1967).
- (5) E. Newman, L. Tamburino and T. Unti, J. Math. Phys. 4, 915, (1963).
- (6) E. P. Wigner, Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra, Academic Press, New York, (1957) (Chapter 15).
- (7) J. A. Schouten, Ricci Calculus, Springer-Verlag, Berlin, (1954) paragraph 10.
- (8) C. W. Misner, J. Math. Phys. 4, 924, (1963).
- (9) D. R. Brill, Nuovo Cimento Suppl. No.1, (1964).
- (10) J. A. Wheeler, "Geometrodynamics and the Issue of Final State" in Relativity Groups and Topology, Gordon and Breach, London, (1964) (paragraph 9 and problem 55).

Works Cited

- J.L. Anderson, in Proceedings of the Royamount Conference on General Relativity
- J.L. Anderson, "On the Consistency of Dirac's Canonical Formulation of General Relativity" in USAF ARL 66-0075 (Technical Report on Research Program in Relativity Physics, Kings College, London, 1966)
- H. Araki, Ann. Phys. Princeton 7, 456, (1959)
- R. Arnowitt, S. Deser, and C.W. Misner, in Gravitation, an Introduction to Current Research, ed L. Witten, J.Wiley and Sons, New York, (1962)
- H. Bondi, Nature 179, 1072, (1957)
- H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner, Proc. Roy. Soc. A269, 21, (1962)
- H. Bondi, Proc. Roy. Soc. A281, 39, (1964)
- D.R. Brill, Ann. Phys. Princeton 7 466, (1959)
- D.R. Brill, Nuovo Cimento Suppl. 2, no.1, (1964)
- D.R. Brill and J.B. Hartle, Phys. Rev. 135, B271, (1964)
- S. Chandrasekhar, Ap.J. 142, 1513, (1965)
- R. Clemens and R. Matzner, Tech. Report #635, University of Maryland Dept. of Physics and Astronomy, (1967)
- E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York, (1955)
- A. Compolattaro and K.S. Thorne, UCRL "Orange Aid" Preprint, (1967)
- W.E. Couch, R.J. Torrence, A.I. Janis, and E.T. Newman, preprint, (1967)
- R. Courant and D. Hilbert, Methods of Mathematical Physics, vol I, Interscience, New York, (1953)
- R. Debever, Compt. rend. 249, 1324, (1959)
- B.S. DeWitt, preprint, (1966)
- P.A.M. Dirac, Proc. Roy. Soc. A246, 326, (1958)

- P.A.M. Dirac, Proc. Roy. Soc. A246, 333, (1958)
- P.A.M. Dirac, Proc. Roy. Soc. 114, 924, (1959)
- P.A.M. Dirac, Can. Jour. Math. 2, 129, (1950)
- G. Dixon, preprint, (1967)
- L. Edelman, Ph. D. thesis, Univ. of Maryland Dept. of Physics and Astronomy, (1967)
- A. Einstein and J. Grommer, Sitzb. Berl. Akad., p.2, (1927)
- A. Einstein and N. Rosen, J. Franklin Inst. 223, 43, (1937)
- A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math., 39, 65, (1938)
- J. Fletcher, Lawrence Radiation Lab. Tech Report UCRL Rept. 14624-t, (1965)
- J. Fletcher, Comm. A. C. M. 9, 552, (1966)
- J. Fletcher, R. Clemens, R. Matzner, K.S. Thorne, B. Zimmerman, Ap. J. 148, L91, (1967)
- V.A. Fock, Zh. Eksp. Teor. Fiz.(J.E.T.P.), 9, 375, (1939) and J. Phys. U.S.S.R. 1, 81, (1939)
- V. A. Fock, Theory of Spacetime and Gravitation, Pergamon Press, London, (second revised edition, 1964)
- J. N. Goldberg and R. K. Sachs, "A Theorem on Petrov Types", in USAF ARL 66-0075, (Technical Report on Research Program in Relativity Physics, Kings College, London, 1962, 1966)
- J. N. Goldberg, "The Equations of Motion" in Gravitation, an Introduction to Current Research, ed. L. Witten, J. Wiley and Sons, New York, (1962)
- A.E. Green and W. Zerna, Theoretical Elasticity, Oxford, (1954)
- A.E. Green and J. E. Adkins, Large Elastic Deformations, Oxford, (1960)
- S. W. Hawking, Ap. J. 140, 1, (1964)
- S. W. Hawking, Ap. J. 145, 544, (1966)
- J. Hély, Compt. rend 249, 1967, (1959)
- R.A. Isaacson, Ph. D. Thesis, University of Maryland Dept. of Physics and Astronomy Technical Report #586, (1966)
- A. Janis and E. Newman, J. Math. Phys. 6, 902, (1965)

- P. Jordan, J. Ehlers, and W. Kundt, Akad. Wiss. Mainz, Abh. Math-Nat. Kl., Jahrgang 1960, Nr.2.
- D.J. Kaup, University of Maryland Dept. of Physics and Astronomy Technical Report #586, (1966).
- D.J. Kaup, University of Maryland Dept. of Physics and Astronomy Technical Report #665, (1967).
- R.P. Kerr, Phys.Rev. Lett. 11 522, (1963).
- R.P. Kerr and A. Schild in Convegno Sulla Relativita Generale: Problemi Dell'Energia e Onde Gravitazionali (Proceedings of Conference in honor of the fourth centenary of the birth of Galileo) published by Comitato Nazionale per le Manifestazioni Celebrative, Rome, (1964).
- K. Kodiyara and D.C. Spencer, Ann. Math. 71, 43, (1960).
- W.T. Koiter, "A Consistent First Approximation in the General Theory of Elastic Shells", in The Theory of Thin Elastic Shells, North-Holland, Amsterdam, (1960).
- A. Komar, P.R. 127 1411, (1962).
- A. Komar, P.R. 129 1873, (1963).
- W. Kundt, Ph.D. dissertation, Hamburg, (1958).
- L. Landau and E.M. Lifshitz, Classical Theory of Fields (second edition) Pergamon Press, Oxford, (1962)
- H. Levy, Ph.D. thesis, King's College, London, (1965).
- T. Lewis, Proc. Roy. Soc. A136, (1932) 176.
- C.C. Lin, L. Mestel, and F.H. Shu, Ap.J. 142, 1431, (1965).
- R.W. Lindquist, C.W. Misner and R.A. Schwartz, Phys. Rev. 137, B 1364, (1965).
- D. Lynden-Bell, On the Origins of Spacetime and Inertia, (in press, M=N.R.A.S.) (1967).
- R.A. Matzner and C.W. Misner, Phys. Rev. 154, 1229, (1967).
- R.A. Matzner, J. Math. Phys., in press, (1967).
- N.W. McLachlan, Theory and Applications of Mathieu Functions, Dover Paperback, New York, (1964).
- S.G. Milhlin, The Problem of the Minimum of a Quadratic Functional, Holden Day, San Francisco, (1965).

- C.W. Misner, Phys. Rev. 130, 1590, (1963).
- C.W. Misner, J. Math. Phys. 4, 924, (1963).
- C.W. Misner and H.S. Zapolsky, preprint (1966).
- C.W. Misner and A.H. Taub, paper submitted to J.E.T.P., (1967).
C.W. Misner, preprint (1967)
- J.F. Nash, Annals of Math 63, 20, (1956).
- E. Newman and R. Penrose, J. Math Phys. 3, 566, (1962).
- E. Newman and T. Unti, J. Math Phys. 3, 891, (1962).
- E. Newman, L. Tamburino, and T. Unti, J. Math. Phys. 4, 915, (1963).
- J.R. Oppenheimer and H. Snyder, P.R. 56, 455, (1939).
- A. Papapetrou, Ann. der Physik 12, (1953) 209.
- A. Papapetrou, Preprint (1965), to be pub. Ann. Inst. H. Poincaré,
Paris.
- A. Papapetrou, Proc. Roy. Soc. A694, 57, (1951).
- R. Penrose, in Perspectives in Geometry and Relativity, ed B. Hoffmann,
Indiana University Press, London (1967).
- R.B. Partridge and D.T. Wilkenson, Phys. Rev. Lett. 18, 557, (1967).
- A. Peres, Phys. Rev. Lett. 3, 571, (1959).
- A. Petrov, Sci. Not. Kazan. State Univ. 114, 55, (1954).
- F.A.E. Pirani and A. Schild, Phys. Rev. 79, 986, (1950).
- C.B. Raynar, Proc. Roy. Soc. A272, 44, (1963).
- I. Robinson and A. Trautman, Phys. Rev. Lett. 4, 431, (1960).
- I. Robinson and A. Trautman, preprint, (1965).
- N. Rosen, Physik Z. Sovyet Union 12, 366, (1937).
- R.K. Sachs, Proc. Roy. Soc. A264, 309, (1961).
- R.K. Sachs, Proc. Roy. Soc. A270, 103, (1962).
- J.A. Schouten, Ricci Calculus, Springer-Verlag, Berlin, (1954),
[paragraph 10].
- E. Schroedinger, Expanding Universes, Cambridge Press, Cambridge,
(1956).
- J.L. Synge, Math. Z. 72, 82, (1959).

- J.L. Synge, Relativity, the General Theory, Amsterdam, North-Holland Publishing Company, (1960).
- H. Takeno, J. Math. Soc., Japan 3, No. 2, (1951).
- A.H. Taub, Ann. Math 53, 472, (1951).
- K.S. Thorne and B.A. Zimmerman, "Albert- A Computer Program for Calculating General Relativistic Curvature Tensors and Equations of Motion," Orange Aid technical report, California Institute of Technology, (1967).
- R. Torrence and A. Janis, J. Math. Phys., (to be published, 1967).
- A. Trautman, "Foundations and Current Problems in General Relativity" in Lectures in General Relativity (Brandeis Summer Institute in Theoretical Physics) Prentice-Hall, Englewood Cliffs, New Jersey), (1964).
- W.J. Van Stockum, Proc. Roy. Soc. Edin. 57, (1937) 135.
- P.C. Vaidya, Nature 171, 260, (1953).
- C.V. Vishveshwara, Technical Report on Grant NSG-436 (U. of Maryland Dept. Physics and Astronomy), Gravitational Radiation from Binary Systems, (1964).
- C.V. Vishveshwara, Ph.D. thesis, University of Maryland Dept. of Physics and Astronomy, (1967).
- J. Weber and J. A. Wheeler, Revs. Mod. Phys. 29, 3, 509, (1957).
- J.A. Wheeler, "Geometrodynamics and the Issue of Final State" in Relativity Groups and Topology, Gordon and Breach, London, (1969).
- E.P. Wigner, Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra, Academic Press, New York, (1959), [Chapter 15] .
- T.J. Willmore, Differential Geometry, Oxford, (1958).
- K. Yano and S. Bochner, Curvature and Betti Numbers, Princeton, (1953).