ABSTRACT

Title of Dissertation: Fuzzy Predicate Product Logic and Embeddings of Ordered Abelian Groups

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We show the embedding provided by the Hahn Embedding Theorem of an ordered Abelian group into a lexicographic functions space is coinitiality preserving. From this we strengthen Hájek's Completeness theorem for predicate product logic. We conclude that the set of tautologies for a lexicographic function space over a set S which is not initially scattered is recursively enumerable. By contrast we conclude that the set of tautologies for a lexicographic function space over a set S which is initially scattered is not arithmetical.

Fuzzy Predicate Product Logic
and Embeddings of Ordered Abelian Groups

by

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DEDICATION

To my beautiful daughters:

Neeloufar and Faranak

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They say “it takes a village to raise a child”. In my case, “it takes a village to write a thesis”. I owe so many thanks to so many people. The lion share of that goes to my advisor, Dr.Chris Laskowski. Without any doubt, I would not have accomplished what I have, had it not been for him. His enthusiasm for Mathematics and new discoveries inspires me. He stuck by me when things were not going well and always had a cheerful outlook. He answered every one of my “quick” questions. There are no words that can express my gratitude to him. So I am just going to keep it simple and say: Chris, thank you from the bottom of my heart.

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Chapter 1

Introduction

In every day life we are used to properties which can not be dealt with satisfactorily on a simple yes or no basis. Whether a house is "small" for example is perhaps best indicated by a shade of gray rather than by the black or white of a simple dichotomy. In 1965 Lotfi Zadeh [Zad65] suggested a modified set theory in which an individual could have a degree of membership which ranged over a continuum of values rather than being 0 or 1. He showed how set operations such as union and intersection could be defined for these "fuzzy sets" and developed a consistent framework for dealing with them. His system allows fuzzy sets to be manipulated in a consistent and reasonable intuitive way.

In [Háj98] Hájek presents a systematic treatment of deductive aspect and structure of fuzzy logic. He gives a set of axioms for fuzzy predicate logic which he calls BL∀-axioms and defines a BL-chain to be a residuated lattice $\mathbf{L} = \langle L, +, \Rightarrow, \leq, 0, 1 \rangle \rangle$. In particular, the framework is general enough to include all three "classical" fuzzy predicate logics, namely Łukawsiewicz, Gödel and product logics. In [LS02], Shashoua and Laskowski algebraically classified BL-chains to be an ordinal sum of certain 'basic forms' derived from ordered Abelian groups.

In this paper we investigate the interplay between fuzzy predicate product logic

$\Pi\forall$ and embeddings of ordered Abelian groups. In the context of predicate fuzzy logic, for a fixed BL-chain $\mathbf{L}$ and countable relational language $\mathcal{L}$, an $\mathbf{L}$-structure $\mathbb{M}$ consists of a non-empty universe M together with functions $r_p : M^n \to \mathbf{L}$ for each $P \in \mathcal{L}$. We define truth evaluation by $||p(\vec{a})||^{\mathbf{L}}_{\mathbb{M}} = r_p(\vec{a})$ for atomic formulas, $||\varphi \& \psi||^{\mathbf{L}}_{\mathbb{M}} = ||\varphi||^{\mathbf{L}}_{\mathbb{M}} + ||\psi||^{\mathbf{L}}_{\mathbb{M}}, ||\varphi \to \psi||^{\mathbf{L}}_{\mathbb{M}} = ||\varphi||^{\mathbf{L}}_{\mathbb{M}} \Rightarrow ||\psi||^{\mathbf{L}}_{\mathbb{M}}$ and $||\forall x\varphi||^{\mathbf{L}}_{\mathbb{M}} = \inf\{||\varphi(a)||^{\mathbf{L}}_{\mathbb{M}} \mid a \in M\}$, $||\exists x\varphi||^{\mathbf{L}}_{\mathbb{M}} = \sup\{||\varphi(a)||^{\mathbf{L}}_{\mathbb{M}} \mid a \in M\}$ provided the infimum and supremum exist. A $\mathbf{L}$-structure, $\mathbb{M}$, is called safe, if $||\varphi||^{\mathbf{L}}_{\mathbb{M}}$ is defined for all φ. α is called an $\mathbf{L}$-tautology if and only if $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$ for all safe $\mathbf{L}$-structures $\mathbb{M}$. Hájek's Completeness Theorem (Theorem 4.2.17) says:

> If T is a theory over any extension of BL$\forall$ by finitely many axioms, then a sentence α is provable from T if and only if $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$ for all BL-chain $\mathbf{L}$ and safe $\mathbf{L}$-structure $\mathbb{M}$.

In this paper we strengthen Hájek's Completeness Theorem for predicate product logic, $\Pi\forall$, substantially. Let $L(\mathbb{R}^{\mathbb{Q}})$ be the BL-chain consisting of the extended negative cone of $\mathbb{R}^{\mathbb{Q}}$ followed by a single point. We proved (Theorem 6.1.6):

> Let T be a theory over $\Pi\forall$ and α a sentence. Then T proves α if and only if $||\alpha||^{L(\mathbb{R}^{\mathbb{Q}})}_{\mathbb{M}} = 1_{L(\mathbb{R}^{\mathbb{Q}})}$, for all closed $L(\mathbb{R}^{\mathbb{Q}})$-structures $\mathbb{M}$.

A closed $\mathbf{L}$-structure is a structure where $||\exists x\varphi||^{\mathbf{L}}_{\mathbb{M}} = ||\varphi(c)||^{\mathbf{L}}_{\mathbb{M}}$ for some $c \in M$ and $||\forall x\varphi||^{\mathbf{L}}_{\mathbb{M}} = ||\varphi(d)||^{\mathbf{L}}_{\mathbb{M}}$ for some $d \in M$, provided $||\forall x\varphi||^{\mathbf{L}}_{\mathbb{M}} \neq 0$. We were actually able to strengthen this result even further. Let $L(\mathbb{R}^S)$ be a BL-chain consisting of the extended negative cone of $\mathbb{R}^S$, where S is not initially scattered (see Definition 2.1.18, followed by a single point. Then for T a theory over $\Pi\forall$ and α a sentence we have (Theorem 6.1.7):

T proves α if and only if $||\alpha||_{\mathbb{M}}^{L(\mathbb{R}^S)} = 1_L(\mathbb{R}^S)$, for all closed $\mathbf{L}$-structures $\mathbb{M}$.

That is the set of sentences provable from $\Pi\forall$ is the same as the set of tautologies for any $\mathbf{L} = L(\mathbb{R}^S)$, where S is not initially scattered. Hence, this set of tautologies are recursively enumerable. On the other hand, in Chapter 7 we extend Montagna's result for $L(\mathbb{R}^1)$ tautologies [Mon01] and we show that if S is initially scattered, then the set of $L(\mathbb{R}^S)$-tautologies is not arithmetical (Theorem 7.3.9).

In order to get our results we show that if $\mathbf{L}$ is a $\Pi\forall$-chain then $\mathbf{L}$ consists of an extended negative cone of an ordered Abelian group followed by a single point, $L(G)$ (Theorem 5.1.6). Therefore we concentrate on ordered Abelian groups. To prove the above results, we extensively use coinitiality preserving embeddings of ordered Abelian groups. We show that the embedding provided by the Hahn Embedding Theorem (Theorem 3.3.6) $f : (G, \leq, +) \to (\mathbb{R}^S, \leq, +)$, where S is the set of Archimedean classes of $N(G)$, the negative cone of G, is coinitiality preserving. On one hand, we prove that if S is countable, then there exists a cointiality preserving embedding or $\mathbb{R}^S$ either into $\mathbb{R}^{\mathbb{Q}}$ or $\mathbb{R}^{1+\mathbb{Q}}$. On the other hand, we show that if S is initially scattered then there exists a coinitiality preserving embedding $h : \mathbb{R}^{\mathbb{Q}} \to \mathbb{R}^S$. It is also proved that if $f : (G, \leq, +) \to (H, \leq, +)$ is a coinitiality preserving embedding of ordered Abelian groups, then if α is not an $L(G)$-tautology then α is not an $L(H)$-tautology (Theorem 5.5.3).

Our Completeness Theorems for fuzzy predicate product logic by combining a modification of Henkin's method with our results on embeddings of ordered Abelian groups We conclude that if S, the set of Archimedean classes of $N(G)$, is initially scattered then $L(\mathbb{R}^S)$-tautologies are not arithmetical. If S is not initially scattered then $L(\mathbb{R}^S)$-tautologies coincide with consequences of $\Pi\forall$ and are recursively enumerable.

Only the notion of initially scattered and Lemmas 2.2.4, 2.2.5, 2.2.6, 2.2.7, 2.2.8 in Chapter 2 are original. In Chapter 3, only the lemmas concerning coinitiality preserving embeddings are original. Material in Chapter 4 is all due to Hájek, Laskowski and Shashoua. The bulk of the original results are in Chapter 5 which is the heart of the thesis. The material in Chapters 6 and 7 follow from Chapter 5 and illustrate the difference between the two cases. Chapter 8 is self contained and is not used anywhere else in the paper.

Chapter 2

Preliminaries for Linear Orderings

In this chapter we will first go over some preliminary definitions and results about linear orderings. Then we will give some specific results for countable linear orderings. The majority of the material in section 2.1 comes from [Ros82].

2.1 Linear Orderings

Definition 2.1.1 (Definition 1.1 of [Ros82]). A *linear ordering of the set* A is a binary relation R on A satisfying the conditions

1. if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$;
2. if $x \neq y$, then either $(x, y) \in R$ or $(y, x) \in R$, but not both;
3. $(x, x) \notin R$.

Example 2.1.2. Here are some examples of linear orderings.

1. $A_n = \{0, 1, 2, \ldots, n-1\}$ and $R_{A_n} = \{(i, j) \mid i < j\}$.
2. ω the set of natural numbers and $R_\omega = \{(i, j) \mid i < j\}$, the natural ordering.
3. ω the set of natural numbers and $R_{\omega^*} = \{(i, j) \mid i > j\}$, the backward ordering.

4. γ any ordinal and $R_\gamma = \{(i, j) \mid i < j\}$, the natural ordering.

5. γ any ordinal and $R_{\gamma^*} = \{(i, j) \mid i > j\}$, the backward ordering.

6. $\mathbb{Z}$ the integers and $R_\mathbb{Z} = \{(i, j) \mid i < j\}$, the natural ordering.

For any ordinal γ, letγ denote $\langle \gamma, R_\gamma \rangle$ and γ^* denote $\langle \gamma, R_{\gamma^*} \rangle$.

Definition 2.1.3 (Definitions 1.8 and 1.9 of [Ros82]). Let R be a linear ordering of A and let S be a linear ordering of B. An *isomorphism* of $\langle A, R \rangle$ *onto* $\langle B, S \rangle$ is an order preserving function f from A onto B which satisfies

$$f(a_1) <_S f(a_2) \text{ if and only if } \quad a_1 <_R a_2.$$

Example 2.1.4. Let $B = \{2n \mid n \in \omega\}$ and R_B the natural ordering. $f : \omega \to \langle B, R_B \rangle$ defined by $f(x) = 2x$ is an isomorphism of ω onto $\langle B, R_B \rangle$ and $\omega \simeq \langle B, R_B \rangle$.

Definition 2.1.5. We say $\langle A, r \rangle$ and $\langle B, S \rangle$ have the *same order type* if $\langle A, r \rangle \simeq \langle B, S \rangle$.

Lemma 2.1.6. *Having the same order type is an equivalence relation on the class of all linear orderings.*

Proof. Obvious. □

Definition 2.1.7. We say that a linear ordering $\langle A, R \rangle$ *has type order* τ if τ is a representative of the equivalence class of $\langle A, R \rangle$.

Definition 2.1.8. Let $\langle A, R \rangle$ and $\langle B, S \rangle$ be linear orderings and assume A and B are disjoint. We define the *sum*

$$\langle A, R \rangle + \langle B, S \rangle$$

to be the linear ordering $\langle C, T\rangle$, where $C = A \cup B$ and

$$\begin{aligned} c_1 <_T c_2 \quad \text{if and only if} \quad & (c_1 \in A \text{ and } c_2 \in B) \text{ or} \\ & (c_1 \in A \text{ and } c_2 \in A \text{ and } \ c_1 <_R c_2) \text{ or} \\ & (c_1 \in B \text{ and } c_2 \in b \text{ and } \ c_1 <_S c_2). \end{aligned}$$

It is easily verified that $\langle C, T\rangle$ is a linear ordering. Intuitively adding $\langle B, S\rangle$ to $\langle A, R\rangle$ means to place the entire linear ordering $\langle B, S\rangle$ to the right of the linear ordering $\langle A, R\rangle$.

Example 2.1.9. 1. $\mathbb{Z}$ has the order type $\omega^* + \omega$.

2. $1 + \mathbb{Q}$ is a linear ordering with a first element followed by a copy of $\mathbb{Q}$ (the rational numbers ordered naturally).

We now turn to the definition of generalized sums of linear orderings.

Definition 2.1.10 (Definition 1.38 of Rosenstein). Let $\langle I, R\rangle$ be a linear ordering and for each $i \in I$ let $\langle A_i, R_i\rangle$ be a linear ordering. We assume that $\{A_i \mid i \in I\}$ are pairwise disjoint. We define the *generalized sum* $\sum\{A_i \mid i \in I\}$ to be the linear ordering $\langle C, T\rangle$ where $C = \cup\{A_i \mid i \in I\}$ and

$$\begin{aligned} x <_T y \quad \text{if} \quad & (\text{for some } i \in I, x \in A_i \text{ and } y \in A_i \text{ and } x <_{R_i} y) \text{ or} \\ & (\text{for some } i \neq j, x \in A_i \text{ and } y \in A_j \text{ and } i <_R j). \end{aligned}$$

Intuitively, imagine I stretched out as a line. Replace each element $i \in I$ by a miniature version of A_i. The resulting linear ordering is $\sum\{A_i \mid i \in I\}$.

Definition 2.1.11. Let A be a linear ordering with more than 1 element. We say A is *dense* if and only if, given any two elements $a_1, a_2 \in A$ such that $a_1 < a_2$, then there exists $a_3 \in A$ such that $a_1 < a_3 < a_2$.

Example 2.1.12. $\mathbb{Q}$, the rational numbers are dense and $\mathbb{Z}$, the integers are not.

Definition 2.1.13. Let A be a linear ordering and let X be a subordering of A.

1. We say X is *cofinal* in A if and only if for all $a \in A$ there is an $x \in X$ such that $a \leq x$.

2. We say X is *coinitial* in A if and only if for all $a \in A$ there is an $x \in X$ such that $x \leq a$.

Example 2.1.14. 1. $\mathbb{Z}$ is both cofinal and coinitial in $\mathbb{R}$.

2. $\mathbb{N}$ is cofinal and not coinitial in $\mathbb{R}$.

Definition 2.1.15. Let A and B be two linear orderings and $f : A \to B$ be an order preserving map. We say f is *cofinality (coinitiality) preserving* if and only if $f(X)$ is cofinal (coinitial) in B when X is cofinal (coinitial) in A.

Definition 2.1.16. A linear ordering $\mathbf{S}$ is called *scattered* if it does not contain a dense subset.

Example 2.1.17. 1. ω, ω^* are scattered.

2. $\mathbb{Q}, \mathbb{R}$ are not scattered.

Definition 2.1.18. Let $(T, <)$ be a linear ordering. We say T is *initially scattered* if and only if there exists $t \in T$ such that $\{s \in T \mid s \leq t\}$ is scattered.

Example 2.1.19. 1. Any scattered set is initially scattered.

2. $\mathbb{Q}$ is not initially scattered but $1 + \mathbb{Q}$ is.

2.2 Countable Linear Orderings

In this section we will focus on countable linear orderings and obtain a few results. For the rest of this section we assume $(T, <)$ is a countable ordered set and let $\mathbb{Q}$ denote the ordered set of the rational numbers.

Definition 2.2.1. Let X and Y be ordered sets. Let $f : X \to Y$ be any order preserving map. We say f *preserves cofinality (coinitiality)* or is *cofinality (coinitiality) preserving* if and only if whenever $\{x_i \mid i \in I\}$ is cofinal (coinitial) in X then $\{f(x_i) \mid i \in I\}$ is cofinal (coinitial) in Y.

Example 2.2.2.

1. Let $f : (\mathbb{R}, \leq) \to (\mathbb{R}^2, \leq)$ defined by $f(x) = (x, 0)$ ($\mathbb{R}^2$ is ordered lexicographically). Then f is cofinality (coinitiality) preserving.

2. Let $f : (\mathbb{R}, \leq) \to (\mathbb{R}^2, \leq)$ defined by $f(x) = (0, x)$. Then f is not cofinality preserving.

The following lemma is a well known result that any countable linear ordering may be embedded into $\mathbb{Q}$.

Lemma 2.2.3. *There exists $f : T \to \mathbb{Q}$ which is order preserving.*

Proof. Let $\{t_n \mid n \in \omega\}$ be an enumeration of T. We will define f inductively. Let $f(t_0) = 0$. Suppose we have decided $f(t_0), \ldots, f(t_n)$ in an order preserving fashion. We need to decide $f(t_{n+1})$. Observe the the relation of t_{n+1} to $t_0, \ldots, t_n$. Since $\mathbb{Q}$ is dense and without end points, there exist $a \in \mathbb{Q} \setminus \{f(t_0), \ldots, f(t_n)\}$ such that a has the same relation to $f(t_0), \ldots, f(t_n)$ as t_{n+1} has to $t_0, \ldots, t_n$. Let $f(t_{n+1}) = a$. By construction f is order preserving. $\square$

The following lemmas gives criteria for existence of cointiality preserving maps between countable linear orderings.

Lemma 2.2.4. *If T does not contain a minimal element, then there exists $f : T \to \mathbb{Q}$ which is coinitiality preserving.*

Proof. We will prove this lemma in a very similar fashion to Lemma 2.2.3. Let $\{t_n \mid n \in \omega\}$ be an enumeration of T. We will define f inductively. Let $f(t_0) = 0$. Suppose we have decided $f(t_0), \ldots, f(t_n)$ in an order preserving fashion. We need to decide $f(t_{n+1})$. Observe the the relation of t_{n+1} to $t_0, \ldots, t_n$. If $t_{n+1} < t_i, i = 1, \ldots, n$, then let $f(t_{n+1}) = \min_i f(t_i) - 1$. Otherwise let $a \in \mathbb{Q} \setminus \{f(t_0), \ldots, f(t_n)\}$ such that a has the same relation to $f(t_0), \ldots, f(t_n)$ as t_{n+1} has to $t_0, \ldots, t_n$. In this case let $f(t_{n+1}) = a$. By construction f is order preserving and cointiality preserving. □

Lemma 2.2.5. *Let T and S be countable sets. Suppose T has a minimal element t_0. Let $f : T \to S$ be an order preserving map. Then f is cointiality preserving if and only if S has a minimal element s_0 and $f(t_0) = s_0$.*

Proof. $\to$: Suppose f is coinitiality preserving. $t_0 \leq t$ for all $t \in T$, therefore $f(t_0) \leq s$ for all $s \in S$. Hence, $f(t_0) = s_0$ the minimal element of S.

$\leftarrow$: Suppose s_0 is the minimal element of S. Suppose X is a coinitial subset of T. Since t_0 is the minimal element of T, $t_0 \in X$. Since f is order preserving and $f(t_0) = s_0 < s$ for all $s \in S$ we have $f(X)$ is coinitial in S.

The proof is complete. □

Lemma 2.2.6. *Let T be a not initially scattered linear ordering. Then for all $t \in T$ there exists $s \in T$ such that $s < t$ and $(s, t) \cap T$ contains a dense subset.*

Proof. Fix $t \in T$. Since T is not initially scattered, $\{s \in T \mid s \leq t\}$ contains a dense subset X. Let $s \in X$ be arbitrary. Then $(s, t) \cap T$ contains a dense subset. □

Lemma 2.2.7. *Let T be a countable not initially scattered linear ordering. Then there is $\{s_n \mid n \in \omega\} \subseteq T$ such that*

1. $s_{n+1} < s_n$ *for all* $n \in \omega$;

2. *For all* $t \in T$, $s_n < t$ *for some* $n \in \omega$;

3. $(s_{n+1}, s_n) \cap T = \{t \in T \mid s_{n+1} < t < s_n\}$ *contains a dense subset.*

Proof. Let $\{t_n \mid n \in \omega\}$ be an enumeration of T. Let $s_0 = t_0$. By Lemma 2.2.6 there is $s_0^* < s_0$ such that $(s_0^*, s_0) \cap T$ contains a dense subset. Let $s_1 = \min\{t_1, s_0^*\}$. Suppose we have chosen $s_0, \ldots s_n$ satisfying the requirements. We need to choose s_{n+1}. By Lemma 2.2.6 there is $s_n^* < s_n$ such that $(s_n^*, s_n) \cap T$ contains a dense subset. Let $s_{n+1} = \min\{t_{n+1}, s_n^*\}$. Then by construction $\{s_n \mid n \in \omega\}$ satisfies the requirements. □

Lemma 2.2.8. *Let T be a countable not initially scattered linear ordering. Then there exists a cointiality preserving embedding $h : \mathbb{Q} \to T$.*

Proof. We first make the following observation that

$$\mathbb{Q} = \bigcup_{n \in \omega} (-n-1, -n) \cup (0, \infty) \cup \{-n \mid n \in \omega\}$$

and $(-n-1, -n) \cong (0, \infty) \cong \mathbb{Q}$ for $n \in \omega$. It is a well known fact that $\mathbb{Q}$ has an order preserving embedding into any countable dense linear ordering. Let $\{s_n \mid n \in \omega\} \subseteq T$ be as in Lemma 2.2.7. Then $(s_{n+1}, s_n) \cap T = \{t \in T \mid s_{n+1} < t < s_n\}$ contains a dense subset. Let $f_n : (-n-1, -n) \to (s_{n+2}, s_{n+1}), n \in \omega$ and $f : (0, \infty) \to (s_1, s_0)$ be order preserving embeddings. Define $h : \mathbb{Q} \to T$ as follows:

$$h(q) = \begin{cases} s_{n+1} & \text{if } q = -n,\ n \in \omega \\ f_n(q) & \text{if } q \in (-n-1, -n),\ n \in \omega \\ f(q) & \text{if } q > 0. \end{cases}$$

By construction, h is cointiality preserving and this completes the proof. □

Chapter 3

Ordered Abelian Groups

In this chapter we will investigate different notions of embedding of ordered Abelian groups. We will show that any ordered Abelian group may be embedded into $\mathbb{R}^T$, a lexicographic function space, in a coinitiality preserving manner. But first we will discuss some properties of countable ordered sets which we will utilize in our embedding discussions.

3.1 Ordered Abelian Groups

Definition 3.1.1. An *ordered Abelian semigroup* $(G, \leq, +)$ is a linear ordering $(G, \leq)$ with a commutative and associative operation $+$ which satisfies

$$x \leq y \quad \text{implies} \quad x + z \leq y + z.$$

for all x, y and $z \in G$.

Definition 3.1.2. $(G, \leq, +)$ is said to be an *ordered Abelian group* if and only if $(G, +)$ is an Abelian group, $(G, \leq)$ is a linear ordering and for all x, y and $z \in G$

$$x \leq y \quad \text{implies} \quad x + z \leq y + z.$$

Note that an ordered Abelian group is either trivial $G = \{0\}$ or G is infinite with no element of finite order. That is, G is torsion free. For the rest of the this paper we will exclude the trivial case.

Definition 3.1.3. Let G be an ordered Abelian group.

1. Let $N(G) = \{g \in G \mid g < 0_G\}$. $N(G)$ is called *the negative cone* of G.

2. Let $P(G) = \{g \in G \mid g > 0_G\}$. $P(G)$ is called *the positive cone* of G.

The negative cone of an ordered Abelian group will be used extensively in the following chapters when we deal with predicate Product Logic. There we would like to to get a "universal" ordered Abelian group. This will be provided by $\mathbb{R}^{\mathbb{Q}}$ which is a lexicographic function space. We define this term formally now.

Definition 3.1.4. Let $(T, <)$ be any non-empty ordered set.

$$\mathbb{R}^T := \{f : T \to \mathbb{R} \mid \{t \in T : f(t) \neq 0\} \text{ is well-ordered}\}$$

$\mathbb{R}^T$ is called a lexicographic function space (LFS) and it is ordered lexicographically.

Example 3.1.5. 1. Let $T = \{t\}$, then $\mathbb{R}^T \cong \mathbb{R}$.

2. If T is well ordered then $\mathbb{R}^T \cong \mathbb{R}^\alpha$ for some ordinal α.

Definition 3.1.6. Let T be an ordered set and let $t \in T$. Let $1_t \in \mathbb{R}^T$ be defined by

$$(1_t)_i = \begin{cases} 1 & i = t \\ 0 & i \neq t. \end{cases}$$

1_t is called the *characteristic function at t*.

Definition 3.1.7. Let $(G, +, \leq)$ be an ordered Abelian group and let $a, b \in G$.

1. We write $a \preceq b$ if and only if $a \leq_G nb$ for some $n \in \omega \setminus \{0\}$.

2. We say $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$.

3. For ease of notation we write $a \ll b$ if and only if $a \preceq b$ and $a \nsim b$.

Definition 3.1.8. Let G be an ordered Abelian group

1. Let $[a] = \{b \in G \mid a \sim b\}$. $[a]$ is called the *Archimedean class of* a.

2. Let T be the collection of Archimedean classes of G. We define order on T by $t < s$ if and only if $a \ll b$, for some $a \in t$ and $b \in s$. (This is well defined by basic properties of $\ll$.)

Example 3.1.9.

1. $(1,1) \preceq (1,0)$ and $(1,0) \preceq (1,1)$, hence they belong to the same Archimedean class of $\mathbb{R}^2$.

2. $(0,1) \ll (1,0)$, hence they belong to different Archimedean classes of $\mathbb{R}^2$.

Definition 3.1.10. Let $(G, +, \leq)$ be an ordered Abelian group. We say G is *Archimedean* if and only if $N(G)$ consists of a single Archimedean class.

Example 3.1.11.

1. $(\mathbb{R}, +, \leq)$ is an Archimedean ordered Abelian group.

2. $(\mathbb{R}^2, +, \leq)$ where $\leq$ is the lexicographic order is not Archimedean; $n \cdot (0,1) < (1,0)$ for all $n \in \omega$.

It is easy to see that, for an ordered Abelian group G, the set of Archimedean classes with $\ll$ form a linear ordering.

Lemma 3.1.12. *Let G and H be ordered Abelian groups. Let $f : G \to H$ be an order preserving group homomorphism. If $[g_1] < [g_2]$, then $[f(g_1)] < [f(g_2)]$.*

Proof. Suppose $[g_1] < [g_2]$. Then $ng_1 < g_2$ for all $n \in \omega$. Therefore $f(ng_1) < f(g_2)$ for all $n \in \omega$. Hence, $[f(g_1)] < [f(g_2)]$. $\square$

Lemma 3.1.13. *Let T be an ordered set. Define $f : N(\mathbb{R}^T) \to T$ by $f(x) = t$ where the first non zero component of x appears at t. For all $x, y \in n(\mathbb{R}^T)$, we have $f(x) = f(y)$ if and only if $[x] = [y]$. Hence, we may identify Archimedean classes of $N(\mathbb{R}^T)$ with the set T.*

Proof. Suppose $x = (x_i)_i, y = (y_i)_i \in N(\mathbb{R}^T)$ so that $x < y < 0$.

$f(x) = f(y)$ Then $x_j = y_j = 0$ for $j < f(x)$ and $x_{f(x)} < y_{f(x)}$. $\mathbb{R}$ is Archimedean, so there exist n a positive integer such that $ny_{f(x)} \leq x_{f(x)}$. Hence, $ny \leq x$ and $[x] = [y]$.

$f(x) \neq f(y)$ Then $nx < y$ for all positive integer n, hence $[x] < [y]$.

Let X be the collection of Archimedean classes of $N(\mathbb{R}^T)$. Then the map $g : X \to T$ defined by $g([x]) = f(x)$ is a well defined order preserving bijection and we have the result. $\square$

3.2 Embeddings of Ordered Abelian Groups

In this section we will investigate certain embeddings of ordered Abelian groups.

Note if $f : G \to H$ is a linear embedding then we have $f(0_G) = 0_H$.

Definition 3.2.1. Let G and H be ordered Abelian groups. Let $f : (G, +, \leq) \to (H, +, \leq)$ be an order preserving map. We say f is an *embedding that preserves cofinality (cointiality)* or a *cofinality (coinitiality) preserving embedding* if f has the following properties:

1. f is a group homomorphism, i.e., $f(g_1 +_G g_2) = f(g_1) +_H f(g_2)$.and

2. $f : (G, \leq) \to (H, \leq)$ is cofinality (coinitiality) preserving.

In the following lemma we show that the notion of cofinality preserving and coinitiality preserving coincides for ordered Abelian groups.

Lemma 3.2.2. *A cofinality preserving embedding, f is coinitiality preserving and vice versa.*

Proof. Suppose $f : G \to H$ is cofinality preserving embedding. Let $\{x_n \mid n \in I\}$ be coinitial in G. Then $\{-x_n \mid n \in I\}$ is cofinal in G and hence $\{f(-x_n) \mid n \in I\} = \{-f(x_n) \mid n \in I\}$ is cofinal in H. Therefore $\{f(x_n) \mid n \in I\}$ is coinitial in H. The opposite direction is symmetric. □

Example 3.2.3.

1. Let $f : (\mathbb{R}, +, \leq) \to (\mathbb{R}^2, +, \leq)$ defined by $f(x) = (x, 0)$. Then f is a cofinality preserving embedding.

2. Let $f : (\mathbb{R}, +, \leq) \to (\mathbb{R}^2, +, \leq)$ defined by $f(x) = (0, x)$. Then f is not a cofinality preserving embedding.

We now turn our attention to the embedding of $\mathbb{R}^T$ into $\mathbb{R}^S$. We will give criteria for existence of coinitiality preserving map between $\mathbb{R}^T$ and $\mathbb{R}^S$. We first make the following definition:

Definition 3.2.4. Let T and S be two ordered sets. Let $h : T \to S$ be an order preserving map. Define $f_h : \mathbb{R}^T \to \mathbb{R}^S$ by

$$f_h((g_i)_{i \in T}) = (\alpha_j)_{j \in S}$$

where

$$\alpha_j = \begin{cases} g_i & \text{if } j = h(i) \\ 0 & \text{otherwise.} \end{cases}.$$

f_h is called *the canonical map induced by h.*

Lemma 3.2.5. *Let T and S be two ordered sets. Then there exist a cointiality preserving map $h : T \to S$ if and only if there exists a coinitiality preserving embedding $f : \mathbb{R}^T \to \mathbb{R}^S$*

Proof. $\to$: Let $f : \mathbb{R}^T \to \mathbb{R}^S$ be the canonical map induced by h. We need to show f is a cofinality preserving embedding. It is very easy to check that f is a group homomorphism. We will show f is order preserving. Let $(g_i)_{i\in T}, (k_i)_{i\in T} \in \mathbb{R}^T$. Suppose $(g_i)_i < (k_i)_i$. Say the first component where they differ is at position t_0, that is $g_{t_0} < k_{t_0}$ and $g_j = k_j = 0$ for $j < t_0$. Then by construction the first position $f((g_i)_i)$ is different from $f((k_i)_i)$ is t_0. h being a order preserving map, gives us $h(g_{t_0}) < h(k_{t_0})$ and hence $f((g_i)_i) < f((k_i)_i)$.

We now show f is cofinality preserving. Let $X = \{x_j \mid j \in I\}$ be a cofinal sequence in $\mathbb{R}^T$. Let $k = (k_i)_{i\in S} \in \mathbb{R}^S$. We need to find $j_0 \in J$ such that $f(x_{j_0}) > k$. Let s be the first non-zero position of k. Since h is coinitiality preserving there is a $t \in T$ such that $h(t) \leq s$. Then we have $n_0 1_{h(t)} > k$, for some $n_0 \in \omega$ and where $1_{h(t)}$ is the characteristic function at $h(t)$. Since X is cofinal in $\mathbb{R}^T$, there is a j_0 such that $x_{j_0} > n_0 1_t$, so $f(x_{j_0}) > f(n_0 1_t) = n_0 1_{h(t)} > k$.

$\leftarrow$: Suppose there exists $f : \mathbb{R}^T \to \mathbb{R}^S$ which is coinitiality preserving. Define $h : T \to S$ by $h(t) = t_s$, where t_s is the first non-zero position of $f(1_t)$, 1_t is the characteristic function at t. We need to show h is coinitiality preserving. h is obviously order preserving. Let $X = \{x_n \mid n \in \omega\}$ be coinitial in T. Let $s \in S$. We need to find $x \in X$ so that $h(x) \leq s$. Since X is coinitial in T, $\{1_x \mid x \in X\}$ is coinitial in $\mathbb{R}^T$. Therefore, since f is coinitiality preserving, there exists $x \in X$ such that $f(1_x) \leq 1_s$. Now $h(x) =$ the first non-zero position of $f(1_x)$, hence $h(x) \leq s$ and the proof is complete.

□

Lemma 3.2.6. *Let T be a countable linear ordering. Then T does not contain a minimal element if and only if there exist a coinitiality preserving embedding $f : \mathbb{R}^T \to \mathbb{R}^{\mathbb{Q}}$.*

Proof. $\to$: If T does not contain a minimal element then by Lemma 2.2.4, there exists $h : T \to \mathbb{Q}$ which is coinitiality preserving and by Lemma 3.2.5, the canonical map induced by h is cofinality preserving.

$\leftarrow$ If T contains a minimal element, then by Lemma 2.2.5, there are no coinitial preserving map $h : T \to \mathbb{Q}$ since $\mathbb{Q}$ has no least element. Hence there is no coinitiality preserving map $f : \mathbb{R}^T \to \mathbb{R}^{\mathbb{Q}}$.

□

Lemma 3.2.7. *Let T be a countable linear ordering with a minimal element. Then there exists a cointiality preserving embedding $f : \mathbb{R}^T \to \mathbb{R}^{1+\mathbb{Q}}$.*

Proof. Let t_0 be the minimal element of T and let $h : T \setminus \{t_0\} \to \mathbb{Q}$ be order preserving. Let $g : T \to 1 + \mathbb{Q}$ be defined by $g(x) = f(x)$ if $x \neq t_0$ and $g(t_0) = 1$. Then g is coinitiality preserving.

□

Lemma 3.2.8. *Let T be a countable linear ordering. Then either there exist $f : \mathbb{R}^T \to \mathbb{R}^{\mathbb{Q}}$ or $f : \mathbb{R}^T \to \mathbb{R}^{1+\mathbb{Q}}$ which is coinitiality preserving.*

Proof. This is a direct corollary to Lemmas 3.2.6 and 3.2.7. □

Example 3.2.9. Let $T = \{1\}$ and $S = \{1, 2\}$.

1. Define $h : T \to S$ by $h(1) = 1$. Then h is cointiality preserving and the corresponding map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, 0)$ is order and cofinality preserving.

2. Note if we define $h : T \to S$ by $h(1) = 2$, then the corresponding map $f : \mathbb{R}^T \to \mathbb{R}^S$ defined by $f(x) = (0, x)$ is not cofinality preserving.

3.3 Hahn Embedding of G into an $\mathbb{R}^T$

The Hahn Embedding Theorem asserts that any ordered Abelian group $(G, +, \leq)$ can be embedded into a lexicographic function space $(\mathbb{R}^T, +, \leq)$, where T is set of Archimedean classes of $N(G)$. This embedding is taken place in two steps. First G is embedded into a ordered vector space and then the ordered vector space is mapped into a lexicographic function space. We will state the results and give references for their proofs. We will show that at each step the embedding is coinitiality preserving. Hence, $(G, +, \leq) \to (\mathbb{R}^T, +, \leq)$ is cointiality preserving.

Definition 3.3.1. Let G be an ordered Abelian group. Let $\mathbf{V}_G = \{(x, n) | x \in G, n \text{ a positive integer}\}$. We define equality by

$$(x, m) \approx (y, n) \text{ if and only if } nx = my,$$

and addition and $\mathbb{Q}$-scalar multiplication by

$$\begin{aligned} (x, m) + (y, n) &\approx (nx + my, mn) \\ \frac{q}{p} \cdot (x, m) &\approx (px, qm). \end{aligned}$$

Proposition 3.3.2. *Let G be an ordered Abelian group. Then*

1. *$\mathbf{V}_G$ is a vector space over the rational numbers, $\mathbb{Q}$. Furthermore, if we define*

$$(x, m) > (y, n) \text{ if and only if } nx > my$$

then $\mathbf{V}_G$ becomes an ordered vector space.

2. *$\mathbf{V}_G$ is divisible as an ordered Abelian group, i.e., for any $\mathbf{v} \in \mathbf{V}_G$ and positive integer n, there exists $\mathbf{x} \in \mathbf{V}_G$ such that $n\mathbf{x} = \mathbf{v}$.*

3. *$\mathbf{V}_G$ is dense.*

Proof. 1. Showing $\mathbf{V}_G$ is an ordered vector space over $\mathbb{Q}$ is a routine exercise. We just note that $0_{\mathbf{V}} = (0, 1)$ and $-(x, m) \approx (-x, m)$.

2. First note that $n(x, m) \approx (nm^{n-1}x, m^n)$ for $(x, m) \in \mathbf{V}_G$ and n a positive natural number. Let $\mathbf{v} = (g, k) \in \mathbf{V}$ and $n \in \mathbb{N}$. Let $\mathbf{x} = (g, kn)$. Hence

$$n\mathbf{x} \approx (n(kn)^{n-1}g, (nk)^n) \approx (n^n k^{n-1} g, n^n k^n) \approx (g, k).$$

3. Any ordered divisible Abelian group is dense. Let $v_1, v_2 \in \mathbf{V}_G$ such that $v_1 < v_2$. Since $\mathbf{V}_G$ is divisible, $\frac{v_1+v_2}{2} \in \mathbf{V}_G$ and furthermore $v_1 < \frac{v_1+v_2}{2} < v_2$. Hence, $\mathbf{V}_G$ is dense.

□

We now show that there exists a natural embedding of $G \to \mathbf{V}_G$ which is coinitiality preserving.

Theorem 3.3.3. *Let G be an ordered Abelian group. Let $f : G \to \mathbf{V}_G$ be defined by $x \mapsto (x, 1)$. Then f is a coinitiality preserving embedding.*

Proof. 1. $f(x + y) = (x + y, 1) = (x, 1) + (y, 1) = f(x) + f(y)$, for all $x, y \in G$.

2. $x < y$ if and only if $f(x) = (x, 1) < (y, 1) = f(y)$, for all $x, y \in G$.

3. Let $X = \{x_i \mid i \in I\} \subseteq G$ be coinitial in G. Let $(y, n) \in \mathbf{V}_G$. There is $x_{i_0} \in X$ such that $x_{i_0} < z$. Therefore, $(x_{i_0}, 1) < (z, n)$ and f is coinitiality preserving.

□

The next lemma shows that the set of the Archimedean classes of $\mathbf{V}_G$ can be identified with that of G.

Lemma 3.3.4. *In the above construction every Archimedean class of* $\mathbf{V}_G$ *contains an element of* G*. Hence we may identify the set* T *of Archimedean equivalence classes of* $\mathbf{V}$ *with that of* G*.*

Proof. Let $[(x, m)]$ be an equivalence class of $\mathbf{V}_G$. We have

$$(x, m) = \frac{1}{m}(x, 1).$$

I.e., $(x, 1) \sim (x, m)$ and therefore $(x, 1) \in [(x, m)]$. We need to show if x, y in G are in different Archimedean classes of G, then $(x, 1)$ and $(y, 1)$ are in different Archimedean classes of $\mathbf{V}_G$. Without loss of generality assume $x \ll y$. Then for all $n \in \omega, nx < y$, hence $(nx, 1) < (y, 1)$ for all $n \in \omega$. Therefore $(x, 1) \ll (y, 1)$. Hence we may identify the set T of Archimedean equivalence classes of $\mathbf{V}_G$ with that of G. □

We will now state the Hahn embedding Theorem as proved by Clifford [Cli54]. In 1907 Hans Hahn showed that every ordered Abelian group can be embedded in a lexicographically ordered real function space. His proof is very lengthy. In 1952, Hausner and Wendel [HW52] gave a much shorter proof of the same theorem for an ordered real vector space. Their work was slightly modified by Clifford

[Cli54]to get the same result for an ordered rational vector space. This provides a more accessible proof of Hahn's fundamental theorem. We will show that the embedding provided in the Hahn embedding Theorem is cofinality preserving. We first need to make a definition so we can give the statement of the Hahn embedding Theorem in full.

Definition 3.3.5. Let $\mathbb{R}^T$ an LFS be given. Let $C_{t_0} : \mathbb{R}^T \to \mathbb{R}^T$ be a linear transformation defined by $C_{t_0}(f)(t) = f(t)$ for $t < t_0$ and $C_{t_0}(f)(t) = 0$ for $t \geq t_0$. C_{t_0} is called *the cut determined by* t_0.

Theorem 3.3.6 (Hahn Embedding Theorem). *Let* $\mathbf{V}$ *be an ordered vector space over* $\mathbb{Q}$*, let* T *be the set of Archimedean classes of* $N(\mathbf{V})$*, and for each* $t \in T$ *let a representative* $e_t \in t$ *be selected. Form the vector space* $\mathbb{R}^T$*, denoting the characteristic function of the point* t *by* 1_t*. There is a mapping* $F : \mathbf{V} \to \mathbb{R}^T$ *satisfying the following requirements:*

1. *F is a group homomorphism;*
2. *F is 1-1;*
3. *F is order preserving;*
4. *$F(qe_t) = q \cdot 1_t$, $t \in T, q \in \mathbb{Q}$;*
5. *If $f \in F(\mathbf{V})$ and C is any cut, then $Cf \in F(\mathbf{V})$.*

We will forgo the proof. The proof can be found in several places including in [Cli54] and [Fuc63]. We have shown that any ordered Abelian group G can be embedded into a vector space $\mathbf{V}_G$ over the rationals and that we can identify the set T of Archimedean equivalence classes of $\mathbf{V}_G$ with that of G. Clifford uses these facts in [Cli54] and modifies the proof of the Theorem 3.3.6 to get Hahn's embedding Theorem for rational vector spaces, hence he has the fundamental Theorem of Hahn that any ordered Abelian group may be embedded into a lexicographically ordered, real function space.

Theorem 3.3.7. *Let G be an ordered Abelian group and let T be the set of Archimedean classes of $N(G)$. Let $F : G \to \mathbb{R}^T$ be the mapping given by the Hahn embedding Theorem. Then F is a coinitiality preserving embedding.*

Proof. The fact that F is a group homomorphism and order preserving is part of the Hahn Theorem. We need to show F is coinitiality preserving. Let $X = \{x_n \mid n \in I\}$ be a coinitial set in G. We will show that $\{F(x_n) \mid n \in I\}$ is coinitial in $\mathbb{R}^T$.

For each $t \in T$, select $e_t \in T$, a representative so that by condition 4 of Hahn Theorem we have $F(e_t) = 1_t$. Let $g \in \mathbb{R}^T$, then there is a t and m_0 a positive integer such that $(-m_0)1_t < g$. Since X is coinitial in G, for each e_t there is an x_{n_0} such that $x_{n_0} < (-m_0)e_t$. We have

$$x_{n_0} < (-m_0)e_t \quad \text{implies} \quad F(x_{n_0}) < F((-m_0)e_t) = (-m_0)1_t \;\; \text{(F is order preserving)}$$
$$(-m_0)1_t < g \quad \text{implies} \quad F(x_{n_0}) < g.$$

We have shown that F is a coinitiality preserving embedding. □

Definition 3.3.8. Let G be an ordered Abelian group. Let $\mathbb{R}^T$ and F be from the Hahn embedding Theorem. $(\mathbb{R}^T, F)$ is called a *Hahn representation* of G.

Chapter 4

Fuzzy Logic

In this chapter we give background information about Fuzzy logic. In section 4.1 we go over the classification of BL-chains. In section 4.2 we introduce fuzzy predicate logic.

4.1 Classification of BL-chains

BL-chains arise naturally in Hájek's analysis of the proof theory of propositional logics. In her Ph.D. thesis, Yvonne Shashoua gave an algebraic classification of BL-chains. The following comes mostly from her thesis [Sha02], [LSar] and [LS02] At the end we will forgo the the rather lengthy proof of the classification of BL-chains theorem.

Definition 4.1.1. $\mathcal{M} = (M, +, \leq, 0, 1)$ is a *BL-Chain* if it satisfies the following:

- The relation $\leq$ is a linear order on A with 1 as the top and 0 as the bottom element;
- $(M, +, \leq)$ is an ordered Abelian semigroup;
- 1 is the identity of $(M, +)$; and

- For all $y \leq x$, there is a largest z such that $x + z = y$.

In particular, this means that BL-algebras form an elementary class.

Definition 4.1.2. Let $\mathcal{M} = (M, +, \leq)$ be given. Let $x, y \in M$, then we define $x \Rightarrow y := z$, where z is the largest element of M such that $x + z = y$ if $y < x$ and $x \Rightarrow y := 1$ if $x \leq y$. Hence $\Rightarrow$ is definable from $+$ and $\leq$. We will use $\Rightarrow$ extensively in our work. It makes the notation and discussions to come have a smoother flow.

Example 4.1.3. The following are the three main examples of BL-chains:

1. Łukasiewicz Logic $[0, 1]_Ł$

$$x + y = \max\{0, x + y - 1\}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if} \quad x \leq y \\ 1 - x + y & \text{if} \quad x > y \end{cases}$$

2. Gödel Logic $[0, 1]_G$

$$x + y = \min\{x, y\}$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if} \quad x \leq y \\ y & \text{if} \quad x > y \end{cases}$$

3. Product logic $[0, 1]_\Pi$

$$x + y = x + y$$

$$x \Rightarrow y = \begin{cases} 1 & \text{if} \quad x \leq y \\ y/x & \text{if} \quad x > y \end{cases}$$

Definition 4.1.4. Let $(G, +, \leq)$ be any ordered Abelian group.

1. The *extended negative cone* $(N_{-\infty}(G), +, \leq)$ is an extension of $N(G)$ with the universe $N(G) \cup \{-\infty\}$, where $+$ and $\leq$ are extended by the definitions: $x + (-\infty) = (-\infty) + x := -\infty$ for all $x \in N_{-\infty}(G)$, $-\infty < x$ for all $x \in N(G)$.

2. Choose and $d \in N(G)$. The *truncation of* $N(G)$ *at* d is the structure $(T(G, d), +_d, \leq_d)$ with universe $\{x \in N(G) : x \geq d\}$, where $\leq_d$ is inherited from $\leq_G$ and $+_d$ is defined by:

$$x +_d y = \begin{cases} x +_G y & \text{if } x +_G y > d \\ d & \text{if } x +_G y \leq d \end{cases}$$

Definition 4.1.5. Let $\mathcal{C} := (C, +, \leq)$ be an ordered Abelian group. We say that $\mathcal{C}$ is a *basic form* if one of the following holds:

1. $\mathcal{C}$ is a singleton $\{p\}$, where $p + p = p$ and $p \leq p$.

2. $\mathcal{C} \cong N(G)$ for some ordered Abelian group G.

3. $\mathcal{C} \cong N_{-\infty}(G)$ for some ordered Abelian group G.

4. $\mathcal{C} \cong T(G, d)$ for some ordered Abelian group and some $d \in N(G)$.

Definition 4.1.6. A *tower of basic forms* is a sequence $\mathcal{T} = \langle \mathcal{C}_i : i \in I \rangle$ indexed by a linearly ordered set $(I, \leq)$ with a first and a last element each such that each $\mathcal{C}_i := (C_i, +, \leq)$ is a basic form, $C_i \cap C_j = \emptyset$ for all $i, j \in I$ such that $i \neq j$, $\mathcal{C}_{\text{first}}$ has a least element, and $\mathcal{C}_{\text{last}}$ is a singleton.

Associated to any tower of basic formulas is a canonical BL-chain $\mathcal{A}_{\mathcal{T}}$ built from $\mathcal{T}$ defined by:

- $\mathcal{A}_{\mathcal{T}} := (A, +, \leq)$

- $A := \bigcup\{C_i : i \in I\}$;
- For $x \in C_i,\ y \in C_j$,

$$x \leq_{\mathcal{T}} y \text{ if and only if } [i \leq_I j \text{ or } (i = j \text{ and } x \leq_{\mathcal{C}_i} y)]$$

- $0_{\mathcal{T}}$:= the least element of C_{first}, and $1_{\mathcal{T}}$:= the unique element of C_{last};
- For $x, y \in A$

$$x +_{\mathcal{T}} y = \begin{cases} x +_{\mathcal{C}_i} y & \text{for } x, y \in C_i \text{ for some } i \in I \\ \min\{x, y\} & \text{for } x \in C_i, y \in C_j, i \neq j \end{cases}$$

Theorem 4.1.7 (Classification of BL-Chains). *For any tower $\mathcal{T}$ of basic forms, the structure $\mathcal{A}_{\mathcal{T}}$ constructed as above is a BL-chain. And every BL-chain is isomorphic to $\mathcal{A}_{\mathcal{T}}$ for some tower $\mathcal{T}$ of basic forms.*

Here is the application of the classification theorem to the three "classical" BL-chains.

Example 4.1.8.

1. $[0,1]_{\text{Ł}} \cong \mathcal{A}_{\mathcal{T}_{\text{Ł}}}$, where $\mathcal{T}_{\text{Ł}} = \langle \mathcal{C}_0, \mathcal{C}_1 \rangle$, where $(C_0, +, \leq) \cong (T(\mathbb{R}, -1), +, \leq)$ and C_1 is a singleton.
2. $[0,1]_G \cong \mathcal{A}_{\mathcal{T}_G}$, where $\mathcal{T}_G = \langle \mathcal{C}_i : i \in [0,1] \rangle$, where C_i is a singleton.
3. $[0,1]_\Pi \cong \mathcal{A}_{\mathcal{T}_\Pi}$, where $\mathcal{T}_\Pi = \langle \mathcal{C}_0, \mathcal{C}_1 \rangle$ where $(C_0, +, \leq) \cong (N_{-\infty}(\mathbb{R}), +, \leq)$ and C_1 is a singleton.

Shahshoua shows that there is only obstruction to the uniqueness of a decomposition. Specifically, whenever a singleton is followed by a copy of $N(G)$ we may "fuse" them together and have $N_{-\infty}(G)$ instead. Hence, we may assume that whenever a singleton is the first component of a tower it is either followed by another singleton or a copy of $T(G, d)$ for some ordered Abelian group. This will be crucial when we investigate models of predicate product logic.

4.2 Fuzzy Predicate Logic

We would like to investigate fuzzy predicate logic. Most of the following material come from Hájek [Háj98].

Definition 4.2.1. A *predicate language* contains the following:

- Predicates: $P, Q, R, \dots$ each together with a positive natural number, its *arity*
- Object constants: $c, d, \dots$
- Object variables: $x, y, z, \dots$
- Connectives: $\&, \to$
- The truth constants: $\bar{0}, \bar{1}$
- Quantifiers $\forall, \exists$.

Definition 4.2.2. *Terms* and *formulas* of predicate logic are defined in the following way:

- Object variables and object constants are *terms*.
- $P(t_1, t_2, \dots t_n)$ where P is a predicate of arity n and $t_1, \dots t_n$ are terms is an *atomic formula*.
- If φ, ψ are formulas, then so are $\varphi \& \psi$ and $\varphi \to \psi$.

- if φ is a formula and x is an object variable, then $(\forall x)\varphi(x)$ and $(\exists x)\varphi(x)$ are formulas.

- $\bar{0}$ and $\bar{1}$ are formulas.

We will assume that predicate languages that we work with are countable and contain only relation symbols and constants. The only place we will deal with the cardinality of $\mathcal{L}$ is when we consider a single sentence in the context of completeness theorem. Since a sentence has finitely many symbols from the language we may already have assumed that $\mathcal{L}$ is countable.

Definition 4.2.3. We define other connectives as follows:

- $\varphi \wedge \psi$ denotes $\varphi \& (\varphi \to \psi)$.

- $\varphi \vee \psi$ denotes $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

- $\neg\varphi$ denotes $\varphi \to \bar{0}$.

- $\varphi \equiv \psi$ denotes $(\varphi \to \psi)\&(\psi \to \varphi)$.

- φ^n denotes $\underbrace{\varphi \& \ldots \& \varphi}_{n \text{ times}}$.

Definition 4.2.4. Let $\mathcal{L}$ be a predicate language and let $\mathbf{L}$ be a BL-chain. An $\mathbf{L}$-structure $\mathbb{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ for $\mathcal{L}$ contains the following:

- Non-empty domain M;

- $r_P : M^n \to \mathbf{L}$, for each n-ary predicate P;

- $m_c \in M$ for each constant object c.

Example 4.2.5. Let $\mathcal{L}$ have one binary predicate R and one object constant c. $\mathbf{L}$ is the BL-algebra $[0,1]_\Pi$. $\mathbb{M} = \langle M, r_R, m_c \rangle$ where $M = \{1,2,3\}$, $m_c = 1$ and r_R is given by the following matrix.

	1	2	3
1	1	.3	.8
2	.5	1	.2
3	.2	1	.3

Definition 4.2.6. 1. An *evaluation* of $\mathcal{L}$ into $\mathbb{M}$ an $\mathbf{L}$-structure, is a map ν from the set object variables into the domain M of $\mathbb{M}$, $\nu : \{x, y, \ldots\} \to M$.

2. Given an evaluation ν we define for every formula φ, the *truth value* $||\varphi||^{\mathbf{L}}_{M,\nu}$ in the following inductive manner ($\Rightarrow$ and $+$ denote the operations of $\mathbf{L}$):

- $||P(x_1, \ldots, x_n)||^{\mathbf{L}}_{M,\nu} = r_P(\nu(x_1), \ldots, \nu(x_n))$, P and n-ary predicate;
- $||\varphi \& \psi||_{M,\nu} = ||\varphi||^{\mathbf{L}}_{M,\nu} + ||\psi||^{\mathbf{L}}_{M,\nu}$;
- $||\varphi \to \psi||^{\mathbf{L}}_{M,\nu} = ||\varphi||^{\mathbf{L}}_{M,\nu} \Rightarrow ||\psi||^{\mathbf{L}}_{M,\nu}$;
- $||\bar{0}||^{\mathbf{L}}_{M,\nu} = 0$; $||\bar{1}||^{\mathbf{L}}_{M,\nu} = 1$;
- Let $E(\nu, x)$ denote the set of all evaluations which coincide with ν on all variables different from x. Then

$$||\forall x \varphi||^{\mathbf{L}}_{M,\nu} = \inf_{\nu' \in E(\nu,x)} ||\varphi||^{\mathbf{L}}_{M,\nu'} \quad \text{and} \quad ||\exists x \varphi||^{\mathbf{L}}_{M,\nu} = \sup_{\nu' \in E(\nu,x)} ||\varphi||^{\mathbf{L}}_{M,\nu'}$$

provided the infimum/supremum exists, otherwise the truth value of the formula in question is undefined.

If the free variables of φ are among $x_1, \ldots, x_n$, and $a_1, \ldots, a_n \in M$, then for all ν and ν' such that for $i = 1, \ldots, n$, $\nu(x_i) = \nu'(x_i)$, we have $||\varphi||_{M,\nu} = ||\varphi||_{M,\nu'}$. Thus we will write $||\varphi(a_1, \ldots, a_n)||_M$ to mean $||\varphi||_{M,\nu}$.

Definition 4.2.7. The structure $\mathbb{M}$ is $\mathbf{L}$-*safe* if all the needed infima and suprema exist, *i.e.* $||\varphi||^{\mathbf{L}}_{M,\nu}$ is defined for all φ, ν.

In particular, each finite structure (with finite domain) is safe. Every $\mathbf{L}$-structure $\mathbb{M}$ where $\mathbf{L} = [0,1]_{\text{L}},\ [0,1]_G,\ [0,1]_\Pi$ is safe, since $[0,1]$ contains the limit points of every monotone sequence.

Example 4.2.8. We verify that in example 4.2.5

$$||\forall x R(x,c)||_{\mathbb{M},\nu} = 0.2$$

and

$$||\exists x \neg R(c,x)||_{\mathbb{M},\nu} = 0.$$

In his work, [Háj98], Hájek uses safe models. In this paper we define a new class of models which has more restrictions than the class of safe models.

Definition 4.2.9. 1. $||\varphi||^{\mathbf{L}}_{\mathbb{M}} = \inf\{||\varphi||^{\mathbf{L}}_{\mathbb{M},\nu} \mid \nu\ M-\text{evaluation}\}$

2. A formula φ of a language $\mathcal{L}$ is an $\mathbf{L}$-*tautology* if and only if $||\varphi||^{\mathbf{L}}_{\mathbb{M}} = 1$ for all $\mathbb{M}$ safe $\mathbf{L}$-model. That is, $||\varphi||^{\mathbf{L}}_{\mathbb{M},\nu} = 1$ for each safe $\mathbf{L}$-structure $\mathbb{M}$ and each $\mathbb{M}$-valuation ν of object variables. Let $\mathbf{L}_{\mathcal{L}}$-TAUT denote the set of $\mathbf{L}$-tautologies for the language $\mathcal{L}$.

We will drop the subscript $\mathcal{L}$ from $\mathbf{L}$-TAUT, whenever the language is clear from the context.

Example 4.2.10. $\forall x \varphi(x) \to \exists x \varphi(x)$ is an $\mathbf{L}_{\mathcal{L}}$-tautology for any $\mathbf{L}$ and $\mathcal{L}$. This is true since for all $\mathbb{M}$ safe $\mathbf{L}$-structures we have

$$\inf_{a_i \in M} ||\varphi(a_i)||_{\mathbb{M}} \leq \sup_{a_i \in M} ||\varphi(a_i)||_{\mathbb{M}}.$$

Definition 4.2.11. Let $\mathcal{L}$ be a predicate language. For any formulas φ, ψ, χ of $\mathcal{L}$, the following are *axioms of basic fuzzy predicate logic*:

$(A1)$ $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

$(A2)$ $(\varphi \& \psi) \to \varphi$

$(A3)$ $(\varphi \& \psi) \to (\psi \& \varphi)$

$(A4)$ $((\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$

$(A5a)$ $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$

$(A5b)$ $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

$(A6)$ $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

$(A7)$ $\bar{0} \to \varphi$

$(\forall 1)$ $\forall x \varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$

$(\exists 1)$ $\varphi(t) \to \exists x \varphi(x)$ (t substitutable for x in $\varphi(x)$

$(\forall 2)$ $\forall x(\psi \to \varphi) \to (\psi \to \forall x \varphi)$ (x not free in ψ)

$(\exists 2)$ $\forall x(\varphi \to \psi) \to (\exists x \varphi \to \psi)$ (x not free in ψ)

$(\forall 3)$ $\forall x(\varphi \vee \psi) \to (\forall x \varphi \vee \psi)$ (x not free in ψ)

$(\forall 1)$ – $(\exists 2)$ are called *Logical axioms on quantifiers:*. We usually denote $(A1)$ – $(\forall 3)$, the logical axioms of quantifiers, by BL$\forall$.

Definition 4.2.12. 1. An *axiom schema* given by a formula $\Phi(p_1, \ldots, p_n)$ is the set of all formulas $\Phi(\varphi_1, \ldots, \varphi_n)$ resulting by the substitution of φ_i for p_i $(i = 1, \ldots, n)$ in $\Phi(p_1, \ldots, p_n)$.

2. A logical calculus $\mathcal{C}\forall$ is a *schematic extension* of BL$\forall$ if it results from BL$\forall$ by adding some (finitely or infinitely many) axiom schemata to its axioms.

3. Let $\mathcal{C}\forall$ be schematic extension of BL$\forall$ and let $\mathbf{L}$ be a BL-chain. $\mathbf{L}$ is a $\mathcal{C}\forall$-*chain* if all axioms of $\mathcal{C}\forall$ are $\mathbf{L}$-tautologies.

Definition 4.2.13. Let $\mathcal{C}\forall$ be a schematic extension of BL$\forall$.

1. A *theory* over $\mathcal{C}\forall$ is a set of formulas. Elements of T are *axioms* of T.

2. The *deduction* rules of Basic Predicate Fuzzy Logic are

   - *modus ponens*: From φ and $\varphi \to \psi$ infer ψ.

   - generalization (from φ infer $\forall x\varphi$).

3. A *proof* from theory T (over $\mathcal{C}\forall$) is a finite sequence of formulas $\varphi_0, \ldots, \varphi_n$ such that for each $i = 0$ to n, φ_i is either an element of $T \cup \mathcal{C}\forall$ or follows from some earlier φ_j and φ_k $(j, k < i)$ by modus ponens or $\varphi_i = \forall x\varphi_j$ for some earlier φ_j.

4. A formula φ is *provable from* T if there exists a proof from $T:\ \varphi_0, \ldots \varphi_n$ with $\varphi = \varphi_n$, the last line of the proof.

5. Let $\mathbb{M}$ be a safe $\mathbf{L}$-structure. $\mathbb{M}$ is called an $\mathbf{L}$-*model of* T if $||\varphi||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$ for each $\varphi \in T$.

Example 4.2.14. The following are three examples of theories:

$$
\begin{aligned}
\text{Ł}\forall &:= \{\text{BL}\forall\text{-axioms}\} \cup \{\neg\neg\varphi : \varphi \text{ is a formula of BL}\forall\} \\
G\forall &:= \{\text{BL}\forall\text{-axioms}\} \cup \{\varphi \to (\varphi \& \varphi) : \varphi \text{ is a formula of BL}\forall\} \\
\Pi\forall &:= \{\text{BL}\forall\text{-axioms}\} \cup \{(\varphi \wedge \neg\varphi) \to \bar{0}) : \varphi \text{ is a formula of BL}\forall\} \\
&\cup \{\neg\neg\chi \to ((\varphi \& \chi \to \psi \& \chi) \to (\varphi \to \psi)) : \varphi, \psi, \chi \text{ are formulas of BL}\forall\}
\end{aligned}
$$

Specifically, we shall be interested in $\Pi\forall$ (predicate Product Logic) in the later chapters.

Lemma 4.2.15. *The axioms* $(\forall 1)-(\exists 2)$ *are all* $\mathbf{L}$*-tautologies for each BL-chain* $\mathbf{L}$.

Proof. Lemma 5.1.9 of [Háj98]. □

Theorem 4.2.16 (Deduction Theorem). *Let* T *be a theory over* $\mathcal{C}\forall$ *and let* φ, ψ *be closed formulas of the language of* T. *Then* $(T \cup \{\varphi\}) \vdash \psi$ *if and only if* $T \vdash \varphi^n \to \psi$ *for some positive integer* n.

Theorem 4.2.17 (Completeness). *Let* $\mathcal{C}\forall$ *be the predicate calculus given by schematic extension* $\mathcal{C}$ *of BL, let* T *be a theory over* $\mathcal{C}\forall$ *and let* α *be a formula of language of* T. *Then* $T \vdash \alpha$ *if and only if for each* $\mathcal{C}$*-chain* $\mathbf{L}$ *and each safe* $\mathbf{L}$*-model* $\mathbb{M}$ *of* T, $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$.

Lemma 4.2.18. *For all* φ, ψ *and* χ *we have* $\vdash (\varphi \to \psi) \to ((\varphi \& \chi) \to (\psi \& \chi))$.

Proof. By the Completeness Theorem it suffices to show

$$||(\varphi \to \psi) \to ((\varphi \& \chi) \to (\psi \& \chi))||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$$

for all BL-chain $\mathbf{L}$ and safe $\mathbf{L}$-structure $\mathbb{M}$. Hence it suffices to show, for all $a, b, c \in \mathbf{L}$, $(a \Rightarrow b) \Rightarrow ((a + c) \Rightarrow (b + c)) = 1_{\mathbf{L}}$.

$a \leq b:$ In this case, $a + c \leq b + c$, hence $a \Rightarrow b = (a + c) \Rightarrow (b + c) = 1$ and $1 \Rightarrow 1 = 1$.

$a > b:$ In this case $a + c \geq b + c$. If a and b are in the same component of $\mathbf{L}$ then $a + c$ and $b + c$ would too. Then $a \Rightarrow b = b - a$ and $a + c \Rightarrow b + c = 1$ or $b + c - a - c = b$. Then, both $(b - a) \Rightarrow 1$ and $(b - a) \Rightarrow 1 = 1$. If a and

b are in different components of $\mathbf{L}$, then $a \Rightarrow b = b$. Considering different position of c with respect to a, b we get $(a+c) \Rightarrow (b+c) = b$ or 1. We have both $b \Rightarrow b = 1$ and $b \Rightarrow 1 = 1$. And the proof is complete.

□

Chapter 5

Predicate Product Logic

In this chapter we will focus on predicate Product Logic. We will use Shashoua's Classification of BL-Chains extensively. Her algebraic classification of BL-chains lets us focus on ordered Abelian groups and their properties. In his work [Háj98], Hájek uses safe structures. We would like to define a new class of structures (called closed) with more restraints. The Completeness Theorem for $\Pi\forall$ as stated by Hájek (Theorem 4.2.17) says $\Pi\forall \vdash \alpha$ if and only if for each $\Pi\forall$-chain $\mathbf{L}$, and $\mathbb{M}$ safe $\mathbf{L}$-structure $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1$. We will be able to improve this theorem to get $\mathbf{L} = L(\mathbb{R}^S)$ a lexicographic function space and $\mathbb{M}$ a closed $\mathbf{L}$-model. We will actually improve the theorem so that S is $\mathbb{Q}$. We will first classify $\Pi\forall$-chains and then define closed structures and motivate their definition.

5.1 Classification of $\Pi\forall$-Chains

In this section we will investigate the possible $\Pi\forall$-chains. As noted by Laskowski and Shashoua BL-chains are a sequence of basic forms (singleton, extended negative cone of an ordered Abelian group and $T(G, d)$ for some ordered Abelian group). We will show that if $\mathbf{L}$ is a $\Pi\forall$-chain then $\mathbf{L}$ consists of an extended negative cone of an ordered Abelian group and a singleton. Remember from chapter

4 that

$$\begin{aligned}\Pi\forall \;:=\; & \{\text{BL}\forall\text{-axioms}\} \cup \{(\varphi \wedge \neg\varphi) \to \bar{0}) : \varphi \text{ is a formula of BL}\forall\} \\ \cup\; & \{\neg\neg\chi \to ((\varphi\&\chi \to \psi\&\chi) \to (\varphi \to \psi)) : \varphi, \psi, \chi \text{ are formulas of BL}\forall\}\end{aligned}$$

and a BL-chain, $\mathbf{L}$, is called a $\Pi\forall$-chain if all axioms of $\Pi\forall$ are $\mathbf{L}$-tautologies.

Lemma 5.1.1. *Let* $\mathbf{L}$ *be any BL-chain. Then for all* $x, y \in \mathbf{L}$, $x + (x \Rightarrow y) = \min\{x, y\}$.

Proof. If $x \leq y$ then $x \Rightarrow y = 1$ and $x + (x \Rightarrow y) = x + 1 = x = \min\{x, y\}$. If $y < x$ then $x \Rightarrow y = z$ where z is the largest such that $x + z = y$, then $x + (x \Rightarrow y) = y = \min\{x, y\}$. □

Lemma 5.1.2. *Let* $\mathbf{L}$ *be a* $\Pi\forall$*-chain. Then for all* $x \in \mathbf{L}$ *we have* $\min\{x, x \Rightarrow 0\} = 0$.

Proof. $\Pi\forall \vdash \varphi \wedge \neg\varphi \to 0$ for all φ. Let $\mathbf{L}$ a $\Pi\forall$-chain and $\mathbb{M}$ a safe $\mathbf{L}$-structure. Then $||\varphi \wedge \neg\varphi \to 0||^{\mathbf{L}}_{\mathbb{M}} = 1$. Hence, $||\varphi\&(p \to \neg\varphi)||^{\mathbf{L}}_{\mathbb{M}} = 0$. Therefore by Lemma 5.1.1 for all $x \in \mathbf{L}$, we get $\min\{x, \neg x\} = \min\{x, x \Rightarrow 0\} = 0$. □

Lemma 5.1.3. *Let* $\mathbf{L}$ *be a* $\Pi\forall$*-chain, where* $\mathbf{L} = \mathcal{A}_{\mathcal{T}}$ *for some tower* $\mathcal{T}$*. Then* $(T(G, d), +, \leq)$ *is not a component of* $\mathcal{T}$ *for any Abelian group* G *and* $d \in G$.

Proof. By way of contradiction suppose $(T(G, d), +, \leq) \in \mathcal{T}$, for some Abelian group G and $d \in G$.

Case 1: $T(G, d)$ is the first component. Note $0_L = d$. Let $x \neq d \in T(G, d)$. Then $\neg x = x \Rightarrow 0_{\mathbf{L}} = d - x$, hence $\min\{x, \neg x\} \neq 0_{\mathbf{L}}$ which is a contradiction to Lemma 5.1.2.

Case 2: $T(G, d) \in \mathcal{T}$ but it is not the first component. Let $a > b > d \in T(G, d)$.

We have that $a + d = b + d = d$. Since $\mathbf{L}$ is a φ-chain we have $\neg\neg\chi \to ((\varphi\&\chi \to \psi\&\chi) \to (\varphi\&\psi))$ is an $\mathbf{L}$-tautology.

$$\begin{aligned} \neg\neg d \Rightarrow ((a + d \to b + d) \Rightarrow (a \Rightarrow b)) &= 1 \\ 1 \Rightarrow ((d \Rightarrow d) \Rightarrow (a \Rightarrow b)) &= 1 \\ 1 \Rightarrow (a \Rightarrow b) &= 1 \\ a \Rightarrow b &= 1. \end{aligned}$$

But $a \Rightarrow b = b - a < 1$ and we have a contradiction.

□

Lemma 5.1.4. *Let* $\mathbf{L}$ *be a* $\Pi\forall$*-chain, where* $\mathbf{L} = \mathcal{A}_{\mathcal{T}}$ *for some tower* $\mathcal{T}$*. Then none of the components are singletons except for possibly the first and last ones.*

Proof. Suppose $\{p\} \in \mathcal{T}$ is neither the first or last component. Then $\neg\neg p = 1$ and $p+p = p$. Since $\mathbf{L}$ is a $\Pi\forall$-chain, we have $\neg\neg\chi \to ((\varphi\&\chi \to \psi\&\chi) \to (\varphi\&\psi))$ is an $\mathbf{L}$-tautology. Hence

$$\begin{aligned} \neg\neg p \Rightarrow ((1 + p \Rightarrow p + p) \Rightarrow (1 \Rightarrow p)) &= 1 \\ 1 \Rightarrow ((p \Rightarrow p) \Rightarrow p) &= 1 \\ (p \Rightarrow p) \Rightarrow p &= 1 \\ 1 \Rightarrow p &= 1 \\ p &= 1. \end{aligned}$$

But $p < 1$ and we have a contradiction. □

Lemma 5.1.5. *Let* $\mathbf{L}$ *be a* $\Pi\forall$*-chain, where* $\mathbf{L} = \mathcal{A}_{\mathcal{T}}$ *for some tower* $\mathcal{T}$*. If* $\mathcal{C}_i, \mathcal{C}_j$ *are components of* $\mathcal{T}$ *and neither is the last component, then* $i = j$*.*

Proof. Without loss of generality, assume $j < i$. By Lemma 5.1.4, we know $\mathcal{C}_i$ and $\mathcal{C}_j$ are not singleton. Let $a, b \in \mathcal{C}_i$ so that $a > b$ and let $c \neq 0 \in \mathcal{C}_j$. Since $\mathbf{L}$ is a $\Pi\forall$-chain, we have $\neg\neg\chi \to ((\varphi\&\chi \to \psi\&\chi) \to (\varphi\&\psi))$ is an $\mathbf{L}$-tautology. Hence

$$\begin{aligned}
\neg\neg c \Rightarrow ((a + c \Rightarrow b + c) \Rightarrow (a \Rightarrow b)) &= 1 \\
1 \Rightarrow ((c \Rightarrow c) \Rightarrow (a \Rightarrow b)) &= 1 \\
(c \Rightarrow c) \Rightarrow (a \Rightarrow b) &= 1 \\
1 \Rightarrow (a \Rightarrow b) &= 1 \\
a \Rightarrow b &= 1.
\end{aligned}$$

But $a \Rightarrow b = b - a < 1_{\mathbf{L}}$ which is a contradiction. Therefore $i = j$. □

Theorem 5.1.6. *Let* $\mathbf{L}$ *be a* $\Pi\forall$*-chain, where* $\mathbf{L} = \mathcal{A}_{\mathcal{T}}$ *for some tower* $\mathcal{T}$*. Then* $\mathcal{T} = \langle \mathcal{C}_0, \mathcal{C}_1 \rangle$ *where* $\mathcal{C}_0 \cong (N_{-\infty}(G), +, \leq)$*, for some ordered Abelian group* G *and* $\mathcal{C}_1$ *is a singleton.*

Proof. This is a direct corollary of the preceding lemmas. If the first component is a singleton by the decomposition theorem it has to be followed either by another singleton or an $T(G, d)$. Lemmas 5.1.4 and 5.1.3 imply that this can not happen. Now Lemma 5.1.5 gives us that $\mathcal{T}$ has two components $\langle \mathcal{C}_0, \mathcal{C}_1 \rangle$, where $\mathcal{C}_0 \cong (N_{-\infty}(G), +, \leq)$, for some ordered Abelian group G and $\mathcal{C}_1$ is a singleton. □

In order to make the notation easy to follow we make the following definition.

Definition 5.1.7. Let G be an ordered Abelian group. Let $\mathbf{L} = \mathcal{A}_{\mathcal{T}}$, where $\mathcal{T} = \langle \mathcal{C}_0, \mathcal{C}_1 \rangle$ and $\mathcal{C}_0 \cong (N_{-\infty}(G), +, \leq)$. We write $L(G)$ for $\mathbf{L}$.

Corollary 5.1.8. *Let* $\mathbf{L}$ *be a* $\Pi\forall$*-chain. Then* $\mathbf{L} = L(G)$ *for some ordered Abelian group* G*.*

Proof. Theorem 5.1.6. □

5.2 Closed Structures

We first remind the reader of the definition of safe structures (Definition 4.2.7). Given $\mathbf{L}$ a BL-chain, the structure $\mathbb{M}$ is $\mathbf{L}$-*safe* if all the needed infima and suprema exist, *i.e.* $||\varphi||^{\mathbf{L}}_{M,\nu}$ is defined for all φ, ν.

Definition 5.2.1. Given $\mathbf{L}$ a BL-chain, the structure $\mathbb{M}$ is called $\mathbf{L}$-*closed* if and only if for every φ a formula

$$\begin{aligned} ||\exists x\varphi(x)||^{\mathbf{L}}_{\mathbb{M}} &= ||\varphi(c)||^{\mathbf{L}}_{\mathbb{M}} \text{ for some } c \in M; \\ ||\forall x\varphi(x)||^{\mathbf{L}}_{\mathbb{M}} &= ||\varphi(c)||^{\mathbf{L}}_{\mathbb{M}} \text{ for some } c \in M, \text{ provided } ||\forall x\varphi(x)||^{\mathbf{L}}_{\mathbb{M}} \neq 0. \end{aligned}$$

Note that a closed $\mathbf{L}$-structure is a safe $\mathbf{L}$-structure. The following example and discussion motivates our definition of closed models, that we may not assume that if 0 is the infimum of a sequence then it is actually an element of the sequence.

Example 5.2.2. $\sigma := (\forall xP \to Q) \to \exists x(P \to Q)$ is not a tautology of $\Pi\forall$.

Let $\mathbf{L} = L(G)$ for some ordered Abelian group, G. Let $\{x_n \mid x_n \in N_{-\infty}(G),\ n \in \omega,\ \inf_n x_n = 0, x_n \neq 0\}$. Let $\mathbb{M} = \langle M, r_P, r_Q\rangle$ where $M = \omega$ and $r_P(n) = x_n$ and $r_Q(n) = 0$ for $n \in \omega$. Then

$$\begin{aligned} ||\sigma||_{\mathbb{M}} &= (\inf_n x_n \Rightarrow y) \Rightarrow \sup_n (x_n \Rightarrow y) \\ &= (0 \Rightarrow 0) \Rightarrow \sup_n (x_n \Rightarrow 0) \\ &= 1 \Rightarrow \sup_n (0) \\ &= 1 \Rightarrow 0 = 0 \end{aligned}$$

In the above example, assume $x_{n_0} = 0$ for some $n_0 \in \omega$. Then we would actually have $||\sigma||_{\mathbb{M}} = 1$.

$$\begin{aligned} ||\sigma||_{\mathbb{M}} &= (\inf_n x_n \Rightarrow y) \Rightarrow \sup_n (x_n \Rightarrow y) \\ &= (0 \Rightarrow 0) \Rightarrow \sup_n (x_n \Rightarrow 0) \\ &= 1 \Rightarrow 1 \\ &= 1. \end{aligned}$$

Here we have a sentence which is not an $L(G)$-tautology, but it will evaluate to 1 if we assume that 0 is attained by a point in the sequence. We shall later see that the obstacle we face in predicate Product Logic is exactly sequences approaching zero. We will divide our $\Pi\forall$-chains, $L(G)$ into two collections. One collection contains those $L(G)$'s that $N(G)$ contains an element with minimal Archimedean class, like $L(\mathbb{R}^{1+\mathbb{Q}})$. The other collection contains those $L(G)$'s that $N(G)$ does not have an element with minimal Archimedean class, like $L(\mathbb{R}^{\mathbb{Q}})$. We will finally see that in evaluating a sentence α in some $L(G)$ and $\mathbb{M}$ $L(G)$-structure, if we do not encounter a sequence approaching zero such that zero is not one of its elements then we may assume that $L(G)$ is in the second collection and is actually $L(\mathbb{R}^{\mathbb{Q}})$.

5.3 Some Results for $\Pi\forall$

In this section we will prove some preliminary result for $\Pi\forall$. For the rest of section we fix $\mathcal{L}$ a countable predicate language.

Lemma 5.3.1. *For each natural $n \geq 1$ and formula φ,*

1. $\Pi\forall \vdash (\forall x \varphi(x))^n \equiv \forall x \varphi^n(x)$,

2. $\Pi\forall \vdash (\exists x\varphi(x))^n \equiv \exists x\varphi^n(x)$.

Proof. 1. Theorem 4.2.17 implies $\Pi\forall \vdash (\forall x\varphi(x))^n \equiv \forall x\varphi^n(x)$ if and only if for all $\Pi\forall$-chains $\mathbf{L}$ and $\mathbf{L}$-safe structure $\mathbb{M}$, we have $||(\forall x\varphi(x))^n \equiv \forall x\varphi^n(x)||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$. We have shown that if $\mathbf{L}$ is a $\Pi\forall$-chain then $\mathbf{L} = L(G)$ for some ordered Abelian group. Let $L(G)$ and $\mathbb{M}$ an $L(G)$-safe structure be given. Let $||\varphi(m_i)||_{\mathbb{M}} = a_i$, for each $m_i \in \mathbb{M}$ and some $a_i \in N(G)$. Then since $+$ is continuous in $L(G)$ we have

$$\inf_i na_i = n(\inf_i a_i).$$

so $||(\forall x\varphi(x))^n \equiv \forall x\varphi^n(x)||^{L(G)}_{\mathbb{M}} = 1_{\mathbf{L}}$.

2. We have the same proof as above, replacing infimum with supremum.

□

Lemma 5.3.2. *For each natural $n \geq 1$ and formulas φ and ψ,*

1. $\Pi\forall \vdash (\varphi \to \psi)^n \equiv (\varphi^n \to \psi^n)$,

2. $\Pi\forall \vdash (\varphi\&\psi)^n \equiv (\varphi^n\&\psi^n)$.

Proof. Again we will use the technique of Lemma 5.3.1. We need to show $||(\varphi \to \psi)^n \equiv (\varphi^n \to \psi^n)||^{L(G)}_{\mathbb{M}} = 1$ and $||(\varphi\&\psi)^n \equiv (\varphi^n\&\psi^n)||^{L(G)}_{\mathbb{M}} = 1$ for all ordered Abelian groups, G and all $L(G)$-safe structures $\mathbb{M}$. Fix $n \geq 1$, $a, b \in N(G)$ for G an ordered Abelian group. We need to show $n(a \Rightarrow b) = na \Rightarrow nb$ and $n(a + b) = na + nb$.

1. If $a \leq b$ then we have $na \leq nb$, hence $n(a \Rightarrow b) = n1 = 1$ and $na \Rightarrow nb = 1$. If $a > b$, then $n(a \Rightarrow b) = n(b - a) = nb - na = na \Rightarrow nb$. In both cases, $n(a \Rightarrow b) = na \Rightarrow nb$.

2. Trivial.

□

Lemma 5.3.3. *Let φ be an arbitrary formula, ν a formula not containing x, then*

1. $\vdash \forall x(\varphi(x) \to \nu) \equiv (\exists x \varphi(x) \to \upsilon)$,

2. $\Pi\forall \vdash \exists x(\nu \to \varphi(x)) \equiv (\nu \to \exists x \varphi(x))$.

Proof. 1. $(\to)$ is $(\exists 2)$.

$(\leftarrow)$, $\vdash \varphi(x) \to \exists x \varphi(x)$;

$\therefore \vdash (\exists x \varphi(x) \to \nu) \to (\varphi(x) \to \nu;)$

$\therefore \vdash \forall x((\exists x \varphi(x) \to \nu) \to (\varphi(x) \to \nu))$ (generalization);

$\therefore \vdash (\exists x \varphi(x) \to \nu) \to \forall x(\varphi(x) \to \nu)$ by $\forall 2$.

2. $(\to)$, $\vdash (\nu \to \varphi(x)) \to (\nu \to \exists x \varphi(x))$;

$\therefore \vdash \forall x((\nu \to \varphi(x)) \to (\nu \to \exists x \varphi(x)))$, (generalization);

$\therefore \vdash \exists x(\nu \to \varphi(x)) \to (\nu \to \exists x \varphi(x))$ by applying $(\exists 2)$.

$(\leftarrow)$, Let $L(G)$ be a $\Pi\forall$-chain, where G is an ordered Abelian group. It suffices to show $a \Rightarrow \sup_i b_i \leq \sup_i (a \Rightarrow b_i)$, for $a, b_i, \sup b_i \in N(G)$. If there is a j such that $a \leq b_j$ then $a \leq \sup_i b_i$. Therefore we get, $a \Rightarrow \sup_i b_i = 1 = a \Rightarrow b_j = \sup_i (a \Rightarrow b_i)$. On the other hand, if $a \geq b_i$ for all i we get $a \Rightarrow \sup_i b_i = \sup_i b_i - a = \sup_i (b_i - a) = \sup_i (a \Rightarrow b_i)$.

□

Lemma 5.3.4. *For each theory T and closed formulas α, φ, ψ, if $T \nvdash \alpha$ then either $T \cup \{\varphi \to \psi\} \nvdash \alpha$ or $T \cup \{\psi \to \varphi\} \nvdash \alpha$.*

Proof. Suppose $T \cup \{\varphi \to \psi\} \vdash \alpha$ and $T \cup \{\psi \to \varphi\} \vdash \alpha$. Then by Deduction Theorem, for some n a positive integer $T \vdash (\varphi \to \psi)^n \to \alpha$ and $T \vdash (\psi \to \varphi)^n \to \alpha$, thus $T \vdash ((\varphi \to \psi)^n \vee (\psi \to \varphi)^n) \to \alpha$. But $\vdash (\varphi \to \psi)^n \vee (\psi \to \varphi)^n$, hence $T \vdash \alpha$ a contradiction.

Let $T' = T \cup \{\varphi \to \psi\}$ in the former case and $T' = T \cup \{\psi \to \varphi\}$ in the latter case. T' is an extension of T deciding (φ, ψ) and keeping α unprovable. □

Lemma 5.3.5. *Let T be a theory over $\Pi\forall$ and assume $T \nvdash \alpha$ for some closed formula α. Let c be a new constant symbol. Then $T \cup \{\exists x \chi(x) \to \chi(c)\} \nvdash \alpha$.*

Proof. By way of contradiction suppose $T \cup \{\exists x \chi(x) \to \chi(c)\} \vdash \alpha$.

$\therefore T \vdash (\exists x \chi(x) \to \chi(c))^k \to \alpha)$, for some positive integer k (Deduction Theorem);

$\therefore T \vdash (\exists x \chi(x) \to \chi(y))^k \to \alpha)$, replacing c by y a new variable;

$\therefore T \vdash \forall y((\exists x \chi(x) \to \chi(y))^k \to \alpha))$ generalization;

$\therefore T \vdash \exists y(\exists x \chi(x) \to \chi(y))^k \to \alpha$, $(\exists 2)$;

$\therefore T \vdash (\exists y(\exists x \chi(x) \to \chi(y)))^k \to \alpha$, Lemma 5.3.1;

$\therefore T \vdash (\exists x \chi(x) \to \exists y \chi(y))^k \to \alpha$, Lemma 5.3.3;

$\therefore T \vdash \alpha$, which is a contradiction.

Hence, $T \cup \{\exists x \chi(x) \to \chi(c)\} \nvdash \alpha$. □

Lemma 5.3.6. *Let T be a theory over $\Pi\forall$ and assume $T \cup \{\chi(c) \to (\forall x)\chi(x)\} \vdash \alpha$ for some new constant symbol c. Then $T \cup \{\exists y(\chi(y) \to \forall x \chi(x))\} \vdash \alpha$.*

Proof. $T \cup \{\chi(c) \to (\forall x)\chi(x)\} \vdash \alpha$

$\therefore T \vdash (\chi(c) \to \forall x \chi(x))^k \to \alpha$, Deduction Theorem;

$\therefore T \vdash (\chi(y) \to \forall x \chi(x))^k \to \alpha$, replacing c by y;

$\therefore T \vdash \forall y((\chi(y) \to \forall x \chi(x))^k \to \alpha)$, generalization;

$\therefore T \vdash \exists y(\chi(y) \to \forall x \chi(x))^k \to \alpha$, Lemma 5.3.3;

$\therefore T \vdash (\exists y(\chi(y) \to \forall x\chi(x))^k \to \alpha$, Lemma 5.3.1;

$T \cup \{\exists y(\chi(y) \to \forall x\chi(x))\} \vdash \alpha$, Deduction Theorem. □

Theorem 5.3.7. *Let T be a theory over $\Pi\forall$ and suppose $T \cup \{\forall x\chi(x) \to 0\} \vdash \alpha$ and $T \cup \{\chi(c) \to (\forall x)\chi(x)\} \vdash \alpha$ (c a new constant symbol). Then $T \vdash \alpha$.*

Proof. Let $\mathbf{L}$ be a BL-chain and let $\mathbb{M}$ be a $\mathbf{L}$-model of T.We need to show $||\alpha||_{\mathbb{M}} = 1$.

Case 1: $||\forall x\chi(x)||^{\mathbf{L}}_{\mathbb{M}} = 0$. Then $||\forall x\chi(x) \to 0||^{\mathbf{L}}_{\mathbb{M}} = 1$. Therefore, $\mathbb{M} \models T \cup \{\forall x\chi(x) \to 0\}$ and $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1$.

Case 2: $||\forall x\chi(x))||^{\mathbf{L}}_{\mathbb{M}} \neq 0$. Therefore, $||\exists y(\chi(y) \to \forall x\chi(x))||^{\mathbf{L}}_{\mathbb{M}} = 1$ On the other hand $T \cup \{\chi(c) \to (\forall x)\chi(x)\} \vdash \alpha$ which implies $T \cup \{\exists y(\chi(y) \to \forall x\chi(x))\} \vdash \alpha$ (Lemma 5.3.6).

$\therefore$ $T \vdash (\exists y(\chi(y) \to \forall x\chi(x)))^k \to \alpha$ (Deduction Theorem).

$\therefore$ $||(\exists y(\chi(y) \to \forall x\chi(x)))^k \to \alpha||^{\mathbf{L}}_{\mathbb{M}} = 1$. But

$$\begin{aligned} ||(\exists y(\chi(y) \to \forall x\chi(x)))^k \to \alpha||^{\mathbf{L}}_{\mathbb{M}} &= ||1 \to \alpha||^{\mathbf{L}}_{\mathbb{M}} \\ &= ||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}. \end{aligned}$$

In both cases we have shown that $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1$ and this completes the proof of the theorem. □

Corollary 5.3.8. *Let T be a theory over $\Pi\forall$ and suppose $T \nvdash \alpha$. Let $\chi(x)$ be a formula of the language of T and c a new constant symbol. Then either $T \cup \{\forall x\chi(x) \to 0\} \nvdash \alpha$ or $T \cup \{\chi(c) \to (\forall x)\chi(x)\} \nvdash \alpha$.*

5.4 Completeness Theorem For $\Pi\forall$

In this section we will strengthen Hájek's Completeness Theorem for predicate product logic. We will use techniques used in [Háj98]. We will expand our language and go through a Henkinization process. We then provide a closed **L**-model for T which is the collection of classes of T-equivalent closed formulas.

Definition 5.4.1. ([Háj98]) Let T be a theory over $\mathcal{C}\forall$ a schematic extension of BL$\forall$.

1. T is *consistent* if there is formula φ unprovable in T.
2. T is *complete* if for each pair φ, ψ of closed formulas, $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$.
3. T is *Henkin* if for each closed formula of the form $\forall x \varphi(x)$ unprovable in T there is a constant c in the language of T such that $\varphi(c)$ is unprovable in T.

Definition 5.4.2. Let $\mathcal{C}\forall$ be a schematic extension of BL$\forall$. Let T be a theory over $\mathcal{C}\forall$. For each closed formula φ let $[\varphi]_T = \{\psi \mid T \vdash \varphi \equiv \psi\}$. Let $\mathbf{L}_T$ be the set of all the classes $[\varphi]_T$. We define

$$[\varphi]_T + [\psi]_T := [\varphi \& \psi]_T.$$

$$[\varphi]_T \Rightarrow [\psi]_T := [\varphi \to \psi]_T$$

Lemma 5.4.3. *1. If T is complete then $\mathbf{L}_T$ is a BL-chain.*

2. If T is Henkin then for each formula $\varphi(x)$ with just one free variable x,

$$[\forall x \varphi]_T = \inf_c [\varphi(c)]_T$$

$$[\exists x\varphi]_T = \sup_c [\varphi(c)]_T$$

(c running over all constants of T).

Proof. 1. We order $\mathbf{L}_T$ by

$$[\varphi]_T \leq [\psi]_T \quad \text{if and only if} \quad T \vdash \varphi \to \psi.$$

(a) Since T is complete the definition of $\leq$ is well defined on $\mathbf{L}_T$ and $[1]_T = 1_{\mathbf{L}_T}$ and $[0]_T = 0_{\mathbf{L}_T}$.

(b) It is a very straightforward application of the definitions to show $(\mathbf{L}_T, +)$ is a semigroup ($[\varphi \& \psi]_T = [\psi \& \varphi]_T$). We now need to show for all φ, ψ and χ formula the following holds.

$$[\varphi]_T \leq [\psi]_T \quad \text{if and only if} \quad [\varphi]_T + [\chi]_T \leq [\psi]_T + [\chi]_T.$$

Suppose $[\varphi]_T \leq [\psi]_T$, then $T \vdash \varphi \to \psi$. But by Lemma 4.2.18 we have $\vdash (\varphi \to \psi) \to ((\varphi \& \chi) \to (\psi \& \chi))$, therefore $T \vdash (\varphi \& \chi) \to (\psi \& \chi)$. Hence $[\varphi]_T + [\chi]_T \leq [\psi]_T + [\chi]_T$.

Now suppose $[\varphi]_T \not\leq [\psi]_T$. Then $[\psi]_T < [\varphi]_T$. A symmetric argument shows that $[\psi]_T + [\chi]_T \leq [\varphi]_T + [\chi]_T$.

(c) $[\varphi]_T + [\varphi \to \psi]_T = [\psi]_T$ by definition.

Therefore $\mathbf{L}_T$ is a BL-chain.

2. Obviously, $[\varphi(c)]_T \leq [\exists x \varphi(x)]_T$ for all c. Assume $[\varphi(c)]_T \leq [\gamma]_T$ for all c. We need to show $[\exists x \varphi(x)]_T \leq [\gamma]_T$.

By way of contradiction, suppose $[\exists x \varphi(x)]_T \not\leq [\gamma]_T$. $\therefore$ $T \nvdash (\exists x \varphi(x)) \to \gamma$.

$\therefore$ $T \nvdash \forall x(\varphi(x) \to \gamma)$.

$\therefore$ $T \nvdash \varphi(c) \to \gamma$ (T is Henkin).

$\therefore \quad [\varphi(c)]_T \not\leq [\gamma]_T$ a contradiction. Therefore $[\exists x\varphi]_T = \sup_c[\varphi(c)]_T$. A similar argument shows that $[\forall x\varphi]_T = \inf_c[\varphi(c)]_T$.

□

Lemma 5.4.4. *For each theory T over $\Pi\forall$ and each closed formula α, if $T \nvdash \alpha$ then there is theory $\hat{T}$ such that*

1. $T \subseteq \hat{T}$;
2. $\hat{T}$ *is Henkin and complete;*
3. $\{\exists x\chi(x) \to \chi(c_\chi)\} \in \hat{T}$ *for all χ formula and some constant c_χ;*
4. *Either $\{\forall x\chi(x) \to 0\} \in \hat{T}$ or $\{\chi(d_\chi) \to (\forall x)\chi(x)\} \in \hat{T}$ for all χ formula and some constant d_χ;*
5. $\hat{T} \nvdash \alpha$.

Proof. We first extend the language $\mathcal{L}$ of T to $\mathcal{L}' = \mathcal{L} \cup \{c_i \mid i \in \omega\}$, where c_i's are new constant symbols. In our construction we need to ensure four properties: completeness, the Henkin property, the $\exists$ and $\forall$ properties. Fix a countable enumeration $\{(\varphi, \psi) \mid \varphi, \psi \text{ formulas of } \mathcal{L}'\}$ of pairs of $\mathcal{L}'$-formulas. For $n \in \omega$, at $4n$ steps we will decide on $(\varphi \to \psi)$ and $(\psi \to \varphi)$ (completeness). At $4n+1$ steps we will take care of formulas of the form $\forall x\chi(x)$, at $4n+2$ steps we will process the $\exists$ property and at $4n+3$ we will process the $\forall$ property.

Let $T_0 = T, \alpha_0 = \alpha$. Assume T_n, α_n have been constructed so that $T_0 \subseteq T_n$, $T_n \vdash \alpha \to \alpha_n$, $T_n \nvdash \alpha_n$. We want to construct T_{n+1}, α_{n+1} so that $T_{n+1} \vdash \alpha_n \to \alpha_{n+1}$, $T_{n+1} \nvdash \alpha_{n+1}$ and T_{n+1} satisfies the nth task.

Case 1: The nth task is deciding (φ, ψ). By Lemma 5.3.4 let T_{n+1} be the extension of T_n deciding (φ, ψ), keeping α_n unprovable. Let $\alpha_{n+1} = \alpha_n$.

Case 2: The nth task is deciding $\forall x\chi(x)$. Let c be one of the new constant symbols of $\mathcal{L}'$ not appearing in T_n.

Subcase (a) $T_n \nvdash \alpha_n \vee \chi(c)$, then $T_n \nvdash \chi(c)$, hence $T_n \nvdash \forall x\chi(x)$. In this case, let $T_{n+1} = T_n$ and $\alpha_{n+1} = \alpha_n \vee \chi(c)$.

Subcase (b) $T_n \vdash \alpha_n \vee \chi(c)$.

$\therefore\ T_n \vdash \alpha_n \vee \chi(x)$ (c does not appear in T_n).

$\therefore\ T_n \vdash \forall x(\alpha_n \vee \chi(x))$ (generalization).

$\therefore\ T_n \vdash (\alpha_n \vee \forall x\chi(x))$ (($\forall$3)).

But $(\alpha_n \vee \forall x\chi(x)) \equiv [(\alpha_n \to \forall x\chi(x)) \to \forall x\chi(x)] \wedge [(\forall x\chi(x) \to \alpha) \to \forall x\chi(x)]$.

$\therefore\ T \cup \{\forall x\chi(x) \to \alpha_n\} \vdash \alpha_n$ (Deduction Theorem).

$\therefore\ T \cup \{\alpha_n \to \forall x\chi(x)\} \nvdash \alpha_n$ (Lemma 5.3.4) and $T \cup \{\alpha_n \to \forall x\chi(x)\} \vdash \forall x\chi(x)$.

In this case, we let $T_{n+1} = T \cup \{\alpha_n \to \forall x\chi(x)\}$ and $\alpha_{n+1} = \alpha_n$.

Case 3: The nth task is deciding $\exists$ property. Let $T_{n+1} = T_n \cup \{\exists x\chi(x) \to \chi(c_\chi)\}$ and $\alpha_{n+1} = \alpha_n$. Note $T_{n+1} \nvdash \alpha_{n+1}$ by Lemma 5.3.5.

Case 4: The nth task is deciding between $\{\forall x\chi(x) \to 0\}$ and $\{\chi(d_\chi) \to (\forall x)\chi(x)\}$. Then by corollary 5.3.8 either $T_n \cup \{\forall x\chi(x) \to 0\} \nvdash \alpha$ or $T_n \cup \{\chi(c) \to (\forall x)\chi(x)\} \nvdash \alpha$. In the former case let $T_{n+1} = T_n \cup \{\forall x\chi(x) \to 0\}$ and in the latter case let $T_{n+1} = T_n \cup \{\chi(c) \to (\forall x)\chi(x)\}$.

Let $\hat{T} = \bigcup_n T_n$. We show $\hat{T}$ has the desired properties.

$\hat{T}$ **is complete:** By construction (φ, ψ) was decided at one of the steps. Hence either $\varphi \to \psi \in \hat{T}$ or $\psi \to \varphi \in \hat{T}$.

$\hat{T}$ **is Henkin:** Suppose $\hat{T} \nvdash \forall x\chi(x)$ and suppose $\forall x\chi(x)$ was handled in step n. Then $T_{n+1} \nvdash \forall x\chi(x)$, so we can apply subcase $(2a)$ and $T_{n+1} \nvdash \alpha_n \vee \chi(c)$. So, $\hat{T} \nvdash \chi(c)$.

$\hat{T} \nvdash \alpha$**:** Suppose $\hat{T} \vdash \alpha$. Then $T_n \vdash \alpha$ for some n. But $T_n \vdash \alpha \to \alpha_n$. Therefore $T_n \vdash \alpha_n$, a contradiction.

We have $\{\exists x\chi(x) \to \chi(c_\chi)\} \in \hat{T}$ for all χ formula and some constant c_χ by construction;

Similarly by construction, either $\{\forall x\chi(x) \to 0\} \in \hat{T}$ or $\{\chi(d_\chi) \to (\forall x)\chi(x)\} \in \hat{T}$ for all χ formula and some constant d_χ;

□

Theorem 5.4.5. *For each theory T over $\Pi\forall$ satisfying the conditions $(2)-(4)$ of Lemma 5.4.4 and each closed formula α such that $T \nvdash \alpha$, there is a $\Pi\forall$-chain $\mathbf{L}$ and closed $\mathbf{L}$-model $\mathbb{M}$ of T such that $||\alpha||^{\mathbf{L}}_{\mathbb{M}} < 1_{\mathbf{L}}$.*

Proof. Let $\mathcal{L}$ be the language of T. Let $\mathbf{L} = \mathbf{L}_T$ Which is a BL-chain by Lemma 5.4.3 . Let $\mathbb{M} = \langle M, (r_P)_P, (m_c)_c \rangle$, where M is the set of all constant symbols of $\mathcal{L}$, $m_c = c$ and for each predicate P define $r_P(\vec{c}) = [P(\vec{c})]_T$.

Claim 1: $||\varphi||^{\mathbf{L}}_{\mathbb{M}} = [\varphi]_T$.

Proof of the claim 1: We will prove this by induction on the formulas. If φ is an atomic formula the claim follows from definition.

$$\begin{aligned} ||\varphi \circ \psi||^{\mathbf{L}}_{\mathbb{M}} &= ||p||^{\mathbf{L}}_{\mathbb{M}} \diamond ||\psi||^{\mathbf{L}}_{\mathbb{M}} \\ &= [\varphi]_T \diamond [\psi]_T \\ &= [\varphi \circ \psi]_T \end{aligned}$$

where $\circ = \&, \rightarrow$ and $\diamond = +, \Rightarrow$.

$$\begin{aligned} ||\forall x\varphi(x)||^{\mathbf{L}}_{\mathbb{M}} &= \inf_c ||\varphi(c)||^{\mathbf{L}}_{\mathbb{M}} \\ &= \inf_c [\varphi(c)]_T, \text{ induction} \\ &= [\forall x\varphi(x)]_T, \text{ Lemma 5.4.3.} \end{aligned}$$

$$\begin{aligned} ||\exists x\varphi(x)||^{\mathbf{L}}_{\mathbb{M}} &= \sup_c ||\varphi(c)||^{\mathbf{L}}_{\mathbb{M}} \\ &= \sup_c [\varphi(c)]_T, \text{ induction} \\ &= [\exists x\varphi(x)]_T, \text{ Lemma 5.4.3.} \end{aligned}$$

This completes the proof of the claim 1.

Claim 2: $\mathbb{M}$ is a closed **L**-model. Let $\chi(x)$ be a formula with one free variable we need to show:

1. $\exists c$ such that $||\exists x\chi(x)||^{\mathbf{L}}_{\mathbb{M}} = ||\chi(c)||^{\mathbf{L}}_{\mathbb{M}}$;

2. If $||\forall x\chi(x)||^{\mathbf{L}}_{\mathbb{M}} \neq 0$ then $\exists d$ such that $||\forall x\chi(x)||^{\mathbf{L}}_{\mathbb{M}} = ||\chi(d)||^{\mathbf{L}}_{\mathbb{M}}$.

Proof of claim 1:Since $\{\exists\chi(x) \rightarrow \chi(c_\chi)\} \in T$, we have $||\exists x\chi(x) \rightarrow \chi(c_\chi)||^{\mathbf{L}}_{\mathbb{M}} = 1$. That is $||\exists x\chi(x)||^{\mathbf{L}}_{\mathbb{M}} \leq ||\chi(c_\chi)||^{\mathbf{L}}_{\mathbb{M}}$. On the other hand, $||\chi(c_\chi)||^{\mathbf{L}}_{\mathbb{M}} \leq ||\exists x\chi(x)||^{\mathbf{L}}_{\mathbb{M}}$ hence we have $||\chi(c_\chi)||^{\mathbf{L}}_{\mathbb{M}} = ||\exists x\chi(x)||^{\mathbf{L}}_{\mathbb{M}}$.
Either $\{\forall x\chi(x) \rightarrow 0\} \in T$ or $\{\chi(d_\chi) \rightarrow (\forall x)\chi(x)\} \in T$. If $||\forall x\chi(x)||^{\mathbf{L}}_{\mathbb{M}} \neq 0$, we have $\{\chi(d_\chi) \rightarrow (\forall x)\chi(x)\} \in T$. We have

$$||\chi(d_\chi) \rightarrow (\forall x)\chi(x)||^{\mathbf{L}}_{\mathbb{M}} = 1.$$

Then

$$||\chi(d_\chi)||^{\mathbf{L}}_{\mathbb{M}} \leq ||(\forall x)\chi(x)||^{\mathbf{L}}_{\mathbb{M}}.$$

But $||(\forall x)\chi(x)||^{\mathbf{L}}_{\mathbb{M}} \leq ||\chi(d_\chi)||^{\mathbf{L}}_{\mathbb{M}}$. Therefore, $||(\forall x)\chi(x)||^{\mathbf{L}}_{\mathbb{M}} = ||\chi(d_\chi)||^{\mathbf{L}}_{\mathbb{M}}$. Hence $\mathbb{M}$ is a closed **L**-model and the proof of claim 2 is complete.

We have shown that $\mathbb{M}$ is a closed model of T, i.e., for every axiom φ of T we have $||\varphi||^{\mathbf{L}}_{\mathbb{M}} = [\varphi]_T = 1_{\mathbf{L}}$. However, $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = [\alpha]_T \neq [1]_T = 1_{\mathbf{L}}$. This completes the proof of the theorem. □

Notice in the above theorem by construction M and $\mathbf{L}$ are countable since we are working with countable languages. We are now ready to tackle Completeness Theorem.

Theorem 5.4.6 (Completeness Theorem for $\Pi\forall$ (A)). *Let T be a theory over $\Pi\forall$. Let φ be a formula of the language of T. Then $T \vdash \varphi$ if and only if $||\varphi||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$ for every countable $\mathbf{L}$ a $\Pi\forall$-chain and every countable closed $\mathbf{L}$-model $\mathbb{M}$.*

Proof. This follows immediately from Theorem 5.4.5 and Lemma 5.4.4. □

Theorem 5.4.7 (Completeness Theorem for $\Pi\forall$ (B)). *Let T be a theory over $\Pi\forall$. Let φ be a formula of the language of T. Then $T \vdash \varphi$ if and only if $||\varphi||^{\mathbf{L}}_{\mathbb{M}} = 1_{L(G)}$ for every countable ordered Abelian group, G, and every countable closed $L(G)$-model $\mathbb{M}$ of T.*

Proof. The proof follows from Theorem 5.4.6 and the fact that if $\mathbf{L}$ is a $\Pi\forall$-chain then $\mathbf{L} = L(G)$ for some ordered Abelian group. □

5.5 Transfer Results to $\mathbb{R}^{\mathbb{Q}}$ or $\mathbb{R}^{1+\mathbb{Q}}$

In this section we will show $\Pi\forall \vdash \alpha$ if and only if for all $\mathbb{M}$ closed $\mathbf{L}$-structures $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1$, where $\mathbf{L} = L(\mathbb{R}^{\mathbb{Q}})$ or $L(\mathbb{R}^{1+\mathbb{Q}})$. This will be done in several steps. We will first show that if α is not an $L(G)$-tautology then α is not an $L(\mathbb{R}^S)$-tautology, where (F, S) is a Hahn representation of G. This will be done using the fact that F is cointiality preserving. We will then show that α is either not an $L(\mathbb{R}^{1+\mathbb{Q}})$-tautology or not an $L(\mathbb{R}^{\mathbb{Q}})$-tautology. This will be achieved using the fact that for any ordered set S there is a coinitiality preserving embedding of $\mathbb{R}^S$ into either $\mathbb{R}^{\mathbb{Q}}$ or $\mathbb{R}^{1+\mathbb{Q}}$.

Definition 5.5.1. Let G and H be two ordered Abelian groups and let $f : G \to H$ be a coinitiality preserving embedding (hence $f : N(G) \hookrightarrow N(H)$) . Let $\tilde{f} : L(G) \to L(H)$ be defined by $\tilde{f}(-\infty) = -\infty$ $\tilde{f}\restriction_{N(G)}= f$ and $\tilde{f}(1) = 1$.

Theorem 5.5.2. *Let G and H be two ordered Abelian groups and let $f : G \to H$ be a coinitiality preserving embedding. Let $\mathbb{M} = \langle M, (r_P)_P, (m_c)_c\rangle$ be a closed $L(G)$-model. Let $\tilde{f}$ be from Definition 5.5.1 and let $\hat{M} = \langle M, (\hat{r}_P)_P, (\hat{m}_c)_c\rangle$, where $\hat{r}_P(\vec{m}) = \tilde{f}(r_P(\vec{m}))$ and $\hat{m}_c = m_c$. Then*

$$||\varphi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} = \tilde{f}(||\varphi||^{\mathbf{L}}_{\mathbb{M},\nu})$$

for all ν evaluation and φ a formula and $\hat{M}$ is a closed $L(H)$-structure.

Proof. . We prove this by induction on the formulas.

$$||x||_{\hat{\mathbb{M}},\nu} = \nu(x) = ||x||_{\mathbb{M},\nu};$$

$$||c||_{\hat{\mathbb{M}},\nu} = \hat{m}_c = m_c = ||c||_{\mathbb{M},\nu};$$

$$||P(\vec{t})||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} = \hat{r}_P(||\vec{t}||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu}) = \tilde{f}(r_P(||\vec{t}||^{\mathbf{L}}_{\mathbb{M},\nu})) = \tilde{f}(||P(\vec{t})||^{\mathbf{L}}_{\mathbb{M},\nu});$$

For connectives we have

$$\begin{aligned} ||\varphi \circ \psi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} &= ||\varphi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} \diamond ||\psi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu}, \\ &= \tilde{f}(||\varphi||^{\mathbf{L}}_{\mathbb{M},\nu}) \diamond \tilde{f}(||\psi||^{\mathbf{L}}_{\mathbb{M},\nu}) \text{ (induction)} \\ &= \tilde{f}(||\varphi \circ \psi||^{\mathbf{L}}_{\mathbb{M},\nu}) \end{aligned}$$

where $\circ = \&, \to$ and $\diamond = +, \Rightarrow$.

For quantifiers we have

$$\begin{aligned}
\tilde{f}(||\exists x\varphi||^{\mathbf{L}}_{\mathbb{M},\nu}) &= \tilde{f}(||\varphi(c)||^{\mathbf{L}}_{\mathbb{M},\nu}) \text{ for some } c \text{ } (\mathbb{M} \text{ is a closed model}) \\
&= ||\varphi(c)||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} \text{ (induction)} \\
&= \sup_{\nu' \in E(\nu,x)} ||\varphi(x)||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu'}
\end{aligned}$$

The last equality holds since $\tilde{f}(||\varphi(c)||^{\mathbf{L}}_{\mathbb{M},\nu}) \geq \tilde{f}(||\varphi(d)||^{\mathbf{L}}_{\mathbb{M},\nu})$ ($\tilde{f}$ is order preserving) and $\sup_{\nu' \in E(\nu,x)} ||\varphi(x)||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu'} = \sup \tilde{f}(||\varphi(d)||^{\mathbf{L}}_{\mathbb{M},\nu})$. Hence $||\exists x\varphi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} = \tilde{f}(||\exists x\varphi||^{\mathbf{L}}_{\mathbb{M},\nu})$.

If $||\forall x\varphi||^{\mathbf{L}}_{\mathbb{M},\nu} \neq 0$, then

$$\begin{aligned}
\tilde{f}(||\forall x\varphi||^{\mathbf{L}}_{\mathbb{M},\nu}) &= \tilde{f}(||\varphi(c)||^{\mathbf{L}}_{\mathbb{M},\nu}) \text{ for some } c \text{ } (\mathbb{M} \text{ is a closed model}) \\
&= ||\varphi(c)||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} \text{ (induction)} \\
&= \inf_{\nu' \in E(\nu,x)} ||\varphi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu'}
\end{aligned}$$

The last equality holds since $\tilde{f}(||\varphi(c)||^{\mathbf{L}}_{\mathbb{M},\nu}) \leq \tilde{f}(||\varphi(d)||^{\mathbf{L}}_{\mathbb{M},\nu})$ ($\tilde{f}$ is order preserving) and $\inf_{\nu' \in E(\nu,x)} ||\varphi(x)||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu'} = \inf \tilde{f}(||\varphi(d)||^{\mathbf{L}}_{\mathbb{M},\nu})$. Hence $||\forall x\varphi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} = \tilde{f}(||\forall x\varphi||^{\mathbf{L}}_{\mathbb{M},\nu})$.

If $||\forall x\varphi||^{\mathbf{L}}_{\mathbb{M},\nu} = 0$, then since $\tilde{f}$ is coinitiality preserving we have $||\forall x\varphi||^{\hat{\mathbf{L}}}_{\hat{\mathbb{M}},\nu} = 0$.

This complete the proof of the claim and that $\hat{\mathbb{M}}$ is a closed $L(H)$-model. □

Corollary 5.5.3. *Suppose $f : G \to H$ is a coinitiality preserving embedding. Let $\mathbb{M}$ be a closed $L(G)$-model such that $||\varphi||^{L(G)}_{\mathbb{M}} < 1$. Then $||\varphi||^{L(H)}_{\hat{\mathbb{M}}} < 1$.*

Proof. This is a direct corollary of Theorem 5.5.2. □

Corollary 5.5.4. *Let G be a countable ordered Abelian group and let $\mathbb{M}$ be a closed $L(G)$-structure. Suppose $||\varphi||^{L(G)}_{\mathbb{M}} < 1$. Then there exist an ordered set S such that $||\varphi||^{L(\mathbb{R}^S)}_{\hat{\mathbb{M}}} < 1$*

Proof. Let $(\mathbb{R}^S, F)$ be a Hahn representation of G. We have shown that F is coinitiality preserving. Therefore by corollary 5.5.3, φ is not an $L(\mathbb{R}^S)$-tautology.

□

Lemma 5.5.5. *Let S be a countable linear ordering and let $\mathbb{M}$ be a closed $L(\mathbb{R}^S)$-structure. Suppose $||\varphi||_{\mathbb{M}}^{L(\mathbb{R}^S)} < 1$ Then either $||\varphi||_{\hat{\mathbb{M}}}^{L(\mathbb{R}^{\mathbb{Q}})} < 1$ or $||\varphi||_{\hat{\mathbb{M}}}^{L(\mathbb{R}^{1+\mathbb{Q}})} < 1$.*

Proof. By Lemma 3.2.8, we have that for any countable ordered set S, $\mathbb{R}^S$ there exists a coinitiality preserving embedding $f : \mathbb{R}^S \to \mathbb{R}^{\mathbb{Q}}$ or $\mathbb{R}^{1+\mathbb{Q}}$. This gives the desired result. □

So far we have shown that if α is not an $L(G)$-tautology for some countable G, then either α is not an $L(\mathbb{R}^{\mathbb{Q}})$-tautology or α is not an $L(\mathbb{R}^{1+\mathbb{Q}})$-tautology. The next two Completeness Theorems follow directly from our earlier versions of Completeness Theorem and the results in section 5.5.

Theorem 5.5.6 (Completeness Theorem for $\Pi\forall$ (C)). *Let T be a theory over $\Pi\forall$. Let φ be a formula of the language of T. Then $T \vdash \varphi$ if and only if $||\alpha||_{\mathbb{M}}^{\mathbf{L}} = 1_{\mathbf{L}}$ where $\mathbf{L} = L(\mathbb{R}^S)$ for all lexicographic function space $\mathbb{R}^S$ (S countable) and every countable closed $\mathbb{R}^S$-model $\mathbb{M}$ of T.*

Proof. ($\rightarrow$) Suppose $T \vdash \alpha$ then for every $\mathbf{L}$, a $\Pi\forall$-chain and every closed $\mathbf{L}$-model $\mathbb{M}$, $||\varphi||_{\mathbb{M}}^{\mathbf{L}} = 1_{\mathbf{L}}$. We have shown that $\mathbf{L} = L(G)$ for some countable ordered Abelian group. And certainly, $\mathbb{R}^S$ is an ordered Abelian group.
($\leftarrow$) Suppose $T \nvdash \alpha$. Then there is an $L(G)$ for some ordered Abelian group and a closed $L(G)$-model $\mathbb{M}$ such that $||\alpha||_{\mathbb{M}}^{L(G)} \neq 1$. Now by corollary 5.5.4 we may assume that $\mathbf{L} = L(\mathbb{R}^S)$, where $(\mathbb{R}^S, F)$ is a Hahn representation for G. And the proof is complete.

□

Theorem 5.5.7 (Completeness Theorem for $\Pi\forall$ (D)). *Let T be a theory over $\Pi\forall$. Let φ be a formula of the language of T. Then $T \vdash \varphi$ if and only if $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1_{\mathbf{L}}$ where $\mathbf{L} = L(\mathbb{R}^{\mathbb{Q}})$ or $L(\mathbb{R}^{1+\mathbb{Q}})$ and every countable closed $\mathbf{L}$-model $\mathbb{M}$ of T.*

Proof. The result directly follows from Theorem 5.5.6 and Lemma 5.5.5. □

Chapter 6

$L(\mathbb{R}^S)$-Tautologies When S Is Not Initially Scattered

In this chapter will show that if α is not provable from $\Pi\forall$, then α is not an $L(\mathbb{R}^S)$-tautology, where S is not initially scattered. We will go through a very similar Henkenization process as in 5.4. We will conclude that $L(\mathbb{R}^S)$-TAUT is recursively enumerable if S is not initially scattered.

6.1 Main Result

Again for the rest of this section let $\mathcal{L}$ be a countable predicate language. We need to deal with sequences that approach but do not equal 0 in evaluation of sentences.

Lemma 6.1.1. *Let T be a theory over $\Pi\forall$ and α a sentence of $\mathcal{L}$, the language of T. Let $\mathbf{L}$ be a $\Pi\forall$-chain and $\mathbb{M}$ a closed $\mathbf{L}$-model of T such that $||\alpha||^{\mathbf{L}}_{\mathbb{M}} < 1$. Let $\chi(x)$ be an $\mathcal{L}$-formula with one single variable and let $c \in M$. Suppose there exists $b^* \in M$ such that $||\chi(b^*)||^{\mathbf{L}}_{\mathbb{M}} \leq m||\chi(c)||^{\mathbf{L}}_{\mathbb{M}}$ for all $m \in \omega$. Then*

$$T \cup \{\chi(e) \to \chi^m(c) : m \in \omega\} \nvdash \alpha,$$

where e is a new constant symbol.

Proof. Note by assumption we have $T \nvdash \alpha$. By way of contradiction suppose $T \cup \{\chi(e) \to \chi^m(c) : m \in \omega\} \vdash \alpha$. Since proofs have finite length, we may assume there is a single m such that $T \cup \{\chi(e) \to \chi^m(c)\} \vdash \alpha$. Now by Deduction theorem $T \vdash (\chi^k(e) \to \chi^{mk}(c)) \to \alpha$, for some positive k. Since e is a constant not appearing in T we may replace it by x and get $T \vdash (\chi^k(x) \to \chi^{mk}(c)) \to \alpha$. Hence by generalization we get $||(\chi^k(b^*) \to \chi^{mk}(c)) \to \alpha||^{\mathbf{L}}_{\mathbb{M}} = 1$. However, we have $||\chi^k(b^*) \to \chi^{mk}(c)||^{\mathbf{L}}_{\mathbb{M}} = 1$. Therefore, $||\alpha||^{\mathbf{L}}_{\mathbb{M}} = 1$ a contradiction and we have the result. □

Lemma 6.1.2. *Let T be a complete Henkin theory over $\Pi\forall$, where $\mathcal{L}$ is the language of T. Suppose $T \nvdash \alpha$. Then there exists $\mathcal{L}' \supseteq \mathcal{L}$ and $T' \supseteq T$ such that*

1. *T' is a complete Henkin theory over $\mathcal{L}'$;*
2. *$T' \nvdash \alpha$;*
3. *For all $\mathcal{L}$-formulas $\chi(x)$ with one free variable, there exist $c \in \mathcal{L}'$ such that $\{\exists x \chi(x) \to \chi(c)\} \in T'$;*
4. *For all $\mathcal{L}$-formulas $\chi(x)$ with one free variable, either there is $d \in \mathcal{L}'$ such that $\{\chi(d) \to \forall x \chi(x)\} \in T'$ or for all $c \in \mathcal{L}$ there is $d_c \in \mathcal{L}'$ such that $\{\chi(d_c) \to \chi(c)^n \mid n \in \omega\} \in T'$.*

Proof. We first extend the language $\mathcal{L}$ of T to $\mathcal{L}' = \mathcal{L} \cup \{c_i \mid i \in \omega\}$, where c_i's are new constants. The proof of this theorem is almost identical to the proof of Lemma 5.4.4. The only slight difference is in ensuring property 4. This is how we do it. If the nth task is deciding between $\{\chi(d) \to \forall x \chi(x)\}$ and $\{\chi(d_c) \to \chi(c)^n \mid n \in \omega\}$. By corollary 5.3.8 we have either $T_n \cup \{\forall x \chi(x) \to 0\} \nvdash \alpha$ or $T_n \cup \{\chi(c) \to (\forall x)\chi(x)\} \nvdash \alpha$. In the former case let $T_{n+1} = T_n \cup \{\chi(d_c) \to \chi(c)^n \mid n \in \omega\}$ and in the latter case let $T_{n+1} = T_n \cup \{\chi(d) \to (\forall x)\chi(x)\}$.

Let $T' = \bigcup_n T_n$. Verifying T' has the desired properties is identical to the proof of Lemma 5.4.4.

$\square$

Theorem 6.1.3. *Let T be a theory over $\Pi\forall$ in the language $\mathcal{L}$. Suppose $T \nvdash \alpha$, α an $\mathcal{L}$-sentence. Then there is $\mathcal{L}^* \supseteq \mathcal{L}$, $\mathbf{L}$ a $\Pi\forall$-chain, $\mathbb{M}$ a closed $\mathbf{L}$-structure such that*

1. *$\mathbb{M}$ is an $\mathbf{L}$-model of T, i.e., $||\sigma||_{\mathbb{M}} = 1_{\mathbf{L}}$ for all $\sigma \in T$.*

2. *$\mathbb{M}$ is super-closed, i.e.,*

   (a) *For all $\mathcal{L}^*$-formula $\varphi(x)$ with one free variable there is $c \in \mathcal{L}^*$ such that $||\varphi(c)||_{\mathbb{M}} \geq ||\exists x\varphi(x)||_{\mathbb{M}}$ (hence we have equality);*

   (b) *For all $\mathcal{L}^*$-formula $\varphi(x)$ with one free variable such that $||\forall x\varphi(x)||_{\mathbb{M}} \neq 0$, there is $d \in \mathcal{L}^*$ such that $||\varphi(d)||_{\mathbb{M}} \leq ||\forall x\varphi(x)||_{\mathbb{M}}$ (hence we have equality).*

   (c) *For all $\mathcal{L}^*$-formula $\varphi(x)$ with one free variable such that $||\forall x\varphi(x)||_{\mathbb{M}} = 0$, but $||\varphi(c)||_{\mathbb{M}} > 0$ for all $c \in \mathcal{L}^*$, there exists $d_c \in \mathcal{L}^*$ such that $||\varphi(d_c)||_{\mathbb{M}} \leq ||\varphi(c)^n||_{\mathbb{M}}$ for all $n \in \omega$ and $c \in \mathcal{L}^*$.*

Proof. Given $T, \mathcal{L}$ and α let $\mathcal{L}_0 \supseteq \mathcal{L} \cup$ Henkin constants. Let T_0 be a complete Henkin theory in $\mathcal{L}_0$ such that $T_0 \nvdash \alpha$. Iterate Lemma 6.1.2 ω times. At step n let $T_{n+1} = (T_n)'$. $\alpha_{n+1} = \alpha$. Let $\mathcal{L}^* = \cup\{\mathcal{L}_n : n \in \omega\}$ and $T^* \cup \{T_n : n \in \omega\}$.

Claim: T^* is a complete Henkin theory in $\mathcal{L}^*$ such that

1. $T^* \nvdash \alpha$;

2. For all $\mathcal{L}$-formulas $\varphi(x)$ with one free variable, there exist $c \in \mathcal{L}'$ such that $\{\exists x\varphi(x) \to \varphi(c)\} \in T'$;

3. For all $\mathcal{L}$-formulas $\varphi(x)$ with one free variable, either there is $d \in \mathcal{L}'$ such that $\{\varphi(d) \to \forall x\varphi(x)\} \in T'$ or for all $c \in \mathcal{L}$ there is $d_c \in \mathcal{L}'$ such that $\{\varphi(d_c) \to \varphi(c)^n \mid n \in \omega\} \in T'$.

Proof of the claim: The completeness and Henkin properties of T^* is proved exactly as in 6.1.2. By way of contradiction suppose $T^* \vdash \alpha$, then $T_n \vdash \alpha$ for some $n \in \omega$ a contradiction to Lemma 6.1.2. By construction we have the other 3 properties involving quantifiers.

Let $\mathbf{L}$ be the $\Pi\forall$-chain and $\mathbb{M}$ be the closed $\mathbf{L}$-model we produced in theorem 5.4.5. We claim that $\mathbb{M}$ has the properties needed by the theorem.

By construction $\mathbb{M}$ is a model of T^*. Again by construction $||\alpha||_{\mathbb{M}} < 1$.

Let $\varphi(x)$ be an $\mathcal{L}^*$ formula with one free variable. Since φ has finitely many symbols we may assume φ is an $\mathcal{L}_n$-formula for some $n \in \omega$. Then by Lemma 6.1.2, there exists $c \in \mathcal{L}_{n+1} \subseteq \mathcal{L}^*$ such that $\{\exists x\varphi(x) \to \varphi(c)\} \in T_{n+1} \subseteq T^*$. Hence $||\exists x\varphi(x) \to \varphi(c)||_{\mathbb{M}} = 1$, that is $||\varphi(c)||_{\mathbb{M}} \geq ||\exists x\varphi(x)||_{\mathbb{M}}$.

Let $\varphi(x)$ be an $\mathcal{L}^*$ formula with one free variable. Again we may assume φ is an $\mathcal{L}_n$-formula for some $n \in \omega$. Then by Lemma 6.1.2, either there is $d \in \mathcal{L}_{n+1} \subseteq \mathcal{L}^*$ such that $\{\varphi(d) \to \forall x\varphi(x)\} \in T_{n+1} \subseteq T^*$ or for all $c \in \mathcal{L}_n$ there is $d_c \in \mathcal{L}_{n+1}$ such that $\{\varphi(d_c) \to \varphi(c)^n \mid n \in \omega\} \in T_{n+1} \subseteq T^*$. In the former case $||\varphi(d) \to \forall x\varphi(x)||_{\mathbb{M}} = 1$, hence $||\varphi(d)||_{\mathbb{M}} \leq ||\forall x\varphi(x)||_{\mathbb{M}}$. In the latter case $||\varphi(d_c) \to \varphi(c)^n||_{\mathbb{M}} = 1$ for all $n \in \omega$, hence $||\varphi(d_c)||_{\mathbb{M}} \leq ||\varphi(c)^n||_{\mathbb{M}} = n||\varphi(c)||_{\mathbb{M}}$ for all $n \in \omega$.

And the proof is complete. □

Definition 6.1.4. Let $\mathcal{L}$ be a predicate language. Let $\mathbf{L}$ be a $\Pi\forall$-chain and let $\mathbb{M}$ be a closed $\mathbf{L}$-structure over $\mathcal{L}$. We say *there is a threat present at* $\mathbb{M}$ if and only if for some $\mathcal{L}$-formula with one free variable, $\varphi(x)$, $||\forall x \varphi(x)||^{\mathbf{L}}_{\mathbb{M}} = 0$, but $||\varphi(c)||^{\mathbf{L}}_{\mathbb{M}} \neq 0$ for all $c \in \mathcal{L}$.

Now we would like to show if $||\alpha||^{L(\mathbb{R}^S)}_{\mathbb{M}} < 1$ then $||\alpha||^{L(\mathbb{R}^{\mathbb{Q}})}_{\mathbb{M}} < 1$. We will use Theorem 5.5.2 which requires a coinitiality preserving embedding. However, we note that if there are no threats present at $\mathbb{M}$, it suffices to have an order preserving embedding $f : S \to \mathbb{Q}$.

Lemma 6.1.5. *Let T be a theory over $\Pi\forall$. Suppose $T \nvdash \alpha$ for some sentence α. Then α is not an $L(\mathbb{R}^{\mathbb{Q}})$-tautology.*

Proof. There exist S an ordered set and $\mathbb{M}$ an $L(\mathbb{R}^S)$-structure such that

$$||\alpha||^{L(\mathbb{R}^S)}_{\mathbb{M}} < 1.$$

There are two cases. Either there is a threat present at $\mathbb{M}$ or not.

Case 1: Suppose there is no threat present at $\mathbb{M}$. Let $h : S \to \mathbb{Q}$ be any order preserving map. Let $f : \mathbb{R}^S \to \mathbb{R}^{\mathbb{Q}}$ be the canonical map induced by h. Since there are no threats present at $\mathbb{M}$ as in the proof of theorem 5.5.2 we get $||\alpha||^{L(\mathbb{R}^{\mathbb{Q}})}_{\mathbb{M}} = f(||\alpha||^{L(\mathbb{R}^S)}_{\mathbb{M}})$.

Case 2: Suppose there is a threat present at $\mathbb{M}$. Hence there exists a φ such that $||\forall x \varphi(x)||^{L(\mathbb{R}^S)}_{\mathbb{M}} = 0$, but $||\varphi(c)||^{L(\mathbb{R}^S)}_{\mathbb{M}} \neq 0$ for all $c \in \mathcal{L}$. Therefore by part $2(c)$ of 6.1.3, we have that $L(\mathbb{R}^S)$ does not contain a smallest non-zero Archimedean class. Therefore S does not contain a smallest element. Let $h : S \to \mathbb{Q}$ be an order and cointiality preserving map given by Lemma 2.2.4. Let $f : \mathbb{R}^S \to \mathbb{R}^{\mathbb{Q}}$ be the canonical map induced by h which is coinitiality preserving. Then $||\alpha||^{L(\mathbb{R}^{\mathbb{Q}})}_{\mathbb{M}} = f(||\alpha||^{L(\mathbb{R}^{\mathbb{Q}})}_{\mathbb{M}}) < 1$.

This completes the proof of the theorem.

□

Theorem 6.1.6 (Completeness theorem for $\Pi\forall$ (E)). *Let T be a theory over $\Pi\forall$. Let φ be a formula of the language of T. Then $T \vdash \varphi$ if and only if $||\alpha||_{\mathbb{M}}^{L(\mathbb{R}^{\mathbb{Q}})} = 1_{L(\mathbb{R}^{\mathbb{Q}})}$, for every countable closed $L(\mathbb{R}^{\mathbb{Q}})$-structure $\mathbb{M}$ of T.*

Proof. This is a direct result of Lemma 6.1.5 and our other completeness theorems. □

Theorem 6.1.7. *Let T be a theory over $\Pi\forall$. Let φ be a formula of the language of T. Let S be a countable, not initially scattered linear ordering. Then $T \vdash \varphi$ if and only if $||\alpha||_{\mathbb{M}}^{L(\mathbb{R}^{S})} = 1_{L(\mathbb{R}^{S})}$, for every countable closed $L(\mathbb{R}^{S})$-structure $\mathbb{M}$ of T.*

Proof. $\rightarrow$ Theorem 5.5.6.

$\leftarrow$ Suppose $T \nvdash \varphi$, then by Theorem 6.1.6 $||\alpha||_{\mathbb{M}}^{L(\mathbb{R}^{\mathbb{Q}})} < 1_{L(\mathbb{R}^{\mathbb{Q}})}$, for every countable closed $L(\mathbb{R}^{\mathbb{Q}})$-structure $\mathbb{M}$. By Lemma 2.2.8 we have a coinitiality preserving map $h : \mathbb{Q} \rightarrow S$. Therefore by Theorem 5.5.2, $||\alpha||_{\hat{\mathbb{M}}}^{L(\mathbb{R}^{S})} < 1_{L(\mathbb{R}^{S})}$, for every countable closed $L(\mathbb{R}^{\mathbb{Q}})$-structure $\hat{\mathbb{M}}$.

□

Corollary 6.1.8. *Let S be a countable not initially scattered linear ordering. Then $L(\mathbb{R}^{S})$-TAUT is recursively enumerable. In particular $L(\mathbb{R}^{S})$-TAUT=$L(\mathbb{R}^{\mathbb{Q}})$-TAUT.*

Proof. $\sigma \in L(\mathbb{R}^{S})$-TAUT if and only if $\Pi\forall \vdash \sigma$ if and only if $\sigma \in L(\mathbb{R}^{\mathbb{Q}})$-TAUT. Hence, $L(\mathbb{R}^{S})$-TAUT is recursively enumerable. □

Chapter 7

$L(\mathbb{R}^S)$-Tautologies When S Is Initially Scattered

In [Mon01], Montagna proves that the set of $L(\mathbb{R}^1)$-tautologies of predicate Product Logic is not arithmetical. We will extend his proof to show that the set of $L(\mathbb{R}^T)$-tautologies is not arithmetical when T is initially scattered.

7.1 Preliminaries

Let $\mathcal{L}_0$ be a predicate language containing $\{Z, S, E, L, A, P\}$, and let $\mathcal{L} = \mathcal{L}_0 \cup \{U\}$ where

- Z and U are unary predicates, $Z(x)$ to be read as $x = 0$ and $U(x)$ has no arithmetical interpretation;
- S, E and L are binary relations. $S(x, y)$ to be read $x + 1 = y$, $E(x, y)$ to be read $x = y$ and $L(x, y)$ to be read $x < y$;
- A and P are ternary relation symbols, $A(x, y, z)$ to be read $x + y = z$ and $P(x, y, z)$ to be read $x \cdot y = z$.

Definition 7.1.1. For every $\mathcal{L}_0$-formula $\varphi(x)$, we write $\varphi(0)$ for

$$\forall x(x = 0 \to \varphi(x)),$$

and for k a positive integer , we write $\varphi(x+k)$ for

$$\forall x_1 \dots \forall x_k((\bigwedge_{i=1}^{k-1}(x_i + 1 = x_{i+1}) \wedge (x_1 = x+1)) \to \varphi(x_k)).$$

Definition 7.1.2. Let **P** denote a finitely axiomatizable system of arithmetic (in the language $\mathcal{L}_0$), whose axioms are:

- The axioms of Robinson's **Q**, including the axioms of equality.
  - $\forall x(\neg S(x) = 0)$;
  - $\forall x \forall y(S(x) = S(y) \to x = y)$;
  - $\forall x(x + 0 = x)$;
  - $\forall x \forall y(x + S(y) = S(x+y))$;
  - $\forall x(x \cdot 0 = 0)$;
  - $\forall x \forall y(x \cdot S(y) = x \cdot y + x)$;
  - $\forall x(\neg x < 0)$;
  - $\forall x \forall y(x < S(y) \leftrightarrow x < y \vee x = y)$;
  - $\forall x \forall y(x < y \vee x = y \vee y < x)$;
- Axioms representing that $<$ is a strict linear order, compatible with $+$ and with $\cdot$.
- The axiom $\forall x(x < x+1 \wedge \neg \exists u(x < u \wedge u < x+1))$.

Let Θ be the conjunction of the axioms of **P**.

Definition 7.1.3. For every $\mathcal{L}$-formula φ we let φ° denote the formula obtained from φ by replacing every atomic formula α of the language $\mathcal{L}_0$ by $\neg\neg\alpha$ and leaving other atomic formulas unchanged.

The above definition basically says in a formula φ of the language $\mathcal{L}$, replace Z, S, E, L, A and P by $\neg\neg Z, \neg\neg S, \neg\neg E, \neg\neg L, \neg\neg A$ and $\neg\neg P$.

Example 7.1.4. 1. $(U(x) \to Z(x))^\circ \equiv U(x) \to \neg\neg Z(x)$.

2. $[\forall x(\exists y S(x,y) \to U(x))]^\circ \equiv \forall x(\exists y \neg\neg S(x,y) \to U(x))$.

In the next section we will utilize some properties of non-standard models of Robinson's **Q**. We will now go over some properties that we will need. For the rest of this chapter let $\mathfrak{N}^*$ be a non-standard model of **Q**. A non-standard model of **Q**, $\mathfrak{N}^*$, contains a copy of the natural numbers and an element which is larger than all natural numbers and models axioms of **Q**. Note, $\mathfrak{N}^*$ is an ordered Abelian semigroup. For $0 < a < b$ we write $a \ll b$ if and only if $na < b$ for all $n \in \omega$ and we write $a \sim b$ if there is a positive integer n such that $b < na$.

Lemma 7.1.5. *$a \sim a + n$ for all $0 < a \in \mathfrak{N}^*$ and $n \in \omega$.*

Proof. By the structure of $\mathfrak{N}^*$, $\exists m \in \omega$ such that $a + n \leq ma$. □

Lemma 7.1.6. *For $a, b \in \mathfrak{N}^*$, if $a \ll b$ then $a + n \ll b$ for all $n \in \omega$.*

Proof. Follows immediately from Lemma 7.1.5. □

Lemma 7.1.7. *For $a, b \in \mathfrak{N}^*$, if $a \ll b$ then $a \ll \frac{a+b}{2} \ll b$.*

Proof. By way of contradiction suppose there is n such that $\frac{a+b}{2} \leq na$. Then $b \leq (2n-1)a$, which is a contradiction to $a < b$. Hence $a \ll \frac{a+b}{2}$. Similar argument shows that $\frac{a+b}{2} \ll b$. □

Lemma 7.1.8. *Let $b \in \mathfrak{N}^*$ be a non-standard element and let m be positive integer. Then $\frac{nb^m + b^{m-1}}{n+1} + 1 \ll b^m$ for all $n \in \omega$.*

Proof. $\frac{nb^m+b^{m-1}}{n+1}+1 \ll b^m$ if and only if $nb^m+b^{m-1}+n+1 \ll (n+1)b^m$ if and only if $b^{m-1}+n+1 \ll b^m$ which is always true by Lemma 7.1.6. $\square$

Lemma 7.1.9. *Let $b \in \mathfrak{N}^*$ be a non-standard element and m a positive integer. Consider the set*

$$B_m = \{b^{m-1}, b^m\} \cup \{\frac{(2^k-1)b^m + b^{m-1}}{2^k} + 1 \mid k \ \ \text{a positive integer}\}.$$

Then

$$b^{m-1} \ll \frac{b^m + b^{m-1}}{2} + 1 \ll \frac{3b^m + b^{m-1}}{4} + 1 \ll$$
$$\ldots \ll \frac{(2^k-1)b^m + b^{m-1}}{2^k} + 1 \ll \ldots \ll b^m + 1.$$

Furthermore, B_m is dense.

Proof. We observe that $\frac{\frac{(2^k-1)b^m+b^{m-1}}{2^k}+1+b^m+1}{2} = \frac{(2^{k+1}-1)b^m+b^{m-1}}{2^{k+1}} + 1$. I.e., each middle term is the average of the previous term and $b^m + 1$. Therefore B_m is dense and the other result follows from Lemma 7.1.7. $\square$

7.2 Montagna's Technique

The basic idea for this section comes from [Mon01], where he focuses on $L(\mathbb{R}^1)$. We were able to extend his result to $L(\mathbb{R}^T)$'s (for initially scattered sets T) which are $\Pi\forall$-chains. Our technique to prove theorem 7.3.5 is new. In his paper, he basically uses the fact that any coinitial sequence in $[0,1]$ converges to 0. For the rest of this section suppose T is a linear ordering and $\mathbb{M}$ is a closed $L(\mathbb{R}^T)$-structure. For ease of notation let $||\cdot||_{\mathbb{M}} = ||\cdot||_{\mathbb{M}}^{L(\mathbb{R}^T)}$.

Lemma 7.2.1. *1. Let $x \in L(\mathbb{R}^T)$ such that $x > 0$. Then $\neg\neg x = 1$ and $\neg\neg 0 = 0$.*

2. *Let* $x, y \in L(\mathbb{R}^T)$ *such that* $x > y$, *then* $x \Rightarrow (y + z) \leq z$ *for all* $z \in L(\mathbb{R}^T)$.

Proof. 1. $\neg\neg x = 0 \Rightarrow (0 \Rightarrow x) = 1$ if $x \neq 0$ and $\neg\neg 0 = 0$.

2. $x > y > y + z$ for all z. Therefore, $x \Rightarrow (y + z) = y + z - x = z + y - x$. $y - x < 0$ implies $z + y - x < z$ and $x \Rightarrow (y + z) \leq z$

□

Lemma 7.2.2. *Let* ν *be an* $\mathbb{M}$*-evaluation. Then for every formula* φ *of the language of* **P**

$$||\varphi^\circ||_{\mathbb{M},\nu} = 0 \textit{ or } 1.$$

Furthermore, $||\varphi^\circ||_{\mathbb{M},\nu} = 1$ *if and only if* $||\varphi||_{\mathbb{M},\nu} > 0$.

Proof. By Lemma 7.2.1, $\neg\neg x$ is either 0 or 1. Now since U does not appear in φ, φ° involves atomic formulas of the form $\neg\neg\alpha$, whose evaluations in $\mathbb{M}$ is either 0 or 1. So we have

$$||\varphi^\circ||_{\mathbb{M},\nu} = 0 \text{ if and only if } ||\varphi||_{\mathbb{M},\nu} = 0$$

and

$$||\varphi^\circ||_{\mathbb{M},\nu} = 1 \text{ if and only if } ||\varphi||_{\mathbb{M},\nu} > 0.$$

□

Lemma 7.2.3. *Suppose* $||\Theta^\circ||_{\mathbb{M}} = 1$ *and* $||E(a,b)^\circ||_{\mathbb{M}} = 1$ *then* $||L(a,b)^\circ||_{\mathbb{M}} = 0$ *and* $||L(b,a)^\circ||_{\mathbb{M}} = 0$.

Proof. Since $||\Theta^\circ||_{\mathbb{M}} = 1$, we have $||E(a,b) \to \neg L(a,b)||_{\mathbb{M}} > 0$. By way of contradiction suppose $||L(a,b)||_{\mathbb{M}} > 0$. Then $||\neg L(a,b)||_{\mathbb{M}} = 0$ and $||E(a,b) \to 0||_{\mathbb{M}} > 0$ implies $||E(a,b)||_{\mathbb{M}} = 0$ which is a contradiction. Therefore, $||L(a,b)||_{\mathbb{M}} = 0$. Switch the place of a and b and we have $||L(b,a)||_{\mathbb{M}} = 0$. □

Lemma 7.2.3, intuitively implies that if a and $b \in M$ have a positive chance of being equal, they have zero chance of not being equal and hence we can not define a linear ordering just yet. To remedy this problem, we define a new relation on M.

Definition 7.2.4. Suppose $||\Theta^\circ||_{\mathbb{M}} = 1$. For a and $b \in M$ let

$$\begin{aligned} a \sim b \quad &\text{if and only if} \quad ||E(a,b)^\circ||_{\mathbb{M}} = 1 \\ &\text{if and only if} \quad ||E(a,b)||_{\mathbb{M}} > 0 \end{aligned}$$

Lemma 7.2.5. *1. $\sim$ is an equivalence relation of M.*

2. The $\sim$-equivalence classes are convex.

3. $\mathbb{M}^\circ := M/\sim, <$ is a linear ordering, where

$$[a] < [b] \quad \textit{if and only if} \quad ||L(a,b)||_{\mathbb{M}} > 0.$$

Proof. 1. $a \sim a$ if and only if $||E(a,a)||_{\mathbb{M}} \neq 0$, which is true since $||\Theta^\circ||_{\mathbb{M}} = 1$. Suppose $a \sim b$, we need to show $b \sim a$. Again since $||\Theta^\circ||_{\mathbb{M}} = 1$, we have that $||E(a,b) \Rightarrow E(b,a)||_{\mathbb{M}} \neq 0$. By way of contradiction suppose $||E(b,a)||_{\mathbb{M}} = 0$ then since $||E(a,b)||_{\mathbb{M}} \neq 0$ we would get

$$||E(a,b) \Rightarrow E(b,a)||_{\mathbb{M}} = 0,$$

which is a contradiction.

Suppose $a \sim b$ and $b \sim c$, we need to show $a \sim c$. We have $||E(a,b)||_{\mathbb{M}} \neq 0$, $||E(b,c)||_{\mathbb{M}} \neq 0$ and $||E(a,b)\&E(b,c) \rightarrow E(a,c)||_{\mathbb{M}} \neq 0$. By way of contradiction suppose $||E(a,c)||_{\mathbb{M}} = 0$. This would imply $||E(a,b)\&E(b,c)||_{\mathbb{M}} = 0$ which in turn implies either $||E(a,b)||_{\mathbb{M}} = 0$ or $||E(b,c)||_{\mathbb{M}} = 0$,a contradiction. Hence $a \sim c$. This completes the proof of part 1.

2. Let $a \sim b$ and suppose $||L(a,c)||_{\mathbb{M}} \neq 0$ and $||L(c,b)||_{\mathbb{M}} \neq 0$, we need to show $a \sim c$ (Intuitively we have $a \sim b$ and $a < c < b$. We want to show $a \sim c$). Since $||\Theta^{\circ}||_{\mathbb{M}} = 1$, we have $||E(a,b)\&L(a,c)\&L(c,b) \rightarrow E(a,c)||_{\mathbb{M}} \neq 0$. Therefore, we have $||E(a,c)||_{\mathbb{M}} \neq 0$ and $a \sim c$.

3. First we show $<$ is well defined. Suppose $a \sim a', b \sim b'$ and $||L(a,b)||_{\mathbb{M}} > 0$. We have $||E(a,a')||_{\mathbb{M}} > 0$ and $||E(b,b')||_{\mathbb{M}} > 0$. Since $||\Theta^{\circ}||_{\mathbb{M}} = 1$, we get $||L(a',b')||_{\mathbb{M}} > 0$. (Read $a = a', b = b', a < b$ then $a' < b'$.)
   We also need to show if $||L(a,b)||_{\mathbb{M}} > 0$ then $a \nsim b$. Suppose $a \sim b$ then by Lemma 7.2.3, $||L(a,b)||_{\mathbb{M}} = 0$. Showing $<$ is a linear ordering of $M/\sim$ follows immediately from the definition.

□

Lemma 7.2.6. *Let* $\mathbb{M}$ *and* $\mathbb{M}^{\circ}$ *be as in Lemma* 7.2.5*. For every* $a \in M$ *let* $[a]$ *denote the equivalence class of* a *modulo* $\sim$*. For every* ν *an* $\mathbb{M}$*-evaluation, let* $[\nu]$ *be an* $\mathbb{M}^{\circ}$*-evaluation defined by* $[\nu](x) = [e(x)]$ *for all variable* x*.*

1. *Then for every formula* φ *of* $\mathbf{P}$ *and for every* $\mathbb{M}$*-evaluation* e*, we have*

$$\mathbb{M}^{\circ}, [\nu] \models \varphi \text{ if and only if } ||\varphi^{\circ}||_{\mathbb{M},\nu} = 1$$

2. *If* $\mathbb{M}^{\circ}$ *is isomorphic to the standard model* $\mathfrak{N}$ *of natural numbers, then for every sentence* φ *in the language of* $\mathbf{P}$ *we have*

$$\mathfrak{N} \models \varphi \text{ if and only if } ||\varphi^{\circ}||_{\mathbb{M}} = 1.$$

Proof. Immediate from Lemma 7.2.5. □

7.3 Main Result

Definition 7.3.1. We introduce the following formulas:

- $\Psi_0 \equiv \neg\forall x U(x)$.
- $\Psi_1 \equiv \forall x\forall y\forall z(((U(x)\&U(z)) \to (U(y)\&U(z))) \to (U(x) \to U(y)))$.
- $\Psi_2 \equiv \forall x\neg\neg U(x)$.
- $\Psi_3 \equiv (\forall x(U(x+1) \to (\forall z((z \leq x) \to U(z)))^3))^\circ$.
- $\Psi \equiv \Theta^\circ\&\Psi_0\&\Psi_1\&\Psi_2\&\Psi_3$.

Intuitively, Ψ_0 says the infimum of $U(a_i)$ is 0, Ψ_1 is one of the axioms of the Product logic. Ψ_2 says no $U(a_i)$ is 0 and Ψ_3 controls the speed at which $U(a_i)$ decreases to 0.

For the rest of this section we will suppose $||\Psi||_{\mathbb{M}} > 0$. Therefore, we have $\mathbb{M}^\circ$ is a model of **P**. So it makes sense to talk about $\leq^{\mathbb{M}^\circ}, \ll^{\mathbb{M}^\circ}$.

Proposition 7.3.2. *Let $\mathbb{M}$ be a closed $L(\mathbb{R}^T)$-structure such that $||\Psi||_{\mathbb{M}} > 0$. Let $\mathbb{M}^\circ$ be defined as before. Then there exists $b \in M$ such that for all $c \in M$, if $\mathbb{M}^\circ \models [b] \leq [c]$, then*

$$||U(c+1)||_{\mathbb{M}} \leq 2(\inf_{[z]\leq^{\mathbb{M}^\circ}[c]} ||U(z)||_{\mathbb{M}})$$

Proof. First note that since $||\Psi||_{\mathbb{M}} > 0$ we have $||\Theta^\circ||_{\mathbb{M}} = 1$. Hence $\mathbb{M}^\circ \models \mathbf{P}$. Let

$$B = \{a \in M \mid ||U(a+1))||_{\mathbb{M}} > (\inf_{[z]\leq^{\mathbb{M}^\circ}[a]} ||U(z)||_{\mathbb{M}})^2\}.$$

Suppose by way of contradiction that for all $a \in M$, there is $b(a) \in B$ such that $[a] \leq^{\mathbb{M}^\circ} [b(a)]$. Then by Lemma 7.2.1

$$||U(a+1))||_{\mathbb{M}} \Rightarrow 3(\inf_{[z]\leq^{\mathbb{M}^\circ}[b(a)]} ||U(z)||_{\mathbb{M}}) \leq \inf_{[z]\leq^{\mathbb{M}^\circ}[b(a)]} ||U(z)||_{\mathbb{M}}.$$

Since $||\Psi_0||_{\mathbb{M}} = 1$ we have $\inf_{a\in M} ||U(a)||_{\mathbb{M}} = 0$. It follows

$$0 = \inf_{a \in M} ||U(a)||_{\mathbb{M}} = \inf_{a \in M} \inf_{[z] \leq^{\mathbb{M}^\circ} [b(a)]} ||U(z)||_{\mathbb{M}} \geq$$

$$||U(a+1)||_{\mathbb{M}} \Rightarrow 3(\inf_{[z] \leq^{\mathbb{M}^\circ} [b(a)]} ||U(z)||_{\mathbb{M}}) \geq ||\Psi_3||_{\mathbb{M}}.$$

So, $||\Psi_3||_{\mathbb{M}} = 0$. This is a contradiction to $||\Psi||_{\mathbb{M}} > 0$. □

Lemma 7.3.3. *Let $\mathbb{M}$ be a closed $L(\mathbb{R}^T)$-structure such that $||\Psi||_{\mathbb{M}} > 0$. Let $\mathbb{M}^\circ$ be defined as before. Let b be from proposition* 7.3.2 . *Then for all $n \in \omega$*

$$||U(b+n)||_{\mathbb{M}} \leq 2^n ||U(b)||_{\mathbb{M}}.$$

Proof. By proposition 7.3.2 we have $||U(b+1)||_{\mathbb{M}} \leq 2(\inf_{[z] \leq^{\mathbb{M}^\circ} [b]} ||U(z)||_{\mathbb{M}}) \leq 2||U(b)||_{\mathbb{M}}$. Inductively, we get the result

$$||U(b+n)||_{\mathbb{M}} \leq 2^n ||U(b)||_{\mathbb{M}}.$$

□

Lemma 7.3.4. *Let $\mathbb{M}$ be a closed $L(\mathbb{R}^T)$-structure such that $||\Psi||_{\mathbb{M}} > 0$. Let $\mathbb{M}^\circ$ be defined as before. Let b be from proposition* 7.3.2. *Then for all $c \in M$ such that $[b] \ll^{\mathbb{M}^\circ} [c]$, $||U(c)||_{\mathbb{M}} \ll ||U(b)||_{\mathbb{M}}$.*

Proof. First note that $[b+n] \ll [c]$ for all $n \in \omega$ by Lemma 7.1.6 ($[b+n] \leq^{\mathbb{M}^\circ} [c]$). Therefore for all $n \in \omega$ we have

$$||U(c+1)||_{\mathbb{M}} \leq 2(\inf_{[z] \leq^{\mathbb{M}^\circ} [c]} ||U(z)||_{\mathbb{M}}) \leq 2||U(b+n)||_{\mathbb{M}} \leq 2^{n+1} ||U(b)||_{\mathbb{M}}.$$

We conclude that for all $n \in \omega$, $||U(c+1)||_{\mathbb{M}} \leq 2^{n+1} ||U(b)||_{\mathbb{M}}$, hence $||U(b)||_{\mathbb{M}} \gg ||U(c)||_{\mathbb{M}}$. □

Theorem 7.3.5. *Let T be an initially scattered set. Let $\mathbb{M}$ be a closed $L(\mathbb{R}^T)$-structure such that $||\Psi||_{\mathbb{M}} > 0$. Let $\mathbb{M}^\circ$ be defined as before. Then $\mathbb{M}^\circ$ is isomorphic with the standard model of natural numbers.*

Proof. By way of contradiction, suppose that $\mathbb{M}^\circ$ contains a non-standard element $[d]$. Without loss of generality we may assume $[d] = [b]$ from proposition 7.3.2. Since $\inf_{a\in M} ||U(a)||_{\mathbb{M}} = 0$, without loss of generality we may assume $\inf_{n\in\omega} ||U(b^n)||_{\mathbb{M}} = 0$. Consider the set

$$B = \bigcup_{m\in\omega\setminus\{0\}} \{b^{m-1}, b^m\} \cup \{\frac{(2^k-1)b^m + b^{m-1}}{2^k} + 1 \mid k \text{ a positive integer}\}.$$

$B = \cup_{m\in\omega\setminus\{0\}} B_m$, B_m from Lemma 7.1.9. Then by Lemma 7.1.9, for each m, we have

$$b^{m-1} \ll \frac{b^m + b^{m-1}}{2} + 1 \ll \frac{3b^m + b^{m-1}}{4} + 1 \ll$$
$$\ldots \ll \frac{(2^k-1)b^m + b^{m-1}}{2^k} + 1 \ll \ldots \ll b^m + 1$$

and B_m is dense. Lemma 7.3.4 implies for each m, that

$$||U(b^{m-1})||_{\mathbb{M}} \gg ||U(\tfrac{b^m+b^{m-1}}{2} + 1)||_{\mathbb{M}} \gg ||U(\tfrac{3b^m+b^{m-1}}{4} + 1)||_{\mathbb{M}} \gg \ldots$$
$$\gg ||\tfrac{U((2^k-1)b^m+b^{m-1}}{2^k} + 1)||_{\mathbb{M}} \gg \ldots \gg ||U(b^m + 1)||_{\mathbb{M}}.$$

Furthermore,

$$\{||U(b^{m-1})||_{\mathbb{M}}, ||U(b^m + 1)||_{\mathbb{M}}\}$$
$$\cup\{||U(\tfrac{(2^k-1)b^m+b^{m-1}}{2^k} + 1)||_{\mathbb{M}} \mid m, k \text{ a positive integer}\}$$

is a coinitial dense subset of $N(\mathbb{R}^T)$, the negative cone of $\mathbb{R}^T$. Hence

$$\{[||U(b^{m-1})||_{\mathbb{M}}], [||U(b^m + 1)||_{\mathbb{M}}]\}$$
$$\cup\{[||U(\tfrac{(2^k-1)b^m+b^{m-1}}{2^k} + 1)||_{\mathbb{M}}] \mid m, k \text{ a positive integer}\}$$

is a coinitial dense subset of Archimedean classes of $N(\mathbb{R}^T)$ which has the same order type of T. However, T is an initially scattered set, and we have arrived at a contradiction. Henceforth, Then $\mathbb{M}^\circ$ is isomorphic with the standard model of natural numbers.

□

Corollary 7.3.6. *Let T be an initially scattered set.*

1. *If Φ is a sentence of arithmetic which is true in the standard model of natural numbers, then $\Psi \to \Phi^\circ$ is an $L(\mathbb{R}^T)$-tautology.*

2. *If Φ is a sentence of arithmetic which is false in the standard model of natural numbers, then $||\Psi \& \Phi^\circ||_{\mathbb{M}} \neq 1$ for any $\mathbb{M}$ closed $L(\mathbb{R}^T)$-structure.*

Proof. Let $\mathbb{M}$ be a closed $L(\mathbb{R}^T)$-structure.

Case 1: $||\Psi||_{\mathbb{M}} = 0$, then $||\Psi \to \Phi^\circ||_{\mathbb{M}} = 1$ and $||\Psi \& \Phi^\circ||_{\mathbb{M}} = 0$

Case 2 $||\Psi||_{\mathbb{M}} > 0$. Then by Theorem 7.3.5 $\mathbb{M}^\circ$ is isomorphic to the standard model of natural numbers. So, if $\mathbb{M}^\circ \models \Phi^\circ$, then $||\Phi^\circ||_{\mathbb{M}} = 1$ and hence $||\Psi \to \Phi^\circ||_{\mathbb{M}} = 1$. If $\mathbb{M}^\circ \not\models \Phi^\circ$, we have $||\Phi^\circ||_{\mathbb{M}} = 0$ and hence $||\Psi \& \Phi^\circ||_{\mathbb{M}} = 0$.

This completes the proof. □

Notice that the choice of formula Ψ is independent of the set T in corollary 7.3.6. That is we use the the same Ψ for different initially scattered sets. Hence we have the following generalized corollary.

Corollary 7.3.7. *Let $\{T_i \mid i \in I\}$ be a collection of initially scattered sets.*

1. *If Φ is a sentence of arithmetic which is true in the standard model of natural numbers, then $\Psi \to \Phi^\circ$ is an $L(\mathbb{R}_i^T)$-tautology, for all $i \in I$.*

2. *If Φ is a sentence of arithmetic which is false in the standard model of natural numbers, then $||\Psi \& \Phi^\circ||_{\mathbb{M}} \neq 1$ for any $\mathbb{M}$ closed $L(\mathbb{R}_i^T)$-structure, $i \in I$.*

Theorem 7.3.8. *Let T be an initially scattered set. Then there exists a closed $L(\mathbb{R}^T)$-structure $\mathbb{M}$ such that $||\Psi||_{\mathbb{M}} = 1$. Hence:*

1. *If Φ is a sentence of arithmetic which is true in the standard model of natural numbers, then $\Psi \to \Phi^\circ$ is an $L(\mathbb{R}^T)$-tautology.*

2. *If Φ is a sentence of arithmetic which is false in the standard model of natural numbers, then $||\Psi \& \Phi^\circ||_{\mathbb{M}} \neq 1$ for any $\mathbb{M}$ closed $L(\mathbb{R}^T)$-structure.*

Proof. Let ω the set of natural numbers be the universe of $\mathbb{M}$. For every atomic $\mathcal{L}_0$-formula $\varphi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in \omega$ let

$$||\varphi(a_1, \ldots, a_n)||_{\mathbb{M}} = 1 \quad \text{if and only if} \quad \mathfrak{N} \models \varphi(a_1, \ldots, a_n).$$

For $k \in \omega$, If T is well ordered let $||U(k)||_{\mathbb{M}} = (\gamma_i)_{i \in T}$, where

$$\gamma_0 = -3^k \text{ and } \gamma_i = 0, i \neq 0.$$

If T is not well ordered, then $\{t_{-k} \mid k \in \omega\} \subseteq T$, where $t_{-k-1} < t_{-k}$ for all $k \in \omega$. In this case let $||U(k)||_{\mathbb{M}} = (\gamma_i)_{i \in T}$, where

$$\gamma_{t_{-k}} = -3^k \text{ and } \gamma_i = 0 \text{ otherwise.}$$

Claim 1: $||\Theta^\circ||_{\mathbb{M}} = ||\Psi_0||_{\mathbb{M}} = ||\Psi_1||_{\mathbb{M}} = ||\Psi_2||_{\mathbb{M}} = ||\Psi_3||_{\mathbb{M}} = 1$.

Proof of the claim 1:

1. $||\Theta^\circ||_{\mathbb{M}} = 1$ by definition.

2. $\inf_{k \in \omega} ||U(k)||_{\mathbb{M}} = 0$, therefore $||\Psi_0||_{\mathbb{M}} = \neg \inf_{k \in \omega} ||U(k)||_{\mathbb{M}} = 1$.

3. $\Psi_1 = \forall x \forall y \forall z(((U(x) \& U(z)) \to (U(y) \& U(z))) \to (U(x) \to U(y)))$ is an axiom of $\Pi\forall$ hence and $||\Psi_0||_{\mathbb{M}} = 1$

4. $||U(k)||_{\mathbb{M}} > 0$ for all k, hence $\neg\neg||U(k)||_{\mathbb{M}} = 1$. Therefore, $||\Psi_2||_{\mathbb{M}} = \inf_{k\in\omega}(\neg\neg||U(k)||_{\mathbb{M}}) = 1$

5. $||\Psi_3||_{\mathbb{M}} = ||(\forall x(U(x+1) \to (\forall z((z \leq x) \to U(z))^3))^{\circ}||_{\mathbb{M}}$. For $l < k+1$ we have $||U(k+1)||_{\mathbb{M}} = ||U(l)||_{\mathbb{M}}$, for both cases of T. An tedious computation notation-wise shows that $||\Psi_3||_{\mathbb{M}} = 1$.

Claim 2: $\mathbb{M}$ is a closed $L(\mathbb{R}^T)$-structure.

Proof of claim 2: First note $||\exists x U(x)||_{\mathbb{M}} = ||U(0)||_{\mathbb{M}}$. By construction, $\mathbb{M}$ is closed. The rest of the theorem is a direct corollary to 7.3.6. □

Theorem 7.3.9. *Fix $L(\mathbb{R}^T)$, T initially scattered. Then for every sentence Φ of arithmetic $\Psi \to \Phi^{\circ}$ is an $L(\mathbb{R}^T)$-tautology if and only if $\mathfrak{N} \models \Phi$. Hence $L(\mathbb{R}^T)$-TAUT, is not arithmetical.*

Proof. This is an immediate result of Theorem 7.3.8 and corollary 7.3.6. □

Example 7.3.10. $L(\mathbb{R}^{1+\mathbb{Q}})$-TAUT is not arithmetical.

Again we have a generalized result since the choice of Ψ is uniform for all T initially scattered set.

Theorem 7.3.11. *Let $\{T_i \mid i \in I\}$ be a collection of initially scattered sets. Then*

$$\bigcap_{i\in I} L(\mathbb{R}_i^T)\text{-}TAUT$$

are not arithmetical.

On one hand the Completeness Theorem for $\Pi\forall$ implies that $L(\mathbb{R}^{\mathbb{Q}})$-TAUT $\cap L(\mathbb{R}^{1+\mathbb{Q}})$-TAUT is recursively enumerable. On the other hand, in this chapter we have shown that $L(\mathbb{R}^{1+\mathbb{Q}})$-TAUT is not arithmetical. Consider the following example.

Example 7.3.12. Let σ be a sentence of arithmetic so that $\mathfrak{N} \models \sigma$ but $\mathbf{Q} \nvdash \alpha$. Let $\theta = \Psi \& \sigma^\circ$. Then, θ is an $L(\mathbb{R}^{1+\mathbb{Q}})$-tautology. On the other hand, $\Pi\forall \nvdash \theta$. Hence θ is not an $L(\mathbb{R}^{\mathbb{Q}})$-tautology. In fact, we have θ is an $L(\mathbb{R}^{T})$-tautology if and only if T is initially scattered.

Chapter 8

Scattered Subsets of Ordered Abelian Group

Remember in chapter 2 we defined a linear ordering $\mathbf{S}$ to be *scattered* if it does not contain a dense subset. In this chapter we will show that if A and B are two convergent and scattered subsets of an ordered Abelian group then $A+B$ is also convergent and scattered. We will formally define $A+B$ and convergent later in thin chapter.

8.1 Main Result

Definition 2.1.16 tells us what a scattered linear ordering is by telling us what it is not. The following definition gives a more positive description of scattered linear orderings by a providing a prescription of how to construct it from "simpler" scattered linear orderings.

Definition 8.1.1. We define the class $\mathbf{L}$ of *ranked linear orderings* by presenting inductively for each ordinal α a class $\mathbf{L}_\alpha$, and then by setting $\mathbf{L} = \cup \mathbf{L}_\alpha$.

1. $\mathbf{0}, \mathbf{1} \in \mathbf{L}_0$.
2. Given a linear ordering I of type γ, γ^* for some ordinal γ and for each $i \in I$ a linear ordering $L_i \in \cup\{\mathbf{L}_\beta \mid \beta < \alpha\}$, then $\sum\{L_i \mid i \in I\} \in \mathbf{L}_\alpha$.

The *rank* of L, a ranked linear ordering, is the smallest ordinal α such that $L \in \mathbf{L}_\alpha$.

Example 8.1.2. ω, ω^* are ranked linear orderings of rank 1.

Theorem 8.1.3 (Hausdorff). *A linear ordering is in* $\mathbf{L}$ *if and only if it is scattered.*

A proof of Hausdorff Theorem may be found in [Ros82].

Definition 8.1.4. Let G be an ordered Abelian group. A scattered linear ordering $\mathbf{S} \subseteq G$ is *convergent* if and only if every countable increasing or decreasing sequence of elements of $\mathbf{S}$ has a limit point in G.

Lemma 8.1.5. *Let A and B be two convergent scattered linear orderings of G an ordered Abelian group. Then $A \cup B$ is also convergent and scattered.*

Proof. **Scattered:** By way of contradiction suppose $X \subseteq A \cup B$ is dense. Then we claim either $X \cap A$ or $X \cap B$ contains a dense subset. Without loss of generality assume $X \cap A$ contains at least two elements $a_1 < a_2$. It suffices to show there exists $a_3 \in X \cap A$ such that $a_1 < a_3 < a_2$. Let $(a_1, a_2) = \{x \in X \mid a_1 < x < a_2\}$. By way of contradiction suppose $(a_1, a_2) \cap X \cap A$ is empty. Then (a_1, a_2) is a dense subset of B which is a contradiction to B being scattered.

Convergent: Let $\{x_n \mid n \in \omega\}$, without loss of generality be an increasing sequence in $A \cup B$. Then either $\{x_n \mid n \in \omega\} \cap A$ or $\{x_n \mid n \in \omega\} \cap B$ is an infinite increasing sequence. By assumption either one has a limit point in G. This completes the proof.

□

Definition 8.1.6. Let G be an ordered Abelian group. Let A and B be subsets of G. Then $A+B$ denotes

$$\{a+b \mid a \in A,\ b \in B\}.$$

Example 8.1.7. Let $G = \mathbb{R}$ and let $\{q_n, n \in \omega\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Let $A = \{10^n + q_n \mid n \in \omega\}$ and $B = \{-10^n \mid n \in \omega\}$. Then $\{10^n + q_n + (-10^n) \mid n \in \omega\} \subseteq A+B$. That is $\mathbb{Q} \cap [0,1] \subseteq A+B$. Hence $A+B$ is not scattered.

The above gives an example of two scattered sets where their sum is not scattered. The problem turns out to be that A and B are not convergent. In the next few theorems we will show that if A and B are scattered and convergent then $A+B$ will be scattered and convergent.

Definition 8.1.8. Let α be an ordinal. Let $A = \sum\{A_i \mid i \in \alpha\}$ and $B = \sum\{B_j \mid j \in \beta\}$ where either $\beta = \alpha$ or $\beta = \alpha^*$.

Let $\odot_I :=$

$$\begin{cases} A + B_j & \text{convergent scattered for all } j \\ A_i + B & \text{convergent scattered for all } \ i \end{cases} \Rightarrow A{+}B \text{ is convergent and scattered.}$$

We will show that $\odot_\alpha$ holds for all ordinals α. We will show $\odot_\omega$ holds and then prove $\odot_\alpha$, $(\omega < \alpha < \omega_1)$ holds by induction. But we first need a preliminary lemma.

Lemma 8.1.9. *Let α be an ordinal. Let $A = \sum\{A_i \mid i \in \alpha\}$ and $B = \sum\{B_j \mid j \in \beta\}$ where either $\beta = \alpha$ or $\beta = \alpha^*$. Suppose $A_i + B$ and $A + B_j$ are convergent and scattered for all $i \in \alpha$ and $j \in \beta$. Then A and B are convergent and scattered.*

Proof. We will show A is convergent and scattered. The proof for B is symmetric. Let $j \in \beta$ and $b \in B_j$. Then by assumption $A + B_j$ is convergent and scattered, hence $A + b \subseteq A + B_j$ is convergent and scattered. Now $A + b \cong A$, hence A is convergent and scattered. □

Note if $\odot_\alpha$ holds then $\odot_\beta$ holds for all $\beta \leq \alpha$. This is true since we may let $A_j = B_j = \emptyset, \beta < j < \alpha$ ($\emptyset$ is assumed to be scattered and convergent). Assume $\odot_\alpha$ holds and β is an initial segment of α. Suppose $A = \sum\{A_i \mid i \in \alpha\}$ and $B = \sum\{B_j \mid j \in \beta\}$ where either $\beta = \alpha$ or $\beta = \alpha^*$. Also suppose that $A_i + B, i \in \alpha$ and $A + B_j, j \in \rho$ is convergent and scattered. Then $A + B$ is convergent and scattered. This is again done by inserting $\emptyset$ into B.

Lemma 8.1.10. *$\odot_\omega$ holds. Hence $\odot_n, n \in \omega$ holds.*

Proof. There are 2 cases to consider. $\alpha = \omega, \beta = \omega^*$ and $\alpha = \omega, \beta = \omega$.

Case 1: $(\alpha = \omega,\ \beta = \omega^*)$ Let $\sup(A) = a \in G$ and $\inf(B) = b \in G$. Consider the interval $I = [\min(A_0) + b, a + \max(B_0)]$. We have $A + B \subseteq I$ and

$$\begin{aligned} I &= \bigcup_{k \in \omega} [\min(A_k) + b, \min(A_{k+1}) + b) \\ \cup \textstyle\bigcup_{k \in \omega} & (a + \max(B_{k+1}), a + \max(B_k)] \\ \cup & \{a + b\} \end{aligned}$$

First we will show that the intersection of $A + B$ with any of these disjoint intervals is scattered.

Claim 1: Fix k. $(A + B) \cap [\min(A_k) + b, \min(A_{k+1}) + b) \subseteq \cup_{t \leq k}(A_t + B)$.

Proof of Claim 1: If $t \geq k + 1$, then for all $x \in A_t$, $x \geq \min(A_{k+1})$. So for all $y \in B$ since $y > b$, we have $x + y > \min(A_{k+1}) + b$. So if $t \geq k + 1$ then $A_t + B \cap [\min(A_k) + b, \min(A_{k+1}) + b) = \emptyset$. So we have shown that

$$(A + B) \cap [\min(A_k) + b, \min(A_{k+1}) + b) \subseteq \cup_{t \leq k}(A_t + B).$$

$\cup_{t\leq k}(A_t + B)$ is a finite union of scattered sets hence it is scattered.

Claim 2: Fix k. $(A+B)\cap(a+\max(B_{k+1}), a+\max(B_k)] \subseteq \cup_{t\leq k}(A+B_t)$.

Proof of Claim 2: The proof of claim 2 is symmetric to that of claim 1. So we have $A+B$ is scattered. We need to show $A+B$ is convergent. Without loss of generality suppose $\{x_n \mid n \in \omega\}$ is an increasing sequence in $A+B$. Then either x_n converges to $a+b$ or for all but finitely many x_n is in one of the above intervals. Now since the intersection of $A+B$ with any of those intervals is convergent, $\{x_n\}$ converges to a point in $A+B$.

Case 2: $(\alpha = \omega,\ \beta = \omega)$ Let $a = \sup(A)$, $b = \sup(B)$.

Claim Let $d < a+b$, then $(A+B)\cap(-\infty, d) \leq \cup_{i<Q}(A_i + B)$ for some $Q \in \omega$.

Proof of the Claim Let $c = \frac{a+b-d}{2}$ and choose $Q \in \omega$ such that if $i, j \geq Q$ then

$$\min(A_i) > a - c \text{ and } \min(B_j) > b - c.$$

So if $i, j \geq Q$ we have $\min(A_i) + \min(B_j) > a - c + b - c = d$. Therefore, if $A_i + B_j$ intersects $(-\infty, d)$ non-trivially, at least one of the i or j is less than Q. Without loss of generality assume $i < Q$. Then $(A+B)\cap(-\infty,d) \leq \cup_{i<Q}(A_i + B)$ which is convergent and scattered. This implies that $A+B$ is scattered. To show $A+B$ is convergent is exactly the same argument as the previous case.

□

Lemma 8.1.11. *$\odot_\delta$ holds for all $\delta < \omega_1$.*

Proof. Since we have already shown $\odot_\omega$, it suffices to show $\odot_\delta$, $\delta < \omega_1$. There are two cases:

δ is a countable limit ordinal $> \omega$: Then $\delta = \cup_{n\in\omega}\beta_n$, $\omega < \beta_n < \delta$ and β_n is strictly increasing. Suppose $\odot_\beta$ holds for all $\gamma < \delta$. We have $A = \sum\{A_i \mid i \in \delta\}$ and $B = \sum\{B_j \mid j \in \delta \text{ or } \delta^*\}$. We need to show $A + B$ is convergent and scattered. Let $\gamma_0 = \beta_0$ and $\gamma_i = \beta_i \backslash \beta_{i-1}$. Then δ is a disjoint union of $\gamma_i, i \in \omega$. We may rewrite A and B as follows:

$$A = \sum\{C_n \mid n \in \omega\} \text{ where } C_n = \sum\{A_i \mid i \in \gamma_n\}$$

$$B = \sum\{D_n \mid n \in \omega\} \text{ where } D_n = \sum\{B_i \mid i \in \gamma_n\}.$$

Now induction hypothesis applies to C_n and D_m. We have $(C_n)_k + B$ and $A + (D_m)_k$ convergent and scattered for all k by the induction hypothesis. Therefore, showing $A + B$ is convergent and scattered which has been reduced to showing $\odot_\omega$, which we already have. Therefore $A+B$ is convergent and scattered. The case where $A = \sum\{A_i \mid i \in \delta\}$ and $B = \sum\{B_j \mid j \in \delta^*\}$ is identical.

$\delta + 1$ is a successor ordinal: Suppose $\odot_\rho$ holds for all $\rho < \delta + 1$. We have $A = \sum\{A_i \mid i \in \delta+1\}$ and $B = \sum\{B_j \mid j \in \delta+1\}$. We may rewrite A and B as follows

$$A = \sum\{A_i \mid i \in \delta\} + A_\delta$$

$$B = \sum\{B_j \mid j \in \delta\} + B_\delta.$$

Now we have

$$\begin{aligned} A + B &= (\sum\{A_i \mid i \in \delta\} + \sum\{B_j \mid j \in \delta\}) \\ &\cup (\sum\{A_i \mid i \in \delta\} + B_\delta) \\ &\cup (A_\delta + \sum\{B_j \mid j \in \delta\}) \cup (A_\delta + B_\delta). \end{aligned}$$

Each part of the above decomposition is convergent and scattered by the induction hypothesis. $A + B$ being a finite union of convergent scattered sets is convergent and scattered. The case where $A = \sum\{A_i \mid i \in \alpha\}$ and $B = \sum\{B_j \mid j \in \alpha^*\}$ is again identical.

$\square$

Theorem 8.1.12. *Let G be an ordered Abelian group and let A and B be countable convergent scattered subsets of G. Then $A + B$ is countable, convergent and scattered.*

Proof. Let G be an ordered Abelian group. We call $A \subseteq G$ ccs if and only if A is countable, convergent and scattered. Let for α and β countable,

$$\begin{aligned} (*)_{(\alpha,\beta)} &:= [\text{if } A \text{ is ccs of rank} < \alpha \text{ and } B \text{ is ccs of rank} < \beta] \Rightarrow \\ &\quad\ A + B \text{ is ccs;} \\ (\#)_\alpha &:= \forall(\beta < \omega_1)(*)_{(\alpha,\beta)} \\ &= (\forall B \text{ ccs})(A \text{ is ccs of rank} < \alpha \Rightarrow A + B \text{ is ccs}). \end{aligned}$$

To prove the theorem we need to prove $(*)_{(\alpha,\beta)}$ for all countable α, β which is equivalent to showing $(\#)_\alpha$ for all countable α. We will do this by induction:

$\alpha = 0$: There are no scattered sets with rank smaller than 0, so $(\#)_0$ is true trivially.

α a limit ordinal: Assume for $\delta < \alpha$, $(\#)_\delta$ holds. Suppose $\text{rank}(A) < \alpha$, then $\text{rank}(A) < \text{rank}(A) + 1 < \alpha$ and by the inductive assumption $(\#)_{\text{rank}(A)+1}$ holds. That is $(\forall B)(A + B)$ is countable, convergent and scattered.

$\alpha + 1$ a successor countable ordinal: Assume $(\#)_\delta$ holds for all $\delta < \alpha + 1$. We need to show $(\#)_{\alpha+1}$. Choose A of rank less than $\alpha+1$. We may assume rank$(A) = \alpha$. So A has the form $A = \sum\{A_i \mid i \in \delta\}$ where rank$(A_i) < \alpha$ for all i. Note since A is countable, $\delta < \omega_1$. We need to prove $A + B$ is convergent and scattered for all B. We will do this by induction on the rank of B.

Let

$$(+)_\beta := \forall B \ \text{ ccs of rank} < \beta,\ A + B \text{ is ccs}$$

$\beta = 0$: Trivially true.

β a limit ordinal: Assume for $\delta < \beta$ $(+)_\delta$ holds. If rank of B is less than β then rank$(B) <$ rank$(B)+1 < \beta$. Therefore by induction hypothesis $(+)_{\text{rank}(B)+1}$ holds. Hence $A + B$ is convergent and scattered.

$\beta + 1$ successor ordinal: Assume $(+)_\delta$ holds for all $\delta < \beta + 1$. we need to show $(+)_{\beta+1}$. Choose B of rank less than $\beta + 1$. We may assume rank$(B) = \beta$. We need to show $A + B$ is convergent and scattered. Say B has the form $B = \sum\{B_j \mid j \in \gamma\}$ where rank$(B_j) < \beta$ for all j and $\gamma < \omega_1$. Now we have both $\odot_\delta$ and $\odot_\gamma$ holds. Without loss of generality suppose $\delta \leq \gamma$. Then by the observation from before and inserting $\emptyset$ into A we get $A + B$ is scattered.

This completes the proof of $(+)_{\beta+1}$, which in turn completes the proof of $(\#)_{\alpha+1}$. This concludes the proof of the double induction and we have the theorem.

□

We now relax the the countability requirement for A and B in Theorem 8.1.12.

Theorem 8.1.13. *Let G be an ordered Abelian group and let A and B be convergent scattered subsets of G. Then $A + B$ is convergent and scattered.*

Proof. The proof mimics the proof of Theorem 8.1.12. We first show $\odot_\alpha$ holds for all ordinal α. We have previously shown that $\odot_\alpha$ holds if α is countable. Let α be an uncountable ordinal. Suppose $A = \sum\{A_i \mid i \in \alpha\}$ and $B = \sum\{B_i \mid i \in \alpha\}$ satisfy the assumptions of $\odot_\alpha$ but $A + B$ is not a convergent scattered subset of G. Let X be a dense subset of $A + B$. We may assume X is countable. Collect all A_i and B_i represented in X and add all the limit points (and corresponding factors) needed from A and B. Call these sets A' and B' respectively. A' and B' are countable subsets of A and B respectively. Then X is a dense subset of $A' + B'$ which by the countable case is scattered. Hence we have a contradiction. Let $\{x_n \mid n \in \omega\}$ be an increasing sequence in $A + B$. Collect all A_i and B_i represented in $\{x_n \mid n \in \omega\}$ and add all the limit points (and corresponding factors) needed from A and B. Call these sets A' and B' respectively. Since the sequence is countable, A' and B' re countable and $\{x_n \mid n \in \omega\} \subseteq A' + B'$. Therefore the sequence has a limit point in $A' + B' \subseteq A + B$ by the countable case. Therefore $\{x_n \mid n \in \omega\}$ is convergent in $A + B$. This completes the proof of $\odot_\alpha$ for all ordinal α. Now we go through the proof of Theorem 8.1.12 and leave out any reference to countability. This shows gives the desired result. □

BIBLIOGRAPHY

[Cli54] A. H. Clifford, *Note on Hahn's theorem on ordered abelian groups*, Proc. Amer. Math. Soc. **5** (1954), 860–863. MR 16,792b

[Fuc63] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford, 1963. MR 30 #2090

[Háj98] Petr Hájek, *Metamathematics of fuzzy logic*, Trends in Logic—Studia Logica Library, vol. 4, Kluwer Academic Publishers, Dordrecht, 1998. MR 2003c:03048

[HW52] M. Hausner and J. G. Wendel, *Ordered vector spaces*, Proc. Amer. Math. Soc. **3** (1952), 977–982. MR 14,566b

[LS02] Michael C. Laskowski and Yvonne V. Shashoua, *A classification of BL-algebras*, Fuzzy Sets and Systems **131** (2002), no. 3, 271–282. MR 2003j:03084

[LSar] ———, *A generalized ordinal sums and th decidability of BL-chains*, Algebra Universalis (To Appear).

[Mon01] Franco Montagna, *Three complexity problems in quantified fuzzy logic*, Studia Logica **68** (2001), no. 1, 143–152, Methods for investigating self-referential truth. MR 2003m:03038

[Ros82] Joseph G. Rosenstein, *Linear orderings*, Pure and Applied Mathematics, vol. 98, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1982. MR 84m:06001

[Sha02] Yvonne Shashoua, *Algebras of basis logic: A classification and related decidability results*, Ph.d. dissertation, University of Maryland, College Park, MD, 2002.

[Zad65] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353. MR 36 #2509