ABSTRACT

Title of dissertation: IDENTIFICATION OF OPERATORS ON ELEMENTARY LOCALLY COMPACT ABELIAN GROUPS

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Measurement of time-variant linear channels is an important problem in communications theory with applications in mobile communications and radar detection. Kailath addressed this problem about half a century ago and developed a spreading criterion for the identifiability of time-variant channels analogous to the band limitation criterion in the classical sampling theory of signals. Roughly speaking, underspread channels are identifiable and overspread channels are not identifiable, where the critical spreading area equals one. Kailath's analysis was later generalized by Bello from rectangular to arbitrary spreading supports.

Modern developments in time-frequency analysis provide a natural and powerful framework in which to study the channel measurement problem from a rigorous mathematical standpoint. Pfander and Walnut, building on earlier work by Kozek and Pfander, have developed a sophisticated theory of "operator sampling" or "operator identification" which not only places the work of Kailath and Bello on rigorous footing, but also takes the subject in new directions, revealing connections with other important problems in time-frequency analysis.

We expand upon the existing work on operator identification, which is restricted to the real line, and investigate the subject on elementary locally compact abelian groups, which are groups built from the real line, the circle, the integers, and finite abelian groups. Our approach is to axiomatize, as it were, the main ideas which have been developed over the real line, working with lattice subgroups. We are thus able to prove the various identifiability results for operators involving both underspread and overspread conditions in both general and specific cases. For example, we provide a finite dimensional example illustrating a necessary and sufficient condition for identifiability of operators, owing to the insight gleaned from the general theory.

In working up to our main results, we set up the quite considerable technical background, bringing some new perspectives to existing ideas and generally filling what we consider to be gaps in the literature.

IDENTIFICATION OF OPERATORS ON ELEMENTARY LOCALLY COMPACT ABELIAN GROUPS

by

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Chapter 1:

Introduction

1.1 Background and Motivation

The main mathematical problem that we address is motivated by a classical problem in communications engineering, that of measurement of time-variant linear channels. In simple mathematical terms, a time-*invariant* linear channel (or operator) A is one that commutes with translations: $T_x A = A T_x$. Translation invariant operators are known to be equivalent to convolution operators; the precise formulation of this statement and the difficulty of its proof depend on the choice of function space. Since the Dirac distribution is the identity of the convolution operation, measurement of time-invariant linear channels is well understood.

Time-variant linear channels arise, for example, in mobile communications [Str06] and radar detection [BGE11]. The time-variant nature of the problem is due to the time delays and Doppler shifts effected during the transmission of a signal. The proper mathematical formulation of a time-variant operator is suggested by the definition of a convolution operator. The convolution operator $g \to \tau * g$ can be expressed as the integral operator

$$g \to \int \tau(\cdot - y) g(y)\, dy.$$

In order to obtain a time-variant operator, we simply let τ vary with time:

$$g \to \int \tau(\cdot, \cdot - y) g(y)\, dy.$$

Setting $\kappa(x, y) = \tau(x, x - y)$, we obtain

$$g \to \int \kappa(\cdot, y) g(y)\, dy.$$

Therefore, time-variant operators are those defined via integration against a kernel function. Although we derived the form a time-variant operator should have, due to the Schwartz kernel theorem, every reasonable operator is necessarily of this form. Moreover, defining σ as the Fourier transform of τ in the second variable, we obtain the expression

$$g \to \int \sigma(\cdot, \omega) \hat{g}(\omega) e^{2\pi i \langle \omega, \cdot \rangle}\, d\omega,$$

making contact with classical pseudodifferential operators. See [Str06] for a detailed discussion of pseudodifferential operators in the context of mobile communications.

The form of a time-variant operator that is most suitable for our purposes is the spreading representation. We define the spreading function η via the change of coordinates $\kappa(x, y) \to \kappa(y, y - x)$ followed by the Fourier transform in y. We then obtain the expression

$$g \to \int \eta(x, \omega) M_\omega T_x g\, dx\, d\omega,$$

where M_ω is the modulation operator $f \to e^{2\pi i \langle \omega, \cdot \rangle} f$. In other words, we take a weighted sum of time-frequency shifts of g. The translation operator represents time delays, and the modulation operator represents Doppler shifts.

In [Kai62], Kailath considered the measurement (or identification) problem for time-variant operators. He proposed a measurement scheme whereby the parameters of a time-variant operator are to be determined by reading its response to a Dirac impulse train (or Dirac comb). More generally, he considered under what conditions a time-variant operator can be identified by reading its response to a single judiciously chosen input signal. Given a family of time-variant operators whose spreading functions are all supported in some fixed rectangular region of the time-frequency plane, Kailath conjectured, based on counting and linear independence arguments, that the family of operators is identifiable if and only if the rectangle has area less than or equal to one. In [Bel69], Bello argued that one can take the common spreading support to be any region of the time-frequency plane, not necessarily a rectangle, and the same identification criterion applies.

More recently, the operator identification problem has been the subject of renewed interest in light of modern developments in time-frequency analysis during the last few decades. In [KP05], Kozek and Pfander rigorously formulated and proved Kailath's conjecture more or less exactly as Kailath had stated it. Specifically, for a rectangular support set of area less than or equal to one, they proved identifiability of a given operator family by a Dirac comb just as Kailath had proposed. On the other hand, they proved that no signal, no matter how cleverly chosen, suffices to identify if the rectangle has area greater than one. Technically, certain continuity criteria are also part of these identification results, and one needs to work with appropriate function spaces.

Shorty thereafter, in [PW06a], Pfander and Walnut generalized the methods in

[KP05] and proved Bello's stronger version of Kailath's conjecture; see also [PW06b]. However, one needs to be careful about the particulars of the spreading support. Here is a convenient formulation of the main results in [PW06a]: If the spreading support is compact with area less than one, then the family of operators is identifiable by a periodically weighted Dirac comb, that is, a signal of the form $\sum_{k\in\mathbb{Z}} c_k\delta_{ka}$, where $c_k = c_{k+L}$ for some positive integer L [PW06a, Theorem 3.1]. If the spreading support is open with area greater than one, then no signal suffices to identify [PW06a, Theorem 4.1].

In recent years, Pfander and Walnut have expanded upon their earlier work and developed a robust theory of sampling of operators [Pfa13b; PW15b]. The term "sampling" reflects the similarity between probing an operator with a weighted Dirac comb and the classical sampling theory of functions. In [WPK15], the two authors together with Kailath give an excellent survey of the subject going back to Kailath's early investigations.

All of the above work has been carried out on the real line and, more generally, on Euclidean space. We are interested in the extension of the theory of operator identification to more general groups. Specifically, we focus here on elementary locally compact abelian (ELCA) groups, that is, groups which are products of any combination of finitely many copies of $\mathbb{R}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{Z}$, and finite abelian groups, e.g., $\mathbb{R}^2 \times \mathbb{T} \times (\mathbb{Z}/4\mathbb{Z})$ or $\mathbb{T} \times \mathbb{Z}^3$. We were originally interested in $\mathbb{T}$, partly encouraged by the possibility that a periodic version of the theory of operator identification could be relevant to applications. Upon resolving the problem on $\mathbb{T}$, motivated by [FK98], we decided that it was logical to try to extend the theory to ELCA groups.

Interestingly, the abstraction that is necessary to carry out the theory in this general setting is very illuminating and renders the theory conceptually simpler.

In Chapter 2, we give a coherent account of Fourier analysis on ELCA groups from the perspective of distributions and with an eye toward time-frequency analysis. This chapter establishes the technical background necessary for our further investigations.

In Chapter 3, we emphasize the special role that the space M^1, known as Feichtinger's algebra, plays in time-frequency analysis, and we develop the theory of operators based on this space and its dual.

In Chapter 4, we discuss the theory of operator identification on ELCA groups and prove our main results.

1.2 Main Results and Technical Contributions

Our main results consist of the extension of the theory of operator identification developed by Pfander and Walnut from the real line to ELCA groups. The starting point is the interplay between two periodization concepts: the Zak transform and quasi-periodization. The latter concept was introduced in [PW15b]. As Proposition 4.3.1 and Proposition 4.3.6 show, the two periodization concepts are closely linked via the action of a pseudodifferential operator. This close link is predicted in Section 4.2, where one sees that the two concepts enjoy parallel properties. Proposition 4.3.6 is the key discretization result which allows reduction of the infinite dimensional theory to the finite dimensional theory discussed in Section 4.1. In Theorem 4.4.1 and Theorem 4.4.5, we characterize spreading supports for which operator identification by a given periodically weighted Dirac comb is possible, generalizing [PW15b, Theorem 2.8]. In Section 4.1, we give a finite dimensional example providing numerical verification of Theorem 4.4.5. It is interesting to note that we were able to think of this example only after formulating and proving Theorem 4.4.5 in general. Corollary 4.4.4 gives a general sufficient condition for operator identification, generalizing [PW06a, Theorem 3.1] and proving that Bello's underspread condition is sufficient for identifiability of operators on ELCA groups.

In Section 4.5, we study the opposite side of the coin, and attempt to generalize [PW06a, Theorem 4.1] and prove that Bello's underspread condition is necessary for identifiability of operators on ELCA groups. The idea essentially is to restrict the identification problem to a subspace synthesized from a very simple class of operators over which we have good control, thereby simplifying the left hand side of the identification problem, and to simplify the right hand side via an appropriate analysis map, thereby obtaining an infinite matrix which is easy to work with. We abstract out the mechanics involved in this scheme. We are then able to obtain the non-identifiability results on both $\mathbb{T}$ and $\mathbb{R}$ (Theorem 4.5.12 and Theorem 4.5.14) by specializing this general scheme. The corresponding result on $\mathbb{Z}$ follows from a duality principle akin to the Plancherel theorem (Theorem 4.5.2). We then consider product groups. As of the writing of this work, we have not fully generalized [PW06a, Theorem 4.1] to ELCA groups, but we do prove a relevant result (Theorem 4.5.17).

Aside from our main results, we perform a substantial amount of work giving a coherent account of harmonic and time-frequency analysis on ELCA groups from the point of view of distributions, a treatment which does not seem to exist in the literature in this form. In the presentation of this material, we offer several new insights and fill some gaps in the literature. We next emphasize the most important aspects of our technical contributions in this regard.

The tensor product construction plays a significant role in our rigorous development of the technical tools needed to address our main objectives. Proposition 2.2.5, an old result of Nachbin, gives a version of the Stone-Weierstrass theorem in the smooth category. We use this theorem to prove the important convergence result in Proposition 2.2.7. We think this result should be well-known, but we were not able to find an existing proof. This convergence result is later used to prove Proposition 2.3.34, which in turn plays a key role in proving Proposition 3.1.11, which itself is vital for the soundness of some of the arguments in Section 3.5 and Section 4.2. In this connection, we also note the entirety of Section 3.1, where we carry out the work of extending many technical results from the setting of the Schwartz space to the setting of M^1, for which we have not found a self-contained treatment in the literature. It is here that we also reconcile various ostensibly different definitions of the short-time Fourier transform encountered throughout the text.

Example 2.4.5 and Proposition 2.6.21 are results where a space which a priori is only known to consist of distributions turns out to be a bona fide function space. It is an oversight we have occasionally come across in the literature whereby a distribution is assumed to be a function throughout the proof of such a result before it is demonstrated as one. For technical correctness, it is necessary to insert an argument confirming that a distribution is indeed a function before one can treat it as such.

We have devised Proposition 2.1.31 to take care of this subtle issue.

Theorem 3.2.1, a result of Bonsall, gives criteria for the decomposability of a Banach space into atomic elements. We have found this result to be of great utility in our investigations in Section 3.4 and Section 3.5 concerning quantization of operators (the Schwartz kernel theorem) and the spreading representation. The approach found in the standard literature utilizes Wilson bases, a construction which we find unintuitive and would therefore like to avoid if possible. Since $\mathbb{T}$ and $\mathbb{Z}$ afford convenient orthonormal bases, the issue here is with $\mathbb{R}$, over which there is no naturally occurring orthonormal basis suitable for time-frequency analysis. However, we found that the atomic decomposition theorem can be a sufficient replacement. It is featured prominently in the proofs of Proposition 3.4.4, Proposition 3.4.5, and Proposition 3.5.2, a perspective we have not seen in the literature. The first two results and the discussion that follows them constitute a complete proof of one direction of the Schwartz kernel theorem in the setting of M^1.

Chapter 2:

Fourier Analysis on Elementary Locally Compact Abelian Groups

In Section 2.1, we give an exposition of the theory of Schwartz functions and tempered distributions. One of the main goals of this section is to establish language and notation. Most of the results and their proofs can be found in [Rud91, Chapters 6 and 7] and [Fol99, Chapters 8 and 9]. We also recommend [Hör90].

In Section 2.2, we discuss tensor products of tempered distributions. The tensor product construction is indispensable to the proper development of time-frequency analysis, and we use it throughout. The main reference for the material in this section is [Hör90, Chapter 5].

In Section 2.3, we begin our study of time-frequency analysis and specifically the short-time Fourier transform, which is the main tool on which all of our work is based. The main reference for this section is [Grö01, Chapters 3 and 11].

In Section 2.4, we study modulation spaces, which are very suitable for time-frequency analysis, owing to their myriad invariance properties. The main reference for this section is [Grö01, Chapter 11].

In Section 2.5, we give a very general account of the critically useful concept of periodization, variants of which will feature most prominently in Chapter 4.

In Section 2.6, we study Wiener amalgam spaces. Although we shall not make use of the full theory, some of its consequences will be relevant. The main references for this section are [Hei03] and [Grö01, Chapter 12].

Generally, we prove results whenever they are nontrivial and not readily available in the literature, or when we are not fully satisfied with existing proofs, or when the inclusion of a proof is warranted for clarity. Otherwise, the reader is referred to the standard literature. We also do not necessarily prove formulas which can be obtained through straightforward algebra.

2.1 Theory of Distributions

Let $G = \mathbb{R}^d \times \mathbb{T}^{d'} \times \mathbb{Z}^{d''} \times \mathbb{A}$, $d, d', d'' \geq 0$. Here, $\mathbb{A}$ is a finite abelian group. By the classification theorem for finite abelian groups, $\mathbb{A}$ is a direct sum of finite cyclic groups where each summand has order the power of a prime. The Haar measure on $\mathbb{R}^d$ will be the standard Lebesgue measure. The Haar measure on $\mathbb{T}^{d'}$ will be normalized to have total measure 1. The Haar measures on $\mathbb{Z}^{d''}$ and $\mathbb{A}$ will be the counting measure. The Haar measure on G will be denoted by μ_G. Recall that $\widehat{\mathbb{R}}^d = \mathbb{R}^d$, $\widehat{\mathbb{T}}^{d'} = \mathbb{Z}^{d'}$, $\widehat{\mathbb{Z}}^{d''} = \mathbb{T}^{d''}$, and $\widehat{\mathbb{A}} = \mathbb{A}$.

Let $(x, z, \iota, \lambda) \in G$ and $(\omega, \xi, y, \tau) \in \widehat{G}$. The pairing between G and $\widehat{G}$ will be defined by $(x, \omega) = e^{2\pi i x\cdot\omega}$, $(z, \xi) = z^{\xi}$, $(\iota, y) = y^{\iota}$, and $(\lambda, \tau) = e^{2\pi i\lambda\tau/N}$ for $\lambda, \tau \in \mathbb{Z}/N\mathbb{Z}$.

The symbols α, β, and γ will denote multi-indices. Multi-indices for differentiation may include any combination of directions along $\mathbb{R}^d$ or $\mathbb{T}^{d'}$. The subscript $\mathbb{R}$ denotes the component along $\mathbb{R}^d$. For example, if $a = (x, z, \iota, \lambda) \in G$, then $a_{\mathbb{R}} = x$.

Similarly for $\mathbb{T}$ and $\mathbb{Z}$.

Let F be a complex function on G. We define the translation operator as $T_aF(t) = F(t-a)$ for $a, t \in G$. We define the modulation operator as $M_{\hat{a}}F(t) = (t, \hat{a})F(t)$ for $\hat{a} \in \widehat{G}$ and $t \in G$. Note that $T_aM_{\hat{a}} = (-a, \hat{a})M_{\hat{a}}T_a$. We define $\check{F}(a) = F(-a)$ for $a \in G$. We define $F^*(a) = \overline{F(-a)}$ for $a \in G$. We define $X^\beta F(a) = a_{\mathbb{R}}^\beta F(a)$ and $K^\gamma F(a) = a_{\mathbb{Z}}^\gamma F(a)$ for $a \in G$.

We state two basic results relating convolution and differentiability.

Proposition 2.1.1. *Let $k \geq 0$. Let $f \in L^1(G)$ and $g \in C^k(G)$. Suppose that $\partial^\alpha g$ is bounded for all $|\alpha| \leq k$. Then $f * g \in C^k(G)$, and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ for all $|\alpha| \leq k$.*

Proposition 2.1.2. *Let $k \geq 0$. Let $f \in L^1_{loc}(G)$ and $g \in C_c^k(G)$. Then $f * g \in C^k(G)$, and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ for all $|\alpha| \leq k$.*

We shall now define the space of Schwartz functions. The utility of Schwartz functions in Fourier analysis stems from the agility that they offer in integration owing to their rapid decay properties together with the following integrability results.

Lemma 2.1.3. *Let $a > 0$ and $1 \leq p < \infty$. If $s > d/p$, then $(a + |x|)^{-s} \in L^p(\mathbb{R}^d)$.*

Lemma 2.1.4. *Let $\epsilon > 0$. The sum $\sum_{\iota \in \mathbb{Z}^d}(1 + |x + \iota|)^{-d-\epsilon}$ is uniformly convergent for $x \in \mathbb{R}^d$.*

For $f \in C^\infty(G)$, let $\|f\|_{\alpha,\beta,\gamma} = \|K^\gamma X^\beta \partial^\alpha f\|_\infty$. Let $\mathcal{S}(G)$ be the set of all $f \in C^\infty(G)$ for which these seminorms are finite. The space $\mathcal{S}(G)$ is a Fréchet space under this separating family of seminorms; we call it the Schwartz space on G. The dual space $\mathcal{S}'(G)$ with its weak* topology is the space of tempered distributions on G. Clearly, $\mathcal{S}(G) \subseteq C_0(G)$ and $\mathcal{S}(G) \subseteq L^p(G)$ $(1 \leq p \leq \infty)$. In particular, Schwartz functions are uniformly continuous.

Remark. We can equally well use the separating family of seminorms

$$\|f\|_{\alpha,\beta,\gamma} = \|K^\gamma \partial^\alpha X^\beta f\|_\infty \qquad (f \in C^\infty(G)).$$

Proposition 2.1.5. *Differentiation is a continuous linear map from $\mathcal{S}(G)$ to $\mathcal{S}(G)$. Let f be a C^∞ function on G all of whose derivatives have polynomial growth (on $\mathbb{R}^d \times \mathbb{Z}^{d''}$). Multiplication by f is a continuous linear map from $\mathcal{S}(G)$ to $\mathcal{S}(G)$. Multiplication by f is a continuous linear map from $\mathcal{S}'(G)$ to $\mathcal{S}'(G)$.*

Proposition 2.1.6. *If $f, g \in \mathcal{S}(G)$, then $f * g \in \mathcal{S}(G)$.*

It follows from a standard theorem on approximate identities that $C_c^\infty(G)$ is dense in $C_0(G)$ and in $L^p(G)$ $(1 \leq p \leq \infty)$. It follows from the next result that $C_c^\infty(G)$ is dense in $\mathcal{S}(G)$; see [Rud91, Theorem 7.10].

Proposition 2.1.7. *Let $\psi \in C_c^\infty(G)$ with $\psi = 1$ on $U \times \mathbb{T}^{d'} \times U'' \times \mathbb{A}$, where U and U'' are open balls about 0. Let $\psi_{\epsilon,n}(x, z, \iota, \lambda) = \psi(\epsilon x, z, \lfloor \iota/n \rfloor, \lambda)$ for $\epsilon > 0$, $n \geq 1$, and $(x, z, \iota, \lambda) \in G$. Here, $\lfloor \cdot \rfloor$ is truncation towards 0. For every $f \in \mathcal{S}(G)$, $\psi_{\epsilon,n} f \to f$ in $\mathcal{S}(G)$ as $\epsilon \to 0$ and $n \to \infty$.*

Let f be a complex function on G. If $f \in L^p(G)$ $(1 \leq p \leq \infty)$, or f is measurable and has polynomial growth, then f defines a tempered distribution via integration. The following result shows that we can consistently identify f with its associated tempered distribution.

Proposition 2.1.8. *Let $f \in L^1_{loc}(G)$. If $\int_G f\phi = 0$ for all $\phi \in C_c^\infty(G)$, then $f = 0$ almost everywhere.*

Similarly, if $\mu \in M(G)$, then μ defines a tempered distribution via integration, and consistency is not an issue.

Proposition 2.1.9. *If $\int_G \phi\, d\mu = 0$ for all $\phi \in C_c^\infty(G)$, then $\mu = 0$.*

Note that the inclusions $\mathcal{S}(G) \subseteq L^p(G) \subseteq \mathcal{S}'(G)$ $(1 \leq p \leq \infty)$ are continuous.

We now define derivatives of tempered distributions. Let $u \in \mathcal{S}'(G)$. We define $\partial^\alpha u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi)$ for $\phi \in \mathcal{S}(G)$. If u is a C^∞ function all of whose derivatives have polynomial growth, integration by parts shows that $\partial^\alpha u$ as just defined is consistent with $\partial^\alpha u$ in the calculus sense.

Proposition 2.1.10. *Differentiation is a continuous linear map from $\mathcal{S}'(G)$ to $\mathcal{S}'(G)$.*

Fourier Transforms

Let $G_1 = \mathbb{R}^{d_1} \times \mathbb{T}^{d_1'} \times \mathbb{Z}^{d_1''} \times \mathbb{A}_1$ and $G_2 = \mathbb{R}^{d_2} \times \mathbb{T}^{d_2'} \times \mathbb{Z}^{d_2''} \times \mathbb{A}_2$. We shall define the partial Fourier transform on $G_1 \times G_2$ with respect to G_1.

Lemma 2.1.11. *Let $a_2 \in G_2$. The linear map $\phi \to \phi(\cdot, a_2)$ from $\mathcal{S}(G_1 \times G_2)$ to $\mathcal{S}(G_1)$ is continuous.*

Lemma 2.1.12. *Let $\phi \in \mathcal{S}(G_1 \times G_2)$.*

(a) The map $a_2 \to \phi(\cdot, a_2)$ from G_2 to $L^1(G_1)$ is uniformly continuous.

(b) The map $a_2 \to \|\phi(\cdot, a_2)\|_1$ on G_2 is in $C_0(G_2)$.

Let $\phi \in \mathcal{S}(G_1 \times G_2)$. We define $\mathcal{F}_1\phi(\hat{a}_1, a_2) = \phi(\cdot, a_2)\hat{}(\hat{a}_1)$ for $\hat{a}_1 \in \widehat{G}_1$ and $a_2 \in G_2$, i.e., we take the Fourier transform in the first variable.

Proposition 2.1.13. *The transform $\mathcal{F}_1$ is a Fréchet space isomorphism from $\mathcal{S}(G_1 \times G_2)$ onto $\mathcal{S}(\widehat{G}_1 \times G_2)$. We have*

$$\mathcal{F}_1\partial^{\alpha_1}\phi(\omega, \xi, \cdots) = (2\pi i\omega)^{\alpha_{1,\mathbb{R}}}(2\pi i\xi)^{\alpha_{1,\mathbb{T}}}\mathcal{F}_1\phi(\omega, \xi, \cdots), \quad \mathcal{F}_1\partial^{\alpha_2}\phi = \partial^{\alpha_2}\mathcal{F}_1\phi,$$

$$\text{and} \quad \partial^{\alpha_1}\mathcal{F}_1\phi = \mathcal{F}_1[(-2\pi iX)^{\alpha_{1,\mathbb{R}}}(-2\pi iK)^{\alpha_{1,\mathbb{T}}}\phi].$$

Corollary 2.1.14. *The transform $\mathcal{F}_1$ of Proposition 2.1.13 extends to a weak* isomorphism from $\mathcal{S}'(G_1 \times G_2)$ onto $\mathcal{S}'(\widehat{G}_1 \times G_2)$.*

Theorem 2.1.15 (Plancherel Theorem)**.** *The transform $\mathcal{F}_1$ of Corollary 2.1.14 is an isometric isomorphism from $L^2(G_1 \times G_2)$ onto $L^2(\widehat{G}_1 \times G_2)$.*

If $\mu \in M(G)$, there are two definitions of the Fourier transform of μ, one in the abstract harmonic analysis sense, and one in the sense of Corollary 2.1.14; these two definitions are consistent with each other.

We state some simple identities involving the Fourier transform. Let F be a complex function on $G_1 \times G_2$. We define $\mathcal{R}_1F(a_1, a_2) = F(-a_1, a_2)$ for $a_1 \in G_1$ and $a_2 \in G_2$.

Proposition 2.1.16. *Let $u \in \mathcal{S}'(G_1 \times G_2)$.*

(a) $\mathcal{F}_1M_{(\hat{a}_1,0)}u = T_{(\hat{a}_1,0)}\mathcal{F}_1u$.

(b) $\mathcal{F}_1T_{(a_1,0)}u = M_{(-a_1,0)}\mathcal{F}_1u$.

(c) $\mathcal{F}_1\mathcal{R}_1u = \mathcal{R}_1\mathcal{F}_1u$.

(d) $\mathcal{R}_1\overline{u} = \overline{\mathcal{R}_1 u}$.

(e) $\mathcal{F}_1\overline{u} = \mathcal{R}_1\overline{\mathcal{F}_1 u}$.

Proof. The proof consists of straightforward calculations which we carry out for the purpose of elucidating some of the definitions that are implicit in our discussion so far. Let $\phi \in \mathcal{S}(\widehat{G}_1 \times G_2)$ and $\psi \in \mathcal{S}(G_1 \times G_2)$.

(a)

$$\mathcal{F}_1 M_{(\hat{a}_1,0)}u(\phi) = M_{(\hat{a}_1,0)}u(\mathcal{F}_1\phi) = u(M_{(\hat{a}_1,0)}\mathcal{F}_1\phi) = u(\mathcal{F}_1 T_{(-\hat{a}_1,0)}\phi)$$

$$= \mathcal{F}_1 u(T_{(-\hat{a}_1,0)}\phi) = T_{(\hat{a}_1,0)}\mathcal{F}_1 u(\phi).$$

(b)

$$\mathcal{F}_1 T_{(a_1,0)}u(\phi) = T_{(a_1,0)}u(\mathcal{F}_1\phi) = u(T_{(-a_1,0)}\mathcal{F}_1\phi) = u(\mathcal{F}_1 M_{(-a_1,0)}\phi)$$

$$= \mathcal{F}_1 u(M_{(-a_1,0)}\phi) = M_{(-a_1,0)}\mathcal{F}_1 u(\phi).$$

(c)

$$\mathcal{F}_1\mathcal{R}_1 u(\phi) = \mathcal{R}_1 u(\mathcal{F}_1\phi) = u(\mathcal{R}_1\mathcal{F}_1\phi) = u(\mathcal{F}_1\mathcal{R}_1\phi)$$

$$= \mathcal{F}_1 u(\mathcal{R}_1\phi) = \mathcal{R}_1\mathcal{F}_1 u(\phi).$$

(d)

$$\mathcal{R}_1\overline{u}(\psi) = \overline{u}(\mathcal{R}_1\psi) = \overline{u(\overline{\mathcal{R}_1\psi})} = \overline{u(\mathcal{R}_1\overline{\psi})}$$

$$= \overline{\mathcal{R}_1 u(\overline{\psi})} = \overline{\mathcal{R}_1 u}(\psi).$$

(e)

$$\begin{aligned}\mathcal{F}_1\overline{u}(\phi) = \overline{u}(\mathcal{F}_1\phi) = \overline{u(\overline{\mathcal{F}_1\phi})} &= \overline{u(\mathcal{R}_1\mathcal{F}_1\overline{\phi})} \\ &= \overline{\mathcal{R}_1 u(\mathcal{F}_1\overline{\phi})} = \overline{\mathcal{F}_1\mathcal{R}_1 u(\overline{\phi})} = \overline{\mathcal{R}_1\mathcal{F}_1 u(\overline{\phi})} \\ &= \overline{\mathcal{R}_1\mathcal{F}_1 u}(\phi) = \mathcal{R}_1\overline{\mathcal{F}_1 u}(\phi).\end{aligned}$$

□

The following Fourier transform is of fundamental importance.

Proposition 2.1.17. *Let 1_G be the constant polynomial 1 on G. Let $\delta_{\widehat{G}}$ be the Dirac distribution on $\widehat{G}$. Then $\hat{\delta}_{\widehat{G}} = 1_G$.*

Proof. Let $\phi \in \mathcal{S}(G)$. We have

$$\hat{\delta}_{\widehat{G}}(\phi) = \delta_{\widehat{G}}(\hat{\phi}) = \hat{\phi}(0) = \int_G \phi(t)(-t,0)\,dt = \int_G \phi(t)\,dt = (1_G,\phi).$$

□

Convolutions

We state an identity, which follows from the binomial theorem, that is useful for establishing certain estimates; see [Grö01, Lemma 11.2.1].

Lemma 2.1.18. *Let $f \in C^\infty(G)$. Then*

$$K^\gamma \partial_{\mathbb{R}}^{\alpha_{\mathbb{R}}} \partial_{\mathbb{T}}^{\alpha_{\mathbb{T}}} X^\beta M_{(\omega_0,\xi_0,y_0,\tau_0)} T_{(x_0,z_0,\iota_0,\lambda_0)} f =$$
$$\sum_{\rho_{1,\mathbb{R}}\leq\alpha_{\mathbb{R}}} \sum_{\rho_{1,\mathbb{T}}\leq\alpha_{\mathbb{T}}} \sum_{\rho_2\leq\beta} \sum_{\rho_3\leq\gamma} \binom{\alpha_{\mathbb{R}}}{\rho_{1,\mathbb{R}}} \binom{\alpha_{\mathbb{T}}}{\rho_{1,\mathbb{T}}} \binom{\beta}{\rho_2} \binom{\gamma}{\rho_3} \iota_0^{\rho_3} x_0^{\rho_2} (2\pi i\omega_0)^{\rho_{1,\mathbb{R}}} (2\pi i\xi_0)^{\rho_{1,\mathbb{T}}} \cdots$$
$$M_{(\omega_0,\xi_0,y_0,\tau_0)} T_{(x_0,z_0,\iota_0,\lambda_0)} K^{\gamma-\rho_3} \partial_{\mathbb{R}}^{\alpha_{\mathbb{R}}-\rho_{1,\mathbb{R}}} \partial_{\mathbb{T}}^{\alpha_{\mathbb{T}}-\rho_{1,\mathbb{T}}} X^{\beta-\rho_2} f.$$

The following result, whose proof uses Lemma 2.1.18, is fundamental to the time-frequency analysis of tempered distributions; see [Grö01, Corollary 11.2.2].

Proposition 2.1.19. *Let $\phi \in \mathcal{S}(G)$. The map $(a, \hat{a}) \to M_{\hat{a}} T_a \phi$ from $G \times \widehat{G}$ to $\mathcal{S}(G)$ is continuous.*

The following technical result is useful; see [Rud91, Lemma 7.17].

Lemma 2.1.20. *Let $\phi \in \mathcal{S}(G)$. Let e_1 be the first standard basis vector in $\mathbb{R}^d$. Then*

$$\frac{T_{-he_1}\phi - \phi}{h} - \partial_{e_1}\phi \to 0$$

in $\mathcal{S}(G)$ as $h \to 0$. There is an analogous result for differentiation on $\mathbb{T}^{d'}$.

We now extend the definition of convolution. Let $u \in \mathcal{S}'(G)$ and $\phi \in \mathcal{S}(G)$. We define $(u * \phi)(a) = u(T_a \check{\phi})$ for $a \in G$.

Proposition 2.1.21. *$u * \phi \in C^\infty(G)$, and $\partial^\alpha(u * \phi) = (\partial^\alpha u) * \phi = u * (\partial^\alpha \phi)$. Moreover, $u * \phi$ has polynomial growth, so $u * \phi \in \mathcal{S}'(G)$.*

Proposition 2.1.22. *Let $u \in \mathcal{S}'(G)$ and $\phi \in \mathcal{S}(G)$. Then $\widehat{u * \phi} = \hat{\phi}\hat{u}$ and $\widehat{\phi u} = \hat{u} * \hat{\phi}$. If $\psi \in \mathcal{S}(G)$, then $(u * \phi) * \psi = u * (\phi * \psi)$.*

Corollary 2.1.23. *Let $u \in \mathcal{S}'(G)$ and $\phi \in \mathcal{S}(G)$. Then $\widehat{\phi u}(\hat{a}) = u(M_{-\hat{a}}\phi)$.*

Let $u \in \mathcal{S}'(G)$. We say that u vanishes on the open set $V \subseteq G$ if $u(\phi) = 0$ for all $\phi \in C_c^\infty(V)$. If W is the union of all such open sets, then a partition of unity argument shows that u vanishes on W. The complement $\operatorname{supp} u = G \setminus W$ is called the support of u.

Compactly supported distributions are of special interest, so we turn to the space $C^\infty(G)$. Let $\{K_j\}$ be a sequence of compact sets in $\mathbb{R}^d$ such that $K_j \subseteq K^o_{j+1}$ and $\mathbb{R}^d = \bigcup K_j$. Let $F_1 \subseteq F_2 \subseteq \cdots \subseteq \mathbb{Z}^{d''}$ be finite sets whose union is $\mathbb{Z}^{d''}$. For $f \in C^\infty(G)$, let

$$\|f\|_{\alpha,N} = \sup\{\partial^\alpha f(b),\ b \in K_N \times \mathbb{T}^{d'} \times F_N \times \mathbb{A}\}.$$

The space $C^\infty(G)$ is a Fréchet space under this separating family of seminorms; the topology is independent of the chosen $\{K_j\}$ and $\{F_j\}$. The inclusion $\mathcal{S}(G) \subseteq C^\infty(G)$ is continuous. The next result shows that a compactly supported distribution on G extends uniquely to a continuous linear functional on $\mathcal{E}(G) = C^\infty(G)$.

Proposition 2.1.24. *$C_c^\infty(G)$ is dense in $C^\infty(G)$. There is a one-to-one correspondence between $\mathcal{E}'(G)$ and the set of all compactly supported distributions on G.*

The Fourier transform of a compactly supported distribution has a simple description.

Proposition 2.1.25. *Let $u \in \mathcal{E}'(G)$. Then $\hat{u}$ is a C^∞ function all of whose derivatives have polynomial growth. Moreover, $\hat{u}(\hat{a}) = u((\cdot, -\hat{a}))$.*

We can now further extend the definition of convolution. Let $u \in \mathcal{S}'(G)$ and $v \in \mathcal{E}'(G)$. We define $u * v$ (or $v * u$) to be that tempered distribution on G whose Fourier transform is $\hat{v}\hat{u}$.

Proposition 2.1.26. *If $u \in \mathcal{E}'(G)$ and $\phi \in \mathcal{S}(G)$, then $u * \phi \in \mathcal{S}(G)$.*

Proposition 2.1.27. *Let $u, v, w \in \mathcal{S}'(G)$.*

(a) If at least one of u and v has compact support, then $\operatorname{supp}(u*v) \subseteq \operatorname{supp} u + \operatorname{supp} v$.

(b) If at least two of u, v, and w have compact support, then $(u*v)*w = u*(v*w)$.

(c) $\partial^\alpha u = (\partial^\alpha \delta_G) * u$.

(d) If at least one of u and v has compact support, then $\partial^\alpha(u*v) = (\partial^\alpha u)*v = u*(\partial^\alpha v)$.

Approximation Results

Proposition 2.1.28. *Let $\{\psi_j\}$ be a sequence in $C_c^\infty(G)$ such that $\psi_j \geq 0$, $\int_G \psi_j = 1$, and* $\operatorname{supp}\psi_j \to 0$.

(a) For every $f \in C(G)$, $f * \psi_j \to f$ uniformly on compact sets.

(b) For every $\phi \in \mathcal{S}(G)$, $\phi * \psi_j \to \phi$ in $\mathcal{S}(G)$.

(c) For every $u \in \mathcal{S}'(G)$, $u * \psi_j \to u$ in $\mathcal{S}'(G)$.

Proof. (a) Let A be a compact subset of G. We have

$$\begin{aligned}
|(f*\psi_j)(a) - f(a)| &\leq \int_G |f(a-t) - f(a)|\psi_j(t)\,dt \\
&= \int_K |f(a-t) - f(a)|\psi_j(t)\,dt \\
&\leq \sup_{t\in K} |f(a-t) - f(a)|.
\end{aligned}$$

Here, K is a compact neighborhood of 0 containing the support of ψ_j for j sufficiently large. Therefore,

$$\sup_{a\in A} |(f*\psi_j)(a) - f(a)| \leq \sup_{a\in A}\sup_{t\in K} |f(a-t) - f(a)|.$$

Since f is uniformly continuous on the compact set $A - K$, the RHS can be made arbitrarily small by making K sufficiently small.

(b) It suffices to show that $\iota^\gamma x^\beta((\phi * \psi_j)(a) - \phi(a)) \to 0$ uniformly for $a = (x, \cdot, \iota, \cdot) \in G$. We have

$$\begin{aligned}
|\iota^\gamma x^\beta((\phi * \psi_j)(a) - \phi(a))| &\leq \int_G |\iota^\gamma x^\beta(\phi(a-t) - \phi(a))| \psi_j(t)\, dt \\
&= \int_K |\iota^\gamma x^\beta(\phi(a-t) - \phi(a))| \psi_j(t)\, dt \\
&\leq \sup_{t \in K} |\iota^\gamma x^\beta(\phi(a-t) - \phi(a))|.
\end{aligned}$$

Here, K is the same as before. By Proposition 2.1.19, this last quantity can be made arbitrarily small, uniformly for $a \in G$, by making K sufficiently small.

(c) Let $\phi \in \mathcal{S}(G)$. We have

$$u(\check{\phi}) = (u * \phi)(0) = \lim (u * \psi_j * \phi)(0) = \lim (u * \psi_j)(\check{\phi}).$$

$\square$

Lemma 2.1.29. *If $u_j \to u$ in $\mathcal{S}'(G)$ and $\phi_j \to \phi$ in $\mathcal{S}(G)$, then $u_j(\phi_j) \to u(\phi)$.*

Proof. Since the sequence $\{u_j(\psi)\}$ is convergent and hence bounded for all $\psi \in \mathcal{S}(G)$, the collection $\{u_j\}$ of continuous linear functionals on $\mathcal{S}(G)$ is equicontinuous by the uniform boundedness principle (the Banach-Steinhaus theorem). In particular, there exist uniform constants C and N, independent of j, such that

$$|u_j(\psi)| \leq C \sum_{|\alpha| \leq N} \sum_{|\beta| \leq N} \sum_{|\gamma| \leq N} \|\psi\|_{\alpha,\beta,\gamma} \qquad (\psi \in \mathcal{S}(G)).$$

Setting $\psi = \phi_j - \phi$, we see that $u_j(\phi_j - \phi) \to 0$. $\square$

The following density result is of fundamental technical importance.

Proposition 2.1.30. *Every tempered distribution on G is the weak* limit of a sequence of functions in $C_c^\infty(G)$.*

Proof. The following proof is inspired by [Hör90, Theorem 4.1.5]. Let $u \in \mathcal{S}'(G)$. Let $\chi_j = \psi_{1/j,j}$, where $\psi_{1/j,j}$ is as in Proposition 2.1.7. Let $\{\psi_j\}$ be as in Proposition 2.1.28. Let $u_j = (\chi_j u) * \psi_j$. Then $u_j \in C_c^\infty(G)$. We claim that $u_j \to u$ in $\mathcal{S}'(G)$.

Let $\phi \in \mathcal{S}(G)$. We have

$$u_j(\phi) = ((\chi_j u) * \psi_j)(\phi) = ((\chi_j u) * \psi_j * \check{\phi})(0) = (\chi_j u)(\check{\psi}_j * \phi).$$

By Proposition 2.1.7, $\chi_j u \to u$ in $\mathcal{S}'(G)$. By Proposition 2.1.28, $\check{\psi}_j * \phi \to \phi$ in $\mathcal{S}(G)$. The claim follows from Lemma 2.1.29. $\square$

The following technical result is useful for determining that a tempered distribution is defined via integration. This result will be used in the proofs of Example 2.4.5 and Proposition 2.6.21.

Proposition 2.1.31. *Let $\{V_j\}$ be a sequence of precompact open sets in G whose union is G. Let $\{\psi_j\}$ be a sequence in $C_c^\infty(G)$ with $\psi_j = 1$ on $\overline{V}_j$. Let $u \in \mathcal{S}'(G)$. Suppose that $\psi_j u$ is a complex measurable function for all j. There exists a locally integrable $f : G \to \mathbb{C}$ such that $u(\phi) = (f, \phi)$ for all $\phi \in C_c^\infty(G)$. In particular, ϕu and ϕf coincide as tempered distributions for all $\phi \in C_c^\infty(G)$.*

Proof. If $\phi \in C_c^\infty(V_j \cap V_k)$, then

$$(\psi_j u)(\phi) = u(\psi_j \phi) = u(\phi) = u(\psi_k \phi) = (\psi_k u)(\phi).$$

It follows that $\psi_j u = \psi_k u$ almost everywhere on $V_j \cap V_k$. Therefore, there exists a complex measurable $f : G \to \mathbb{C}$ such that $f = \psi_j u$ almost everywhere on V_j. Let $\{g_j\}$ be a C^∞ partition of unity on G subordinate to $\{V_j\}$. We have

$$\begin{aligned}
u(\phi) &= u(\sum g_j\phi) \\
&= \sum u(g_j\phi) \\
&= \sum u(\psi_j g_j \phi) \\
&= \sum (\psi_j u)(g_j\phi) \\
&= \sum (f, g_j\phi) \\
&= (f, \sum g_j\phi) \\
&= (f, \phi) \qquad (\phi \in C_c^\infty(G)).
\end{aligned}$$

Since ϕ is compactly supported, the sums run over a fixed finite index set. $\square$

2.2 Tensor Products

Let $f_1 \in C(G_1)$ and $f_2 \in C(G_2)$. We define $(f_1 \otimes f_2)(a_1, a_2) = f_1(a_1)f_2(a_2)$ for $a_1 \in G_1$ and $a_2 \in G_2$.

Proposition 2.2.1. *The bilinear pairing $(\phi_1, \phi_2) \to \phi_1 \otimes \phi_2$ from $\mathcal{S}(G_1) \times \mathcal{S}(G_2)$ to $\mathcal{S}(G_1 \times G_2)$ is continuous.*

Lemma 2.2.2. *Let $u \in \mathcal{S}'(G_1 \times G_2)$. Let V_1 and V_2 be open subsets of G_1 and G_2, respectively. If $u(\psi_1 \otimes \psi_2) = 0$ for all $\psi_1 \in C_c^\infty(V_1)$ and $\psi_2 \in C_c^\infty(V_2)$, then $u = 0$ on $V_1 \times V_2$.*

Proof. Let $\{\psi_j^1\}$ and $\{\psi_j^2\}$ be sequences in $C_c^\infty(G_1)$ and $C_c^\infty(G_2)$, respectively, as in Proposition 2.1.28. Let $\psi_j = \psi_j^1 \otimes \psi_j^2$. Then $\{\psi_j\}$ satisfies the hypotheses of Proposition 2.1.28 on $G_1 \times G_2$, so $u * \psi_j \to u$ in $\mathcal{S}'(G_1 \times G_2)$. However,

$$(u * \psi_j)(a_1, a_2) = u(T_{(a_1,a_2)}\check{\psi}_j) = u(T_{a_1}\check{\psi}_j^1 \otimes T_{a_2}\check{\psi}_j^2) = 0 \qquad (a_1 \in V_1,\ a_2 \in V_2)$$

for j sufficiently large. $\square$

Lemma 2.2.3. *Let $u \in \mathcal{S}'(G_1)$ and $\phi \in \mathcal{S}(G_1 \times G_2)$. The map $a_2 \to u(\phi(\cdot, a_2))$ on G_2 is in $\mathcal{S}(G_2)$, and $\partial_{G_2}^\alpha u(\phi(\cdot, a_2)) = u(\partial_{G_2}^\alpha \phi(\cdot, a_2))$.*

Proposition 2.2.4. *Let $u_1 \in \mathcal{S}'(G_1)$ and $u_2 \in \mathcal{S}'(G_2)$. There exists a unique $u = u_1 \otimes u_2 \in \mathcal{S}'(G_1 \times G_2)$ such that $u(\psi_1 \otimes \psi_2) = u_1(\psi_1)u_2(\psi_2)$ for all $\psi_1 \in C_c^\infty(G_1)$ and $\psi_2 \in C_c^\infty(G_2)$. We have $u(\phi) = u_1(u_2(\phi)) = u_2(u_1(\phi))$ for all $\phi \in \mathcal{S}(G_1 \times G_2)$. Moreover,* $\operatorname{supp} u = \operatorname{supp} u_1 \times \operatorname{supp} u_2$. *Let C_1 and N_1 be such that*

$$|u_1(\phi_1)| \leq C_1 \sum_{|\alpha_1| \leq N_1} \sum_{|\beta_1| \leq N_1} \sum_{|\gamma_1| \leq N_1} \|\phi_1\|_{\alpha_1,\beta_1,\gamma_1} \qquad (\phi_1 \in \mathcal{S}(G_1)).$$

Let C_2 and N_2 be such that

$$|u_2(\phi_2)| \leq C_2 \sum_{|\alpha_2| \leq N_2} \sum_{|\beta_2| \leq N_2} \sum_{|\gamma_2| \leq N_2} \|\phi_2\|_{\alpha_2,\beta_2,\gamma_2} \qquad (\phi_2 \in \mathcal{S}(G_2)).$$

Then

$$|u(\phi)| \leq C_1 C_2 \sum_{\substack{|\alpha_1| \leq N_1 \\ |\alpha_2| \leq N_2}} \sum_{\substack{|\beta_1| \leq N_1 \\ |\beta_2| \leq N_2}} \sum_{\substack{|\gamma_1| \leq N_1 \\ |\gamma_2| \leq N_2}} \|\phi\|_{(\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2)} \qquad (\phi \in \mathcal{S}(G_1 \times G_2)).$$

Differentiable Stone-Weierstrass Theorem

We now state a differentiable version of the Stone-Weierstrass theorem suitably phrased for our purposes.

Proposition 2.2.5 (Nachbin [Nac49])**.** *Let $\mathcal{A}$ be a (not necessarily unital) subalgebra of $C^\infty(G)$ with the following properties:*

(a) $\mathcal{A}$ is closed under complex conjugation.

(b) For every $a \in G$, there exists $f \in \mathcal{A}$ with $f(a) \neq 0$.

(c) For every $a, b \in G$ with $a \neq b$, there exists $f \in \mathcal{A}$ with $f(a) \neq f(b)$.

(d) For every $a \in G$ and direction e along $\mathbb{R}^d \times \mathbb{T}^{d'}$, there exists $f \in \mathcal{A}$ with $\partial_e(a) \neq 0$.

In this case, for every $g \in C^\infty(G)$, $\epsilon > 0$, compact $K \subseteq G$, and N, there exists $f \in \mathcal{A}$ such that

$$\sum_{|\alpha| \leq N} \sup_{a \in K} |\partial^\alpha (g - f)(a)| < \epsilon.$$

Compare the next result to Proposition 3.2.5.

Corollary 2.2.6. *Let $\mathcal{A}$ be the linear span of*

$$\{\phi_1 \otimes \phi_2 : \phi_1 \in C_c^\infty(G_1),\ \phi_2 \in C_c^\infty(G_2)\}.$$

Then $\mathcal{A}$ is dense in $\mathcal{S}(G_1 \times G_2)$.

Proof. Let $\phi \in \mathcal{S}(G_1 \times G_2)$. Since $C_c^\infty(G_1 \times G_2)$ is dense in $\mathcal{S}(G_1 \times G_2)$, we can choose $\psi \in C_c^\infty(G_1 \times G_2)$ arbitrarily close to ϕ in the topology of $\mathcal{S}(G_1 \times G_2)$. Let V_1 and V_2

be precompact open subsets of G_1 and G_2, respectively, such that $\operatorname{supp}\psi \subseteq V_1 \times V_2$. Let $\mathcal{A}_{V_1,V_2}$ be the linear span of

$$\{\psi_1 \otimes \psi_2 : \psi_1 \in C_c^\infty(V_1),\ \psi_2 \in C_c^\infty(V_2)\}.$$

It is clear that $\mathcal{A}_{V_1,V_2}$ is a (not necessarily unital) subalgebra of $C^\infty(G_1 \times G_2)$ satisfying the hypotheses of Proposition 2.2.5. It follows that we can choose $f \in \mathcal{A}_{V_1,V_2}$ arbitrarily close to ψ in the topology of $\mathcal{S}(G_1 \times G_2)$. □

The following result will be used in the proof of Proposition 2.3.34. Note the similarity between the proofs of this result and Proposition 3.2.7.

Proposition 2.2.7. *If $u_{1,j} \to u_1$ in $\mathcal{S}'(G_1)$ and $u_{2,j} \to u_2$ in $\mathcal{S}'(G_2)$, then $u_{1,j} \otimes u_{2,j} \to u_1 \otimes u_2$ in $\mathcal{S}'(G_1 \times G_2)$.*

Proof. In view of the identity

$$\begin{aligned} u_{1,j} \otimes u_{2,j} - u_1 \otimes u_2 = (u_{1,j} - u_1) \otimes (u_{2,j} - u_2) \cdots \\ + u_1 \otimes (u_{2,j} - u_2) + (u_{1,j} - u_1) \otimes u_2, \end{aligned}$$

it suffices to consider the cases $u_1 = u_2 = 0$, $u_1 = 0$, and $u_2 = 0$.

We can argue as in the proof of Lemma 2.1.29 and appeal to the last part of Proposition 2.2.4 to conclude that there exist uniform constants C and N, independent of j, such that

$$|(u_{1,j} \otimes u_{2,j})(\psi)| \leq C \sum_{\substack{|\alpha_1| \leq N \\ |\alpha_2| \leq N}} \sum_{\substack{|\beta_1| \leq N \\ |\beta_2| \leq N}} \sum_{\substack{|\gamma_1| \leq N \\ |\gamma_2| \leq N}} \|\psi\|_{(\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2)} \qquad (\psi \in \mathcal{S}(G_1 \times G_2)).$$

Let $\phi \in \mathcal{S}(G_1 \times G_2)$. Let $f = \sum_{k=1}^n \phi_{1,k} \otimes \phi_{2,k}$ for some $\phi_{1,k} \in C_c^\infty(G_1)$ and

$\phi_{2,k} \in C_c^\infty(G_2)$. We have

$$\begin{aligned}
|(u_{1,j} \otimes u_{2,j})(\phi)| &\le |(u_{1,j} \otimes u_{2,j})(\phi - f)| + |(u_{1,j} \otimes u_{2,j})(f)| \\
&\le C \sum_{\substack{|\alpha_1| \le N \\ |\alpha_2| \le N}} \sum_{\substack{|\beta_1| \le N \\ |\beta_2| \le N}} \sum_{\substack{|\gamma_1| \le N \\ |\gamma_2| \le N}} \|\phi - f\|_{(\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2)} \cdots \\
&\quad + \sum_{k=1}^{n} |u_{1,j}(\phi_{1,k}) u_{2,j}(\phi_{2,k})|.
\end{aligned}$$

Let $\epsilon > 0$. By Corollary 2.2.6, we can choose f so that the first term is less than $\epsilon/2$. Since f is now fixed, the second term can be made less than $\epsilon/2$ by choosing j sufficiently large. □

2.3 The Short-Time Fourier Transform

Let X and Y be complex vector spaces. Let $\langle \cdot, \cdot \rangle$ be a pairing on $X \times Y$ which is linear on X and conjugate-linear on Y. Suppose that there exist translation operators T_a ($a \in G$) and modulation operators $M_{\hat{a}}$ ($\hat{a} \in \widehat{G}$) on X and Y satisfying the canonical commutation relations $T_a M_{\hat{a}} = (-a, \hat{a}) M_{\hat{a}} T_a$. Suppose that $\langle T_a f, g \rangle = \langle f, T_{-a} g \rangle$ and $\langle M_{\hat{a}} f, g \rangle = \langle f, M_{-\hat{a}} g \rangle$ for all $f \in X$ and $g \in Y$. We define the short-time Fourier transform (STFT) as $V_g f(a, \hat{a}) = \langle f, M_{\hat{a}} T_a g \rangle$ for $f \in X$ and $g \in Y$. The prototypical examples for (X, Y) are $(\mathcal{S}'(G), \mathcal{S}(G))$ and $(L^2(G), L^2(G))$.

The following identity is called the covariance property of the STFT.

Proposition 2.3.1. *Let $f \in X$ and $g \in Y$. Then*

$$V_g M_{\hat{b}} T_b f(a, \hat{a}) = (-b, \hat{a} - \hat{b}) V_g f(a - b, \hat{a} - \hat{b}).$$

We shall need the following more general version of the covariance property.

Proposition 2.3.2. *Let $f \in X$ and $g \in Y$. Then*

$$V_{M_{\hat{c}}T_c g} M_{\hat{b}} T_b f(a, \hat{a}) = (-b, \hat{a} - \hat{b} + \hat{c})(a, \hat{c}) V_g f(a - b + c, \hat{a} - \hat{b} + \hat{c}).$$

Corollary 2.3.3. *Let $f \in X$ and $g \in Y$. Then*

$$V_{M_{\hat{c}}T_c g} M_{\hat{b}} T_b f = (-b, -\hat{b} + \hat{c}) M_{(\hat{c}, -b)} T_{(b-c, \hat{b}-\hat{c})} V_g f$$

and

$$V_{T_c M_{\hat{c}} g} T_b M_{\hat{b}} f = (-b + c, \hat{c}) M_{(\hat{c}, -b)} T_{(b-c, \hat{b}-\hat{c})} V_g f.$$

Corollary 2.3.4. *Let $f \in X$ and $g \in Y$. Then*

$$T_{(b,\hat{b})} V_g f = V_{T_{-b} g} M_{\hat{b}} f$$

and

$$M_{(\hat{b},b)} V_g f = V_{M_{\hat{b}} T_{-b} g} M_{\hat{b}} T_{-b} f.$$

Proposition 2.3.5. *Let $(X, Y) = (\mathcal{S}'(G), \mathcal{S}(G))$ or $(X, Y) = (L^2(G), L^2(G))$. Let $f \in X$ and $g \in Y$.*

(a) $V_g \overline{f}(a, \hat{a}) = \overline{V_{\overline{g}} f(a, -\hat{a})}$.

(b) $V_g \check{f}(a, \hat{a}) = V_{\check{g}} f(-a, -\hat{a})$.

L^2 Theory

Proposition 2.3.6. *Let $f, g \in L^2(G)$. Then $V_g f$ is uniformly continuous and $\|V_g f\|_\infty \leq \|f\|_2 \|g\|_2$.*

The following identity is called the fundamental identity of time-frequency analysis.

Proposition 2.3.7. *Let $f, g \in L^2(G)$. Then $V_g f(a, \hat{a}) = (-a, \hat{a}) V_{\hat{g}} \hat{f}(\hat{a}, -a)$.*

We shall need the following more general version of the fundamental identity involving partial Fourier transforms.

Proposition 2.3.8. *Let $f, g \in L^2(G_1 \times G_2)$. Then*

$$V_g f(a_1, a_2, \hat{a}_1, \hat{a}_2) = (-a_1, \hat{a}_1) V_{\mathcal{F}_1 g} \mathcal{F}_1 f(\hat{a}_1, a_2, -a_1, \hat{a}_2).$$

We state some alternate formulas for the STFT.

Proposition 2.3.9. *Let $f, g \in L^2(G)$. Then*

$$V_g f(a, \hat{a}) = (f T_a \overline{g})\hat{\ }(\hat{a}) = (-a, \hat{a})(f * M_{\hat{a}} g^*)(a) = (-a, \hat{a})(\hat{f} T_{\hat{a}} \overline{\hat{g}})\hat{\ }(-a).$$

We shall obtain yet one more description of the STFT which is necessary for establishing certain results.

Lemma 2.3.10. *The sesquilinear pairing $(f, g) \to V_g f$ from $L^2(G) \times L^2(G)$ to $L^\infty(G \times \widehat{G})$ is continuous.*

Lemma 2.3.11. *The sesquilinear pairing $(f, g) \to f \otimes \overline{g}$ from $L^2(G) \times L^2(G)$ to $L^2(G \times G)$ is continuous.*

Let F be a complex function on $G \times G$. We define the asymmetric coordinate transform as $\mathcal{T}_G F(a, t) = F(t, t - a)$ for $a, t \in G$. Note that $\mathcal{T}_G^{-1} F(a, t) = F(a - t, a)$. We define $\mathcal{I}_G F(a, b) = F(b, a)$ for $a, b \in G$.

Proposition 2.3.12. *Let $f, g \in L^2(G)$. Then $V_g f = \mathcal{F}_2 \mathcal{T}_G(f \otimes \overline{g})$ almost everywhere.*

Proof. The equality is obtained by direct calculation when $f, g \in \mathcal{S}(G)$. By Lemma 2.3.10, the pairing on the left is continuous into $\mathcal{S}'(G \times \widehat{G})$. By Lemma 2.3.11 and the Plancherel theorem, the pairing on the right is continuous into $\mathcal{S}'(G \times \widehat{G})$. Since $\mathcal{S}(G)$ is dense in $L^2(G)$, the result follows. $\square$

Corollary 2.3.13. *The sesquilinear pairing $(f, g) \to V_g f$ from $\mathcal{S}(G) \times \mathcal{S}(G)$ to $\mathcal{S}(G \times \widehat{G})$ is continuous.*

Corollary 2.3.14. *Let $f_1, f_2, g_1, g_2 \in L^2(G)$. Then $\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$.*

Corollary 2.3.15 (STFT Inversion Theorem)**.** *Let $f, g, h \in L^2(G)$. Suppose that $\langle h, g \rangle \neq 0$. Then*

$$f = \frac{1}{\langle h, g \rangle} \int_{G \times \widehat{G}} V_g f(a, \hat{a}) M_{\hat{a}} T_a h \, da \, d\hat{a},$$

where the right hand side is an $L^2(G)$ valued integral.

Proof. Let $\phi \in L^2(G)$. We have

$$\begin{aligned}
\frac{1}{\langle h, g \rangle} \int_{G \times \widehat{G}} V_g f(a, \hat{a}) \langle M_{\hat{a}} T_a h, \phi \rangle \, da \, d\hat{a} &= \frac{1}{\langle h, g \rangle} \int_{G \times \widehat{G}} V_g f(a, \hat{a}) \overline{V_h \phi(a, \hat{a})} \, da \, d\hat{a} \\
&= \frac{1}{\langle h, g \rangle} \langle V_g f, V_h \phi \rangle \\
&= \frac{1}{\langle h, g \rangle} \langle f, \phi \rangle \overline{\langle g, h \rangle} \\
&= \langle f, \phi \rangle.
\end{aligned}$$

$\square$

Corollary 2.3.16. *Let $g \in L^2(G)$. Then V_g is a bounded linear map from $L^2(G)$ to*

$L^2(G \times \widehat{G})$. *We have*

$$V_g^* F = \int_{G\times\widehat{G}} F(a,\hat{a}) M_{\hat{a}} T_a g \, da \, d\hat{a}$$

for all $F \in L^2(G \times \widehat{G})$, *where the right hand side is an* $L^2(G)$ *valued integral.*

Corollary 2.3.17. *Let* $g, h \in L^2(G)$. *Suppose that* $\langle h, g\rangle \neq 0$. *Then*

$$\frac{1}{\langle h, g\rangle} V_h^* V_g = I.$$

Distributional Theory

For the rest of this section, $a = (x, \cdot, \iota, \cdot) \in G$ and $\hat{a} = (\omega, \xi, \cdot, \cdot) \in \widehat{G}$.

Proposition 2.3.18 ([Grö01, Theorem 11.2.3])**.** *Let* $f \in \mathcal{S}'(G)$ *and* $g \in \mathcal{S}(G)$. *Then* $V_g f$ *is continuous. Moreover,* $|V_g f(a,\hat{a})| \leq C(1+|x|+|\omega|+|\iota|+|\xi|)^N$. *In particular,* $V_g f$ *is a tempered distribution on* $G \times \widehat{G}$. *The constants* C *and* N *can be chosen uniformly for* f *in a pointwise bounded collection of tempered distributions on* G.

Proposition 2.3.19. *Let* $f \in \mathcal{S}'(G_1 \times G_2)$ *and* $g \in \mathcal{S}(G_1 \times G_2)$. *Then*

$$V_g f(a_1, a_2, \hat{a}_1, \hat{a}_2) = (-a_1, \hat{a}_1) V_{\mathcal{F}_1 g} \mathcal{F}_1 f(\hat{a}_1, a_2, -a_1, \hat{a}_2).$$

Proposition 2.3.20. *Let* $f \in \mathcal{S}'(G)$ *and* $g \in \mathcal{S}(G)$. *Then*

$$V_g f(a,\hat{a}) = (f T_a \overline{g})\widehat{\ }(\hat{a}) = (-a,\hat{a})(f * M_{\hat{a}} g^*)(a) = (-a,\hat{a})(\hat{f} T_{\hat{a}} \overline{\hat{g}})\widehat{\ }(-a).$$

We shall obtain the distributional version of Proposition 2.3.12.

Lemma 2.3.21. *Let* $g \in \mathcal{S}(G)$. *If* $f_j \to f$ *in* $\mathcal{S}'(G)$, *then* $V_g f_j \to V_g f$ *in* $\mathcal{S}'(G \times \widehat{G})$.

Proof. We can assume without loss of generality that $f = 0$. Let $\phi \in \mathcal{S}(G \times \widehat{G})$. Then

$$(V_g f_j, \phi) = \int_{G\times\widehat{G}} \langle f_j, M_{\hat{a}} T_a g\rangle \phi(a, \hat{a})\, da\, d\hat{a}.$$

It is clear that the integrand goes to 0. By Proposition 2.3.18, the dominated convergence theorem applies. □

Lemma 2.3.22. *Let $g \in \mathcal{S}(G)$. If $f_j \to f$ in $\mathcal{S}'(G)$, then $f_j \otimes \overline{g} \to f \otimes \overline{g}$ in $\mathcal{S}'(G\times G)$.*

Proposition 2.3.23. *Let $f \in \mathcal{S}'(G)$ and $g \in \mathcal{S}(G)$. Then $V_g f = \mathcal{F}_2 \mathcal{T}_G(f \otimes \overline{g})$.*

Proof. The equality is true when $f \in \mathcal{S}(G)$ as already established in Proposition 2.3.12. The general case follows by taking a sequence in $\mathcal{S}(G)$ converging in $\mathcal{S}'(G)$ to f (Proposition 2.1.30), and then appealing to Lemma 2.3.21 and Lemma 2.3.22. □

Corollary 2.3.24. *Let $f_1 \in \mathcal{S}'(G)$ and $f_2, g_1, g_2 \in \mathcal{S}(G)$. Then $\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}$.*

Corollary 2.3.25 (STFT Inversion Theorem)**.** *Let $f \in \mathcal{S}'(G)$ and $g, h \in \mathcal{S}(G)$. Suppose that $\langle h, g\rangle \neq 0$. Then*

$$f = \frac{1}{\langle h, g\rangle} \int_{G\times\widehat{G}} V_g f(a, \hat{a}) M_{\hat{a}} T_a h\, da\, d\hat{a},$$

where the right hand side is an $\mathcal{S}'(G)$ valued integral.

The next result gives a way to manufacture Schwartz functions.

Proposition 2.3.26. *Let $g \in \mathcal{S}(G)$. Let F be a complex measurable function on $G \times \widehat{G}$ such that $|F(a, \hat{a})| \le C_k(1 + |x| + |\omega| + |\iota| + |\xi|)^{-k}$ for all $k \ge 0$. The map*

$$\varphi : t \to \int_{G\times\widehat{G}} F(a, \hat{a}) M_{\hat{a}} T_a g(t)\, da\, d\hat{a}$$

on G is in $\mathcal{S}(G)$. We have the estimate

$$\|K^\gamma \partial^\alpha X^\beta \varphi\|_\infty \leq C_{\alpha,\beta,\gamma} \int_{G\times\widehat{G}} |F(a,\hat{a})|(1+|x|+|\omega|+|\iota|+|\xi|)^{N_{\alpha,\beta,\gamma}}\, da\, d\hat{a}.$$

Moreover,

$$\langle f, \varphi\rangle = \int_{G\times\widehat{G}} \overline{F(a,\hat{a})} V_g f(a,\hat{a})\, da\, d\hat{a}$$

for all $f \in \mathcal{S}'(G)$. In other words,

$$\varphi = \int_{G\times\widehat{G}} F(a,\hat{a}) M_{\hat{a}} T_a g\, da\, d\hat{a},$$

where the right hand side is an $\mathcal{S}(G)$ valued integral.

Proof. See [Grö01, Proposition 11.2.4] for the proof of the estimate. We indicate the proof of the last assertion. The equality is obtained by direct calculation when $f \in \mathcal{S}(G)$. The general case follows from Proposition 2.1.30, Proposition 2.3.18, and Lemma 2.3.21. □

The following result characterizes Schwartz functions; see [Grö01, Theorem 11.2.5].

Corollary 2.3.27. *Let $g \in \mathcal{S}(G)$ be nonzero. Let $f \in \mathcal{S}'(G)$. The following statements are equivalent:*

(a) $f \in \mathcal{S}(G)$.

(b) $V_g f \in \mathcal{S}(G \times \widehat{G})$.

(c) $|V_g f(a,\hat{a})| \leq C_k(1+|x|+|\omega|+|\iota|+|\xi|)^{-k}$ *for all* $k \geq 0$.

The next result gives a way to manufacture tempered distributions.

Proposition 2.3.28. *Let $g \in \mathcal{S}(G)$. Let F be a complex measurable function on $G \times \widehat{G}$ such that $|F(a, \hat{a})| \leq C(1 + |x| + |\omega| + |\iota| + |\xi|)^N$. The linear map*

$$f : \overline{\phi} \to \int_{G\times\widehat{G}} F(a, \hat{a})\langle M_{\hat{a}} T_a g, \phi\rangle \, da \, d\hat{a}$$

on $\mathcal{S}(G)$ is continuous. In other words, $f \in \mathcal{S}'(G)$, and

$$f = \int_{G\times\widehat{G}} F(a, \hat{a}) M_{\hat{a}} T_a g \, da \, d\hat{a},$$

where the right hand side is an $\mathcal{S}'(G)$ valued integral. For $h \in \mathcal{S}(G)$, we have the pointwise estimate $|V_h f| \leq |F| * |V_h g|$.

Proof. Choose M large enough so that $F(a, \hat{a})(1+|x|+|\omega|+|\iota|+|\xi|)^{-M}$ is integrable. Suppose that $\phi_j \to 0$ in $\mathcal{S}(G)$. By Corollary 2.3.13, $V_g \phi_j \to 0$ in $\mathcal{S}(G \times \widehat{G})$. In particular, $|V_g \phi_j(a, \hat{a})| \leq (1+|x|+|\omega|+|\iota|+|\xi|)^{-M}$ for j sufficiently large. It follows from the dominated convergence theorem that $\langle f, \phi_j\rangle \to 0$. The pointwise estimate is obtained by direct calculation. □

Additional Useful Properties and Formulas

The following result shows that the STFT preserves tensor products.

Proposition 2.3.29. *Let $f_1 \in \mathcal{S}'(G_1)$, $f_2 \in \mathcal{S}'(G_2)$, $g_1 \in \mathcal{S}(G_1)$, and $g_2 \in \mathcal{S}(G_2)$. Then*

$$V_{g_1 \otimes g_2}(f_1 \otimes f_2) = (V_{g_1} f_1) \otimes (V_{g_2} f_2).$$

Proof.

$$\begin{aligned} V_{g_1 \otimes g_2}(f_1 \otimes f_2)(a_1, a_2, \hat{a}_1, \hat{a}_2) &= \langle f_1 \otimes f_2, M_{(\hat{a}_1, \hat{a}_2)} T_{(a_1, a_2)}(g_1 \otimes g_2) \rangle \\ &= \langle f_1 \otimes f_2, (M_{\hat{a}_1} T_{a_1} g_1) \otimes (M_{\hat{a}_2} T_{a_2} g_2) \rangle \\ &= \langle f_1, M_{\hat{a}_1} T_{a_1} g_1 \rangle \langle f_2, M_{\hat{a}_2} T_{a_2} g_2 \rangle \\ &= V_{g_1} f_1(a_1, \hat{a}_1) V_{g_2} f_2(a_2, \hat{a}_2). \end{aligned}$$

□

The following three formulas can be found in [CG03]. Proposition 2.3.31 will be used in the proof of Proposition 2.6.28. Proposition 2.3.32 will be used in the proof of Proposition 3.1.13.

Proposition 2.3.30. *Let $f_1 \in \mathcal{S}'(G)$ and $f_2, g_1, g_2 \in \mathcal{S}(G)$. Then*

$$(V_{g_1} f_1 \overline{V_{g_2} f_2})\widehat{\ }(\hat{b}, b) = (V_{f_2} f_1 \overline{V_{g_2} g_1})(-b, \hat{b}).$$

Proof.

$$\begin{aligned} (V_{g_1} f_1 \overline{V_{g_2} f_2})\widehat{\ }(\hat{b}, b) &= \langle V_{g_1} f_1, M_{(\hat{b}, b)} V_{g_2} f_2 \rangle \\ &= \langle V_{g_1} f_1, V_{M_{\hat{b}} T_{-b} g_2} M_{\hat{b}} T_{-b} f_2 \rangle \\ &= \langle f_1, M_{\hat{b}} T_{-b} f_2 \rangle \overline{\langle g_1, M_{\hat{b}} T_{-b} g_2 \rangle} \\ &= (V_{f_2} f_1 \overline{V_{g_2} g_1})(-b, \hat{b}). \end{aligned}$$

□

Proposition 2.3.31. *Let $f \in \mathcal{S}'(G)$ and $g, \varphi \in \mathcal{S}(G)$. Then*

$$V_{V_\varphi \varphi} V_g f(a, \hat{a}, \hat{b}, b) = (-b, \hat{a}) V_\varphi f(-b, \hat{a} + \hat{b}) \overline{V_\varphi g(-a - b, \hat{b})}.$$

Proof.

$$\begin{aligned}
V_{V_\varphi \varphi} V_g f(a, \hat{a}, \hat{b}, b) &= \langle V_g f, M_{(\hat{b}, b)} T_{(a, \hat{a})} V_\varphi \varphi \rangle \\
&= \langle V_g f, M_{(\hat{b}, b)} V_{T_{-a}\varphi} M_{\hat{a}} \varphi \rangle \\
&= (V_g f \overline{V_{T_{-a}\varphi} M_{\hat{a}} \varphi})\widehat{\ }(\hat{b}, b) \\
&= V_{M_{\hat{a}}\varphi} f(-b, \hat{b}) \overline{V_{T_{-a}\varphi} g(-b, \hat{b})} \\
&= (-b, \hat{a}) V_\varphi f(-b, \hat{a} + \hat{b}) \overline{V_\varphi g(-a - b, \hat{b})}.
\end{aligned}$$

□

Proposition 2.3.32. *Let $f \in \mathcal{S}'(G)$ and $g, \varphi \in \mathcal{S}(G)$. Then*

$$V_{\varphi * \varphi}(f * g)(a, \hat{a}) = (-a, \hat{a})(f * M_{\hat{a}} \varphi^* * g * M_{\hat{a}} \varphi^*)(a).$$

Proof.

$$\begin{aligned}
V_{\varphi * \varphi}(f * g)(a, \hat{a}) &= (-a, \hat{a})(f * g * M_{\hat{a}}(\varphi * \varphi)^*)(a) \\
&= (-a, \hat{a})(f * g * M_{\hat{a}}(\varphi^* * \varphi^*))(a) \\
&= (-a, \hat{a})(f * g * M_{\hat{a}} \varphi^* * M_{\hat{a}} \varphi^*)(a) \\
&= (-a, \hat{a})(f * M_{\hat{a}} \varphi^* * g * M_{\hat{a}} \varphi^*)(a).
\end{aligned}$$

□

We now expand the scope of the STFT. We define $V_g f = \mathcal{F}_2 \mathcal{T}_G (f \otimes \overline{g})$ for

$f, g \in \mathcal{S}'(G)$.

Lemma 2.3.33. *Let $f, g \in L^2(G)$. Then $V_g f(a, \hat{a}) = (-a, \hat{a})\overline{V_f g(-a, -\hat{a})}$.*

Proof. We have

$$
\begin{aligned}
V_g f(a, \hat{a}) &= \langle f, M_{\hat{a}} T_a g\rangle \\
&= \langle T_{-a} M_{-\hat{a}} f, g\rangle \\
&= (-a, \hat{a})\langle M_{-\hat{a}} T_{-a} f, g\rangle \\
&= (-a, \hat{a})\overline{V_f g(-a, -\hat{a})}.
\end{aligned}
$$

□

Note the subtlety in the proof of the following seemingly obvious result. This result will be important when we study the spreading representation in Section 3.5.

Proposition 2.3.34. *Let $\chi(a, \hat{a}) = (a, \hat{a})$ for $a \in G$ and $\hat{a} \in \widehat{G}$. Let $f, g \in \mathcal{S}'(G)$. Then*

$$\chi V_g f = \widecheck{\overline{V_f g}}.$$

Proof. The result holds when $f, g \in \mathcal{S}(G)$ by Lemma 2.3.33. The general case follows by taking sequences in $\mathcal{S}(G)$ converging in $\mathcal{S}'(G)$ to f and g, and then appealing to Proposition 2.2.7. □

2.4 Modulation Spaces

The theory of modulation spaces in full generality depends on the theory of mixed-norm L^p-spaces. Much of the theory of mixed-norm L^p-spaces parallels the theory of ordinary L^p-spaces. We refer to [BP61] for an extensive treatment and for notation. Here, we shall content ourselves with the definition of the mixed-norm. Let $(X_1,\mu_1),\ldots,(X_n,\mu_n)$ be σ-finite measure spaces. Let $1 \le p_1,\ldots,p_n \le \infty$ and $p=(p_1,\ldots,p_n)$. Let $f : X_1\times\cdots\times X_n \to [0,\infty]$ be measurable. We define $\|f\|_p = \|\|\|\|f\|_{L^{p_1}(X_1)}\|_{L^{p_2}(X_2)}\cdots\|_{L^{p_n}(X_n)}$. For example, if $p_1,\ldots,p_n<\infty$, then

$$\|f\|_p = \left(\int_{X_n}\cdots\left(\int_{X_2}\left(\int_{X_1} f^{p_1}\,d\mu_1\right)^{p_2/p_1} d\mu_2\right)^{p_3/p_2}\cdots d\mu_n\right)^{1/p_n}.$$

In the sequel, we shall not need the full scope of the theory of modulation spaces. Nevertheless, our account will be as general as possible without obscuring the essential ideas.

Definition 2.4.1. A submultiplicative weight function on G is a continuous function $v : G \to (0,\infty)$ such that $v(a_1+a_2) \le v(a_1)v(a_2)$ for all $a_1,a_2 \in G$. A v-moderate weight function on G is a continuous function $m : G \to (0,\infty)$ such that $m(a_1+a_2) \le Cv(a_1)m(a_2)$ for all $a_1,a_2\in G$. We shall consider only weight functions with the property that both the weight function and its reciprocal have polynomial growth; this restriction allows us to stay within the framework of Schwartz functions and tempered distributions. In much of the sequel, we shall dispense with weight functions altogether in order to keep the discussion focused on the applications that we have in mind.

Let $g \in \mathcal{S}(G)$ be nonzero. Let v and m be weight functions on $G\times\widehat{G}$ as in Definition 2.4.1. Let $1\le p,q\le\infty$. Here, p and q are tuples with as many components as the number of factors of G; how one chooses to factorize G is flexible. For the rest of this section, g, p, and q will be fixed unless otherwise specified. For example, certain arguments will require p and q to be numbers. (In this case, G will have only one factor, namely, G.)

We define $\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}}$ for $f \in \mathcal{S}'(G)$. Let $M_m^{p,q}(G)$ be the set of all $f \in \mathcal{S}'(G)$ such that $\|f\|_{M_m^{p,q}} < \infty$. We establish below that the definition of $M_m^{p,q}(G)$ is independent of the chosen window function g up to norm equivalence.

Remark. In the definition of the modulation space norm, we take the p-norm on G ("time" variable) followed by the q-norm on $\widehat{G}$ ("frequency" variable). For example, if p and q are finite numbers, then

$$\|f\|_{M_m^{p,q}} = \left(\int_{\widehat{G}} \left(\int_G |V_g f(a,\hat{a}) m(a,\hat{a})|^p \, da\right)^{q/p} d\hat{a}\right)^{1/q}.$$

We shall denote by $W_m^{p,q}$ the norm where we take the p-norm on $\widehat{G}$ ("frequency" variable) followed by the q-norm on G ("time" variable). All of the results in this section involving $M_m^{p,q}$ hold for $W_m^{p,q}$ with little or no modification. The space $M_m^{p,q}(G)$ is called a modulation space whereas the space $W_m^{p,q}(G)$ is called a Wiener amalgam space. We define $M_m^p = M_m^{p,p}$ and $W_m^p = W_m^{p,p}$. Note that $M_m^p = W_m^p$.

The following result generalizes Young's inequality and sheds some light on Definition 2.4.1.

Proposition 2.4.2. *Let $\tilde{v}$ and $\tilde{m}$ be weight functions on $G_1 \times \cdots \times G_n$ as in Definition 2.4.1. Let $1 \leq P, Q, R \leq \infty$ with $1/P + 1/Q = 1/R + 1$. Here, P, Q, and R are n-tuples. Let $\tilde{f} \in L_{\tilde{v}}^P(G_1 \times \cdots \times G_n)$ and $\tilde{g} \in L_{\tilde{m}}^Q(G_1 \times \cdots \times G_n)$. Then $\tilde{f} * \tilde{g}$ is defined almost everywhere and $\|\tilde{f} * \tilde{g}\|_{L_{\tilde{m}}^R} \leq \widetilde{C} \|\tilde{f}\|_{L_{\tilde{v}}^P} \|\tilde{g}\|_{L_{\tilde{m}}^Q}$.*

Remark. Note that the inclusions $\mathcal{S}(G_1 \times \cdots \times G_n) \subseteq L^P_{\tilde{m}}(G_1 \times \cdots \times G_n) \subseteq \mathcal{S}'(G_1 \times \cdots \times G_n)$ are continuous.

Proposition 2.4.3. *The definition of $M^{p,q}_m(G)$ is independent of the chosen window function g up to norm equivalence.*

Proof. Let h be another window function. Let $f \in \mathcal{S}'(G)$. By Proposition 2.3.28 and the STFT inversion theorem for tempered distributions, $|V_h f| \leq \|g\|_2^{-2} |V_g f| * |V_h g|$. By Proposition 2.4.2,

$$\|V_h f\|_{L^{p,q}_m} \leq C \frac{1}{\|g\|_2^2} \|V_h g\|_{L^1_v} \|V_g f\|_{L^{p,q}_m}.$$

The result is obtained by reversing the roles of g and h. □

Example 2.4.4. The function $v_s(a, \hat{a}) = (1+|a_{\mathbb{R}}|+|\hat{a}_{\mathbb{R}}|+|a_{\mathbb{Z}}|+|\hat{a}_{\mathbb{Z}}|)^s$, $s \geq 0$, on $G \times \widehat{G}$ is a submultiplicative weight function. By Proposition 2.3.18 and Corollary 2.3.27,

$$\mathcal{S}(G) = \bigcap_{s \geq 0} M^{\infty}_{v_s}(G) \qquad \text{and} \qquad \mathcal{S}'(G) = \bigcup_{s \geq 0} M^{\infty}_{1/v_s}(G).$$

Example 2.4.5. $M^2(G) = L^2(G)$ up to norm equivalence.

Proof. We can assume that g is compactly supported and $g = 1$ on $\overline{B(0,1)} \times \mathbb{T}^{d'} \times \{0\} \times \mathbb{A}$. Let $f \in M^2(G)$. We have

$$\begin{aligned}
\|f\|^2_{M^2} &= \int_{G \times \widehat{G}} |V_g f(a, \hat{a})|^2 \, da \, d\hat{a} \\
&= \int_{G \times \widehat{G}} |(f T_a \overline{g})\widehat{\ }(\hat{a})|^2 \, da \, d\hat{a} \\
&= \int_G \int_{\widehat{G}} |(f T_a \overline{g})\widehat{\ }(\hat{a})|^2 \, d\hat{a} \, da.
\end{aligned}$$

Since $\|f\|_{M^2}$ is finite, $\int_{\widehat{G}} |(fT_a\overline{g})\widehat{\ }(\hat{a})|^2 \, d\hat{a}$ is finite for almost every $a \in G$. In other words, $(fT_a\overline{g})\widehat{\ } \in L^2(\widehat{G})$ for almost every $a \in G$. By the Plancherel theorem, $fT_a\overline{g} \in L^2(G)$ for almost every $a \in G$. In particular, f satisfies the hypotheses of Proposition 2.1.31. Let $\tilde{f}$ be a locally integrable function on G as in the conclusion of Proposition 2.1.31. We now have

$$\begin{aligned}
\|f\|_{M^2}^2 &= \int_G \int_{\widehat{G}} |(\tilde{f}T_a\overline{g})\widehat{\ }(\hat{a})|^2 \, d\hat{a} \, da \\
&= \int_G \int_G |(\tilde{f}T_a\overline{g})(t)|^2 \, dt \, da \\
&= \int_G \int_G |\tilde{f}(t)|^2 |g(t-a)|^2 \, dt \, da \\
&= \|\tilde{f}\|_2^2 \|g\|_2^2.
\end{aligned}$$

Since $\|f\|_{M^2}$ is finite, $\tilde{f} \in L^2(G)$. In particular, $\tilde{f}$ is a tempered distribution, so f and $\tilde{f}$ coincide as tempered distributions.

Conversely, if $f \in L^2(G)$, we obtain $\|f\|_{M^2} = \|f\|_2 \|g\|_2$ by the same calculation.

□

Remark. Note that the inclusion $L^2(G) \subseteq M^\infty(G)$ is continuous by Proposition 2.3.6.

The next result refines the STFT inversion theorem in the context of modulation spaces.

Proposition 2.4.6 (STFT Inversion Theorem)**.** *Let $h \in \mathcal{S}(G)$.*

(a) For $F \in L_m^{p,q}(G \times \widehat{G})$, the linear map

$$V_h^* F : \overline{\phi} \to \int_{G \times \widehat{G}} F(a, \hat{a}) \langle M_{\hat{a}} T_a h, \phi \rangle \, da \, d\hat{a}$$

on $\mathcal{S}(G)$ is continuous. In other words, $V_h^* F \in \mathcal{S}'(G)$, and

$$V_h^* F = \int_{G \times \widehat{G}} F(a, \hat{a}) M_{\hat{a}} T_a h \, da \, d\hat{a},$$

where the right hand side is an $\mathcal{S}'(G)$ valued integral. We have the pointwise estimate $|V_g V_h^* F| \leq |F| * |V_g h|$. Moreover, $V_h^* F \in M_m^{p,q}(G)$ and $\|V_h^* F\|_{M_m^{p,q}} \leq C\|F\|_{L_m^{p,q}} \|h\|_{M_v^1}$. In particular, V_h^* is a bounded linear map from $L_m^{p,q}(G \times \widehat{G})$ to $M_m^{p,q}(G)$.

(b) Suppose that $\langle h, g \rangle \neq 0$. Then

$$\frac{1}{\langle h, g \rangle} V_h^* V_g = I.$$

Proof. (a) Suppose that $\phi_j \to 0$ in $\mathcal{S}(G)$. By Corollary 2.3.13, $V_h \phi_j \to 0$ in $\mathcal{S}(G \times \widehat{G})$. It follows from Hölder's inequality and the dominated convergence theorem that $\langle V_h^* F, \phi_j \rangle \to 0$. The remaining assertions follow by direct calculation and Proposition 2.4.2. □

Note that the inclusions $\mathcal{S}(G) \subseteq M_m^{p,q}(G) \subseteq \mathcal{S}'(G)$ are continuous.

Proposition 2.4.7. *If $1 \leq p, q < \infty$, then $\mathcal{S}(G)$ is dense in $M_m^{p,q}(G)$.*

The completeness and duality properties enjoyed by L^p-spaces have analogues for modulation spaces.

Proposition 2.4.8. *$M_m^{p,q}(G)$ is a Banach space.*

Proposition 2.4.9. *If $1 \leq p, q < \infty$, then $M_{1/m}^{p',q'}(G)$ and $M_m^{p,q}(G)^*$ are isomorphic* *up to norm equivalence under the pairing*

$$\langle f, F\rangle = \langle V_g f, V_g F\rangle \qquad (f \in M_m^{p,q}(G),\ F \in M_{1/m}^{p',q'}(G)).$$

Moreover,

$$|\langle f, F\rangle| \leq \|f\|_{M_m^{p,q}}\|F\|_{M_{1/m}^{p',q'}} \qquad (f \in M_m^{p,q}(G),\ F \in M_{1/m}^{p',q'}(G)).$$

Proof. It follows from Hölder's inequality that the pairing just defined induces a bounded linear map from $M_{1/m}^{p',q'}(G)$ to $M_m^{p,q}(G)^*$. By the open mapping theorem, it suffices to show that this map is one-to-one and onto.

Let $F \in M_{1/m}^{p',q'}(G)$. Suppose that $\langle f, F\rangle = 0$ for all $f \in M_m^{p,q}(G)$. By Corollary 2.3.24, $F = 0$.

Let $u \in M_m^{p,q}(G)^*$. By the Hahn-Banach theorem, there exists $\tilde{u} \in L_m^{p,q}(G \times \widehat{G})^*$ extending u. By duality, $\tilde{u}$ is induced by some $H \in L_{1/m}^{p',q'}(G \times \widehat{G})$. Let $h = \|g\|_2^{-2} V_g^* H$. We have

$$\begin{aligned}
\langle f, h\rangle &= \langle V_g f, V_g h\rangle \\
&= \langle V_g f, \|g\|_2^{-2} V_g V_g^* H\rangle \\
&= \langle V_g f, H\rangle \\
&= \tilde{u}(V_g f) \\
&= u(f) \qquad (f \in \mathcal{S}(G)).
\end{aligned}$$

The third equality makes use of the STFT inversion theorem for tempered distributions, hence the restriction $f \in \mathcal{S}(G)$. Since $\mathcal{S}(G)$ is dense in $M_m^{p,q}(G)$, $\langle f, h\rangle = u(f)$ for all $f \in M_m^{p,q}(G)$. $\square$

Remark. Note that the hypothesis $1 \leq p, q < \infty$ is only relevant to the proof that the map is onto and that its inverse is continuous.

The following result elucidates the dependence of the duality pairing on the chosen window function.

Proposition 2.4.10. *Let $h \in \mathcal{S}(G)$ be nonzero. Then*

$$\|h\|_2^2 \langle V_g f, V_g F\rangle = \|g\|_2^2 \langle V_h f, V_h F\rangle \qquad (f \in M_m^{p,q}(G),\ F \in M_{1/m}^{p',q'}(G)).$$

Proof. The equality holds when $f \in \mathcal{S}(G)$ by Corollary 2.3.24. Since $\mathcal{S}(G)$ is dense in $M_m^{p,q}(G)$, the result follows from the norm estimate of Proposition 2.4.9. □

Remark. Let $f \in M_m^{p,q}(G)$ and $\phi \in \mathcal{S}(G)$. If $\|g\|_2 = 1$, then $\langle V_g f, V_g \phi\rangle = \langle f, \phi\rangle$ by Corollary 2.3.24. In this case, the duality pairing is consistent with the standard pairing.

The duality pairing satisfies the following sequential form of continuity; this result is the analogue of Lemma 2.1.29.

Proposition 2.4.11. *If $f_j \to f$ in $M^1(G)$ and $F_j \to F$ in the weak* topology of $M^\infty(G)$, then $\langle f_j, F_j\rangle \to \langle f, F\rangle$.*

Proof. Since the sequence $\{\langle g, F_j\rangle\}$ is convergent and hence bounded for all $g \in M^1(G)$, the collection $\{F_j\}$ of continuous linear functionals on $M^1(G)$ is equicontinuous by the uniform boundedness principle. In particular, there exists a uniform constant C', independent of j, such that

$$|\langle g, F_j\rangle| \leq C' \|g\|_{M^1} \qquad (g \in M^1(G)).$$

Setting $g = f_j - f$, we see that $\langle f_j - f, F_j \rangle \to 0$. $\square$

Invariance Properties

It is straightforward to check that unweighted modulation spaces are strongly invariant under complex conjugation and coordinate reflection. The duality pairing satisfies the expected identities

$$\langle \overline{f}, F \rangle = \overline{\langle f, \overline{F} \rangle} \quad \text{and} \quad \langle \check{f}, F \rangle = \langle f, \check{F} \rangle$$

for $f \in M^{p,q}(G)$ and $F \in M^{p',q'}(G)$. The next result shows that modulation spaces are invariant under translations and modulations.

Proposition 2.4.12. *Let $f \in \mathcal{S}'(G)$. We have $\|M_{\hat{a}} T_a f\|_{M_m^{p,q}} \leq Cv(a, \hat{a}) \|f\|_{M_m^{p,q}}$.*

The duality pairing satisfies the expected identities $\langle T_a f, F \rangle = \langle f, T_{-a} F \rangle$ and $\langle M_{\hat{a}} f, F \rangle = \langle f, M_{-\hat{a}} F \rangle$ for $f \in M_m^{p,q}(G)$ and $F \in M_{1/m}^{p',q'}(G)$. The proof involves a straightforward application of the fundamental identity of time-frequency analysis.

The following result extends Proposition 2.1.19.

Proposition 2.4.13. *Suppose that $1 \leq p, q < \infty$. Let $f \in M^{p,q}(G)$. The map $(a, \hat{a}) \to M_{\hat{a}} T_a f$ from $G \times \widehat{G}$ to $M^{p,q}(G)$ is continuous.*

Proof. We have

$$\begin{aligned}
\|M_{\hat{b}}T_b f - M_{\hat{a}}T_a f\|_{M^{p,q}} &= \|V_g M_{\hat{b}}T_b f(t,\hat{t}) - V_g M_{\hat{a}}T_a f(t,\hat{t})\|_{L^{p,q}} \\
&= \|(-b,\hat{t}-\hat{b})V_g f(t-b,\hat{t}-\hat{b})\cdots \\
&\qquad - (-a,\hat{t}-\hat{a})V_g f(t-a,\hat{t}-\hat{a})\|_{L^{p,q}} \\
&\le \|(-b,\hat{t}-\hat{b})V_g f(t-b,\hat{t}-\hat{b})\cdots \\
&\qquad - (-b,\hat{t}-\hat{b})V_g f(t-a,\hat{t}-\hat{a})\|_{L^{p,q}}\cdots \\
&\quad + \|(-b,\hat{t}-\hat{b})V_g f(t-a,\hat{t}-\hat{a})\cdots \\
&\qquad\quad - (-a,\hat{t}-\hat{a})V_g f(t-a,\hat{t}-\hat{a})\|_{L^{p,q}} \\
&= \|V_g f(t-b,\hat{t}-\hat{b}) - V_g f(t-a,\hat{t}-\hat{a})\|_{L^{p,q}}\cdots \\
&\quad + \|((-b,\hat{t}-\hat{b}) - (-a,\hat{t}-\hat{a}))V_g f(t-a,\hat{t}-\hat{a})\|_{L^{p,q}}.
\end{aligned}$$

Since translation is continuous in $L^{p,q}(G\times\widehat{G})$, the first quantity becomes arbitrarily small as $(b,\hat{b})\to(a,\hat{a})$. By the dominated convergence theorem, the second quantity becomes arbitrarily small as $(b,\hat{b})\to(a,\hat{a})$. □

Corollary 2.4.14. *Let $F\in M^\infty(G)$. The map $(a,\hat{a})\to M_{\hat{a}}T_aF$ from $G\times\widehat{G}$ to $M^\infty(G)$ is continuous, where $M^\infty(G)$ is endowed with the weak* topology.*

Lemma 2.4.15. *The asymmetric coordinate transform $\mathcal{T}_G$ satisfies the following identities:*

(a) $\mathcal{T}_G M_{(\hat{a},\hat{b})} = M_{(-\hat{b},\hat{a}+\hat{b})}\mathcal{T}_G$.

(b) $\mathcal{T}_G^{-1} M_{(\hat{a},\hat{b})} = M_{(\hat{a}+\hat{b},-\hat{a})}\mathcal{T}_G^{-1}$.

(c) $\mathcal{T}_G T_{(a,b)} = T_{(a-b,a)}\mathcal{T}_G$.

(d) $\mathcal{T}_G^{-1} T_{(a,b)} = T_{(b,b-a)} \mathcal{T}_G^{-1}$.

Lemma 2.4.16. *Let $f \in \mathcal{S}'(G \times G)$ and $g \in \mathcal{S}(G \times G)$. Then*

$$V_g \mathcal{T}_G f(a, b, \hat{a}, \hat{b}) = V_{\mathcal{T}_G^{-1} g} f(b, b-a, \hat{a}+\hat{b}, -\hat{a}).$$

Proposition 2.4.17. *Let $1 \leq p \leq \infty$. Here, p is a number. The asymmetric coordinate transform $\mathcal{T}_G$ is an isomorphism from $M^p(G \times G)$ onto $M^p(G \times G)$ up to norm equivalence.*

Proposition 2.4.18. *Let v and m be weight functions on $G \times \widehat{G}$ as in Definition 2.4.1. Suppose that $m(a, \hat{a}) = m(-a, \hat{a})$. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. The Fourier transform is an isomorphism from $M_m^{p,q}(G)$ onto $W_m^{p,q}(\widehat{G})$ up to norm equivalence.*

Proof. Let $g \in \mathcal{S}(G)$ be nonzero. Let $f \in \mathcal{S}'(G)$. We have

$$\begin{aligned}
\|\hat{f}\|_{W_m^{p,q}} &= \|\|V_{\hat{g}} \hat{f}(\hat{a}, a) m(\hat{a}, a)\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&= \|\|V_g f(-a, \hat{a}) m(a, \hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&= \|\|V_g f(a, \hat{a}) m(-a, \hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&= \|\|V_g f(a, \hat{a}) m(a, \hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&= \|f\|_{M_m^{p,q}}.
\end{aligned}$$

$\square$

Proposition 2.4.19. *Let v and m be weight functions on $G_1 \times G_2 \times \widehat{G}_1 \times \widehat{G}_2$ as in Definition 2.4.1. Suppose that $m(a_1, a_2, \hat{a}_1, \hat{a}_2) = m(-a_1, a_2, \hat{a}_1, \hat{a}_2)$. Let $1 \leq p \leq$*

∞. *Here, p is a number. The partial Fourier transform with respect to G_1 is an isomorphism from $M^p_m(G_1 \times G_2)$ onto $M^p_m(\widehat{G}_1 \times G_2)$ up to norm equivalence.*

Proof. Let $g \in \mathcal{S}(G_1 \times G_2)$ be nonzero. Let $f \in \mathcal{S}'(G_1 \times G_2)$. We have

$$\begin{aligned}
\|\mathcal{F}_1 f\|_{M^p_m} &= \|V_{\mathcal{F}_1 g}\mathcal{F}_1 f(\hat{a}_1, a_2, a_1, \hat{a}_2) m(\hat{a}_1, a_2, a_1, \hat{a}_2)\|_{L^p} \\
&= \|V_g f(-a_1, a_2, \hat{a}_1, \hat{a}_2) m(a_1, a_2, \hat{a}_1, \hat{a}_2)\|_{L^p} \\
&= \|V_g f(a_1, a_2, \hat{a}_1, \hat{a}_2) m(-a_1, a_2, \hat{a}_1, \hat{a}_2)\|_{L^p} \\
&= \|V_g f(a_1, a_2, \hat{a}_1, \hat{a}_2) m(a_1, a_2, \hat{a}_1, \hat{a}_2)\|_{L^p} \\
&= \|f\|_{M^p_m}.
\end{aligned}$$

$\square$

Proposition 2.4.20. *Let v and m be weight functions on $G_1 \times G_2 \times \widehat{G}_1 \times \widehat{G}_2$ as in Definition 2.4.1. Suppose that $m(a_1, a_2, \hat{a}_1, \hat{a}_2) = m(-a_1, a_2, \hat{a}_1, \hat{a}_2)$. Let $1 \leq p \leq \infty$. Here, p is a number. Then*

$$\langle f, F\rangle = \langle \mathcal{F}_1 f, \mathcal{F}_1 F\rangle \qquad (f \in M^p_m(G_1 \times G_2),\ F \in M^{p'}_{1/m}(G_1 \times G_2)).$$

Proof. Let $g \in \mathcal{S}(G_1 \times G_2)$ be nonzero. We have

$$\begin{aligned}
\langle \mathcal{F}_1 f, \mathcal{F}_1 F\rangle &= \langle V_{\mathcal{F}_1 g}\mathcal{F}_1 f(\hat{a}_1, a_2, a_1, \hat{a}_2), V_{\mathcal{F}_1 g}\mathcal{F}_1 F(\hat{a}_1, a_2, a_1, \hat{a}_2)\rangle \\
&= \langle (-a_1, \hat{a}_1) V_g f(-a_1, a_2, \hat{a}_1, \hat{a}_2), (-a_1, \hat{a}_1) V_g F(-a_1, a_2, \hat{a}_1, \hat{a}_2)\rangle \\
&= \langle V_g f(a_1, a_2, \hat{a}_1, \hat{a}_2), V_g F(a_1, a_2, \hat{a}_1, \hat{a}_2)\rangle \\
&= \langle f, F\rangle.
\end{aligned}$$

$\square$

Remark. Note that the equality holds under the condition that the chosen window functions are related via the partial Fourier transform. Otherwise, the correct equality is furnished by Proposition 2.4.10.

Proposition 2.4.21. *Let $1 \leq p \leq \infty$. Here, p is a number. Then*

$$\langle f, F\rangle = \langle \mathcal{T}_G f, \mathcal{T}_G F\rangle \qquad (f \in M^p(G \times G),\ F \in M^{p'}(G \times G)).$$

Proof. Let $g \in \mathcal{S}(G \times G)$ be nonzero. We have

$$\begin{aligned}
\langle \mathcal{T}_G f, \mathcal{T}_G F\rangle &= \langle V_{\mathcal{T}_G g} \mathcal{T}_G f(a, b, \hat{a}, \hat{b}), V_{\mathcal{T}_G g} \mathcal{T}_G F(a, b, \hat{a}, \hat{b})\rangle \\
&= \langle V_g f(b, b - a, \hat{a} + \hat{b}, -\hat{a}), V_g F(b, b - a, \hat{a} + \hat{b}, -\hat{a})\rangle \\
&= \langle V_g f(a, b, \hat{a}, \hat{b}), V_g F(a, b, \hat{a}, \hat{b})\rangle \\
&= \langle f, F\rangle.
\end{aligned}$$

□

Remark. Note that the equality holds under the condition that the chosen window functions are related via the asymmetric coordinate transform. Otherwise, the correct equality is furnished by Proposition 2.4.10.

We note the following tensor product property of modulation spaces.

Proposition 2.4.22. *Let $1 \leq p \leq \infty$. Here, p is a number. Let $f_1 \in M^p(G_1)$ and $f_2 \in M^p(G_2)$. Then $f_1 \otimes f_2 \in M^p(G_1 \times G_2)$ and $\|f_1 \otimes f_2\|_{M^p} = \|f_1\|_{M^p}\|f_2\|_{M^p}$.*

Proof. Let $g_1 \in \mathcal{S}(G_1)$ and $g_2 \in \mathcal{S}(G_2)$ be nonzero. We have

$$\begin{aligned}
\|f_1 \otimes f_2\|_{M^p} &= \|V_{g_1 \otimes g_2}(f_1 \otimes f_2)\|_{L^p} \\
&= \|(V_{g_1} f_1) \otimes (V_{g_2} f_2)\|_{L^p} \\
&= \|V_{g_1} f_1\|_{L^p} \|V_{g_2} f_2\|_{L^p} \\
&= \|f_1\|_{M^p} \|f_2\|_{M^p}.
\end{aligned}$$

□

Remark. Note that the equality holds under the condition that the window function on $G_1 \times G_2$ is the tensor product of the window functions on G_1 and G_2.

The following result shows that the duality pairing commutes with tensor products.

Proposition 2.4.23. *Let $1 \leq p \leq \infty$. Here, p is a number. Let $f_1 \in M^p(G_1)$ and $f_2 \in M^p(G_2)$. Let $F_1 \in M^{p'}(G_1)$ and $F_2 \in M^{p'}(G_2)$. Then $\langle f_1 \otimes f_2, F_1 \otimes F_2 \rangle = \langle f_1, F_1 \rangle \langle f_2, F_2 \rangle$.*

Proof. Let $g_1 \in \mathcal{S}(G_1)$ and $g_2 \in \mathcal{S}(G_2)$ be nonzero. We have

$$\begin{aligned}
\langle f_1 \otimes f_2, F_1 \otimes F_2 \rangle &= \langle V_{g_1 \otimes g_2}(f_1 \otimes f_2), V_{g_1 \otimes g_2}(F_1 \otimes F_2) \rangle \\
&= \langle (V_{g_1} f_1) \otimes (V_{g_2} f_2), (V_{g_1} F_1) \otimes (V_{g_2} F_2) \rangle \\
&= \langle V_{g_1} f_1, V_{g_1} F_1 \rangle \langle V_{g_2} f_2, V_{g_2} F_2 \rangle \\
&= \langle f_1, F_1 \rangle \langle f_2, F_2 \rangle.
\end{aligned}$$

□

Remark. Note that the equality holds under the condition that the window function on $G_1 \times G_2$ is the tensor product of the window functions on G_1 and G_2.

Compact Supports

We next study modulation space norms of compactly supported distributions. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. The following result generalizes [Oko09, Lemma 1].

Proposition 2.4.24. *Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G.*

(a) Let K be a compact subset of G with nonempty interior. Then $\|f\|_{M^{p,q}} \asymp \|\hat{f}\|_{L^q}$ for all $f \in \mathcal{S}'(G)$ with $\operatorname{supp} f \subseteq K$.

(b) Let L be a compact subset of $\widehat{G}$ with nonempty interior. Then $\|f\|_{W^{p,q}} \asymp \|f\|_{L^q}$ for all $f \in \mathcal{S}'(G)$ with $\operatorname{supp} \hat{f} \subseteq L$.

Proof. *(a)* Let $g \in C_c^\infty(G)$ be nonzero with $\operatorname{supp} g \subseteq K$. Since $\operatorname{supp} M_{\hat{a}} T_a g \subseteq a + K$,

$V_g f(a,\hat{a}) = 0$ when $a \notin K - K$. We have

$$\begin{aligned}
\|f\|_{M^{p,q}} &= \|\|V_g f(a,\hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&= \|\|\mathbb{1}_{K-K}(a) V_g f(a,\hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&\leq \|\mathbb{1}_{K-K}\|_{L^p(G)} \|\|V_g f(a,\hat{a})\|_{L^\infty(G)}\|_{L^q(\widehat{G})} \\
&= \|\mathbb{1}_{K-K}\|_{L^p(G)} \|\|(\hat{f} T_{\hat{a}} \overline{\hat{g}})\widehat{\ }(a)\|_{L^\infty(G)}\|_{L^q(\widehat{G})} \\
&\leq \|\mathbb{1}_{K-K}\|_{L^p(G)} \|\|(\hat{f} T_{\hat{a}} \overline{\hat{g}})(\hat{t})\|_{L^1(\widehat{G})}\|_{L^q(\widehat{G})} \\
&= \|\mathbb{1}_{K-K}\|_{L^p(G)} \||\hat{f}| * |\check{\hat{g}}|\|_{L^q(\widehat{G})} \\
&\leq \|\mathbb{1}_{K-K}\|_{L^p(G)} \|\hat{g}\|_{L^1(\widehat{G})} \|\hat{f}\|_{L^q(\widehat{G})}.
\end{aligned}$$

For the converse, let $\psi \in C_c^\infty(G)$ with $\psi = 1$ on an open neighborhood of K. Let $h \in C_c^\infty(G)$ be nonnegative with $h = 1$ on $\operatorname{supp}\psi - \operatorname{supp}\psi$. Note that

$$\psi(a) f = \psi(a)\psi f = \psi(a)(T_a h)\psi f = \psi(a)(T_a h) f$$

for all $a \in G$. Then

$$\psi(a)\hat{f}(\hat{a}) = \psi(a)(f T_a h)\widehat{\ }(\hat{a}) = \psi(a) V_h f(a,\hat{a})$$

for all $a \in G$ and $\hat{a} \in \widehat{G}$. We now have

$$\begin{aligned}
\|\psi\|_{L^p(G)} \|\hat{f}\|_{L^q(\widehat{G})} &= \|\|\psi(a)\hat{f}(\hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&= \|\|\psi(a) V_h f(a,\hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&\leq \|\psi\|_{L^\infty(G)} \|\|V_h f(a,\hat{a})\|_{L^p(G)}\|_{L^q(\widehat{G})} \\
&= \|\psi\|_{L^\infty(G)} \|f\|_{M^{p,q}}.
\end{aligned}$$

(b) The result follows from Proposition 2.4.18 and *(a)*. $\square$

Example 2.4.25. Let $1 \leq p, q \leq \infty$. Here, p and q are numbers. By Proposition 2.4.24, $M^{p,q}(\mathbb{T}^d \times \mathbb{A})$, $W^{p,q}(\mathbb{Z}^d \times \mathbb{A})$, and $\ell^q(\mathbb{Z}^d \times \mathbb{A})$ are isomorphic up to norm equivalence.

Example 2.4.26. Let $1 \leq p \leq \infty$. Here, p is a number. Since $\hat{\delta}_G = 1_{\widehat{G}} \in \ell^\infty(\widehat{G})$, $\delta_G \in M^{p,\infty}(G)$.

2.5 Periodization

In the first half of this section, we consider general locally compact abelian groups. We refer to [Rei68; Fol95] for a detailed treatment of the material that follows.

Let G be a locally compact abelian group. Let H be a closed subgroup of G. We fix Haar measures on G and H. The linear map $\mathcal{P}_H : C_c(G) \to C_c(G/H)$ defined by

$$\mathcal{P}_H f(x + H) = \int_H f(x + \xi)\, d\xi$$

is surjective. The Haar measure on G/H can be suitably normalized so that

$$\begin{aligned} \int_G f(x)\, dx &= \int_{G/H} \mathcal{P}_H f(x + H)\, d(x + H) \\ &= \int_{G/H} \int_H f(x + \xi)\, d\xi\, d(x + H) \qquad (f \in C_c(G)). \end{aligned} \tag{2.5.1}$$

In this case, we say that the Haar measures on G, H, and G/H are canonically related; any choice of two normalizations forces the third normalization. We have the L^1 estimate $\|\mathcal{P}_H f\|_1 \leq \|f\|_1$ for $f \in C_c(G)$.

Let N be the null space of $\mathcal{P}_H$. Then $\mathcal{P}_H$ descends to an L^1 isometry from $C_c(G)/N$ onto $C_c(G/H)$. Let $\overline{N}$ be the closure of N in $L^1(G)$, i.e., the L^1 completion of N. It follows from functional analytic generalities that $\mathcal{P}_H$ extends to an isometric isomorphism from $L^1(G)/\overline{N}$ onto $L^1(G/H)$. Moreover, (2.5.1) holds for $f \in L^1(G)$. More precisely, $f(x+\cdot) \in L^1(H)$ for almost every $x+H \in G/H$, and $\mathcal{P}_H f(x+H) = \int_H f(x+\xi)\,d\xi$ for almost every $x+H \in G/H$.

The set $H^\perp = \{\gamma \in \widehat{G} : (x,\gamma) = 1 \text{ for all } x \in H\}$ is a closed subgroup of $\widehat{G}$. We have $(H^\perp)^\perp = H$. Moreover, $H^\perp$ is the dual group of G/H, and $\widehat{G}/H^\perp$ is the dual group of H.

Proposition 2.5.2. *Let $f \in L^1(G)$.*

(a) $\mathcal{P}_H T_y f = T_{y+H}\mathcal{P}_H f$ *for all* $y \in G$.

(b) $\mathcal{P}_H M_\nu f = M_\nu \mathcal{P}_H f$ *for all* $\nu \in H^\perp$.

Proof. Suppose first that $f \in C_c(G)$.

(a)

$$
\begin{aligned}
\mathcal{P}_H T_y f(x+H) &= \int_H T_y f(x+\xi)\,d\xi \\
&= \int_H f(x+\xi-y)\,d\xi \\
&= \mathcal{P}_H f(x-y+H) \\
&= T_{y+H}\mathcal{P}_H f(x+H).
\end{aligned}
$$

(b)

$$\begin{aligned}
\mathcal{P}_H M_\nu f(x+H) &= \int_H M_\nu f(x+\xi)\,d\xi \\
&= \int_H f(x+\xi)(x+\xi,\nu)\,d\xi \\
&= (x,\nu)\int_H f(x+\xi)\,d\xi \\
&= M_\nu \mathcal{P}_H f(x+H).
\end{aligned}$$

The general case follows from the fact that $C_c(G)$ is dense in $L^1(G)$. □

The Fourier inversion formula requires that the Haar measures on a group and its dual be suitably normalized. For dual pairs where one group is compact and the other group is discrete, this compatibility requirement is satisfied if the Haar measure on the compact group is normalized to have total measure 1, and the Haar measure on the discrete group is the counting measure.

We now state a general Poisson summation formula.

Theorem 2.5.3 (Poisson Summation Formula)**.** *In the following, the Haar measure on $H^\perp$ is the dual of the Haar measure on G/H which is suitably normalized so that* (2.5.1) *holds.*

(a) Let $f \in C_c(G)$. Then $\widehat{\mathcal{P}_H f} = \hat{f}|H^\perp$. If $\hat{f}|H^\perp \in L^1(H^\perp)$, then

$$\int_H f(x+\xi)\,d\xi = \int_{H^\perp} \hat{f}(\nu)(x,\nu)\,d\nu \qquad (x \in G).$$

(b) Let $f \in L^1(G)$. Then $\widehat{\mathcal{P}_H f} = \hat{f}|H^\perp$. If $\hat{f}|H^\perp \in L^1(H^\perp)$, then

$$\int_H f(x+\xi)\,d\xi = \int_{H^\perp} \hat{f}(\nu)(x,\nu)\,d\nu \qquad (a.e.\ x+H \in G/H).$$

The following result is relevant to the hypotheses of the Poisson summation formula.

Lemma 2.5.4 ([Rei68, p. 120])**.** *Let $f \in C_c(G)$. If $\hat{f} \in L^1(\widehat{G})$, then $\hat{f}|H^\perp \in L^1(H^\perp)$.*

The following compatibility result is a consequence of the Poisson summation formula.

Proposition 2.5.5 ([Rei68, p. 122])**.** *If the Haar measures on G, H, and G/H are canonically related, then the dual Haar measures on $\widehat{G}$, $H^\perp = \widehat{G/H}$, and $\widehat{G}/H^\perp = \widehat{H}$ are canonically related.*

If H is a discrete subgroup of G such that G/H is compact, then H is called a lattice. In this case, $H^\perp$ is also a lattice by the duality between subgroups and quotient groups discussed above. Since H is discrete, the Haar measure on H will be the counting measure. Since G/H is compact, the Haar measure on G/H will be normalized to have total measure 1. With this last normalization, the Haar measures on G, H, and G/H might no longer be canonically related. Therefore, (2.5.1) becomes

$$\int_G f(x)\,dx = s(H)\int_{G/H} \mathcal{P}_H f(x+H)\,d(x+H).$$

Here, $s(H)$ is the measure of G/H if the Haar measure on G/H were normalized to be canonically related to the Haar measures on G and H. Similarly, $s(H^\perp)$ is the measure of $\widehat{G}/H^\perp$ if the Haar measure on $\widehat{G}/H^\perp$ were normalized to be canonically related to the dual Haar measure on $\widehat{G}$ and the counting measure on $H^\perp$.

Proposition 2.5.6. $s(H)s(H^\perp) = 1$.

Proof. Suppose that the Haar measures on G, H, and G/H are canonically related. Recall that the Haar measure on H is the counting measure. Then the measure of G/H is $s(H)$, and the dual Haar measure on $\widehat{G}/H^{\perp}$ has total measure 1. Since the measure of G/H is $s(H)$, the dual Haar measure on $H^{\perp}$ is so normalized that every point has measure $1/s(H)$. By Proposition 2.5.5, the dual Haar measures on $\widehat{G}$, $H^{\perp}$, and $\widehat{G}/H^{\perp}$ are canonically related. However, the dual Haar measure on $H^{\perp}$ might not be the counting measure. If we normalize the dual Haar measure on $H^{\perp}$ to be the counting measure, then the dual Haar measure on $\widehat{G}/H^{\perp}$ must be normalized to have total measure $1/s(H)$. In other words, $s(H^{\perp}) = 1/s(H)$. $\square$

If the Haar measures on both H and $H^{\perp}$ are the counting measure, then the Poisson summation formula becomes

$$\int_H f(x+\xi)\,d\xi = \frac{1}{s(H)} \int_{H^{\perp}} \hat{f}(\nu)(x,\nu)\,d\nu.$$

Example 2.5.7. It is well known that every lattice in $\mathbb{R}^d$ is of the form $A\mathbb{Z}^d$, where A is an invertible real $d \times d$ matrix. Let f be the characteristic function of $A[0,1)^d$. Then (2.5.1) shows that $s(A\mathbb{Z}^d) = m(A[0,1)^d) = |\det(A)|$. The dual lattice is $(A^T)^{-1}\mathbb{Z}^d$.

Proof. Let B be an invertible real $d \times d$ matrix such that $B\mathbb{Z}^d$ is the dual lattice. Then $|\det(A)\det(B)| = s(A\mathbb{Z}^d)s(B\mathbb{Z}^d) = 1$. Since $A\mathbb{Z}^d$ and $B\mathbb{Z}^d$ annihilate each other, $U = A^T B$ must be an integer matrix. Since $\det(A^T B) = \det(A)\det(B) = \pm 1$, U is an invertible integer matrix (unimodular matrix). Then $B\mathbb{Z}^d = (A^T)^{-1}U\mathbb{Z}^d = (A^T)^{-1}\mathbb{Z}^d$. $\square$

Example 2.5.8. The lattices in $\mathbb{T}^d$ are precisely the finite subgroups of $\mathbb{T}^d$. By duality, these are in one-to-one correspondence with the finite index subgroups of $\mathbb{Z}^d$. The preimage of a lattice in $\mathbb{T}^d$ under the exponential map is a lattice in $\mathbb{R}^d$. Therefore, every lattice in $\mathbb{T}^d$ is the image of a lattice in $\mathbb{R}^d$ under the exponential map. However, not every lattice in $\mathbb{R}^d$ gives a lattice in $\mathbb{T}^d$. For example, if α is irrational, then the image of $\alpha\mathbb{Z}$ under the exponential map is dense in S^1.

Let H be a lattice in $\mathbb{Z}^d$. Let f be the characteristic function of $\{0\}$. Then (2.5.1) shows that $s(H) = [\mathbb{Z}^d : H]$. By duality, $|H^\perp| = [\mathbb{Z}^d : H] = s(H) = 1/s(H^\perp)$.

Example 2.5.9. Let H be a subgroup of $\mathbb{A}$. Let f be the characteristic function of $\{0\}$. Then (2.5.1) shows that $s(H) = [\mathbb{A} : H]$. By duality, $|H^\perp| = [\mathbb{A} : H] = s(H) = 1/s(H^\perp)$.

We now return to the setting where $G = \mathbb{R}^d \times \mathbb{T}^{d'} \times \mathbb{Z}^{d''} \times \mathbb{A}$. We take the lattice $\Gamma_\mathbb{R} = A\mathbb{Z}^d$ in $\mathbb{R}^d$, where A is an invertible real $d \times d$ matrix. Let $m_1, \ldots, m_{d'}$ be nonnegative integers. Let $\Gamma_{\mathbb{T},j}$ be the group of m_jth roots of unity. We take the lattice $\Gamma_\mathbb{T} = \Gamma_{\mathbb{T},1} \times \cdots \times \Gamma_{\mathbb{T},d'}$ in $\mathbb{T}^{d'}$. Let $n_1, \ldots, n_{d''}$ be positive integers. We take the lattice $\Gamma_\mathbb{Z} = n_1\mathbb{Z} \times \cdots \times n_{d''}\mathbb{Z}$ in $\mathbb{Z}^{d''}$. Let $\Gamma_\mathbb{A}$ be a subgroup of $\mathbb{A}$. Let $\Gamma = \Gamma_\mathbb{R} \times \Gamma_\mathbb{T} \times \Gamma_\mathbb{Z} \times \Gamma_\mathbb{A}$. Note that $\Gamma_\mathbb{R}^\perp$, $\Gamma_\mathbb{T}^\perp$, and $\Gamma_\mathbb{Z}^\perp$ are of the same type as $\Gamma_\mathbb{R}$, $\Gamma_\mathbb{Z}$, and $\Gamma_\mathbb{T}$, respectively.

We fix the following fundamental domains for the lattices described above: $D_\mathbb{R} = A[0,1)^d$ for $\Gamma_\mathbb{R}$, $D_\mathbb{T} = [0, 1/m_1) \times \cdots \times [0, 1/m_{d'})$ for $\Gamma_\mathbb{T}$, $D_\mathbb{Z} = [0, n_1) \times \cdots \times [0, n_{d''})$ for $\Gamma_\mathbb{Z}$, and any choice of coset representatives $D_\mathbb{A}$ for $\Gamma_\mathbb{A}$. Let $D = D_\mathbb{R} \times D_\mathbb{T} \times D_\mathbb{Z} \times D_\mathbb{A}$. We define $D^\perp$ similarly for $\Gamma^\perp$. Note that $\mu_G(D) = s(\Gamma)$ and $\mu_{\widehat{G}}(D^\perp) = s(\Gamma^\perp)$.

Note that we haven't been particular with our choice of $\Gamma_\mathbb{A}$ and $D_\mathbb{A}$, the reason being that what choice we make has no bearing on much of our discussion in the sequel. In fact, we could have been quite arbitrary with our choice of $D_{\mathbb{R}}$, $D_{\mathbb{T}}$, and $D_{\mathbb{Z}}$ as well. We made the above choices for the sake of definiteness and ease of presentation. However, we will need to be much more particular with our choice of $\Gamma_{\mathbb{A}}$ and $D_{\mathbb{A}}$ in some parts of Chapter 4. In fact, our choices will be limited to the trivial ones.

Let $P : G \to G/\Gamma$ be the quotient map. Note that D and G/Γ are isomorphic as measure spaces via P. However, the Haar measure on G/Γ must be normalized to have total measure $\mu_G(D)$.

The following series of results up to the end of Proposition 2.5.16 is inspired by the discussion in [Fol99, p. 298] and [Fol99, p. 299, Exercise 24].

Lemma 2.5.10. *Let $\phi \in \mathcal{S}(G)$. Then $\sum_{w\in\Gamma} T_w\phi$ converges in $C^\infty(G)$.*

Proof. Let $N > 0$. We have

$$\sum_{w\in\Gamma} |T_w\phi(a)| = \sum_{w\in\Gamma} |\phi(a-w)| \leq C_N \sum_{w\in\Gamma} (1 + |a_{\mathbb{R}} - w_{\mathbb{R}}| + |a_{\mathbb{Z}} - w_{\mathbb{Z}}|)^{-N}.$$

By Lemma 2.1.4, the last sum is uniformly convergent if we choose N sufficiently large. It follows that $\sum_{w\in\Gamma} |T_w\phi|$ converges uniformly. Since differentiation commutes with translation, the same conclusion applies to $\sum_{w\in\Gamma} |\partial^\alpha T_w\phi|$. $\square$

Lemma 2.5.11. *The linear map $\phi \to \sum_{w\in\Gamma} T_w\phi$ from $\mathcal{S}(G)$ to $C^\infty(G)$ is continuous.*

Proof. Suppose that $\phi_j \to 0$ in $\mathcal{S}(G)$. It suffices to show that $\sum_{w\in\Gamma} T_w\phi_j \to 0$ uniformly. Let $\epsilon, N > 0$. We have

$$\sum_{w\in\Gamma} |T_w\phi_j(a)| = \sum_{w\in\Gamma} |\phi_j(a-w)| \leq \epsilon \sum_{w\in\Gamma} (1 + |a_{\mathbb{R}} - w_{\mathbb{R}}| + |a_{\mathbb{Z}} - w_{\mathbb{Z}}|)^{-N}$$

for j sufficiently large. By Lemma 2.1.4, the last sum is uniformly bounded for $a \in G$ if we choose N sufficiently large. $\square$

Lemma 2.5.12. *Let $\phi \in C_c^\infty(G)$ with $\int_G \phi = 1$. Then $\sum_{w\in\Gamma} T_w(\phi * \mathbb{1}_D) = 1$.*

Proof.

$$\begin{aligned}\sum_{w\in\Gamma} T_w(\phi * \mathbb{1}_D)(a) &= \sum_{w\in\Gamma} (\phi * \mathbb{1}_D)(a-w) \\ &= \sum_{w\in\Gamma} \int_G \phi(t)\mathbb{1}_D(a-w-t)\,dt \\ &= \sum_{w\in\Gamma} \int_{a-w-D} \phi(t)\,dt \\ &= \int_G \phi(t)\,dt.\end{aligned}$$

$\square$

Remark. Note that $\phi * \mathbb{1}_D \in C_c^\infty(G)$ by Proposition 2.1.2.

Proposition 2.5.13. *Let K be a compact subset of G. There exists $\vartheta \in C_c^\infty(G)$ such that $\vartheta \geq 0$, ϑ is constant and positive on K, and $\sum_{w\in\Gamma} T_w\vartheta = 1$.*

Proof. Let $\phi \in C_c^\infty(G)$ such that $\phi \geq 0$, ϕ is constant and positive on $K - D$, and $\int_G \phi = 1$. Let C be the constant value of ϕ on $K - D$. Let $\vartheta = \phi * \mathbb{1}_D$. Then $\vartheta \geq 0$

and $\sum_{w\in\Gamma} T_w\vartheta = 1$. We have

$$\begin{aligned}\vartheta(a) &= (\phi * \mathbb{1}_D)(a)\\ &= \int_G \phi(t)\mathbb{1}_D(a-t)\,dt\\ &= \int_{a-D} \phi(t)\,dt\\ &= \int_{a-D} C\,dt\\ &= C\mu_G(D) \qquad (a\in K).\end{aligned}$$

$\square$

Proposition 2.5.14. *The linear map $\mathcal{P}_\Gamma : \mathcal{S}(G) \to C^\infty(G/\Gamma)$ defined by*

$$\mathcal{P}_\Gamma\phi(a+\Gamma) = \sum_{w\in\Gamma} \phi(a-w)$$

is continuous and surjective. In particular, the dual map $\mathcal{P}'_\Gamma : \mathcal{S}'(G/\Gamma) \to \mathcal{S}'(G)$ is injective.

Proof. By Lemma 2.5.11 and periodicity, $\mathcal{P}_\Gamma$ is well-defined and continuous. Let $\vartheta \in C_c^\infty(G)$ with $\sum_{w\in\Gamma} T_w\vartheta = 1$. Let $\psi \in C^\infty(G/\Gamma)$. Then $\mathcal{P}_\Gamma(\vartheta(\psi\circ P)) = \psi$. $\square$

Let $\mathcal{S}'_\Gamma(G)$ be the set of all Γ-periodic distributions on G, i.e., $u \in \mathcal{S}'(G)$ such that $T_w u = u$ for all $w \in \Gamma$. Clearly, the image of $\mathcal{P}'_\Gamma$ is contained in $\mathcal{S}'_\Gamma(G)$.

Proposition 2.5.15. *The image of $\mathcal{P}'_\Gamma$ coincides with $\mathcal{S}'_\Gamma(G)$.*

Proof. Let $\vartheta \in C_c^\infty(G)$ with $\sum_{w\in\Gamma} T_w\vartheta = 1$. Let $u \in \mathcal{S}'_\Gamma(G)$. Define $v \in \mathcal{S}'(G/\Gamma)$ by

$v(\psi) = u(\vartheta(\psi \circ P))$. We have

$$\begin{aligned}
\mathcal{P}'_\Gamma v(\phi) &= v(\mathcal{P}_\Gamma \phi) \\
&= u(\vartheta(\mathcal{P}_\Gamma \phi \circ P)) \\
&= u(\sum_{w \in \Gamma} \vartheta T_w \phi) \\
&= \sum_{w \in \Gamma} u(\vartheta T_w \phi) \\
&= \sum_{w \in \Gamma} u(\phi T_{-w} \vartheta) \\
&= u(\sum_{w \in \Gamma} \phi T_{-w} \vartheta) \\
&= u(\phi) \qquad (\phi \in C_c^\infty(G)).
\end{aligned}$$

Since $C_c^\infty(G)$ is dense in $\mathcal{S}(G)$, $\mathcal{P}'_\Gamma v = u$. $\square$

Proposition 2.5.16. *Let $f \in L^1(G/\Gamma)$. Then $\mathcal{P}'_\Gamma f = \mu_G(D)^{-1} f \circ P$.*

Proof. Let $\phi \in \mathcal{S}(G)$. Let $N > 0$. We have

$$
\begin{aligned}
\int_G |(f \circ P)(t)\phi(t)|\, dt &= \sum_{w\in\Gamma} \int_{w+D} |(f \circ P)(t)\phi(t)|\, dt \\
&= \sum_{w\in\Gamma} \int_D |(f \circ P)(t+w)\phi(t+w)|\, dt \\
&= \sum_{w\in\Gamma} \int_D |(f \circ P)(t)\phi(t+w)|\, dt \\
&= \int_D |(f \circ P)(t)| \sum_{w\in\Gamma} |\phi(t+w)|\, dt \\
&= \mu_G(D) \int_{G/\Gamma} |(f \circ P)(t)| \sum_{w\in\Gamma} |\phi(t+w)|\, d(t+\Gamma) \\
&\le \mu_G(D) \int_{G/\Gamma} |(f \circ P)(t)| \cdots \\
&\qquad\qquad C_N \sum_{w\in\Gamma} (1 + |t_{\mathbb{R}} + w_{\mathbb{R}}| + |t_{\mathbb{Z}} + w_{\mathbb{Z}}|)^{-N}\, d(t+\Gamma).
\end{aligned}
$$

By Lemma 2.1.4, the last sum is uniformly bounded for $t \in G$ if we choose N sufficiently large. We have shown that $(f \circ P)\phi$ is integrable. We now have

$$
\begin{aligned}
(\mathcal{P}'_\Gamma f, \phi) &= (f, \mathcal{P}_\Gamma \phi) \\
&= \int_{G/\Gamma} (f \circ P)(t) \sum_{w\in\Gamma} \phi(t-w)\, d(t+\Gamma) \\
&= \int_{G/\Gamma} \sum_{w\in\Gamma} (f \circ P)(t)\phi(t-w)\, d(t+\Gamma) \\
&= \int_{G/\Gamma} \sum_{w\in\Gamma} (f \circ P)(t-w)\phi(t-w)\, d(t+\Gamma) \\
&= \mu_G(D)^{-1} \int_G (f \circ P)(t)\phi(t)\, dt.
\end{aligned}
$$

The last equality follows from (2.5.1). □

We shall obtain the distributional version of the Poisson summation formula.

For $\psi \in \mathcal{S}(G/\Gamma)$,

$$\hat{\psi} = \sum_{w^\perp \in \Gamma^\perp} \hat{\psi}(w^\perp) T_{w^\perp} \delta_{\widehat{G}}$$

and

$$\psi = \sum_{w^\perp \in \Gamma^\perp} \hat{\psi}(w^\perp) M_{w^\perp} 1_G$$

with convergence in $\mathcal{S}(\Gamma^\perp)$ and $\mathcal{S}(G/\Gamma)$, respectively. For $v \in \mathcal{S}'(G/\Gamma)$,

$$\hat{v} = \sum_{w^\perp \in \Gamma^\perp} \hat{v}(w^\perp) T_{w^\perp} \delta_{\widehat{G}}$$

and

$$v = \sum_{w^\perp \in \Gamma^\perp} \hat{v}(w^\perp) M_{w^\perp} 1_G$$

with convergence in $\mathcal{S}'(\Gamma^\perp)$ and $\mathcal{S}'(G/\Gamma)$, respectively. For $f \in L^2(G/\Gamma)$,

$$\hat{f} = \sum_{w^\perp \in \Gamma^\perp} \hat{f}(w^\perp) T_{w^\perp} \delta_{\widehat{G}}$$

and

$$f = \sum_{w^\perp \in \Gamma^\perp} \hat{f}(w^\perp) M_{w^\perp} 1_G$$

with convergence in $L^2(\Gamma^\perp)$ and $L^2(G/\Gamma)$, respectively.

Remark. By Proposition 2.1.25, $\hat{v}$ is a complex function of polynomial growth. Therefore, it makes sense to evaluate $\hat{v}$ at elements of $\Gamma^\perp$.

By Proposition 2.5.16,

$$\mathcal{P}'_\Gamma v = \sum_{w^\perp \in \Gamma^\perp} \hat{v}(w^\perp) \mathcal{P}'_\Gamma M_{w^\perp} 1_G = \frac{1}{\mu_G(D)} \sum_{w^\perp \in \Gamma^\perp} \hat{v}(w^\perp) M_{w^\perp} 1_G$$

with convergence in $\mathcal{S}'(G)$. In particular,

$$\sum_{w\in\Gamma} T_w\delta_G = \mathcal{P}'_\Gamma\delta_G = \frac{1}{\mu_G(D)}\sum_{w^\perp\in\Gamma^\perp} M_{w^\perp}1_G$$

with convergence in $\mathcal{S}'(G)$. Evaluating both sides of this expression at $\phi\in\mathcal{S}(G)$ gives

$$\sum_{w\in\Gamma}\phi(w) = \frac{1}{\mu_G(D)}\sum_{w^\perp\in\Gamma^\perp}\hat{\phi}(w^\perp).$$

The tempered distribution $\sum_{w\in\Gamma} T_w\delta_G$ is often called a Dirac comb. More generally, we can construct weighted Dirac combs.

Lemma 2.5.17. *The linear map $\phi\to\phi|\Gamma$ from $\mathcal{S}(G)$ to $\mathcal{S}(\Gamma)$ is continuous.*

Proof. The noncompact factors of G correspond to the noncompact factors of Γ, i.e., $A\mathbb{Z}^d$ is a subgroup of $\mathbb{R}^d$, Υ is a subgroup of $\mathbb{Z}^{d''}$, and these are all the noncompact factors. Therefore, the inequalities that characterize Schwartz functions on Γ are all restrictions of inequalities satisfied by Schwartz functions on G. For example, for $\phi\in\mathcal{S}(G)$, $|\phi(a)|\leq C_N(1+|a_\mathbb{R}|+|a_\mathbb{Z}|)^{-N}$ for all $N>0$. It follows that $\phi|\Gamma\in\mathcal{S}(\Gamma)$. The continuity of the map follows by the same reasoning. □

Proposition 2.5.18. *Let $f\in\mathcal{S}'(\Gamma)$. Let u_f be the image of f under the dual of the restriction map of Lemma 2.5.17. Then $u_f=\sum_{w\in\Gamma} f(w)T_w\delta_G$ with convergence in $\mathcal{S}'(G)$.*

Proof. Let $\phi\in\mathcal{S}(G)$. We have

$$u_f(\phi) = (f,\phi|\Gamma) = \sum_{w\in\Gamma} f(w)\phi(w).$$

□

Example 2.5.19. Let $f \in \mathcal{S}'(\Gamma)$ and $g \in \mathcal{S}(G)$. We have

$$\begin{aligned} V_g u_f(a, \hat{a}) &= \sum_{w \in \Gamma} f(w) V_g T_w \delta_G(a, \hat{a}) \\ &= \sum_{w \in \Gamma} f(w)(-w, \hat{a}) V_g \delta_G(a - w, \hat{a}) \\ &= \sum_{w \in \Gamma} f(w)(-w, \hat{a}) \overline{g(w-a)}. \end{aligned}$$

Let $N > 0$. We have

$$\begin{aligned} |V_g u_f(a, \hat{a})| &\leq \sum_{w \in \Gamma} |f(w) g(w-a)| \\ &\leq C_N \|f\|_\infty \sum_{w \in \Gamma} (1 + |w_{\mathbb{R}} - a_{\mathbb{R}}| + |w_{\mathbb{Z}} - a_{\mathbb{Z}}|)^{-N}. \end{aligned}$$

By Lemma 2.1.4, the last sum is uniformly bounded for $a \in G$ if we choose N sufficiently large. It follows that $u_f \in M^\infty(G)$ if f is bounded.

2.6 Wiener Amalgam Spaces

We have already encountered the Wiener amalgam space $W^{p,q}(G)$. We now look at the definition of $W^{p,q}(G)$ from a slightly different perspective to make contact with the more general notion of a Wiener amalgam space, a term originally coined by Benedetto. Our treatment of the material in this section borrows mainly from [Hei03] in addition to [Fei80; Grö01]. Throughout, Γ is a lattice in G as described in Section 2.5.

Frequency Domain Approach

Let $g \in \mathcal{S}(G)$ be nonzero. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. We have

$$\|f\|_{W^{p,q}} = \| \|(fT_a g)\widehat{\ }(\hat{a})\|_{L^p(\widehat{G})}\|_{L^q(G)} \qquad (f \in \mathcal{S}'(G)).$$

Our goal is to discretize the outer norm.

Definition 2.6.1. Let U be a precompact open neighborhood of 0. Let $\{a_j\}$ be a subset of G such that

$$\sup_j |\{k : (a_j + K) \cap (a_k + K) \neq \emptyset\}| = C_K < \infty$$

for every compact $K \subseteq G$. We require that the index set that j runs over has as many factors as the number of factors of G. Note that $\{a_j\}$ is necessarily closed and discrete. Let $\{\psi_j\}$ be a corresponding subset of $C_c^\infty(G)$ such that $\sup \|\hat{\psi}_j\|_1 = M < \infty$, $\operatorname{supp} \psi_j \subseteq a_j + U$, and $\sum \psi_j = 1$. The collection $\{\psi_j\}$ is called a Fourier bounded uniform partition of unity (FBUPU) on G.

Remark. Let K be a compact subset of G. Every point of G has an open neighborhood intersecting only finitely many of $a_j + K$. Note also that

$$\sup_{a \in G} |\{j : a \in a_j + K\}| \leq C_K.$$

Example 2.6.2. Let $\vartheta \in C_c^\infty(G)$ with $\sum_{w \in \Gamma} T_w \vartheta = 1$. Then $\{T_w \vartheta\}_{w \in \Gamma}$ is a FBUPU.

Let $\{\psi_j\}$ be a FBUPU on G. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. Let K be a compact subset of G

containing U. We define

$$\|f\|_K = \|\sum \|\widehat{\psi_j f}\|_{L^p} \mathbb{1}_{a_j+K}\|_{L^q} \qquad (f \in \mathcal{S}'(G)).$$

Lemma 2.6.3. *Let $\{\psi_j\}$ be a FBUPU on G. Let $1 \le p, q \le \infty$. Here, p and q are tuples with as many components as the number of factors of G. Let K_1 and K_2 be compact subsets of G containing U. Then $\|\cdot\|_{K_1} \asymp \|\cdot\|_{K_2}$.*

Proof. There exist $b_1, \dots, b_n \in K_2$ such that $K_2 \subseteq \bigcup_{k=1}^n (b_k + K_1)$. We have

$$\begin{aligned}\sum \|\widehat{\psi_j f}\|_{L^p} \mathbb{1}_{a_j+K_2} &\le \sum \|\widehat{\psi_j f}\|_{L^p} \sum_{k=1}^n \mathbb{1}_{a_j+b_k+K_1} \\ &= \sum_{k=1}^n \sum \|\widehat{\psi_j f}\|_{L^p} \mathbb{1}_{a_j+b_k+K_1} \\ &= \sum_{k=1}^n T_{b_k} \sum \|\widehat{\psi_j f}\|_{L^p} \mathbb{1}_{a_j+K_1}.\end{aligned}$$

Then

$$\begin{aligned}\|f\|_{K_2} &\le \sum_{k=1}^n \|T_{b_k} \sum \|\widehat{\psi_j f}\|_{L^p} \mathbb{1}_{a_j+K_1}\|_{L^q} \\ &= \sum_{k=1}^n \|\sum \|\widehat{\psi_j f}\|_{L^p} \mathbb{1}_{a_j+K_1}\|_{L^q} \\ &= n\|f\|_{K_1}.\end{aligned}$$

The result is obtained by reversing the roles of K_1 and K_2. $\square$

The following result is the first step towards the intended discretization of the outer norm.

Proposition 2.6.4. *Let $\{\psi_j\}$ be a FBUPU on G. Let $1 \le p, q \le \infty$. Here, p and q are tuples with as many components as the number of factors of G. Let K be a* *compact subset of G containing U. Then $\|\cdot\|_K \asymp \|\cdot\|_{W^{p,q}}$.*

Proof. Let $g \in C_c^\infty(G)$ with $g = 1$ on $K - K$. Note that $\psi_j f = (T_a g)\psi_j f$ for all $a \in a_j + K$. Then

$$\begin{aligned}
\|\widehat{\psi_j f}\|_{L^p} &= \|((T_a g)\psi_j f)\hat{}\,\|_{L^p} \\
&= \|(f T_a g)\hat{}\, * \hat{\psi}_j\|_{L^p} \\
&\le \|\hat{\psi}_j\|_{L^1}\|(f T_a g)\hat{}\,\|_{L^p} \\
&\le M\|(f T_a g)\hat{}\,\|_{L^p}
\end{aligned}$$

for all $a \in a_j + K$. We now have

$$\sum \|\widehat{\psi_j f}\|_{L^p}\mathbb{1}_{a_j+K}(a) = \sum_{j : a \in a_j + K} \|\widehat{\psi_j f}\|_{L^p} \le C_K M\|(f T_a g)\hat{}\,\|_{L^p}$$

for all $a \in G$, so

$$\|f\|_K \le C_K M\|f\|_{W^{p,q}}.$$

For the converse, let $h \in C_c^\infty(G)$ be nonzero. Let A be a compact subset of G containing $U \cup (U - \operatorname{supp} h)$. For $a \in G$, let

$$S_a = \{j : (a_j + U) \cap (a + \operatorname{supp} h) \neq \emptyset\}.$$

We have

$$\begin{aligned}
\|(fT_ah)\widehat{\ }\|_{L^p} &= \|(\sum_{j\in S_a}(T_ah)\psi_j f)\widehat{\ }\|_{L^p} \\
&\le \sum_{j\in S_a}\|((T_ah)\psi_j f)\widehat{\ }\|_{L^p} \\
&= \sum_{j\in S_a}\|\widehat{\psi_j f}*\widehat{T_ah}\|_{L^p} \\
&\le \sum_{j\in S_a}\|\widehat{T_ah}\|_{L^1}\|\widehat{\psi_j f}\|_{L^p} \\
&= \|\hat{h}\|_{L^1}\sum_{j\in S_a}\|\widehat{\psi_j f}\|_{L^p} \\
&= \|\hat{h}\|_{L^1}\sum_{j\in S_a}\|\widehat{\psi_j f}\|_{L^p}\mathbb{1}_{a_j+A}(a) \\
&\le \|\hat{h}\|_{L^1}\sum\|\widehat{\psi_j f}\|_{L^p}\mathbb{1}_{a_j+A}(a)
\end{aligned}$$

for all $a\in G$, so

$$\|f\|_{W^{p,q}}\le\|\hat{h}\|_{L^1}\|f\|_A.$$

The result follows from Lemma 2.6.3. $\square$

We shall need the following technical results; see [Hei03].

Lemma 2.6.5. *Let (X,μ) be a measure space. Let $\{E_j\}_{j\in J}$ be a sequence of measurable sets in X such that*

$$0<\sup_{j\in J}|\{k\in J:\mu(E_j\cap E_k)>0\}|=N<\infty.$$

There exists a partition $\{J_1,\dots,J_N\}$ of J such that $\mu(E_j\cap E_k)=0$ for all distinct $j,k\in J_r$, $1\le r\le N$.

Lemma 2.6.6. *Let (X, μ) be a measure space. Let $\{f_j : X \to [0, \infty]\}_{j\in J}$ be a sequence of measurable functions such that*

$$0 < \sup_{j\in J} |\{k \in J : \mu(\operatorname{supp} f_j \cap \operatorname{supp} f_k) > 0\}| = N < \infty.$$

Let $1 \le p \le \infty$. Define $F : J \to [0, \infty]$ by $F(j) = \|f_j\|_p$. Then

$$\|F\|_p \le \|\sum_{j\in J} f_j\|_p \le N^{1/p'}\|F\|_p.$$

We finally achieve the intended discretization of the outer norm.

Proposition 2.6.7. *Let $\{\psi_j\}$ be a FBUPU on G. Let $1 \le p, q \le \infty$. Here, p and q are tuples with as many components as the number of factors of G. Then*

$$\|f\|_{W^{p,q}} \asymp \|\|\widehat{\psi_j f}\|_{L^p}\|_{\ell^q} \qquad (f \in \mathcal{S}'(G)).$$

Proof. Let K be a compact subset of G containing U. We have

$$\begin{aligned}
\|f\|_{W^{p,q}} &\asymp \|f\|_K \\
&= \|\sum \|\widehat{\psi_j f}\|_{L^p}\mathbb{1}_{a_j+K}\|_{L^q} \\
&\asymp \|\|\|\|\widehat{\psi_j f}\|_{L^p}\mathbb{1}_{a_j+K}\|_{L^q}\|_{\ell^q} \\
&= \|\|\widehat{\psi_j f}\|_{L^p}\|\mathbb{1}_{a_j+K}\|_{L^q}\|_{\ell^q} \\
&= \|\mathbb{1}_K\|_{L^q}\|\|\widehat{\psi_j f}\|_{L^p}\|_{\ell^q}.
\end{aligned}$$

□

Time Domain Approach

We next study a different class of Wiener amalgam spaces. Let $1 \leq p \leq \infty$. Here, p is a tuple with as many components as the number of factors of G. The key ingredient of our discussion above is the fact that $\mathcal{F}L^p(\widehat{G})$ is a Banach module over $\mathcal{F}L^1(\widehat{G})$ under convolution. The following discussion will be entirely similar but based on a different Banach module, namely, $L^p(G)$ as a Banach module over $C_0(G)$ under pointwise multiplication.

Let $g \in C_c^\infty(G)$ be nonzero. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. We define

$$\|f\|_{W(L^p,L^q)} = \| \| f(a) T_b g(a) \|_{L^p} \|_{L^q} \qquad (\text{measurable } f : G \to \mathbb{C}).$$

Here, we take the p-norm over $a \in G$ followed by the q-norm over $b \in G$. For example, if p and q are finite numbers, then

$$\|f\|_{W(L^p,L^q)} = \left(\int_G \left(\int_G |f(a) g(a-b)|^p \, da \right)^{q/p} db \right)^{1/q}.$$

Let $W(L^p(G), L^q(G))$ be the set of all measurable $f : G \to \mathbb{C}$ such that $\|f\|_{W(L^p,L^q)} < \infty$.

Proposition 2.6.8. *The definition of $W(L^p(G), L^q(G))$ is independent of the chosen window function g up to norm equivalence.*

Proof. Let h be another window function. Let V be a precompact open subset of G such that $\overline{V} \subseteq \{a \in G : h(a) \neq 0\}$. Let $m = \min_{a \in \overline{V}} |h(a)|$. Let $C = \|g\|_\infty / m$. There exist $a_1, \ldots, a_n \in G$ such that $\operatorname{supp} g \subseteq \bigcup_{k=1}^n (a_k + V)$. Then $|g| \leq C \sum_{k=1}^n T_{a_k} |h|$.

Let $f : G \to \mathbb{C}$ be measurable. We have

$$\begin{aligned}
\|\|f(a)T_b g(a)\|_{L^p}\|_{L^q}\| &\leq \|\|f(a)C\sum_{k=1}^{n} T_{b+a_k}|h|(a)\|_{L^p}\|_{L^q} \\
&\leq C\sum_{k=1}^{n} \|\|f(a)T_{b+a_k}h(a)\|_{L^p}\|_{L^q} \\
&= C\sum_{k=1}^{n} \|\|f(a)T_b h(a)\|_{L^p}\|_{L^q} \\
&= nC\|\|f(a)T_b h(a)\|_{L^p}\|_{L^q}.
\end{aligned}$$

The result is obtained by reversing the roles of g and h. □

Proposition 2.6.9 ([Hei03, Proposition 11.3.2])**.** *$W(L^p(G), L^q(G))$ is a Banach space.*

We now discretize the outer norm. The mathematics is essentially identical to the case of $W^{p,q}$ apart from the representative Banach module. Therefore, we shall merely state the relevant definitions and results.

Definition 2.6.10. Let U be a precompact open neighborhood of 0. Let $\{a_j\}$ be a subset of G such that

$$\sup_j |\{k : (a_j + K) \cap (a_k + K) \neq \emptyset\}| = C_K < \infty$$

for every compact $K \subseteq G$. We require that the index set that j runs over has as many factors as the number of factors of G. Note that $\{a_j\}$ is necessarily closed and discrete. Let $\{\psi_j\}$ be a corresponding subset of $C_c^\infty(G)$ such that $\sup \|\psi_j\|_\infty = M < \infty$, $\operatorname{supp} \psi_j \subseteq a_j + U$, and $\sum \psi_j = 1$. The collection $\{\psi_j\}$ is called a bounded uniform partition of unity (BUPU) on G.

Example 2.6.11. Let $\vartheta \in C_c^\infty(G)$ with $\sum_{w\in\Gamma} T_w\vartheta = 1$. Then $\{T_w\vartheta\}_{w\in\Gamma}$ is a BUPU.

Let $\{\psi_j\}$ be a BUPU on G. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. Let K be a compact subset of G containing U. We define

$$\|f\|_K = \|\sum \|\psi_j f\|_{L^p} \mathbb{1}_{a_j+K}\|_{L^q} \qquad (\text{measurable } f : G \to \mathbb{C}).$$

Lemma 2.6.12. *Let $\{\psi_j\}$ be a BUPU on G. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. Let K_1 and K_2 be compact subsets of G containing U. Then $\|\cdot\|_{K_1} \asymp \|\cdot\|_{K_2}$.*

Proposition 2.6.13. *Let $\{\psi_j\}$ be a BUPU on G. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. Let K be a compact subset of G containing U. Then $\|\cdot\|_K \asymp \|\cdot\|_{W(L^p, L^q)}$.*

Proposition 2.6.14. *Let $\{\psi_j\}$ be a BUPU on G. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. Then*

$$\|f\|_{W(L^p,L^q)} \asymp \|\|\psi_j f\|_{L^p}\|_{\ell^q} \qquad (\textit{measurable } f : G \to \mathbb{C}).$$

Proposition 2.6.15. *Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. Then*

$$\|f\|_{W(L^p,L^q)} \asymp \|\|f \mathbb{1}_{w+D}\|_{L^p}\|_{\ell^q} \qquad (\textit{measurable } f : G \to \mathbb{C}).$$

Proof. Let $\vartheta \in C_c^\infty(G)$ such that $\vartheta \geq 0$, ϑ is constant and positive on D, and $\sum_{w \in \Gamma} T_w \vartheta = 1$. Let C be the constant value of ϑ on D. Then

$$\|\|f \mathbb{1}_{w+D}\|_{L^p}\|_{\ell^q} \leq \frac{1}{C} \|\|f T_w \vartheta\|_{L^p}\|_{\ell^q}.$$

For the converse, let $w_1, \ldots, w_n \in \Gamma$ such that $\operatorname{supp} \vartheta \subseteq \bigcup_{k=1}^n (w_k + D)$. Then $\vartheta \leq \|\vartheta\|_\infty \sum_{k=1}^n \mathbb{1}_{w_k+D}$. We have

$$\begin{aligned}
\|\|fT_w\vartheta\|_{L^p}\|_{\ell^q} &\leq \|\vartheta\|_\infty \sum_{k=1}^n \|\|f\mathbb{1}_{w+w_k+D}\|_{L^p}\|_{\ell^q} \\
&= \|\vartheta\|_\infty \sum_{k=1}^n \|\|f\mathbb{1}_{w+D}\|_{L^p}\|_{\ell^q} \\
&= n\|\vartheta\|_\infty \|\|f\mathbb{1}_{w+D}\|_{L^p}\|_{\ell^q}.
\end{aligned}$$

□

Inclusion Relations

Proposition 2.6.16. *Let $1 \leq p \leq \infty$. Here, p is a number. Then $W(L^p(G), L^p(G)) = L^p(G)$ up to norm equivalence.*

Proof. Let $g \in C_c^\infty(G)$ be nonzero. Let $f : G \to \mathbb{C}$ be measurable. We have

$$\|f\|_{W(L^p,L^p)} = \|\|f(a)T_b g(a)\|_{L^p}\|_{L^p} = \|\|g\|_{L^p} f(a)\|_{L^p} = \|g\|_{L^p}\|f\|_{L^p}.$$

Here, we take the p-norm over $b \in G$ followed by the p-norm over $a \in G$; we are able to switch the order of integration only because p is a number. □

Proposition 2.6.17. *Let $1 \leq p \leq q \leq \infty$. Here, p and q are numbers. The inclusions $L^p(G) \subseteq W(L^p(G), L^q(G))$ and $L^q(G) \subseteq W(L^p(G), L^q(G))$ are continuous.*

Proof. Let $\{\psi_j\}$ be a BUPU on G. We have

$$\|\|\psi_j f\|_{L^p}\|_{\ell^q} \leq \|\|\psi_j f\|_{L^p}\|_{\ell^p} \asymp \|f\|_{L^p} \qquad (\text{measurable } f : G \to \mathbb{C}).$$

Let K be a compact subset of G containing U. We have

$$\|\|\psi_j f\|_{L^p}\|_{\ell^q} \leq |K|^{1/p-1/q}\|\|\psi_j f\|_{L^q}\|_{\ell^q} \asymp \|f\|_{L^q} \qquad (\text{measurable } f : G \to \mathbb{C}).$$

□

Proposition 2.6.18. *Let $1 \leq q \leq p \leq \infty$. Here, p and q are numbers. The inclusions $W(L^p(G), L^q(G)) \subseteq L^p(G)$ and $W(L^p(G), L^q(G)) \subseteq L^q(G)$ are continuous.*

Proof. Essentially identical to the proof of Proposition 2.6.17. □

Proposition 2.6.19. *Let $1 \leq p \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Here, p, q_1, and q_2 are tuples with as many components as the number of factors of G. The inclusion $W(L^p(G), L^{q_1}(G)) \subseteq W(L^p(G), L^{q_2}(G))$ is continuous.*

Proposition 2.6.20. *Let $1 \leq p \leq \infty$. Here, p is a tuple with as many components as the number of factors of G. The inclusion $W(L^\infty(G), L^p(G)) \subseteq L^p(G)$ is continuous.*

Proof.

$$\begin{aligned}
\|f\|_{L^p} &= \|\sum_{w\in\Gamma} |f|\mathbb{1}_{w+D}\|_{L^p} \\
&\leq \|\sum_{w\in\Gamma} \|f\mathbb{1}_{w+D}\|_{L^\infty}\mathbb{1}_{w+D}\|_{L^p} \\
&\asymp \|\|\|f\mathbb{1}_{w+D}\|_{L^\infty}\mathbb{1}_{w+D}\|_{L^p}\|_{\ell^p} \\
&= \|\|f\mathbb{1}_{w+D}\|_{L^\infty}\|\mathbb{1}_{w+D}\|_{L^p}\|_{\ell^p} \\
&= \|\mathbb{1}_D\|_{L^p}\|\|f\mathbb{1}_{w+D}\|_{L^\infty}\|_{\ell^p} \qquad (\text{measurable } f : G \to \mathbb{C}).
\end{aligned}$$

□

Proposition 2.6.21. *Let $1 \leq p \leq \infty$. Here, p is a tuple with as many components as the number of factors of G. The inclusion $W^{1,p}(G) \subseteq W(L^\infty(G), L^p(G))$ is continuous. In particular, the inclusion $W^{1,p}(G) \subseteq L^q(G)$ is continuous for all $q \geq p$. If $p < \infty$, then the inclusion $W^{1,p}(G) \subseteq C_0(G)$ is continuous. Otherwise, $W^{1,\infty}(G) \subseteq C(G)$.*

Proof. Let V be a precompact open subset of G such that $\bigcup_{w\in\Gamma}(w+V) = G$. Let $\vartheta \in C_c^\infty(G)$ such that $\vartheta \geq 0$, ϑ is constant and positive on $\overline{V}$, and $\sum_{w\in\Gamma} T_w\vartheta = 1$. Let $f \in W^{1,p}(G)$. Since

$$\|\|(fT_w\vartheta)\widehat{\ }\|_{L^1}\|_{\ell^\infty} \leq \|\|(fT_w\vartheta)\widehat{\ }\|_{L^1}\|_{\ell^p} < \infty,$$

$(fT_w\vartheta)\widehat{\ } \in L^1(\widehat{G})$ for all $w \in \Gamma$. Then $fT_w\vartheta \in C_0(G)$ for all $w \in \Gamma$. Clearly, f satisfies the hypotheses of Proposition 2.1.31. Let $\tilde{f}$ be a locally integrable function on G as in the conclusion of Proposition 2.1.31. We have

$$\|\|\tilde{f}T_w\vartheta\|_{L^\infty}\|_{\ell^p} \leq \|\|(\tilde{f}T_w\vartheta)\widehat{\ }\|_{L^1}\|_{\ell^p} < \infty.$$

It follows that $\tilde{f}$ is a tempered distribution, so f and $\tilde{f}$ coincide as tempered distributions. Since $fT_w\vartheta$ is continuous for all $w \in \Gamma$, f is continuous. Suppose that $p < \infty$. We have already established that the inclusion $W^{1,p}(G) \subseteq L^\infty(G)$ is continuous. Since $\mathcal{S}(G)$ is dense in $W^{1,p}(G)$, the inclusion $W^{1,p}(G) \subseteq C_0(G)$ is continuous. $\square$

Corollary 2.6.22. *The inclusion $M^1(G) \subseteq C_0(G)$ is continuous.*

We note the following continuity result.

Proposition 2.6.23. *The function $(a, f) \to f(a)$ on $G \times C_0(G)$ is continuous.*

Proof. Let $a, b \in G$ and $f, g \in C_0(G)$. We have

$$|g(b) - f(a)| \leq |g(b) - f(b)| + |f(b) - f(a)| \leq \|g - f\|_\infty + |f(b) - f(a)|$$

The right hand side becomes arbitrarily small as (b, g) approaches (a, f). $\square$

Corollary 2.6.24. *The function $(a, f) \to f(a)$ on $G \times M^1(G)$ is continuous.*

We define $W(G) = W(L^\infty(G), L^1(G))$. By Proposition 2.6.18, the inclusions $W(G) \subseteq L^1(G)$ and $W(G) \subseteq L^\infty(G)$ are continuous. By [Fol99, Proposition 6.10], the inclusion $W(G) \subseteq L^p(G)$ $(1 \leq p \leq \infty)$ is continuous. Here, p is a tuple with as many components as the number of factors of G.

Corollary 2.6.25. *The inclusions $M^1(G) \subseteq W(G)$ and $M^1(G) \subseteq \mathcal{F}W(\widehat{G})$ are continuous. In particular,*

$$M^1(G) \subseteq W(G) \cap \mathcal{F}W(\widehat{G}) \subseteq L^1(G) \cap \mathcal{F}L^1(\widehat{G}),$$

and the inclusion $M^1(G) \subseteq L^2(G)$ is continuous.

Proposition 2.6.26 ([Grö01, Proposition 12.1.7])**.** *$\|h * f\|_{M^1} \leq \|h\|_{L^1}\|f\|_{M^1}$ for all $h \in L^1(G)$ and $f \in M^1(G)$. In particular, $M^1(G)$ is a Banach algebra under both convolution and pointwise multiplication.*

Some Important Consequences

The following result refines the Poisson summation formula in the context of Wiener amalgam spaces.

Proposition 2.6.27. *Let $f \in W(G) \cap \mathcal{F}W(\widehat{G})$. Then*

$$\sum_{w \in \Gamma} f(a+w) = \frac{1}{\mu_G(D)} \sum_{w^\perp \in \Gamma^\perp} \hat{f}(w^\perp)(a, w^\perp) \qquad (a \in G)$$

with uniform absolute convergence on both sides.

Proof. Uniform absolute convergence follows immediately from the definition of $W(G)$. In particular, both sides are continuous. Our discussion of the general Poisson summation formula shows that the two sides are equal almost everywhere. Since both sides are continuous, they are equal everywhere. □

We introduced modulation spaces to quantify the decay properties of the STFT. The following result, together with Proposition 2.6.21, shows that such quantification gives something more refined than what is apparent from the definition; see [CG03, Lemma 4.1].

Proposition 2.6.28. *Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. Let $f \in M^{p,q}(G)$ and $g \in \mathcal{S}(G)$. Then $V_g f \in W^{1,(p,q)}(G \times \widehat{G})$ and*

$$\|V_g f\|_{W^{1,(p,q)}} \leq C \|f\|_{M^{p,q}} \|g\|_{M^1}.$$

The constant C does not depend on f and g.

Proof. Let $\varphi \in \mathcal{S}(G)$ be nonzero. We have

$$\begin{aligned}
\|V_g f\|_{W^{1,(p,q)}} &= \|\|V_{V_\varphi \varphi} V_g f(a, \hat{a}, \hat{b}, b)\|_{L^1(\widehat{G}\times G)}\|_{L^{p,q}(G\times\widehat{G})} \\
&= \|\|V_\varphi f(-b, \hat{a}+\hat{b}) V_\varphi g(-a-b, \hat{b})\|_{L^1(\widehat{G}\times G)}\|_{L^{p,q}(G\times\widehat{G})} \\
&= \|\|V_\varphi f(b, \hat{a}+\hat{b}) V_\varphi g(-a+b, \hat{b})\|_{L^1(\widehat{G}\times G)}\|_{L^{p,q}(G\times\widehat{G})} \\
&= \|\|V_\varphi f(b, \hat{b}) V_\varphi g(-a+b, -\hat{a}+\hat{b})\|_{L^1(\widehat{G}\times G)}\|_{L^{p,q}(G\times\widehat{G})} \\
&= \|\|V_\varphi f(b, \hat{b}) \widetilde{V_\varphi g}(a-b, \hat{a}-\hat{b})\|_{L^1(\widehat{G}\times G)}\|_{L^{p,q}(G\times\widehat{G})} \\
&= \||V_\varphi f| * |\widetilde{V_\varphi g}|\|_{L^{p,q}(G\times\widehat{G})} \\
&\leq \|V_\varphi f\|_{L^{p,q}(G\times\widehat{G})} \|V_\varphi g\|_{L^1(G\times\widehat{G})} \\
&= \|f\|_{M^{p,q}} \|g\|_{M^1}.
\end{aligned}$$

Here, we take the 1-norm over $(\hat{b}, b) \in \widehat{G} \times G$ followed by the (p,q)-norm over $(a, \hat{a}) \in G \times \widehat{G}$. The second equality follows from Proposition 2.3.31. □

The following result concerning the nestedness of modulation spaces is in stark contrast to the case of L^p-spaces.

Proposition 2.6.29. *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$ Here, p_1, p_2, q_1, and q_2 are tuples with as many components as the number of factors of G. The inclusion $M^{p_1,q_1}(G) \subseteq M^{p_2,q_2}(G)$ is continuous.*

Proof. Let $g \in \mathcal{S}(G)$ be nonzero. We have

$$\begin{aligned}
\|f\|_{M^{p_2,q_2}} &= \|V_g f\|_{L^{p_2,q_2}} \\
&\leq C\|V_g f\|_{W^{1,(p_1,q_1)}} \\
&\leq C'\|f\|_{M^{p_1,q_1}}\|g\|_{M^1} \qquad (f \in \mathcal{S}'(G)).
\end{aligned}$$

The first inequality follows from Proposition 2.6.21. $\square$

Chapter 3:

The Space M^1 and Quantization of Operators

3.1 Window Functions

In this section, we enlarge the class of window functions that can be used in the definition of the modulation space norm. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. For the rest of this section, p and q will be fixed unless otherwise specified. We endow $\mathcal{S}(G)$ with the topology of $M^1(G)$. The discussion below up to the end of Proposition 3.1.4 elaborates on [Grö01, Theorem 11.3.7].

Let $\tilde{g} \in \mathcal{S}(G)$ with $\|\tilde{g}\|_2 = 1$. Let $f \in M^{p,q}(G)$. By Proposition 2.3.28 and the STFT inversion theorem for tempered distributions, $|V_g f| \leq |V_{\tilde{g}} f| * |V_g \tilde{g}|$ for all $g \in \mathcal{S}(G)$. By Proposition 2.4.2,

$$\|V_g f\|_{L^{p,q}} \leq \|V_{\tilde{g}} f\|_{L^{p,q}} \|V_g \tilde{g}\|_{L^1} = \|f\|_{M^{p,q}} \|g\|_{M^1}$$

for all $g \in \mathcal{S}(G)$. In other words, the linear map $\overline{g} \to V_g f$ from $\mathcal{S}(G)$ to $L^{p,q}(G \times \widehat{G})$ has operator norm bounded by $\|f\|_{M^{p,q}}$. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, and $L^{p,q}(G \times \widehat{G})$ is complete, we get a unique extension to a bounded linear map from $M^1(G)$ to $L^{p,q}(G \times \widehat{G})$ whose operator norm is bounded by $\|f\|_{M^{p,q}}$. In particular, $\|V_g f\|_{L^{p,q}} \leq$

$\|f\|_{M^{p,q}}\|g\|_{M^1}$ for all $g \in M^1(G)$. We have shown:

Proposition 3.1.1. *Let $g \in M^1(G)$. Then V_g is a bounded linear map from $M^{p,q}(G)$ to $L^{p,q}(G \times \widehat{G})$ whose operator norm is bounded by $\|g\|_{M^1}$.*

Let $F \in L^{p,q}(G \times \widehat{G})$. By Proposition 2.4.6, the linear map $h \to V_h^* F$ from $\mathcal{S}(G)$ to $M^{p,q}(G)$ has operator norm bounded by $\|F\|_{L^{p,q}}$. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, and $M^{p,q}(G)$ is complete, we get a unique extension to a bounded linear map from $M^1(G)$ to $M^{p,q}(G)$ whose operator norm is bounded by $\|F\|_{L^{p,q}}$. In particular, $\|V_h^* F\|_{M^{p,q}} \leq \|F\|_{L^{p,q}}\|h\|_{M^1}$ for all $h \in M^1(G)$. We have shown:

Proposition 3.1.2. *Let $h \in M^1(G)$. Then V_h^* is a bounded linear map from $L^{p,q}(G \times \widehat{G})$ to $M^{p,q}(G)$ whose operator norm is bounded by $\|h\|_{M^1}$.*

Recall that the inclusion $M^1(G) \subseteq L^2(G)$ is continuous. Let $g, h \in M^1(G)$ with $\langle h, g\rangle \neq 0$. Let $\{g_j\}$ and $\{h_j\}$ be sequences in $\mathcal{S}(G)$ such that $g_j \to g$ and $h_j \to h$ in $M^1(G)$. Then $\langle h_j, g_j\rangle \to \langle h, g\rangle$. Let $f \in M^{p,q}(G)$. We have

$$
\begin{aligned}
\|V_h^* V_g f - V_{h_j}^* V_{g_j} f\|_{M^{p,q}} &\leq \|V_h^* V_g f - V_h^* V_{g_j} f\|_{M^{p,q}} + \|V_h^* V_{g_j} f - V_{h_j}^* V_{g_j} f\|_{M^{p,q}} \\
&\leq \|V_g f - V_{g_j} f\|_{L^{p,q}}\|h\|_{M^1} + \|V_{g_j} f\|_{L^{p,q}}\|h - h_j\|_{M^1} \\
&\leq \|f\|_{M^{p,q}}\|g - g_j\|_{M^1}\|h\|_{M^1} + \|f\|_{M^{p,q}}\|g_j\|_{M^1}\|h - h_j\|_{M^1} \\
&\to 0.
\end{aligned}
$$

It follows that

$$\frac{1}{\langle h, g\rangle} V_h^* V_g f = \lim \frac{1}{\langle h_j, g_j\rangle} V_{h_j}^* V_{g_j} f = f.$$

We have shown:

Proposition 3.1.3 (STFT Inversion Theorem)**.** *Let $g, h \in M^1(G)$ with $\langle h, g\rangle \neq 0$. Then*

$$\frac{1}{\langle h, g\rangle} V_h^* V_g = I.$$

Proposition 3.1.4. *Let $g \in M^1(G)$ be nonzero. Then*

$$\|f\|_{M^{p,q}} \asymp \|V_g f\|_{L^{p,q}} \qquad (f \in M^{p,q}(G)).$$

Proof. By Proposition 3.1.1, $\|V_g f\|_{L^{p,q}} \leq \|g\|_{M^1}\|f\|_{M^{p,q}}$. By Proposition 3.1.2 and Proposition 3.1.3,

$$\|f\|_{M^{p,q}} = \|g\|_2^{-2}\|V_g^* V_g f\|_{M^{p,q}} \leq \|g\|_2^{-2}\|g\|_{M^1}\|V_g f\|_{L^{p,q}}.$$

□

The continuity of the STFT holds in the case of window functions in $M^1(G)$.

Proposition 3.1.5. *Let $f \in M^{p,q}(G)$ and $g \in M^1(G)$. Then $V_g f$ is continuous.*

Proof. By Proposition 2.6.29, $f \in M^\infty(G)$. Let $\{g_j\}$ be a sequence in $\mathcal{S}(G)$ such that $g_j \to g$ in $M^1(G)$. By Proposition 3.1.1, $V_{g_j} f \to V_g f$ uniformly. Since $V_{g_j} f$ is continuous, the result follows. □

The following result extends Proposition 2.4.10 to include window functions in $M^1(G)$.

Proposition 3.1.6. *Let $g, h \in M^1(G)$ be nonzero. Then*

$$\|h\|_2^2\langle V_g f, V_g F\rangle = \|g\|_2^2\langle V_h f, V_h F\rangle \qquad (f \in M^{p,q}(G),\ F \in M^{p',q'}(G)).$$

Proof. The result holds when $g, h \in \mathcal{S}(G)$ by Proposition 2.4.10. By Proposition 3.1.1 and Hölder's inequality, the sesquilinear pairing $\langle \overline{k}, \overline{k'} \rangle \to \langle V_k f, V_{k'} F \rangle$ on $M^1(G) \times M^1(G)$ is continuous. Recall that the inclusion $M^1(G) \subseteq L^2(G)$ is continuous. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, the general case follows by taking sequences in $\mathcal{S}(G)$ converging in $M^1(G)$ to g and h. $\square$

We see by Proposition 3.1.6 that the duality pairing does not depend on the chosen window function as long as the window function has unit L^2 norm. Therefore, whenever a duality pairing is used, the window function shall be assumed to have unit L^2 norm. The remark following Proposition 2.4.10 is also relevant here.

We note the following special cases of the duality pairing.

Proposition 3.1.7. *Let $f \in L^p(G)$. Then $f \in M^\infty(G)$, and $\langle f, g \rangle = \int_G f\overline{g}$ for all $g \in M^1(G)$.*

Proof. Recall that the inclusion $M^1(G) \subseteq L^{p'}(G)$ is continuous. By Hölder's inequality, the linear map $\overline{g} \to \int_G f\overline{g}$ on $M^1(G)$ is continuous. By duality, there exists $u \in M^\infty(G)$ such that $\langle u, g \rangle = \int_G f\overline{g}$ for all $g \in M^1(G)$. In particular, $\langle u, \phi \rangle = \int_G f\overline{\phi}$ for all $\phi \in \mathcal{S}(G)$. It follows that $u = f$. $\square$

Proposition 3.1.8. *Let $f \in M^1(G)$. Then $\langle f, \delta_G \rangle = f(0)$.*

Proof. The equality holds by definition when $f \in \mathcal{S}(G)$. The general case follows from the density of $\mathcal{S}(G)$ in $M^1(G)$. $\square$

We originally defined the STFT via the pairing between $\mathcal{S}'(G)$ and $\mathcal{S}(G)$. In this section, we extended the definition of the STFT to include window functions in

$M^1(G)$ using a standard metric space argument. The following result shows that we could have used the duality pairing to carry out this extension.

Proposition 3.1.9. *Let $f \in M^{p,q}(G)$ and $g \in M^1(G)$. Then $V_g f(a, \hat{a}) = \langle f, M_{\hat{a}} T_a g\rangle$.*

Proof. The result holds by definition when $g \in \mathcal{S}(G)$. By Proposition 2.6.29, $f \in M^\infty(G)$. Let $\{g_j\}$ be a sequence in $\mathcal{S}(G)$ such that $g_j \to g$ in $M^1(G)$. By Proposition 3.1.1, $V_{g_j} f \to V_g f$ uniformly. In particular, $V_{g_j} f(a, \hat{a}) \to V_g f(a, \hat{a})$. By Proposition 2.4.12, $M_{\hat{a}} T_a g_j \to M_{\hat{a}} T_a g$ in $M^1(G)$. By Proposition 2.6.29, $M_{\hat{a}} T_a g_j \to M_{\hat{a}} T_a g$ in $M^{p',q'}(G)$. By Hölder's inequality, $\langle f, M_{\hat{a}} T_a g_j\rangle \to \langle f, M_{\hat{a}} T_a g\rangle$. □

The following result extends Proposition 2.3.23 to include window functions in $M^1(G)$. Recall that $\mathcal{T}_G$ is the asymmetric coordinate transform defined in Section 2.3.

Proposition 3.1.10. *Let $1 \le p \le \infty$. Here, p is a number. Let $f \in M^p(G)$ and $g \in M^1(G)$. Then $V_g f = \mathcal{F}_2 \mathcal{T}_G (f \otimes \overline{g})$.*

Proof. The result holds when $g \in \mathcal{S}(G)$ by Proposition 2.3.23. By Proposition 3.1.1, the linear map $\overline{h} \to V_h f$ from $M^1(G)$ to $L^p(G \times \widehat{G})$ is continuous. Since the inclusion $L^p(G \times \widehat{G}) \subseteq \mathcal{S}'(G \times \widehat{G})$ is continuous, we have a continuous linear map from $M^1(G)$ to $\mathcal{S}'(G \times \widehat{G})$. By Proposition 2.4.17, Proposition 2.4.19, Proposition 2.4.22, and Proposition 2.6.29, the linear map $\overline{h} \to \mathcal{F}_2 \mathcal{T}_G (f \otimes \overline{h})$ from $M^1(G)$ to $M^p(G \times \widehat{G})$ is continuous. Since the inclusion $M^p(G \times \widehat{G}) \subseteq \mathcal{S}'(G \times \widehat{G})$ is continuous, we have a continuous linear map from $M^1(G)$ to $\mathcal{S}'(G \times \widehat{G})$. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, the two maps we have described coincide. □

Proposition 3.1.11. *Let $1 \le p \le \infty$. Here, p is a number. Let $f \in M^1(G)$ and $g \in M^p(G)$. Then $V_g f \in L^p(G \times \widehat{G})$ and $V_g f(a, \hat{a}) = \langle f, M_{\hat{a}} T_a g\rangle$.*

Proof. The result follows from Proposition 2.3.34, Proposition 3.1.9, and Proposition 3.1.10. ☐

We have encountered a few ostensibly different definitions of the STFT based on various pairings between function spaces. The last few results reconcile all of these definitions.

The following result extends Proposition 2.6.28 to include window functions in $M^1(G)$.

Proposition 3.1.12. *Let $f \in M^{p,q}(G)$ and $g \in M^1(G)$. Then $V_g f \in W^{1,(p,q)}(G \times \widehat{G})$ and*

$$\|V_g f\|_{W^{1,(p,q)}} \leq C\|f\|_{M^{p,q}}\|g\|_{M^1}.$$

The constant C does not depend on f and g.

Proof. By Proposition 2.6.28, the linear map $\overline{h} \to V_h f$ from $\mathcal{S}(G)$ to $W^{1,(p,q)}(G \times \widehat{G})$ has operator norm bounded by $C\|f\|_{M^{p,q}}$. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, and $W^{1,(p,q)}(G \times \widehat{G})$ is complete, we get a unique extension to a bounded linear map from $M^1(G)$ to $W^{1,(p,q)}(G \times \widehat{G})$ whose operator norm is bounded by $C\|f\|_{M^{p,q}}$. By Proposition 2.6.21, the inclusion $W^{1,(p,q)}(G \times \widehat{G}) \subseteq L^{p,q}(G \times \widehat{G}))$ is continuous, so we have a bounded linear map from $M^1(G)$ to $L^{p,q}(G \times \widehat{G}))$. By Proposition 3.1.1, the linear map $\overline{h} \to V_h f$ from $M^1(G)$ to $L^{p,q}(G \times \widehat{G}))$ is continuous. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, this map coincides with the extension described above. ☐

Convolutions

We next study convolutions in the setting of M^1 and M^∞. Let $f \in M^\infty(G)$ and $g \in M^1(G)$. We define $(f * g)(a) = \langle f, T_a g^* \rangle = V_{g^*} f(a, 0)$ for $a \in G$. This definition of convolution is consistent with the definition in Section 2.1 when $g \in \mathcal{S}(G)$.

The following result is a special case of [CG03, Proposition 2.4].

Proposition 3.1.13. *$f * g \in M^{\infty,1}(G)$ and $\|f * g\|_{M^{\infty,1}} \leq \|f\|_{M^\infty} \|g\|_{M^1}$.*

Proof. Let $\varphi \in \mathcal{S}(G)$ be nonzero. Suppose first that $g \in \mathcal{S}(G)$. By Proposition 2.3.32,

$$V_{\varphi * \varphi}(f * g)(a, \hat{a}) = (-a, \hat{a})(f * M_{\hat{a}} \varphi^* * g * M_{\hat{a}} \varphi^*)(a).$$

Then

$$\begin{aligned}
\|f * g\|_{M^{\infty,1}} &= \|V_{\varphi * \varphi}(f * g)\|_{L^{\infty,1}} \\
&= \big\|\|(f * M_{\hat{a}} \varphi^* * g * M_{\hat{a}} \varphi^*)(a)\|_{L^\infty(G)}\big\|_{L^1(\widehat{G})} \\
&\leq \big\|\|f * M_{\hat{a}} \varphi^*(a)\|_{L^\infty(G)} \|g * M_{\hat{a}} \varphi^*(a)\|_{L^1(G)}\big\|_{L^1(\widehat{G})} \\
&\leq \big\|\|f * M_{\hat{a}} \varphi^*(a)\|_{L^\infty(G)}\big\|_{L^\infty(\widehat{G})} \big\|\|g * M_{\hat{a}} \varphi^*(a)\|_{L^1(G)}\big\|_{L^1(\widehat{G})} \\
&= \|f\|_{M^\infty} \|g\|_{M^1}.
\end{aligned}$$

We now lift the restriction that $g \in \mathcal{S}(G)$. We have just shown that the linear map $h \to f * h$ from $\mathcal{S}(G)$ to $M^{\infty,1}(G)$ has operator norm bounded by $\|f\|_{M^\infty}$. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, and $M^{\infty,1}(G)$ is complete, we get a unique extension to a bounded linear map from $M^1(G)$ to $M^{\infty,1}(G)$ whose operator norm is bounded by $\|f\|_{M^\infty}$. Since the inclusion $M^{\infty,1}(G) \subseteq \mathcal{S}'(G)$ is continuous, we have a continuous linear map from $M^1(G)$ to $\mathcal{S}'(G)$. On the other hand, by Hölder's inequality, the linear map $h \to f * h$ from $M^1(G)$ to $L^\infty(G)$ is continuous. Since the inclusion $L^\infty(G) \subseteq \mathcal{S}'(G)$ is continuous, we have a continuous linear map from $M^1(G)$ to $\mathcal{S}'(G)$. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, the two maps we have described coincide. $\square$

We can now define $\hat{g}\hat{f} = \widehat{f * g} \in W^{\infty,1}(\widehat{G}) \cap M^\infty(\widehat{G})$. Equivalently, we can define $gf = \mathcal{F}(\mathcal{R}\mathcal{F}f * \mathcal{R}\mathcal{F}g) \in W^{\infty,1}(G) \cap M^\infty(G)$. By Proposition 2.1.22, this definition of multiplication is consistent with the definition in Section 2.1 when $g \in \mathcal{S}(G)$. Note that $\widehat{gf} = \hat{f} * \hat{g}$.

Suppose that $f \in L^\infty(G)$. Then $fg \in L^1(G)$. We claim that gf as defined above coincides with fg. Indeed, for $\phi \in \mathcal{S}(G)$, we have

$$
\begin{aligned}
\langle gf, \phi \rangle &= \langle \widehat{gf}, \hat{\phi} \rangle \\
&= \langle \hat{f} * \hat{g}, \hat{\phi} \rangle \\
&= (\hat{f} * \hat{g} * (\hat{\phi})^*)(0) \\
&= \langle \hat{f}, (\hat{g})^* * \hat{\phi} \rangle \\
&= \langle \hat{f}, \hat{\overline{g}} * \hat{\phi} \rangle \\
&= \langle \hat{f}, \widehat{\overline{g}\phi} \rangle \\
&= \langle f, \overline{g}\phi \rangle \\
&= \int_G fg\overline{\phi} \\
&= \langle fg, \phi \rangle.
\end{aligned}
$$

In particular, $fg \in W^{\infty,1}(G) \cap M^\infty(G)$.

The following result on the associativity of convolution has been used above.

Proposition 3.1.14. *If $f \in M^\infty(G)$ and $g, h \in M^1(G)$, then $(f * g) * h = f * (g * h)$.*

Proof. The result holds when $g, h \in \mathcal{S}(G)$. By Proposition 2.6.26, Proposition 2.6.29, and Proposition 3.1.13, the bilinear map $(k, k') \to (f * k) * k'$ from $M^1(G) \times M^1(G)$ to $M^{\infty,1}(G)$ is continuous. Similarly, the bilinear map $(k, k') \to f * (k * k')$ from $M^1(G) \times M^1(G)$ to $M^{\infty,1}(G)$ is continuous. Since $\mathcal{S}(G)$ is dense in $M^1(G)$, the general case follows by taking sequences in $\mathcal{S}(G)$ converging in $M^1(G)$ to g and h. $\square$

We can now establish the following result on approximations of the identity.

Proposition 3.1.15. *Let $\{\psi_j\}$ be a sequence in $C_c^\infty(G)$ such that $\psi_j \geq 0$, $\int_G \psi_j = 1$, and* $\operatorname{supp} \psi_j \to 0$.

(a) For every $f \in M^1(G)$, $f * \psi_j \to f$ in $M^1(G)$.

(b) For every $F \in M^\infty(G)$, $F * \psi_j \to F$ in the weak topology of $M^\infty(G)$.*

Proof. (a) Suppose first that $f \in \mathcal{S}(G)$. By Proposition 2.1.28, $f * \psi_j \to f$ in $\mathcal{S}(G)$. Since the inclusion $\mathcal{S}(G) \subseteq M^1(G)$ is continuous, $f * \psi_j \to f$ in $M^1(G)$. We now lift the restriction that $f \in \mathcal{S}(G)$. Let $g \in \mathcal{S}(G)$. We have

$$\begin{aligned} \|f * \psi_j - f\|_{M^1} &\leq \|f * \psi_j - g * \psi_j\|_{M^1} + \|g * \psi_j - g\|_{M^1} + \|g - f\|_{M^1} \\ &\leq \|f - g\|_{M^1} \|\psi_j\|_{L^1} + \|g * \psi_j - g\|_{M^1} + \|g - f\|_{M^1} \\ &= \|f - g\|_{M^1} + \|g * \psi_j - g\|_{M^1} + \|g - f\|_{M^1}, \end{aligned}$$

where we have used Proposition 2.6.26. The result now follows form the density of $\mathcal{S}(G)$ in $M^1(G)$.

(b) Let $f \in M^1(G)$. We have

$$\langle F, f^* \rangle = (F * f)(0) = \lim (F * \psi_j * f)(0) = \lim \langle F * \psi_j, f^* \rangle.$$

$\square$

The following result is the analogue of Proposition 2.1.7.

Proposition 3.1.16. *Let $\psi \in C_c^\infty(G)$ with $\psi = 1$ on $U \times \mathbb{T}^{d'} \times U'' \times \mathbb{A}$, where U and U'' are open balls about 0. Let $\psi_{\epsilon,n}(x, z, \iota, \lambda) = \psi(\epsilon x, z, \lfloor \iota/n \rfloor, \lambda)$ for $\epsilon > 0$, $n \geq 1$, and $(x, z, \iota, \lambda) \in G$. Here, $\lfloor \cdot \rfloor$ is truncation towards 0. For every $f \in M^1(G)$, $\psi_{\epsilon,n} f \to f$ in $M^1(G)$ as $\epsilon \to 0$ and $n \to \infty$.*

Proof. We shall prove the result for the case $G = \mathbb{R}^d$. Suppose first that $f \in \mathcal{S}(G)$. By Proposition 2.1.7, $\psi_\epsilon f \to f$ in $\mathcal{S}(G)$ as $\epsilon \to 0$. Since the inclusion $\mathcal{S}(G) \subseteq M^1(G)$ is continuous, $\psi_\epsilon f \to f$ in $M^1(G)$ as $\epsilon \to 0$. We now lift the restriction that $f \in \mathcal{S}(G)$. Let $g \in \mathcal{S}(G)$. We have

$$\begin{aligned}
\|\psi_\epsilon f - f\|_{M^1} &\leq \|\psi_\epsilon f - \psi_\epsilon g\|_{M^1} + \|\psi_\epsilon g - g\|_{M^1} + \|g - f\|_{M^1} \\
&\leq \|\hat{f} * \hat{\psi}_\epsilon - \hat{g} * \hat{\psi}_\epsilon\|_{M^1} + \|\psi_\epsilon g - g\|_{M^1} + \|g - f\|_{M^1} \\
&\leq \|\hat{f} - \hat{g}\|_{M^1} \|\hat{\psi}_\epsilon\|_{L^1} + \|\psi_\epsilon g - g\|_{M^1} + \|g - f\|_{M^1} \\
&= \|\hat{f} - \hat{g}\|_{M^1} \|\hat{\psi}\|_{L^1} + \|\psi_\epsilon g - g\|_{M^1} + \|g - f\|_{M^1},
\end{aligned}$$

where we have used Proposition 2.6.26. The result now follows form the density of $\mathcal{S}(G)$ in $M^1(G)$. $\square$

The following important result is the analogue of Proposition 2.1.30.

Proposition 3.1.17. *Every distribution in $M^\infty(G)$ is the weak* limit of a sequence of functions in $C_c^\infty(G)$.*

Proof. Let $F \in M^\infty(G)$. Let $\chi_j = \psi_{1/j,j}$, where $\psi_{1/j,j}$ is as in Proposition 3.1.16. Let $\{\psi_j\}$ be as in Proposition 3.1.15. Let $F_j = (\chi_j F) * \psi_j$. Then $F_j \in C_c^\infty(G)$. We claim that $F_j \to F$ in the weak* topology of $M^\infty(G)$.

Let $f \in M^1(G)$. We have

$$\langle F_j, f\rangle = \langle(\chi_j F) * \psi_j, f\rangle = ((\chi_j F) * \psi_j * f^*)(0) = \langle \chi_j F, \psi_j^* * f\rangle.$$

By Proposition 3.1.16, $\chi_j F \to F$ in the weak* topology of $M^\infty(G)$. By Proposition 3.1.15, $\psi_j^* * f \to f$ in $M^1(G)$. The claim follows from Proposition 2.4.11. $\square$

3.2 Atomic Decompositions

In this section, we study how functions in $M^1(G)$ can be decomposed into simpler "atoms". The decomposition results below are discussed in [FK98; FZ98; Grö01]. We shall obtain these results as special cases of the general atomic decomposition theorem of Bonsall; see [Bon91]. We state the theorem here for convenience.

Let X be a Banach space. Let B be the open unit ball of X. Let E be a nonempty subset of X. The absolutely convex hull of E is $\operatorname{abco} E = \{\sum_{k=1}^n a_k u_k : u_k \in E,\ a_k \in \mathbb{C},\ \sum_{k=1}^n |a_k| \le 1\}$. For $f \in X$, let $\Lambda(E, f)$ be the set of all sequences $\{\lambda_j\} \in \ell^1$ such that $f = \sum \lambda_j u_j$ for some $u_j \in E$.

Theorem 3.2.1 (Bonsall [Bon91])**.** *Let $m, M > 0$. The following statements are equivalent:*

(a) *For every* $\phi \in X^*$, $m\|\phi\| \le \sup\{|\phi(u)| : u \in E\} \le M\|\phi\|$.

(b) $m\overline{B} \subseteq \overline{\operatorname{abco} E} \subseteq M\overline{B}$.

(c) *For every* $f \in X$, $\Lambda(E, f)$ *is nonempty, and*

$$M^{-1}\|f\| \le \inf\{\|\lambda\|_1 : \lambda \in \Lambda(E, f)\} \le m^{-1}\|f\|.$$

Proposition 3.2.2. *Let* $g \in M^1(G)$ *be nonzero. For every* $f \in M^1(G)$, *there exist sequences* $\{a_j\} \subseteq G$, $\{\hat{a}_j\} \subseteq \widehat{G}$, *and* $\{c_j\} \in \ell^1$ *such that* $f = \sum c_j M_{\hat{a}_j} T_{a_j} g$ *with convergence in* $M^1(G)$. *Moreover, the norm defined by* $\|f\| = \inf\{\|\{c_j\}\|_1\}$, *where the infimum is taken over all such representations of* f, *is equivalent to the modulation space norm.*

Proof. Let $E = \{M_{\hat{a}} T_a g : (a, \hat{a}) \in G \times \widehat{G}\}$. We have

$$\sup\{|\langle u, h\rangle| : h \in E\} = \|V_g u\|_{L^\infty} \asymp \|u\|_{M^\infty} \qquad (u \in M^\infty(G)).$$

By Theorem 3.2.1, this statement is equivalent to the assertion of the proposition. ☐

Corollary 3.2.3. *For every* $f \in M^1(G)$, *there exists a sequence* $\{g_j\} \subseteq C_c^\infty(G)$ *such that* $f = \sum g_j$ *with convergence in* $M^1(G)$. *In particular,* $C_c^\infty(G)$ *is dense in* $M^1(G)$.

Proof. Take $g \in C_c^\infty(G)$ in Proposition 3.2.2. ☐

The next result is the important minimality property of $M^1(G)$ originally discovered by Feichtinger; see [Grö01, Theorem 12.1.9].

Proposition 3.2.4. *Let* X *be a Banach space that is continuously embedded in* $\mathcal{S}'(G)$, *and is strongly invariant under translations and modulations. If* $M^1(G) \cap X \neq 0$, *then* *the inclusion $M^1(G) \subseteq X$ is continuous.*

Proof. Let $g \in M^1(G) \cap X$ be nonzero. Let $f \in M^1(G)$. Let $\{a_j\} \subseteq G$, $\{\hat{a}_j\} \subseteq \widehat{G}$, and $\{c_j\} \in \ell^1$ be sequences such that $f = \sum c_j M_{\hat{a}_j} T_{a_j} g$ with convergence in $M^1(G)$. Since $M_{\hat{a}_j} T_{a_j} g \in X$ and $\|M_{\hat{a}_j} T_{a_j} g\|_X = \|g\|_X$, we have $\sum |c_j| \|M_{\hat{a}_j} T_{a_j} g\|_X \leq \|\{c_j\}\|_1 \|g\|_X$. In particular, $\sum c_j M_{\hat{a}_j} T_{a_j} g$ converges absolutely with respect to the norm of X. Since X is complete, there exists $u \in X$ such that $u = \sum c_j M_{\hat{a}_j} T_{a_j} g$ in X. Since the inclusions $M^1(G) \subseteq \mathcal{S}'(G)$ and $X \subseteq \mathcal{S}'(G)$ are continuous, $u = f$. We have shown that $M^1(G) \subseteq X$. Since $\|f\|_X \leq \|\{c_j\}\|_1 \|g\|_X$, taking the infimum over all representations of f, the continuity of the inclusion follows from Proposition 3.2.2. $\square$

We next obtain the following tensor product property of M^1.

Proposition 3.2.5. *For every $f \in M^1(G_1 \times G_2)$, there exist sequences $\{f_{1,j}\} \subseteq M^1(G_1)$ and $\{f_{2,j}\} \subseteq M^1(G_2)$ such that $f = \sum f_{1,j} \otimes f_{2,j}$ with convergence in $M^1(G_1 \times G_2)$. Moreover, the norm defined by*

$$\|f\| = \inf\{\sum \|f_{1,j}\|_{M^1} \|f_{2,j}\|_{M^1}\},$$

where the infimum is taken over all such representations of f, is equivalent to the modulation space norm.

Proof. Let $g_1 \in \mathcal{S}(G_1)$ and $g_2 \in \mathcal{S}(G_2)$ be nonzero. For every $f \in M^1(G_1 \times G_2)$, there exist sequences $\{(a_{1,j}, a_{2,j})\} \subseteq G_1 \times G_2$, $\{(\hat{a}_{1,j}, \hat{a}_{2,j})\} \subseteq \widehat{G}_1 \times \widehat{G}_2$, and $\{c_j\} \in \ell^1$ such that

$$
\begin{aligned}
f &= \sum c_j M_{(\hat{a}_{1,j},\hat{a}_{2,j})} T_{(a_{1,j},a_{2,j})}(g_1 \otimes g_2) \\
&= \sum c_j (M_{\hat{a}_{1,j}} T_{a_{1,j}} g_1) \otimes (M_{\hat{a}_{2,j}} T_{a_{2,j}} g_2)
\end{aligned}
$$

with convergence in $M^1(G_1 \times G_2)$. This proves the existence claim. We need to prove that $\|\cdot\|$ and $\|\cdot\|_{M^1}$ are equivalent. Let $\|\cdot\|_*$ be the norm corresponding to $g_1 \otimes g_2$ as defined in Proposition 3.2.2. Since $\|\cdot\|_*$ and $\|\cdot\|_{M^1}$ are equivalent, it suffices to find $m, M > 0$ such that $m\|\cdot\|_{M^1} \leq \|\cdot\| \leq M\|\cdot\|_*$. It is clear that $\|f\|_{M^1} \leq \|f\|$ for all $f \in M^1(G_1 \times G_2)$. Therefore, we can take $m = 1$. On the other hand,

$$
\|f\| \leq \sum |c_j| \|g_1\|_{M^1} \|g_2\|_{M^1} = \sum |c_j| \|g_1 \otimes g_2\|_{M^1},
$$

so we can take $M = \|g_1 \otimes g_2\|_{M^1}$. $\square$

We now state a result from functional analysis that will be used a number of times in the sequel. In fact, we shall only need *(b)*.

Proposition 3.2.6. *Let X be a Banach space.*

(a) If $\{x_j\}$ is convergent in the weak topology of X, then $\{x_j\}$ is bounded in the norm topology of X.

(b) If $\{x_j^*\}$ is convergent in the weak topology of X^*, then $\{x_j^*\}$ is bounded in the norm topology of X^*.*

Proof. (a) By duality theory, X is isometrically embedded in X^{**}. Since $\{x^*(x_j)\}$ is bounded for all $x^* \in X^*$, $\{x_j\}$ is uniformly bounded by the Banach-Steinhaus theorem.

(b) Since $\{x_j^*(x)\}$ is bounded for all $x \in X$, $\{x_j^*\}$ is uniformly bounded by the Banach-Steinhaus theorem. $\square$

Remark. In *(a)*, the completeness of X is superfluous since X^* is complete irrespective of whether X is complete.

The following result is the analogue of Proposition 2.2.7.

Proposition 3.2.7. *If $F_{1,j} \to F_1$ in the weak* topology of $M^\infty(G_1)$ and $F_{2,j} \to F_2$ in the weak* topology of $M^\infty(G_2)$, then $F_{1,j} \otimes F_{2,j} \to F_1 \otimes F_2$ in the weak* topology of $M^\infty(G_1 \times G_2)$.*

Proof. In view of the identity

$$\begin{aligned} F_{1,j} \otimes F_{2,j} - F_1 \otimes F_2 = (F_{1,j} - F_1) \otimes (F_{2,j} - F_2) \cdots \\ + F_1 \otimes (F_{2,j} - F_2) + (F_{1,j} - F_1) \otimes F_2, \end{aligned}$$

it suffices to consider the cases $F_1 = F_2 = 0$, $F_1 = 0$, and $F_2 = 0$.

By Proposition 3.2.6, $\|F_{1,j}\|_{M^\infty} \leq C_1$ and $\|F_{2,j}\|_{M^\infty} \leq C_2$.

Let $f \in M^1(G_1 \times G_2)$. Let $\{f_{1,k}\} \subseteq M^1(G_1)$ and $\{f_{2,k}\} \subseteq M^1(G_2)$ be sequences such that $f = \sum f_{1,k} \otimes f_{2,k}$ with convergence in $M^1(G_1 \times G_2)$. We have

$$\begin{aligned}
|\langle f, F_{1,j} \otimes F_{2,j}\rangle| &\leq |\langle f - \sum_{k=1}^{n} f_{1,k} \otimes f_{2,k}, F_{1,j} \otimes F_{2,j}\rangle| \cdots \\
&\quad + |\langle \sum_{k=1}^{n} f_{1,k} \otimes f_{2,k}, F_{1,j} \otimes F_{2,j}\rangle| \\
&\leq \|f - \sum_{k=1}^{n} f_{1,k} \otimes f_{2,k}\|_{M^1} \|F_{1,j} \otimes F_{2,j}\|_{M^\infty} \cdots \\
&\quad + |\sum_{k=1}^{n} \langle f_{1,k}, F_{1,j}\rangle \langle f_{2,k}, F_{2,j}\rangle| \\
&\leq \|f - \sum_{k=1}^{n} f_{1,k} \otimes f_{2,k}\|_{M^1} C_1 C_2 \cdots \\
&\quad + \sum_{k=1}^{n} |\langle f_{1,k}, F_{1,j}\rangle \langle f_{2,k}, F_{2,j}\rangle|.
\end{aligned}$$

Let $\epsilon > 0$. The first term can be made less than $\epsilon/2$ by choosing n sufficiently large. Since n is now fixed, the second term can be made less than $\epsilon/2$ by choosing j sufficiently large. $\square$

3.3 Sampling on Modulation Spaces

Let Γ be a lattice in G as described in Section 2.5.

Proposition 3.3.1. *Let $1 \leq p \leq \infty$. Here, p is a tuple with as many components as the number of factors of G. The linear map $f \to f|\Gamma$ from $W(L^\infty(G), L^p(G)) \cap C(G)$ to $\ell^p(\Gamma)$ is continuous. In particular, the linear map $f \to f|\Gamma$ from $W^{1,p}(G)$ to $\ell^p(\Gamma)$ is continuous.*

Proof. Let $f \in W(L^\infty(G), L^p(G)) \cap C(G)$. We have $\|f|\Gamma\|_{\ell^p} \leq \|\|f\mathbb{1}_{w+D}\|_{L^\infty}\|_{\ell^p}$. $\square$

Corollary 3.3.2. *The linear map $g \to g|\Gamma$ from $M^1(G)$ to $M^1(\Gamma) = \ell^1(\Gamma)$ is continuous.*

The following result is the analogue of Proposition 2.5.18. It also provides an alternate proof of the claim in Example 2.5.19.

Proposition 3.3.3. *Let $f \in \ell^\infty(\Gamma)$. Let u_f be the image of f under the dual of the restriction map of Corollary 3.3.2. Then $u_f = \sum_{w\in\Gamma} f(w)T_w\delta_G$ with convergence in the weak* topology of $M^\infty(G)$.*

Proof. Let $g \in M^1(G)$. We have

$$\langle g, u_f\rangle = \langle g|\Gamma, f\rangle = \sum_{w\in\Gamma} g(w)\overline{f(w)} = \sum_{w\in\Gamma}\langle g, T_w\delta_G\rangle\overline{f(w)} = \sum_{w\in\Gamma}\langle g, f(w)T_w\delta_G\rangle.$$

□

We have already established the periodization maps $\mathcal{P}_\Gamma : C_c(G) \to C_c(G/\Gamma)$, $\mathcal{P}_\Gamma : L^1(G) \to L^1(G/\Gamma)$, and $\mathcal{P}_\Gamma : \mathcal{S}(G) \to C^\infty(G/\Gamma)$. Therefore, the following result is expected, and is easy to prove using the minimality property of M^1; see [Fei81, Theorem 7].

Proposition 3.3.4. *The linear map $\mathcal{P}_\Gamma : M^1(G) \to M^1(G/\Gamma)$ defined by*

$$\mathcal{P}_\Gamma f(a+\Gamma) = \sum_{w\in\Gamma} f(a-w)$$

is continuous and surjective.

Proof. Note that the series defines a continuous function on G by Proposition 2.6.27. Let X be the image of $M^1(G)$ under the periodization map $\mathcal{P}_\Gamma : L^1(G) \to L^1(G/\Gamma)$.

We endow X with the quotient norm induced by the quotient

$$M^1(G)/(M^1(G) \cap \ker \mathcal{P}_\Gamma).$$

Since the inclusion $M^1(G) \subseteq L^1(G)$ is continuous, $M^1(G) \cap \ker \mathcal{P}_\Gamma$ is closed in $M^1(G)$. In particular, X is a Banach space. Since $\mathcal{P}_\Gamma$ descends to a continuous linear map from $M^1(G)/(M^1(G) \cap \ker \mathcal{P}_\Gamma)$ to $L^1(G/\Gamma)$, X is continuously embedded in $L^1(G/\Gamma)$. By Proposition 2.5.2 and the strong invariance of $M^1(G)$ under translations and modulations, X is strongly invariant under translations and modulations. By Proposition 3.2.4, the inclusion $M^1(G/\Gamma) \subseteq X$ is continuous. It remains to show that $M^1(G/\Gamma) = X$; the continuity assertion then follows form the open mapping theorem. Since G/Γ is compact, it suffices to show that $\widehat{\mathcal{P}_\Gamma f} = s(\Gamma)^{-1} \hat{f}|\Gamma^\perp \in \ell^1(\Gamma^\perp)$ for all $f \in M^1(G)$. Since $\hat{f} \in M^1(\widehat{G})$, this follows from Corollary 3.3.2. $\square$

Let Λ be a lattice in G as described in Section 2.5.

Proposition 3.3.5. *Let $g \in M^1(G)$. Let $1 \leq p, q \leq \infty$. Here, p and q are tuples with as many components as the number of factors of G. The linear map $C_g : M^{p,q}(G) \to \ell^{p,q}(\Gamma \times \Lambda)$ defined by $C_g(f)(w, \upsilon) = \langle f, M_\upsilon T_w g \rangle$ is continuous.*

Proof. The result follows immediately from Proposition 3.1.12 and Proposition 3.3.1.

$\square$

Corollary 3.3.6. *Let $g \in M^1(G)$. The linear map $C_g : L^2(G) \to \ell^2(\Gamma \times \Lambda)$ defined by $C_g(f)(w, \upsilon) = \langle f, M_\upsilon T_w g \rangle$ is continuous.*

3.4 Kernels and Operators

In this section, we study operators of the (presently imprecise) form

$$f \to \int \kappa(\cdot, t) f(t)\, dt,$$

where κ is the kernel. See [FK98] and [Grö01, Chapter 14] for a comprehensive discussion of the material in this section and the next.

Lemma 3.4.1. *Let $a_1 \in G_1$. The linear map $\kappa \to \kappa(a_1, \cdot)$ from $M^1(G_1 \times G_2)$ to $M^1(G_2)$ is continuous.*

Proof. Let $g_1 \in \mathcal{S}(G_1)$ and $g_2 \in \mathcal{S}(G_2)$ be nonzero. Let $\kappa \in M^1(G_1 \times G_2)$. Let $\{(b_{1,j}, b_{2,j})\} \subseteq G_1 \times G_2$, $\{(\hat{b}_{1,j}, \hat{b}_{2,j})\} \subseteq \widehat{G}_1 \times \widehat{G}_2$, and $\{c_j\} \in \ell^1$ be sequences such that

$$\kappa = \sum c_j M_{(\hat{b}_{1,j}, \hat{b}_{2,j})} T_{(b_{1,j}, b_{2,j})} (g_1 \otimes g_2)$$

with convergence in $M^1(G_1 \times G_2)$. Since the inclusion $M^1(G_1 \times G_2) \subseteq C_0(G_1 \times G_2)$ is continuous,

$$\begin{aligned} \kappa(t_1, t_2) &= \sum c_j M_{(\hat{b}_{1,j}, \hat{b}_{2,j})} T_{(b_{1,j}, b_{2,j})} (g_1 \otimes g_2)(t_1, t_2) \\ &= \sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(t_1) M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2(t_2). \end{aligned}$$

In particular,

$$\kappa(a_1, t_2) = \sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1) M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2(t_2).$$

Since

$$\|c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1) M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2\|_{M^1} \le |c_j| \|g_1\|_\infty \|g_2\|_{M^1},$$

the series

$$\sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1) M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2$$

converges in $M^1(G_2)$. Since the inclusion $M^1(G_2) \subseteq C_0(G_2)$ is continuous, the sum of this series is $\kappa(a_1, \cdot)$. The continuity assertion follows from the inequality

$$\|\{c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1)\}\|_1 \leq \|\{c_j\}\|_1 \|g_1\|_\infty.$$

□

Corollary 3.4.2. *Let $\kappa \in M^1(G_1 \times G_2)$ and $a_1 \in G_1$. Then $\mathcal{F}_2\kappa(a_1, \cdot) = \kappa(a_1, \cdot)\widehat{\ }$.*

Proof. We have

$$\mathcal{F}_2\kappa = \sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1 \otimes \mathcal{F} M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2$$

with convergence in $M^1(G_1 \times G_2)$. By Lemma 3.4.1

$$\mathcal{F}_2\kappa(a_1, \cdot) = \sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1) \mathcal{F} M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2$$

and

$$\kappa(a_1, \cdot) = \sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1) M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2$$

with convergence in $M^1(G_2)$. The result follows by taking the Fourier transform of the latter series. □

Lemma 3.4.3. *Let $f \in M^\infty(G_2)$ and $\kappa \in M^1(G_1 \times G_2)$. The map $a_1 \to \langle \kappa(a_1, \cdot), f \rangle$ on G is in $M^1(G_1)$.*

Proof. Let $g_1 \in \mathcal{S}(G_1)$ and $g_2 \in \mathcal{S}(G_2)$ be nonzero. Let $\{(b_{1,j}, b_{2,j})\} \subseteq G_1 \times G_2$,

$\{(\hat{b}_{1,j}, \hat{b}_{2,j})\} \subseteq \widehat{G}_1 \times \widehat{G}_2$, and $\{c_j\} \in \ell^1$ be sequences such that

$$\kappa = \sum c_j M_{(\hat{b}_{1,j}, \hat{b}_{2,j})} T_{(b_{1,j}, b_{2,j})} (g_1 \otimes g_2)$$

with convergence in $M^1(G_1 \times G_2)$. We have previously shown that

$$\kappa(a_1, \cdot) = \sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1) M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2$$

with convergence in $M^1(G_2)$. Then

$$\langle \kappa(a_1, \cdot), f \rangle = \sum c_j M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1(a_1) \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f \rangle.$$

Since

$$\| c_j \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f \rangle M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1 \|_{M^1} \leq |c_j| \|f\|_{M^\infty} \|g_1\|_{M^1} \|g_2\|_{M^1},$$

the series

$$\sum c_j \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f \rangle M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1$$

converges in $M^1(G_1)$. Since the inclusion $M^1(G_1) \subseteq C_0(G_1)$ is continuous, the sum of this series is $\langle \kappa(\cdot, t_2), f(t_2) \rangle$. $\square$

Proposition 3.4.4. *Let $\kappa \in M^1(G_1 \times G_2)$. The operator $\mathcal{K} : M^\infty(G_2) \to M^1(G_1)$ defined by $\mathcal{K}\overline{f}(a_1) = \langle \kappa(a_1, \cdot), f \rangle$ has operator norm bounded by $C\|\kappa\|_{M^1}$. The constant C does not depend on κ.*

Proof. Let $g_1 \in \mathcal{S}(G_1)$ and $g_2 \in \mathcal{S}(G_2)$ be nonzero. Let $\{(b_{1,j}, b_{2,j})\} \subseteq G_1 \times G_2$, $\{(\hat{b}_{1,j}, \hat{b}_{2,j})\} \subseteq \widehat{G}_1 \times \widehat{G}_2$, and $\{c_j\} \in \ell^1$ be sequences such that

$$\kappa = \sum c_j M_{(\hat{b}_{1,j}, \hat{b}_{2,j})} T_{(b_{1,j}, b_{2,j})} (g_1 \otimes g_2)$$

with convergence in $M^1(G_1 \times G_2)$. Let $f \in M^\infty(G_2)$. We have previously shown that

$$\mathcal{K}\overline{f} = \sum c_j \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f\rangle M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1$$

with convergence in $M^1(G_1)$. Then

$$\begin{aligned}
\|\mathcal{K}\overline{f}\|_{M^1} &\leq \sum \|c_j \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f\rangle M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1\|_{M^1} \\
&\leq \sum |c_j| \|f\|_{M^\infty} \|g_1\|_{M^1} \|g_2\|_{M^1} \\
&= \|\{c_j\}\|_1 \|f\|_{M^\infty} \|g_1\|_{M^1} \|g_2\|_{M^1}.
\end{aligned}$$

By Proposition 3.2.2, taking the infimum over all representations of κ,

$$\|\mathcal{K}\overline{f}\|_{M^1} \leq C' \|\kappa\|_{M^1} \|f\|_{M^\infty} \|g_1\|_{M^1} \|g_2\|_{M^1}.$$

Therefore, we can take $C = C' \|f\|_{M^\infty} \|g_1\|_{M^1} \|g_2\|_{M^1}$. $\square$

Lemma 3.4.1, Lemma 3.4.3, and Proposition 3.4.4 together show that every $\kappa \in M^1(G_1 \times G_2)$ defines a bounded operator $\mathcal{K} : M^\infty(G_2) \to M^1(G_1)$. However, we actually have the following stronger form of continuity:

Proposition 3.4.5. *If $f_j \to 0$ in the weak* topology of $M^\infty(G_2)$, then $\mathcal{K} f_j \to 0$ in $M^1(G_1)$.*

Proof. Let $g_1 \in \mathcal{S}(G_1)$ and $g_2 \in \mathcal{S}(G_2)$ be nonzero. Let $\{(b_{1,j}, b_{2,j})\} \subseteq G_1 \times G_2$, $\{(\hat{b}_{1,j}, \hat{b}_{2,j})\} \subseteq \widehat{G}_1 \times \widehat{G}_2$, and $\{c_j\} \in \ell^1$ be sequences such that

$$\kappa = \sum c_j M_{(\hat{b}_{1,j}, \hat{b}_{2,j})} T_{(b_{1,j}, b_{2,j})} (g_1 \otimes g_2)$$

with convergence in $M^1(G_1 \times G_2)$. We have previously shown that

$$\|\mathcal{K}\overline{f_j}\|_{M^1} \leq \sum \|c_j \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f_j \rangle M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1\|_{M^1}.$$

Since

$$\|c_j \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f_j \rangle M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1\|_{M^1} \leq \|\{c_j\}\|_\infty |\langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f_j \rangle| \|g_1\|_{M^1},$$

every term of this series converges to 0. The result will follow once we show that the dominated convergence theorem applies. By Proposition 3.2.6, $\|f_j\|_{M^\infty} \leq C$. We then have

$$\begin{aligned} \|c_j \langle M_{\hat{b}_{2,j}} T_{b_{2,j}} g_2, f_j \rangle M_{\hat{b}_{1,j}} T_{b_{1,j}} g_1\|_{M^1} &\leq |c_j| \|f_j\|_{M^\infty} \|g_1\|_{M^1} \|g_2\|_{M^1} \\ &\leq |c_j| C \|g_1\|_{M^1} \|g_2\|_{M^1}, \end{aligned}$$

and

$$\{|c_j| C \|g_1\|_{M^1} \|g_2\|_{M^1}\} \in \ell^1.$$

$\square$

The next result provides an alternate description of $\mathcal{K}$ when we restrict it to $M^1(G_2)$.

Proposition 3.4.6. *$\langle \mathcal{K} g, f \rangle = \langle \kappa, f \otimes \overline{g} \rangle$ for all $f \in M^1(G_1)$ and $g \in M^1(G_2)$.*

Proof.

$$\begin{aligned}
\langle \mathcal{K}g, f\rangle &= \int_G \mathcal{K}g(t_1)\overline{f(t_1)}\,dt_1 \\
&= \int_G \langle \kappa(t_1,\cdot), \overline{g}\rangle \overline{f(t_1)}\,dt_1 \\
&= \int_G \left(\int_G \kappa(t_1,t_2)g(t_2)\,dt_2\right)\overline{f(t_1)}\,dt_1 \\
&= \int_G \int_G \kappa(t_1,t_2)\overline{f(t_1)}g(t_2)\,dt_2\,dt_1 \\
&= \langle \kappa, f\otimes \overline{g}\rangle.
\end{aligned}$$

□

Corollary 3.4.7. *$\langle \mathcal{K}g, f\rangle = \langle \kappa, f\otimes\overline{g}\rangle$ for all $f \in M^\infty(G_1)$ and $g \in M^\infty(G_2)$.*

Proof. Let $\{f_j\}$ be a sequence in $M^1(G_1)$ such that $f_j \to f$ in the weak* topology of $M^\infty(G_1)$. Let $\{g_j\}$ be a sequence in $M^1(G_2)$ such that $g_j \to g$ in the weak* topology of $M^\infty(G_2)$. By Proposition 3.4.5, $\mathcal{K}g_j \to \mathcal{K}g$ in $M^1(G_1)$. By Proposition 2.4.11, $\langle \mathcal{K}g_j, f_j\rangle \to \langle \mathcal{K}g, f\rangle$. By Proposition 3.2.7, $\langle \kappa, f_j\otimes\overline{g}_j\rangle \to \langle \kappa, f\otimes\overline{g}\rangle$. □

Let $\mathcal{O}^{\infty,1}(G_1, G_2)$ be the set of all operators from $M^\infty(G_2)$ to $M^1(G_1)$ which are continuous in the weak* sense of Proposition 3.4.5. In light of Lemma 2.2.2 and Proposition 3.4.6, we have an injective map from $M^1(G_1\times G_2)$ to $\mathcal{O}^{\infty,1}(G_1, G_2)$ mapping a kernel to its corresponding operator. The next result shows that this map is a bijection; see [FK98, Theorem 7.4.1]. Therefore, $\mathcal{O}^{\infty,1}(G_1, G_2) \cong M^1(G_1\times G_2)$.

Proposition 3.4.8. *Every operator in $\mathcal{O}^{\infty,1}(G_1, G_2)$ is induced by a kernel in $M^1(G_1\times G_2)$.*

3.5 The Spreading Representation

We now take $G_1 = G_2 = G$. Let $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)$ with kernel κ. The spreading function of $\mathcal{K}$ is $\eta = \mathcal{F}_2\mathcal{T}_G\kappa$. Note that $\eta \in M^1(G \times \widehat{G})$. Applying $\mathcal{F}_2\mathcal{T}_G$ to κ and $f \otimes \overline{g}$ in Proposition 3.4.6, we get

$$\langle \mathcal{K}g, f\rangle = \langle \eta, V_g f\rangle \qquad (f, g \in M^1(G)).$$

In other words,

$$\mathcal{K}g = \int_{G\times\widehat{G}} \eta(a,\hat{a}) M_{\hat{a}} T_a g \, da \, d\hat{a} \qquad (g \in M^1(G)),$$

where the right hand side is an $M^\infty(G)$ valued integral given that $M^\infty(G)$ is endowed with the weak* topology. In fact, applying $\mathcal{F}_2\mathcal{T}_G$ to κ and $f \otimes \overline{g}$ in Corollary 3.4.7, we get

$$\langle \mathcal{K}g, f\rangle = \langle \eta, V_g f\rangle \qquad (f, g \in M^\infty(G)).$$

By Proposition 3.1.11, $V_g f$ is bounded for $f \in M^1(G)$ and $g \in M^\infty(G)$. It follows that

$$\mathcal{K}g = \int_{G\times\widehat{G}} \eta(a,\hat{a}) M_{\hat{a}} T_a g \, da \, d\hat{a} \qquad (g \in M^\infty(G)), \tag{3.5.1}$$

where the right hand side is an $M^\infty(G)$ valued integral given that $M^\infty(G)$ is endowed with the weak* topology. However, we actually have the following more concrete equation. Note the similarity between this result and Proposition 2.3.26. Also note the technical subtlety of the proof in spite of the plausibility of the result in light of (3.5.1).

Proposition 3.5.2.

$$\mathcal{K}g(t) = \int_{G\times\widehat{G}} \eta(a,\hat{a}) M_{\hat{a}} T_a g(t)\, da\, d\hat{a} \qquad (g \in M^1(G)).$$

Proof. By Proposition 2.4.13 and Corollary 2.6.24, the function $(a,\hat{a},t) \to M_{\hat{a}} T_a g(t)$ on $G \times \widehat{G} \times G$ is continuous and bounded. It follows that the integrand is integrable and the integral is well-defined.

We next show that the map

$$\varphi : t \to \int_{G\times\widehat{G}} \eta(a,\hat{a}) M_{\hat{a}} T_a g(t)\, da\, d\hat{a}$$

on G is in $M^1(G)$. Let $h \in \mathcal{S}(G)$ be nonzero. Let $\{(x_j,\hat{x}_j)\} \subseteq G \times \widehat{G}$, $\{(\hat{y}_j, y_j)\} \subseteq \widehat{G} \times G$, and $\{c_j\} \in \ell^1$ be sequences such that

$$\eta = \sum c_j M_{(\hat{y}_j,y_j)} T_{(x_j,\hat{x}_j)} (h \otimes \hat{h})$$

with convergence in $M^1(G\times\widehat{G})$. Let $\{s_k\} \subseteq G$, $\{\hat{s}_k\} \subseteq \widehat{G}$, and $\{d_k\} \in \ell^1$ be sequences such that

$$g = \sum d_k M_{\hat{s}_k} T_{s_k} h$$

with convergence in $M^1(G)$. Since the inclusions $M^1(G \times \widehat{G}) \subseteq C_0(G \times \widehat{G})$ and $M^1(G) \subseteq C_0(G)$ are continuous,

$$\eta(a,\hat{a}) = \sum c_j M_{(\hat{y}_j,y_j)} T_{(x_j,\hat{x}_j)} (h \otimes \hat{h})(a,\hat{a})$$

and

$$M_{\hat{a}} T_a g(t) = \sum d_k M_{\hat{a}} T_a M_{\hat{s}_k} T_{s_k} h(t).$$

Since

$$\sum |c_j M_{(\hat{y}_j, y_j)} T_{(x_j, \hat{x}_j)} (h \otimes \hat{h})(a, \hat{a})| \leq \|\{c_j\|_1 \|h\|_\infty \|\hat{h}\|_\infty$$

and

$$\sum |d_k M_{\hat{a}} T_a M_{\hat{s}_k} T_{s_k} h(t)| \leq \|\{d_k\}\|_1 \|h\|_\infty,$$

$$\eta(a, \hat{a}) M_{\hat{a}} T_a g(t) = \sum_{j,k} c_j d_k M_{(\hat{y}_j, y_j)} T_{(x_j, \hat{x}_j)} (h \otimes \hat{h})(a, \hat{a}) M_{\hat{a}} T_a M_{\hat{s}_k} T_{s_k} h(t).$$

Since

$$\sum_{j,k} \|c_j d_k M_{(\hat{y}_j, y_j)} T_{(x_j, \hat{x}_j)} (h \otimes \hat{h})(a, \hat{a}) M_{\hat{a}} T_a M_{\hat{s}_k} T_{s_k} h(t)\|_{L^1(G \times \widehat{G})} \leq$$

$$\|\{c_j\}\|_1 \|\{d_k\}\|_1 \|h\|_1 \|\hat{h}\|_1 \|h\|_\infty,$$

the dominated convergence theorem applies. Here, we take the 1-norm over $(a, \hat{a}) \in G \times \widehat{G}$. Let

$$\varphi_{j,k}(t) = \int_{G \times \widehat{G}} M_{(\hat{y}_j, y_j)} T_{(x_j, \hat{x}_j)} (h \otimes \hat{h})(a, \hat{a}) M_{\hat{a}} T_a M_{\hat{s}_k} T_{s_k} h(t) \, da \, d\hat{a}$$

for $t \in G$. By Proposition 2.3.26, $\varphi_{j,k} \in \mathcal{S}(G)$. Integrating term by term over $(a, \hat{a}) \in G \times \widehat{G}$, we obtain

$$\varphi(t) = \sum_{j,k} c_j d_k \varphi_{j,k}(t).$$

By Proposition 2.4.6,

$$\|\varphi_{j,k}\|_{M^1} \leq \|M_{(\hat{y}_j, y_j)} T_{(x_j, \hat{x}_j)} (h \otimes \hat{h})\|_{L^1} \|M_{\hat{s}_k} T_{s_k} h\|_{M^1} = \|h\|_{L^1} \|\hat{h}\|_{L^1} \|h\|_{M^1}.$$

It follows that

$$\sum_{j,k} \|c_j d_k \varphi_{j,k}\|_{M^1} \leq \|\{c_j\}\|_1 \|\{d_k\}\|_1 \|h\|_{L^1} \|\hat{h}\|_{L^1} \|h\|_{M^1},$$

and the series

$$\sum_{j,k} c_j d_k \varphi_{j,k}$$

converges in $M^1(G)$. Since the inclusion $M^1(G) \subseteq C_0(G)$ is continuous, the sum of this series is φ. We have shown that $\varphi \in M^1(G)$.

For $f \in M^1(G)$, we have

$$\begin{aligned}
\langle \varphi, f\rangle &= \int_G \left(\int_{G\times\widehat{G}} \eta(a,\hat{a}) M_{\hat{a}} T_a g(t)\, da\, d\hat{a} \right) \overline{f(t)}\, dt \\
&= \int_{G\times\widehat{G}} \eta(a,\hat{a}) \left(\int_G M_{\hat{a}} T_a g(t) \overline{f(t)}\, dt \right) da\, d\hat{a} \\
&= \langle \eta, V_g f\rangle \\
&= \langle \mathcal{K} g, f\rangle.
\end{aligned}$$

It follows that $\varphi = \mathcal{K} g$. □

Let $\chi(a,\hat{a}) = (a,\hat{a})$ for $a \in G$ and $\hat{a} \in \widehat{G}$. Recall the definitions of $\mathcal{T}_{\widehat{G}}$ and $\mathcal{I}_{\widehat{G}}$ from Section 2.3.

Proposition 3.5.3. *Let $u \in \mathcal{S}'(G \times \widehat{G})$. Then*

$$\chi u = \mathcal{F}_1^{-1} \mathcal{I}_{\widehat{G}} \mathcal{T}_{\widehat{G}}^{-1} \mathcal{F}_1^{-1} u.$$

Proof. Let $\phi \in \mathcal{S}(G \times \widehat{G})$. We have

$$\begin{aligned}
\mathcal{I}_{\widehat{G}} \mathcal{T}_{\widehat{G}}^{-1} \mathcal{F}_1^{-1} \phi(\hat{t}, \hat{a}) &= \mathcal{T}_{\widehat{G}}^{-1} \mathcal{F}_1^{-1} \phi(\hat{a}, \hat{t}) \\
&= \mathcal{F}_1^{-1} \phi(\hat{a} - \hat{t}, \hat{a}) \\
&= \int_G \phi(a, \hat{a})(a, \hat{a} - \hat{t}) \, da \\
&= \int_G \phi(a, \hat{a})(a, \hat{a})(-a, \hat{t}) \, da \\
&= \mathcal{F}_1(\chi\phi)(\hat{t}, \hat{a}).
\end{aligned}$$

The general case follows from the sequential density of $\mathcal{S}(G)$ in $\mathcal{S}'(G)$. □

Corollary 3.5.4. *Let $1 \leq p \leq \infty$. Here, p is a number. Multiplication by χ is an isomorphism of $M^p(G \times \widehat{G})$ up to norm equivalence.*

We define $\eta_{\mathcal{F}}(\hat{a}, a) = (-a, \hat{a})\eta(-a, \hat{a})$ for $a \in G$ and $\hat{a} \in \widehat{G}$. Note that $\eta_{\mathcal{F}} \in M^1(\widehat{G} \times G)$. Let $\mathcal{K}_{\mathcal{F}}$ be the operator in $\mathcal{O}^{\infty,1}(\widehat{G})$ with spreading function $\eta_{\mathcal{F}}$.

Proposition 3.5.5.

$$\widehat{\mathcal{K}g} = \mathcal{K}_{\mathcal{F}} \hat{g} \qquad (g \in M^\infty(G)).$$

Proof. Suppose first that $g \in M^1(G)$. We have

$$\begin{aligned}
\widehat{\mathcal{K}g}(\hat{t}) &= \int_G \left(\int_{G\times\widehat{G}} \eta(a,\hat{a}) M_{\hat{a}} T_a g(t)\, da\, d\hat{a} \right) (-t,\hat{t})\, dt \\
&= \int_{G\times\widehat{G}} \eta(a,\hat{a}) \left(\int_G M_{\hat{a}} T_a g(t)(-t,\hat{t})\, dt \right) da\, d\hat{a} \\
&= \int_{G\times\widehat{G}} \eta(a,\hat{a}) \widehat{M_{\hat{a}} T_a g}(\hat{t})\, da\, d\hat{a} \\
&= \int_{G\times\widehat{G}} \eta(a,\hat{a}) T_{\hat{a}} M_{-a} \hat{g}(\hat{t})\, da\, d\hat{a} \\
&= \int_{G\times\widehat{G}} (a,\hat{a}) \eta(a,\hat{a}) M_{-a} T_{\hat{a}} \hat{g}(\hat{t})\, da\, d\hat{a} \\
&= \int_{G\times\widehat{G}} (-a,\hat{a}) \eta(-a,\hat{a}) M_a T_{\hat{a}} \hat{g}(\hat{t})\, da\, d\hat{a} \\
&= \mathcal{K}_{\mathcal{F}} \hat{g}(\hat{t}).
\end{aligned}$$

The general case follows from the sequential density of $M^1(G)$ in $M^\infty(G)$, where $M^\infty(G)$ is endowed with the weak* topology. □

The following result explains the meaning of the term "spreading".

Proposition 3.5.6. *Let $g \in M^1(G)$.*

(a) $\operatorname{supp} \mathcal{K}g \subseteq \operatorname{supp} g + \pi_G(\operatorname{supp} \eta)$, *where $\pi_G : G \times \widehat{G} \to G$ is the projection map.*

(b) $\operatorname{supp} \widehat{\mathcal{K}g} \subseteq \operatorname{supp} \hat{g} + \pi_{\widehat{G}}(\operatorname{supp} \eta)$, *where $\pi_{\widehat{G}} : G \times \widehat{G} \to \widehat{G}$ is the projection map.*

Proof. *(a)* Let $\phi \in C_c^\infty(G)$ with $\operatorname{supp}\phi \cap (\operatorname{supp} g + \pi_G(\operatorname{supp}\eta)) = \emptyset$. We have

$$\begin{aligned}
\langle \mathcal{K}g, \phi\rangle &= \langle \eta, V_g\phi\rangle \\
&= \int_{G\times\widehat{G}} \eta(a,\hat{a})\langle M_{\hat{a}}T_a g, \phi\rangle \, da\, d\hat{a} \\
&= \int_{G\times\widehat{G}} \eta(a,\hat{a}) \left(\int_G (t,\hat{a}) g(t-a)\overline{\phi(t)}\, dt\right) da\, d\hat{a} \\
&= \int_{\pi_G(\operatorname{supp}\eta)\times\widehat{G}} \eta(a,\hat{a}) \left(\int_{\operatorname{supp} g + a} (t,\hat{a}) g(t-a)\overline{\phi(t)}\, dt\right) da\, d\hat{a} \\
&= \int_{\pi_G(\operatorname{supp}\eta)\times\widehat{G}} \eta(a,\hat{a}) \left(\int_{\operatorname{supp} g + \pi_G(\operatorname{supp}\eta)} (t,\hat{a}) g(t-a)\overline{\phi(t)}\, dt\right) da\, d\hat{a} = 0.
\end{aligned}$$

(b) The result follows from *(a)* and Proposition 3.5.5. □

The spreading representation has the following tensor product property.

Proposition 3.5.7. *Let $\eta_1 \in M^1(G_1\times\widehat{G}_1)$ and $\eta_2 \in M^1(G_2\times\widehat{G}_2)$. Let $\mathcal{K}_1$ be the operator in $\mathcal{O}^{\infty,1}(G_1)$ with spreading function η_1. Let $\mathcal{K}_2$ be the operator in $\mathcal{O}^{\infty,1}(G_2)$ with spreading function η_2. Let $\mathcal{K}$ be the operator in $\mathcal{O}^{\infty,1}(G_1\times G_2)$ with spreading function $\eta_1\otimes\eta_2$. Then $\mathcal{K}(g_1\otimes g_2) = (\mathcal{K}_1 g_1)\otimes(\mathcal{K}_2 g_2)$ for all $g_1 \in M^\infty(G_1)$ and $g_2 \in M^\infty(G_2)$.*

Proof. Suppose first that $g_1 \in M^1(G_1)$ and $g_2 \in M^1(G_2)$. We have

$$\begin{aligned}
\mathcal{K}(g_1\otimes g_2)(t_1,t_2) &= \int_{G_1\times G_2\times\widehat{G}_1\times\widehat{G}_2} \eta_1(a_1,\hat{a}_1)\eta_2(a_2,\hat{a}_2)\cdots \\
&\qquad\qquad M_{(\hat{a}_1,\hat{a}_2)}T_{(a_1,a_2)}(g_1\otimes g_2)(t_1,t_2)\, da_1\, da_2\, d\hat{a}_1\, d\hat{a}_2 \\
&= \left(\int_{G_1\times\widehat{G}_1} \eta_1(a_1,\hat{a}_1) M_{\hat{a}_1}T_{a_1} g_1(t_1)\, da_1\, d\hat{a}_1\right)\cdots \\
&\quad \left(\int_{G_2\times\widehat{G}_2} \eta_2(a_2,\hat{a}_2) M_{\hat{a}_2}T_{a_2} g_2(t_2)\, da_2\, d\hat{a}_2\right) \\
&= \mathcal{K}_1 g_1(t_1)\mathcal{K}_2 g_2(t_2).
\end{aligned}$$

For the general case, take a sequence in $M^1(G_1)$ converging in the weak* topology of $M^\infty(G_1)$ to g_1. Similarly, take a sequence in $M^1(G_2)$ converging in the weak* topology of $M^\infty(G_2)$ to g_2. The result follows from Proposition 2.4.22 and Proposition 3.2.7.

$\square$

Chapter 4:

Identification of Operators

4.1 The Identification Problem

Let X and Y be Banach spaces. Let $\mathcal{O}$ be a Banach space of bounded linear maps $\mathcal{K} : X \to Y$. Let $g \in X$. Consider the evaluation map $e_g : \mathcal{O} \to Y$ defined by $e_g\mathcal{K} = \mathcal{K}g$. We say that $\mathcal{O}$ is weakly identifiable by g if e_g is injective [PW15a]. We say that $\mathcal{O}$ is strongly identifiable by g if e_g is continuous with a bounded inverse [PW15a]. We shall also use the term "stable" to mean "having a bounded inverse" [PW06a].

We first study the finite dimensional instance of the operator identification problem. Apart from being of interest in its own right, the finite dimensional theory forms the basis of the general infinite dimensional theory via a discretization scheme.

Consider a finite abelian group $\mathbb{A}$. Recall that $\widehat{\mathbb{A}} = \mathbb{A}$. Observe that all of the function spaces that we have studied on ELCA groups coincide in this case with the $|\mathbb{A}|$ dimensional vector space $\mathbb{C}^{\mathbb{A}}$.

Let $\eta \in \mathbb{C}^{\mathbb{A}\times\widehat{\mathbb{A}}}$. The operator $\mathcal{K}$ corresponding to the spreading function η is defined as follows. Let $g \in \mathbb{C}^{\mathbb{A}}$. Let $A(g)$ be the matrix whose columns consist of the Gabor system generated by g, i.e., the $|\mathbb{A}|\times|\mathbb{A}|^2$ matrix with columns $\{M_\tau T_\lambda g\}_{\lambda\in\mathbb{A},\tau\in\widehat{\mathbb{A}}}$.

If we specialize (3.5.1) to the present case, we see that $\mathcal{K}g = |\mathbb{A}|^{-1}A(g)\eta$. The factor $|\mathbb{A}|^{-1}$ appears because the measure of $\mathbb{A} \times \widehat{\mathbb{A}}$ is $|\mathbb{A}|$.

Let $S \subseteq \mathbb{A} \times \widehat{\mathbb{A}}$. Let $\mathcal{O}_S$ be the set of all $\eta \in \mathbb{C}^{\mathbb{A}\times\widehat{\mathbb{A}}}$ with $\operatorname{supp}\eta \subseteq S$. Let $g \in \mathbb{C}^{\mathbb{A}}$. Consider the evaluation map $e_g : \mathcal{O}_S \to \mathbb{C}^{\mathbb{A}}$. The matrix representation of e_g is precisely $|\mathbb{A}|^{-1}A(g)_S$, where $A(g)_S$ is obtained from $A(g)$ by removing those columns corresponding to $(\mathbb{A} \times \widehat{\mathbb{A}}) \setminus S$. In particular, $\mathcal{O}_S$ is identifiable by g only if $|S| \leq |\mathbb{A}|$ or, equivalently, $\mu_{\mathbb{A}\times\widehat{\mathbb{A}}}(S) \leq 1$. Therefore, the condition $\mu_{\mathbb{A}\times\widehat{\mathbb{A}}}(S) \leq 1$ is necessary for the identifiability of $\mathcal{O}_S$. We next study to what extent this condition is also sufficient.

We interrupt our main discussion to make some definitions. Let R be a complex $n \times p$ matrix. The spark of R is $q+1$, where q is the largest $m \leq p$ such that every set of m columns of R is linearly independent [DE03]. The matrix R is called full spark if the spark is $n+1$ or, equivalently, $p \geq n$ and every $n \times n$ minor of R is invertible [PW15b].

Let $g \in \mathbb{C}^{\mathbb{Z}/N\mathbb{Z}}$. The matrix $A(g)$ is defined as follows. Let $\omega_N = e^{2\pi i/N}$. Let W_N be the $N \times N$ discrete Fourier transform matrix

$$(\omega_N^{pq})_{p,q=0}^{N} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \cdots & \omega_N^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \omega_N^{N-1} & \cdots & \omega_N^{(N-1)^2} \end{pmatrix}.$$

Let $T_k(g)$ be the $N \times N$ diagonal matrix

$$\operatorname{diag}(g(k), g(k+1), \ldots, g(k-1)).$$

Then

$$A(g) = (T_0(g)W_N \mid T_1(g)W_N \mid \cdots \mid T_{N-1}(g)W_N).$$

The matrix $A(g)$ has the following remarkable properties.

Theorem 4.1.1 (Lawrence-Pfander-Walnut [LPW05])**.** *Suppose that N is prime. The product of all $K \times K$ ($1 \leq K \leq N$) determinants of $A(g)$, interpreted as a polynomial in the indeterminates $g(0), \ldots, g(N-1)$, does not vanish identically.*

Theorem 4.1.2 (Malikiosis [Mal15])**.** *The product of all $N \times N$ determinants of $A(g)$, interpreted as a polynomial in the indeterminates $g(0), \ldots, g(N-1)$, does not vanish identically.*

The complement of the zero set of the polynomial in Theorem 4.1.2 is a dense open set of full measure. For every g in this complementary set, every $N \times N$ minor of $A(g)$ is invertible, i.e., $A(g)$ is full spark. In particular, for $S \subseteq \mathbb{Z}/N\mathbb{Z} \times (\mathbb{Z}/N\mathbb{Z})^{\widehat{}}$, the condition $|S| \leq N$ is sufficient for the identifiability of $\mathcal{O}_S$.

If $\mathbb{A}$ is not cyclic, then, for $S \subseteq \mathbb{A} \times \widehat{\mathbb{A}}$, the condition $\mu_{\mathbb{A}\times\widehat{\mathbb{A}}}(S) \leq 1$ may not be sufficient for the identifiability of $\mathcal{O}_S$. Counterexamples exist even for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ [Pfa13a]. The discrete Fourier transform matrix for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

where the elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ have been ordered as

$$(0,0),(0,1),(1,0),(1,1).$$

Let $g \in \mathbb{C}^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$. Let

$$(c_1, c_2, c_3, c_4) = (g(0,0), g(0,1), g(1,0), g(1,1)).$$

The translations of g as we run through

$$(0,0),(0,1),(1,0),(1,1)$$

correspond to the columns of the matrix

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_1 & c_4 & c_3 \\ c_3 & c_4 & c_1 & c_2 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix}.$$

Then

$$A(g) = \begin{pmatrix} c_1 & c_1 & c_1 & c_1 & c_2 & c_2 & c_2 & c_2 & \cdots \\ c_2 & -c_2 & c_2 & -c_2 & c_1 & -c_1 & c_1 & -c_1 & \cdots \\ c_3 & c_3 & -c_3 & -c_3 & c_4 & c_4 & -c_4 & -c_4 & \cdots \\ c_4 & -c_4 & -c_4 & c_4 & c_3 & -c_3 & -c_3 & c_3 & \cdots \end{pmatrix.$$

$$\begin{matrix} \cdots & c_3 & c_3 & c_3 & c_3 & c_4 & c_4 & c_4 & c_4 \\ \cdots & c_4 & -c_4 & c_4 & -c_4 & c_3 & -c_3 & c_3 & -c_3 \\ \cdots & c_1 & c_1 & -c_1 & -c_1 & c_2 & c_2 & -c_2 & -c_2 \\ \cdots & c_2 & -c_2 & -c_2 & c_2 & c_1 & -c_1 & -c_1 & c_1 \end{matrix}\right).$$

Of the $\binom{16}{4} = 1820$ 4×4 determinants of $A(g)$, 240 of them are identically zero. For example, the determinant of the matrix corresponding to columns $1, 2, 5, 8$ is identically zero. Therefore, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\hat{}$ has 240 subsets S with $|S| = 4$ for which $\mathcal{O}_S$ is not identifiable. However, there are additional conditions we can impose on $S \subseteq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\hat{}$ to guarantee the identifiability of $\mathcal{O}_S$. We give one example.

Let $c \in \mathbb{C}^{\mathbb{Z}/2\mathbb{Z}}$. Then

$$A(c) = \begin{pmatrix} c_0 & c_0 & c_1 & c_1 \\ c_1 & -c_1 & c_0 & -c_0 \end{pmatrix}.$$

The matrix $A(c)$ is full spark if and only if $c_0 c_1 (c_0 - c_1)(c_0 + c_1) \neq 0$. Let c be chosen so that $A(c)$ is full spark. Let $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \{0\}$ and $\Lambda = \{0\} \times \{0\}$. Then

$\Gamma^\perp = \{0\} \times \mathbb{Z}/2\mathbb{Z}$ and $\Lambda^\perp = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Note that $\Lambda^\perp/\Gamma \cong \mathbb{Z}/2\mathbb{Z}$. Let

$$g = \sum_{v^\perp+\Gamma \in \Lambda^\perp/\Gamma} c_{v^\perp+\Gamma} T_{v^\perp} \sum_{w\in\Gamma} T_w \delta_G.$$

Then

$$(g(0,0), g(0,1), g(1,0), g(1,1)) = (c_0, c_1, c_0, c_1).$$

Let $S \subseteq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\widehat{}}$. By Theorem 4.4.5, $\mathcal{O}_S$ is identifiable by g if and only if *(a)* the translations of S by $\Gamma \times \Lambda$ are disjoint, and *(b)* no three of the translations of S by $\Lambda^\perp \times \Gamma^\perp$ have nonempty intersection. We used Mathematica to numerically verify Theorem 4.4.5 in this particular case. Note that

$$A(g) = \left(\begin{matrix} c_0 & c_0 & c_0 & c_0 & c_1 & c_1 & c_1 & c_1 & \cdots \\ c_1 & -c_1 & c_1 & -c_1 & c_0 & -c_0 & c_0 & -c_0 & \cdots \\ c_0 & c_0 & -c_0 & -c_0 & c_1 & c_1 & -c_1 & -c_1 & \cdots \\ c_1 & -c_1 & -c_1 & c_1 & c_0 & -c_0 & -c_0 & c_0 & \cdots \end{matrix}\right.$$

$$\left.\begin{matrix} \cdots & c_0 & c_0 & c_0 & c_0 & c_1 & c_1 & c_1 & c_1 \\ \cdots & c_1 & -c_1 & c_1 & -c_1 & c_0 & -c_0 & c_0 & -c_0 \\ \cdots & c_0 & c_0 & -c_0 & -c_0 & c_1 & c_1 & -c_1 & -c_1 \\ \cdots & c_1 & -c_1 & -c_1 & c_1 & c_0 & -c_0 & -c_0 & c_0 \end{matrix}\right).$$

Of the $\binom{16}{4} = 1820$ subsets of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{\widehat{}}$ of size 4, 576 of them satisfy both *(a)* and *(b)*. Each of the corresponding 4×4 determinants belongs to the list

$$\pm 16c_0^2c_1^2, \quad \pm 8c_0c_1(c_0-c_1)(c_0+c_1), \quad \pm 8c_0c_1(c_0^2+c_1^2),$$

$$\pm 4(c_0-c_1)^2(c_0+c_1)^2, \quad \pm 4(c_0-c_1)(c_0+c_1)(c_0^2+c_1^2), \quad \pm 4(c_0^2+c_1^2)^2.$$

Since $A(c)$ is full spark, none of these are equal to zero. Therefore, for each of these 576 subsets S, $\mathcal{O}_S$ is identifiable by g, as predicted by Theorem 4.4.5. For the remaining 1244 subsets of size 4, the corresponding 4×4 determinants are all zero. Therefore, for each of these 1244 subsets S, $\mathcal{O}_S$ is not identifiable by g, as predicted by Theorem 4.4.5.

4.2 The Zak Transform and Quasi-Periodization

Let Γ be a lattice in G as described in Section 2.5. Let D be the canonical fundamental domain of Γ as described in Section 2.5.

Let $f \in M^1(G)$. We define the Zak transform of f as

$$Z_\Gamma f(a, \hat{a}) = \sum_{w \in \Gamma} f(a+w)(-w, \hat{a})$$

for $a \in G$ and $\hat{a} \in \widehat{G}$. (See [Grö01, Chapter 8] and [Grö98] for a more comprehensive discussion of the Zak transform.) By Proposition 2.6.27, this series converges uniformly absolutely. Therefore, $Z_\Gamma f$ is continuous.

Proposition 4.2.1. *The Zak transform has the following quasi-periodicity property:*

(a) $Z_\Gamma f(a+k, \hat{a}) = (k, \hat{a}) Z_\Gamma f(a, \hat{a}) \qquad (k \in \Gamma).$

(b) $Z_\Gamma f(a, \hat{a}+k^\perp) = Z_\Gamma f(a, \hat{a}) \qquad (k^\perp \in \Gamma^\perp).$

Proof. *(a)*

$$\begin{aligned} Z_\Gamma f(a+k,\hat{a}) &= \sum_{w\in\Gamma} f(a+k+w)(-w,\hat{a}) \\ &= \sum_{w\in\Gamma} f(a+w+k)(-w-k,\hat{a})(k,\hat{a}) \\ &= (k,\hat{a})Z_\Gamma f(a,\hat{a}). \end{aligned}$$

(b)

$$\begin{aligned} Z_\Gamma f(a,\hat{a}+k^\perp) &= \sum_{w\in\Gamma} f(a+w)(-w,\hat{a}+k^\perp) \\ &= \sum_{w\in\Gamma} f(a+w)(-w,\hat{a})(-w,k^\perp) \\ &= \sum_{w\in\Gamma} f(a+w)(-w,\hat{a}) \\ &= Z_\Gamma f(a,\hat{a}). \end{aligned}$$

□

Since $Z_\Gamma f$ is determined by its values on $D \times D^\perp$, we identify $Z_\Gamma f$ with its restriction to $D \times D^\perp$.

Proposition 4.2.2. *The Zak transform has the following diagonalization property:*

$$Z_\Gamma T_k M_{k^\perp} f = M_{(k^\perp,-k)} Z_\Gamma f \qquad (k \in \Gamma,\ k^\perp \in \Gamma^\perp).$$

Proof.

$$
\begin{aligned}
Z_\Gamma T_k M_{k^\perp} f(a,\hat{a}) &= \sum_{w\in\Gamma} T_k M_{k^\perp} f(a+w)(-w,\hat{a}) \\
&= \sum_{w\in\Gamma} M_{k^\perp} f(a+w-k)(-w,\hat{a}) \\
&= \sum_{w\in\Gamma} (a+w-k,k^\perp) f(a+w-k)(-w,\hat{a}) \\
&= \sum_{w\in\Gamma} (a,k^\perp) f(a+w-k)(-w+k,\hat{a})(-k,\hat{a}) \\
&= (a,k^\perp)(-k,\hat{a}) \sum_{w\in\Gamma} f(a+w-k)(-w+k,\hat{a}) \\
&= M_{(k^\perp,-k)} Z_\Gamma f(a,\hat{a}).
\end{aligned}
$$

□

For $a \in G$, the function $w \to f(a+w)$ on Γ is in $\ell^1(\Gamma) \subseteq \ell^2(\Gamma)$. By the Plancherel theorem,

$$
\begin{aligned}
\sum_{w\in\Gamma} |f(a+w)|^2 &= \int_{\widehat{G}/\Gamma^\perp} |\sum_{w\in\Gamma} f(a+w)(-w,\hat{a})|^2 \, d(\hat{a}+\Gamma^\perp) \\
&= \int_{\widehat{G}/\Gamma^\perp} |Z_\Gamma f(a,\hat{a})|^2 \, d(\hat{a}+\Gamma^\perp)
\end{aligned}
$$

for all $a \in G$. Then

$$
\begin{aligned}
& s(\Gamma) \int_{G/\Gamma} \int_{\widehat{G}/\Gamma^\perp} |Z_\Gamma f(a,\hat{a})|^2 \, d(\hat{a}+\Gamma^\perp) \, d(a+\Gamma) = \\
& s(\Gamma) \int_{G/\Gamma} \sum_{w\in\Gamma} |f(a+w)|^2 \, d(a+\Gamma) = \\
& \int_G |f(a)|^2 \, da.
\end{aligned}
$$

If we normalize the Haar measure on $G/\Gamma \times \widehat{G}/\Gamma^\perp$ to have total measure $s(\Gamma)$, we have shown that $\|f\|_2 = \|Z_\Gamma f\|_2$. Since $M^1(G)$ is dense in $L^2(G)$, Z_Γ extends uniquely to an isometric linear map from $L^2(G)$ to $L^2(G/\Gamma \times \widehat{G}/\Gamma^\perp)$.

In fact, this map is surjective as follows. Since G/Γ is compact, $\{M_{k^\perp} 1_{G/\Gamma}\}_{k^\perp \in \Gamma^\perp}$ is an orthonormal basis for $L^2(G/\Gamma)$. In other words, $\{M_{k^\perp} \mathbb{1}_D\}_{k^\perp \in \Gamma^\perp}$ is an orthogonal basis for $L^2(D)$. Then $\{T_k M_{k^\perp} \mathbb{1}_D\}_{k \in \Gamma, k^\perp \in \Gamma^\perp}$ is an orthogonal basis for $L^2(G)$. It follows from the diagonalization property of the Zak transform and the following lemma that the Zak transform maps $\{T_k M_{k^\perp} \mathbb{1}_D\}_{k \in \Gamma, k^\perp \in \Gamma^\perp}$ onto $\{M_{(k, -k^\perp)} \mathbb{1}_{D \times D^\perp}\}_{k \in \Gamma, k^\perp \in \Gamma^\perp}$, and this latter set is an orthogonal basis for $L^2(D \times D^\perp)$.

Lemma 4.2.3. $Z_\Gamma \mathbb{1}_D = \mathbb{1}_{D \times D^\perp}$.

Proof. Let $K_1 \subseteq K_2 \subseteq \cdots \subseteq D^o$ be compact sets with $D^o = \bigcup K_j$. Let $\psi_j \in C_c^\infty(D^o)$ with $0 \le \psi_j \le 1$ and $\psi_j = 1$ on K_j. Clearly, $\psi_j \to \mathbb{1}_D$ in $L^2(G)$. Then $Z_\Gamma \psi_j \to Z_\Gamma \mathbb{1}_D$ in $L^2(D \times D^\perp)$. Passing to a subsequence, we can assume that $Z_\Gamma \psi_j \to Z_\Gamma \mathbb{1}_D$ almost everywhere. Since $\operatorname{supp} \psi_j \subseteq D^o$, $Z_\Gamma \psi_j(a, \hat{a}) = \psi_j(a)$ for all $a \in D$ and $\hat{a} \in D^\perp$. The result is now immediate. $\square$

Example 4.2.4. Let $g = \sum_{w \in \Gamma} T_w \delta_G$. By Proposition 3.3.3, this series converges in the weak* topology of $M^\infty(G)$. By Proposition 3.1.11,

$$\begin{aligned} V_g f(a,\hat{a}) &= \langle f, M_{\hat{a}} T_a g\rangle \\ &= \langle f, M_{\hat{a}} T_a \sum_{w\in\Gamma} T_w \delta_G\rangle \\ &= \langle f, \sum_{w\in\Gamma} M_{\hat{a}} T_a T_w \delta_G\rangle \\ &= \sum_{w\in\Gamma} \langle f, M_{\hat{a}} T_a T_w \delta_G\rangle \\ &= \sum_{w\in\Gamma} f(a+w)(a+w,-\hat{a}) \\ &= (-a,\hat{a}) Z_\Gamma f(a,\hat{a}). \end{aligned}$$

In particular, $Z_\Gamma f$ is bounded.

Quasi-Periodization

We next study the concept of quasi-periodization introduced in [PW15b]. Let Λ be a lattice in $\widehat{G}$ as described in Section 2.5 such that $\Lambda \subseteq \Gamma^\perp$. Let Ξ be the canonical fundamental domain of Λ as described in Section 2.5. In thinking of Γ and Λ, one should keep in mind Figure 4.1 and refer to it as needed for the remainder of this chapter.

Let $\eta \in M^1(G\times\widehat{G})$. We define the quasi-periodization of η as

$$\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a}) = \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda} \eta(a+w,\hat{a}+\upsilon)(-w,\hat{a})$$

for $a\in G$ and $\hat{a}\in\widehat{G}$. By Proposition 2.6.27, this series converges uniformly absolutely. Therefore, $\mathcal{QP}_{\Gamma,\Lambda}\eta$ is continuous. It is immediate from the definition that

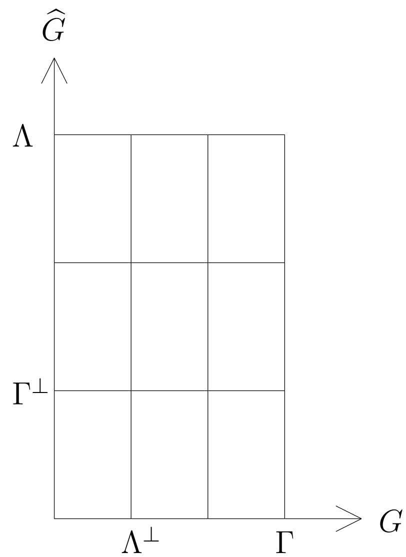

Figure 4.1: The lattices Γ and Λ.

$\mathcal{QP}_{\Gamma,\Lambda}\eta$ is quasi-periodic, i.e.,

$$\mathcal{QP}_{\Gamma,\Lambda}\eta(a+k,\hat{a}) = (k,\hat{a})\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a}) \qquad (k \in \Gamma)$$

and

$$\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a}+\ell) = \mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a}) \qquad (\ell \in \Lambda).$$

Note that the inclusion $\Lambda \subseteq \Gamma^\perp$ is crucial here. Since $\mathcal{QP}_{\Gamma,\Lambda}\eta$ is determined by its values on $D \times \Xi$, we identify $\mathcal{QP}_{\Gamma,\Lambda}\eta$ with its restriction to $D \times \Xi$.

Proposition 4.2.5 and Proposition 4.2.6 below will be used in the proof of Theorem 4.4.5.

Proposition 4.2.5. *Quasi-periodization has the following diagonalization property:*

$$\mathcal{QP}_{\Gamma,\Lambda}M_{(k^\perp,\ell^\perp)}T_{(k,\ell)}\eta = M_{(k^\perp,-k+\ell^\perp)}\mathcal{QP}_{\Gamma,\Lambda}\eta \qquad (k \in \Gamma,\ k^\perp \in \Gamma^\perp,\ \ell \in \Lambda,\ \ell^\perp \in \Lambda^\perp).$$

Proof.

$$
\begin{aligned}
\mathcal{QP}_{\Gamma,\Lambda} M_{(k^\perp,\ell^\perp)} T_{(k,\ell)} \eta(a,\hat{a}) &= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda} M_{(k^\perp,\ell^\perp)} T_{(k,\ell)} \eta(a+w,\hat{a}+\upsilon)(-w,\hat{a}) \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda} (a+w,k^\perp)(\ell^\perp,\hat{a}+\upsilon)\cdots \\
&\qquad \eta(a+w-k,\hat{a}+\upsilon-\ell)(-w,\hat{a}) \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda} (a,k^\perp)(\ell^\perp,\hat{a})\cdots \\
&\qquad \eta(a+w-k,\hat{a}+\upsilon-\ell)(-w+k,\hat{a})(-k,\hat{a}) \\
&= (a,k^\perp)(-k+\ell^\perp,\hat{a})\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a}) \\
&= M_{(k^\perp,-k+\ell^\perp)}\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a}).
\end{aligned}
$$

□

Proposition 4.2.6.

$$
\chi M_{(\hat{c},c)} T_{(b,\hat{b})}\eta = (-b,\hat{b}) M_{(\hat{b}+\hat{c},b+c)} T_{(b,\hat{b})}(\chi\eta).
$$

Proposition 4.2.7. *Suppose that* $\mu_{G\times\widehat{G}}(\operatorname{supp}\eta \cap (\operatorname{supp}\eta + (k,\ell))) = 0$ *for all* $k\in\Gamma$ *and* $\ell\in\Lambda$ *with* $(k,\ell)\neq(0,0)$*. Then*

$$
\|\eta\|_2^2 = \int_{D\times\Xi} |\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a})|^2\, da\, d\hat{a}.
$$

Proof.

$$\begin{aligned}\int_{D\times\Xi}|\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a})|^2\,da\,d\hat{a} &= \int_{D\times\Xi}|\sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\eta(a+w,\hat{a}+\upsilon)(-w,\hat{a})|^2\,da\,d\hat{a}\\ &= \int_{D\times\Xi}\sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}|\eta(a+w,\hat{a}+\upsilon)|^2\,da\,d\hat{a}\\ &= \int_{G\times\widehat{G}}|\eta(a,\hat{a})|^2\,da\,d\hat{a}\\ &= \|\eta\|_2^2.\end{aligned}$$

□

Example 4.2.8. Let $h=\sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}T_{(w,\upsilon)}\delta_{G\times\widehat{G}}$. By Proposition 3.3.3, this series converges in the weak* topology of $M^\infty(G\times\widehat{G})$. By Proposition 3.1.11,

$$\begin{aligned}V_h\eta(a,\hat{a},\hat{a},0) &= \langle\eta, M_{(\hat{a},0)}T_{(a,\hat{a})}h\rangle\\ &= \langle\eta, M_{(\hat{a},0)}T_{(a,\hat{a})}\sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}T_{(w,\upsilon)}\delta_{G\times\widehat{G}}\rangle\\ &= \langle\eta, \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}M_{(\hat{a},0)}T_{(a,\hat{a})}T_{(w,\upsilon)}\delta_{G\times\widehat{G}}\rangle\\ &= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\langle\eta, M_{(\hat{a},0)}T_{(a,\hat{a})}T_{(w,\upsilon)}\delta_{G\times\widehat{G}}\rangle\\ &= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\eta(a+w,\hat{a}+\upsilon)(a+w,-\hat{a})\\ &= (-a,\hat{a})\mathcal{QP}_{\Gamma,\Lambda}\eta(a,\hat{a}).\end{aligned}$$

In particular, $\mathcal{QP}_{\Gamma,\Lambda}\eta$ is bounded.

4.3 Discretization of Operators

Let Γ be a lattice in G as described in Section 2.5. Let D be the canonical fundamental domain of Γ as described in Section 2.5. Let $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)$. Recall that $\chi(a, \hat{a}) = (a, \hat{a})$ for $a \in G$ and $\hat{a} \in \widehat{G}$. The following result generalizes [PW15b, Lemma 3.2].

Proposition 4.3.1. *Let $g = \sum_{w\in\Gamma} T_w \delta_G$. Then $Z_\Gamma \mathcal{K} g = \mu_{\widehat{G}}(D^\perp)\mathcal{Q}\mathcal{P}_{\Gamma,\Gamma^\perp}(\chi\eta_{\mathcal{K}})$.*

Proof. Let $f \in M^1(G)$. We have

$$\begin{aligned}
\langle \mathcal{K}g, f\rangle &= \langle \eta_{\mathcal{K}}, V_g f\rangle \\
&= \langle \chi\eta_{\mathcal{K}}, Z_\Gamma f\rangle && (4.3.2)\\
&= \int_{G\times\widehat{G}} \chi(a,\hat{a})\eta_{\mathcal{K}}(a,\hat{a})\overline{Z_\Gamma f(a,\hat{a})}\, da\, d\hat{a} \\
&= \int_{G/\Gamma\times\widehat{G}/\Gamma^\perp} \sum_{w\in\Gamma}\sum_{w^\perp\in\Gamma^\perp} \chi(a+w,\hat{a}+w^\perp)\eta_{\mathcal{K}}(a+w,\hat{a}+w^\perp)\cdots && (4.3.3)\\
&\qquad\qquad \overline{Z_\Gamma f(a+w,\hat{a}+w^\perp)}\, d(a+\Gamma)\, d(\hat{a}+\Gamma^\perp) \\
&= \int_{G/\Gamma\times\widehat{G}/\Gamma^\perp} \sum_{w\in\Gamma}\sum_{w^\perp\in\Gamma^\perp} \chi(a+w,\hat{a}+w^\perp)\eta_{\mathcal{K}}(a+w,\hat{a}+w^\perp)\cdots && (4.3.4)\\
&\qquad\qquad (-w,\hat{a})\overline{Z_\Gamma f(a,\hat{a})}\, d(a+\Gamma)\, d(\hat{a}+\Gamma^\perp) \\
&= \int_{G/\Gamma\times\widehat{G}/\Gamma^\perp} \mathcal{Q}\mathcal{P}_{\Gamma,\Gamma^\perp}(\chi\eta_{\mathcal{K}})(a,\hat{a})\overline{Z_\Gamma f(a,\hat{a})}\, d(a+\Gamma)\, d(\hat{a}+\Gamma^\perp) \\
&= \langle \mathcal{Q}\mathcal{P}_{\Gamma,\Gamma^\perp}(\chi\eta_{\mathcal{K}}), Z_\Gamma f\rangle,
\end{aligned}$$

where (4.3.2) follows from Example 4.2.4, (4.3.3) follows from (2.5.1), and (4.3.4) follows from the quasi-periodicity of the Zak transform. Since the Zak transform is an L^2 isometry, we have

$$\langle Z_\Gamma \mathcal{K} g, Z_\Gamma f\rangle = \mu_{\widehat{G}}(D^\perp)\langle \mathcal{Q}\mathcal{P}_{\Gamma,\Gamma^\perp}(\chi\eta_\mathcal{K}), Z_\Gamma f\rangle.$$

Here, the inner products are taken with respect to the unit Haar measure on $G/\Gamma \times \widehat{G}/\Gamma^\perp$. The factor $\mu_{\widehat{G}}(D^\perp)$ appears because the Haar measure on $G/\Gamma \times \widehat{G}/\Gamma^\perp$ must be normalized for the Zak transform to be an L^2 isometry. Since $M^1(G)$ is dense in $L^2(G)$, $Z_\Gamma M^1(G)$ is dense in $L^2(G/\Gamma \times \widehat{G}/\Gamma^\perp)$. The result is now immediate. □

Lemma 4.3.5. $\eta_{\mathcal{K}T_a} = T_{(a,0)}\eta_\mathcal{K}$.

Proof. Let $f, g \in M^1(G)$. We have

$$\begin{aligned}\langle \mathcal{K}T_a g, f\rangle &= \langle \eta_\mathcal{K}, V_{T_a g} f\rangle \\ &= \langle \eta_\mathcal{K}, T_{(-a,0)} V_g f\rangle \\ &= \langle T_{(a,0)}\eta_\mathcal{K}, V_g f\rangle.\end{aligned}$$

□

Let Λ be a lattice in $\widehat{G}$ as described in Section 2.5 such that $\Lambda \subseteq \Gamma^\perp$. Observe that the annihilator subgroup of $\Gamma^\perp/\Lambda$ is $\Gamma \subseteq \Lambda^\perp$. It follows that the dual group of $\Gamma^\perp/\Lambda$ is $\Lambda^\perp/\Gamma$. Since $\Gamma^\perp/\Lambda$ is finite, $\Gamma^\perp/\Lambda \cong \Lambda^\perp/\Gamma$. The following result generalizes [PW15b, Lemma 3.7].

Proposition 4.3.6. *Let* $c \in \mathbb{C}^{\Lambda^\perp/\Gamma}$ *and*

$$g = \sum_{v^\perp+\Gamma \in \Lambda^\perp/\Gamma} c_{v^\perp+\Gamma} T_{v^\perp} \sum_{w\in\Gamma} T_w \delta_G.$$

Let $\mathbf{Z}_\Gamma \mathcal{K} g \in \mathbb{C}^{(\Gamma^\perp/\Lambda)\times G\times\widehat{G}}$ *be defined by*

$$\mathbf{Z}_\Gamma \mathcal{K} g(\ell^\perp + \Gamma, a, \hat{a}) = (-\ell^\perp, \hat{a}) Z_\Gamma \mathcal{K} g(a + \ell^\perp, \hat{a}).$$

Let $\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda} \in \mathbb{C}^{(\Lambda^\perp/\Gamma)\times(\Gamma^\perp/\Lambda)\times G\times\widehat{G}}$ *be defined by*

$$\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}(\upsilon^\perp + \Gamma, w^\perp + \Lambda, a, \hat{a}) = (-\upsilon^\perp, \hat{a} + w^\perp) \mathcal{Q}\mathcal{P}_{\Gamma,\Lambda}(\chi \eta_{\mathcal{K}})(a + \upsilon^\perp, \hat{a} + w^\perp).$$

Then

$$\mathbf{Z}_\Gamma \mathcal{K} g = \mu_{\widehat{G}}(D^\perp) A(c) \boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}.$$

Proof. We have

$$\begin{aligned}
Z_\Gamma \mathcal{K} g &= \sum_{\upsilon^\perp + \Gamma \in \Lambda^\perp/\Gamma} c_{\upsilon^\perp+\Gamma} Z_\Gamma \mathcal{K} T_{\upsilon^\perp} \sum_{w\in\Gamma} T_w \delta_G \\
&= \mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp + \Gamma \in \Lambda^\perp/\Gamma} c_{\upsilon^\perp+\Gamma} \mathcal{Q}\mathcal{P}_{\Gamma,\Gamma^\perp}(\chi \eta_{\mathcal{K}T_{\upsilon^\perp}}) \\
&= \mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp + \Gamma \in \Lambda^\perp/\Gamma} c_{\upsilon^\perp+\Gamma} \mathcal{Q}\mathcal{P}_{\Gamma,\Gamma^\perp}(\chi T_{(\upsilon^\perp,0)} \eta_{\mathcal{K}}) \\
&= \mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp + \Gamma \in \Lambda^\perp/\Gamma} c_{\upsilon^\perp+\Gamma} \mathcal{Q}\mathcal{P}_{\Gamma,\Gamma^\perp}(M_{(0,\upsilon^\perp)} T_{(\upsilon^\perp,0)}(\chi \eta_{\mathcal{K}})).
\end{aligned}$$

Then

$$
\begin{aligned}
Z_\Gamma \mathcal{K} g(a,\hat{a}) &= \mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} c_{\upsilon^\perp+\Gamma}\mathcal{QP}_{\Gamma,\Gamma^\perp}(M_{(0,\upsilon^\perp)}T_{(\upsilon^\perp,0)}(\chi\eta_\mathcal{K}))(a,\hat{a}) \\
&= \mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} c_{\upsilon^\perp+\Gamma}\cdots \\
&\qquad \sum_{w\in\Gamma}\sum_{w^\perp\in\Gamma^\perp}(\upsilon^\perp,\hat{a}+w^\perp)\chi(a+w-\upsilon^\perp,\hat{a}+w^\perp)\cdots \\
&\qquad\qquad \eta_\mathcal{K}(a+w-\upsilon^\perp,\hat{a}+w^\perp)(-w,\hat{a}) \\
&= \mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} c_{\upsilon^\perp+\Gamma}\cdots \\
&\qquad \sum_{w\in\Gamma}\sum_{w^\perp+\Lambda\in\Gamma^\perp/\Lambda}\sum_{\upsilon\in\Lambda}(\upsilon^\perp,\hat{a}+w^\perp+\upsilon)\cdots \\
&\qquad\qquad \chi(a+w-\upsilon^\perp,\hat{a}+w^\perp+\upsilon)\cdots \\
&\qquad\qquad \eta_\mathcal{K}(a+w-\upsilon^\perp,\hat{a}+w^\perp+\upsilon)(-w,\hat{a}) \\
&= \mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma}\sum_{w^\perp+\Lambda\in\Gamma^\perp/\Lambda} c_{\upsilon^\perp+\Gamma}(\upsilon^\perp,\hat{a}+w^\perp)\cdots \\
&\qquad\qquad \mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_\mathcal{K})(a-\upsilon^\perp,\hat{a}+w^\perp).
\end{aligned}
$$

Let $\ell^\perp + \Gamma \in \Lambda^\perp/\Gamma$. We have

$$
\begin{aligned}
&Z_\Gamma \mathcal{K} g(a+\ell^\perp, \hat{a}) \\
=&\mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} \sum_{w^\perp+\Lambda\in\Gamma^\perp/\Lambda} c_{\upsilon^\perp+\Gamma}(\upsilon^\perp, \hat{a}+w^\perp)\cdots \\
&\qquad \mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_{\mathcal{K}})(a+\ell^\perp-\upsilon^\perp, \hat{a}+w^\perp) \\
=&\mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} \sum_{w^\perp+\Lambda\in\Gamma^\perp/\Lambda} c_{-\upsilon^\perp+\Gamma}(-\upsilon^\perp, \hat{a}+w^\perp)\cdots \\
&\qquad \mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_{\mathcal{K}})(a+\ell^\perp+\upsilon^\perp, \hat{a}+w^\perp) \\
=&\mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} \sum_{w^\perp+\Lambda\in\Gamma^\perp/\Lambda} c_{\ell^\perp-\upsilon^\perp+\Gamma}(\ell^\perp-\upsilon^\perp, \hat{a}+w^\perp)\cdots \\
&\qquad \mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_{\mathcal{K}})(a+\upsilon^\perp, \hat{a}+w^\perp) \\
=&\mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} \sum_{w^\perp+\Lambda\in\Gamma^\perp/\Lambda} c_{\ell^\perp-\upsilon^\perp+\Gamma}(\ell^\perp, w^\perp)(\ell^\perp, \hat{a})(-\upsilon^\perp, \hat{a}+w^\perp)\cdots \\
&\qquad \mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_{\mathcal{K}})(a+\upsilon^\perp, \hat{a}+w^\perp) \\
=&\mu_{\widehat{G}}(D^\perp) \sum_{\upsilon^\perp+\Gamma\in\Lambda^\perp/\Gamma} \sum_{w^\perp+\Lambda\in\Gamma^\perp/\Lambda} (M_{w^\perp+\Lambda}T_{\upsilon^\perp+\Gamma}c)_{\ell^\perp+\Gamma}(\ell^\perp, \hat{a})(-\upsilon^\perp, \hat{a}+w^\perp)\cdots \\
&\qquad \mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_{\mathcal{K}})(a+\upsilon^\perp, \hat{a}+w^\perp)
\end{aligned}
$$

□

4.4 Sufficient Conditions for Identification of Operators

Recall the abstract operator identification problem described in Section 4.1, where we studied the finite dimensional instance of the problem. We now formulate the infinite dimensional theory. Recall that $\mathcal{O}^{\infty,1}(G) \cong M^1(G\times\widehat{G}) \subseteq L^2(G\times\widehat{G})$. We endow $\mathcal{O}^{\infty,1}(G)$ with the L^2 norm induced by $L^2(G\times\widehat{G})$. In other words, for $\mathcal{K} \in$

$\mathcal{O}^{\infty,1}(G)$, $\|\mathcal{K}\|_2 = \|\eta_{\mathcal{K}}\|_2$. For $S \subseteq G \times \widehat{G}$, let $\mathcal{O}^{\infty,1}(G)|S$ be the set of all $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)$ with $\operatorname{supp} \eta_{\mathcal{K}} \subseteq S$. We consider the evaluation map $e_g : \mathcal{O}^{\infty,1}(G) \to M^1(G) \subseteq L^2(G)$ and its restriction $e_g|S : \mathcal{O}^{\infty,1}(G)|S \to L^2(G)$ for $S \subseteq G \times \widehat{G}$. We emphasize that both sides of e_g are endowed with the L^2 norm.

Let Γ be a lattice in G as described in Section 2.5. Let D be the canonical fundamental domain of Γ as described in Section 2.5. Let Λ be a lattice in $\widehat{G}$ as described in Section 2.5 such that $\Lambda \subseteq \Gamma^{\perp}$. Let Ξ be the canonical fundamental domain of Λ as described in Section 2.5. Recall that in Section 2.5 we were not particular about our choice of $\Gamma_{\mathbb{A}}$ and $D_{\mathbb{A}}$. We now make the trivial choice: If $\mathbb{A}$ is cyclic, let $\Gamma_{\mathbb{A}} = \{0\}$ or $\Gamma_{\mathbb{A}} = \mathbb{A}$. In general, make the trivial choice for each cyclic summand. The geometry of Γ is now as simple as possible, which is necessary for the arguments in the following proof to go through. But also note that this is a very minor technical point. We similarly specify Λ.

The following result generalizes part of [PW15b, Theorem 2.8].

Theorem 4.4.1. *Suppose that $\Lambda^{\perp}/\Gamma$ is cyclic. Choose $c \in \mathbb{C}^{\Lambda^{\perp}/\Gamma}$ so that $A(c)$ is full spark. Let*

$$g = \sum_{v^{\perp}+\Gamma \in \Lambda^{\perp}/\Gamma} c_{v^{\perp}+\Gamma} T_{v^{\perp}} \sum_{w \in \Gamma} T_w \delta_G.$$

Let $S \subseteq G \times \widehat{G}$ be measurable. Suppose that

$$\sum_{k \in \Gamma} \sum_{\ell \in \Lambda} \mathbb{1}_{S+(k,\ell)} \leq 1 \tag{4.4.2}$$

and

$$\sum_{\ell^{\perp} \in \Lambda^{\perp}} \sum_{k^{\perp} \in \Gamma^{\perp}} \mathbb{1}_{S+(\ell^{\perp},k^{\perp})} \leq |\Lambda^{\perp}/\Gamma|, \tag{4.4.3}$$

where the inequalities hold almost everywhere. Then $\mathcal{O}^{\infty,1}(G)|S$ is strongly identifiable by g.

Proof. We first elaborate on (4.4.2) and (4.4.3). The statement that (4.4.2) holds pointwise everywhere is equivalent to the statement that, if we partition S into pieces, one piece for each square of $\Gamma \times \Lambda$, and translate all the pieces by $\Gamma \times \Lambda$ to collect them in $D \times \Xi$, the canonical square of $\Gamma \times \Lambda$, there is no overlap between the translated pieces. If (4.4.2) holds almost everywhere, then the set of all points where there is overlap of translated pieces is a set of measure zero. Let $\mathcal{S}_{\Gamma,\Lambda}$ be the indexed collection of all the translated pieces described above.

The statement that (4.4.3) holds pointwise everywhere is equivalent to the statement that, if we partition S into pieces, one piece for each square of $\Lambda^\perp \times \Gamma^\perp$, and translate all the pieces by $\Lambda^\perp \times \Gamma^\perp$ to collect them in $\Xi^\perp \times D^\perp$, the canonical square of $\Lambda^\perp \times \Gamma^\perp$, there are at most $|\Lambda^\perp/\Gamma|$ translated pieces overlapping at any point. If (4.4.3) holds almost everywhere, then the set of all points where there is overlap of more than $|\Lambda^\perp/\Gamma|$ translated pieces is a set of measure zero. Let $\mathcal{S}_{\Lambda^\perp,\Gamma^\perp}$ be the indexed collection of all the translated pieces described above.

Note that $D \times \Xi$ is the disjoint union of $|\Lambda^\perp/\Gamma||\Gamma^\perp/\Lambda| = |\Lambda^\perp/\Gamma|^2$ translations of $\Xi^\perp \times D^\perp$, one translation for each index in $(\Lambda^\perp/\Gamma)\times(\Gamma^\perp/\Lambda)$. For $J \subseteq (\Lambda^\perp/\Gamma)\times(\Gamma^\perp/\Lambda)$ with $|J| \leq |\Lambda^\perp/\Gamma|$, let V_J be the set of all $(a, \hat{a}) \in \Xi^\perp \times D^\perp$ such that *(i)* if $(\ell^\perp, k^\perp) \in J$, then $(a, \hat{a}) + (\ell^\perp, k^\perp)$ is contained in a unique translated piece in $\mathcal{S}_{\Gamma,\Lambda}$, and *(ii)* if $(\ell^\perp, k^\perp) \in (\Lambda^\perp/\Gamma) \times (\Gamma^\perp/\Lambda) \setminus J$, then $(a, \hat{a}) + (\ell^\perp, k^\perp)$ is not contained in any translated piece in $\mathcal{S}_{\Gamma,\Lambda}$. In particular, $(a, \hat{a})$ is contained in exactly $|J|$ translated pieces in $\mathcal{S}_{\Lambda^\perp,\Gamma^\perp}$, but we are also keeping track of where in $D\times\Xi$ each such translated piece would lie were it translated by the coarser lattice $\Gamma\times\Lambda$ instead of the finer lattice $\Lambda^\perp\times\Gamma^\perp$. Note that $V_J\cap V_{J'}=\emptyset$ for distinct $J,J'\subseteq(\Lambda^\perp/\Gamma)\times(\Gamma^\perp/\Lambda)$ with $|J|,|J'|\leq|\Lambda^\perp/\Gamma|$, and $\Xi^\perp\times D^\perp\setminus\bigcup V_J$, where the union is over all J as described above, is a set of measure zero.

By Proposition 4.3.6,

$$\mathbf{Z}_\Gamma\mathcal{K}g=\mu_{\widehat{G}}(D^\perp)A(c)\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\qquad(\mathcal{K}\in\mathcal{O}^{\infty,1}(G)|S).$$

Consider J as described above. By construction, we can choose $|\Lambda^\perp/\Gamma|^2-|J|$ entries of $\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}$ which necessarily vanish on V_J independent of $\mathcal{K}\in\mathcal{O}^{\infty,1}(G)|S$. Since $|J|\leq|\Lambda^\perp/\Gamma|$, we can in fact choose $|\Lambda^\perp/\Gamma|^2-|\Lambda^\perp/\Gamma|$ such entries. For $\mathcal{K}\in\mathcal{O}^{\infty,1}(G)|S$, let $\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}$ be $\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}$ with $|\Lambda^\perp/\Gamma|^2-|\Lambda^\perp/\Gamma|$ such entries removed. Let $A(c)_J$ be $A(c)$ with the corresponding columns removed. Since $A(c)$ is full spark, $A(c)_J$ is invertible. We now have

$$\mathbf{Z}_\Gamma\mathcal{K}g=\mu_{\widehat{G}}(D^\perp)A(c)_J\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\qquad(\mathcal{K}\in\mathcal{O}^{\infty,1}(G)|S)$$

and

$$\mu_{\widehat{G}}(D^\perp)\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}=A(c)_J^{-1}\mathbf{Z}_\Gamma\mathcal{K}g\qquad(\mathcal{K}\in\mathcal{O}^{\infty,1}(G)|S)$$

on V_J. Let $a_J=\|A(c)_J^{-1}\|_2^{-1}$ and $b_J=\|A(c)_J\|_2$. Here, the norm is the Frobenius norm. Then

$$\mu_{\widehat{G}}(D^\perp)^2a_J^2\|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\|_2^2\leq\|\mathbf{Z}_\Gamma\mathcal{K}g\|_2^2\leq\mu_{\widehat{G}}(D^\perp)^2b_J^2\|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\|_2^2\qquad(\mathcal{K}\in\mathcal{O}^{\infty,1}(G)|S)$$

on V_J. Since $\|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda,J}\|_2^2 = \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2$ on V_J,

$$\mu_{\widehat{G}}(D^\perp)^2 a_J^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \leq \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 \leq \mu_{\widehat{G}}(D^\perp)^2 b_J^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \qquad (\mathcal{K} \in \mathcal{O}^{\infty,1}(G)|S)$$

on V_J.

Let $a = \min a_J$ and $b = \min b_J$, where the minimum is over all J as described above. Then

$$\mu_{\widehat{G}}(D^\perp)^2 a^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \leq \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 \leq \mu_{\widehat{G}}(D^\perp)^2 b^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \qquad (\mathcal{K} \in \mathcal{O}^{\infty,1}(G)|S)$$

on $\bigcup V_J$, where the union is over all J as described above. In particular,

$$\mu_{\widehat{G}}(D^\perp)^2 a^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \leq \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 \leq \mu_{\widehat{G}}(D^\perp)^2 b^2 \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 \qquad (\mathcal{K} \in \mathcal{O}^{\infty,1}(G)|S),$$

where the inequality holds almost everywhere on $\Xi^\perp \times D^\perp$. By Proposition 4.2.7,

$$\int_{\Xi^\perp \times D^\perp} \|\boldsymbol{\eta}_{\mathcal{K},\Gamma,\Lambda}\|_2^2 = \int_{D \times \Xi} |\mathcal{Q}\mathcal{P}_{\Gamma,\Lambda}(\chi \eta_{\mathcal{K}})(a,\hat{a})|^2 \, da \, d\hat{a} = \|\eta_{\mathcal{K}}\|_2^2 \qquad (\mathcal{K} \in \mathcal{O}^{\infty,1}(G)|S).$$

Since the Zak transform is an L^2 isometry,

$$\int_{\Xi^\perp \times D^\perp} \|\mathbf{Z}_\Gamma \mathcal{K} g\|_2^2 = \int_{D \times D^\perp} |Z_\Gamma \mathcal{K} g(a,\hat{a})|^2 \, da \, d\hat{a} = \mu_{\widehat{G}}(D^\perp) \|\mathcal{K} g\|_2^2 \qquad (\mathcal{K} \in \mathcal{O}^{\infty,1}(G)|S).$$

It follows that

$$\mu_{\widehat{G}}(D^\perp) a^2 \|\eta_{\mathcal{K}}\|_2^2 \leq \|e_g \mathcal{K}\|_2^2 \leq \mu_{\widehat{G}}(D^\perp) b^2 \|\eta_{\mathcal{K}}\|_2^2 \qquad (\mathcal{K} \in \mathcal{O}^{\infty,1}(G)|S).$$

We have shown that $e_g|S$ is bounded and stable. $\square$

Remark. Note that

$$\mu_{G \times \widehat{G}}(\Xi^\perp \times D^\perp) = \mu_G(\Xi^\perp) \mu_{\widehat{G}}(D^\perp) = \mu_G(\Xi^\perp)/\mu_G(D) = 1/|\Lambda^\perp/\Gamma|.$$

It follows that $\mu_{G\times\widehat{G}}(S) \leq 1$ in Theorem 4.4.1.

The following result generalizes [PW06a, Theorem 3.1].

Corollary 4.4.4. *Suppose that G has at most one finite cyclic summand. Let $S \subseteq G \times \widehat{G}$ be compact with $\mu_{G\times\widehat{G}}(S) < 1$. Then $\mathcal{O}^{\infty,1}(G)|S$ is strongly identifiable.*

Proof. It suffices to specify Γ and Λ so that $\Lambda^\perp/\Gamma$ is cyclic, and (4.4.2) and (4.4.3) are satisfied. Since S is compact, we can satisfy (4.4.2) by making Γ and Λ sufficiently coarse. Note that as Γ and Λ become coarser, $\Gamma^\perp$ and $\Lambda^\perp$ become finer. Since S is compact with $\mu_{G\times\widehat{G}}(S) < 1$, and $\mu_{G\times\widehat{G}}$ is outer regular, we can make $\Gamma^\perp$ and $\Lambda^\perp$ even finer so that S is covered by at most $|\Lambda^\perp/\Gamma|$ translations of $\Xi^\perp \times D^\perp$ by $\Lambda^\perp \times \Gamma^\perp$. We thus satisfy (4.4.3). It remains to ensure that $\Lambda^\perp/\Gamma$ is cyclic. Since there are infinitely many primes, we can ensure that the elementary divisors of $(\Lambda^\perp_{\mathbb{R}}/\Gamma_{\mathbb{R}}) \times (\Lambda^\perp_{\mathbb{T}}/\Gamma_{\mathbb{T}}) \times (\Lambda^\perp_{\mathbb{Z}}/\Gamma_{\mathbb{Z}})$ are distinct primes. The key observation here is that the sizes of $\Lambda^\perp_{\mathbb{R}}/\Gamma_{\mathbb{R}}$, $\Lambda^\perp_{\mathbb{T}}/\Gamma_{\mathbb{T}}$, and $\Lambda^\perp_{\mathbb{Z}}/\Gamma_{\mathbb{Z}}$ are unconstrained as $\Gamma^\perp$ and $\Lambda^\perp$ become finer. If G has a finite cyclic summand, then we also have to ensure that none of these distinct primes divide the order of the finite cyclic summand. $\square$

The following result generalizes [PW15b, Theorem 2.8] in full.

Theorem 4.4.5. *Suppose that $\Lambda^\perp/\Gamma$ is cyclic. Choose $c \in \mathbb{C}^{\Lambda^\perp/\Gamma}$ so that $A(c)$ is full spark. Let*

$$g = \sum_{v^\perp+\Gamma\in\Lambda^\perp/\Gamma} c_{v^\perp+\Gamma} T_{v^\perp} \sum_{w\in\Gamma} T_w \delta_G.$$

Let $S \subseteq G \times \widehat{G}$ be open. The following statements are equivalent:

(a) (4.4.2) *and* (4.4.3) *hold pointwise everywhere.*

(b) $\mathcal{O}^{\infty,1}(G)|S$ *is strongly identifiable by* g*.*

(c) $\mathcal{O}^{\infty,1}(G)|S$ *is weakly identifiable by* g*.*

Proof. We have already shown that *(a)* implies *(b)*. That *(b)* implies *(c)* is trivial. We show that *(c)* implies *(a)* via proof by contradiction. Suppose that (4.4.2) does not hold pointwise everywhere. Then there exist $(s, \hat{s}) \in S$ and $(k, \ell) \in \Gamma \times \Lambda \setminus \{(0, 0)\}$ such that $(s, \hat{s}) + (k, \ell) \in S$. Let $\eta \in C_c^\infty(S)$ with $\eta(s, \hat{s}) = 1$, $\operatorname{supp} \eta + (k, \ell) \subseteq S$, and $\operatorname{supp} \eta \cap (\operatorname{supp} \eta + (k, \ell)) = \emptyset$. Let $\mathcal{K}$ be the operator in $\mathcal{O}^{\infty,1}(G)|S$ with spreading function $\eta_{\mathcal{K}} = \eta - (k, \ell) M_{(-\ell,0)} T_{(k,\ell)} \eta$. By Proposition 4.2.6,

$$\chi \eta_{\mathcal{K}} = \chi \eta - (k, \ell) \chi M_{(-\ell,0)} T_{(k,\ell)} \eta = \chi \eta - M_{(0,k)} T_{(k,\ell)} (\chi \eta).$$

By the diagonalization property of quasi-periodization,

$$\mathcal{QP}_{\Gamma,\Lambda}(\chi \eta_{\mathcal{K}}) = \mathcal{QP}_{\Gamma,\Lambda}(\chi \eta) - \mathcal{QP}_{\Gamma,\Lambda}(\chi \eta) = 0.$$

By Proposition 4.3.6, $Z_\Gamma \mathcal{K} g = 0$ and hence $\mathcal{K} g = 0$. Since $\mathcal{K} \neq 0$, we have a contradiction. Therefore, (4.4.2) holds pointwise everywhere.

Suppose now that (4.4.3) does not hold pointwise everywhere. Then there exist $(t, \hat{t}) \in (\Xi^\perp \times D^\perp)^o$ and $J \subseteq \Lambda^\perp \times \Gamma^\perp$ with $|J| = |\Lambda^\perp / \Gamma| + 1$ such that $(t, \hat{t}) + (\ell^\perp, k^\perp) \in S$ for all $(\ell^\perp, k^\perp) \in J$. Moreover, since (4.4.2) holds pointwise everywhere, the elements of J belong to distinct equivalence classes in $(\Lambda^\perp / \Gamma) \times (\Gamma^\perp / \Lambda)$. Let $A(c)_J$ be $A(c)$ with those columns corresponding to $(\Lambda^\perp / \Gamma) \times (\Gamma^\perp / \Lambda) \setminus J$ removed. Since $A(c)$ is full spark and $A(c)_J$ has $|\Lambda^\perp / \Gamma| + 1$ columns, $\dim \ker A(c)_J = 1$. Let $\alpha \in \ker A(c)_J$ be nonzero. Let $\psi \in C_c^\infty((\Xi^\perp \times D^\perp)^o)$ with $\psi(t, \hat{t}) = 1$. Let $\mathcal{H}$ be the operator in $\mathcal{O}^{\infty,1}(G)|S$ with spreading function

$$\eta_{\mathcal{H}} = \sum_{(\ell^\perp,k^\perp)\in J} \alpha(\ell^\perp,k^\perp)(\ell^\perp,k^\perp)M_{(-k^\perp,0)}T_{(\ell^\perp,k^\perp)}\psi.$$

By Proposition 4.2.6,

$$\begin{aligned}\chi\eta_{\mathcal{H}} &= \sum_{(\ell^\perp,k^\perp)\in J} \alpha(\ell^\perp,k^\perp)(\ell^\perp,k^\perp)\chi M_{(-k^\perp,0)}T_{(\ell^\perp,k^\perp)}\psi \\ &= \sum_{(\ell^\perp,k^\perp)\in J} \alpha(\ell^\perp,k^\perp)M_{(0,\ell^\perp)}T_{(\ell^\perp,k^\perp)}(\chi\psi).\end{aligned}$$

Let $(a,\hat{a}) \in \Xi^\perp \times D^\perp$. If $(v^\perp+\Gamma, w^\perp+\Lambda) \in (\Lambda^\perp/\Gamma)\times(\Gamma^\perp/\Lambda)\setminus J$, then $(-v^\perp,\hat{a}+w^\perp)\mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_{\mathcal{H}})(a+v^\perp,\hat{a}+w^\perp)=0$. On the other hand,

$$\begin{aligned}&(-\ell^\perp,\hat{a}+k^\perp)\mathcal{QP}_{\Gamma,\Lambda}(\chi\eta_{\mathcal{H}})(a+\ell^\perp,\hat{a}+k^\perp) \\ =&\alpha(\ell^\perp,k^\perp)(-\ell^\perp,\hat{a}+k^\perp)M_{(0,\ell^\perp)}T_{(\ell^\perp,k^\perp)}(\chi\psi)(a+\ell^\perp,\hat{a}+k^\perp) \\ =&\alpha(\ell^\perp,k^\perp)(\chi\psi)(a,\hat{a})\end{aligned}$$

for all $(\ell^\perp,k^\perp)\in J$. By construction, $A(c)\boldsymbol{\eta}_{\mathcal{H},\Gamma,\Lambda}=0$. By Proposition 4.3.6, $Z_\Gamma\mathcal{H}g = 0$ and hence $\mathcal{H}g=0$. Since $\mathcal{H}\neq 0$, we have a contradiction. Therefore, (4.4.3) holds pointwise everywhere. □

4.5 Necessary Conditions for Identification of Operators

Our goal in this section is to formulate and prove a partial converse to Corollary 4.4.4 as best as we can. We begin with a duality result for identification of operators in the spirit of the Plancherel theorem. Recall that, for $\mathcal{K}\in\mathcal{O}^{\infty,1}(G)$, $\mathcal{K}_{\mathcal{F}}$ is the operator in $\mathcal{O}^{\infty,1}(\widehat{G})$ with spreading function $\eta_{\mathcal{K}_{\mathcal{F}}}$, where $\eta_{\mathcal{K}_{\mathcal{F}}}(\hat{a},a) =$

$(-a,\hat{a})\eta_{\mathcal{K}}(-a,\hat{a})$. Let $S \subseteq G \times \widehat{G}$. Let $S_{\mathcal{F}} = \{(\hat{a},a) \in \widehat{G} \times G : (-a,\hat{a}) \in S\}$. The map $\mathcal{K} \to \mathcal{K}_{\mathcal{F}}$ from $\mathcal{O}^{\infty,1}(G)|S$ to $\mathcal{O}^{\infty,1}(\widehat{G})|S_{\mathcal{F}}$ is an L^2 isometric isomorphism. Let $g \in M^{\infty}(G)$. By Proposition 3.5.5, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}^{\infty,1}(G)|S & \xrightarrow{e_g} & L^2(G) \\ {\scriptstyle \mathcal{K}\to\mathcal{K}_{\mathcal{F}}}\big\downarrow & & {\scriptstyle \mathcal{F}}\big\downarrow \\ \mathcal{O}^{\infty,1}(\widehat{G})|S_{\mathcal{F}} & \xrightarrow{e_{\hat{g}}} & L^2(\widehat{G}) \end{array} \tag{4.5.1}$$

Therefore, we have the following result.

Theorem 4.5.2. *$\mathcal{O}^{\infty,1}(G)|S$ is strongly identifiable by g if and only if $\mathcal{O}^{\infty,1}(\widehat{G})|S_{\mathcal{F}}$ is strongly identifiable by $\hat{g}$.*

Let $S \subseteq G \times \widehat{G}$ and $g \in M^{\infty}(G)$. We would like to study under what conditions $\mathcal{O}^{\infty,1}(G)|S$ is not strongly identifiable by g. To show that $\mathcal{O}^{\infty,1}(G)|S$ is not strongly identifiable by g, it suffices to show that a subspace $\mathcal{V}$ of $\mathcal{O}^{\infty,1}(G)|S$ is not strongly identifiable by g, where $\mathcal{V}$ is constructed so as to be much easier to work with. We now carry out this program.

Lemma 4.5.3. *Let $\mathcal{K} \in \mathcal{O}^{\infty,1}(G)$. Then*

$$\eta_{M_{\hat{b}}T_b\mathcal{K}T_aM_{\hat{a}}} = (b,\hat{a}+\hat{b})M_{(-\hat{a},-b)}T_{(a+b,\hat{a}+\hat{b})}\eta_{\mathcal{K}}.$$

Proof. Let $f, g \in M^1(G)$. We have

$$\begin{aligned}
\langle M_{\hat{b}} T_b \mathcal{K} T_a M_{\hat{a}} g, f\rangle &= \langle \mathcal{K} T_a M_{\hat{a}} g, T_{-b} M_{-\hat{b}} f\rangle \\
&= \langle \eta_{\mathcal{K}}, V_{T_a M_{\hat{a}} g} T_{-b} M_{-\hat{b}} f\rangle \\
&= \langle \eta_{\mathcal{K}}, (a+b, \hat{a}) M_{(\hat{a}, b)} T_{(-a-b, -\hat{a}-\hat{b})} V_g f\rangle \\
&= \langle (-a-b, \hat{a}) T_{(a+b, \hat{a}+\hat{b})} M_{(-\hat{a}, -b)} \eta_{\mathcal{K}}, V_g f\rangle \\
&= \langle (b, \hat{a}+\hat{b}) M_{(-\hat{a}, -b)} T_{(a+b, \hat{a}+\hat{b})} \eta_{\mathcal{K}}, V_g f\rangle,
\end{aligned}$$

where we have used the covariance property of the STFT. □

Let Γ be a lattice in G as described in Section 2.5. Let D be the canonical fundamental domain of Γ as described in Section 2.5. Let Γ_c be a lattice in G as described in Section 2.5. Let D_c be the canonical fundamental domain of Γ_c as described in Section 2.5. Suppose that there exists $\theta \in G$ such that $D_c + \theta \subseteq D^o$. Let $\eta_{\Gamma,\Gamma_c} \in C_c^\infty(G)$ with $0 \le \eta_{\Gamma,\Gamma_c} \le 1$, $\eta_{\Gamma,\Gamma_c} = 1$ on $D_c + \theta$, and $\eta_{\Gamma,\Gamma_c} = 0$ outside D^o. We denote by $\mathcal{D}_G$ the data that have just been described.

Let Λ be a lattice in $\widehat{G}$ as described in Section 2.5. Let Ξ be the canonical fundamental domain of Λ as described in Section 2.5. Let Λ_c be a lattice in $\widehat{G}$ as described in Section 2.5. Let Ξ_c be the canonical fundamental domain of Λ_c as described in Section 2.5. Suppose that there exists $\hat{\theta} \in \widehat{G}$ such that $\Xi_c + \hat{\theta} \subseteq \Xi^o$. Let $\eta_{\Lambda,\Lambda_c} \in C_c^\infty(\widehat{G})$ with $0 \le \eta_{\Lambda,\Lambda_c} \le 1$, $\eta_{\Lambda,\Lambda_c} = 1$ on $\Xi_c + \hat{\theta}$, and $\eta_{\Lambda,\Lambda_c} = 0$ outside Ξ^o. We denote by $\mathcal{D}_{\widehat{G}}$ the data that have just been described. (See Figure 4.2.)

Let $\mathcal{P}$ be the operator in $\mathcal{O}^{\infty,1}(G)$ with spreading function $\eta_{\mathcal{P}} = \eta_{\Gamma,\Gamma_c} \otimes \eta_{\Lambda,\Lambda_c}$.

The following result generalizes [PW06a, Lemma 4.5(a)].

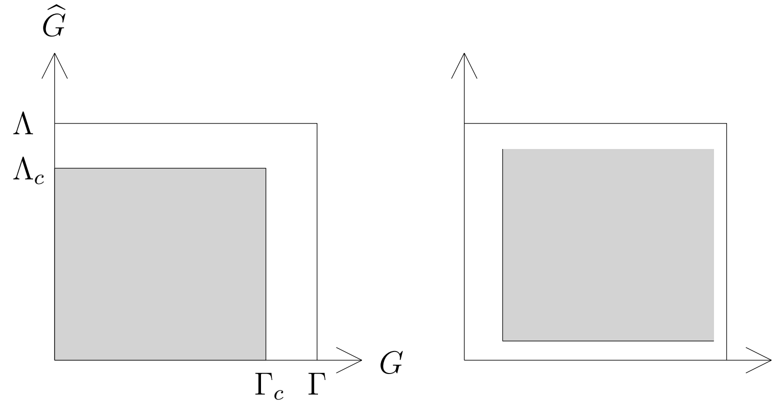

Figure 4.2: $D_c \times \Xi_c$ and its translation by $(\theta, \hat{\theta})$.

Proposition 4.5.4. *The linear map $U : \ell_c(\Gamma \times \Lambda \times \Gamma_c^\perp \times \Lambda_c^\perp) \to \mathcal{O}^{\infty,1}(G)$ defined by*

$$U\sigma = \sum_{w \in \Gamma} \sum_{\upsilon \in \Lambda} \sum_{w_c^\perp \in \Gamma_c^\perp} \sum_{\upsilon_c^\perp \in \Lambda_c^\perp} \sigma(w, \upsilon, w_c^\perp, \upsilon_c^\perp) M_{\upsilon + w_c^\perp} T_{-\upsilon_c^\perp} \mathcal{P} T_{w + \upsilon_c^\perp} M_{-w_c^\perp}$$

is bounded and stable. Here, $\ell_c(\Gamma \times \Lambda \times \Gamma_c^\perp \times \Lambda_c^\perp)$ is endowed with the L^2 norm. Therefore, U extends uniquely to a bounded and stable linear map $U : \ell^2(\Gamma \times \Lambda \times \Gamma_c^\perp \times \Lambda_c^\perp) \to \mathcal{O}^2(G)$. Equivalently,

$$\{M_{\upsilon + w_c^\perp} T_{-\upsilon_c^\perp} \mathcal{P} T_{w + \upsilon_c^\perp} M_{-w_c^\perp}\}_{(w, \upsilon, w_c^\perp, \upsilon_c^\perp) \in \Gamma \times \Lambda \times \Gamma_c^\perp \times \Lambda_c^\perp}$$

is a Riesz basis for its closed linear span in $\mathcal{O}^2(G)$. Here, $\mathcal{O}^2(G)$ is the set of all Hilbert-Schmidt operators on $L^2(G)$; see [Shu01, Appendix 3].

Proof. By Lemma 4.5.3,

$$\eta_{M_{\upsilon + w_c^\perp} T_{-\upsilon_c^\perp} \mathcal{P} T_{w + \upsilon_c^\perp} M_{-w_c^\perp}} = (-\upsilon_c^\perp, \upsilon) M_{(w_c^\perp, \upsilon_c^\perp)} T_{(w, \upsilon)} \eta_{\mathcal{P}}.$$

Let $\rho(w, \upsilon, w_c^\perp, \upsilon_c^\perp) = (w, w_c^\perp)\sigma(w, \upsilon, w_c^\perp, \upsilon_c^\perp)$. We have

$$
\begin{aligned}
\|U\sigma\|_2^2 &= \|\eta_{U\sigma}\|_2^2 \\
&= \|\sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp} \sigma(w, \upsilon, w_c^\perp, \upsilon_c^\perp)(-\upsilon_c^\perp, \upsilon)M_{(w_c^\perp,\upsilon_c^\perp)}T_{(w,\upsilon)}\eta_{\mathcal{P}}\|_2^2 \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\|\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp} \sigma(w, \upsilon, w_c^\perp, \upsilon_c^\perp)(-\upsilon_c^\perp, \upsilon)M_{(w_c^\perp,\upsilon_c^\perp)}T_{(w,\upsilon)}\eta_{\mathcal{P}}\|_2^2 \quad (4.5.5)\\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\|\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp} \rho(w, \upsilon, w_c^\perp, \upsilon_c^\perp)T_{(w,\upsilon)}M_{(w_c^\perp,\upsilon_c^\perp)}\eta_{\mathcal{P}}\|_2^2 \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\|\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp} \rho(w, \upsilon, w_c^\perp, \upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}\eta_{\mathcal{P}}\|_2^2 \quad (4.5.6)\\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\|\eta_{\mathcal{P}}\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp} \rho(w, \upsilon, w_c^\perp, \upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}1_{G\times\widehat{G}}\|_2^2 \quad (4.5.7)\\
&\geq \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\|\mathbb{1}_{(D_c+\theta)\times(\Xi_c+\hat{\theta})}\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp} \rho(w, \upsilon, w_c^\perp, \upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}1_{G\times\widehat{G}}\|_2^2 \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\|\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp} \rho(w, \upsilon, w_c^\perp, \upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}\mathbb{1}_{(D_c+\theta)\times(\Xi_c+\hat{\theta})}\|_2^2 \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp}\|\sigma(w, \upsilon, w_c^\perp, \upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}\mathbb{1}_{(D_c+\theta)\times(\Xi_c+\hat{\theta})}\|_2^2 \quad (4.5.8)\\
&= \mu_{G\times\widehat{G}}(D_c\times\Xi_c)\|\sigma\|_2^2,
\end{aligned}
$$

where (4.5.5) follows from the fact that the translations of $\eta_{\mathcal{P}}$ by $\Gamma\times\Lambda$ have disjoint supports, (4.5.6) follows from the translation invariance of the L^2 norm, and (4.5.8) follows from the Pythagorean theorem. Let K be a finite subset of G such that $D \subseteq \bigcup_{k\in K}(D_c + k)$. Let L be a finite subset of $\widehat{G}$ such that $\Xi \subseteq \bigcup_{\ell\in L}(\Xi_c + \ell)$. Note that

$$|\eta_{\mathcal{P}}|^2 \leq \sum_{k\in K}\sum_{\ell\in L}\mathbb{1}_{(D_c+k)\times(\Xi_c+\ell)}.$$

We now go back to (4.5.7). We have

$$\begin{aligned}
\|U\sigma\|_2^2 &= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\|\eta_{\mathcal{P}}\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp}\rho(w,\upsilon,w_c^\perp,\upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}1_{G\times\widehat{G}}\|_2^2 \\
&\le \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\sum_{k\in K}\sum_{\ell\in L}\|\mathbb{1}_{(D_c+k)\times(\Xi_c+\ell)}\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp}\rho(w,\upsilon,w_c^\perp,\upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}1_{G\times\widehat{G}}\|_2^2 \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\sum_{k\in K}\sum_{\ell\in L}\|\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp}\rho(w,\upsilon,w_c^\perp,\upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}\mathbb{1}_{(D_c+k)\times(\Xi_c+\ell)}\|_2^2 \\
&= \sum_{w\in\Gamma}\sum_{\upsilon\in\Lambda}\sum_{k\in K}\sum_{\ell\in L}\sum_{w_c^\perp\in\Gamma_c^\perp}\sum_{\upsilon_c^\perp\in\Lambda_c^\perp}\|\sigma(w,\upsilon,w_c^\perp,\upsilon_c^\perp)M_{(w_c^\perp,\upsilon_c^\perp)}\mathbb{1}_{(D_c+k)\times(\Xi_c+\ell)}\|_2^2 \\
&= |K||L|\mu_{G\times\widehat{G}}(D_c\times\Xi_c)\|\sigma\|_2^2.
\end{aligned}$$

We have shown that

$$\mu_{G\times\widehat{G}}(D_c\times\Xi_c)\|\sigma\|_2^2 \le \|U\sigma\|_2^2 \le |K||L|\mu_{G\times\widehat{G}}(D_c\times\Xi_c)\|\sigma\|_2^2.$$

In other words, U is bounded and stable. Since $\ell_c(\Gamma\times\Lambda\times\Gamma_c^\perp\times\Lambda_c^\perp)$ is dense in $\ell^2(\Gamma\times\Lambda\times\Gamma_c^\perp\times\Lambda_c^\perp)$, and both $\ell^2(\Gamma\times\Lambda\times\Gamma_c^\perp\times\Lambda_c^\perp)$ and $\mathcal{O}^2(G)$ are complete, U extends uniquely to a bounded and stable linear map $U:\ell^2(\Gamma\times\Lambda\times\Gamma_c^\perp\times\Lambda_c^\perp)\to\mathcal{O}^2(G)$. □

Let J be a finite subset of $\Gamma\times\Lambda$. Let $i_J:\ell_c(J\times\Gamma_c^\perp\times\Lambda_c^\perp)\to\ell_c(\Gamma\times\Lambda\times\Gamma_c^\perp\times\Lambda_c^\perp)$ be the inclusion map. Let $\mathcal{V}_J$ be the image of $U\circ i_J$ in $\mathcal{O}^{\infty,1}(G)$. Note that J determines the maximal spreading support of any operator in $\mathcal{V}_J$. Let $S\subseteq G\times\widehat{G}$ and $g\in M^\infty(G)$. Suppose that $\mathcal{V}_J\subseteq\mathcal{O}^{\infty,1}(G)|S$. Then we can restrict the identification problem to $\mathcal{V}_J$. More specifically, we can consider the stability of $e_g\circ U\circ i_J = e_g|S\circ U\circ i_J$ rather than the stability of $e_g|S$. Of course, if we wish to obtain a negative result, J cannot be too small. To simplify matters even further, we define, if possible, a bounded and stable analysis map $V:L^2(G)\to\ell^2(\mathbb{Z})$, and we consider the stability of

$V \circ e_g | S \circ U \circ i_J$, which is controlled by how the entries of the matrix representation of $V \circ e_g | S \circ U \circ i_J$ decay.

Lemma 4.5.9 ([KP05, Lemma 3.4]). *Let $g \in M^\infty(G)$. There exists a nonnegative continuous function r on G, decreasing faster than any polynomial, such that $|\mathcal{P} M_{\hat{b}} T_b g| \leq r$.*

Proof. Suppose first that $g \in M^1(G)$. We have

$$\begin{aligned}
|\mathcal{P}g(t)| &= \left| \int_{G \times \widehat{G}} \eta_{\mathcal{P}}(a, \hat{a}) M_{\hat{a}} T_a g(t)\, da\, d\hat{a} \right| \\
&= \left| \int_{G \times \widehat{G}} \eta_{\Gamma,\Gamma_c}(a) \eta_{\Lambda,\Lambda_c}(\hat{a})(t, \hat{a}) g(t-a)\, da\, d\hat{a} \right| \\
&= \left| \int_{G} \eta_{\Gamma,\Gamma_c}(a) g(t-a)\, da \right| \left| \int_{\widehat{G}} \eta_{\Lambda,\Lambda_c}(\hat{a})(t, \hat{a})\, d\hat{a} \right| \\
&= |\langle \eta_{\Gamma,\Gamma_c}, T_t \check{g} \rangle| |\hat{\eta}_{\Lambda,\Lambda_c}(-t)| \\
&\leq \|\eta_{\Gamma,\Gamma_c}\|_{M^1} \|g\|_{M^\infty} |\hat{\eta}_{\Lambda,\Lambda_c}(-t)|.
\end{aligned}$$

In particular,

$$\begin{aligned}
|\mathcal{P} M_{\hat{b}} T_b g(t)| &\leq \|\eta_{\Gamma,\Gamma_c}\|_{M^1} \|M_{\hat{b}} T_b g\|_{M^\infty} |\hat{\eta}_{\Lambda,\Lambda_c}(-t)| \\
&= \|\eta_{\Gamma,\Gamma_c}\|_{M^1} \|g\|_{M^\infty} |\hat{\eta}_{\Lambda,\Lambda_c}(-t)|.
\end{aligned}$$

We now lift the restriction that $g \in M^1(G)$. Let $\{g_j\}$ be a sequence in $M^1(G)$ such that $g_j \to g$ in the weak* topology of $M^\infty(G)$. Then $\mathcal{P} M_{\hat{b}} T_b g_j \to \mathcal{P} M_{\hat{b}} T_b g$ in $M^1(G)$. Since the inclusion $M^1(G) \subseteq C_0(G)$ is continuous, $\mathcal{P} M_{\hat{b}} T_b g_j(t) \to \mathcal{P} M_{\hat{b}} T_b g(t)$. By Proposition 3.2.6, $\|g_j\|_{M^\infty} \leq C$. It follows that

$$|\mathcal{P} M_{\hat{b}} T_b g(t)| \leq C \|\eta_{\Gamma,\Gamma_c}\|_{M^1} |\hat{\eta}_{\Lambda,\Lambda_c}(-t)|.$$

We now define $r(t) = C\|\eta_{\Gamma,\Gamma_c}\|_{M^1}|\hat{\eta}_{\Lambda,\Lambda_c}(-t)|$ for $t \in G$, and the result follows. $\square$

Lemma 4.5.10 ([KP05, Lemma 3.4]). *Let $g \in M^\infty(G)$. There exists a nonnegative continuous function $r_{\mathcal{F}}$ on $\widehat{G}$, decreasing faster than any polynomial, such that $|(\mathcal{P}M_{\hat{b}}T_b g)^\wedge| \leq r_{\mathcal{F}}$.*

Proof. Suppose first that $g \in M^1(G)$. By Proposition 3.5.5,

$$\begin{aligned}
|\widehat{\mathcal{P}g}(\hat{t})| &= |\mathcal{P}_{\mathcal{F}}\hat{g}(\hat{t})| \\
&= \left| \int_{G\times\widehat{G}} (-a,\hat{a})\eta_{\mathcal{P}}(-a,\hat{a})M_a T_{\hat{a}}\hat{g}(\hat{t})\, da\, d\hat{a} \right| \\
&= \left| \int_{G\times\widehat{G}} (-a,\hat{a})\eta_{\Gamma,\Gamma_c}(-a)\eta_{\Lambda,\Lambda_c}(\hat{a})(a,\hat{t})\hat{g}(\hat{t}-\hat{a})\, da\, d\hat{a} \right|.
\end{aligned}$$

Note that the term $(-a,\hat{a})$ precludes us from proceeding as in the proof of Lemma 4.5.9. We have

$$\begin{aligned}
|\widehat{\mathcal{P}g}(\hat{t})| &= \left| \int_{G\times\widehat{G}} (-a,\hat{a})\eta_{\Gamma,\Gamma_c}(-a)\eta_{\Lambda,\Lambda_c}(\hat{a})(a,\hat{t})\hat{g}(\hat{t}-\hat{a})\, da\, d\hat{a} \right| \\
&= \left| \int_{\widehat{G}} \left(\int_G \eta_{\Gamma,\Gamma_c}(-a)(a,\hat{t}-\hat{a})\, da \right) \eta_{\Lambda,\Lambda_c}(\hat{a})\hat{g}(\hat{t}-\hat{a})\, d\hat{a} \right| \\
&= \left| \int_{\widehat{G}} \left(\int_G \eta_{\Gamma,\Gamma_c}(a)(-a,\hat{t}-\hat{a})\, da \right) \eta_{\Lambda,\Lambda_c}(\hat{a})\hat{g}(\hat{t}-\hat{a})\, d\hat{a} \right| \\
&= \left| \int_{\widehat{G}} \hat{\eta}_{\Gamma,\Gamma_c}(\hat{t}-\hat{a})\eta_{\Lambda,\Lambda_c}(\hat{a})\hat{g}(\hat{t}-\hat{a})\, d\hat{a} \right| \\
&= \left| \int_{\widehat{G}} \hat{\eta}_{\Gamma,\Gamma_c}(\hat{a})\eta_{\Lambda,\Lambda_c}(\hat{t}-\hat{a})\hat{g}(\hat{a})\, d\hat{a} \right| \\
&= |\langle \hat{\eta}_{\Gamma,\Gamma_c}T_{\hat{t}}\check{\eta}_{\Lambda,\Lambda_c}, \hat{g}\rangle| \\
&\leq \|\hat{\eta}_{\Gamma,\Gamma_c}T_{\hat{t}}\check{\eta}_{\Lambda,\Lambda_c}\|_{M^1}\|\hat{g}\|_{M^\infty} \\
&= \|\hat{\eta}_{\Gamma,\Gamma_c}T_{\hat{t}}\check{\eta}_{\Lambda,\Lambda_c}\|_{M^1}\|g\|_{M^\infty}.
\end{aligned}$$

We can now proceed exactly as in the proof of Lemma 4.5.9 to obtain

$$|(\mathcal{P}M_{\hat{b}}T_b g)\widehat{\ }(\hat{t})| \leq C_{\mathcal{F}}\|\hat{\eta}_{\Gamma,\Gamma_c}T_{\hat{t}}\check{\eta}_{\Lambda,\Lambda_c}\|_{M^1}$$

even without the restriction that $g \in M^1(G)$. We define $r_{\mathcal{F}}(\hat{t}) = C_{\mathcal{F}}\|\hat{\eta}_{\Gamma,\Gamma_c}T_{\hat{t}}\check{\eta}_{\Lambda,\Lambda_c}\|_{M^1}$ for $\hat{t} \in \widehat{G}$. See [KP05, Lemma 3.4] for the proof that $r_{\mathcal{F}}$ decreases faster than any polynomial. $\square$

The following result is proved in [KP05, Lemma 3.5] and [Pfa08, Theorem 2.1].

Proposition 4.5.11. *Let $A : \ell_c(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ be a (not necessarily bounded) linear map. Let $(a_{k',k})_{k',k\in\mathbb{Z}^d}$ be the matrix representation of A with respect to the orthonormal bases $\{T_{k'}\delta_{\mathbb{Z}^d}\}_{k'\in\mathbb{Z}^d}$ and $\{T_k\delta_{\mathbb{Z}^d}\}_{k\in\mathbb{Z}^d}$. Let $\tilde{r}$ be a nonnegative Borel measurable function on $\mathbb{R}$, decreasing faster than any polynomial. Let $\lambda > 1$. Suppose that $|a_{k',k}| \leq \tilde{r}(\|\lambda k' - k\|_\infty)$. In this case, there does not exist a bounded linear map $B : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ with $BA = I$.*

We next illustrate the abstract procedure described above in specific cases.

The Circle

The following calculations first appeared in [Civ15].

Let Γ be the group of Kth roots of unity. Let Γ_c be the group of Lth roots of unity, where $L > K$. Note that $D = [0, 1/K)$ and $D_c = [0, 1/L)$. Let $\theta = e^{\pi i(1/K-1/L)}$. Let $\eta_{\Gamma,\Gamma_c} \in C^\infty(\mathbb{T})$ with $0 \leq \eta_{\Gamma,\Gamma_c} \leq 1$, $\eta_{\Gamma,\Gamma_c} = 1$ on $D_c + \theta$, and $\eta_{\Gamma,\Gamma_c} = 0$ outside D^o.

Let $\Lambda = \Lambda_c = \mathbb{Z}$. Note that $\Xi = \Xi_c = \{0\}$. Let $\hat{\theta} = 0$. Let $\eta_{\Lambda,\Lambda_c} = \delta_{\mathbb{Z}}$.

Let $\mathcal{P}$ be the operator in $\mathcal{O}^{\infty,1}(\mathbb{T})$ with spreading function $\eta_{\mathcal{P}} = \eta_{\Gamma,\Gamma_c} \otimes \eta_{\Lambda,\Lambda_c} = \eta_{\Gamma,\Gamma_c} \otimes \delta_{\mathbb{Z}}$.

Note that $\Gamma \cong \mathbb{Z}/K\mathbb{Z}$, $\Gamma_c^{\perp} = L\mathbb{Z}$, and $\Lambda_c^{\perp} = 0$. By Proposition 4.5.4, the linear map $U : \ell_c(\mathbb{Z}/K\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \to \mathcal{O}^{\infty,1}(\mathbb{T})$ defined by

$$U\sigma = \sum_{k=0}^{K-1} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \sigma(k,p,q) M_{p+qL} \mathcal{P} T_{\omega_K^k} M_{-qL}$$

is bounded and stable. Note that $\eta_{M_{p+qL}\mathcal{P}T_{\omega_K^k}M_{-qL}} = M_{(qL,1)} T_{(\omega_K^k, p)} \eta_{\mathcal{P}}$.

Let $g \in M^{\infty}(\mathbb{T})$. Let $A_g = \mathcal{F} \circ e_g \circ U$. Recall that $e_g : \mathcal{O}^{\infty,1}(\mathbb{T}) \to L^2(\mathbb{T})$ is the evaluation map. Let $(a_{\xi,(k,p,q)})_{\xi \in \mathbb{Z},(k,p,q) \in \mathbb{Z}/K\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}$ be the matrix representation of A_g with respect to the orthonormal bases $\{T_\xi \delta_{\mathbb{Z}}\}_{\xi \in \mathbb{Z}}$ and $\{T_{(k,p,q)} \delta_{\mathbb{Z}/K\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}\}_{(k,p,q) \in \mathbb{Z}/K\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}$. We have

$$\begin{aligned} a_{\xi,(k,p,q)} &= (A_g T_{(k,p,q)} \delta_{\mathbb{Z}/K\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}})(\xi) \\ &= (M_{p+qL} \mathcal{P} T_{\omega_K^k} M_{-qL} g)\widehat{\ }(\xi) \\ &= T_{p+qL} (\mathcal{P} T_{\omega_K^k} M_{-qL} g)\widehat{\ }(\xi) \\ &= (\mathcal{P} T_{\omega_K^k} M_{-qL} g)\widehat{\ }(\xi - p - qL). \end{aligned}$$

By Lemma 4.5.10, there exists a nonnegative function $r_{\mathcal{F}}$ on $\mathbb{Z}$, decreasing faster than any polynomial, such that $|a_{\xi,(k,p,q)}| \leq r_{\mathcal{F}}(\xi - p - qL)$. We extend $r_{\mathcal{F}}$ from $\mathbb{Z}$ to $\mathbb{R}$ by defining $r_{\mathcal{F}}(x) = r_{\mathcal{F}}(\lceil x \rceil)$ for $x \in \mathbb{R}$.

Let $J = \{(k_0, p_0), (k_1, p_1), \ldots\}$ be a finite subset of $\Gamma \times \Lambda$ such that $\lambda = |J|/L > 1$. Let $\tilde{r}(x) = \max_{j=0}^{|J|-1} r_{\mathcal{F}}(\lambda^{-1}(x - \lambda p_j + j))$ for $x \in \mathbb{R}$. Note that $\tilde{r}$ decreases faster than any polynomial. For $0 \leq j \leq |J|-1$,

$$\begin{aligned}|a_{\xi,(k_j,p_j,q)}| &\leq r_{\mathcal{F}}(\xi - p_j - qL) \\ &= r_{\mathcal{F}}(\lambda^{-1}(\lambda\xi - \lambda p_j + j - (q|J| + j))) \\ &\leq \tilde{r}(\lambda\xi - (q|J| + j)).\end{aligned}$$

Let $i_J : \ell_c(J \times \mathbb{Z}) \to \ell_c(\mathbb{Z}/K\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$ be the inclusion map. By Proposition 4.5.11, $A_g \circ i_J$ is not stable. Equivalently, $e_g \circ U \circ i_J$ is not stable.

Theorem 4.5.12. *Let $S \subseteq \mathbb{T} \times \mathbb{Z}$ be open with $\mu_{\mathbb{T}\times\mathbb{Z}}(S) > 1$. There exists no $g \in M^\infty(\mathbb{T})$ for which $e_g|S$ is stable.*

Proof. In the above discussion, choose K large enough with $L = K + 1$ so that S contains $L + 1 = K + 2$ translations of $D \times \Xi$ by $\Gamma \times \Lambda$. Let J be the corresponding subset of $\Gamma \times \Lambda$. Then $e_g \circ U \circ i_J = e_g|S \circ U \circ i_J$ is not stable for any $g \in M^\infty(\mathbb{T})$. Since $U \circ i_J$ is stable, $e_g|S$ is not stable for any $g \in M^\infty(\mathbb{T})$. □

Corollary 4.5.13. *Let $S \subseteq \mathbb{Z} \times \mathbb{T}$ be open with $\mu_{\mathbb{Z}\times\mathbb{T}}(S) > 1$. There exists no $g \in \ell^\infty(\mathbb{Z})$ for which $e_g|S$ is stable.*

Proof. The result follows from Theorem 4.5.12 and (4.5.1). □

The Integers

As we saw above, the analysis over $\mathbb{Z}$ follows from the analysis over $\mathbb{T}$ via the duality principle. Nonetheless, it is instructive to carry out the computations anew.

Let $\Gamma = \Gamma_c = \mathbb{Z}$. Note that $D = D_c = \{0\}$. Let $\theta = 0$. Let $\eta_{\Gamma,\Gamma_c} = \delta_{\mathbb{Z}}$.

Let Λ be the group of Kth roots of unity. Let Λ_c be the group of Lth roots of unity, where $L > K$. Note that $\Xi = [0, 1/K)$ and $\Xi_c = [0, 1/L)$. Let $\hat{\theta} = e^{\pi i(1/K - 1/L)}$. Let $\eta_{\Lambda,\Lambda_c} \in C^\infty(\mathbb{T})$ with $0 \le \eta_{\Lambda,\Lambda_c} \le 1$, $\eta_{\Lambda,\Lambda_c} = 1$ on $\Xi_c + \hat{\theta}$, and $\eta_{\Lambda,\Lambda_c} = 0$ outside Ξ^o.

Let $\mathcal{P}$ be the operator in $\mathcal{O}^{\infty,1}(\mathbb{Z})$ with spreading function $\eta_{\mathcal{P}} = \eta_{\Gamma,\Gamma_c} \otimes \eta_{\Lambda,\Lambda_c} = \delta_{\mathbb{Z}} \otimes \eta_{\Lambda,\Lambda_c}$.

Note that $\Lambda \cong \mathbb{Z}/K\mathbb{Z}$, $\Gamma_c^\perp = 0$, and $\Lambda_c^\perp = L\mathbb{Z}$. By Proposition 4.5.4, the linear map $U : \ell_c(\mathbb{Z} \times \mathbb{Z}/K\mathbb{Z} \times \mathbb{Z}) \to \mathcal{O}^{\infty,1}(\mathbb{Z})$ defined by

$$U\sigma = \sum_{p\in\mathbb{Z}} \sum_{k=0}^{K-1} \sum_{q\in\mathbb{Z}} \sigma(p,k,q) M_{\omega_K^k} T_{-qL} \mathcal{P} T_{p+qL}$$

is bounded and stable. Note that $\eta_{M_{\omega_K^k} T_{-qL} \mathcal{P} T_{p+qL}} = \omega_K^{-kqL} M_{(0,qL)} T_{(p,\omega_K^k)} \eta_{\mathcal{P}}$.

Let $g \in \ell^\infty(\mathbb{Z})$. Let $A_g = e_g \circ U$. Recall that $e_g : \mathcal{O}^{\infty,1}(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the evaluation map. Let $(a_{\xi,(p,k,q)})_{\xi\in\mathbb{Z},(p,k,q)\in\mathbb{Z}\times\mathbb{Z}/K\mathbb{Z}\times\mathbb{Z}}$ be the matrix representation of A_g with respect to the orthonormal bases $\{T_\xi \delta_{\mathbb{Z}}\}_{\xi\in\mathbb{Z}}$ and $\{T_{(p,k,q)} \delta_{\mathbb{Z}\times\mathbb{Z}/K\mathbb{Z}\times\mathbb{Z}}\}_{(p,k,q)\in\mathbb{Z}\times\mathbb{Z}/K\mathbb{Z}\times\mathbb{Z}}$. We have

$$\begin{aligned} a_{\xi,(p,k,q)} &= (A_g T_{(p,k,q)} \delta_{\mathbb{Z}\times\mathbb{Z}/K\mathbb{Z}\times\mathbb{Z}})(\xi) \\ &= (M_{\omega_K^k} T_{-qL} \mathcal{P} T_{p+qL} g)(\xi) \\ &= \omega_K^{k\xi} (\mathcal{P} T_{p+qL} g)(\xi + qL). \end{aligned}$$

By Lemma 4.5.9, there exists a nonnegative function r on $\mathbb{Z}$, decreasing faster than any polynomial, such that $|a_{\xi,(p,k,q)}| \le r(\xi + qL)$. We extend r from $\mathbb{Z}$ to $\mathbb{R}$ by defining $r(x) = r(\lceil x \rceil)$ for $x \in \mathbb{R}$.

Let $J = \{(p_0, k_0), (p_1, k_1), \ldots\}$ be a finite subset of $\Gamma \times \Lambda$ such that $\lambda = |J|/L > 1$. Let $\tilde{r}(x) = \max_{j=0}^{|J|-1} r(\lambda^{-1}(x - j))$ for $x \in \mathbb{R}$. Note that $\tilde{r}$ decreases faster than any polynomial. For $0 \le j \le |J|-1$,

$$
\begin{aligned}
|a_{\xi,(p_j,k_j,q)}| &\le r(\xi + qL) \\
&= r(\lambda^{-1}(\lambda\xi - j + q|J| + j)) \\
&\le \tilde{r}(\lambda\xi + q|J| + j).
\end{aligned}
$$

Let $i_J : \ell_c(J \times \mathbb{Z}) \to \ell_c(\mathbb{Z} \times \mathbb{Z}/K\mathbb{Z} \times \mathbb{Z})$ be the inclusion map. By Proposition 4.5.11, $A_g \circ i_J$ is not stable. Equivalently, $e_g \circ U \circ i_J$ is not stable.

The Real Line

Let $\alpha > 0$ and $\lambda > 1$. Let $\Gamma = \Lambda = \alpha\mathbb{Z}$ and $\Gamma_c = \Lambda_c = (\alpha/\lambda)\mathbb{Z}$. Let $\theta = \hat{\theta} = (\lambda - 1)\alpha/(2\lambda)$.

Let $\eta_{\Gamma,\Gamma_c} \in C_c^\infty(\mathbb{R})$ with $0 \le \eta_{\Gamma,\Gamma_c} \le 1$, $\eta_{\Gamma,\Gamma_c} = 1$ on $D_c + \theta$, and $\eta_{\Gamma,\Gamma_c} = 0$ outside D^o. Let $\eta_{\Lambda,\Lambda_c} \in C_c^\infty(\widehat{\mathbb{R}})$ with $0 \le \eta_{\Lambda,\Lambda_c} \le 1$, $\eta_{\Lambda,\Lambda_c} = 1$ on $\Xi_c + \hat{\theta}$, and $\eta_{\Lambda,\Lambda_c} = 0$ outside Ξ^o. Let $\mathcal{P}$ be the operator in $\mathcal{O}^{\infty,1}(\mathbb{R})$ with spreading function $\eta_{\mathcal{P}} = \eta_{\Gamma,\Gamma_c} \otimes \eta_{\Lambda,\Lambda_c}$.

By Proposition 4.5.4, the linear map $U : \ell_c(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) \to \mathcal{O}^{\infty,1}(\mathbb{R})$ defined by

$$
U\sigma = \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} \sigma(k,\ell,p,q) M_{\ell\alpha + p\lambda/\alpha} T_{-q\lambda/\alpha} \mathcal{P} T_{k\alpha + q\lambda/\alpha} M_{-p\lambda/\alpha}
$$

is bounded and stable. Note that

$$
\eta_{M_{\ell\alpha+p\lambda/\alpha} T_{-q\lambda/\alpha} \mathcal{P} T_{k\alpha+q\lambda/\alpha} M_{-p\lambda/\alpha}} = e^{-2\pi i \ell q \lambda} M_{(p\lambda/\alpha, q\lambda/\alpha)} T_{(k\alpha,\ell\alpha)} \eta_{\mathcal{P}}.
$$

Let $\phi(x) = e^{-\pi x^2}$ for $x \in \mathbb{R}$. Note that $\hat{\phi} = \phi$. Let $N \in \mathbb{Z}$ with $N > \lambda^4/\alpha^2$. By Corollary 3.3.6, the linear map $C_\phi : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z} \times \mathbb{Z})$ defined by $C_\phi(f)(s,t) =$

$\langle f, M_{s\lambda^2/\alpha} T_{t\lambda^2/(\alpha N)} \phi \rangle$ is bounded. In fact, since $\lambda^4/(\alpha^2 N) < 1$, C_ϕ is bounded and stable; see [Grö01, Theorem 7.5.3].

Let $g \in M^\infty(\mathbb{R})$. Let $A_g = C_\phi \circ e_g \circ U$. Recall that $e_g : \mathcal{O}^{\infty,1}(\mathbb{R}) \to L^2(\mathbb{R})$ is the evaluation map. Let $(a_{(s,t),(k,\ell,p,q)})_{(s,t)\in\mathbb{Z}\times\mathbb{Z},(k,\ell,p,q)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}$ be the matrix representation of A_g with respect to the orthonormal bases $\{T_{(s,t)}\delta_{\mathbb{Z}\times\mathbb{Z}}\}_{(s,t)\in\mathbb{Z}\times\mathbb{Z}}$ and $\{T_{(k,\ell,p,q)}\delta_{\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}\}_{(k,\ell,p,q)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}}$.

Let $J = \{(k_0, \ell_0), \ldots, (k_{N-1}, \ell_{N-1})\} \subseteq \Gamma \times \Lambda$. It can be shown via calculations similar to the ones in [PW06a, p. 4819] that there exists a nonnegative Borel measurable function $\tilde{r}$ on $\mathbb{R}$, decreasing faster than any polynomial, such that, for $0 \leq j \leq N-1$,

$$|a_{(s,t),(k_j,\ell_j,p,q)}| \leq \tilde{r}(\max\{|\lambda s - p|, |\lambda t + qN + j|\}).$$

Let $i_J : \ell_c(J\times\mathbb{Z}\times\mathbb{Z}) \to \ell_c(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})$ be the inclusion map. By Proposition 4.5.11, $A_g \circ i_J$ is not stable. Equivalently, $e_g \circ U \circ i_J$ is not stable.

Theorem 4.5.14 (Pfander-Walnut [PW06a, Theorem 4.1])**.** *Let $S \subseteq \mathbb{R} \times \widehat{\mathbb{R}}$ be open with $\mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(S) > 1$. There exists no $g \in M^\infty(\mathbb{R})$ for which $e_g|S$ is stable.*

Proof. In the above discussion, choose $\alpha > 0$ and $\lambda > 1$ small enough with $2\alpha^2 + \lambda^4 < \mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(S)$ and $N = 1 + \lceil \lambda^4/\alpha^2 \rceil$ so that S contains N translations of $D \times \Xi$ by $\Gamma \times \Lambda$. Let J be the corresponding subset of $\Gamma\times\Lambda$. Then $e_g \circ U \circ i_J = e_g|S \circ U \circ i_J$ is not stable for any $g \in M^\infty(\mathbb{R})$. Since $U \circ i_J$ is stable, $e_g|S$ is not stable for any $g \in M^\infty(\mathbb{R})$. $\square$

Product Groups

Let $\mathcal{D}_{G_1}$ and $\mathcal{D}_{\widehat{G}_1}$ be as described above. Let $\mathcal{P}_1$ be the operator in $\mathcal{O}^{\infty,1}(G_1)$ with spreading function $\eta_{\mathcal{P}_1} = \eta_{\Gamma_1,\Gamma_{1,c}} \otimes \eta_{\Lambda_1,\Lambda_{1,c}}$.

Let $\mathcal{D}_{G_2}$ and $\mathcal{D}_{\widehat{G}_2}$ be as described above. Let $\mathcal{P}_2$ be the operator in $\mathcal{O}^{\infty,1}(G_2)$ with spreading function $\eta_{\mathcal{P}_2} = \eta_{\Gamma_2,\Gamma_{2,c}} \otimes \eta_{\Lambda_2,\Lambda_{2,c}}$.

Let $\mathcal{P}$ be the operator in $\mathcal{O}^{\infty,1}(G_1 \times G_2)$ with spreading function $\eta_{\mathcal{P}} = \eta_{\mathcal{P}_1} \otimes \eta_{\mathcal{P}_2}$.

By Proposition 4.5.4, the linear map

$$U : \ell_c(\Gamma_1 \times \Gamma_2 \times \Lambda_1 \times \Lambda_2 \times \Gamma_{1,c}^{\perp} \times \Gamma_{2,c}^{\perp} \times \Lambda_{1,c}^{\perp} \times \Lambda_{2,c}^{\perp}) \to \mathcal{O}^{\infty,1}(G_1 \times G_2)$$

defined by

$$\begin{aligned} U\sigma = &\sum_{w_1\in\Gamma_1} \sum_{w_2\in\Gamma_2} \sum_{\upsilon_1\in\Lambda_1} \sum_{\upsilon_2\in\Lambda_2} \sum_{w_{1,c}^{\perp}\in\Gamma_{1,c}^{\perp}} \sum_{w_{2,c}^{\perp}\in\Gamma_{2,c}^{\perp}} \sum_{\upsilon_{1,c}^{\perp}\in\Lambda_{1,c}^{\perp}} \sum_{\upsilon_{2,c}^{\perp}\in\Lambda_{2,c}^{\perp}} \cdots \\ &\sigma(w_1, w_2, \upsilon_1, \upsilon_2, w_{1,c}^{\perp}, w_{2,c}^{\perp}, \upsilon_{1,c}^{\perp}, \upsilon_{2,c}^{\perp}) \cdots \\ &M_{(\upsilon_1+w_{1,c}^{\perp},\upsilon_2+w_{2,c}^{\perp})} T_{(-\upsilon_{1,c}^{\perp},-\upsilon_{2,c}^{\perp})} \mathcal{P} T_{(w_1+\upsilon_{1,c}^{\perp},w_2+\upsilon_{2,c}^{\perp})} M_{(-w_{1,c}^{\perp},-w_{2,c}^{\perp})} \end{aligned}$$

is bounded and stable. Note that

$$\begin{aligned} &\eta_{M_{(\upsilon_1+w_{1,c}^{\perp},\upsilon_2+w_{2,c}^{\perp})} T_{(-\upsilon_{1,c}^{\perp},-\upsilon_{2,c}^{\perp})} \mathcal{P} T_{(w_1+\upsilon_{1,c}^{\perp},w_2+\upsilon_{2,c}^{\perp})} M_{(-w_{1,c}^{\perp},-w_{2,c}^{\perp})}} \\ =&(-\upsilon_{1,c}^{\perp}, \upsilon_1)(-\upsilon_{2,c}^{\perp}, \upsilon_2) M_{(w_{1,c}^{\perp},w_{2,c}^{\perp},\upsilon_{1,c}^{\perp},\upsilon_{2,c}^{\perp})} T_{(w_1,w_2,\upsilon_1,\upsilon_2)} \eta_{\mathcal{P}} \\ =&(-\upsilon_{1,c}^{\perp}, \upsilon_1)(M_{(w_{1,c}^{\perp},\upsilon_{1,c}^{\perp})} T_{(w_1,\upsilon_1)} \eta_{\mathcal{P}_1}) \otimes (-\upsilon_{2,c}^{\perp}, \upsilon_2)(M_{(w_{2,c}^{\perp},\upsilon_{2,c}^{\perp})} T_{(w_2,\upsilon_2)} \eta_{\mathcal{P}_2}). \end{aligned}$$

Let $g_1 \in M^\infty(G_1)$ and $g_2 \in M^\infty(G_2)$. By Proposition 3.5.7,

$$
\begin{aligned}
&(UT_{(w_1,w_2,v_1,v_2,w_{1,c}^\perp,w_{2,c}^\perp,v_{1,c}^\perp,v_{2,c}^\perp)}\delta_{\Gamma_1\times\Gamma_2\times\Lambda_1\times\Lambda_2\times\Gamma_{1,c}^\perp\times\Gamma_{2,c}^\perp\times\Lambda_{1,c}^\perp\times\Lambda_{2,c}^\perp})(g_1\otimes g_2)\\
=&(U_1T_{(w_1,v_1,w_{1,c}^\perp,v_{1,c}^\perp)}\delta_{\Gamma_1\times\Lambda_1\times\Gamma_{1,c}^\perp\times\Lambda_{1,c}^\perp})g_1\otimes(U_2T_{(w_2,v_2,w_{2,c}^\perp,v_{2,c}^\perp)}\delta_{\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp})g_2.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&U(\sigma_1\otimes T_{(w_2,v_2,w_{2,c}^\perp,v_{2,c}^\perp)}\delta_{\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp})(g_1\otimes g_2)\\
=&(U_1\sigma 1)g_1\otimes(U_2T_{(w_2,v_2,w_{2,c}^\perp,v_{2,c}^\perp)}\delta_{\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp})g_2.
\end{aligned}
$$

for all $\sigma_1 \in \ell_c(\Gamma_1\times\Lambda_1\times\Gamma_{1,c}^\perp\times\Lambda_{1,c}^\perp)$.

Lemma 4.5.15. *Let J_1 be a finite subset of $\Gamma_1\times\Lambda_1$. Let $i_{J_1} : \ell_c(J_1\times\Gamma_{1,c}^\perp\times\Lambda_{1,c}^\perp)\to\ell_c(\Gamma_1\times\Lambda_1\times\Gamma_{1,c}^\perp\times\Lambda_{1,c}^\perp)$ be the inclusion map. Let $(k_2,\ell_2,k_{2,c}^\perp,\ell_{2,c}^\perp)\in\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp$. Let $A_2=\{(k_2,\ell_2,k_{2,c}^\perp,\ell_{2,c}^\perp)\}$. Let $i_{A_2} : \ell_c(A_2)\to\ell_c(\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp)$ be the inclusion map. Let $g_1\in M^\infty(G_1)$ and $g_2\in M^\infty(G_2)$. If $e_{g_1}\circ U_1\circ i_{J_1}$ is not stable, then $e_{g_1\otimes g_2}\circ U\circ(i_{J_1}\otimes i_{A_2})$ is not stable.*

Proof. Suppose that $e_{g_1\otimes g_2}\circ U\circ(i_{J_1}\otimes i_{A_2})$ is stable. Then there exists $C>0$ such that

$$
\begin{aligned}
C\|\sigma_1\|_2 &= C\|\sigma_1\otimes T_{(k_2,\ell_2,k_{2,c}^\perp,\ell_{2,c}^\perp)}\delta_{\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp}\|_2\\
&\le \|U(\sigma_1\otimes T_{(k_2,\ell_2,k_{2,c}^\perp,\ell_{2,c}^\perp)}\delta_{\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp})(g_1\otimes g_2)\|_2\\
&= \|(U_1\sigma 1)g_1\|_2\|(U_2T_{(w_2,v_2,w_{2,c}^\perp,v_{2,c}^\perp)}\delta_{\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp})g_2\|_2
\end{aligned}
$$

for all $\sigma_1\in\ell_c(J_1\times\Gamma_{1,c}^\perp\times\Lambda_{1,c}^\perp)$. Dividing by $\|(U_2T_{(w_2,v_2,w_{2,c}^\perp,v_{2,c}^\perp)}\delta_{\Gamma_2\times\Lambda_2\times\Gamma_{2,c}^\perp\times\Lambda_{2,c}^\perp})g_2\|_2$, we obtain that $e_{g_1}\circ U_1\circ i_{J_1}$ is stable, a contradiction. ◻

Definition 4.5.16. We say that G has the finely tuned overspreading property if for every open set $S \subseteq G \times \widehat{G}$ with $\mu_{G\times\widehat{G}}(S) > 1$, one can specify $\mathcal{D}_G$ and $\mathcal{D}_{\widehat{G}}$, and find a positive integer N so that *(a)* there exists $J \subseteq \Gamma \times \Lambda$ with $|J| = N$ and $\mathcal{V}_J \subseteq \mathcal{O}^{\infty,1}(G)|S$, and *(b)* for every such J, $e_g \circ U \circ i_J = e_g|S \circ U \circ i_J$ is not stable for any $g \in M^\infty(G)$.

We have shown above that $\mathbb{R}$, $\mathbb{T}$, and $\mathbb{Z}$ have the finely tuned overspreading property. It easily follows from the discussion in Section 4.1 that finite abelian groups have the finely tuned overspreading property.

Theorem 4.5.17. *Suppose that G_1 has the finely tuned overspreading property. Let $S \subseteq G_1 \times G_2 \times \widehat{G}_1 \times \widehat{G}_2$ be open. Suppose that there exists $(a_2, \hat{a}_2) \in G_2 \times \widehat{G}_2$ such that $\mu_{G_1\times\widehat{G}_1}(S_{(a_2,\hat{a}_2)}) > 1$, where*

$$S_{(a_2,\hat{a}_2)} = \{(a_1, \hat{a}_1) \in G_1 \times \widehat{G}_1 : (a_1, a_2, \hat{a}_1, \hat{a}_2) \in S\}.$$

In this case, there exist no $g_1 \in M^\infty(G_1)$ and $g_2 \in M^\infty(G_2)$ for which $e_{g_1\otimes g_2}|S$ is stable.

Proof. Suppose that there exist $g_1 \in M^\infty(G_1)$ and $g_2 \in M^\infty(G_2)$ such that $e_{g_1\otimes g_2}|S$ is stable. Since G_1 has the finely tuned overspreading property, we can specify $\mathcal{D}_{G_1}$, $\mathcal{D}_{\widehat{G}_1}$, and $J_1 \subseteq \Gamma_1 \times \Lambda_1$ finite so that $\mathcal{V}_{J_1} \subseteq \mathcal{O}^{\infty,1}(G_1)|S_{(a_2,\hat{a}_2)}$ and $e_{g_1} \circ U_1 \circ i_{J_1} = e_{g_1}|S_{(a_2,\hat{a}_2)} \circ U_1 \circ i_{J_1}$ is not stable. Let K_1 be the maximal spreading support of any operator in $\mathcal{V}_{J_1}$. By the tube lemma of topology concerning finite products of compact spaces, there exist open sets $W_1 \subseteq G_1 \times \widehat{G}_1$ and $W_2 \subseteq G_2 \times \widehat{G}_2$ such that $W_1 \times W_2 \subseteq S$, $K_1 \subseteq W_1$, and $(a_2, \hat{a}_2) \in W_2$. We can specify Γ_2 and Λ_2 so that there exists $(k_2, \ell_2) \in \Gamma_2 \times \Lambda_2$ with $D_2 \times \Xi_2 + (k_2, \ell_2) \subseteq W_2$. Let $A_2 = \{(k_2, \ell_2, 0, 0)\}$. Since $e_{g_1 \otimes g_2}|S$ is stable, $e_{g_1 \otimes g_2} \circ U \circ (i_{J_1} \otimes i_{A_2}) = e_{g_1 \otimes g_2}|S \circ U \circ (i_{J_1} \otimes i_{A_2})$ is stable, contradicting Lemma 4.5.15. $\square$

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