ABSTRACT

Title of Dissertation: AUTONOMOUS STOCHASTIC PERTURBATIONS OF HAMILTONIAN SYSTEMS

Huaizhong Ren, Doctor of Philosophy, 2004

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We consider autonomous stochastic perturbations $\dot{X}^\varepsilon(t) = \overline{\nabla} H(X^\varepsilon(t)) + \varepsilon b(X^\varepsilon(t))$ of Hamiltonian systems of one degree of freedom whose Hamiltonian H is quadratic in a neighborhood of the only saddle point of H. Assume that $b = b_1 + \xi b_2$ for some random fields b_i, $i = 1, 2$, and ξ is a random variable, and that $\mathrm{div} b_i < 0$ and $\xi > 0$ with probability 1. Also assume that H has only one saddle point and two minima. To consider the effects of the perturbations, we consider the graph Γ homeomorphic to the space of connected components of the level curves of H and the processes Y_t^ε on Γ which represent the slow component of the motion of the perturbed system. We show that as $\varepsilon \to 0$, the processes Y_t^ε tend to a certain stochastic process Y_t on Γ which can be determined inside the edges by a version of the averaging principle and branches at the interior vertex into adjacent edges with certain probabilities which can be calculated by H and

the perturbation b. Also our result can be used to regularize some deterministic perturbations, partially coinciding with the results obtained by Brin and Freidlin in a earlier work.

AUTONOMOUS STOCHASTIC PERTURBATIONS OF HAMILTONIAN SYSTEMS

by

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DEDICATION

To my parents, Hongsheng Ren and Qiwu Chen,

my wife Qiong Li,

for their love and support,

and Professor Danyan Gan,

from whom I learned free and independent thinking

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It is impossible to remember all, and I apologize to those I have inadvertently left out. Thanks to all of you who have made my graduate experience in University of Maryland more cherishable for the years to come.

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Chapter 1

Introduction

1.1 Hamiltonian systems

1.1.1 Hamiltonian systems

A *Hamiltonian system with n degrees of freedom* is a system of differential equations of the form

$$\dot{X}_t = \overline{\nabla} H(X_t), \quad X_t = (p_1, \dots, p_n, q_1, \dots, q_n) \in \mathbb{R}^{2n}, \tag{1.1}$$

where $\overline{\nabla} H(x) = \left(-\frac{\partial H}{\partial q_1}, \dots, -\frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n} \right)$ is the skew-gradient, $H(x) = H(p_1, \dots, p_n, q_1, \dots, q_n)$ is the *Hamiltonian*, and n is the number of *degrees of freedom*. An important example is provided by an oscillator with one degree of freedom, described by the following equation

$$\ddot{q}(t) + f(q(t)) = 0, \quad q(0) = q_0, \quad \dot{q}(0) = p_0, \tag{1.2}$$

which can be transformed into a Hamiltonian system by the transformation $p = \dot{q}$ with the Hamiltonian defined by

$$H(p, q) = \frac{1}{2} p^2 + F(q),$$

where $F(q) = \int_0^q f(u) du$ is the potential.

H is a first integral of motion of the system (1.1), i.e., $H(X_t) = H(X_0)$, for all t.

It is well known that the flow of (1.1) preserves the standard symplectic structure defined by the 2-form

$$\Omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

and hence the Lebesgue measure. From this we know that there is an invariant measure on the level sets of (1.1).

1.1.2 Description of the trajectories

We study the case of one degree of freedom only. The systems we are considering satisfy the following conditions:

(1) $H \in C^3$;

(2) $\lim_{|x|\to\infty} H(x) = +\infty$;

(3) H is generic, i.e., H has only finite number of non-degenerate critical points, and the critical values are pairwise distinct. Also H has no local maxima.

The trajectories of the Hamiltonian system (1.1) are level sets of H. By our assumptions, there are three types of level sets, periodic, single points(minima), or homoclinic separatrices(∞-shaped curves).

Let $C(z) = H^{-1}(z) = \bigcup_{i=1}^{n(z)} C_i(z), \quad z \in \mathbb{R}$, where $C_i(z)$ is a component of $C(z)$.

The set of all the connected components of the level curves $\{C(z) : z \in \mathbf{R}\}$, with the natural topology, is homeomorphic to a graph Γ with vertices $\mathcal{O}_i$ and edges I_k, where an interior vertex $\mathcal{O}_i$ corresponds to a component containing a saddle point and an exterior vertex corresponds to an extremal point. Let $Y : \mathbb{R}^2 \to \Gamma$, $Y(x) = (H(x), e(x)) \in C_k(H(x))$, be the projection of the phase space to the graph, where $e(x) = k$ if x belongs to the component corresponding to a point in I_k. Both $H(x)$ and $e(x)$ are first integrals of the unperturbed system.

A typical illustration of the phase curves and the corresponding graph is shown in Figure 1.1.

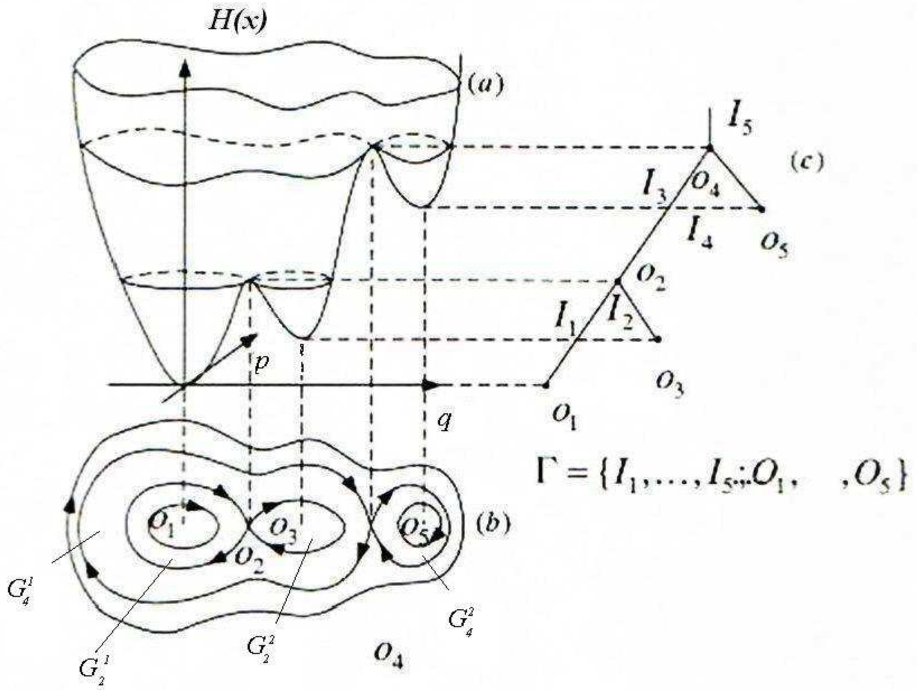

Figure 1.1

Let x_0 be a point in the phase space such that $Y^{-1}(Y(x_0))$ contains no critical point of H and has only one component. Let $O_k, k = i_1, i_2, \ldots, i_r$, be the saddle points of $H(x)$ inside the region bounded by $Y^{-1}(Y(x_0))$ ($k = 2, 4$ in Figure1.1) and $\gamma_k = Y^{-1}(Y(O_k))$ the homoclinic loops, which has the ∞-shape and bounds two domains, denoted by G_k^i $(i = 1, 2)$(G_2^1, G_2^2, G_4^1, G_4^2 in Figure 1.1). We shall call these domains "basins".

1.2 Perturbations of Hamiltonian systems

1.2.1 Autonomous perturbations

We are interested in the long-time behavior of the perturbed system with one degree of freedom

$$\dot{X}_t^\varepsilon = \overline{\nabla} H(X_t^\varepsilon) + \varepsilon b(X_t^\varepsilon), \quad X_0^\varepsilon = X_0, \quad 0 < \varepsilon << 1, \tag{1.3}$$

where $b(x)$ is an autonomous random field on $\mathbb{R}^2$ of class C^2. We will focus on the asymptotic properties with the time interval of order $1/\varepsilon$ when $\varepsilon \to 0$.

For the oscillator example, we may consider the following perturbation

$$\ddot{q}^\varepsilon(t) + f(q^\varepsilon(t)) = \varepsilon b(\dot{q}^\varepsilon(t), q^\varepsilon(t)), \quad 0 \le \varepsilon \ll 1, \tag{1.4}$$

where b is a smooth function with bounded first and second derivatives. For example, we may let $b(x) = -\xi\beta(p^\varepsilon(t), q^\varepsilon(t))\dot{q}^\varepsilon(t)$, with $\beta > 0$ and $\xi > 0$ both random.

Since $\overline{\nabla} H(x)$ is orthogonal to $\nabla H(x)$, the trajectories of the unperturbed system are level curves of the Hamiltonian $H(x)$. As for the perturbed system, the motion along the trajectories decomposes into two components, a fast component along the direction of the level curves of $H(x)$, and a slow component along the direction of the gradient of H, i.e., shift between different level curves of H. Roughly speaking, the slow motion is the effect of the perturbation and hence our main concern. This slow motion is better described by a process on the graph which is the set of all the connected components of the level curves of $H(x)$.

The structure of the phase curves of the system is changed by the perturbation. If we assume div$b < 0$, the saddles persist but their location will be changed by a distance of order ε. The centers, however, will become foci, i.e., stable spiral points due to the assumption that $\mathrm{div} b(x) < 0$. The trajectories of the perturbed system, except the separatrices, are attracted to one of the foci.

The projection $Y(X_t^\varepsilon)$ represents the slow component of X^ε, which captures the evolution of the system caused by the perturbation. Since the evolution of the system is expressed mainly in terms of this slow component, we consider the process on Γ.

1.2.2 The method of averaging

Averaging principle for the perturbations is a method to simplify a system when its components can be separated into two groups according to their rate of change, one fast and the other slow. In the case that the fast components are quasi-periodic or ergodic, the slow components are distributed uniformly on the trajectories of the unperturbed system, thus can be approximated by the "averaged" system which is the average of the slow components over the trajectories of the unperturbed system. Generally speaking, the validity of the approximation requires careful examination. When the perturbations are stochastic, it is appropriate to consider the weak convergence.

The idea of averaging was first used by Clairaut, Lagrange and Laplace, and later by Jacobi, Poincaré, and Van der Pol (see Sanders and Verhulst[15], also see Arnold, Kozlov, and Neishdadt[3]). Fatou proved the first asymptotic validity for averaging method. Krylov and Bogolyubov developed averaging method in the almost periodic case and Bogolyubov proved the averaging principle in the general case for the system of the form

$$\dot{x} = \varepsilon f(t, x)$$

where the time-average

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t,x)dt$$

exists. Bogolyubov and Mitropolskii [4] studied nonlinear oscillation extensively. But it was Anosov who first proved a general version of the averaging principle for the system with no saddle points under the ergodicity assumption(see Lochak and Meunier [13]).

The averaging principle for random perturbations was studied by R. Khs'minskii[12], M. Freidlin and A. Wentzell[9], Yu. Kifer, and others. In their work, Freidlin and Wentzell considered the perturbations of the white noise type. Also they considered the case for the Hamiltonian systems with one degree of freedom[8]. M. Freidlin and M. Weber[6, 7] considered the white-noise type perturbations for nonlinear oscillators and a nonlinear pendulum. But no one has ever studied the averaging principle for autonomous random perturbations.

The presence of the saddle points complicates the problem. If the perturbation $b(x)$ is deterministic, the limit does not exist in the classical sense, because as $\varepsilon \downarrow 0$, the initial point x will belong to the strips leading to the left and right basins alternatively. Instead, we must consider the convergence in a weaker sense, e.g., weak convergence or convergence in distribution. Also we need to regularize the system by some means. Arnold[1] and later Neishtadt[14] studied the system with saddle points and formulated the averaging principle in this case. But the first proof was given by M. Brin and M. Freidlin in an independent work[5], while Neishtadt only gives the statement of part of the results in [5]. Brin and Freidlin [5] used an additional perturbation of the white noise-type $\kappa\sigma(x)\dot{W}_t$ to regularize the perturbation for the oscillator (1.4) and proved the weak convergence of the processes Y_t^ε to a limit process Y_t on Γ, which is deterministic in the edges and branches at the interior vertices with certain probabilities determined by H and the perturbation. They showed that the limit does not depend on $\sigma(x)$ as $\kappa \to 0$, which justifies the use of the additional perturbation. They also considered the case when the initial point is perturbed to regularize the perturbed system (1.3) and proved the weak convergence.

G. Wolandsky[17] also obtained the result in a special case, using white noise $\dot{W}_t$ as the additional perturbation. But his approach is unable to show the independence of the limit of the additional perturbation, and hence did not justify the regularization.

In this work, we consider a different type of perturbations of the system (1.1). Instead of a deterministic perturbation, we consider a random field $b(x)$ as the perturbing term. It seems natural that a system in reality would be affected by some random factors. Also there are perturbations that do not come from white noise and not depend on time. The approach also has the advantage in dealing with limits, when the randomness leads naturally to weak convergence. On the technical side, the key is the behavior of the system when it approaches a saddle point corresponding to an interior vertex of the graph Γ. This is the main part of our study. The random field can represent a wide class of perturbations. In particular, it can be used to regularize the system with deterministic perturbations as well.

1.2.3 The processes on the graph Γ

Since it is easier to consider the perturbed systems (1.3) on a finite time interval than on $[\,0,\, T/\varepsilon\,]$, we scale the time by the following transformation $\widetilde{X}_t^\varepsilon = X_{t/\varepsilon}^\varepsilon$. Then

$$\dot{\widetilde{X}}_t^\varepsilon = \frac{1}{\varepsilon}\overline{\nabla} H(\widetilde{X}_t^\varepsilon) + b(\widetilde{X}_t^\varepsilon), \quad t \in [0, T] \tag{1.5}$$

and it follows that

$$H(\widetilde{X}_t^\varepsilon) - H(\widetilde{X}_0^\varepsilon) = \int_0^t \nabla H(\widetilde{X}_s^\varepsilon) \cdot b(\widetilde{X}_s^\varepsilon) ds. \tag{1.6}$$

Let $Y_t^\varepsilon = Y(\widetilde{X}_t^\varepsilon)$, then Y_t^ε is a random process on the graph Γ.

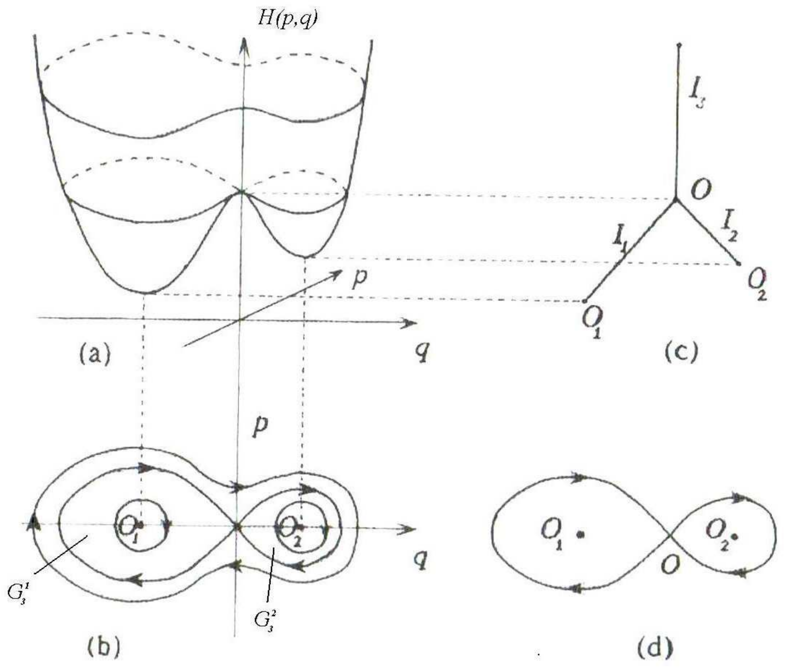

Figure 1.2

On the time interval $[\,t,\, t+h\,]$, where h is small but independent of ε, before $H(\widetilde{X}_t^\varepsilon)$ can make a change of an amount of order h, the number of rotations of the fast component along the level set will be of order h/ε. Therefore, inside each edge of the graph Γ, the averaging principle holds and the slow component $H(\widetilde{X}_t^\varepsilon)$ converges uniformly on any finite interval to an averaged motion. More precisely, let $C_i(z)$ be the family of components corresponding to the edge I_i, and $G_i(z)$ be the domain in $\mathbb{R}^2$ bounded by $C_i(z)$(see Figure 1.2).

The averaging principle says that under some conditions, as $\varepsilon \downarrow 0$ the processes Y_t^ε inside the edge I_i will converge to the averaged process $Y_t = (H_i(t), i)$ defined in the interior of the edge I_i of Γ by

$$\dot{H}_i(t) = B_i(H_i(t)) \cdot T_i(H_i(t))^{-1}, \quad H_i(t) = H(x_0), \tag{1.7}$$

where

$$T_i(z) = \oint_{C_i(z)} \frac{dl}{|\nabla H(x)|}, \quad \text{and} \quad B_i(z) = \int_{G_i(z)} \mathrm{div} b(x) dx. \tag{1.8}$$

where dl is the length element. Since $\mathrm{div} b(x) < 0$, the limit process Y_t decreases with time.

At the vertices, however, the situation is more complicated. The exterior vertices are generally inaccessible, while interior vertices can be reached in finite time. With the assumption that $\mathrm{div} b < 0$, the process Y_t has a decreasing H-component and therefore at an interior vertex, which corresponds to a saddle point, there is one entrance edge and two exit edges. We show that when the process Y_t reaches the vertex, it will spend no time there, and will enter one of the two exit edges with certain probability determined by H and b.

1.3 Main results

1.3.1 Conditions and conclusions

We say that a random field $b(x)$ satisfies **Condition 1** if there are random fields $b_1(x)$, $b_2(x) \in C^2$, and a random variable $\xi > 0$ such that $b(x) = b_1(x) + \xi b_2(x)$ with probability 1, and $\mathrm{E}|\xi|^2 < \infty$, $\mathrm{E}|b_1(x)|^2, \mathrm{E}|b_2(x)|^2, \mathrm{E}|\frac{\partial b_1}{\partial p}|^2, \mathrm{E}|\frac{\partial b_2}{\partial p}|^2, \mathrm{E}|\frac{\partial b_1}{\partial q}|^2,$

$\mathrm{E}|\frac{\partial b_2}{\partial q}|^2 \leq M < \infty$, and that ξ has a continuous conditional density $p(z|b_1, b_2)$ given $b_1(x)$ and $b_2(x)$.

We say that the random field $b(x)$ satisfies **Condition 2** if $\mathrm{div}(b_1(x) + \xi b_2(x)) < 0, \mathrm{div} b_1(x) < 0$ for all x, $\xi > 0$, with probability 1.

The assumption about the perturbation b makes it possible to include varies types of perturbations in this general form. For example, the case of the deterministic perturbation, which we may denote as b_1, can be regularized by adding the small perturbation $\xi b_2(x)$ where ξ and/or b_2 is random.

Since we assume that $\mathrm{div} b(x) < 0$, , for small ε, the energy level $H(X_t^\varepsilon)$ is very close to Y_t. Eventually the process will approach one of the minima of $H(x)$ as both $t \to \infty$ and $\varepsilon \downarrow 0$. But at an interior vertex $\mathcal{O}_i$ of Γ corresponding to a saddle point, the trajectory is very sensitive to the perturbation even if its magnitude is very small. This raises the following question: how will the trajectory behave at the level of a saddle point? Or equivalently, which edge will the process Y_t enter after reaching $\mathcal{O}$?

Define a random process $Y_t = (H_i(t), i)$ inside the edge I_i of Γ by (1.7) and at the interior vertices we have the following

Conjecture 1.3.1. The process Y_t approaches the vertex corresponding to a minimum after consecutively passing through the vertices $Y(x_{i_5}), \ldots, Y(x_{i_l})$ corresponding to the saddle points $x_{i_1}, \ldots, x_{i_l}$ with the probability

$$\prod_{j=1}^{l} \frac{\beta_{i_j}^{k_j}}{\beta_{i_j}^1 + \beta_{i_j}^2},$$

where $\beta_k^i = -\int_{G_k^i} \mathrm{div} b(x) dx$, $i = 1, 2$.

Nevertheless, after reaching the vertex $\mathcal{O}_i = Y(x_i)$ in finite time, Y_t leaves $\mathcal{O}_i$ instantly, and enters I_{i_1} or I_{i_2} with probabilities $\beta_{i_1}\beta_{i_3}^{-1}$ and $\beta_{i_2}\beta_{i_3}^{-1}$, respectively, where $\beta_{i_3} = \beta_{i_1} + \beta_{i_2}$. Also Y_t is determined uniquely by the conditions 1 and 2.

However, we are not going to prove the result in this generality. Rather we restrict ourselves to the special case that there are only one saddle point and two minima for H, and that H is quadratic in a neighborhood of the saddle point.

THEOREM. (Weak Convergence) *Assume that H satisfies the basic conditions in (1.1.2), and H has one saddle point at the origin and two local minima, H is quadratic in a neighborhood of the origin, and that $b(x)$ satisfies Conditions 1 and 2. Then the processes Y_t^ε converge weakly in $C([0,T],\Gamma)$, the space of continuous functions taking values in Γ, to the process Y_t as $\varepsilon \downarrow 0$, which is defined inside each edge I_i, $i = 0, 1, 2$, by (1.7), and branching at the interior vertex. Moreover, starting from a point in I_0, Y_t reaches the interior vertex in finite time and leaves instantly, entering one of the edges I_1, I_2 with the probabilities*

$$p_l^\xi = \frac{\beta_1(\xi)}{\beta_1(\xi) + \beta_2(\xi)}, \quad p_r^\xi = \frac{\beta_2(\xi)}{\beta_1(\xi) + \beta_2(\xi)}.$$

.

Remark 1.3.2. (1) The assumption that H is locally quadratic is due to a technical reason. We are working on the general case to eliminate this extra assumption.

(2) When $b_1(x)$ is deterministic, our approach can be used to regularize the problem for the deterministic perturbation. This is done in Chapter 3.

1.3.2 Idea of the proof

Now we explain briefly our plan of carrying out the proof.

We consider only the simplest case, in which the system has only two minima and one saddle point.

Consider the two stable separatrices coming toward the saddle point when time increases, that is, the stable invariant manifolds converge at the saddle point as time $t \to \infty$, denoted by γ_l^ε and γ_r^ε, respectively. They bound two strips leading to the left and right basins (domains bounded by the components of the homoclinic separatrices of the Hamiltonian system (1.1)), each being a neighborhood of a minimum. We shall call them "flow ribbons". If a point x belongs to one of the flow ribbons, then the flow line passing through x will enter the corresponding basin. The basic assumption is that if we fix $b_1(x)$ and $b_2(x)$, then ξ has a continuous density, which can be shown to imply that for a small change of ξ, the position of x relative to the boundaries of the flow ribbon will have a small change of the same order as that of ξ. Therefore it is distributed almost uniformly. It follows that the probability that a trajectory passing through x enters the left or right well is determined by the ratio of the "H-width" of the flow ribbons corresponding to the two wells.

More precisely, fixing a value ξ_0, we can write

$$\varepsilon(b_1(x) + \xi b_2(x)) = \varepsilon(b_1(x) + \xi_0 b_2(x)) + \varepsilon(\xi - \xi_0) b_2(x)$$

with the change of $\xi - \xi_0 = \alpha\varepsilon$ for some α. So we can rewrite this as

$$\varepsilon b(x) + \varepsilon^2 \beta(x).$$

We are going to show that for the time interval $[0, T/\varepsilon]$, the second order perturbation $\varepsilon^2\beta(x)$ gives rise to a change of order ε of the level of H-value of the trajectory. This justifies the almost uniformity of the distribution of a point in the flow ribbon, which is the key to our argument.

From this we can conclude that the ratio of the probabilities that the trajectory enters the left and right basins is almost the same as ratio of the H-width of the corresponding flow ribbons. Taking limit as $\varepsilon \to 0$, we find the ratio for the limiting process branching at the interior vertex. The weak convergence inside the edges is routine.

Chapter 2

Autonomous Perturbations: One Saddle Point

We begin the study of the autonomous stochastic perturbation of the Hamiltonian system (1.3) in this chapter, focusing on the behavior near the saddle point. As a strategy we first condition on b_1 and b_2, and consider the randomness caused by ξ only. We establish the almost uniformity and calculate the ratio of the probabilities that a trajectory enters one of the two basins L and R.

2.1 Trajectories under the perturbations

In this section we consider the behavior of the perturbed systems in a neighborhood of the ∞-shaped level curve, which is the homoclinic separatrix of the unperturbed system. We assume first that the perturbation $b(x)$ is not random. Although our main results are restricted to the case when the Hamiltonian is quadratic in a neighborhood of the origin (the saddle point of the unperturbed systems), we will start our preparation in a general setting.

2.1.1 Change of the trajectories

Since the perturbation is small in magnitude, the fixed points of the perturbed systems are close to those of the original system by a small distance. More precisely we have the following result.

Lemma 2.1.1. (change of the fixed points) *Let $B(x)$ and $b(x)$ be C^2 vector fields. Let x_0 be a non-degenerate fixed point of the system*

$$\dot{X}_t = B(X_t),$$

and $b(x_0) \neq 0$, then for ε small enough, the perturbed system

$$\dot{X}_t^\varepsilon = B(X_t^\varepsilon) + \varepsilon b(X_t^\varepsilon)$$

has a non-degenerate fixed point x_ε with $\|x_\varepsilon - x_0\| = O(\varepsilon)$ of order ε.

Proof. We need only to show that there exist $h_1, h_2 > 0$ such that

$$h_1\varepsilon \leq \|x_\varepsilon - x_0\| \leq h_2\varepsilon.$$

Let

$$F(x, \varepsilon) = B(x) + \varepsilon b(x),$$

then

$$DF(x, \varepsilon) = (DB(x) + \varepsilon Db(x), b(x)).$$

Since $DB(x_0)$ is non-degenerate,

$$DF(x_0, 0) = (DB(x_0), b(x_0))$$

is of full rank. By the Implicit Function Theorem, there exists a $\delta > 0$ and a neighborhood $U_\delta(\mathcal{O})$ of $\mathcal{O}$, such that $F(x, \varepsilon) = 0$ has a solution $x_\varepsilon \in U_\delta(\mathcal{O})$ for each $\varepsilon < \delta$. If we write $F_\varepsilon(x) = F(x, \varepsilon)$ for any given ε, then F_ε has an inverse in $U_\delta(\mathcal{O})$ and $x_\varepsilon = F_\varepsilon^{-1}(0)$.

Now we go to estimate the distance between x_ε and x_0.

By a variation of the Mean Value Theorem in the multidimensional case, there is a ξ_ε such that

$$\|B(x_\varepsilon) - B(x_0)\| \leq \|DB(\xi_\varepsilon)\| \cdot \|x_\varepsilon - x_0\|.$$

But $B(x_\varepsilon) + \varepsilon b(x_\varepsilon) = F(x_\varepsilon, \varepsilon) = 0$. Thus

$$\|\varepsilon b(x_\varepsilon)\| \leq \|DB(\xi_\varepsilon)\| \cdot \|x_\varepsilon - x_0\|,$$

or

$$\|x_\varepsilon - x_0\| \geq \varepsilon \frac{\|b(x_\varepsilon)\|}{\|DB(\xi_\varepsilon)\|}.$$

Here $\|DB(\xi_\varepsilon)\|$ cannot be zero since if it were, then we would have $\|b(x_\varepsilon)\| = 0$ and hence $\|B(x_\varepsilon)\| = 0$ or $B(x_\varepsilon) = 0$, which is impossible as the non-degeneracy of $DB(x_0)$ ensures that x_0 is the only zero of $B(x)$ in a neighborhood of x_0.

Now let $g(y) = B^{-1}(y)$ be the local inverse of $B(x)$ in a neighborhood of x_0, whose existence follows also from the non-degeneracy of $DB(x_0)$ by the Inverse Function Theorem. Let $y_\varepsilon = B(x_\varepsilon)$, then again by the variation of the Mean Value Theorem, there exists an η_ε such that

$$\|g(y_\varepsilon) - g(0)\| \leq \|Dg(\eta_\varepsilon)\| \cdot \|y_\varepsilon\|.$$

But $g(y_\varepsilon) = x_\varepsilon$, and $g(0) = x_0$. Thus we have

$$\begin{aligned} \|x_\varepsilon - x_0\| &\leq \|Dg(\eta_\varepsilon)\| \cdot \|B(x_\varepsilon)\| \\ &= \varepsilon \|Dg(\eta_\varepsilon)\| \cdot \|b(x_\varepsilon)\| \\ &= \varepsilon \|DB^{-1}(\eta_\varepsilon)\| \cdot \|b(x_\varepsilon)\|. \end{aligned}$$

To summarize, we have proved that there is an estimation

$$\varepsilon \frac{\|b(x_\varepsilon)\|}{\|DB(\xi_\varepsilon)\|} \leq \|x_\varepsilon - x_0\| \leq \varepsilon \|DB^{-1}(\eta_\varepsilon)\| \cdot \|b(x_\varepsilon)\|.$$

□

Let $B(x) = \overline{\nabla} H(x)$. By the lemma, the saddle points of the perturbed system are away from those of the unperturbed system by a distance of order ε. Nevertheless, we can find a sufficiently large compact set K containing a neighborhood of the separatrices or the "∞-shaped curve" such that in K $\|DB^{-1}(x)\| \cdot \|b(x)\|$ has a maximum M_K. Thus there is an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, $\|x_\varepsilon - x_0\| \leq M_K \varepsilon$. .

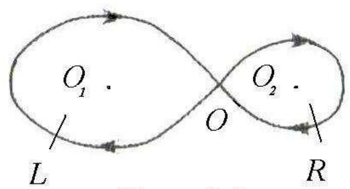

Figure 2.1

To describe the trajectories of the systems, we adopt the notations used in Brin and Freidlin[5]. Let $\mathcal{O}$ be the saddle point of the system (1.1), and $\mathcal{O}^\varepsilon$ the saddle point of the perturbed system (1.3), which tends to $\mathcal{O}$ as $\varepsilon \downarrow 0$. The two separatrices γ_l, γ_r issued from $\mathcal{O}$ form an ∞-shaped figure which bounds the region consisting of two domains L and R(see Figure 2.1). For ε small enough, the trajectories for the perturbed system spiral into the corresponding domains to the left or right of $\mathcal{O}$ [11].

Let $\Psi^t(\cdot, \varepsilon)$ denote the time-t map of (1.3), and G^t the time-t map of the gradient flow of the system

$$\dot{X} = \nabla H(X). \tag{2.1}$$

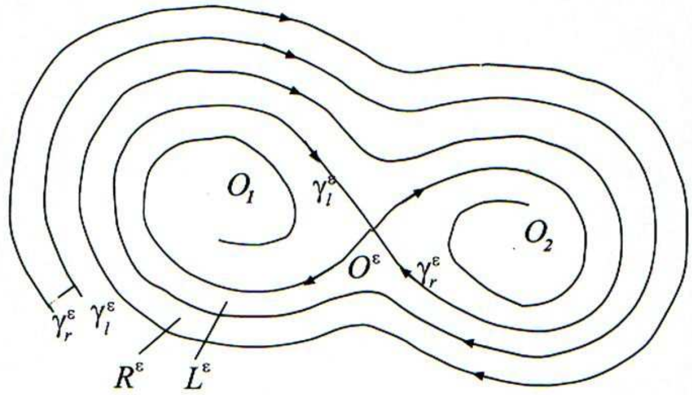

Figure 2.2

and $G(z)$ the trajectory of the point z under this flow. Let L^ε and R^ε be the basins of the attraction of L and R, respectively, i.e.,

$$L^\varepsilon = \{x : \Psi^t(x, \varepsilon) \in L, \quad \text{for} \quad t \geq T(x) \geq 0\},$$

$$R^\varepsilon = \{x : \Psi^t(x, \varepsilon) \in R, \quad \text{for} \quad t \geq T(x) \geq 0\}.$$

Then L^ε and R^ε consists of the central parts that are close to L and R, respectively, and thin ribbons which we refer to as *flow ribbons.* The boundaries of the flow ribbons are the stable separatrices γ_l^ε and γ_r^ε of $\mathcal{O}^\varepsilon$(see Figure 2.2).

To show that a change of ξ of order ε will cause a change of the same order at a distant point x, we need to estimate the number of rotations the separatrix will take when going from a neighborhood of x to a neighborhood of the saddle point, and the change of H- value for every rotation it takes. Let x_1 and x_2 be two points on the same separatrix, corresponding to time t_1 and t_2, then the time duration can be calculated by the following formula,

$$t_2 - t_1 = \int_{C(x_1, x_2)} \frac{dl}{|\overline{\nabla} H(z) + \varepsilon b(z)|},$$

where $C(x_1, x_2)$ is the part of separatrix from x_1 to x_2 and dl is the line element. When x is close to the saddle point, the velocity of the motion will be close to 0, and the time duration will be very large. However, if we can choose a neighborhood of the saddle point that is large enough to ensure the existence of some lower bound of the velocity, then the estimation should hold.

2.1.2 H-width of the flow ribbons

First we need to estimate the width or rather "H-width" of the flow ribbons for one rotation of the separatrices. We consider the left separatrix as the case for the right separatrix is the same.

Lemma 2.1.2. (H-width of the flow ribbon) *Given $\delta > 0$. Let x be a point on the separatrix γ_l^ε, outside the neighborhood $U_\delta = \{x \in \mathbb{R}^2, |x| < \delta\}$. Let $\tau_x^\varepsilon = \min\{t > 0, G^t(x) \in \gamma_l^\varepsilon\}$, and $y = G^{\tau_x^\varepsilon}(x)$. Then there exist an $\varepsilon_0 > 0$, and $0 < C_1 < C_2$ such that for all $\varepsilon < \varepsilon_0$, andb all x satisfying the assumption,*

$$C_1\varepsilon \leq H(y) - H(x) \leq C_2\varepsilon.$$

Proof. Let x be a point on the separatrix γ_l^ε and not in a neighborhood of the saddle point $\mathcal{O}^\varepsilon$. Let $G^t(x)$ be the flow with $G^0(x) = x$, $\tau = \min\{t > 0; G^t(x) \in \gamma_l^\varepsilon\}$, and $y = G^\tau(x)$. Let $G_\varepsilon(x, y)$ denote the region bounded by the flow $G^t(x), 0 \leq t \leq \tau$, from x to y and γ_l^ε.(see Figure 2.3). Let $C(x, y)$ be the part of the separatrix from y to x. Let $\mathbf{n}(z)$ be the unit outward normal vector of the boundary of $G_\varepsilon(x, y)$. The flux of the flow of the system X_t^ε along $\partial G_\varepsilon(x, y)$ is given by

$$\oint_{\partial G_\varepsilon(x,y)} (\overline{\nabla} H(z) + \varepsilon b(z)) \cdot \mathbf{n}(z) dl = \varepsilon \int_{G_\varepsilon(x,y)} \text{div} b(z) dz. \tag{2.2}$$

as $\text{div}\overline{\nabla} H(z) = 0$. But we know $(\overline{\nabla} H(z) + \varepsilon b(z)) \cdot \mathbf{n}(z) = 0$ along $C(x, y)$ and thus

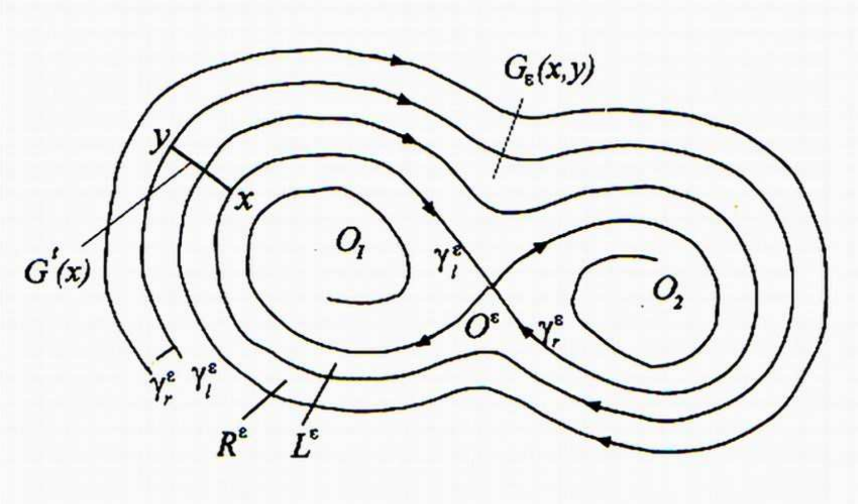

Figure 2.3

$$\int_y^x (\overline{\nabla} H(z) + \varepsilon b(z)) \cdot \mathbf{n}(z) dl = \varepsilon \int_{G_\varepsilon(x,y)} \mathrm{div} b(z) dz. \tag{2.3}$$

Since

$$(\overline{\nabla} H(z) + \varepsilon b(z)) \cdot \mathbf{n}(z) = -|\overline{\nabla} H(z)| + \varepsilon b(z) \cdot \mathbf{n}(z),$$

$$-|\overline{\nabla} H(z)| - \varepsilon |b(z)| \leq (\overline{\nabla} H(z) + \varepsilon b(z)) \cdot \mathbf{n}(z) \leq -|\overline{\nabla} H(z)| + \varepsilon |b(z)|.$$

Denote the quantity in (2.2) by $F^\varepsilon(x,y)$, then

$$-\int_y^x (|\overline{\nabla} H(z)| + \varepsilon |b(z)|) dl \leq F^\varepsilon(x,y) \leq -\int_y^x (|\overline{\nabla} H(z)| - \varepsilon |b(z)|) dl.$$

For the neighborhood U_δ of the saddle point $\mathcal{O}^\varepsilon$, when ε is small enough, say less than a ε_0, we can find positive numbers m_0 and M_0, with $\varepsilon << m_0 \leq M_0$, such that for x and y outside U_δ,

$$-M_0 \int_y^x dl \leq F^\varepsilon(x,y) \leq -m_0 \int_y^x dl.$$

In fact, we may take a sufficiently large compact set K containing a neighborhood of the homoclinic separatrix $\gamma_{l,r}^\varepsilon = C(H(\mathcal{O})) = Y^{-1}(H(\mathcal{O}))$. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$,

$$M_0 = \max_{z \in K \setminus U_\delta} (|\overline{\nabla} H(z)| + \varepsilon |b(z)|) \geq m_0 = \min_{z \in K \setminus U_\delta} (|\overline{\nabla} H(z)| - \varepsilon |b(z)|) \gg \varepsilon.$$

Since for ε small, we have approximately

$$\int_y^x dl = |x - y|,$$

thus

$$-M_0 |x - y| \leq \varepsilon \int_{G_\varepsilon(x,y)} \text{div} b(z) dz \leq -m_0 |x - y|,$$

or

$$-\frac{\varepsilon \beta}{M_0} \leq |x - y| \leq -\frac{\varepsilon \beta}{m_0},$$

where

$$\beta = \int_{G_\varepsilon(x,y)} \text{div} b(z) dz.$$

Easy to see that we can find an upper and a lower bound for β independent of x and y, thus we can find $0 < c_1 < c_2$ independent of x and y such that

$$c_1 \varepsilon \leq |x - y| \leq c_2 \varepsilon.$$

Since $H \in C^3$, by the Mean Value Theorem, there exist a $\xi = \xi_{x,y}$ between x and y, such that

$$H(x) - H(y) = \nabla H(\xi) \cdot (x - y).$$

Since $\nabla H \neq 0$ except at the critical points of H, we can also find positive lower and upper bounds for $|\nabla H(x)|$ outside U_δ. Therefore we can find $0 < C_1 < C_2$ independent of x and y such that

$$C_1 \varepsilon \leq H(y) - H(x) \leq C_2 \varepsilon.$$

$\square$

We need to know how much the separatrices of the perturbed system deviate from those of the original. We estimate the deviation in a neighborhood of $\mathcal{O}^\varepsilon$ first. Since near the saddle point, the velocity of the motion is near zero, the time duration is very large. Because of this, we cannot use the differentiable dependence of the solution to the initial value problem for the system on the initial conditions and parameters, as we usually do for an ordinary differential equation. Instead we use a small perturbation and calculate the asymptotic expansion of the solution to estimate the change in H-value of the separatrices.

2.1.3 Exit from the δ-neighborhood of the separatrices

Let $U_\delta = U(\mathcal{O}, \delta)$ and $U_\varepsilon^{1/k} = U(\mathcal{O}, \varepsilon^{1/k})$ be the open balls with center $\mathcal{O}$ and radii δ and $\varepsilon^{1/k}$, respectively.

Lemma 2.1.3. *For any $\delta \in (0,1)$, there exists an $\varepsilon_1 > 0$ such that whenever $0 < \varepsilon < \varepsilon_1$, and any two points x, y other than $\mathcal{O}^\varepsilon$ in the same component of $\gamma_l^\varepsilon \cap U_\delta$,*

$$H(x) - H(y) = o(\varepsilon).$$

Proof. Since $\delta < 1$, we can always find an $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$,

$$|\mathcal{O} - \mathcal{O}^\varepsilon| \leq M_0 \varepsilon < \frac{1}{2}\sqrt{\varepsilon} < \frac{1}{2}\delta.$$

Also, there is an integer $k \geq 2$ such that $\varepsilon^{1/k} \geq \delta$ and therefore $U_\delta \subset U_\varepsilon^k$.

Now for any two points other than $\mathcal{O}^\varepsilon$ in the same component of $\gamma_l^\varepsilon \cap U_\delta$, there are t_1, t_2 such that $X_{t_1}^\varepsilon = x$, $X_{t_2}^\varepsilon = y$. We have

$$H(y) - H(x) = \varepsilon \int_{t_1}^{t_2} \nabla H(X_s^\varepsilon) \cdot b(X_s^\varepsilon) ds.$$

Since $X_s^\varepsilon \in U_\varepsilon^k$, $|\nabla H(X_s^\varepsilon)| \leq C\varepsilon^{1/k}$ for some constant C, while $t_2 - t_1 = O(\ln \varepsilon)$. Therefore, noting that $b(x)$ is bounded in a compact set(say K),

$$|H(x) - H(y)| \leq C\varepsilon^{1+1/k} |\ln \varepsilon|.$$

□

For convenience we use the time-reversed system X_{-t}^ε, whose trajectories are the same as those of X_t^ε except for the orientation. Then the stable separatrices of X_t^ε become unstable ones of X_{-t}^ε. Let $-\gamma_l^\varepsilon$(or $-\gamma_r^\varepsilon$) denote the separatrix γ_l^ε(or γ_r^ε) with the opposite orientation. Also let $\Phi^t(\cdot, \varepsilon)$ denote the time-t map of the flow of X_{-t}^ε.

Let $a = H(\mathcal{O})$, and $H_\delta = H^{-1}((-\infty, a+\delta))$ be the *δ-neighborhood by H-value* of the separatrices of the original system (1.1).

Lemma 2.1.4. *Given $\delta \in (0,1)$, when ε is sufficiently small, there exists an $N = N_{\varepsilon,\delta}$ of order δ/ε, such that after N rotations, the unstable separatrices $-\gamma_l^\varepsilon$ (or $-\gamma_r^\varepsilon$) of the time-reversed system X_{-t}^ε will leave H_δ forever.*

Proof. Let $X_0 \in -\gamma_l^\varepsilon$ be a point close to $\mathcal{O}^\varepsilon$ such that $|X_0 - \mathcal{O}| = A\varepsilon$. Again we can find an $\varepsilon_4 \leq \varepsilon_3$ such that for $0 < \varepsilon < \varepsilon_4$,

$$|X_0 - \mathcal{O}| = A\varepsilon < \frac{1}{2}\sqrt{\varepsilon} \leq \frac{1}{2}\delta.$$

Also there is an integer $k \geq 2$ such that $\delta \leq \varepsilon^{1/k}$ and $U_\delta \subset U_\varepsilon^k$. Let $X_0^\varepsilon = X_0$, and $t_1 = \min\{t > 0 : X_{-t}^\varepsilon \in \partial U_\delta\}$, and $X_1 = X_{-t_1}^\varepsilon$. For $k \geq 1$, let $t_k = \min\{t > t_{k-1} : X_{-t}^\varepsilon \in \partial U_\delta\}$, and $X_k = X_{-t_k}^\varepsilon$(see Figure 2.4). By Lemma 2.2, for $k \geq 0$, $H(X_{2k}) - H(X_{2k+1}) = o(\varepsilon)$.

Let $H(a-\delta, a+\delta) = H^{-1}([a-\delta, a+\delta])$. Since $K_\delta = H(a-\delta, a+\delta) - U_\delta$ is compact, $|\overline{\nabla} H(x) + \varepsilon b(x)|$ has a lower bound $m_\delta > 0$. When ε is small

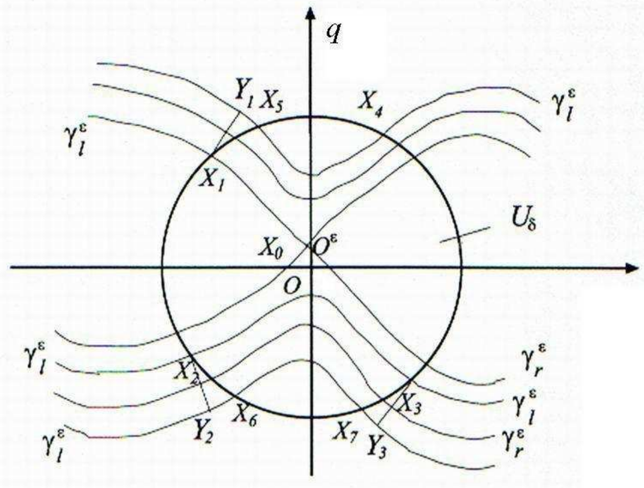

Figure 2.4

enough, $m_\delta \gg \varepsilon$. Thus the time duration from X_{2k+1} to X_{2k+2} is finite. Let $C(X_{2k+1}, X_{2k+2})$ be the part of $-\gamma_r^\varepsilon$ from X_{2k+1} to X_{2k+2}. Since $\nabla H(x)$ and $b(x)$ have also maxima in K_δ,

$$|H(X_{2k+1}) - H(X_{2k+2})| = \varepsilon \left| \int_{t_{2k+1}}^{t_{2k+2}} \nabla H(X_s^\varepsilon) \cdot b(X_s^\varepsilon) ds \right| = O(\varepsilon).$$

Therefore, we can conclude that for any two points x, $y \in -\gamma_l^\varepsilon$ within a rotation, i.e. between X_k and X_{k+4} for some k, $H(x) - H(y) = O(\varepsilon)$. In particular, for any point x between X_0 and X_4, $H(x) - H(X_0) = O(\varepsilon)$, and hence $H(x) - a = O(\varepsilon)$.

Now consider the time t-map $G^t(x)$ of the gradient flow of H starting at x. Let $\tau_k = \min\{t > 0 : G^t(X_k) \in -\gamma_l^\varepsilon\}$, and $Y_k = G^{\tau_k}(X_k)$.

Since $|Y_k - X_{k+4}| \leq M\varepsilon$ for some constant M, we have

$$T_k = \int_{C(X_{k+4}, Y_k)} \frac{dl}{|\overline{\nabla} H(x) + \varepsilon b(x)|} \leq M\varepsilon / m_0.$$

Thus

$$|H(X_{k+4}) - H(Y_k)| \leq \varepsilon \int_0^{T_k} |\nabla H(X_s^\varepsilon) \cdot b(X_s^\varepsilon)| ds \leq \varepsilon^2 \frac{M \|\nabla H\| \cdot \|b\|}{m_0}.$$

Now since we have the estimation $C_1\varepsilon \le H(Y_k) - H(X_k) \le C_2\varepsilon$ (note that Y_k is outside U_ε^2), we know that there is a $C_3 \le C_1$ such that $C_3\varepsilon \le H(X_{k+4}) - H(X_k) \le C_2$.

Let $x \in C(X_k, X_{k+4})$, and $\tau_x = \min\{t > 0 : G^t(x) \in -\gamma_l^\varepsilon\}$, and $y = G^{\tau_x}(x)$. Then since $C(X_k, X_{k+4})$ is compact, there is a minimum $h_k = \min\{H(y)-H(x) : x \in C(X_k, X_{k+4})\} \ge C_3\varepsilon$. Also since $C(X_0, X_4)$ is compact, there is a minimum $H_0 = \min\{H(x) : x \in C(X_0, X_4)\} \ge a - c\varepsilon$ for some $c > 0$, where $a = H(\mathcal{O})$. Now for any point $x \in C(X_0, X_4)$, let $x_0 = x$, $\tau_0(x) = 0$, $\tau_k(x) = \min\{t > \tau_{k-1}(x) : G^t(x) \in -\gamma_l^\varepsilon\}$, and $x_k = G^{\tau_k(x)}(x)$. Then $H(x_k) > kC_3\varepsilon + a - c\varepsilon$. Therefore there is an $N = N_{\varepsilon,\delta}$ of order δ/ε such that $H(x_k) > a + \delta$ for all $k \ge N$, and after N rotations $-\gamma_l^\varepsilon$ will no longer touch H_δ.

$\square$

2.2 Deviations inside the δ-neighborhood

To establish the almost-uniformity of the distribution of the point x within the flow ribbon, we consider the change of the random variable ξ of order ε. As shown in Chapter 1, we can rewrite this as an additional perturbation term $\varepsilon^2\beta(x)$. Therefore we consider the following system instead.

$$\dot{\tilde{X}}_t^\varepsilon = \overline{\nabla}H(\tilde{X}_t^\varepsilon) + \varepsilon b(\tilde{X}_t^\varepsilon) + \varepsilon^2\beta(\tilde{X}_t^\varepsilon), \tag{2.4}$$

In the same fashion as for the perturbed system (1.3), we can show that this new system (2.4) has a saddle point $\tilde{\mathcal{O}}^\varepsilon$ which is away from $\mathcal{O}^\varepsilon$ by a distance of order ε^2 and from $\mathcal{O}$ by a distance of order ε, that outside a neighborhood of $\mathcal{O}$, the H-width of the flow ribbons is of order ε, that it will take $N_{\varepsilon,\delta}$ number of rotatbions for the corresponding time-reversed separatrices, denoted by $-\tilde{\gamma}_l^\varepsilon$(or

$-\tilde{\gamma}_r^\varepsilon$), to get out of the δ-neighborhood by H-value H_δ, and that for any two points x and y in the same component of $\tilde{\gamma}_l^\varepsilon \cap U_\delta$, $H(x) - H(y) = o(\varepsilon)$.

We need to consider the effect of this additional perturbation term $\beta(x)$ in terms of the deviation of its separatrices $-\tilde{\gamma}_l^\varepsilon$(or $-\tilde{\gamma}_r^\varepsilon$) from $-\gamma_l^\varepsilon$ (or $-\gamma_r^\varepsilon$, resp.) of X_t^ε when both of them get out of H_δ after $N_{\varepsilon,\delta}$ rotations. Due to the delicate nature of the separatrices in a neighborhood of the saddle point, we need some additional assumption to simplify the calculation when trajectories are close to the saddle points. This will of course reduce the significance of our result, but since the general case is more complicated, we would rather deal with this special case first before attacking the general one. The estimation will consists of there steps. First, we show that the separatrices will get out of the homoclinic separatrix $\gamma_{l,r}^\varepsilon$, the ∞-shaped curve in a few rotations and the deviation of $-\tilde{\gamma}_l^\varepsilon$ from $-\gamma_l^\varepsilon$ is of order ε^2. Secondly, we estimate the increment of that deviation when the separatrices just get out of the neighborhood U of the origin. Then we estimate the deviation when the separatrices pass through a square neighborhood of the origin. When these results are put together, we have the estimation of the deviation outside a δ-neighborhood.

2.2.1 Finite time estimation

Consider the square $D_1 = \{(p,q) : |p| + |q| < 1\}$. Let $I_\delta = \{(1,q) : |q| \leq \delta\}$. For any point $x \in I_\delta$, we may identify x with its q-coordinate. Let $J = \{(p,1) : p \in [-1,1]\}$, the lower edge of D_1. Without loss of generality, we may assume that the flow of the unperturbed system is transverse to ∂D_1.

Consider the vector field

$$F(x,\xi,\varepsilon) = \overline{\nabla} H(x) + \varepsilon(b_1(x) + \xi b_2(x)).$$

Define $\varphi : \mathbb{R}^+ \times I_\delta \times \mathbb{R}^+ \times [0, \varepsilon_0) \to \mathbb{R}^2$ by $\varphi(t, q, \xi, \varepsilon) = X^{\varepsilon,\xi}_{-t,q}$, where $X^{\varepsilon,\xi}_{t,q}$ is the solution of

$$\dot{X}^{\varepsilon,\xi}_{t,q} = F(X^{\varepsilon,\xi}_{t,q}, \xi, \varepsilon), \quad X^{\varepsilon,\xi}_{0,q} = (1, q).$$

Also define $\tau^{\varepsilon,\xi}_q = \min\{t > 0 : X^{\varepsilon,\xi}_{t,q} \in \partial D_1\}$. We need to show that $\tau^{\varepsilon,\xi}_q$ is finite.

By assumption, the flow of the unperturbed system, which has the form $\varphi(t, q, \xi, 0)$, is transverse to J. Since J is away from the fixed points of the vector field F, for ε small enough, the flow of the system defined by $-F(x, \xi, \varepsilon)$ is also transverse to J, and we can find a lower bound $m > 0$ for $\|F\|$ around J. Now let $y = (1, q)$, $y' = \varphi(\tau^{\varepsilon,\xi}_q, q, \xi, \varepsilon)$, and $C(y, y')$ denote the part of trajectory from y to y'. Then

$$\tau^{\varepsilon,\xi}_q = \int_{C(y,y')} \frac{dl}{|F(x, \xi, \varepsilon)|} \leq \int_{C(y,y')} \frac{ds}{m}$$

is finite.

Note that locally, $\tau^{\varepsilon,\xi}_q$ is a function defined by the equation $\varphi_2(\tau^{\varepsilon,\xi}_q, q, \xi, \varepsilon) = -1$, where φ_2 is the second component of φ. The unperturbed flow is a local C^2 diffeomorphism onto a neighborhood of J. This is because we have an extension of the differentiability of the solution of the initial value problem of an ODE w.r.t. the initial condition and parameters. Easy to see that $\partial\varphi_2/\partial t \neq 0$ around ∂D_1. By the Implicit Function Theorem, there exists a C^2 function $\tau = \tau^{\varepsilon,\xi}_q = \tau(q, \xi, \varepsilon)$ such that $\varphi_2(\tau, q, \xi, \varepsilon) = -1$ in a open neighborhood V of $I_\delta \times \{\xi\} \times \{0\}$ for any $\xi \in \mathbb{R}^+$. Define $\Phi : V \to J$ by

$$\Phi(q, \xi, \varepsilon) = \varphi_1(\tau(q, \xi, \varepsilon), q, \xi, \varepsilon).$$

Then Φ is C^2.

Now let $\tilde{q}, q \in [-\delta, \delta]$, and $h = c\varepsilon$ for some $c \gg \varepsilon$. Let $\tilde{x} = \Phi(\tilde{q}, \xi + h, 0)$, and $x = \Phi(q, \xi, 0)$.

Lemma 2.2.1. *The increment of the difference between $\tilde{q}$ and q after the trajectories hit J is given by*

$$\Delta(\tilde{q}, q) = (\tilde{q} - q)C\varepsilon + \varepsilon^2 M$$

for some constants C and M.

Proof.

$$\begin{aligned}
\Delta(\tilde{q}, q) &= \Phi(\tilde{q}, \xi + h, \varepsilon) - \Phi(q, \xi, \varepsilon) - \tilde{x} + x \\
&= \Phi(\tilde{q}, \xi + h, \varepsilon) - \Phi(\tilde{q}, \xi + h, 0) + \Phi(q, \xi, 0) - \Phi(q, \xi, \varepsilon) \\
&= \varepsilon \frac{\partial \Phi}{\partial \varepsilon}(\tilde{q}, \xi + h, 0) + \varepsilon^2 \frac{\partial^2 \Phi}{\partial \varepsilon^2}(\tilde{q}, \xi + h, \theta_{\tilde{q}}\varepsilon) - \varepsilon \frac{\partial \Phi}{\partial \varepsilon}(q, \xi, 0) - \varepsilon^2 \frac{\partial^2 \Phi}{\partial \varepsilon^2}(q, \xi, \theta_q \varepsilon) \\
&= \varepsilon \frac{\partial^2 \Phi}{\partial \varepsilon \partial q}(q', \xi + h, 0)(\tilde{q} - q) + \varepsilon h \frac{\partial^2 \Phi}{\partial \varepsilon \partial \xi}(q, \xi + h\theta_h, 0) + \varepsilon^2 M \\
&= (\tilde{q} - q)C\varepsilon + \varepsilon^2 M,
\end{aligned}$$

where $\theta_{\tilde{q}} \in (0, 1)$, $\theta_h \in (0, 1)$, q' is between q and $\tilde{q}$. $\square$

2.2.2 Exit from the ∞-shaped curve

For the estimation inside the square, we assume that the Hamiltonian H is quadratic in a neighborhood U of the origin $\mathcal{O}$. Without loss of generality, we may assume that U contains the square $D_1 = \{(p, q) : |p| + |q| < 1\}$. After a suitable coordinate change, the corresponding Hamiltonian system is of the form

$$\dot{X}_t = AX_t, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.5}$$

in U. The corresponding perturbed system then becomes

$$\dot{X}_t^\varepsilon = AX_t^\varepsilon + \varepsilon b(X_t^\varepsilon), \tag{2.6}$$

and the system with the second perturbation term

$$\dot{\widetilde{X}}_t^\varepsilon = A\widetilde{X}_t^\varepsilon + \varepsilon b(\widetilde{X}_t^\varepsilon) + \varepsilon^2 \beta(\widetilde{X}_t^\varepsilon). \tag{2.7}$$

Again we consider the time-reversed systems. Suppose the systems X_{-t}^ε and $\widetilde{X}_{-t}^\varepsilon$ start at the points $X_0 \in -\gamma_l^\varepsilon$ and $\widetilde{X}_0 \in \tilde{\gamma}_l^\varepsilon$, respectively, with $|\tilde{x}_0 - x_0| = C_0\varepsilon^2$. We will estimate the distance between them when they both get out of the δ-neighborhood H_δ. We may assume that both X_0 and $\widetilde{X}_0$ are in the first quadrant.

Write $X_{-t}^\varepsilon = (p_t, q_t)$ and $\widetilde{X}_{-t}^\varepsilon = (\tilde{p}_t, \tilde{q}_t)$. Then we have

$$\begin{aligned} \dot{p}_t &= p_t - \varepsilon b_1(p_t, q_t) \\ \dot{q}_t &= -q_t - \varepsilon b_2(p_t, q_t), \end{aligned}$$

with $X_0^\varepsilon = (p_0, q_0)$ and

$$\begin{aligned} \dot{\tilde{p}}_t &= \tilde{p}_t - \varepsilon b_1(\tilde{p}_t, \tilde{q}_t) - \varepsilon^2 \beta_1(\tilde{p}_t, \tilde{q}_t) \\ \dot{\tilde{q}}_t &= -\tilde{q}_t - \varepsilon b_2(\tilde{p}_t, \tilde{q}_t) - \varepsilon^2 \beta_2(\tilde{p}_t, \tilde{q}_t), \end{aligned}$$

$\widetilde{X}_0^\varepsilon = (\tilde{p}_0, \tilde{q}_0)$.

Note that here we use $b_i(x)$ to denote the i-th component of $b(x)$, $i = 1, 2$. This is not the same as the one we used in Chapter 1, where b_1 and b_2 are two different vector fields.

Let $\tau_1 = \min\{t > 0 : p_t = 1\}$, $q_1 = q_{\tau_1}$. Similarly, let $\tilde{\tau}_1 = \min\{t > 0 : \tilde{p}_t = 1\}$, $\tilde{q}_1 = \tilde{q}_{\tilde{\tau}_1}$.

Lemma 2.2.2. *If* $|\tilde{p}_0 - p_0| = a\varepsilon^2$, $|\tilde{q}_0 - q_0| = b\varepsilon^2$, *then* $|\tilde{q}_1 - q_1| \le C\varepsilon^2$.

The proof involves lengthy and tedious calculation. We put it in an appendix.

Corollary 2.2.3. *If* $\tilde{q}_0 = q_0 = 1$, *then* $|\tilde{q}_1 - q_1| \le C\varepsilon^2$.

This is because in the proof of the lemma, we would have $|\tilde{q}_0 - q_0| = 0$ and the estimation is a little bit easier.

By the Corollary, we can have the estimation when the two separatrices enter the square D_1 when they are mostly in the fourth quadrant. Combine this with Lemma 2.2.1 and the one we just proved, we know that when the two separatrices get out of the ∞-shaped curve in a few rotations, the distance between them is of order ε^2.

2.2.3 Estimation near the saddle point

When the orientation-reversed separatrix $-\gamma_i^\varepsilon$ (or $-\tilde{\gamma}_i^\varepsilon$) takes the n-th rotation, it enters the square D_1 at a point on the upper edge and exit at a point on the right edge of D_1. Suppose it hits the upper edge at $(p_n, 1)$ (or $((\tilde{p}_n, 1)$, resp.) and the right edge at $(1, q_n)$ (or $(\tilde{q}_n, 1)$, resp.). We need to estimate the increment of the distance between the two separatrices $-\gamma_i^\varepsilon$ and $-\tilde{\gamma}_i^\varepsilon$, i.e., we need to compare $\tilde{q}_n - q_n$ and $\tilde{p}_n - p_n$.

By the estimation of the H-width of the flow ribbon, approximately we have $p_n = (a_0 + na)\varepsilon$ and $\tilde{p}_n = (\tilde{a}_0 + n\tilde{a})\varepsilon$, where $a\varepsilon(\tilde{a}\varepsilon$, resp.) is the average H-width of the flow ribbon for $-\gamma_i^\varepsilon(-\tilde{\gamma}_i^\varepsilon$, resp.).

Again we consider the ε-expansion of the time-reversed systems

$$\begin{aligned}
\dot{p}_t &= p_t - \varepsilon b_1(p_t, q_t), \\
\dot{q}_t &= -q_t - \varepsilon b_2(p_t, q_t),
\end{aligned}$$

with $p_0 = p_n$, $q_0 = 1$. Write

$$\dot{X}_{-t}^\varepsilon = X_{-t}^{(0)} + \varepsilon X_{-t}^{(1)} + \varepsilon^2 X_{-t}^{(2)} + h.o.t., \quad X_0^\varepsilon = (p_n.1),$$

with $X^\varepsilon_{-t} = (p_t, q_t)$. Similarly, we have

$$\begin{aligned}
\dot{\tilde{p}}_t &= \tilde{p}_t - \varepsilon b_1(\tilde{p}_t, \tilde{q}_t) - \varepsilon^2 \beta_1(\tilde{p}_t, \tilde{q}_t), \\
\dot{\tilde{q}}_t &= -\tilde{q}_t - \varepsilon b_2(\tilde{p}_t, \tilde{q}_t) - \varepsilon^2 \beta_2(\tilde{p}_t, \tilde{q}_t),
\end{aligned}$$

with $\tilde{p}_0 = \tilde{p}_n$, $\tilde{q}_0 = 1$, or

$$\dot{\widetilde{X}}^\varepsilon_{-t} = \widetilde{X}^{(0)}_{-t} + \varepsilon \widetilde{X}^{(1)}_{-t} + \varepsilon^2 \widetilde{X}^{(2)}_{-t} + h.o.t., \quad \widetilde{X}^\varepsilon_0 = (\tilde{p}_n, 1),$$

with $\widetilde{X}^\varepsilon_{-t} = (\tilde{p}_t, \tilde{q}_t)$. By omitting the higher order terms, we have approximately the solutions as follows,

$$\begin{aligned}
p_t &= e^t(p_n - \varepsilon \int_0^t e^{-s} b_1(e^s p_n, e^{-s}) ds - \varepsilon^2 \int_0^t e^{-s} Db_1(e^s p_n, e^{-s}) \cdot X^{(1)}_{-s} ds), \\
q_t &= e^{-t}(1 - \varepsilon \int_0^t e^{s} b_2(e^s p_n, e^{-s}) ds - \varepsilon^2 \int_0^t e^{s} Db_2(e^s p_n, e^{-s}) \cdot X^{(1)}_{-s} ds) \\
\tilde{p}_t &= e^t(\tilde{p}_n - \varepsilon \int_0^t e^{-s} b_1(e^s \tilde{p}_n, e^{-s}) ds - \varepsilon^2 \int_0^t e^{-s} [Db_1(e^s \tilde{p}_n, e^{-s}) \cdot \widetilde{X}^{(1)}_{-s} + \beta_1(e^s \tilde{p}_n, e^{-s})] ds), \\
\tilde{q}_t &= e^{-t}(1 - \varepsilon \int_0^t e^{s} b_2(e^s \tilde{p}_n, e^{-s}) ds - \varepsilon^2 \int_0^t e^{s} [Db_2(e^s \tilde{p}_n, e^{-s}) \cdot \widetilde{X}^{(1)}_{-s} + \beta_2(e^s \tilde{p}_n, e^{-s})] ds)
\end{aligned}$$

Let $\tau_n = \min\{t > 0 : p_t = 1\}$, $\tilde{\tau}_n = \min\{t > 0 : \tilde{p}_t = 1\}$, then $q_n = q_{\tau_n}$, $\tilde{q}_n = \tilde{q}_{\tilde{\tau}_n}$.

Lemma 2.2.4. *With the above notation, we have the difference of the two separatrices after passing through the square neighborhood D_1 in the first quadrant, given by*

$$(\tilde{q}_n - q_n) = (\tilde{p}_n - p_n)(1 + \frac{C^1_n}{n^2} + C_{n,0}\varepsilon\tilde{\tau}_n + C_{n,1}\varepsilon + C_{n,2}\varepsilon^2) + \varepsilon^2 M_n + o(\varepsilon^2),$$

where $C_{n,i}$ are constants depending on n and $b(x)$ but not ε.

The proof is put in an appendix.

2.2.4 Deviation after getting out of the δ-neighborhood

Now we can combine all the results obtained so far to estimate the deviation of the separatrices caused by the change of ξ when both separatrices, the one with the the additional perturbation and the one without, get out of the δ-neighborhood. Since the H-width of the flow ribbon is of order ε, roughly it takes $N = N_{\varepsilon,\delta}$ rotations for the separatrix to get out of the δ-neighborhood of the ∞-shaped curve. Here N is of order $\frac{\delta}{\varepsilon}$. Within each rotation, the distance between the two separatrices increases according to the three lemmas we have proved in this section. Let $\delta_n = |\tilde{p}_n - p_n|$, then we have

Lemma 2.2.5.

$$\delta_{n+1} = \delta_n(1 + \frac{C_0}{n^2} + C_1\varepsilon\tilde{\tau}_n + C_2\varepsilon + C_3\varepsilon^2) + M\varepsilon^2 + o(\varepsilon^2).$$

Proof. We need only to note that outside the square neighborhood D_1, the distance increases by a factor $1 + C\varepsilon$ and plus something of order ε^2. After some simple calculation we can see the result easily. $\square$

Remark 2.2.6. This actually holds in general outside the δ-neighborhood. If δ_n is the distance between the two separatrices $-\gamma_i^\varepsilon$ and $-\tilde{\gamma}_i^\varepsilon$ then outside that square neighborhood we still have the equality above, since the increment of the distance is given by Lemma 2.2.1., which holds because of the finite time duration.

We are yet to show that after the separatrices get out of the δ-neighborhood, the distance between the two separatrices is still of order ε. More precisely, we have

Lemma 2.2.7. *Given any $\delta > 0$, there exists an ε_δ such that for $\varepsilon < \varepsilon_\delta$, after both separatrices leave the δ-neighborhood, the distance between the two separatrices*

$-\gamma_i^\varepsilon$ and $-\tilde{\gamma}_i^\varepsilon$ will be approximately $c(\varepsilon,\delta)\varepsilon$, where $c(\varepsilon,\delta)$ can be made smaller than any given $\eta > 0$ by choosing suitable δ and ε_δ.

Proof. Since the average H-width for the flow ribbon is $a\varepsilon$ for $-\gamma_i^\varepsilon$, it will take about N rotations for it to leave the δ-neighborhood of the ∞-shaped curve, where N is of order $\delta/a\varepsilon$. Now by the estimation in Lemma 2.2.4., we have

$$\delta_{n+1} = \delta_n(1 + \frac{C_0}{n^2} + C_1\varepsilon\tilde{\tau}_n + C_2\varepsilon + C_3\varepsilon^2) + M\varepsilon^2 + o(\varepsilon^2).$$

When n is large enough, $\tilde{\tau}_n = \ln(\tilde{p}_n)^{-1}$ approximately. Let $a_n = 1 + C_0/n^2 + C_1\varepsilon\ln(\tilde{p}_n)^{-1} + C_2\varepsilon + C_3\varepsilon^2$. Define a sequence e_n as follows, $e_0 = 1$, $e_1 = a_n$, $e_k = a_n a_{n-1}\cdot\ldots\cdot a_{n-k+1}$ for $k > 1$. Then by recursion we have

$$\delta_{n+1} = \delta_0\prod_{k=1}^{n} a_k + M\varepsilon^2\sum_{k=0}^{n-1}\frac{e_n}{e_{n-k}}.$$

Now

$$\begin{aligned}
\prod_{n=1}^{N} a_n &= \prod_{n=1}^{N}(1 + \frac{C_0}{n^2} + C_1\varepsilon\ln(\tilde{p}_n)^{-1} + C_2\varepsilon + C_3\varepsilon^2) \\
&= \exp\left(\sum_{n=1}^{N}\ln(1 + \frac{C_0}{n^2} + C_1\varepsilon\ln(\tilde{p}_n)^{-1} + C_2\varepsilon + C_3\varepsilon^2)\right) \\
&= \exp\left(\sum_{n=1}^{N}\left(\frac{C_0}{n^2} + C_1\varepsilon\ln(\tilde{p}_n)^{-1} + C_2\varepsilon + C_3\varepsilon^2\right) + o(1)\right)
\end{aligned}$$

It is well known that

$$\sum_{n=1}^{N}\frac{1}{n^2} \le \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

For we can roughly write $\tilde{p}_n = n\tilde{a}\varepsilon$,

$$\sum_{1}^{N}\varepsilon\ln(\tilde{p}_n)^{-1} = -\sum_{1}^{N}\varepsilon\ln(n\tilde{a}\varepsilon) \le -\frac{1}{\tilde{a}}\int_0^\delta \ln(x)dx = \frac{\delta}{\tilde{a}}(1 - \ln\delta).$$

Thus by adjusting the constants, we may write

$$\prod_{n=1}^{N} a_n = \exp\left(\frac{C_0\pi^2}{6} + \frac{C_1\delta}{\tilde{a}}(1-\ln\delta) + \frac{C_2\delta}{\tilde{a}} + \frac{C_3\delta\varepsilon}{\tilde{a}}\right).$$

Let $\sigma(\delta)$ denote $\frac{C_0\pi^2}{6} + \frac{C_1\delta}{\tilde{a}}(1-\ln\delta) + \frac{C_2\delta}{\tilde{a}} + \frac{C_3\delta\varepsilon}{\tilde{a}}$, again adjusting the constants yields

$$M\varepsilon^2 \sum_{k=0}^{n-1} \frac{e_n}{e_{n-k}} = M\varepsilon^2 \frac{\delta}{\tilde{a}\varepsilon} e^{\sigma(\delta)} = \frac{M\delta\varepsilon}{\tilde{a}} e^{\sigma(\delta)}.$$

Note that $\delta_0 = \alpha\varepsilon^2$ for some α, we have

$$\delta_N = \alpha\varepsilon^2 e^{\sigma(\delta)} + \frac{M\delta\varepsilon}{\tilde{a}} e^{\sigma(\delta)} = \varepsilon\left(\alpha\varepsilon + \frac{M\delta}{\tilde{a}}\right) e^{\sigma(\delta)} = c(\varepsilon,\delta)\varepsilon.$$

Given any $\eta > 0$, easy to see that we can make $c(\varepsilon,\delta)$ smaller than η by choosing δ and ε_δ small enough. □

2.2.5 The action-angle variables

Outside the ∞-shaped curve, the trajectories of the unperturbed system are periodic. We can introduce the so-called "action-angle variables" to simplify the system. We follow the description of Arnold [2].

For any $h > H(\mathcal{O}) + \delta$, let $M_h = H^{-1}(h)$ denote the closed trajectory on which H has the constant value h.

Theorem 2.2.8. (Liouville[2]) *With the above assumption, there exists a canonical transformation $(p,q) \mapsto (I,\varphi)$, such that*

$$I = I(h), \qquad \oint_{M_h} d\varphi = 2\pi,$$

and in the new coordinate system (I,φ) the unperturbed system has the form

$$\begin{aligned} \dot{I}_t &= 0, \\ \dot{\varphi}_t &= \omega(I_t). \end{aligned}$$

Remark 2.2.9. In fact,

$$I(h) = \frac{1}{2\pi}\oint_{M_h} pdq, \quad \varphi = \frac{\partial S(I,q)}{\partial I},$$

where $S = S(I,q)$ is the generating function defined by

$$S(I,q) = \int_{q_0}^{q} pdq|_{H=h(I)},$$

with $h(I)$ the inverse function of $I(h)$. Note that φ is multi-valued.

For the perturbed system, suppose that in the new coordinate system, the vector field F has the form $F(I,\varphi,\xi,\varepsilon) = (\varepsilon B_1(I,\varphi,\xi), \omega(I)+\varepsilon B_2(I,\varphi,\xi))$. Then the perturbed system then has the form

$$\begin{aligned}
\dot{I}_t^{\varepsilon,\xi} &= \varepsilon B_1(I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi}, \xi),\\
\dot{\varphi}_t^{\varepsilon,\xi} &= \omega(I_t^{\varepsilon,\xi}) + \varepsilon B_2(I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi}, \xi).
\end{aligned}$$

Note that the functions B_i are periodic in φ with period 2π.

2.2.6 Estimate outside the δ-neighborhood

Now for a point x outside the δ-neighborhood, we may assume that it has the coordinates (I_x, φ_0). We want to compare the distance between the two separatrices $-\gamma_i^\varepsilon$ and $-\tilde{\gamma}_i^\varepsilon$, which correspond to the solution $(I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi})$ with initial point (I_0, φ_0), and $(I_t^{\varepsilon,\xi+h}, \varphi_t^{\varepsilon,\xi+h})$ with initial point $(\tilde{I}_0, \varphi_0)$ when they both reach a neighborhood of x, intersecting the flow line $G^t(x)$ of the gradient of H containing x. Here we assume both separatrices start with the same angle φ_0 as that of the point x for convenience. The difference between the two initial points is $c(\delta,\varepsilon)\varepsilon$ from the result of last section. Since the velocity of I is of order ε, the amount of time it takes for the system to reach a neighborhood of x is of order

$1/\varepsilon$. We need to work on a finite time interval for using the dependence of the solution to an ODE on initial data and parameters.

Consider the systems with the time scaling $t \to t/\varepsilon$, the perturbed systems are of the form

$$\begin{aligned}
\dot{I}_t^{\varepsilon,\xi} &= B_1(I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi}, \xi), \quad I_0^{\varepsilon,\xi} = I_0, \\
\dot{\varphi}_t^{\varepsilon,\xi} &= \frac{1}{\varepsilon}\omega(I_t^{\varepsilon,\xi}) + B_2(I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi}, \xi), \quad \varphi_0^{\varepsilon,\xi} = \varphi_0,
\end{aligned}$$

and

$$\begin{aligned}
\dot{I}_t^{\varepsilon,\xi+h} &= B_1(I_t^{\varepsilon,\xi+h}, \varphi_t^{\varepsilon,\xi+h}, \xi+h), \quad I_0^{\varepsilon,\xi+h} = \tilde{I}_0, \\
\dot{\varphi}_t^{\varepsilon,\xi+h} &= \frac{1}{\varepsilon}\omega(I_t^{\varepsilon,\xi+h}) + B_2(I_t^{\varepsilon,\xi+h}, \varphi_t^{\varepsilon,\xi+h}), \quad \varphi_0^{\varepsilon,\xi+h} = \varphi_0.
\end{aligned}$$

In the following we will work with these scaled systems on a finite time interval.

Let $G^t(x)$ denote the flow line of the gradient field of H passing through x. Let $n = \max\{k : \varphi_t^{\varepsilon,\xi} = \varphi_0 + 2k\pi, I_t^{\varepsilon,\xi} \leq I_x\}$, $\tau = \min\{t > 0 : \varphi_t^{\varepsilon,\xi} = \varphi_0 + 2n\pi\}$. Similarly, define $\tilde{\tau} = \min\{t > 0 : \varphi_t^{\varepsilon,\xi+h} = \varphi_0 + 2n\pi\}$.

Define $\psi : \mathbb{R}^+ \times \mathbb{R} \times S^1 \times \mathbb{R}^+ \to \mathbb{R}^2$ by $\psi(t, I^*, \varphi^*, \xi) = (\psi_1(t, I^*, \varphi^*, \xi), \psi_2(t, I^*, \varphi^*, \xi)) = (I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi})$ where $I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi}$ satisfy the system

$$\dot{X}_t^{\varepsilon,\xi} = F(I_t^{\varepsilon,\xi}, \varphi_t^{\varepsilon,\xi}, \xi, \varepsilon), \quad I_0^{\varepsilon,\xi} = I^*, \varphi_0^{\varepsilon,\xi} = \varphi^*.$$

Define $\tau_{I^*,\varphi^*}^{\xi,n} = \min\{t > 0 : \psi_2(t, I^*, \varphi^*, \xi) = \varphi^* + 2n\pi\}$. This is the time that $\psi(t, I^*, \varphi^*, \xi)$ intersects the flow line of the gradient of H the n-th time after starting with (I^*, φ^*). Now we know that for fixed ε and φ_0, the flow ψ is a local C^2 diffeomorphism for the initial condition $(I, \varphi) = (I_0, \varphi_0)$. Since $\frac{\partial \psi_2}{\partial t} \neq 0$, by the Implicit Function Theorem, there exists a C^2-function $\tau = \tau_{I,\varphi_0}^{\xi,n} = \tau(I, \xi)$ such that $\psi_2(\tau(I,\xi), I, \varphi_0, \xi) = \varphi_0 + 2n\pi$ in an open neighborhood $U \subset \mathbb{R} \times \{\varphi_0\} \times \mathbb{R}^+$ of the point $(\tau, I_0, \varphi_0, \xi)$. Define $\Psi : U \to \mathbb{R}^2$ by

$$\Psi(I,\xi) = \psi(\tau(I,\xi), I, \varphi_0, \xi),$$

then Ψ is a local C^2-diffeomorphism and in particular, Ψ_1 is C^2. Now

$$\begin{aligned}
I_{\tilde{\tau}}^{\varepsilon,\xi+h} - I_{\tau}^{\varepsilon,\xi} &= \Psi_1(\tilde{I}_0, \xi+h) - \Psi_1(I_0,\xi) \\
&= \frac{\partial \Psi_1}{\partial I}(I^0, \xi+h)(\tilde{I}_0 - I_0) + \frac{\partial \Psi}{\partial \xi}(I_0, \xi + h\theta)h \\
&= \varepsilon\left(\frac{\partial \Psi_1}{\partial I}(I^0, \xi+h)c(\delta,\varepsilon) + \frac{\partial \Psi}{\partial \xi}(I_0, \xi+h\theta)\alpha\right) + o(\varepsilon) \\
&= C(\varepsilon,\xi,h)\varepsilon + o(\varepsilon),
\end{aligned}$$

where $h = \alpha\varepsilon$, $\theta \in (0,1)$, and I^0 is between I_0 and $\tilde{I}_0$.

Therefore we have proved the following

Theorem 2.2.10. *If the increment of ξ is $h = \alpha\varepsilon$, then at a distant point x, the change of the left (resp. right) separatrix γ_l^ε (resp. γ_r^ε) in the gradient direction of H is $C(\varepsilon,\xi,h)\varepsilon + o(\varepsilon)$.*

Remark 2.2.11. Note that this is actually true for all the trajectories outside the ∞-shaped curve going into one of the basins L and R.

2.3 Branching at the interior vertex

In this section we will prove the distribution formula of the probability that a trajectory passing through a point x will enter the left or right basin.

2.3.1 Almost uniformity

Since ξ has a continuous conditional density given b_1 and b_2, locally it has an almost uniform distribution, which means the rate of change for ξ is almost constant. By Theorem 2.3.3., the increment of change of the separatrices in the gradient direction of H also changes at an almost constant rate. Let $G^t(x)$ be the flow line of the gradient field of H containing x, and the separatrix γ_i^ε intersect $G^t(x)$ at $y(\xi)$ and $z(\xi)$ with x between them, then we have the following theorem.

Theorem 2.3.1. (Almost uniformity) *Condition on b_1 and b_2, the distribution of x in the segment from y to z is almost uniform. More precisely, let $w = w(\xi)$ be the intersection point of the other separatrix γ_j^ε ($j \neq i$) with $G^t(x)$ between y and z. Denote the conditional probability measure of ξ condition on b_1 and b_2 by $\mathrm{P}^\varepsilon_{\xi|b_1,b_2}$. Then*

$$\mathrm{P}^\varepsilon_{\xi|b_1,b_2}\{x \in (w, y) | x \in (z, y)\} = \frac{I(y) - I(w)}{I(y) - I(z)} + o(1).$$

Proof. Since ξ has a continuous conditional density given b_1 and b_2, in a neighborhood of ξ, the distribution of ξ is almost uniform. By Theorem 2.3.3., when ξ has a changes $h = \alpha\varepsilon$ in this neighborhood, the corresponding separatrices will have a change $C(\varepsilon, \xi, h)\varepsilon + o(\varepsilon)$ in the gradient direction of H near x. Thus the distances the separatrices move in the gradient direction are proportional to the corresponding changes of ξ. Hence the almost uniform distribution. □

Remark 2.3.2. We can replace I by H in the theorem as from the construction of the action-angle coordinates we see that I and H are inverse to each other.

2.3.2 Ratio of the H-widths of the flow ribbons

Now since the distribution of the point x is almost uniform in the segment from y to z, the probability that a trajectory passing through x enters the left or right basin is proportional to the relative H-width of the flow ribbon leading to the left or right basin. If $y \in \gamma_l^\varepsilon$, then $w \in \gamma_r^\varepsilon$ and $z \in \gamma_l^\varepsilon$, with $H(y) < H(w) < H(z)$.

If we denote the solution to the system (1.3) with initial point x by $\varphi(t, x, \xi, \varepsilon)$, and the probability that this system enters the left basin L or right basin R by

$$p^{\varepsilon}_{l,x} = \mathrm{P}^{\varepsilon}_{\xi|b_1,b_2}\{\lim_{t\to\infty} \varphi(t, x, \xi, \varepsilon) \in L\}$$

or

$$p^{\varepsilon}_{r,x} = \mathrm{P}^{\varepsilon}_{\xi|b_1,b_2}\{\lim_{t\to\infty} \varphi(t, x, \xi, \varepsilon) \in R\},$$

then

$$\frac{p^{\varepsilon}_{r,x}}{p^{\varepsilon}_{l,x}} = \frac{H(y) - H(w)}{H(w) - H(z)} + o(1).$$

Theorem 2.3.3. (Brin and Freidlin[5]) *Let*

$$\beta_l(\xi) = \int_L \mathrm{div}(b_1 + \xi b_2), \quad \beta_r(\xi) = \int_R \mathrm{div}(b_1 + \xi b_2).$$

Then

$$\lim_{\varepsilon\to 0} \frac{p^{\varepsilon,\xi}_{r,x}}{p^{\varepsilon,\xi}_{l,x}} = \frac{\beta_r(\xi)}{\beta_l(\xi)}.$$

Corollary 2.3.4. *Condition on b_1 and b_2, as $\varepsilon \to 0$, the probability that the system (1.3) enters the left (resp. right) basin L (resp. R) as time $t \to \infty$ is given by the following formula*

$$p^{\xi}_l = \lim_{\varepsilon\to 0} p^{\varepsilon,\xi}_{l,x} = \frac{\beta_l(\xi)}{\beta_l(\xi) + \beta_r(\xi)}$$

$$(\text{resp.} \quad p^{\xi}_r = \lim_{\varepsilon\to 0} p^{\varepsilon,\xi}_{r,x} = \frac{\beta_r(\xi)}{\beta_l(\xi) + \beta_r(\xi)}.)$$

Note that the limits are independent of x, the initial point of the system.

The proof of the theorem we present here is essentially the one given in Brin and Freidlin[5], with some more details and also minor corrections here. We will more or less follow their notations.

Let $F^\varepsilon(y,w)$ denote the flux of $\overline{\nabla}H+\varepsilon b$ through the segment of $G^t(x)$ between y and w, and let t be the time that $G^t(y)=w$, then

$$F^\varepsilon(y,w)=\int_0^t \big(\overline{\nabla}H(G^s(y))+\varepsilon b(G^s(y))\big)\cdot\overline{\nabla}H(G^s(y))ds. \tag{2.8}$$

Lemma 2.3.5. (Brin and Freidlin[5]) *Let $H(O^\varepsilon)=a$ and let $b>a$ be such that $(a,b]$ does not contain any critical values of H. Then there exists a $C>0$ with the following property. Suppose that w, $z\in G^t(y)$, with $y\in\gamma_l^\varepsilon$, $w\in\gamma_r^\varepsilon$, $z\in\gamma_l^\varepsilon$ and w is the only intersection of $G^t(y)$ with the separatrices between y and z. Also assume that $b>H(z)>H(w)>H(y)=H(O^\varepsilon)+\delta$ with $\delta>0$ independent of ε. Then*

$$\lim_{\varepsilon\to0}\left|\frac{H(z)-H(w)}{H(w)-H(y)}-\frac{\beta_l(\xi)}{\beta_r(\xi)}\right|<C\sqrt{\delta}.$$

Proof. Since O^ε is a saddle point, the area of $H^{-1}(a-\delta,a+\delta)$ does not exceeds $C_1\sqrt{\delta}$ for some constant $C_1>0$. This is because near the saddle point, the level curve $H^{-1}(a+\delta)$ is away from the saddle point O^ε, and hence the ∞-shaped curve by a distance of order $\sqrt{\delta}$, while away from the saddle point, the distance between this level curve and the separatrices of the unperturbed system is of order ε. Similar for the level curve $H^{-1}(a-\delta)$. Thus we have an upper bound for the integral of the divergence of b over the set $H^{-1}(a-\delta,a+\delta)$

$$\left|\int_{H^{-1}(a-\delta,a+\delta)}\mathrm{div}b\right|\le C_2(H,b)\sqrt{\delta}. \tag{2.9}$$

For $\zeta\notin H^{-1}(a-\delta,a+\delta)$, $|\nabla H(\zeta)|>C_2(H)\sqrt{\delta}$. Therefore,

$$\begin{aligned}
|H(w)-H(y)-F^\varepsilon(y,w)| &= \left|\int_0^t|\overline{\nabla}H(G^s(y))|^2ds-F^\varepsilon(y,w)\right|\\
&= \varepsilon\left|\int_0^t b(G^s(y))\cdot\overline{\nabla}H(G^s(y))ds\right|\\
&= C(y,w)\varepsilon.
\end{aligned}$$

Similarly,

$$|H(z) - H(w) - F^{\varepsilon}(w,z)| = C(w,z)\varepsilon.$$

By the divergence theorem,

$$F^{\varepsilon}(y,w) = -\varepsilon \int_{R^{\varepsilon}(y,w)} \mathrm{div} b, \qquad F^{\varepsilon}(w,z) = -\varepsilon \int_{L^{\varepsilon}(w,z)} \mathrm{div} b,$$

where $L^{\varepsilon}(y,w)$ is the part of L^{ε} with the tail flow ribbon cut off along the segment of $G^t(y)$ from y to w, while $R^{\varepsilon}(w,z)$ is the part with the tail flow ribbon cut off along the segment from w to z. Now by (2.9)

$$\begin{aligned} |-F^{\varepsilon}(y,w) - \varepsilon\beta_r(\xi)| &\leq C_3\varepsilon\sqrt{\delta}|\beta_r(\xi)| \\ |-F^{\varepsilon}(w,z) - \varepsilon\beta_l(\xi)| &\leq C_4\varepsilon\sqrt{\delta}|\beta_l(\xi)|, \end{aligned}$$

for some constants C_3 and C_4.

Let $\rho_{\varepsilon,\delta} = \left|\dfrac{H(z)-H(w)}{H(w)-H(y)} - \dfrac{\beta_l(\xi)}{\beta_r(\xi)}\right|$, then

$$\begin{aligned} \rho_{\varepsilon,\delta} &= \left|\frac{H(z)-H(w)}{H(w)-H(y)} - \frac{F^{\varepsilon}(w,z)}{F^{\varepsilon}(y,w)} + \frac{F^{\varepsilon}(w,z)}{F^{\varepsilon}(y,w)} - \frac{\beta_l(\xi)}{\beta_r(\xi)}\right| \\ &= \left|\frac{\int_{L^{\varepsilon}(w,z)} \mathrm{div} b}{\int_{R^{\varepsilon}(w,y)} \mathrm{div} b} - \frac{\int_L \mathrm{div} b}{\int_R \mathrm{div} b} + O(\varepsilon)\right| \\ &\leq C\sqrt{\delta} + O(\varepsilon). \end{aligned}$$

□

Now that the ratio of the H-width of the flow ribbons near the δ-neighborhood is close to the ratio of the integrals of divergence of b, we need to check that this ratio is almost the same far away from the δ-neighborhood. To this end, we construct a coordinate system outside the δ-neighborhood and apply the averaging principle to the variational equation for (1.3).

For $-\infty < h_1 < h_2 < \infty$, let $K(h_1,h_2)$ be a component of $H^{-1}([h_1,h_2])$ containing no critical points of H. Recall that G^t is the time-t map of the gradient flow of H and $\Psi^t(\cdot,\varepsilon)$ the time-t map of (1.3). We say that a solution to (1.1) or (1.3) is *regular* if it is not a fixed point and does not tend to any fixed point in either direction. For a point $x \in K(h_1,h_2)$, the regular solution of (1.1) starting at x is periodic with period $T(x)$ given by

$$T(x) = \oint_{C(x)} \frac{dl}{|\overline{\nabla} H(u)|}.$$

Let $y \in K(h_1,h_2)$ be a point with $H(y) = h_1$ and let $\varphi(z) = 2\pi t(z)/T(y)$ be the time parameter rescaled to length 2π on the solution $S(y)$ of (1.1) starting at y, where

$$t(z) = \int_y^z \frac{dl}{|\overline{\nabla} H(x)|}.$$

Let $\Pi : K(h_1,h_2) \to S(y)$ be the projection along the gradient flow (2.1). Then $\varphi(\Pi(\Psi^t(x,\varepsilon)))$ is a smooth function and $(H(x),\varphi(\Pi(x)))$ is a smooth coordinate system in $K(h_1,h_2)$. Let

$$h(x,\varepsilon) = \{\|dG^\tau(x)(\overline{\nabla} H(x) + \varepsilon b(x))\| l(S(y))\}^{-1},$$

where $l(S(y))$ is the length of $S(y)$, $dG^t(x)$ is the derivative of $G^t(x)$ with respect to x and τ is such that $G^\tau(x) = \Pi(x)$. Then in coordinates (H,φ),

$$\dot{X}^\varepsilon(t) = (\overline{\nabla} H(X^\varepsilon(t)) + \varepsilon b(X^\varepsilon(t)))h(X^\varepsilon(t),\varepsilon) \tag{2.10}$$

has the form

$$\begin{cases} \dot{H} = \varepsilon u(\varepsilon, H, \varphi) \\ \dot{\varphi} = 1, \end{cases} \tag{2.11}$$

where u is a smooth uniformly bounded function whose derivatives are uniformly bounded for $x \in K(h_1,h_2)$ and $\varepsilon \in (0,\varepsilon_0)$. In fact,

$$\begin{aligned} \frac{d}{dt} H(X^\varepsilon(t)) &= \nabla H(X^\varepsilon(t))(\overline{\nabla} H(X^\varepsilon(t)) + \varepsilon b(X^\varepsilon(t)))h(X^\varepsilon(t),\varepsilon) \\ &= \varepsilon \nabla H(X^\varepsilon(t)) b(X^\varepsilon(t)))h(X^\varepsilon(t),\varepsilon) \\ &= \varepsilon u(\varepsilon, H, \varphi). \end{aligned}$$

As for φ, note that

$$\frac{d}{dt}\Pi(X^\varepsilon(t)) = \frac{dG^\tau(x)(\overline{\nabla}H(x)+\varepsilon b(x))}{\|dG^\tau(x)(\overline{\nabla}H(x)+\varepsilon b(x))\|}l(S(y))^{-1}$$

Note that changing the velocity by a factor h in (2.10) does not change the trajectories of (1.3) and also L^ε and R^ε.

Now we consider the variational equation of (2.11), which is the system for the derivative of the solution of (2.11) with respect to the initial data in the direction of H.

$$\begin{cases} \dfrac{d}{dt}\Delta H = \varepsilon\dfrac{\partial u(\varepsilon,H,\varphi)}{\partial H}\Delta H \\ \dot{H} = \varepsilon u(\varepsilon,H,\varphi) \\ \dot{\varphi} = 1, \end{cases} \tag{2.12}$$

To work with a finite time interval, we make the scaling $t \mapsto t/\varepsilon$ again and obtain the system

$$\begin{cases} \dfrac{d}{dt}\Delta H = \dfrac{\partial u(\varepsilon,H,\varphi)}{\partial H}\Delta H \\ \dot{H} = u(\varepsilon,H,\varphi) \\ \dot{\varphi} = \varepsilon^{-1}. \end{cases} \tag{2.13}$$

Let $\mathcal{B}$ be the preimage of I_0 under the projection Y, I_0 the entrance edge of the graph Γ, that is, $H(Y^{-1}(z))$ decreases along I_0 as $z \in \Gamma$ approaches $\mathcal{O}$.

Lemma 2.3.6. (Brin and Freidlin[5]) *As $\varepsilon \to 0$, the solutions of (2.13) converge uniformly in $\mathcal{B}\setminus(L\cup R)$, with first derivatives with respect to H, on bounded time intervals, to the solutions of the following averaged system*

$$\begin{cases} \dfrac{d}{dt}\Delta H = \dfrac{1}{2\pi}\left(\displaystyle\int_0^{2\pi}\dfrac{\partial u(0,H,\varphi)}{\partial H}d\varphi\right)\Delta H \\ \dot{H} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(0,H,\varphi)d\varphi. \end{cases} \tag{2.14}$$

Proof. Apply the classical averaging principle to the system (2.13). □

Now by Lemma 2.4.5. when $\delta > 0$ is small, the ratio of the H-widths of the L^ε flow ribbon and the R^ε flow ribbon is close to $\beta_l(\xi)/\beta_r(\xi)$ in the δ-neighborhood of $\mathcal{O}^\varepsilon$. By Lemma 2.4.6. as $\varepsilon \to 0$, this ratio tends to a constant. The theorem thus follows from the two lemmas.

Chapter 3

Averaging Principle and Weak Convergence

In this chapter we prove our main theorem. The proof consists of two steps: first we establish the weak convergence of the processes Y_t^ε to the limit Y_t inside the edges of the graph Γ, which is a version of the averaging principle when the slow motion is stochastic; then we determine the probability distribution of the limit process going into one of the edges of the graph at the interior vertex, which was done in Chapter 2. Also we mention the application of our result to the regularization of deterministic perturbations.

3.1 Weak convergence in $C([0,T],\Gamma)$

We need some preparation in probability, especially the weak convergence of stochastic processes on a graph.

3.1.1 Weak convergence via the averaging principle

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let (S, ρ) be a metric space with Borel σ-field $\mathcal{S}$, and μ, μ_1, $\mu_2, \ldots$ probability measures on $(S, \mathcal{S})$. We say that μ_n *converges weakly* to μ, denoted by $\mu_n \xrightarrow{w} \mu$, if $\mu_n f \to \mu f$ for every $f \in C_b(S)$, the class of bounded continuous functions $f : S \to \mathbb{R}$. If ξ, ξ_1, $\xi_2, \ldots$ are random elements in S, we say that ξ_n *converges in distribution to* ξ, denoted by $\xi_n \xrightarrow{d} \xi$, if $P \circ \xi_n^{-1} \xrightarrow{w} P \circ \xi^{-1}$, that is, $Ef(\xi_n) \to Ef(\xi)$ for all $f \in C_b(S)$. Here we use $P \circ \xi^{-1}$ to denote the probability measure defined on $(S, \mathcal{S})$ by $P(\xi^{-1}(A))$ for an random element ξ in S. If (ξ_α) is a continuous family of random elements in (S, ρ), then we say that ξ_α converges to ξ in distribution if for any sequence (α_n), the sequence ξ_{α_n} converges to ξ in distribution. Therefore we need only to deal with the convergence of sequences of random elements.

The graph Γ as a metric space is equipped with the distance defined as $d(z_1, z_2) = |H(Y^{-1}(z_1)) - H(Y^{-1}(z_2))|$ for two points in the same edge of Γ.

We start by looking at the perturbed system (1.3). After the scaling $t \mapsto t/\varepsilon$, the perturbed system is of the form

$$\dot{\widetilde{X}}_t^\varepsilon = \frac{1}{\varepsilon}\overline{\nabla} H(\widetilde{X}_t^\varepsilon) + b(\widetilde{X}_t^\varepsilon), \tag{3.1}$$

Consider the projection $Y : \mathbb{R}^2 \to \Gamma$ defined in §1.1.3. Recall that $Y_t^\varepsilon = Y(\widetilde{X}_t^\varepsilon) \in C([0,T], \Gamma)$ for some $T > 0$. $(Y_t^\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ is a family of processes on Γ. Inside an edge I_i of the graph Γ, $Y_t^\varepsilon = (H(\widetilde{X}_t^\varepsilon), i)$, where i is the number of the edge. Conditioning on b_1, b_2, for every ξ, as $\varepsilon \to 0$, by the classical averaging principle, the processes (Y_t^ε) converge uniformly to a limiting process $Y_t = (H_i(t), i)$ inside the edge I_i of Γ defined by the following equation:

$$\dot{H}_i(t) = B_i(H_i(t)) T_i(H_i(t))^{-1} \tag{3.2}$$

where

$$T_i(z) = \oint_{C_i(z)} \frac{dl}{|\nabla H(x)|}, \quad \text{and} \quad B_i(z) = \int_{G_i(z)} \mathrm{div} b(x) dx. \tag{3.3}$$

where dl is the length element, $C_i(z)$ is the family of components corresponding to the edge I_k, and $G_i(z)$ is the domain in $\mathbb{R}^2$ bounded by $C_i(z)$.

Lemma 3.1.1. *Starting from I_0, the process Y_t^ε reaches the vertex $\mathcal{O}$ in finite time, and leaves immediately, entering one of the edges I_i's, $i = 1, 2$.*

Proof. This follows from our estimation for the separatrices in Chapter 2. In fact, although the speed dY_t^ε/dt approaches zero as Y_t^ε approaches the vertex $\mathcal{O}$, in a neighborhood of the vertex, the order of the zero is $(\big|\ln|z - H(\mathcal{O})|\big|)^{-1}$ as $|z - H(\mathcal{O})| \to 0$. More precisely, from Chapter 2, we know that at a point x in a neighborhood of the ∞-shaped curve with $Y(x) = (z, 0) \in I_0$, given b_i, $i = 1, 2$, for fixed ξ, in one rotation a separatrix will have an increment in H-value approximately $\alpha\varepsilon$ in the time $\big|\ln|H(x) - H(\mathcal{O})|\big| = \big|\ln|z - H(\mathcal{O})|\big|$, with $z - H(\mathcal{O}) = Cn\alpha\varepsilon$ for some $C > 0$. Thus as $\varepsilon \to 0$, the mean velocity at x is approximately

$$v^\varepsilon(x) = \frac{\alpha\varepsilon}{\big|\ln|Cn\alpha\varepsilon|\big|}.$$

After time scaling, this becomes

$$\tilde{v}^\varepsilon(x) = \frac{\alpha}{\big|\ln|Cn\alpha\varepsilon|\big|},$$

which tends to zero but is of order $|\ln|z - H(\mathcal{O})||^{-1}$. From this we can calculate the time $\tilde{\tau}_h^{\varepsilon,\xi}$ it takes for $Y_t^{\varepsilon,\xi}$ to reach the vertex $\mathcal{O}$ from $Y(x) = (z, 0)$, with $h = z - H(\mathcal{O})$. Integrating with respect to z from 0 to h, we have approximately

$$\tilde{\tau}_h^{\varepsilon,\xi} = ch(1 - \ln h)$$

for some $c > 0$, which tends to zero as $h \to 0$. Therefore we have a finite time for Y_t^ε to reach $\mathcal{O}$. Similarly we can show that it takes a finite time for the process to exit from an h-neighborhood of $\mathcal{O}$ along either one of the exit edges I_i, $i = 1, 2$.

Letting $h \to 0$, we see that this time tends to zero, too. Thus the process Y_t^ε leaves the vertex $\mathcal{O}$ without delay. $\square$

Corollary 3.1.2. *Starting from some point in I_0, the limiting process Y_t reaches the vertex $\mathcal{O}$ in finite time, and enters the edges I_i, $i = 1, 2$, without delay.*

Note that the point $\xi = 0$ causes no problem because ξ has a continuous conditional density and we are considering weak convergence.

The convergence holds if we apply any bounded linear functional and so taking the expectation with respect to ξ and b_i's would result in weak convergence. However, this convergence is valid only inside an edge I_i of Γ. To complete the picture, we need to consider the branching of the limiting process at the interior vertex $\mathcal{O}$ which corresponds to the saddle point of the unperturbed system.

3.1.2 The branching at the interior vertex

The graph Γ has three edges I_0, I_1, I_2, and three vertices, an interior vertex $\mathcal{O}$, and two exterior ones $\mathcal{O}_i$, $i = 1, 2$. Since $\mathrm{div} b < 0$, Y_t is decreasing and reaches the vertex O from I_0 in finite time, it then leaves O immediately and enters one of the I_i's, $i = 1$, 2. From results in Chapter 2, we have the following

Lemma 3.1.3. *The limiting system Y_t enters the left (resp. right) edge I_1 (resp. I_2) with the probability*

$$p_l^\xi = \frac{\int_L \mathrm{div}(b_1(x) + \xi b_2(x))dx}{\int_L \mathrm{div}(b_1(x) + \xi b_2(x))dx + \int_R \mathrm{div}(b_1(x) + \xi b_2(x))dx}$$

$$(resp. \quad p_r^\xi = \frac{\int_R \mathrm{div}(b_1(x) + \xi b_2(x))dx}{\int_L \mathrm{div}(b_1(x) + \xi b_2(x))dx + \int_R \mathrm{div}(b_1(x) + \xi b_2(x))dx})$$

This follows from the corollary for Theorem 2.4.3.

Now that we have the probabilities then it is the time to prove the weak convergence. By Lemma 3.1.1., if the system Y_t^ε starts from $(H(x), 0) \in I_0$ at time $t = 0$, then there is a $\tilde{\tau}_0^{\varepsilon,\xi} > 0$ such that $Y^\varepsilon_{\tilde{\tau}_0^{\varepsilon,\xi}} = \mathcal{O}$, the interior vertex corresponding to the saddle point. Taking the limit results in a finite time $\tilde{\tau}_0^\xi > 0$ such that $Y_{\tilde{\tau}_0^\xi} = \mathcal{O}$. Let Φ be a bounded linear functional on $C([0,T],\Gamma)$, then

$$\begin{aligned}
\lim_{\varepsilon\to 0} E\{\Phi(Y_t^\varepsilon)|b_1,b_2\} &= \lim_{\varepsilon\to 0} E_{0\le t\le \tilde{\tau}_0^{\varepsilon,\xi}}\{\Phi(Y_t^\varepsilon)|b_1,b_2\} + \lim_{\varepsilon\to 0} E_{t>\tilde{\tau}_0^{\varepsilon,\xi}}\{\Phi(Y_t^\varepsilon)|b_1,b_2\} \\
&= \lim_{\varepsilon\to 0} E_{0\le t\le \tilde{\tau}_0^{\varepsilon,\xi}}\{\Phi(Y_t^\varepsilon)|b_1,b_2\} \\
&+ \lim_{\varepsilon\to 0} E_{t>\tilde{\tau}_0^{\varepsilon,\xi}}\{I_{[Y_t^\varepsilon\in I_1]}\Phi(Y_t^\varepsilon)|b_1,b_2\} \\
&+ \lim_{\varepsilon\to 0} E_{t>\tilde{\tau}_0^{\varepsilon,\xi}}\{I_{[Y_t^\varepsilon\in I_2]}\Phi(Y_t^\varepsilon)|b_1,b_2\} \\
&= E_{0\le t\le \tilde{\tau}_0^{\xi}}\{\Phi(Y_t)\} \\
&+ \lim_{\varepsilon\to 0} p_{l,x}^{\varepsilon,\xi} E_{t>\tilde{\tau}_0^{\varepsilon,\xi}}\{\Phi(Y_t^\varepsilon)|b_1,b_2\} + \lim_{\varepsilon\to 0} p_{r,x}^{\varepsilon,\xi} E_{t>\tilde{\tau}_0^{\varepsilon,\xi}}\{\Phi(Y_t^\varepsilon)|b_1,b_2\} \\
&= E_{0\le t\le \tilde{\tau}_0^{\xi}}\{\Phi(Y_t)\} \\
&+ p_l^\xi E_{t>\tilde{\tau}_0^{\xi}}\{\Phi(Y_t)|b_1,b_2\} + p_r^\xi E_{t>\tilde{\tau}_0^{\xi}}\{\Phi(Y_t)|b_1,b_2\}.
\end{aligned}$$

Note that

$$E\{\Phi(Y_t^\varepsilon)\} = E\{E\{\Phi(Y_t^\varepsilon)|b_1,b_2\}\}$$

and that the limit and expectation are exchangeable, we have

$$\lim_{\varepsilon\to 0} E\{\Phi(Y_t^\varepsilon)\} = E\{\Phi(Y_t)\}.$$

Thus the processes (Y_t^ε) converges weakly to the process Y_t defined as the averaged system in (3.2) in the edges but with the probabilities specified above entering the edges I_1 or I_2.

Thus we have proved our main theorem.

THEOREM. (Weak Convergence) *Assume that H satisfies the basic conditions in (1.1.2), and H has one saddle point at the origin and two local minima, H is quadratic in a neighborhood of the origin, and that $b(x)$ satisfies Conditions 1 and 2. Then the processes Y_t^ε converge weakly in $C([0,T],\Gamma)$, the space of continuous functions taking values in Γ, to the process Y_t as $\varepsilon \downarrow 0$, which is defined inside each edge I_i, $i = 0,1,2$, by (1.7), and branching at the interior vertex. Moreover, starting from a point in I_0, Y_t reaches the interior vertex in finite time and leaves instantly, entering one of the edges I_1, I_2 with the probabilities*

$$p_l^\xi = \frac{\beta_1(\xi)}{\beta_1(\xi)+\beta_2(\xi)}, \quad p_r^\xi = \frac{\beta_2(\xi)}{\beta_1(\xi)+\beta_2(\xi)}.$$

3.2 Regularization of deterministic perturbations

Our approach can be used to regularize the deterministic perturbations for which the perturbed system and the corresponding processes on the graph Γ have no limit in the classical sense, as we mentioned in the Introduction. However, in this case we can add an additional perturbation term to regularize it.

Let the perturbed system has the form (after time scaling)

$$\dot{X}_t^\varepsilon = \frac{1}{\varepsilon}\overline{\nabla} H(X^\varepsilon) + b_1(X_t^\varepsilon), \tag{3.4}$$

where $b(x)$ is deterministic. Consider a second perturbation term $\kappa\xi b_2(x)$, with $b_2(x)$ also deterministic but $\xi > 0$ a random variable. Consider the new system

$$\dot{X}_t^{\varepsilon,\kappa} = \frac{1}{\varepsilon}\overline{\nabla} H(X_t^{\varepsilon,\kappa}) + b_1(X_t^{\varepsilon,\kappa}) + \kappa\xi b_2(X_t^{\varepsilon,\kappa}). \tag{3.5}$$

If we denote the perturbation term in this system by $b(\kappa, x)$, then by the result of Chapter 2, the probability that the limiting process on the graph Γ go from the edge I_0 to I_1 or I_2 is given by

$$\lim_{\varepsilon\to 0} p_l^{\varepsilon,\kappa} = \frac{\int_L \mathrm{div} b(\kappa, x)dx}{\int_L \mathrm{div} b(\kappa, x)dx + \int_R \mathrm{div} b(\kappa, x)dx},$$

$$\lim_{\varepsilon\to 0} p_r^{\varepsilon,\kappa} = \frac{\int_R \mathrm{div} b(\kappa, x)dx}{\int_L \mathrm{div} b(\kappa, x)dx + \int_R \mathrm{div} b(\kappa, x)dx}.$$

Letting $\kappa \to 0$, we get the probabilities

$$p_l = \frac{\int_L \mathrm{div} b_1(x)dx}{\int_L \mathrm{div} b_1(x)dx + \int_R \mathrm{div} b_1(x)dx}, \qquad p_r = \frac{\int_L \mathrm{div} b_1(x)dx}{\int_R \mathrm{div} b_1(x)dx + \int_R \mathrm{div} b_1(x)dx}.$$

Note that they are independent of κ, ξ, or b_2. Our result is the same as that obtained in Brin and Freidlin[5] for the oscillator, using a white noise-type perturbation for the regularization. The difference is that we have a restriction for the form of the Hamiltonian, which reduces the scope of application of our result. We are trying to remove this restriction.

Appendix A

Some results from analysis and differential equations

A.1 Dependence of solution on initial condition and parameters

The first is an extended version of the dependence of solution to an ODE on the initial data and parameters. Let

$$F : I \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n, \quad I \subset \mathbb{R}$$

be a vector field on $\mathbb{R}^n$. Consider the system

$$\dot{y} = F(t, \tau, x), \quad t \in I, \quad \tau \in \mathbb{R}^k, \quad x \in \mathbb{R}^n.$$

Let $y = y(t, \tau, x)$ be the solution of the system with $y(0, \tau, x) = y_0$.

This can be deduced from the usual dependence on initial data and parameters. See Taylor[16], §1.6.

Theorem A.1.1. *If F is C^r in an open set U in $I \times \mathbb{R}^k \times \mathbb{R}^n$, then $y(t, \tau, x)$ is C^r in (t, τ, x) jointly.*

A.2 The Implicit Function Theorem

The Implicit Function Theorem is an important tool and has varies version in different context. We list a version which is adequate for our use here.

Theorem A.2.1. (Implicit Function Theorem) *Let $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^l$ be open sets with $e_0 \in U$ and $z_0 \in V$, and*

$$F : D \times V \to \mathbb{R}^l$$

a C^r-map with $F(x_0, z_0) = u_0$. If $D_z F(x_0, z_0)$ is invertible, then the equation $F(x, z) = u_0$ defines a function $z = f(x)$ in a neighborhood of x_0 with f a C^r-map.

A.3 The Gronwall inequality

Since we need to compare two solutions of differential equations, we will use a special version of Gronwall inequality, which is given in Guckenheimer and Holmes[10]. Since they did not provide a complete proof, we give one here.

Lemma A.3.1. (Gronwall Inequality[10]) *If $u(t)$, $v(t)$, and $c(t) \geq 0$ on $[0, T]$, c is differentiable, and*

$$v(t) \leq c(t) + \int_0^t u(s)v(s)ds,$$

then

$$v(t) \leq c(0) \exp \int_0^t u(s)ds + \int_0^t c'(s) \left(\exp \int_s^t u(\tau)d\tau \right) ds, \quad t \in [0, T].$$

Proof. Let $R(t) = \int_0^t u(s)v(s)ds$ and it is easy to see that $R' - uR \leq uc$. Let $U(t) = \int_0^t u(s)ds$. Then $R' \exp\{-U(t)\} - Ru \exp\{-U(t)\} \leq c(t)u(t) \exp\{-U(t)\}$, or $(R(t) \exp\{-U(t)\})' \leq -c(t)(\exp\{-U(t)\})'$. Integrating on both sides,

$$\begin{aligned}
R(t)\exp\{-U(t)\} &\leq -c(s)\exp\{-U(s)\}|_0^t + \int_0^t c'(s)\exp\{-U(s)\}ds \\
&= -c(t)\exp\{-U(t)\} + c(0)\exp\{-U(0)\} + \int_0^t c'(s)\exp\{-U(s)\}ds \\
&= -c(t)\exp\{-U(t)\} + c(0) + \int_0^t c'(s)\exp\{-U(s)\}ds
\end{aligned}$$

Thus

$$\begin{aligned}
v(t) &\leq R(t) + c(t) \\
&\leq c(0)\exp\{U(t)\} + \int_0^t c'(s)\exp\{U(t) - U(s)\}ds \\
&= c(0)\exp\int_0^t u(s)ds + \int_0^t c'(s)\left(\exp\int_s^t u(\tau)d\tau\right)ds.
\end{aligned}$$

□

A.4 Variational equation

Variational equations arise when we consider derivatives of solutions of an ODE with respect to the initial condition or parameters. For an ODE of the form

$$\dot{x} = v(t, x).$$

Let φ_ξ be the solution with the initial condition $\varphi_\xi(0) = \xi$. Let

$$X(t) = \left.\frac{\partial \varphi_\xi}{\partial \xi}\right|_{\xi = x_0};$$

$X(t)$ is a linear map depending on t and satisfies the following variational equation

$$\dot{X}(t) = A(t)X(t), \quad \text{where} \quad A(t) = \left.\frac{\partial v}{\partial x}\right|_{x = \varphi_{x_0}(t)}.$$

Also, suppose that the original differential equation depends on parameters α. Let $\varphi_{\alpha,\xi}$ be the solution with initial condition $\varphi_{\alpha,\xi}(0) = \xi$. Let

$$Y(t) = \left.\frac{\partial \varphi_{\alpha,\xi}(t)}{\partial \alpha}\right|_{\xi = x_0, \alpha = \alpha_0}.$$

Then $Y(t)$ satisfies the variation equation

$$\dot{Y}(t) = A(t)Y(t) + b(t),$$

where

$$A(t) = \left.\frac{\partial v}{\partial x}\right|_{x = \varphi_{\alpha_0}(t), \alpha = \alpha_0}, \quad b(t) = \left.\frac{\partial v}{\partial \alpha}\right|_{x = \varphi_{\alpha_0}(t), \alpha = \alpha_0}.$$

Appendix B

The proof of Lemma 2.2.2.

The proof has a few steps. We start with the asymptotic solutions of the systems in the form of the ε-expansion:

$$X_t^\varepsilon = X_t^{(0)} + \varepsilon X_t^{(1)} + \varepsilon^2 X_t^{(2)} + h.o.t. \tag{B.1}$$

$$\widetilde{X}_t^\varepsilon = \widetilde{X}_t^{(0)} + \varepsilon \widetilde{X}_t^{(1)} + \varepsilon^2 \widetilde{X}_t^{(2)} + h.o.t. \tag{B.2}$$

Then we have the equations

$$\begin{aligned}
\dot{X}_t^{(0)} &= AX_t^{(0)}, \quad X_0^{(0)} = X_0, \\
\dot{X}_t^{(1)} &= AX_t^{(1)} + b(X_t^{(0)}), \quad X_0^{(1)} = (0,0), \\
\dot{X}_t^{(2)} &= AX_t^{(2)} + Db(X_t^{(0)}) \cdot X_t^{(1)}, \quad X_0^{(2)} = (0,0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\dot{\widetilde{X}}_t^{(0)} &= A\widetilde{X}_t^{(0)}, \quad \widetilde{X}_0^{(0)} = \widetilde{X}_0, \\
\dot{\widetilde{X}}_t^{(1)} &= A\widetilde{X}_t^{(1)} + b(\widetilde{X}^{(0)}), \quad \widetilde{X}_0^{(1)} = (0,0), \\
\dot{\widetilde{X}}_t^{(2)} &= A\widetilde{X}_t^{(2)} + Db(\widetilde{X}_t^{(0)}) \cdot \widetilde{X}_t^{(1)} + \beta(\widetilde{X}_t^{(0)}), \quad \widetilde{X}_0^{(2)} = (0,0).
\end{aligned}$$

The corresponding solutions can be written as

$$
\begin{aligned}
X_t^{(0)} &= (e^{-t}p_0, e^t q_0),\\
X_t^{(1)} &= (e^{-t}\int_0^t e^s b_1(e^{-s}p_0, e^s q_0)ds, e^t \int_0^t e^{-s} b_2(e^{-s}p_0, e^s q_0)ds)\\
X_t^{(2)} &= (e^{-t}\int_0^t e^s Db_1(e^{-s}p_0, e^s q_0)\cdot X_s^{(1)}ds, e^t \int_0^t e^{-s} Db_2(e^{-s}p_0, e^s q_0)\cdot X_s^{(1)}ds),
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{X}_t^{(0)} &= (e^{-t}\tilde{p}_0, e^t \tilde{q}_0),\\
\widetilde{X}_t^{(1)} &= (-e^t\int_0^t e^s b_1(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)ds, -e^{-t}\int_0^t e^s b_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)ds)\\
\widetilde{X}_t^{(2)} &= (-e^t\int_0^t e^{-s}[Db_1(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)\cdot \widetilde{X}_s^{(1)} + \beta_1(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)]ds,\\
&\quad -e^{-t}\int_0^t e^s[Db_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)\cdot \widetilde{X}_s^{(1)} + \beta_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)]ds).
\end{aligned}
$$

Since the higher order terms can be ignored, we have approximately

$$
\begin{aligned}
p_t &= p_0 e^t - \varepsilon e^t \int_0^t e^{-s} b_1(e^s p_0, e^{-s} q_0)ds\\
&- \varepsilon^2 e^t \int_0^t e^{-s} Db_1(e^s p_0, e^{-s} q_0)\cdot X_s^{(1)} ds\\
q_t &= q_0 e^{-t} - \varepsilon e^{-t} \int_0^t e^s b_2(e^s p_0, e^{-s} q_0)ds\\
&- \varepsilon^2 e^{-t} \int_0^t e^s Db_2(e^s p_0, e^{-s} q_0)\cdot X_s^{(1)} ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{p}_t &= \tilde{p}_0 e^t + \varepsilon e^t \int_0^t e^{-s} b_1(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)ds\\
&- \varepsilon^2 e^t \int_0^t e^{-s}[Db_1(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)\cdot \widetilde{X}_s^{(1)} + \beta_1(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)]ds\\
\tilde{q}_t &= \tilde{q}_0 e^{-t} + \varepsilon e^{-t} \int_0^t e^s b_2(e^s\tilde{p}_0, e^{\varepsilon}\tilde{q}_0)ds\\
&- \varepsilon^2 e^{-t} \int_0^t e^s[Db_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)\cdot \widetilde{X}_s^{(1)} + \beta_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)]ds,
\end{aligned}
$$

Since $\tau_1 = \min\{t > 0 : p_t = 1\}$, $q_1 = q_{\tau_1}$, $p_1 = p_{\tau_1}$,we have

$$\begin{aligned}
1 &= p_0 e^{\tau_1} - \varepsilon e^{\tau_1} \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0) ds \\
&- \varepsilon^2 e^{\tau_1} \int_0^{\tau_1} e^{-s} Db_2(e^s p_0, e^{-s} q_0) \cdot X_s^{(1)} ds \\
&= [p_0 - (\varepsilon b_1^0 + \varepsilon^2 b_1^1)(1 - e^{-\tau_1})] e^{\tau_1} \\
&= e^{\tau_1}(p_0 - \varepsilon b_1^0 - \varepsilon^2 b_1^1) + \varepsilon b_1^0 + \varepsilon^2 b_1^1,
\end{aligned}$$

where

$$\int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0) ds = b_1^0 \int_0^{\tau_1} e^{-s} ds$$

for some number b_1^0 by the mean value theorem for integrals, and

$$\int_0^{\tau_1} e^{-s} Db_1(e^s p_0, e^{-s} q_0) \cdot X_s^{(1)} ds = b_1^1 \int_0^{\tau_1} e^{-s} ds.$$

Thus

$$e^{\tau_1} = \frac{1 - \varepsilon b_1^0 - \varepsilon^2 b_1^1}{q_0 - \varepsilon b_1^0 - \varepsilon^2 b_1^1}$$

and

$$\begin{aligned}
q_1 &= q_0 e^{-\tau_1} - \varepsilon e^{-\tau_1} \int_0^{\tau_1} e^{s} b_2(e^s p_0, e^{-s} q_0) ds \\
&- \varepsilon^2 e^{\tau_1} \int_0^{\tau_1} e^{s} Db_1(e^s p_0, e^{-s} q_0) \cdot X_s^{(1)} ds
\end{aligned}$$

Similarly, let $\tilde{\tau}_1 = \min\{t > 0 : \tilde{p}_t = 1\}$, $\tilde{q}_1 = \tilde{q}_{\tilde{\tau}_1}$, $\tilde{p}_1 = \tilde{p}_{\tilde{\tau}_1}$. Then

$$\begin{aligned}
1 &= \tilde{p}_0 e^{\tilde{\tau}_1} - \varepsilon e^{\tilde{\tau}_1} \int_0^{\tilde{\tau}_1} e^{-s} b_1(e^s \tilde{p}_0, e^{-s} \tilde{q}_0) ds \\
&- \varepsilon^2 e^{\tilde{\tau}_1} \int_0^{\tilde{\tau}_1} e^{-s} [Db_1(e^s \tilde{p}_0, e^{-s} \tilde{q}_0) \cdot \widetilde{X}_s^{(1)} + \beta_1(e^s \tilde{p}_0, e^{-s} \tilde{q}_0)] ds
\end{aligned}$$

where again $\tilde{b}_1^0$ and $\tilde{\beta}_1^0$ are the numbers coming from the mean value theorem.

Thus we have

$$e^{\tilde{\tau}_1} = \frac{1 - \varepsilon\tilde{b}_1^0 - \varepsilon^2(\tilde{b}_1^1 + \beta_1^0)}{\tilde{p}_0 - \varepsilon\tilde{b}_1^0 - \varepsilon^2(\tilde{b}_1^1 + \beta_1^0)} = \frac{\widetilde{C}_1}{\varepsilon},$$

and

$$\tilde{\tau}_1 = -\ln\varepsilon + \ln\widetilde{C}_1.$$

Accordingly we have

$$\begin{aligned}
\tilde{q}_1 &= \tilde{q}_0 e^{-\tilde{\tau}_1} - \varepsilon e^{-\tilde{\tau}_1}\int_0^{\tilde{\tau}_1} e^s b_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)ds \\
&- \varepsilon^2 e^{-\tilde{\tau}_1}\int_0^{\tilde{\tau}} e^s[Db_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)\cdot\widetilde{X}_s^{(1)} + \beta_2(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)]ds \\
&= e^{-\tilde{\tau}_1}(\tilde{q}_0 + \varepsilon\tilde{b}_2^0 + \varepsilon^2\tilde{b}_2^1) - \varepsilon(\tilde{b}_2^0 + \varepsilon^2\tilde{b}_2^1)
\end{aligned}$$

where $\tilde{b}_2^0$ and $\tilde{b}_2^1$ are numbers from the mean value theorem.

Now $\tilde{q}_1 - q_1 = \tilde{q}_{\tilde{\tau}_1} - q_{\tau_1} = \tilde{q}_{\tilde{\tau}_1} - \tilde{q}_{\tau_1} + \tilde{q}_{\tau_1} - q_{\tau_1}$. We will estimate the two parts one by one. But first we need the estimation for $e^{-\tilde{\tau}_1} - e^{-\tau_1}$.

$$\begin{aligned}
e^{-\tilde{\tau}_1} - e^{-\tau_1} &= \frac{\tilde{p}_0 - \varepsilon\tilde{b}_1^0 + \varepsilon^2(\tilde{b}_1^1 + \beta_1^0)}{1 - \varepsilon\tilde{b}_1^0 - \varepsilon^2(\tilde{b}_1^1 + \beta_1^0)} - \frac{p_0 - \varepsilon b_1^0 - \varepsilon^2 b_1^1}{1 - \varepsilon b_1^0 - \varepsilon^2 b_1^1} \\
&= \frac{\tilde{p}_0 - p_0 - \varepsilon(\tilde{b}_1^0 - b_1^0) - \varepsilon(\tilde{b}_1^0 p_0 - b_1^0\tilde{p}_0) + \varepsilon^2 R}{(1 - \varepsilon\tilde{b}_1^0 - \varepsilon^2(\tilde{b}_1^1 + \beta_1^0))(1 - \varepsilon b_1^0 - \varepsilon^2 b_1^1)} \\
&= \tilde{p}_0 - p_0 - \varepsilon(\tilde{b}_1^0 - b_1^0) - \varepsilon(\tilde{b}_1^0 p_0 - b_1^0\tilde{p}_0) + \varepsilon^2 R' \\
&= \tilde{p}_0 - p_0 - \varepsilon(\tilde{b}_1^0 - b_1^0) - \varepsilon(\tilde{b}_1^0 p_0 - b_1^0 p_0 + b_1^0 p_0 - b_1^0\tilde{p}_0) \\
&= C_0\varepsilon^2 - \varepsilon(1 + p_0)(\tilde{b}_1^0 - b_1^0)
\end{aligned}$$

where R and R' are some constants.

$$\begin{aligned}
\tilde{b}_1^0 - b_1^0 &= \frac{\int_0^{\tilde{\tau}_1} e^{-s} b_1(e^s\tilde{p}_0, e^{-s}\tilde{q}_0)ds}{1 - e^{\tilde{\tau}_1}} - \frac{\int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds}{1 - e^{\tau_1}} \\
&= \{(1 - e^{-\tilde{\tau}_1})(1 - e^{-\tau_1})\}^{-1}(\mathrm{I} + \mathrm{II})
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I} &= \int_0^{\tilde{\tau}_1} e^{-s} b_1(e^s \tilde{p}_0, e^{-s}\tilde{q}_0)ds - \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds \\
\mathrm{II} &= e^{-\tilde{\tau}_1} \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds - e^{-\tau_1} \int_0^{\tilde{\tau}_1} e^{-s} b_1(e^s \tilde{p}_0, e^{-s}\tilde{q}_0)ds
\end{aligned}$$

Now

$$\begin{aligned}
\mathrm{I} &= \int_0^{\tilde{\tau}_1} e^{-s}[b_1(e^s \tilde{p}_0, e^{-s}\tilde{q}_0) - b_1(e^s p_0, e^{-s} q_0)]ds + \int_{\tilde{\tau}_1}^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds \\
&= \int_0^{\tilde{\tau}_1} e^{-s}[b_1(e^s \tilde{p}_0, e^{-s}\tilde{q}_0) - b_1(e^s p_0, e^{-s}\tilde{q}_0) + b_1(e^s p_0, e^{-s}\tilde{q}_0) - b_1(e^s p_0, e^{-s} q_0)]ds \\
&\quad +(e^{-\tilde{\tau}_1} - e^{-\tau_1})C_{b_1} \\
&= \int_0^{\tilde{\tau}_1} e^{-s}[\frac{\partial b_1}{\partial p}(e^s p_0', e^{-s}\tilde{q}_0)(\tilde{p}_0 - p_0) + \frac{\partial b_1}{\partial q}(e^s p_0, e^{-s} q_0')(\tilde{q}_0 - q_0)]ds + (e^{-\tilde{\tau}_1} - e^{-\tau_1})C_{b_1} \\
&= (e^{-\tilde{\tau}_1} - e^{-\tau_1})C_{b_1} + (\tilde{p}_0 - p_0)C'_{b_1} + (\tilde{q}_0 - q_0)C''_{b_1} \\
&= C\varepsilon^2 + C'(e^{-\tilde{\tau}_1} - e^{-\tau_1})
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} &= e^{-\tilde{\tau}_1} \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds - e^{-\tau_1} \int_0^{\tilde{\tau}_1} e^{-s} b_1(e^s \tilde{p}_0, e^{-s}\tilde{q}_0)ds \\
&= e^{-\tilde{\tau}_1} \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds - e^{-\tau_1} \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds \\
&+ e^{-\tau_1} \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds - e^{-\tau_1} \int_0^{\tilde{\tau}_1} e^{-s} b_1(e^s \tilde{p}_0, e^{-s}\tilde{q}_0)ds \\
&= (e^{-\tilde{\tau}_1} - e^{-\tau_1}) \int_0^{\tau_1} e^{-s} b_1(e^s p_0, e^{-s} q_0)ds - e^{-\tau_1}(\mathrm{I}) \\
&= (e^{-\tilde{\tau}_1} - e^{-\tau_1})C'''_{b_1} - e^{-\tau_1}(\mathrm{I})
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{b}_1^0 - b_1^0 &= \frac{\mathrm{I} + (e^{-\tilde{\tau}_1} - e^{-\tau_1})C'''_{b_1} - e^{-\tau_1}(\mathrm{I})}{(1 - e^{-\tilde{\tau}_1})(1 - e^{-\tau_1})} \\
&= ((e^{-\tilde{\tau}_1} - e^{-\tau_1})C'''_{b_1} + (1 - e^{-\tau_1})(\mathrm{I})) \\
&\quad \cdot(1 + \tilde{p}_1 + p_1 - \varepsilon(\tilde{b}_1^0 + b_1^0 - \tilde{p}_1\tilde{b}_1^0 - p_1 b_1^0) + M\varepsilon^2) \\
&= C(e^{-\tilde{\tau}_1} - e^{-\tau_1}) + C'\varepsilon^2
\end{aligned}$$

Now

$$e^{-\tilde{\tau}_1} - e^{-\tau_1} = C_0\varepsilon^2 - C_1\varepsilon(e^{-\tilde{\tau}_1} - e^{-\tau_1})$$

and thus

$$e^{-\tilde{\tau}_1} - e^{-\tau_1} = C\varepsilon^2$$

and hence

$$\tilde{b}_1^0 - b_1^0 = C'\varepsilon^2.$$

Now

$$\begin{aligned}
\tilde{q}_{\tilde{\tau}_1} - q_{\tau_1} &= e^{-\tilde{\tau}_1}(\tilde{q}_0 + \varepsilon\tilde{b}_2^0 + \varepsilon^2\tilde{b}_2^1) - \varepsilon\tilde{b}_1^0 - e^{-\tau_1}(q_0 + \varepsilon b_2^0 + \varepsilon^2 b_2^1) + \varepsilon b_2^0 \\
&= e^{-\tilde{\tau}_1}(\tilde{q}_0 - q_0 + \varepsilon(\tilde{b}_2^0 - b_2^0)) + (e^{-\tilde{\tau}_1} - e^{-\tau_1})(q_0 + \varepsilon b_2^0) - \varepsilon(\tilde{b}_1^0 - b_1^0) + C\varepsilon^2 \\
&= C\varepsilon^2
\end{aligned}$$

for some constant C.

Appendix C

The proof of Lemma 2.2.4

From

$$
\begin{aligned}
1 &= e^{\tau_n}(p_n - \varepsilon \int_0^{\tau_n} e^{-s} b_1(e^s p_n, e^{-s}) ds - \varepsilon^2 \int_0^t e^{-s} Db_1(e^s p_n, e^{-s}) \cdot X_{-s}^{(1)} ds) \\
&= e^{\tau_n}(p_n - \varepsilon b_{1,n}^0 - \varepsilon^2 b_{1,n}^1) + \varepsilon b_{1,n}^0 + \varepsilon^2 b_{1,n}^1,
\end{aligned}
$$

where $b_{1,n}^0$ and $b_{1,n}^1$ are the numbers from the mean value theorem, we have

$$
e^{\tau_n} = \frac{1 - \varepsilon b_{1,n}^0 - \varepsilon^2 b_{1,n}^1}{p_n - \varepsilon b_{1,n}^0 - \varepsilon^2 b_{1,n}^1},
$$

and

$$
\begin{aligned}
e^{-\tau_n} &= \frac{p_n - \varepsilon b_{1,n}^0 - \varepsilon^2 b_{1,n}^1}{1 - \varepsilon b_{1,n}^0 - \varepsilon^2 b_{1,n}^1} \\
&= (p_n - \varepsilon b_{1,n}^0 - \varepsilon^2 b_{1,n}^1)(1 + \varepsilon b_{1,n}^0 + \varepsilon^2 b_{1,n}^1 + o(\varepsilon)) \\
&= p_n - \varepsilon b_{1,n}^0 + o(\varepsilon).
\end{aligned}
$$

Also

$$
\begin{aligned}
(1 - e^{-\tau_n})^{-1} &= (1 - p_n + \varepsilon b_{1,n}^0 + o(\varepsilon))^{-1} \\
&= 1 + p_n - \varepsilon b_{1,n}^0 + o(\varepsilon).
\end{aligned}
$$

Similarly for the system $(\tilde{p}_t, \tilde{q}_t)$,

$$
\begin{aligned}
1 &= e^{\tilde{\tau}_n}(\tilde{p}_n - \varepsilon \int_0^{\tilde{\tau}_n} e^{-s} b_1(e^s \tilde{p}_n, e^{-s}) ds - \varepsilon^2 \int_0^t e^{-s} Db_1(e^s \tilde{p}_n, e^{-s}) \cdot \widetilde{X}_{-s}^{(1)} ds) \\
&= e^{\tilde{\tau}_n}(\tilde{p}_n - \varepsilon \tilde{b}_{1,n}^0 - \varepsilon^2 \tilde{b}_{1,n}^1) + \varepsilon \tilde{b}_{1,n}^0 + \varepsilon^2 \tilde{b}_{1,n}^1,
\end{aligned}
$$

where $\tilde{b}^0_{1,n}$ and $\tilde{b}^1_{1,n}$ are the numbers from the mean value theorem. From this we see that

$$e^{\tilde{\tau}_n} = \frac{1 - \varepsilon \tilde{b}^0_{1,n} - \varepsilon^2 \tilde{b}^1_{1,n}}{\tilde{p}_n - \varepsilon \tilde{b}^0_{1,n} - \varepsilon^2 \tilde{b}^1_{1,n}},$$

and

$$\begin{aligned} e^{-\tilde{\tau}_n} &= \frac{\tilde{p}_n - \varepsilon \tilde{b}^0_{1,n} - \varepsilon^2 (\tilde{b}^1_{1,n} + \beta^0_{1,n})}{1 - \varepsilon \tilde{b}^0_{1,n} - \varepsilon^2 (\tilde{b}^1_{1,n} + \beta^0_{1,n})} \\ &= (\tilde{p}_n - \varepsilon \tilde{b}^0_{1,n} - \varepsilon^2 (\tilde{b}^1_{1,n} + \beta^0_{1,n}))(1 + \varepsilon \tilde{b}^0_{1,n} + \varepsilon^2 (\tilde{b}^1_{1,n} + \beta^0_{1,n}) + o(\varepsilon^2)) \\ &= \tilde{p}_n - \varepsilon \tilde{b}^0_{1,n} + o(\varepsilon). \end{aligned}$$

Also

$$\begin{aligned} (1 - e^{-\tilde{\tau}_n})^{-1} &= (1 - \tilde{p}_n + \varepsilon \tilde{b}^0_{1,n} + o(\varepsilon))^{-1} \\ &= 1 + \tilde{p}_n - \varepsilon \tilde{b}^0_{1,n} + o(\varepsilon). \end{aligned}$$

We have approximately

$$\begin{aligned} q_n &= e^{-\tau_n}(1 - \varepsilon \int_0^{\tau_n} e^s b_2(e^s p_n, e^{-s}) ds - \varepsilon^2 \int_0^{\tau_n} e^s D b_2(e^s p_n, e^{-s}) \cdot X^{(1)}_{-s} ds) \\ &= e^{-\tau_n}(1 - \varepsilon b^0_{2,n} - \varepsilon^2 b^1_{2,n}) + \varepsilon b^0_{2,n} + \varepsilon^2 b^1_{2,n} \end{aligned}$$

where

$$\begin{aligned} b^0_{2,n}(e^{\tau_n} - 1) &= \int_0^{\tau_n} e^s b_2(e^s p_n, e^{-s}) ds, \\ b^1_{2,n}(e^{\tau_n} - 1) &= \int_0^{\tau_n} e^s D b_2(e^s p_n, e^{-s}) \cdot X^{(1)}_{-s} ds. \end{aligned}$$

Similarly we have

$$\begin{aligned} \tilde{q}_n &= e^{-\tilde{\tau}_n}(1 - \varepsilon \int_0^{\tilde{\tau}_n} e^s b_2(e^s \tilde{p}_n, e^{-s}) ds - \varepsilon^2 \int_0^{\tilde{\tau}_n} e^s D b_2(e^s \tilde{p}_n, e^{-s}) \cdot \widetilde{X}^{(1)}_{-s} ds) \\ &= e^{-\tilde{\tau}_n}(1 - \varepsilon \tilde{b}^0_{2,n} - \varepsilon^2 \tilde{b}^1_{2,n}) + \varepsilon \tilde{b}^0_{2,n} + \varepsilon^2 \tilde{b}^1_{2,n} \end{aligned}$$

where

$$
\begin{aligned}
\tilde{b}^0_{2,n}(e^{\tilde{\tau}_n}-1) &= \int_0^{\tilde{\tau}_n} e^s b_2(e^s\tilde{p}_n, e^{-s})ds, \\
\tilde{b}^1_{2,n}(e^{\tilde{\tau}_n}-1) &= \int_0^{\tilde{\tau}_n} e^s Db_2(e^s\tilde{p}_n, e^{-s})\cdot \widetilde{X}^{(1)}_{-s}ds.
\end{aligned}
$$

To estimate $\tilde{q}_n - q_n$, we need to estimate $\tilde{b}^0_{1,n} - b^0_{1,n}$ first.

$$
\begin{aligned}
\tilde{b}^0_{1,n} - b^0_{1,n} &= \frac{\int_0^{\tilde{\tau}_n} e^{-s}b_1(e^s\tilde{p}_n, e^{-s})ds}{1-e^{-\tilde{\tau}_n}} - \frac{\int_0^{\tau_n} e^{-s}b_1(e^s p_n, e^{-s})ds}{1-e^{-\tau_n}} \\
&= \{(1-e^{-\tilde{\tau}_n})(1-e^{-\tau_n})\}^{-1}(\mathrm{I_n}+\mathrm{II_n})
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{I_n} &= \int_0^{\tilde{\tau}_n} e^{-s}b_1(e^s\tilde{p}_n, e^{-s})ds - \int_0^{\tau_n} e^{-s}b_1(e^s p_n, e^{-s})ds \\
&= \int_0^{\tilde{\tau}_n} e^{-s}[b_1(e^s\tilde{p}_n, e^{-s}) - b_1(e^s p_n, e^{-s})]ds + \int_{\tau_n}^{\tilde{\tau}_n} e^{-s}b_1(e^s p_n, e^{-s})ds \\
&= \int_0^{\tilde{\tau}_n} \frac{\partial b_1}{\partial p}(e^s p_n', e^{-s})(\tilde{p}_n - p_n)ds + C_n(e^{-\tilde{\tau}_n} - e^{-\tau_n}) \\
&= C_n(e^{-\tilde{\tau}_n} - e^{-\tau_n}) + C_n'(\tilde{p}_n - p_n)\tilde{\tau}_n \\
\mathrm{II_n} &= e^{-\tilde{\tau}_n}\int_0^{\tau_n} e^{-s}b_1(e^s p_n, e^{-s})ds - e^{-\tau_n}\int_0^{\tilde{\tau}_n} e^{-s}b_1(e^s\tilde{p}_n, e^{-s})ds \\
&= (e^{-\tilde{\tau}_n} - e^{-\tau_n})\int_0^{\tau_n} e^{-s}b_1(e^s p_n, e^{-s})ds - e^{-\tau_n}(\mathrm{I_n}) \\
&= (e^{-\tilde{\tau}_n} - e^{-\tau_n})C_n(b_1)(1-e^{-\tau_n}) - e^{-\tau_n}(\mathrm{I_n})
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}^0_{1,n} - b^0_{1,n} &= \frac{\mathrm{I_n}+\mathrm{II_n}}{(1-e^{-\tilde{\tau}_n})(1-e^{-\tau_n})} \\
&= \frac{C_n(b_1)(e^{-\tilde{\tau}_n} - e^{-\tau_n}) + \mathrm{I_n}}{1-e^{-\tilde{\tau}_n}} \\
&= (C_n'(b_1)(e^{-\tilde{\tau}_n} - e^{-\tau_n}) + C_n'(\tilde{p}_n - p_n)\tilde{\tau}_n)(1+\tilde{p}_n - \varepsilon\tilde{b}^0_{1,n}(1-\tilde{p}_n) + \varepsilon^2 M)
\end{aligned}
$$

from this we can estimate

$$
\begin{aligned}
e^{-\tilde{\tau}_n} - e^{-\tau_n} &= \tilde{p}_n - p_n - \varepsilon(\tilde{b}^0_{1,n} - b^0_{1,n}) + \varepsilon(\tilde{p}_n\tilde{b}_{1,n} - p_n b_{1,n}) + \varepsilon^2 M' \\
&= (1 + \varepsilon\tilde{b}_{1,n})(\tilde{p}_n - p_n) - \varepsilon(1 - p_n)(\tilde{b}^0_{1,n} - b^0_{1,n}) + \varepsilon^2 M' \\
&= (\tilde{p}_n - p_n)(1 + \varepsilon b^0_{1,n} - \varepsilon C'_n(1 - p_n)(1 + \tilde{p}_n - \varepsilon\tilde{b}^0_{1,n}(1 - \tilde{p}_n) + \varepsilon^2 M)\tilde{\tau}_n) \\
&- \varepsilon(1 - p_n)(1 + \tilde{p}_n - \varepsilon\tilde{b}^0_{1,n}(1 - \tilde{p}_n) + \varepsilon^2 M)C_n(b_1)'(e^{-\tilde{\tau}_n} - e^{-\tau_n}) + \varepsilon^2 M'
\end{aligned}
$$

and thus

$$
e^{-\tilde{\tau}_n} - e^{-\tau_n} = (\tilde{p}_n - p_n)(1 + \varepsilon(C_{n,0}\tilde{\tau}_n + C_{n,1} + C_{n,2}\varepsilon) + \varepsilon^2 M_n.
$$

In turn we have

$$
\tilde{b}^0_{1,n} - b^0_{1,n} = (\tilde{p}_n - p_n)(C^{(0)}_n(b) + C^{(1)}_n(b)\varepsilon\tau_n + C^{(2)}_n\varepsilon + C^{(3)}_n\varepsilon^2) + \varepsilon^2 M_n(b)
$$

Recall that $p_n = (a_0 + na)\varepsilon$ and $\tilde{p}_n = (\tilde{a}_0 + n\tilde{a})\varepsilon$, approximately we have

$$
\frac{1}{\tilde{p}_n} - \frac{1}{p_n} = \frac{p_n - \tilde{p}_n}{\tilde{p}_n p_n} = (\tilde{p}_n - p_n)\frac{C}{\varepsilon^2 n^2}
$$

Also

$$
\frac{1}{p_n^2} - \frac{1}{p_n^2 e^{2\tau_n}} = \frac{C}{\varepsilon^2 n^2} - 1 + \frac{C'}{n} + \frac{C''}{n^2} + o(1).
$$

Now

$$
\begin{aligned}
\frac{b^0_{1,n}}{p_n} &= \frac{1}{1 - e^{-\tau_n}}\frac{1}{p_n}\int_0^{\tau_n} e^{-s} b_1(e^s p_n, e^{-s})ds \\
&= (1 + p_n - \varepsilon b^0_{1,n} + o(\varepsilon))\frac{1}{p_n}\int_0^{\tau_n} e^{-s} b_1(e^s p_n, e^{-s})ds \\
&= \frac{1}{p_n}\int_0^{\tau_n} e^{-s} b_1(e^s p_n, e^{-s})ds + \int_0^{\tau_n} e^{-s} b_1(e^s p_n, e^{-s})ds \\
&\quad -\varepsilon\frac{b^0_{1,n}}{p_n}\int_0^{\tau_n} e^{-s} b_1(e^s p_n, e^{-s})ds \\
&= \int_{e^{-\tau_n}/p_n}^{1/p_n} b_1(1/u, up_n)du + b_1(e^{w_n} p_n, e^{-w_n})(1 - e^{-\tau_n})\left(1 - \varepsilon\frac{b^0_{1,n}}{p_n}\right).
\end{aligned}
$$

Thus

$$\frac{b_{1,n}^0}{p_n} = \int_{e^{-\tau_n}/p_n}^{1/p_n} b_1(1/u, up_n)du + C(b_1, p_n),$$

where $C(b_1, p_n)$ is a number depending on b_1 and p_n. Similarly we have

$$\frac{\tilde{b}_{1,n}^0}{\tilde{p}_n} = \int_{e^{-\tilde{\tau}_n}/\tilde{p}_n}^{1/\tilde{p}_n} b_1(1/u, u\tilde{p}_n)du + C(b_1, \tilde{p}_n).$$

$$\begin{aligned}
\frac{\tilde{b}_{1,n}^0}{\tilde{p}_n} - \frac{b_{1,n}^0}{p_n} &= \overbrace{\int_{e^{-\tilde{\tau}_n}/\tilde{p}_n}^{1/\tilde{p}_n} b_1(1/u, u\tilde{p}_n)du - \int_{e^{-\tau_n}/p_n}^{1/p_n} b_1(1/u, up_n)du}^{\mathrm{IV}} \\
&\quad + C(b_1, p_n, \tilde{p}_n),
\end{aligned}$$

where we have combined the two constants into a single one.

Now we try to look at the difference

$$\begin{aligned}
\tilde{q}_n - q_n &= e^{-\tilde{\tau}_n}(1 - \varepsilon \int_0^{\tilde{\tau}_n} e^s b_2(e^s\tilde{p}_n, e^{-s})ds \\
&\quad -\varepsilon^2 \int_0^{\tilde{\tau}_n} e^s[Db_2(e^s\tilde{p}_n, e^{-s}) \cdot \widetilde{X}_{-s}^{(1)} + \beta_2(e^s\tilde{p}_n, e^{-s})]ds) \\
&\quad -e^{-\tau_n}(1 - \varepsilon \int_0^{\tau_n} e^s b_2(e^s p_n, e^{-s})ds - \varepsilon^2 \int_0^{\tau_n} e^s Db_2(e^s p_n, e^{-s}) \cdot X_{-s}^{(1)} ds) \\
&= e^{-\tilde{\tau}_n} - e^{-\tau_n} - \varepsilon(\mathrm{III_n}) + \varepsilon^2 M_n'',
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{III_n} &= e^{-\tilde{\tau}_n} \int_0^{\tilde{\tau}_n} e^s b_2(e^s\tilde{p}_n, e^{-s})ds - e^{-\tau_n} \int_0^{\tau_n} e^s b_2(e^s p_n, e^{-s})ds \\
&= \frac{e^{-\tilde{\tau}_n}}{\tilde{p}_n} \int_{\tilde{p}_n}^{\tilde{p}_n e^{\tilde{\tau}_n}} b_2(u, \tilde{p}_n/u)du - \frac{e^{-\tau_n}}{p_n} \int_{p_n}^{p_n e^{\tau_n}} b_2(u, p_n/u)du \\
&= (1 + \varepsilon \tilde{b}_{1,n}^0 - \varepsilon \frac{\tilde{b}_{1,n}^0}{\tilde{p}_n} + o(\varepsilon)) \int_{\tilde{p}_n}^{\tilde{p}_n e^{\tilde{\tau}_n}} b_2(u, \tilde{p}_n/u)du \\
&\quad -(1 + \varepsilon b_{1,n}^0 - \varepsilon \frac{b_{1,n}^0}{p_n} + o(\varepsilon)) \int_{p_n}^{p_n e^{\tau_n}} b_2(u, p_n/u)du \\
&= \mathrm{III_n^{(1)}} + \varepsilon \mathrm{III_n^{(2)}} - \varepsilon \mathrm{III_n^{(3)}} + o(\varepsilon)
\end{aligned}$$

where

$$
\begin{aligned}
\mathrm{III_n^{(1)}} &= \int_{\tilde{p}_n}^{\tilde{p}_n e^{\tilde{\tau}_n}} b_2(u, \tilde{p}_n/u)du - \int_{p_n}^{p_n e^{\tau_n}} b_2(u, p_n/u)du \\
&= \int_{p_n e^{\tau_n}}^{\tilde{p}_n e^{\tilde{\tau}_n}} b_2(u, \tilde{p}_n/u)du + \int_{p_n}^{p_n e^{\tau_n}} [b_2(u, \tilde{p}_n/u) - b_2(u, p_n/u)]du + \int_{\tilde{p}_n}^{p_n} b_2(u, \tilde{p}_n/u)du \\
&= (\tilde{p}_n e^{\tilde{\tau}_n} - p_n e^{\tau_n}) b_2(u_n, \tilde{p}_n/u_n) + \frac{\partial b_2}{\partial q}(u_n', p_n'/u_n')(\tilde{p}_n - p_n)\tau_n + b_2(v_n, \tilde{p}_n/v_n)(p_n - \tilde{p}_n) \\
&= b_2(u_n, \tilde{p}_n/u_n)(\varepsilon(\tilde{b}^0_{1,n} - b^0_{1,n}) - \varepsilon\left(\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n} - \frac{b^0_{1,n}}{p_n}\right) + o(\varepsilon)) \\
&\quad + (\tilde{p}_n - p_n)\left(\frac{\partial b_2}{\partial q}(u_n', p_n'/u_n')\tau_n - b_2(v_n, \tilde{p}_n/v_n)\right) \\
&= C\varepsilon\left(\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n} - \frac{b^0_{1,n}}{p_n}\right) + (\tilde{p}_n - p_n)(C'\tau_n + C'').
\end{aligned}
$$

Here the term $\varepsilon(\tilde{b}^0_{1,n} - b^0_{1,n})$ is absorbed into

$$
(\tilde{p}_n - p_n)\left(\frac{\partial b_2}{\partial q}(u_n', p_n'/u_n')\tau_n - b_2(v_n, \tilde{p}_n/v_n)\right) = (\tilde{p}_n - p_n)(C'\tau_n + C'').
$$

We go estimate the term $\tilde{b}^0_{1,n}/\tilde{p}_n - b^0_{1,n}/p_n$, which depends on IV, the difference of the two integrals, as shown above. Note that there will be a factor ε^2 in front of this term when it appears in the expression for $\tilde{q}_n - q_n$, we need only consider the terms of order 1 or lower. Now

$$
\begin{aligned}
\mathrm{IV} &= \int_{1/p_n}^{1/\tilde{p}_n} b_1(1/u, u\tilde{p}_n)du + \int_{e^{-\tilde{\tau}_n}/\tilde{p}_n}^{e^{-\tau_n}/p_n} b_1(1/u, u\tilde{p}_n)du \\
&\quad + \int_{e^{-\tau_n}/p_n}^{1/p_n} [b_1(1/u, u\tilde{p}_n) - b_1(1/u, up_n)]du \\
&= b_1(1/u_n', u_n'\tilde{p}_n)\left(\frac{1}{\tilde{p}_n} - \frac{1}{p_n}\right) + b_1(1/w_n', w_n'\tilde{p}_n)\left(\frac{1}{\tilde{p}_n e^{\tilde{\tau}_n}} - \frac{1}{p_n e^{\tau_n}}\right) \\
&\quad + \frac{1}{2}\frac{\partial b_1}{\partial q}(1/v_n', v_n'\bar{p}_n)(\tilde{p}_n - p_n)\left(\frac{1}{p_n^2} - \frac{1}{p_n^2 e^{2\tau_n}}\right) \\
&= (\tilde{p}_n - p_n)\left(\frac{C_0}{\varepsilon^2 n^2} + C_1 + \frac{C_2}{n} + \frac{C_3}{n^2}\right) + C\varepsilon\left(\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n} - \frac{b^0_{1,n}}{p_n}\right) + o(1)
\end{aligned}
$$

Thus

$$\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n}-\frac{b^0_{1,n}}{p_n}=(\tilde{p}_n-p_n)\left(\frac{C_0}{\varepsilon^2 n^2}+\frac{C_0'}{\varepsilon n^2}+C_1+\frac{C_2}{n}+\frac{C_3}{n^2}+o(1)\right)+C(b_1,\tilde{p}_n,p_n)+o(1).$$

Easy to see that

$$\mathrm{III}_{\mathrm{n}}^{(2)}=\tilde{b}^0_{1,n}\int_{\tilde{p}_n}^{\tilde{p}_n e^{\tilde{\tau}_n}} b_2(u,\tilde{p}_n/u)du-b^0_{1,n}\int_{p_n}^{p_n e^{\tau_n}} b)2(u,p_n/u)du=C+o(1),$$

while

$$\begin{aligned}
\mathrm{III}_{\mathrm{n}}^{(3)} &= \frac{\tilde{b}^0_{1,n}}{\tilde{p}_n}\int_{\tilde{p}_n}^{\tilde{p}_n e^{\tilde{\tau}_n}} b_2(u,\tilde{p}_n/u)du-\frac{b^0_{1,n}}{p_n}\int_{p_n}^{p_n e^{\tau_n}} b_2(u,p_n/u)du\\
&= \frac{\tilde{b}^0_{1,n}}{\tilde{p}_n}(\mathrm{III}_{\mathrm{n}}^{(1)})+\left(\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n}-\frac{b^0_{1,n}}{p_n}\right)\int_{p_n}^{p_n e^{\tau_n}} b_2(u,p_n/u)du\\
&= C(\mathrm{III}_{\mathrm{n}}^{(1)})+C'\left(\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n}-\frac{b^0_{1,n}}{p_n}\right).
\end{aligned}$$

Note that this term has a factor ε in the expression for $\mathrm{III}_{\mathrm{n}}$ and is absorbed into $\mathrm{III}_{\mathrm{n}}^{(1)}$. Therefore

$$\mathrm{III}_{\mathrm{n}}=C\varepsilon\left(\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n}-\frac{b^0_{1,n}}{p_n}\right)+(\tilde{p}_n-p_n)(C'\tau_n+C'')+o(\varepsilon).$$

Now

$$\begin{aligned}
(\tilde{q}_n-q_n) &= e^{-\tilde{\tau}_n}-e^{-\tau_n}-\varepsilon(\mathrm{III}_{\mathrm{n}})+\varepsilon^2 M_n''\\
&= (\tilde{p}_n-p_n)(1+\varepsilon(C_{n,0}\tilde{\tau}_n+C_{n,1}+C_{n,2}\varepsilon))+\varepsilon^2 M_n\\
&\quad -C\varepsilon^2\left(\frac{\tilde{b}^0_{1,n}}{\tilde{p}_n}-\frac{b^0_{1,n}}{p_n}\right)+o(\varepsilon^2)\\
&= (\tilde{p}_n-p_n)(1+\frac{C_n^1}{n^2}+C_{n,0}\varepsilon\tilde{\tau}_n+C_{n,1}\varepsilon+C_{n,2}\varepsilon^2)+\varepsilon^2 M_n+o(\varepsilon^2).
\end{aligned}$$

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