ABSTRACT

Title of dissertation: ASYMPTOTIC PROBLEMS FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Stochastic partial differential equations (SPDEs) can be used to model systems in a wide variety of fields including physics, chemistry, and engineering. The main SPDEs of interest in this dissertation are the semilinear stochastic wave equations which model the movement of a material with constant mass density that is exposed to both determinstic and random forcing. Cerrai and Freidlin have shown that on fixed time intervals, as the mass density of the material approaches zero, the solutions of the stochastic wave equation converge uniformly to the solutions of a stochastic heat equation, in probability. This is called the Smoluchowski-Kramers approximation. In Chapter 2, we investigate some of the multi-scale behaviors that these wave equations exhibit. In particular, we show that the Freidlin-Wentzell exit place and exit time asymptotics for the stochastic wave equation in the small noise regime can be approximated by the exit place and exit time asymptotics for the stochastic heat equation. We prove that the exit time and exit place asymptotics are characterized by quantities called quasipotentials and we prove that the

quasipotentials converge. We then investigate the special case where the equation has a gradient structure and show that we can explicitly solve for the quasipotentials, and that the quasipotentials for the heat equation and wave equation are equal. In Chapter 3, we study the Smoluchowski-Kramers approximation in the case where the material is electrically charged and exposed to a magnetic field. Interestingly, if the system is frictionless, then the Smoluchowski-Kramers approximation does not hold. We prove that the Smoluchowski-Kramers approximation is valid for systems exposed to both a magnetic field and friction. Notably, we prove that the solutions to the second-order equations converge to the solutions of the first-order equation in an L^p sense. This strengthens previous results where convergence was proved in probability.

ASYMPTOTIC PROBLEMS FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

by

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I thank my parents, my wife, and my advisor.

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Chapter 1: Introduction

1.1 Semilinear stochastic wave equations

By Newton's law, the movement of a material with constant mass density $\mu > 0$ in the spatial region $D \subset \mathbb{R}^d$ can be modeled by the damped semilinear stochastic wave equation

$$\begin{cases} \mu \dfrac{\partial^2 u_\epsilon^\mu}{\partial t^2}(t,\xi) = \Delta u_\epsilon^\mu(t,\xi) - \dfrac{\partial u_\epsilon^\mu}{\partial t}(t,\xi) + B(u_\epsilon^\mu(t))(\xi) + \sqrt{\epsilon}\,\dfrac{\partial w^Q}{\partial t}(t,\xi), \quad \xi \in D, \\ u_\epsilon^\mu(0,\xi) = u_0(\xi), \quad \dfrac{\partial u_\epsilon^\mu}{\partial t}(0,\xi) = v_0(\xi), \ \ \xi \in D, \quad u_\epsilon^\mu(t,\xi) = 0, \ \ \xi \in \partial D. \end{cases} \tag{1.1}$$

In the above equation, the Laplacian Δ models the forces neighboring particles exert on each other, $-\partial u_\epsilon^\mu/\partial t$ models friction, and B models some nonlinear forcing. Stochastic perturbations of this system are modeled by $\partial w^Q/\partial t$ which is a noise that is white in time and Q-correlated in space. We also impose Dirichlet boundary conditions and initial conditions.

In this dissertation, we are interested in the limiting behaviors of $u_\epsilon^\mu(t,\xi)$ as the mass density μ and noise intensity ϵ go to zero. It is not surprising that on a fixed time interval as the noise intensity ϵ goes to zero, the perturbed solutions u_ϵ^μ converge to u_0^μ, the solution to the unperturbed deterministic PDE. As for the small-mass asymptotics, Cerrai and Freidlin [2, 3] showed that if $\epsilon > 0$ is fixed and a time horizon $T > 0$ is fixed, then u_ϵ^μ converge to u_ϵ the solution of the following semilinear stochastic heat equation

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,\xi) = \Delta u_\epsilon(t,\xi) + B(u_\epsilon(t))(\xi) + \sqrt{\epsilon}\dfrac{\partial w^Q}{\partial t}(t,\xi), \quad \xi \in D, \\ u_\epsilon(0,\xi) = u_0(\xi), \quad \xi \in D, \quad u_\epsilon(t,\xi) = 0, \quad \xi \in \partial D. \end{cases} \tag{1.2}$$

Formally, equation (1.2) is derived by setting $\mu = 0$ in (1.1). Specifically, they showed that for any $\delta > 0$, $\epsilon > 0$, and $T > 0$,

$$\lim_{\mu \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} |u_\epsilon^\mu(t,\cdot) - u_\epsilon(t,\cdot)|_{L^2(D)} > \delta \right) = 0. \tag{1.3}$$

In fact, by using the methods developed in Chapter 3 of this dissertation, we can strengthen this result to prove that for any $p \geq 1$,

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} |u_\epsilon^\mu(t,\cdot) - u_\epsilon(t,\cdot)|_{L^2(D)}^p = 0. \tag{1.4}$$

This small mass asymptotic, called the Smoluchowski-Kramers approximation, is useful because in practice the second-order equation (1.1) is somewhat cumbersome to study. For example, u_ϵ^μ is not a Markov process because its dynamics depend not only on the position of the particles, but also on their velocities. It is most natural to study the pair $(u_\epsilon^\mu, \partial u_\epsilon^\mu/\partial t)$ in phase space, as this pair is a Markov process. On the other hand, the solution u_ϵ of the first-order equation (1.2) is a Markov process and has other nice features because of the regularizing properties of the heat equation.

While the solutions u_ϵ^μ and u_ϵ are close over finite time intervals for small μ, on an infinite time horizon the solutions deviate substantially. Because of the presence of random noise, we can show that for $\epsilon > 0$ and $\mu > 0$ fixed,

$$\mathbb{P}\left(\sup_{t \geq 0} |u_\epsilon^\mu(t,\cdot) - u_\epsilon(t,\cdot)|_{L^2(D)} = +\infty \right) = 1. \tag{1.5}$$

In light of (1.3) and (1.5), we see that u_ϵ^μ exhibits multi-scale behaviors that depend on the relationship between the mass density μ and the time horizon T. In fact, this multi-scale behavior also depends on ϵ because when there is less random noise, it takes more time for u_ϵ^μ to deviate from u_ϵ.

In Chapter 2, we study the relationship between the Smoluchowski-Kramers approximation and the exit problems from a domain of attraction. In particular, we fix an open bounded subset $G \subset L^2(D)$ and study the exit times

$$\begin{aligned} \tau^{\mu,\epsilon}_{u_0,v_0} &= \inf\{t > 0 : u^\mu_\epsilon(t,\cdot) \notin G\} \\ \tau^{\epsilon}_{u_0} &= \inf\{t > 0 : u_\epsilon(t,\cdot) \notin G\}. \end{aligned} \tag{1.6}$$

Because of the non-degeneracy of the noise terms, these exit times are finite with probability 1 for any $\epsilon > 0$. The unperturbed solutions, on the other hand, will dissipate and will not leave a bounded set G. The wave equations dissipate because they are exposed to friction. If the unperturbed solutions have the property that $u_0^\mu(t,\cdot) \in G$ and $u_0(t,\cdot) \in G$ for all $t > 0$, then $\tau^{\mu,\epsilon}_{u_0,v_0}$ and $\tau^{\epsilon}_{u_0}$ will diverge as $\epsilon \to 0$. By using the theory of large deviations, we show that these exit times diverge approximately exponentially and that

$$\begin{aligned} &\lim_{\epsilon\to 0} \epsilon \log\left(\mathbb{E}\tau^{\mu,\epsilon}_{u_0,v_0}\right) = \inf_{x\in\partial G} V_\mu(x), \\ &\lim_{\epsilon\to 0} \epsilon \log\left(\mathbb{E}\tau^{\epsilon}_{u_0}\right) = \inf_{x\in\partial G} V(x), \\ &\lim_{\epsilon\to 0} \epsilon \log\left(\tau^{\mu,\epsilon}_{u_0,v_0}\right) = \inf_{x\in\partial G} V_\mu(x) \text{ in probability}, \\ &\lim_{\epsilon\to 0} \epsilon \log\left(\tau^{\epsilon}_{u_0}\right) = \inf_{x\in\partial G} V(x) \text{ in probability}. \end{aligned} \tag{1.7}$$

The functionals V_μ and V map $L^2(D)$ to $[0,+\infty]$ and are called the quasipotentials. These functionals also characterize the exit place. We study the $L^2(D)$-valued random variables $u^\mu_\epsilon(\tau^{\mu,\epsilon}_{u_0,v_0})$ and $u_\epsilon(\tau^{\epsilon}_{u_0})$ and show that these converge to the minimizers of the quasipotentials on the boundary of G. That is, if $N \subset G$ is closed and

$$\inf_{x \in N} V_\mu(x) > \inf_{x \in \partial G} V_\mu(x),$$

then

$$\lim_{\epsilon \to 0} \mathbb{P}\left(u_\epsilon^\mu(\tau_{u_0,v_0}^{\mu,\epsilon}) \in N\right) = 0. \tag{1.8}$$

Such an N is far from the minimizers of V_μ on ∂G. The above equation means that the exit place $u_\epsilon^\mu(\tau_{u_0,v_0}^{\mu,\epsilon})$ cannot be far from the minimizers of V_μ. We also have the analogous result for the heat equation. If $N \subset \partial G$ has the property that

$$\inf_{x \in N} V(x) > \inf_{x \in \partial G} V(x),$$

then

$$\lim_{\epsilon \to 0} \mathbb{P}\left(u_\epsilon(\tau_{u_0}^{\epsilon}) \in N\right) = 0. \tag{1.9}$$

The main result of Chapter 2 is that the quasipotentials converge as μ goes to zero. More specifically, for any closed $N \subset \partial G$,

$$\lim_{\mu \to 0} \inf_{x \in N} V_\mu(x) = \inf_{x \in N} V(x). \tag{1.10}$$

Consequently, when μ is small, the exponential divergence rate of the wave equation and the heat equation are close. Namely,

$$\lim_{\mu \to 0} \lim_{\epsilon \to 0} \epsilon \log(\mathbb{E}\tau_{u_0,v_0}^{\mu,\epsilon}) = \lim_{\epsilon \to 0} \epsilon \log(\mathbb{E}\tau_{u_0}^{\epsilon})$$

and

$$\lim_{\mu \to 0} \lim_{\epsilon \to 0} \epsilon \log(\tau_{u_0,v_0}^{\mu,\epsilon}) = \lim_{\epsilon \to 0} \epsilon \log(\tau_{u_0}^{\epsilon}) \text{ in probability.}$$

Another consequence is that when μ is small, the exit place for the wave equation is close to the exit place for the heat equation. If $N \subset \partial G$ is such that

$$\inf_{x \in N} V(x) > \inf_{x \in \partial G} V(x),$$

Then it follows from (1.10) that there exists $\mu_0 > 0$ so that if $\mu < \mu_0$,

$$\inf_{x \in N} V_\mu(x) > \inf_{x \in \partial G} V_\mu(x).$$

Therefore, for $\mu < \mu_0$,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(u_\epsilon^\mu(\tau_{u_0,v_0}^{\mu,\epsilon}) \in N\right) = \lim_{\epsilon \to 0} \mathbb{P}\left(u_\epsilon(\tau_{u_0}^\epsilon) \in N\right) = 0.$$

Finally, Chapter 2 concludes with the analysis of some special cases where (1.1) and (1.2) are gradient systems. That is, we assume that the nonlinearity is of the form $B(x) = -Q^2 DF(x)$ where $DF(x)$ denotes the Frechet derivative of a sufficient differentiable $F : L^2(D) \to [0, +\infty)$. In this particular case one can explicitly solve for the quasipotentials. In fact, in this case the wave equation and heat equation quasipotentials are equal for all $x \in L^2(D)$ and

$$V_\mu(x) = V(x) = |Q^{-1}(-\Delta)^{1/2} x|^2 + 2F(x).$$

Therefore, in the gradient case, the exit time and exit place asymptotics for the heat equation and wave equation match for all $\mu > 0$.

In Chapter 3 we study the Smoluchowski-Kramers approximation for a slightly different stochastic wave equation that models the movement of an electrically charged material that is exposed to a magnetic field as well as deterministic and random forcing. Consider, for example, an electrically charged one-dimensional string in three-dimensional space. At rest, the string has finite length L forms a line segment from $(0,0,0)$ to $(0,0,L)$. The string can move freely through the other two spatial dimensions, but its endpoints are fixed. This is a different situation than (1.1) where the string only moved through one other spatial dimension. This string has constant mass density $\mu > 0$ and is exposed to several different forces. The forces neighboring particles exert on each other can be modeled by a Laplace operator $\frac{\partial^2}{\partial t^2}$. The electrically charged string is exposed to a constant uniform magnetic field that is parallel to the string's rest position $\vec{m} = (0,0,1)$. The string is also exposed to a deterministic nonlinear forcing b that depends only on the position of the string and a stochastic forcing whose intensity also depends on the position of the string. By Newton's law, the position of the string at time t is parameterized by $\xi \in [0,L] \mapsto (u^\mu(\xi,t),\xi) \in \mathbb{R}^3$, where $u_\mu : [0,L] \times [0,+\infty) \to \mathbb{R}^2$ solves the following SPDE

$$\begin{cases} \mu u_\mu(\xi,t) = \dfrac{\partial^2 u_\mu}{\partial \xi^2}(\xi,t) + \vec{m} \times \left(\dfrac{\partial u_\mu}{\partial t}(\xi,t), 0\right) + b(u_\mu(\xi,t),\xi,t) \\ \qquad\qquad + g(u_\mu(t,\xi),\xi,t)\dfrac{\partial w}{\partial t}(\xi,t), \\ u_\mu(0,t) = u_\mu(L,t) = 0, \\ u_\mu(\xi,0) = u_0(\xi), \quad \dfrac{\partial u_\mu}{\partial t}(\xi,0) = v_0(\xi). \end{cases} \tag{1.11}$$

The cross product can be modeled as

$$\vec{m} \times \left(\frac{\partial u_\mu}{\partial t}(\xi,t), 0\right) = -J_0 \frac{\partial u_\mu}{\partial t}(\xi,t)$$

where J_0 is the 2×2 real matrix

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We can generalize this problem to any spatial dimension. Let $D \subset \mathbb{R}^d$ be bounded and sufficiently regular. Consider the following SPDE where
$u_\mu : D \times [0, +\infty) \to \mathbb{R}^2$.

$$\begin{cases} \mu u_\mu(\xi,t) = \Delta u_\mu(\xi,t) - J_0 \dfrac{\partial u_\mu}{\partial t}(\xi,t) + b(u_\mu(\xi,t),\xi,t) + g(u_\mu(t,\xi),\xi,t)\dfrac{\partial w^Q}{\partial t}(\xi,t) \\ u_\mu(\xi,t) = 0, \quad \xi \in \partial D, \\ u_\mu(\xi,0) = u_0(\xi), \quad \dfrac{\partial u_\mu}{\partial t}(\xi,0) = v_0(\xi). \end{cases} \tag{1.12}$$

We would like to prove a Smoluchowski-Kramers approximation for this system. That is we want to find the limit of u_μ as $\mu \to 0$. By formally replacing μ by 0 in (1.12), one might guess that the limit of u_μ is the solution of the first order equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) = J_0^{-1}\left(\Delta u(\xi,t) + (u(\xi,t),\xi,t) + g(u(\xi,t),\xi,t)\dfrac{\partial w^Q}{\partial t}(\xi,t)\right) \\ u(\xi,t) = 0, \quad \xi \in \partial D, \\ u(\xi,0) = u_0(\xi). \end{cases} \tag{1.13}$$

Unfortunately, as in the finite dimensional case [4], u_μ does not converge to u. Actually, if the stochastic term were replaced by a continuous function, then u_μ would converge uniformly on $[0,T]$ to u. As observed in [4], this phenomenon is related to the fact that if φ is any continuous function,

$$\lim_{\mu \to 0} \int_0^t \sin(s/\mu)\varphi(s)ds = 0,$$

but if this continuous function is replaced by an Ito integral, then

$$\lim_{\mu \to 0} \int_0^t \sin(s/\mu)dB(s) \neq 0$$

because

$$\mathbb{E}\left|\int_0^t \sin(s/\mu)dB(s)\right|^2 = \int_0^t \sin^2(s/\mu)ds \to t/2.$$

In [4], the authors consider various regularizations of (1.12) under which Smoluchowski-Kramers approximations are valid. In [4] as well as [17] the authors consider a finite dimensional system that is exposed to regularized noise using the Wong-Zakai approximation. They show that in this situation, a Smoluchowski-Kramers approximation does hold. In Chapter 3 of this dissertation we regularize (1.12) by adding a small amount of friction. That is, we consider for any $\epsilon > 0$,

$$\begin{cases} \mu u_\mu^\epsilon(\xi,t) = \Delta u_\mu^\epsilon(\xi,t) - J_0 \dfrac{\partial u_\mu^\epsilon}{\partial t}(\xi,t) - \epsilon \dfrac{\partial u_\mu^\epsilon}{\partial t}(\xi,t) + b(u_\mu^\epsilon(\xi,t),\xi,t) \\ \qquad\qquad + g(u_\mu^\epsilon(t,\xi),\xi,t)\dfrac{\partial w^Q}{\partial t}(\xi,t) \\ u_\mu^\epsilon(\xi,t) = 0, \quad \xi \in \partial D, \\ u_\mu^\epsilon(\xi,0) = u_0(\xi), \quad \dfrac{\partial u_\mu^\epsilon}{\partial t}(\xi,0) = v_0(\xi). \end{cases} \tag{1.14}$$

and the associated first order equation

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(\xi,t) = (J_0+\epsilon I)^{-1}\left(\Delta u_\epsilon(\xi,t) + b(u_\epsilon(\xi,t),\xi,t) + g(u_\epsilon(t,\xi),\xi,t)\dfrac{\partial w^Q}{\partial t}(\xi,t)\right) \\ u_\epsilon(\xi,t) = 0, \quad \xi \in \partial D, \\ u_\epsilon(\xi,0) = u_0(\xi). \end{cases} \tag{1.15}$$

The addition of friction is reasonable from a physical point of view because very few real-world systems are frictionless. Mathematically, this approximation is useful because we show in Chapter 3 that for any fixed $\epsilon > 0$, $T > 0$, $p \geq 1$,

$$\lim_{\mu\to 0} \sup_{t\in[0,T]} \mathbb{E}\left|u_\mu^\epsilon(\cdot,t) - u_\epsilon(\cdot,t)\right|^p_{L^2(D;\mathbb{R}^2)} = 0. \tag{1.16}$$

The proof of (1.16) is based on the stochastic factorization lemma and some explicit estimates of the linear semigroups associated with equation (1.14). We separate the proofs of these results into the additive and multiplicative noise cases. If $G(u,t,\xi) \equiv Q$ is a constant linear operator, then we can prove (1.16) for any spatial dimension $d \geq 1$ as long as Q is sufficiently regular. If G depends on u, then our methods suffice to prove the Smoluchowski-Kramers approximation only in the case that the spatial dimension $d = 1$. In fact, these methods will also work in the case where there is only friction and no magnetic field. In this sense, the results of Chapter 3 strengthen the Smoluchowski-Kramers results of [2] and [3]. In these papers, Cerrai and Freidlin showed the solutions to (1.1) converge to the solutions of (1.2) uniformly on bounded time intervals in probability. In fact, the convergence is in $L^p(\Omega; C([0,T]; L^2(D)))$.

Once we have established that the Smoluchowski-Kramers approximation is valid for systems exposed to small friction, we show that the approximations (1.14) and (1.15) are close to (1.12) and (1.13). Namely, we show that

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |u_\mu^\epsilon(\cdot,t) - u_\mu(\cdot,t)|^p_{L^2(D;\mathbb{R}^2)} = 0 \tag{1.17}$$

and

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |u_\epsilon(\cdot,t) - u(\cdot,t)|^p_{L^2(D;\mathbb{R}^2)} = 0. \tag{1.18}$$

We note, however, that the limit (1.16) is not uniform with respect to $\epsilon > 0$ and the limit does not hold for $\epsilon = 0$.

1.2 Sobolev spaces and semigroups

Let $D \subset \mathbb{R}^d$ be an open connected region with sufficiently smooth boundary. Let $H = L^2(D)$ be the Hilbert space of square integrable functions from $D \to \mathbb{R}$ endowed with the inner product

$$\langle f, g \rangle_H = \int_D f(\xi) g(\xi) d\xi.$$

Let $A : \mathrm{Dom}(A) \subset H \to H$ be the realization of the Laplace operator with Dirichlet boundary conditions. There exists a complete orthonormal basis of H consisting of eigenvalues of A. We order this basis in such a way that

$$Ae_k = -\alpha_k e_k, \quad 0 < \alpha_k \leq \alpha_{k+1}. \tag{1.19}$$

Also we note that $\lim_{k \to +\infty} \alpha_k = +\infty$ so the operator A is unbounded. For example, in the one-dimensional case $D = (0, \pi)$, the eigenfunctions and eigenvalues of the Lapace operator $\Delta f = \frac{\partial^2 f}{\partial \xi^2}$ are $e_k(\xi) = \frac{2}{\pi} \sin(k\xi)$ and $\alpha_k = k^2$. In any spatial dimension, every $f \in H$ can be written in its Fourier expansion

$$f = \sum_{k=1}^{\infty} \langle f, e_k \rangle_H e_k.$$

Let $C_0^\infty(D)$ denote the space of infinitely differentiable functions whose support is compactly contained in D. For any $\delta \in \mathbb{R}$ we define H^δ as the closure of $C_0^\infty(D)$ under the norm

$$|f|_{H^\delta}^2 = \sum_{k=1}^{\infty} \alpha_k^\delta \langle f, e_k \rangle_H^2 . \tag{1.20}$$

H^δ is a Hilbert space with inner product

$$\langle f, h \rangle_{H^\delta} = \sum_{k=1}^{\infty} \alpha_k^\delta \langle f, e_k \rangle_H \langle h, e_k \rangle_H .$$

The space H^δ is the fractional Sobolev space $W_0^{\delta,2}(D)$. When $\delta = n \in \mathbb{N}$, H^n is the space of square integrable functions with zero trace and square integrable weak derivatives up to degree n. As an example, we will show that $H^1 \equiv W_0^{1,2}(D)$.

Proposition 1.2.1. *The spaces H^1 and $W_0^{1,2}(D)$ are equivalent.*

Proof. Suppose that $f \in C_0^\infty(D)$. Such an f can be written as a Fourier series

$$f(\xi) = \sum_{k=1}^{\infty} \langle f, e_k \rangle_H \, e_k(\xi).$$

Notice that by linearity and the fact that e_k are eigenvalues of A, that

$$Af(\xi) = \sum_{k=1}^{\infty} \langle f, e_k \rangle_H \, Ae_k(\xi) = -\sum_{k=1}^{\infty} \alpha_k \langle f, e_k \rangle_H \, e_k(\xi).$$

By the integration by parts formula,

$$\int_D |\nabla f(\xi)|^2 \, d\xi = -\int_D f(\xi)\Delta f(\xi) d\xi + \int_{\partial D} f(\xi) \frac{\partial f}{\partial \nu}(\xi) d\xi$$

where ν denotes the outward pointing normal on the boundary of D. Because f has zero trace on the boundary, we see that

$$\begin{aligned} \int_D |\nabla f(\xi)|^2 \, d\xi &= -\int_D f(\xi)\Delta f(\xi) d\xi \\ &= -\langle f, \Delta f \rangle_H^2 = \sum_{k=1}^{\infty} \alpha_k \langle f, e_k \rangle_H^2 = |f|_{H^1}^2. \end{aligned} \tag{1.21}$$

Furthermore,

$$\int_D |f(\xi)|^2 d\xi = |f|_H^2 \leq \frac{1}{\alpha_1} |f|_{H^1}.$$

Therefore, the norm in $W_0^{1,2}(D)$ and the norm in H^1 are equivalent. This means that the completion of C_0^∞ in the H^1 and $W_0^{1,2}(D)$ norms coincide. □

We define the fractional powers $(-A)$ by

$$(-A)^{\delta} f = \sum_{k=1}^{\infty} \alpha_k^{\delta} \langle f, e_k \rangle e_k.$$

The space $H^{\delta} = \mathrm{Dom}(-A)^{\delta/2}$ and

$$|f|_{H^{\delta}} = \left|(-A)^{\delta/2} f\right|_{H}.$$

Proposition 1.2.2. *For $\delta < \eta$, the closed unit ball of H^{η} is a compact subset of H^{δ}.*

Proof. Let $|x_n|_{H^{\eta}} \leq 1$. Because H^{η} is a Hilbert space, there is a subsequence which we label as x_n that converges to a limit x in the weak topology. In particular, for any $k \in \mathbb{N}$,

$$\lim_{n \to +\infty} \langle x_n, e_k \rangle_{H^{\eta}} = \langle x, e_k \rangle_{H^{\eta}}.$$

Then for $N \in \mathbb{N}$ to be chosen later

$$\begin{aligned}
|x_n - x|_{H^{\delta}}^2 &= \sum_{k=1}^{\infty} \alpha_k^{\delta} \langle x_n - x, e_k \rangle_H^2 = \sum_{k=1}^{\infty} \alpha_k^{\delta-\eta} \alpha_k^{\eta} \langle x_n - x, e_k \rangle_H^2 \\
&\leq \sum_{k=1}^{N} \alpha_k^{\delta-\eta} \langle x_n - x, e_k \rangle_{H^{\eta}}^2 + 2\alpha_{N+1}^{\delta-\eta}.
\end{aligned}$$

By first choosing N large and then n large we can make the above expression arbitrarily small because $\delta - \eta < 0$. □

Let $S(t) : H^{\delta} \to H^{\delta}$ be the semigroup generated by A. That is for any $x \in H^{\delta}$, $u(t) = S(t)x$ is the solution of the linear differential equation

$$\frac{\partial u}{\partial t}(t) = Au(t), \quad u(0) = x.$$

The semigroup $S(t)$ is independent of the specified domain H^δ. For any Fourier coefficient,

$$\frac{d}{dt}\left\langle S(t)x, e_k\right\rangle_H = \left\langle AS(t)x, e_k\right\rangle_H = -\alpha_k \left\langle S(t)x, e_k\right\rangle$$

and it follows that

$$\left\langle S(t)x, e_k\right\rangle_H = e^{-\alpha_k t}\left\langle x, e_k\right\rangle_H .$$

Therefore the Fourier series of $S(t)x$ is

$$S(t)x = \sum_{k=1}^{\infty} e^{-\alpha_k t}\left\langle x, e_k\right\rangle_H e_k. \tag{1.22}$$

We also want to define the semigroup related to the damped wave equation (1.1). Define the phase spaces $\mathcal{H}_\delta = H^\delta \times H^{\delta-1}$. $\mathcal{H}_\delta$ is a Hilbert space endowed with the inner product

$$|(x,y)|^2_{\mathcal{H}^\delta} = |x|^2_{H^\delta} + |y|^2_{H^{\delta-1}}.$$

We set $\mathcal{H} = \mathcal{H}_0$. Define the operator $\mathcal{A}_\mu : \mathrm{Dom}(\mathcal{A}_\mu) \subset \mathcal{H}_\delta \to \mathcal{H}_\delta$ by

$$A_\mu(u,v) = (v, \mu^{-1}(Au - v)). \tag{1.23}$$

Then Let $S_\mu(t)$ be the semigroup generated by A_μ. Then if $(u(t), v(t)) = S_\mu(t)(u_0, v_0)$, it follows that

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t) = v(t)\\
&\frac{\partial v}{\partial t}(t) = \mu^{-1}(Au(t) - v(t)).\\
&u(0) = u_0, \quad v(0) = v_0.
\end{aligned}$$

Therefore, such a $u(t)$ solves the damped wave equation

$$\mu\frac{\partial^2 u}{\partial t^2}(t) + \frac{\partial u}{\partial t}(t) = Au(t).$$

An explicit representation of $S_\mu(t)$ is given in [2].

There is an analogous way of building the Sobolev spaces and semigroups related to problems (1.12), (1.13), (1.14), and (1.15) which we describe in Chapter 3.

1.3 Noise and mild solutions

Let $D \subset \mathbb{R}^d$. In this section we describe space-time white noise. Space-time white noise is a Schwartz-distribution-valued Gaussian random variable. It is Gaussian in the sense that the (probability) distribution of white noise integrated against any test function is Gaussian. For any deterministic test function $\varphi \in C_0^\infty(D \times [0, +\infty))$, the real valued random variable

$$\int\limits_{D\times[0,+\infty)} \varphi(\xi, t)\frac{\partial w}{\partial t} d\xi dt \stackrel{\text{dist}}{=} \mathcal{N}\left(0, \int\limits_{D\times[0,+\infty)} \varphi^2(\xi, t) d\xi dt\right).$$

These random variables have covariance

$$\mathbb{E}\left(\int\limits_{D\times[0,+\infty)} \varphi(\xi, t)\frac{\partial w}{\partial t} d\xi dt\right)\left(\int\limits_{D\times[0,+\infty)} \psi(\xi, t)\frac{\partial w}{\partial t} d\xi dt\right) = \int\limits_{D\times[0,+\infty)} \varphi(\xi, t)\psi(\xi, t) d\xi dt.$$

Space-time white noise is transition and rotation invariant in law. Furthermore, if the support of φ and the support of ψ are disjoint, then the above equation implies that

$$\left(\int\limits_{D\times[0,+\infty)} \varphi(\xi, t)\frac{\partial w}{\partial t} d\xi dt\right) \text{ and } \left(\int\limits_{D\times[0,+\infty)} \psi(\xi, t)\frac{\partial w}{\partial t} d\xi dt\right)$$

are independent. In fact, up to a multiplicative constant the distribution of white noise is characterized by its Gaussianity, independence on disjoint subsets of space-time and translation invariance in law. When considering random perturbations on space-time, these three features are completely natural. In many applications, white noise models difficult-to-predict microscopic behaviors such as the collision of molecules. By the central limit theorem, large quantities of random collisions should average to a behavior that is approximately Gaussian. The translation invariance means that the nature of the noise is similar in all regions of space-time. No regions of space-time are noisier than others. Finally, the independence on disjoint regions of space-time is a very natural assumption. If the noise is the consequence of microscopic collisions, then those collisions should be approximately independent on regions of space that do not overlap, and the future noise should be independent of the past noise.

The Ito "derivative" of a Brownian motion $\beta(t)$ is an example of a time-only white noise in the sense that if φ and ψ are deterministic test functions

$$\int_0^T \varphi(s)d\beta(s) \text{ and } \int_0^T \psi(s)d\beta(s)$$

are Gaussian random variables with covariance

$$\int_0^T \varphi(s)\psi(s)ds.$$

This is why continuous finite dimensional SDEs are often driven by $d\beta$. Of course, the Ito integral is not the only type of stochastic integration that has this property. The Stratovich integral and Ito integral coincide for deterministic integrands. This example demonstrates that there is some ambiguity in the original description of a space-time white noise. We choose to work with Ito integrals so that time increments are independent, and we can explicitly build a space-time white noise from one-dimensional Ito integrals. Specifically, let $\{\beta_k(t)\}$ be a sequence of independent identically distributed one-dimensional Brownian motions defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a filtration $\mathcal{F}_t$. Let $H = L^2(D)$ and let $\{e_k\}$ be a complete orthonormal basis of H.

Definition 1.3.1. *Space-time white noise is given by the formal sum*

$$\partial w(t,\xi) = \sum_{k=1}^{\infty} e_k(\xi) d\beta_k(t). \tag{1.24}$$

which means that for any $\varphi(t,\xi)$ that is adapted to $\mathcal{F}_t$,

$$\begin{aligned} \int\limits_{D\times[0,T]} \varphi(t,\xi)\frac{\partial w}{\partial t}(t,\xi)d\xi dt &= \sum_{k=1}^{\infty}\int_0^T \int\limits_D \varphi(t,\xi)e_k(\xi)d\xi d\beta_k(t) \\ &= \sum_{k=1}^{\infty}\int_0^T \langle\varphi(t,\cdot), e_k\rangle_H \, d\beta_k(t) \end{aligned} \tag{1.25}$$

where the above integrals are taken in the Ito sense.

This definition of white noise has the properties that we desire. If φ and ψ are predictable,

$$\begin{aligned} &\mathbb{E}\left(\sum_{k=1}^{\infty}\int_0^T \langle\varphi(t,\cdot), e_k\rangle_H \, d\beta_k(t)\right)\left(\sum_{k=1}^{\infty}\int_0^T \langle\psi(t,\cdot), e_k\rangle_H \, d\beta_k(t)\right) \\ &= \sum_{k=1}^{\infty}\mathbb{E}\int_0^T \langle\varphi(t,\cdot), e_k\rangle_H \langle\psi(t,\cdot), e_k\rangle_H \, dt = \mathbb{E}\int_0^T\int_D \varphi(t,\xi)\psi(t,\xi)d\xi dt. \end{aligned}$$

Definition 1.3.2. *The mild solution of the abstract stochastic equation*

$$\frac{\partial u}{\partial t}(t) = Au(t) + B(u(t)) + \frac{\partial w}{\partial t}(t), \quad u(0) = u_0 \tag{1.26}$$

is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)B(u(s))ds + \int_0^t S(t-s)dw(t). \tag{1.27}$$

By (1.22) and (1.24), the stochastic convolution

$$\int_0^t S(t-s)dw(s) = \sum_{k=1}^{\infty}\int_0^t e^{-\alpha_k(t-s)} e_k d\beta_k(s).$$

Therefore, we can calculate that for $\delta > 0$

$$\mathbb{E}\left|\int_0^t S(t-s)dw(s)\right|_{H^\delta}^2 = \sum_{k=1}^\infty \alpha_k^\delta \int_0^t e^{-2\alpha_k(t-s)}ds = \sum_{k=1}^\infty \sum_{k=1}^\infty \frac{1-e^{-2\alpha_k(t-s)}}{2\alpha_k^{1-\delta}}.$$

By Weyl's law for the asymptotics of the eigenvalues of the Laplacian,

$$\alpha_k \sim k^{2/d}$$

where d is the spatial dimension. This means that the stochastic convolution is H^δ-valued for any $\delta < \frac{2-d}{2}$. If $d = 1$, then the stochastic convolution is H^δ valued for $\delta < 1/2$ and is therefore function valued. On the other hand, if $d \geq 2$, then the solutions only exist in negative Sobolev spaces and are therefore genuine Schwartz-distributions. In the linear case ($B(u) \equiv 0$), the mild solution (1.27) is simply H^δ valued for $\delta < \frac{2-d}{2}$. If, however, the domain of the nonlinearity B is functions, then the mild solution has no meaning when $d \geq 2$. For example, consider for any function $u(\xi)$, the composition operator $B(u)(\xi) = b(u(\xi))$ for some $b : \mathbb{R} \to \mathbb{R}$. Such a B has no meaning when $u \in H^\delta$ for negative δ.

It is therefore necessary to study SPDEs exposed to regularizations of white noise that guarantee that the mild solution is function valued.

Definition 1.3.3. *Let $Q \in \mathcal{L}(H)$. A noise that is white in time and Q^2-correlated in space is given formally by*

$$dw^Q(t,\xi) = \sum_{k=1}^\infty (Qe_k)(\xi)d\beta_k(t). \tag{1.28}$$

Proposition 1.3.4. *If $Qe_k = \lambda_k e_k$ and $Ae_k = -\alpha_k e_k$, then the stochastic convolution*

$$\int_0^t S(t-s)dw^Q(s)$$

is H valued if

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k} < +\infty.$$

Proof. We take the expectation of

$$\mathbb{E}\left|\int_0^t S(t-s)dw^Q\right|_H^2 = \sum_{k=1}^{\infty} \int_0^t e^{-\alpha_k(t-s)}\lambda_k^2 ds \leq \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k}.$$

$\square$

Such a noise is still white in time in the sense dw^Q is translation invariant (in distribution) in time and the future is independent of the past. On the other hand, dw^Q no longer is translation invariant in space, and it is not independent on disjoint subsets of space. Despite these drawbacks, dw^Q can be a suitable model of noise in cases where dw is too rough for solution of the SPDE to exist.

Chapter 2: Smoluchowski-Kramers approximation of the exit problem

2.1 Introduction

In the present chapter, we are dealing with the following stochastic wave equation in a bounded regular domain $D \subset \mathbb{R}^d$, with $d \geq 1$,

$$\begin{cases} \mu \dfrac{\partial^2 u_\epsilon^\mu}{\partial t^2}(t,\xi) = \Delta u_\epsilon^\mu(t,\xi) - \dfrac{\partial u_\epsilon^\mu}{\partial t}(t,\xi) + B(u_\epsilon^\mu(t))(\xi) + \sqrt{\epsilon}\,\dfrac{\partial w^Q}{\partial t}(t,\xi), \quad \xi \in D, \\ u_\epsilon^\mu(0,\xi) = u_0(\xi), \qquad \dfrac{\partial u_\epsilon^\mu}{\partial t}(0,\xi) = v_0(\xi), \quad \xi \in D, \qquad u_\epsilon^\mu(t,\xi) = 0, \quad \xi \in \partial D. \end{cases} \tag{2.1}$$

Here $\partial w^Q/\partial t$ is a cylindrical Wiener process, white in time and colored in space, with covariance Q^2, and μ and ϵ are small positive constants.

As a consequence of the Newton law, we may interpret the solution $u_\epsilon^\mu(t,\xi)$ of equation (2.1) as the displacement field of the particles of a material continuum in the domain D, subject to a random external force field $\sqrt{\epsilon}\partial w^Q/\partial t(t,\xi)$ and a damping force proportional to the velocity field $\partial u_\epsilon^\mu/\partial t(t,\xi)$. The Laplacian describes interaction forces between neighboring particles, in presence of a non-linear reaction described by B. The constant μ represents the constant density of the particles.

In [2] and [3], it has been proven that, for fixed $\epsilon > 0$, as the density μ converges to 0, the solution $u_\epsilon^\mu(t)$ of problem (2.1) converges to the solution $u_\epsilon(t)$ of the stochastic first order equation

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,\xi) = \Delta u_\epsilon(t,\xi) + B(u_\epsilon(t))(\xi) + \sqrt{\epsilon}\,\dfrac{\partial w^Q}{\partial t}(t,\xi), \quad \xi \in D, \\ u_\epsilon(0,\xi) = u_0(\xi), \quad \xi \in D, \quad u_\epsilon(t,\xi) = 0, \quad \xi \in \partial D, \end{cases} \tag{2.2}$$

uniformly for t on fixed intervals. More precisely, they have shown that for any $\eta > 0$ and $T > 0$

$$\lim_{\mu \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} |u_\epsilon^\mu(t) - u_\epsilon(t)|_H > \eta \right) = 0. \tag{2.3}$$

Such an approximation is known as the Smoluchowski-Kramers approximation. In fact, in Chapter 3 of this dissertation, we show that the solutions converge in $L^p(\Omega; C([0,T]; H))$ for any $p \geq 1$. That is

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} |u_\epsilon^\mu(t) - u_\epsilon(t)|_H^p = 0. \tag{2.4}$$

Once one has proved the validity of (2.3), an important question arises: how do some relevant asymptotic properties of the second and the first order systems compare, with respect to the small mass asymptotic? In [14] and [10] finite dimensional analogues of this problem were studied. The authors investigated the interactions between the small mass asymptotic ($\mu \to 0$) and other asymptotic behviors including large deviation estimates, the exit problem from a domain, various averaging procedures, the Wong-Zakai approximation, and homogenization. It has been proven that in some cases the two asymptotics do match together properly and in other cases they exhibit non-trivial multi-scale behaviors.

In [2], where the validity of the Smoluchowski-Kramers approximation for SPDEs has been approached for the first time, the long time behavior of equations (2.1) and (2.2) has been compared, under the assumption that the two systems are of gradient type. Actually, in the case of white noise in space and time (that is $Q = I$ and hence $d = 1$) an explicit expression for the Boltzman distribution of the process $z^\mu_\epsilon(t) := (u^\mu_\epsilon(t), \partial u^\mu_\epsilon/\partial t(t))$ in the phase space $\mathcal{H} := L^2(0,1) \times H^{-1}(0,1)$ has been given. Of course, since in the functional space $\mathcal{H}$ there is no translation invariant measure analogous to the Lebesgue measure in finite dimensional space, an auxiliary Gaussian measure has been introduced, with respect to which the density of the Boltzman distribution has been written down. This auxiliary Gaussian measure is the stationary measure of the linear wave equation related to problem (2.1). In particular, it has been shown that the first marginal of the invariant measure associated with the process $z^\mu_\epsilon(t)$ does not depend on μ and coincides with the invariant measure of the process $u_\epsilon(t)$, defined as the unique solution of the heat equation (2.2).

In the present chapter, we are interested in comparing the small noise asymptotics, as $\epsilon \downarrow 0$, for system (2.1) and system (2.2). Actually, we want to show that the Smoluchowski-Kramers approximation, which implies pathwise convergence on finite time intervals, also implies a certain convergence in the large deviations regime. More precisely, we want to compare the quasi-potential $V^\mu(x,y)$ associated with (2.1), with the quasi-potential $V(x)$ associated with (2.2). We to show that for any closed set $N \subset L^2(D)$ it holds

$$\lim_{\mu\to 0} \inf_{x\in N} V_\mu(x) = \inf_{x\in N} V(x) \tag{2.5}$$

where

$$V_\mu(x) := \inf_{y\in H^{-1}(D)} V^\mu(x,y). \tag{2.6}$$

This means that taking first the limit as $\epsilon \downarrow 0$ (large deviation) and then taking the limit as $\mu \downarrow 0$ (Smoluchowski-Kramers approximation) is the same as first taking the limit as $\mu \downarrow 0$ and then as $\epsilon \downarrow 0$. In particular, this result provides a rigorous mathematical justification of what is done in applications, when, in order to study rare events and transitions between metastable states for the more complicated system (2.1), as well as exit times from basins of attraction and the corresponding exit places, the relevant quantities associated with the large deviations for system (2.2) are considered.

The first key idea in order to prove (2.5) is to characterize $V^\mu(x,y)$ as the minimum value for a suitable functional. We recall that the quasi-potential $V^\mu(x,y)$ is defined as the minimum energy required to the system to go from the asymptotically stable equilibrium 0 to the point $(x,y) \in \mathcal{H}$, in any time interval. Namely

$$V^\mu(x,y) = \inf\left\{ I^\mu_{0,T}(z)\ ;\ z(0)=0,\ z(T)=(x,y),\ T>0 \right\},$$

where

$$I^\mu_{0,T}(z) = \frac{1}{2}\inf\left\{ |\psi|^2_{L^2((0,T);H)}\ :\ z = z^\mu_\psi \right\},$$

is the large deviation action functional and $z^\mu_\psi = (u^\mu_\psi, \partial u^\mu_\psi/\partial t)$ is a mild solution of the skeleton equation associated with equation (2.1), with control $\psi \in L^2((0,T);H)$,

$$\mu \frac{\partial^2 u^\mu_\psi}{\partial t^2}(t) = \Delta u^\mu_\psi(t) - \frac{\partial u^\mu_\psi}{\partial t}(t) + B(u^\mu_\psi(t)) + Q\psi(t), \qquad t \in [0,T]. \tag{2.7}$$

By working thoroughly with the skeleton equation (2.7), we show that, for small enough $\mu > 0$,

$$V^\mu(x,y) = \min\left\{ I^\mu_{-\infty,0}(z) : \lim_{t\to-\infty} |z(t)|_{\mathcal{H}} = 0,\ z(0) = (x,y) \right\}. \tag{2.8}$$

In particular, we get that the level sets of V^μ and V_μ are compact in $\mathcal{H}$ and $L^2(D)$, respectively. Moreover, we show that both V^μ and V_μ are well defined and continuous in suitable Sobolev spaces of functions. We would like to stress that in [6] a result analogous to (2.8) has been proved for equation (2.2) and $V(x)$, in terms of the corresponding functional $I_{-\infty,0}$. In both cases, the proof is highly non trivial, due to the degeneracy of the associated control problems, and requires a detailed analysis of the optimal regularity of the solution of the skeleton equation (2.7).

The second key idea is based on the fact that, as in [10] where the finite dimensional case is studied, for all functions $z \in C((-\infty,0];\mathcal{H})$ that are regular enough,

$$\begin{aligned}&I^\mu_{-\infty}(z) = I_{-\infty}(\varphi) + \frac{\mu^2}{2}\int_{-\infty}^0 \left|Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t)\right|_H^2 dt\\&+\mu\int_{-\infty}^0 \left\langle Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t), Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t) - A\varphi(t) - B(\varphi(t))\right)\right\rangle_H dt =: I_{-\infty}(\varphi) + J^\mu_{-\infty}(z),\end{aligned} \tag{2.9}$$

where $\varphi(t) = \Pi_1 z(t)$. Thus, if $\bar{z}^\mu$ is the minimizer of $V_\mu(x)$, whose existence is guaranteed by (2.8), and if $\bar{z}^\mu$ has enough regularity to guarantee that all terms in (2.9) are meaningful, we obtain

$$V_\mu(x) = I_{-\infty}(\bar{\varphi}_\mu) + J^\mu_{-\infty}(\bar{z}^\mu) \geq V(x) + J^\mu_{-\infty}(\bar{z}^\mu). \tag{2.10}$$

In the same way, if $\bar{\varphi}$ is a minimizer for $V(x)$ and is regular enough, then

$$V_\mu(x) \leq I^\mu_{-\infty}(\bar{\varphi}, \partial\bar{\varphi}/\partial t) = V(x) + J^\mu_{-\infty}((\bar{\varphi}, \partial\bar{\varphi}/\partial t)). \tag{2.11}$$

If we could prove that

$$\liminf_{\mu\to 0} J^\mu_{-\infty}(\bar{z}^\mu) = \limsup_{\mu\to 0} J^\mu_{-\infty}((\bar{\varphi}, \partial\bar{\varphi}/\partial t)) = 0, \tag{2.12}$$

from (2.10) and (2.11) we could conclude that (2.5) holds true. But unfortunately, neither $\bar{z}^\mu$ nor $\bar{\varphi}$ have the required regularity to justify (2.12). Thus, we have to proceed with suitable approximations, which, among other things, require us to prove the continuity of the mappings $V_\mu : D((-\Delta)^{1/2}Q^{-1}) \to \mathbb{R}$, uniformly with respect to $\mu \in (0,1]$.

In Section 2.9 we want to apply (2.5) to the study of the exit time and of the exit place of u^μ_ϵ from a given domain in $L^2(D)$. For any open and bounded domain $G \subset L^2(D)$, containing the asymptotically stable equilibrium 0, and for any $z_0 \in G \times H^{-1}(D)$ we define the exit time

$$\tau^{\mu,\epsilon}_{z_0} := \inf \left\{ t \geq 0 \, : \, u^\mu_{\epsilon,z_0}(t) \in \partial G \right\}.$$

Our first goal is to show that, for fixed $\mu > 0$ and $z_0 \in G$,

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\tau^{\mu,\epsilon}_{z_0} = \inf_{x \in \partial G} V_\mu(x), \tag{2.13}$$

and

$$\lim_{\epsilon \to 0} \epsilon \log (\tau^{\mu,\epsilon}_{z_0}) = \inf_{x \in \partial G} V_\mu(x), \qquad \text{in probability.} \tag{2.14}$$

We also want to prove that if $N \subset \partial G$ has the property that $\inf\limits_{x \in N} V_\mu(x) > \inf\limits_{x \in \partial G} V_\mu(x)$, then

$$\lim_{\epsilon \to 0} \mathbb{P}\left(u^\mu_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0}) \in N \right) = 0. \tag{2.15}$$

We would like to stress that the method we are using here in our infinite dimensional setting has several considerable differences compared to the classical finite dimensional argument developed in [15] (see also [13]). The most fundamental difference between the two settings is that, unlike in the finite dimensional case, in the infinite dimensional case the quasi-potentials V_μ are not continuous in $L^2(D)$. Nevertheless, we show here that the lower-semi-continuity of V_μ in $L^2(D)$ along with a convex type regularity assumption for the domain G are sufficient to prove our results. Another important difference is that u_ϵ^μ is not a Markov process, but the pair $(u_\epsilon^\mu, \partial u_\epsilon^\mu/\partial t)$ in the phase space $\mathcal{H}$ is. For this reason, the exit time problem should be considered as the exit from the cylinder $G \times H^{-1} \subset \mathcal{H}$. But, unfortunately, this is an unbounded domain, and as we show in Section 2.3, the unperturbed trajectories are not uniformly attracted to zero from this cylinder. We show that the unboundedness of this cylinder does not prevent us from proving the exit time and exit place asymptotics. We believe that the methods we use to prove the exit time and exit place results should be applicable to most stochastic equations with second-order time derivatives.

In a similar manner, one can show that if

$$\tau_{u_0}^\epsilon = \inf\{t > 0 : u_\epsilon(t) \notin G\}$$

is the exit time from G for the solution of (2.2), and $V(x)$ is the quasipotential associated with this system, the exit time and exit place results for the first-order system are analogous to (2.13), (2.14), and (2.15).

In view of (2.5), the exit time and exit place asymptotics of (2.1) can be approximated by V. Namely

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\tau_{z_0}^{\mu,\epsilon} = \inf_{x\in \partial G} V(x) = \lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\tau_{u_0}^\epsilon,$$

and

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0} \epsilon \log \tau_{z_0}^{\mu,\epsilon} = \inf_{x\in \partial G} V(x) = \lim_{\epsilon\to 0} \epsilon \log \tau_{u_0}^\epsilon, \qquad \text{in probability.}$$

Furthermore, if there exists a unique $\tilde{x} \in \partial G$ such that $V(\tilde{x}) = \inf_{x\in\partial G} V(x)$, then

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0} u_\epsilon^\mu(\tau^{\mu,\epsilon}) = \tilde{x} = \lim_{\epsilon\to 0} u_\epsilon(\tau^\epsilon), \qquad \text{in probability.}$$

We conclude this chapter in Section 2.10 with the discussion of a special case in which the quasipotentials V^μ, V_μ, and V have explicit representations. Specifically we assume that the non-linearity B in (2.1) has the gradient form

$$B(x) = -Q^2 DF(x)$$

where Q^2 is the covariance operator of the noise w^Q and DF denotes the Frechet derivative of a suitably regular non-negative function $F : H \to \mathbb{R}$. In this case

$$V^\mu(x,y) = \left|(-A)^{\frac{1}{2}} Q^{-1} x\right|_H^2 + 2F(x) + \mu|Q^{-1}y|_H^2 \tag{2.16}$$

and

$$V(x) = \left|(-A)^{\frac{1}{2}} Q^{-1} x\right|_H^2 + 2F(x). \tag{2.17}$$

In particular, for any $\mu > 0$,

$$V_\mu(x) = V(x).$$

Therefore, if follows from (2.13) and (2.14) that for any $\mu > 0$,

$$\lim_{\epsilon\to 0} \epsilon \log(\mathbb{E}\tau_{z_0}^{\mu,\epsilon}) = \lim_{\epsilon\to 0} \epsilon \log(\mathbb{E}\tau_{u_0}^{\epsilon})$$

and

$$\lim_{\epsilon\to 0} \epsilon \log(\tau_{z_0}^{\mu,\epsilon}) = \lim_{\epsilon\to 0} \epsilon \log(\tau_{u_0}^{\epsilon}) \text{ in probability.}$$

Furthermore, it follows from (2.15) that the exit place asymptotics also match for the solutions of (2.1) and (2.2) as $\epsilon \to 0$.

2.2 Preliminaries and assumptions

Let D be an open, bounded, regular domain in $\mathbb{R}^d$, with $d \geq 1$ and let H denote the Hilbert space $L^2(D)$. In what follows, we shall denote by A the realization in H of the Laplace operator, endowed with Dirichlet boundary conditions, and we shall denote by $\{e_k\}_{k\in\mathbb{N}}$ and $\{-\alpha_k\}_{k\in\mathbb{N}}$ the corresponding sequence of eigenfunctions and eigenvalues, with $0 < \alpha_1 \leq \alpha_k \leq \alpha_{k+1}$, for any $k \in \mathbb{N}$. Here, we assume that the domain D is regular enough so that

$$\alpha_k \sim k^{2/d}, \qquad k \in \mathbb{N}. \tag{2.18}$$

For any $\delta \in \mathbb{R}$, we shall denote by H^δ the completion of $C_0^\infty(D)$ with respect to the norm

$$|x|^2_{H^\delta} = \sum_{k=1}^{+\infty} \alpha_k^\delta \left\langle x, e_k\right\rangle_H^2$$

H^δ is a Hilbert space, endowed with the scalar product

$$\left\langle x, y\right\rangle_{H^\delta} = \sum_{k=1}^{+\infty} \alpha_k^\delta \left\langle x, e_k\right\rangle_H \left\langle y, e_k\right\rangle_H, \quad x, y \in H^\delta(D).$$

Finally, we shall denote by $\mathcal{H}_\delta$ the Hilbert space $H^\delta \times H^{\delta-1}$ and in the case $\delta = 0$ we shall set $\mathcal{H}_0 = \mathcal{H}$. Moreover, we shall denote

$$\Pi_1 : \mathcal{H}_\delta \to H^\delta, \ \ (u,v) \mapsto u, \qquad \Pi_2 : \mathcal{H}_\delta \to H^{\delta-1}, \ \ (u,v) \mapsto v.$$

Sometimes, for the sake of simplicity, we will denote for any $\mu > 0$ and $\delta \in \mathbb{R}$

$$\mathcal{I}_\mu(u,v) = (u, \sqrt{\mu} v), \qquad (u,v) \in \mathcal{H}_\delta. \tag{2.19}$$

The stochastic perturbation is given by a cylindrical Wiener process $w^Q(t,\xi)$, for $t \geq 0$ and $\xi \in \mathcal{O}$, which is assumed to be white in time and colored in space, in the case of space dimension $d > 1$. Formally, it is defined as the infinite sum

$$w^Q(t,\xi) = \sum_{k=1}^{+\infty} Qe_k(\xi)\,\beta_k(t), \tag{2.20}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal basis in $L^2(D)$ which diagonalizes A and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions defined on the same complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Hypothesis 1. *The linear operator Q is bounded in H and diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$ which diagonalizes A. Moreover, if $\{\lambda_k\}_{k\in\mathbb{N}}$ is the corresponding sequence of eigenvalues, we have*

$$\frac{1}{c}\alpha_k^{-\beta} \leq \lambda_k \leq c\,\alpha_k^{-\beta}, \qquad k \in \mathbb{N}, \tag{2.21}$$

for some $c > 0$ and $\beta > (d-2)/4$.

Remark 2.2.1.

1. If $d = 1$, according to Hypothesis 1 we can consider space-time white noise ($Q = I$).

2. Thanks to (2.18), condition (2.21) implies that if $d \geq 2$, then there exists $\gamma < 2d/(d-2)$ such that
$$\sum_{k=1}^{\infty} \lambda_k^{\gamma} < \infty.$$
Moreover
$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k} < \infty.$$

3. As a consequence of (2.21), for any $\delta \in \mathbb{R}$
$$D((-A)^{\delta/2}Q^{-1}) = H^{\delta+2\beta}$$

and there exists $c_\delta > 0$ such that for any $x \in H^{\delta+2\beta}$

$$\frac{1}{c_\delta}\,|(-A)^{\delta/2}Q^{-1}x|_H \leq |x|_{\delta+2\beta} \leq c_\delta\,|(-A)^{\delta/2}Q^{-1}x|_H.$$

Concerning the nonlinearity B, we shall assume the following conditions.

Hypothesis 2. *For any $\delta \in [0, 1+2\beta]$, the mapping $B : H^\delta \to H^\delta$ is Lipschitz continuous, with*

$$[B]_{\mathrm{Lip}(H^\delta)} =: \gamma_\delta < \alpha_1.$$

Moreover $B(0) = 0$. We also assume that B is differentiable in the space $H^{2\beta}$, with

$$\sup_{z\in\mathcal{H}} \|DB(z)\|_{\mathcal{L}(H^{2\beta})} = \gamma_{2\beta}.$$

Remark 2.2.2. 1. The assumption that B is differentiable is made for convenience to simplify the proof of lower bounds in Theorem 2.8.2. We believe that by approximating the Lipschitz continuous B with a sequence of differentiable functions whose C^1 semi-norm is controlled by the Lipschitz semi-norm of B, the results proved in Theorem 2.8.2 should remain true.

2. If we define for any $x \in H$

$$B(x)(\xi) = b(\xi, x(\xi)), \quad \xi \in D,$$

and we assume that $b(\xi,\cdot) \in C^{2k}(\mathbb{R})$, for $k \in [\beta + \delta/2 - 5/4, \beta + \delta/2 - 1/4]$, and

$$\frac{\partial^j b}{\partial \sigma^j}(\xi,\sigma)_{|\sigma=0} = 0, \quad \xi \in \overline{D},$$

then B maps H^δ into itself, for any $\delta \in [0, 1+2\beta]$. The Lipschitz continuity of B in H^δ and the bound on the Lipschitz norm, are satisfied if the derivatives of $b(\xi,\cdot)$ are small enough.

With these notations, equation (2.2) can be written as the following abstract evolution equation in H

$$du_\epsilon(t) = [Au_\epsilon(t) + B(u_\epsilon(t))]\, dt + \sqrt{\epsilon}\, dw^Q(t), \qquad u(0) = u_0. \tag{2.22}$$

Definition 2.2.3. *A predictable process* $u_\epsilon \in L^2(\Omega; C([0,T]; H))$ *is a* mild solution *to equation* (2.22) *if*

$$u_\epsilon(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} B(u_\epsilon(s))\, ds + \sqrt{\epsilon} \int_0^t e^{(t-s)A} dw^Q(s).$$

Now, for each $\mu > 0$ and $\delta \in \mathbb{R}$ we define $A_\mu : D(A_\mu) \subset \mathcal{H}_\delta \to \mathcal{H}_\delta$ by setting

$$A_\mu(u,v) = \left(-v, \frac{1}{\mu}Au - \frac{1}{\mu}v\right), \quad (u,v) \in D(A_\mu) = \mathcal{H}_{1+\delta}, \tag{2.23}$$

and we denote by $S_\mu(t)$ the semigroup on $\mathcal{H}_\delta$ generated by A_μ. In [2, Proposition 2.4], it is proved that for each $\mu > 0$ there exist $\omega_\mu > 0$ and $M_\mu > 0$ such that

$$\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H})} \leq M_\mu\, e^{-\omega_\mu t}, \qquad t \geq 0. \tag{2.24}$$

Notice that, since for any $\delta \in \mathbb{R}$ and $(u,v) \in \mathcal{H}_\delta$

$$\big((-A)^\delta \Pi_1 S_\mu(t)(u,v), (-A)^\delta \Pi_2 S_\mu(t)(u,v)\big) = S_\mu(t)((-A)^\delta u, (-A)^\delta v), \quad t \geq 0,$$

(2.24) implies that for any $\delta \in \mathbb{R}$

$$\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H}_\delta)} \leq M_\mu\, e^{-\omega_\mu t}, \qquad t \geq 0. \tag{2.25}$$

Next, for any $\mu > 0$ we denote

$$B_\mu(u,v) = \frac{1}{\mu}(0, B(u)), \qquad (u,v) \in \mathcal{H},$$

and

$$Q_\mu u = \frac{1}{\mu}(0, Qu), \qquad u \in H.$$

With these notations, equation (2.1) can be written as the following abstract evolution equation in the space $\mathcal{H}$

$$dz(t) = [A_\mu z(t) + B_\mu(z(t))]\, dt + \sqrt{\epsilon}\, Q_\mu\, dw(t), \qquad z(0) = (u_0, v_0). \tag{2.26}$$

Definition 2.2.4. *A predictable process u_ϵ^μ is a* mild solution *of* (3.12) *if*

$$u_\epsilon^\mu \in L^2(\Omega; C([0,T]; H)), \qquad v_\epsilon^\mu =: \frac{\partial\, u_\epsilon^\mu}{\partial t} \in L^2(\Omega; C([0,T]; H^{-1})),$$

for any $T > 0$, and

$$z_\epsilon^\mu(t) = S_\mu(t)z(0) + \int_0^t S_\mu(t-s)B_\mu(z_\epsilon^\mu(s))ds + \sqrt{\epsilon}\int_0^t S_\mu(t-s)Q_\mu dw(s), \tag{2.27}$$

where $z(0) = (u_0, v_0)$ and $z_\epsilon^\mu = (u_\epsilon^\mu, v_\epsilon^\mu)$.

In view of Hypothesis 1 and of the fact that $B : H \to H$ is Lipschitz continuous, for any $\mu > 0$ and any initial condition $z_0 = (u_0, v_0) \in \mathcal{H}$, there exists a unique mild solution u_ϵ^μ for equation (2.1), (for a proof see e.g. [2]). In [2, Theorem 4.6] we have proved that for any fixed $\epsilon > 0$ and $T > 0$ the solution u_ϵ^μ of equation (2.1) converges in $C([0,T]; H)$, in probability, to the solution u_ϵ of equation (2.2), as $\mu \downarrow 0$. Namely, for any $\eta > 0$

$$\lim_{\mu \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} |u_\epsilon^\mu(t) - u_\epsilon(t)|_H > \eta \right) = 0.$$

2.3 The unperturbed equation

We consider here equation (3.12), for $\epsilon = 0$. Namely,

$$\frac{dz}{dt}(t) = A_\mu z(t) + B_\mu(z(t)), \qquad z(0) = z_0 = (u_0, v_0). \tag{2.28}$$

The solution to (2.28) will be denoted by $z_{z_0}^\mu(t)$. We recall here that γ_0 denotes the Lipschitz constant of B in H (see Hypothesis 4).

Lemma 2.3.1. *If $\mu < (\alpha_1 - \gamma_0)\gamma_0^{-2}$, there exists a constant $c_1(\mu) > 0$ such that*

$$\sup_{t \geq 0} \left| z_{z_0}^\mu(t) \right|_{\mathcal{H}} + \left| z_{z_0}^\mu \right|_{L^2((0,+\infty);\mathcal{H})} \leq c_1(\mu) |z_0|_{\mathcal{H}}, \qquad z_0 \in \mathcal{H}. \tag{2.29}$$

Proof. If $\varphi(t) = \Pi_1 z_{z_0}^\mu(t)$ then

$$\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) = A\varphi(t) + B(\varphi(t)). \tag{2.30}$$

By taking the inner product of (2.30) with $\frac{\partial \varphi}{\partial t}$ in H^{-1}, and by using the Lipschitz continuity of B in H, we see that

$$\mu \frac{d}{dt} \left| \frac{\partial \varphi}{\partial t}(t) \right|_{H^{-1}}^2 + 2 \left| \frac{\partial \varphi}{\partial t}(t) \right|_{H^{-1}}^2 \leq -\frac{d}{dt} |\varphi(t)|_H^2 + \left| \frac{\partial \varphi}{\partial t}(t) \right|_{H^{-1}}^2 + \frac{\gamma_0^2}{\alpha_1} |\varphi(t)|_H^2. \tag{2.31}$$

By integrating this expression in time, we see that

$$\mu \left| \frac{\partial \varphi}{\partial t}(t) \right|_{H^{-1}}^2 + |\varphi(t)|_H^2 + \int_0^t \left| \frac{\partial \varphi}{\partial s}(s) \right|_{H^{-1}}^2 ds \leq \mu |v_0|_{H^{-1}}^2 + |u_0|_H^2 + \frac{\gamma_0^2}{\alpha_1} \int_0^t |\varphi(s)|_H^2 \, ds. \tag{2.32}$$

Next, by taking the inner product of (2.30) with $\varphi(t)$ in H^{-1}, since

$$\left\langle \frac{\partial^2 \varphi}{\partial t^2}(t), \varphi(t) \right\rangle_{H^{-1}} = \frac{1}{2} \frac{d^2}{dt^2} |\varphi(t)|_{H^{-1}}^2 - \left| \frac{\partial \varphi}{\partial t}(t) \right|_{H^{-1}}^2,$$

we have

$$\mu\frac{d^2}{dt^2}\,|\varphi(t)|^2_{H^{-1}}+\frac{d}{dt}|\varphi(t)|^2_{H^{-1}}\leq -2|\varphi(t)|^2_H+\frac{2\gamma_0}{\alpha_1}|\varphi(t)|^2_H+2\mu\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}.$$

By (2.31), this yields

$$\begin{aligned}&\mu\frac{d^2}{dt^2}\,|\varphi(t)|^2_{H^{-1}}+\frac{d}{dt}|\varphi(t)|^2_{H^{-1}}\leq\\&-2|\varphi(t)|^2_H+\frac{2\gamma_0}{\alpha_1}|\varphi(t)|^2_H-2\mu^2\frac{d}{dt}\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}-2\mu\frac{d}{dt}|\varphi(t)|^2_H+\frac{2\gamma_0^2\mu}{\alpha_1}|\varphi(t)|^2_H.\end{aligned}\tag{2.33}$$

Now, if $\mu<(\alpha_1-\gamma_0)\gamma_0^{-2}$, it follows

$$\rho_\mu:=2-\frac{2\gamma_0}{\alpha_1}-\frac{2\mu\gamma_0^2}{\alpha_1}>0.$$

Then, by integrating both sides in (2.33), we see

$$\begin{aligned}&\mu\frac{d}{dt}\,|\varphi(t)|^2_{H^{-1}}+|\varphi(t)|^2_{H^{-1}}+\rho_\mu\int_0^t|\varphi(s)|^2_H\,ds\\&\leq 2\mu\left\langle v_0,u_0\right\rangle_{H^{-1}}+|u_0|^2_{H^{-1}}+2\mu^2|v_0|^2_{H^{-1}}+2\mu|u_0|^2_H,\end{aligned}\tag{2.34}$$

and this implies that

$$\int_0^\infty|\varphi(t)|^2_H ds\leq\frac{1}{\rho_\mu}\left(2\mu\left\langle v_0,u_0\right\rangle_{H^{-1}}+|u_0|^2_{H^{-1}}+2\mu^2|v_0|^2_{H^{-1}}+2\mu|u_0|^2_H\right).\tag{2.35}$$

Actually, if there exists $t_0>0$ and $\delta>0$ such that

$$\int_0^{t_0}|\varphi(t)|^2_H ds>\frac{1}{\rho_\mu}\left(2\mu\left\langle v_0,u_0\right\rangle_{H^{-1}}+|u_0|^2_{H^{-1}}+2\mu^2|v_0|^2_{H^{-1}}+2\mu|u_0|^2_H\right)+\delta,$$

then, in view of (2.34), for any $t>t_0$

$$\mu\frac{d}{dt}\,|\varphi(t)|^2_{H^{-1}}<-\delta.$$

This would imply that for any $t>t_0$

$$|\varphi(t)|^2_{H^{-1}}<|\varphi(t_0)|^2_{H^{-1}}-(t-t_0)\delta,$$

which is impossible because $|\varphi(t)|^2_{H^{-1}}$ is always nonnegative.

We conclude the proof by combining (2.32) and (2.35), to see that

$$\mu\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}+|\varphi(t)|^2_H+\int_0^t\left|\frac{\partial\varphi}{\partial s}(s)\right|^2_{H^{-1}}ds+\int_0^t|\varphi(s)|_H\,ds\leq c|z_0|^2_{\mathcal{H}}.$$

□

Lemma 2.3.2. *Assume $\mu<(\alpha_1-\gamma_0)\gamma_0^{-2}$, then for any $R>0$,*

$$\lim_{t\to+\infty}\sup_{|z_0|_{\mathcal{H}}\leq R}\left|z^\mu_{z_0}(t)\right|_{\mathcal{H}}=0. \tag{2.36}$$

Proof. Let us fix $R,\rho>0$ and for any $\mu>0$ let us define

$$T=\frac{(c_1(\mu))^4R^2}{\rho^2}.$$

Let $|z_0|_{\mathcal{H}}\leq R$. Since

$$\left|z^\mu_{z_0}\right|_{L^2((0,T);\mathcal{H})}\geq\sqrt{T}\min_{s\leq T}|z^\mu_{z_0}(s)|_{\mathcal{H}},$$

according to (2.29) there must exists $t_0<T$ such that

$$|z^\mu_{z_0}(t_0)|_{\mathcal{H}}\leq\frac{\rho}{c_1(\mu)}.$$

By using again (2.29), this implies

$$\sup_{t\geq T}\left|z^\mu_{z_0}(t)\right|_{\mathcal{H}}=\sup_{t\geq T}\left|z^\mu_{z^\mu_{z_0}(t_0)}(t-t_0)\right|_{\mathcal{H}}\leq\rho.$$

Notice that T is independent of our choice of z_0 so we can conclude that

$$\sup_{t\geq T}\sup_{|z_0|_{\mathcal{H}}\leq R}\left|z^\mu_{z_0}(t)\right|_{\mathcal{H}}\leq\rho.$$

□

Now that we have shown that the unperturbed system is uniformly attracted to 0 from any bounded set in $\mathcal{H}$, we show that if the initial velocity is large enough, $\Pi_1 z^\mu_{z_0}$ will leave any bounded set.

Lemma 2.3.3. *For any $\mu > 0$ and $t > 0$, there exists $c_2(\mu, t) > 0$ such that*

$$\sup_{s\leq t} |\Pi_1 S_\mu(s)\,(0, v_0)|_H \geq c_2(\mu, t)\,|v_0|_{H^{-1}}\,, \qquad v_0 \in H^{-1}. \tag{2.37}$$

Proof. Let $\varphi(t) = \Pi_1 S_\mu(t)(0, v_0)$. Then

$$\mu\frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) = A\varphi(t), \qquad \varphi(0) = 0, \quad \frac{\partial\varphi}{\partial t}(0) = v_0.$$

By taking the inner product of this equation with $\frac{\partial\varphi}{\partial t}(t)$ in H^{-1}, we see that

$$\mu\frac{d}{dt}\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} + 2\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} = -\frac{d}{dt}\,|\varphi(t)|^2_H\,.$$

Therefore, by standard calculations,

$$\begin{aligned}\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} &= e^{-\frac{2t}{\mu}}|v|^2_{H^{-1}} - \frac{1}{\mu}\int_0^t e^{-\frac{2(t-s)}{\mu}}\frac{d}{ds}\,|\varphi(s)|^2_H\,ds\\ &= e^{-\frac{2t}{\mu}}|v_0|^2_{H^{-1}} - \frac{1}{\mu}|\varphi(t)|^2_H + \frac{2}{\mu^2}\int_0^t e^{-\frac{2(t-s)}{\mu}}\,|\varphi(s)|^2_H\,ds,\end{aligned}$$

so that

$$\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} \leq e^{-\frac{2t}{\mu}}|v_0|^2_{H^{-1}} + \frac{1}{\mu}\sup_{s\leq t}|\varphi(s)|^2_H. \tag{2.38}$$

Next, since

$$\begin{aligned}\frac{d}{dt}\left|\varphi(t) + \mu\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} &= 2\left\langle\varphi(t) + \mu\frac{\partial\varphi}{\partial t}(t), \frac{\partial\varphi}{\partial t}(t) + \mu\frac{\partial^2\varphi}{\partial t^2}(t)\right\rangle_{H^{-1}}\\ = 2\left\langle\varphi(t) + \mu\frac{\partial\varphi}{\partial t}(t), A\varphi(t)\right\rangle_{H^{-1}} &= -2|\varphi(t)|^2_H - \mu\frac{d}{dt}\,|\varphi(t)|_H\,,\end{aligned} \tag{2.39}$$

if we integrate in time we get

$$\mu^2|v_0|^2_{H^{-1}} = \left|\varphi(t) + \mu\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} + 2\int_0^t |\varphi(s)|^2_H ds + \mu|\varphi(t)|^2_H.$$

For any $a > 0$ to be chosen later, we have

$$\left|\varphi(t) + \mu\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} \leq \left(1 + a^{-1}\right)\frac{1}{\alpha_1}\left|\varphi(t)\right|^2_H + \mu^2(1+a)\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}$$

and therefore,

$$\mu^2|v_0|^2_{H^{-1}} \leq \left(\mu + 2t + \left(1 + a^{-1}\right)\frac{1}{\alpha_1}\right)\sup_{s\leq t}|\varphi(s)|^2_H + \mu^2(1+a)\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}$$

Thanks to (2.38), this yields

$$\mu^2\left(1 - (1+a)e^{-\frac{2t}{\mu}}\right)|v_0|^2_{H^{-1}} \leq \left(\mu + 2t + \left(1 + a^{-1}\right)\frac{1}{\alpha_k} + (1+a)\mu\right)\sup_{s\leq t}|\varphi(s)|^2_H\,,$$

and our conclusion follows with if we pick $a < e^{\frac{2t}{\mu}} - 1$.

□

As a consequence of the previous lemma, we can conclude that the following lower bound estimate holds for the solution of (2.28).

Lemma 2.3.4. *For any $\mu > 0$ and $t > 0$ there exists $c(\mu, t) > 0$ such that*

$$\sup_{s\leq t}\left|\Pi_1 z^\mu_{z_0}(s)\right|_H \geq c(\mu,t)|\Pi_2 z_0|_{H^{-1}}, \qquad z_0 \in \mathcal{H}. \tag{2.40}$$

Proof. Let $z_0 = (u_0, v_0)$. Since

$$\Pi_1 z^\mu_{z_0}(t) = \Pi_1 S_\mu(t)(u_0, 0) + \Pi_1 S_\mu(t)(0, v_0) + \Pi_1\int_0^t S_\mu(t-s)B_\mu(z^\mu_{z_0}(s))ds,$$

from Hypothesis 4 and (2.24), for any $s > 0$

$$|\Pi_1 S_\mu(s)(0, v_0)|_H \leq \left(2M_\mu + \frac{\gamma_0 M_\mu}{\omega_\mu \mu}\right)\sup_{r\leq s}\left|\Pi_1 z^\mu_{z_0}(r)\right|_H.$$

According to (2.37), this implies that for any $t > 0$,

$$c_2(\mu,t)|v_0|_{H^{-1}} \leq \sup_{s\leq t}|\Pi_1 S_\mu(t)(0, v_0)|_H \leq \left(2M_\mu + \frac{\gamma_0 M_\mu}{\omega_\mu \mu}\right)\sup_{s\leq t}\left|\Pi_1 z^\mu_{z_0}(s)\right|_H.$$

Therefore, the result follows with

$$c(\mu, t) = c_1(\mu, t)\left(2M_\mu + \frac{\gamma_0 M_\mu}{\omega_\mu \mu}\right)^{-1}.$$

$\square$

2.4 The skeleton equation

For any $\mu > 0$ and $s < t$ and for any $\psi \in L^2((s,t); H)$ we define

$$L^\mu_{s,t}\psi = \int_s^t S_\mu(t-r)Q_\mu\psi(r)dr.$$

Clearly $L^\mu_{s,t}$ is a continuous bounded linear operator from $L^2([s,t]; H)$ into $\mathcal{H}$. If we define the pseudo-inverse of $L^\mu_{s,t}$ as

$$(L^\mu_{s,t})^{-1}(x) = \text{argmin}\left\{|(L^\mu_{s,t})^{-1}(\{x\})|_{L^2([s,t];H)}\right\}, \qquad x \in \text{Im }(L^\mu_{s,t}),$$

we have the following bounds.

Theorem 2.4.1. *For any $\mu > 0$ and $s < t$, it holds*

$$\left|\left(L^\mu_{s,t}\right)^{-1} z\right|_{L^2((s,t);H)} = \sqrt{2}\left|(C_\mu - S_\mu(t-s)C_\mu S^\star_\mu(t-s))^{-1/2} z\right|_{\mathcal{H}}, \qquad z \in \text{Im }(L^\mu_{s,t}), \tag{2.41}$$

where

$$C_\mu(u,v) = \left(Q^2(-A)^{-1}u, \frac{1}{\mu}Q^2(-A)^{-1}v\right), \qquad (u,v) \in \mathcal{H}. \tag{2.42}$$

Moreover, for every $\mu > 0$ there exists $T_\mu > 0$ such that

$$\text{Im }(L^\mu_{s,t}) = \text{Im }((C_\mu)^{1/2}) = \mathcal{H}_{1+2\beta}, \qquad t - s \geq T_\mu, \tag{2.43}$$

and

$$\left|(L^\mu_{s,t})^{-1}z\right|_{L^2((s,t);H)} \leq c(\mu,t-s)\,|z|_{\mathcal{H}_{1+2\beta}}\,, \quad z\in \mathcal{H}_{1+2\beta}, \tag{2.44}$$

for some constant $c(\mu,r)>0$, with $r\geq T_\mu$.

Proof. It is immediate to check that for any $z\in\mathcal{H}$

$$\left|(L^\mu_{s,t})^\star z\right|^2_{L^2((s,t);H)} = \frac{1}{\mu^2}\int_0^{t-s}\left|Q(-A)^{-1}\Pi_2 S^\star_\mu(r)z\right|^2_H\,dr. \tag{2.45}$$

Now, if we expand $S^\star_\mu(t)(u,v)$ in Fourier series, we have (see [2, Proposition 2.3])

$$S^\star_\mu(t)(u,v)=\sum_{k=1}^\infty\left(\hat{f}^\mu_k(t)e_k,\hat{g}^\mu_k(t)e_k\right),$$

where $\hat{f}^\mu_k$ and $\hat{g}^\mu_k$ solve the system

$$\begin{cases}\mu(\hat{f}^\mu_k)'(t)=-\hat{g}^\mu_k(t), & \hat{f}^\mu_k(0)=u_k,\\[2mm] \mu(\hat{g}^\mu_k)'(t)=\mu\alpha_k\hat{f}^\mu_k(t)-\hat{g}^\mu_k(t), & \hat{g}^\mu_k(0)=v_k.\end{cases} \tag{2.46}$$

In particular,

$$|\hat{g}^\mu_k(t)|^2=-\frac{\mu^2\alpha_k}{2}\frac{d}{dt}|\hat{f}^\mu_k(t)|^2-\frac{\mu}{2}\frac{d}{dt}|\hat{g}^\mu_k(t)|^2 \tag{2.47}$$

Due to (2.45), we get

$$\begin{aligned}&\left|(L^\mu_{s,t})^\star z\right|^2_{L^2([s,t];H)}=\frac{1}{2}\sum_{k=1}^\infty\int_0^{t-s}\left(-\frac{\lambda_k^2}{\alpha_k}\frac{d}{dr}|\hat{f}^\mu_k(r)|^2-\frac{\lambda_k^2}{\mu\alpha_k^2}\frac{d}{dr}|\hat{g}^\mu_k(r)|^2\right)dr\\ &=\frac{1}{2}\sum_{k=1}^\infty\left(-\frac{\lambda_k^2}{\alpha_k}|\hat{f}^\mu_k(t-s)|^2-\frac{\lambda_k^2}{\alpha_k^2\mu}|\hat{g}^\mu_k(t-s)|^2+\frac{\lambda_k^2}{\alpha_k}|u_k|^2+\frac{\lambda_k^2}{\alpha_k^2\mu}|v_k|^2\right)\\ &=\frac{1}{2}\left(|C_\mu^{1/2}z|^2_{\mathcal{H}}-|C_\mu^{1/2}S^\star_\mu(t-s)z|^2_{\mathcal{H}}\right)=\frac{1}{2}\left\langle(C_\mu-S_\mu(t-s)C_\mu S^\star_\mu(t-s))z,z\right\rangle_{\mathcal{H}}.\end{aligned} \tag{2.48}$$

This implies that

$$\mathrm{Im}\,(L^\mu_{s,t})=\mathrm{Im}\,((C_\mu-S_\mu(t-s)C_\mu S^\star_\mu(t-s))^{1/2},$$

and (2.41) follows.

Next, in order to prove (2.43), we notice that

$$C_1^{1/2} S_\mu^\star(t) = S_\mu^\star(t) C_1^{1/2}, \qquad t \geq 0, \tag{2.49}$$

and that

$$(1 \wedge \sqrt{\mu}) \left|C_\mu^{1/2} z\right|_{\mathcal{H}} \leq \left|C_1^{1/2} z\right|_{\mathcal{H}} \leq (1 + \sqrt{\mu}) \left|C_\mu^{1/2} z\right|_{\mathcal{H}}$$

so that, due to (2.24), we have

$$|C_\mu^{1/2} S_\mu^\star(t) z|_{\mathcal{H}} \leq c_\mu M_\mu e^{-\omega_\mu t} |C_\mu^{1/2} z|_{\mathcal{H}}, \qquad t \geq 0.$$

According to (2.48), this implies

$$\left|(L_{s,t}^\mu)^* z\right|_{\mathcal{H}}^2 = \frac{1}{2}|C_\mu^{1/2} z|_{\mathcal{H}}^2 - \frac{1}{2}|C_\mu^{1/2} S_\mu(t-s) z|_{\mathcal{H}} \geq \frac{1}{2}(1 - c_\mu^2\, M_\mu^2 e^{-2\omega_\mu(t-s)})|C_\mu^{1/2} z|_{\mathcal{H}}^2.$$

Therefore, if we pick $T_\mu > 0$ large enough so that $c_\mu^2\, M_\mu e^{-\omega_\mu T_\mu} < 1$, we obtain that

$$\text{Im } (L_{s,t}^\mu) = \text{Im } ((C_\mu)^{1/2}),$$

and

$$\left|(L_{s,t}^\mu)^{-1} z\right|_{L^2((s,t);H)} \leq \sqrt{2}\, \left(1 - c_\mu^2\, M_\mu^2 e^{-2\omega_\mu r}\right)^{-1/2}\, |(C_\mu)^{-1/2} z|_{\mathcal{H}}.$$

Now, as for any $\mu > 0$ we have Im $((C_\mu)^{1/2}) = \mathcal{H}_{1+2\beta}$, and

$$(1 \wedge \mu)\, |z|_{\mathcal{H}_{1+2\beta}} \leq |(C_\mu)^{-1/2} z|_{\mathcal{H}} \leq (1 + \mu)\, |z|_{\mathcal{H}_{1+2\beta}}, \tag{2.50}$$

(2.43) and (2.44) follow immediately, with

$$c(\mu, r) = (1 + \mu)\, \sqrt{2}\, \left(1 - c_\mu^2\, M_\mu^2 e^{-2\omega_\mu r}\right)^{-1/2}.$$

□

Remark 2.4.2. 1. In fact, it is possible to show that Im $(L^{\mu}_{s,t}) =$ Im $((C_\mu)^{1/2})$, for all $t-s>0$, by using the explicit representation of $S^\star_\mu(t)$.

2. From (2.24) and (2.41), it easily follows that

$$|(L^{\mu}_{-\infty,t})^{-1}z|_{L^2((-\infty,t);H)} = \sqrt{2}\,|C_\mu^{-1/2}z|_{\mathcal{H}}, \quad z \in \mathrm{Im}(L^{\mu}_{-\infty,t}). \tag{2.51}$$

Lemma 2.4.3. *Let us fix $\psi \in L^2((-\infty,0);H^{2\alpha})$, with $\alpha \in [0,1/2]$, and $\mu > 0$ and let $z^\mu_\psi \in C((-\infty,0);\mathcal{H})$ solve the equation*

$$z^\mu_\psi(t) = \int_{-\infty}^t S_\mu(t-s)B_\mu(z^\mu_\psi(s))\,ds + \int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)\,ds, \quad t\in\mathbb{R}. \tag{2.52}$$

Then, if

$$\lim_{t\to-\infty}|z^\mu_\psi(t)|_{\mathcal{H}} = 0, \tag{2.53}$$

we have $z^\mu_\psi \in C((-\infty,0];\mathcal{H}_{1+2(\alpha+\beta)})$ and

$$\lim_{t\to-\infty}\left|z^\mu_\psi(t)\right|_{\mathcal{H}_{1+2(\alpha+\beta)}} = 0. \tag{2.54}$$

Proof. According to (2.25), for any $\delta>0$ we have

$$\left|\int_{-\infty}^t S_\mu(t-s)B_\mu(z^\mu_\psi(s))ds\right|_{\mathcal{H}_\delta} \leq \frac{M_\mu}{\mu}\sup_{s\leq t}\left|B(\Pi_1 z^\mu_\psi(s))\right|_{H^{\delta-1}}\int_{-\infty}^t e^{-\omega_\mu(t-s)}ds$$
$$\leq \frac{M_\mu}{\mu\,\omega_\mu}\sup_{s\leq t}\left|B(\Pi_1 z^\mu_\psi(s))\right|_{H^{\delta-1}}.$$

Therefore, due to Hypothesis 4, if we take $\delta = 1$

$$\left|\int_{-\infty}^t S_\mu(t-s)B_\mu(z^\mu_\psi(s))ds\right|_{\mathcal{H}_1} \leq \frac{M_\mu\gamma_0}{\mu\,\omega_\mu}\sup_{s\leq t}\left|\Pi_1 z^\mu_\psi(s)\right|_H. \tag{2.55}$$

For the second term in (2.52), if $\psi\in L^2(-\infty,0;H^{2\alpha})$, then $Q_\mu\psi \in L^2((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$, with

$$|Q_\mu\psi|_{L^2((-\infty,t);\mathcal{H}_{1+2(\alpha+\beta)})} \leq \frac{c}{\mu}\,|\psi|_{L^2((-\infty,t);H^{2\alpha})}, \quad t\leq 0.$$

Due to (2.25), this yields

$$\left|\int_{-\infty}^{t} S_\mu(t-s)Q_\mu\psi(s)ds\right|_{\mathcal{H}_{1+2(\alpha+\beta)}} \leq \frac{M_\mu}{\mu}\left(\int_0^\infty e^{-2\omega_\mu s}\,ds\right)^{1/2} |\psi|_{L^2((-\infty,t);H^{2\alpha})}. \tag{2.56}$$

Therefore, from (2.52), (2.55) and (2.56), we get

$$\left|z_\psi^\mu(t)\right|_{\mathcal{H}_1} \leq c_\mu\left(\sup_{s\leq t}\left|\Pi_1 z_\psi^\mu(s)\right|_H + |\psi|_{L^2((-\infty,t);H^{2\alpha})}\right).$$

In particular, we have $z_\psi^\mu \in L^\infty((-\infty,0);\mathcal{H}_1)$ and

$$\lim_{t\to-\infty}\left|z_\psi^\mu(t)\right|_{\mathcal{H}_1} = 0.$$

Now, by repeating the same arguments, we can prove that for any $n \in \mathbb{N}$, with $n \leq [1+2\beta]$, if

$$z_\psi^\mu \in L^\infty((-\infty,0);\mathcal{H}_n), \quad \text{and} \quad \lim_{t\to-\infty}\left|z_\psi^\mu(t)\right|_{\mathcal{H}_n} = 0,$$

then

$$z_\psi^\mu \in L^\infty((-\infty,0);\mathcal{H}_{n+1}), \quad \text{and} \quad \lim_{t\to-\infty}\left|z_\psi^\mu(t)\right|_{\mathcal{H}_{n+1}} = 0.$$

Since there exists $\bar{n} \in \mathbb{N}$ such that $\mathcal{H}_{1+2(\alpha+\beta)} \supset \mathcal{H}_{\bar{n}}$, we can conclude that z_ψ^μ belongs to $L^\infty((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$ and (2.54) holds. Continuity follows easily, by standard arguments. $\square$

Remark 2.4.4.

1. From the previous lemma, we have that if $z_\psi^\mu \in C((-\infty,0);\mathcal{H})$ solves equation (2.52) and limit (2.53) holds, then $z_\psi^\mu(t) \in \mathcal{H}_{1+2\beta}$, for any $t \leq 0$. In particular $z_\psi^\mu(0) \in \mathcal{H}_{1+2\beta}$.

2. In [6, Lemma 3.5], it has been proven that the same holds for equation (2.22).

Actually, if $\varphi_\psi \in C((-\infty,0);H)$ is the solution to

$$\varphi_\psi(t) = \int_{-\infty}^t e^{(t-s)A} B(\varphi_\psi(s))ds + \int_{-\infty}^t e^{(t-s)A} Q\psi(s)ds,$$

for $\psi \in L^2((-\infty,0);H)$, and

$$\lim_{t\to-\infty} |\varphi_\psi(t)|_H = 0,$$

then $\varphi_\psi \in C((-\infty,0);H^{1+2\beta})$ and there exists a constant such that for all $t \leq 0$,

$$|\varphi_\psi(t)|_{H^{1+2\beta}} \leq c\,|\psi|_{L^2((-\infty,0;H)}. \tag{2.57}$$

Moreover,

$$\lim_{t\to-\infty} |\varphi_\psi(t)|_{H^{1+2\beta}} = 0. \tag{2.58}$$

Lemma 2.4.5. *Let $\alpha \in [0,1/2]$ and let $\psi_1,\psi_2 \in L^2((-\infty,0);H^{2\alpha})$. In correspondence of each ψ_i, let $z^\mu_{\psi_i} \in C((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$ be a solution of equation* (2.52), *verifying* (2.53). *Then, $z^\mu_{\psi_i} \in L^2((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$, for $i=1,2$, and there exist $\mu_0 > 0$ and $c > 0$ such that for any $\mu \leq \mu_0$ and $\tau \leq 0$*

$$|z^\mu_{\psi_1} - z^\mu_{\psi_2}|^2_{L^2((-\infty,\tau);\mathcal{H}_{1+2(\alpha+\beta)})} + \sup_{t\leq\tau} \left|\mathcal{I}_\mu(z^\mu_{\psi_1}(t) - z^\mu_{\psi_2}(t))\right|^2_{\mathcal{H}_{1+2(\alpha+\beta)}} \leq c\,|\psi_1-\psi_2|^2_{L^2((-\infty,\tau);H^{2\alpha})}, \tag{2.59}$$

where $\mathcal{I}_\mu$ is defined in (2.19).

Proof. If we define

$$u(t) = (-A)^{\alpha+\beta}\Pi_1\left(z^\mu_{\psi_1}(t) - z^\mu_{\psi_2}(t)\right), \quad t \leq 0,$$

and

$$\psi(t) = (-A)^{\alpha+\beta}Q(\psi_1(t)-\psi_2(t)), \quad t \leq 0,$$

we have

$$\mu\frac{\partial^2 u}{\partial t^2}(t)+\frac{\partial u}{\partial t}(t)=Au(t)+(-A)^{\alpha+\beta}\left(B(\Pi_1 z^\mu_{\psi_1}(t))-B(\Pi_1 z^\mu_{\psi_2}(t))\right)+\psi(t). \tag{2.60}$$

According to Hypothesis 4, $B : H^{2(\alpha+\beta)} \to H^{2(\alpha+\beta)}$ is Lipschitz-continuous, and then

$$\left|(-A)^{\alpha+\beta}\left(B(\Pi_1 z^\mu_{\psi_1}(t))-B(\Pi_1 z^\mu_{\psi_2}(t))\right)\right|_H=|B(\Pi_1 z^\mu_{\psi_1}(t))-B(\Pi_1 z^\mu_{\psi_2}(t))|_{H^{2(\alpha+\beta)}}$$
$$\leq \gamma_{2(\alpha+\beta)}\,|\Pi_1(z^\mu_{\psi_1}(t))-z^\mu_{\psi_2}(t))|_{H^{2(\alpha+\beta)}}=\gamma_{2(\alpha+\beta)}\,|u(t)|_H.$$

Therefore, by taking the scalar product of both sides with $\partial u/\partial t$, we get

$$\begin{aligned}&\left|\frac{\partial u}{\partial t}(t)\right|^2_H+\frac{\mu}{2}\frac{d}{dt}\left|\frac{\partial u}{\partial t}(t)\right|^2_H+\frac{1}{2}\frac{d}{dt}\left|(-A)^{\frac{1}{2}}u(t)\right|^2_H\\&\leq \gamma_{2(\alpha+\beta)}\,|u(t)|_H\left|\frac{\partial u}{\partial t}(t)\right|_H+|\psi(t)|_H\left|\frac{\partial u}{\partial t}(t)\right|_H.\end{aligned} \tag{2.61}$$

Now, since

$$\gamma_{2(\alpha+\beta)}\,|u(t)|_H\left|\frac{\partial u}{\partial t}(t)\right|_H+|\psi(t)|_H\left|\frac{\partial u}{\partial t}(t)\right|_H\leq\frac{1}{2}\left|\frac{\partial u}{\partial t}(t)\right|^2_H+\gamma^2_{2(\alpha+\beta)}\,|u(t)|^2_H+|\psi(t)|^2_H\,,$$

(2.61) implies

$$\left|\frac{\partial u}{\partial t}(t)\right|^2_H+\mu\frac{d}{dt}\left|\frac{\partial u}{\partial t}(t)\right|^2_H+\frac{d}{dt}\left|(-A)^{\frac{1}{2}}u(t)\right|^2_H\leq 2\gamma^2_{2(\alpha+\beta)}\,|u(t)|^2_H+2\,|\psi(t)|^2_H\,. \tag{2.62}$$

Therefore, integrating this expression with respect to $t\in(-\infty,\tau)$, we obtain

$$\begin{aligned}&\int_{-\infty}^{\tau}\left|\frac{\partial u}{\partial t}(t)\right|^2_H dt+|u(\tau)|^2_{H^1}+\mu\left|\frac{\partial u}{\partial t}(t)\right|^2_H\\&\leq 2\gamma^2_{2(\alpha+\beta)}\int_{-\infty}^{\tau}|u(t)|^2_H\,dt+2\int_{-\infty}^{\tau}|\psi(t)|^2_H\,dt,\end{aligned} \tag{2.63}$$

since, due to Lemma 2.4.3,

$$\int_{-\infty}^{\tau}\frac{d}{dt}\left(\mu\left|\frac{\partial u}{\partial t}(t)\right|^2_H+\left|(-A)^{\frac{1}{2}}u(t)\right|^2_H\right)dt$$
$$=\mu\left|\frac{\partial u}{\partial t}(\tau)\right|^2_H+|u(\tau)|^2_{H^1}-\lim_{T\to-\infty}\left(\mu\left|\frac{\partial u}{\partial t}(T)\right|^2_H+|u(T)|^2_{H^1}\right)=\mu\left|\frac{\partial u}{\partial t}(\tau)\right|^2_H+|u(\tau)|^2_{H^1}.$$

Next we take the inner product of each side of (2.60) with $u(t)$ and use the fact that

$$\left\langle \frac{\partial^2 u}{\partial t^2}(t), u(t) \right\rangle_H = \frac{1}{2}\frac{d^2}{dt^2}\left|u(t)\right|_H^2 - \left|\frac{\partial u}{\partial t}(t)\right|_H^2$$

and again the Lipschitz-continuity of B in $H^{2(\alpha+\beta)}$ to get

$$\begin{aligned}
&\frac{\mu}{2}\frac{d^2}{dt^2}\left|u(t)\right|_H^2 + \frac{1}{2}\frac{d}{dt}\left|u(t)\right|_H^2 + \hat{\gamma}\left|u(t)\right|_{H^1}^2 \leq \mu\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + \langle \psi(t), u(t)\rangle_H \\
&\leq \mu\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + \frac{\hat{\gamma}}{2}\left|u(t)\right|_{H^1}^2 + c\left|\psi(t)\right|_H^2,
\end{aligned}$$

where $\hat{\gamma} := 1 - \gamma_{2(\alpha+\beta)}/\alpha_1 > 0$. This yields

$$\mu\frac{d^2}{dt^2}\left|u(t)\right|_H^2 + \frac{d}{dt}\left|u(t)\right|_H^2 + \hat{\gamma}\left|u(t)\right|_{H^1}^2 \leq 2\mu\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + c\left|\psi(t)\right|_H^2. \tag{2.64}$$

Combining together (2.62) and (2.64), we get

$$\begin{aligned}
&\mu\frac{d^2}{dt^2}\left|u(t)\right|_H^2 + \frac{d}{dt}\left|u(t)\right|_H^2 + \hat{\gamma}\left|u(t)\right|_{H^1}^2 + 2\,\mu^2\frac{d}{dt}\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + 2\mu\frac{d}{dt}\left|u(t)\right|_{H^1}^2 \\
&\leq c_1\,\mu\left|u(t)\right|_H^2 + c_2(1+\mu)\left|\psi(t)\right|_H^2.
\end{aligned}$$

If we take

$$\mu < \frac{\hat{\gamma}\alpha_1}{2c_1},$$

and integrate both sides with respect to $t \in (-\infty, \tau)$, as a consequence of (2.54), we get

$$\frac{1}{2}\int_{-\infty}^{\tau}\left|u(t)\right|_{H^1}^2 dt \leq -2\mu\left\langle u(\tau), \frac{\partial u}{\partial t}(\tau)\right\rangle_H + c_2(1+\mu)\int_{-\infty}^{\tau}\left|\psi(t)\right|_H^2 dt. \tag{2.65}$$

Substituting this back into (2.63), we have

$$\begin{aligned}
&\int_{-\infty}^{\tau}\left(\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + \left|u(t)\right|_{H^1}^2\right)dt + \mu\left|\frac{\partial u}{\partial t}(\tau)\right|_H^2 + \left|u(\tau)\right|_{H^1}^2 dt \\
&\leq -c\,\mu\left\langle u(\tau), \frac{\partial u}{\partial t}(\tau)\right\rangle_H + c\int_{-\infty}^{\tau}\left|\psi(t)\right|_H^2 dt. \\
&\leq c\sqrt{\mu}\left(\mu\left|\frac{\partial u}{\partial t}(\tau)\right|_H^2 + \left|u(\tau)\right|_{H^1}^2\right) + c\int_{-\infty}^{\tau}\left|\psi(t)\right|_H^2 dt
\end{aligned}$$

Therefore, since

$$|\psi(t)|_H \leq c\,|\psi_1(t)-\psi_2(t)|_{H^{2\alpha}},$$

and

$$\left|\mathcal{I}_\mu(z^\mu_{\psi_1}(\tau)-z^\mu_{\psi_2}(\tau))\right|_{\mathcal{H}_{1+2(\alpha+\beta)}} = \mu\left|\frac{\partial u}{\partial t}(\tau)\right|^2_H + |u(\tau)|^2_{H^1}$$

if we choose μ_0 small enough this yields (2.59). □

Remark 2.4.6. 1. Notice that, since $B(0)=0$, we have $z^\mu_0=0$, so that from (2.59) we get

$$|z^\mu_\psi|^2_{L^2((-\infty,\tau);\mathcal{H}_{1+2(\alpha+\beta)})} + \sup_{t\leq\tau}\left|\mathcal{I}_\mu z^\mu_\psi(t)\right|^2_{\mathcal{H}_{1+2(\alpha+\beta)}} \leq c\,|\psi|^2_{L^2((-\infty,\tau);H^{2\alpha})}, \tag{2.66}$$

for any $\mu\leq\mu_0$ and $\tau\leq 0$.

2. By proceeding as in the proof of Lemma 2.4.5, we can prove that

$$|z^\mu_{\psi_1}-z^\mu_{\psi_2}|^2_{L^2((-\infty,\tau);\mathcal{H}_{2\beta})} + \sup_{t\leq\tau}\left|\mathcal{I}_\mu(z^\mu_{\psi_1}(t)-z^\mu_{\psi_2}(t))\right|^2_{\mathcal{H}_{2\beta}} \leq c\,|\psi_1-\psi_2|^2_{L^2((-\infty,\tau);H^{-1})}. \tag{2.67}$$

and

$$|z^\mu_\psi|^2_{L^2((-\infty,\tau);\mathcal{H}_{2\beta})} + \sup_{t\leq\tau}\left|\mathcal{I}_\mu z^\mu_\psi(t)\right|^2_{\mathcal{H}_{2\beta}} \leq c\,|\psi|^2_{L^2((-\infty,\tau);H^{-1})}.$$

2.5 A characterization of the quasi-potential

For any $t_1<t_2$, $\mu>0$ and $z\in C((t_1,t_2);\mathcal{H})$, we define

$$I^\mu_{t_1,t_2}(z) = \frac{1}{2}\inf\left\{|\psi|^2_{L^2((t_1,t_2);H)}\,:\,z=z^\mu_{\psi,z_0}\right\}, \tag{2.68}$$

where z^μ_{ψ,z_0} is a mild solution of the skeleton equation associated with equation (3.12), with deterministic control $\psi\in L^2((t_1,t_2);H)$ and initial conditions z_0, namely

$$\frac{dz^\mu_{\psi,z_0}}{dt}(t) = A_\mu z^\mu_{\psi,z_0}(t) + B_\mu(z^\mu_{\psi,z_0}(t)) + Q_\mu\psi(t), \qquad t_1 \le t \le t_2. \tag{2.69}$$

As in Definition 2.2.4, for $\epsilon, \mu > 0$ and $z_0 \in \mathcal{H}$ we denote by $z^\mu_{\epsilon,z_0} \in L^2(\Omega; C([0,T];\mathcal{H}))$ the mild solution of equation (3.12). Since the mapping $B_\mu : \mathcal{H} \to \mathcal{H}$ is Lipschitz-continuous and the noisy perturbation in (3.12) is of additive type, as an immediate consequence of the contraction lemma, for any fixed $\mu > 0$ the family $\{\mathcal{L}(z^\mu_{\epsilon,z_0})\}_{\epsilon>0}$ satisfies a large deviation principle in $C([t_1,t_2];\mathcal{H})$, with action functional $I^\mu_{t_1,t_2}$. In particular, for any $\delta > 0$ and $T > 0$,

$$\liminf_{\epsilon\to 0} \epsilon \log\left(\inf_{z_0\in\mathcal{H}} \mathbb{P}\left(\left|z^\mu_{\epsilon,z_0} - z^\mu_{\psi,z_0}\right|_{C([0,T];\mathcal{H})} < \delta\right)\right) \ge -\frac{1}{2}|\psi|^2_{L^2((0,T);H)} \tag{2.70}$$

and, if $K^\mu_{0,T}(r) = \{z \in C([0,T];\mathcal{H}) : I^\mu_{0,T}(z) \le r\}$,

$$\limsup_{\epsilon\to 0} \epsilon \log\left(\sup_{z_0\in\mathcal{H}} \mathbb{P}\left(\mathrm{dist}_{\mathcal{H}}(z^\mu_{\epsilon,z_0}, K^\mu_{0,T}(r)) > \delta\right)\right) \le -r. \tag{2.71}$$

Analogously, if for any $\epsilon > 0$ u_ϵ denotes the mild solution of equation (2.22), the family $\{\mathcal{L}(u_\epsilon)\}_{\epsilon>0}$ satisfies a large deviation principle in $C([t_1,t_2];H)$ with action functional

$$I_{t_1,t_2}(\varphi) = \inf\left\{\frac{1}{2}\,|\psi|^2_{L^2([t_1,t_2];H)}\ :\ \varphi = \varphi_\psi\right\}, \tag{2.72}$$

where φ_ψ is a mild solution of the skeleton equation associated with equation (2.22)

$$\frac{du}{dt}(t) = Au(t) + B(u(t)) + Q\psi(t), \qquad t_1 \le t \le t_2.$$

In particular, the functionals $I^\mu_{t_1,t_2}$ and I_{t_1,t_2} are lower semi-continuous and have compact level sets. Moreover, it is not difficult to show that for any compact sets

$E \subset H$ and $\mathcal{E} \subset \mathcal{H}$, the level sets

$$K_{E,t_1,t_2}(r) = \{\varphi \in C([t_1,t_2];H)\ ;\ I_{t_1,t_2}(\varphi) \leq r,\ \varphi(t_1) \in E\}$$

and

$$K^\mu_{\mathcal{E},t_1,t_2}(r) = \left\{z \in C([t_1,t_2];\mathcal{H})\ ;\ I^\mu_{t_1,t_2}(z) \leq r,\ z(t_1) \in \mathcal{E}\right\}$$

are compact.

In what follows, for the sake of brevity, for any $\mu > 0$ and $t \in (0,+\infty]$ we shall define $I^\mu_t := I^\mu_{0,t}$ and $I^\mu_{-t} := I^\mu_{-t,0}$ and, analogously, for any $t \in (0,+\infty]$ we shall define $I_t := I_{0,t}$ and $I_{-t} := I_{-t,0}$. In particular, we shall set

$$I^\mu_{-\infty}(z) = \sup_{t>0} I^\mu_{-t}(z), \qquad I_{-\infty}(\varphi) = \sup_{t>0} I_{-t}(\varphi).$$

Moreover, for any $r > 0$ we shall set

$$K^\mu_{-\infty}(r) = \left\{z \in C((-\infty,0];\mathcal{H})\ ;\ \lim_{t\to-\infty} |z(t)|_{\mathcal{H}} = 0,\ I^\mu_{-\infty}(z) \leq r\right\}$$

and

$$K_{-\infty}(r) = \left\{\varphi \in C((-\infty,0];H)\ ;\ \lim_{t\to-\infty} |\varphi(t)|_H = 0,\ I_{-\infty}(\varphi) \leq r\right\}.$$

Once we have introduced the action functionals $I^\mu_{t_1,t_2}$ and I_{t_1,t_2}, we can introduce the corresponding *quasi-potentials*, by setting for any $\mu > 0$ and $(x,y) \in \mathcal{H}$

$$V^\mu(x,y) = \inf\left\{I^\mu_{0,T}(z)\ ;\ z(0) = 0,\ z(T) = (x,y), T > 0\right\},$$

and for any $x \in H$

$$V(x) = \inf\left\{I_{0,T}(\varphi)\ ;\ \varphi(0) = 0,\ \varphi(T) = x,\ T \geq 0\right\}.$$

Moreover, for any $\mu > 0$ and $x \in H$, we shall define

$$V_\mu(x) = \inf_{y \in H^{-1}} V^\mu(x, y). \tag{2.73}$$

In [6, Proposition 5.1] it has been proved that the level set $K_{-\infty}(r)$ is compact in the space $C((-\infty, 0]; H)$, endowed with the uniform convergence on bounded sets, and in [6, Proposition 5.4] it has been proven that

$$V(x) = \min \left\{ I_{-\infty}(\varphi) \ ; \ \varphi \in C((-\infty, 0]; H), \ \lim_{t \to -\infty} |\varphi(t)|_H = 0, \ \varphi(0) = x \right\}.$$

In what follows we want to prove an analogous result for $K^\mu_{-\infty}$, $V^\mu(x, y)$ and $V_\mu(x)$.

Theorem 2.5.1. *For small enough $\mu > 0$, the level sets $K^\mu_{-\infty}(r)$ are compact in the topology of uniform convergence on bounded intervals.*

Proof. Suppose that z_n is a sequence in $K^\mu_{-\infty}(r)$ where $\mu \leq \mu_0$ and μ_0 is the constant introduced in Lemma 2.4.5. Let c be the constant from that lemma and let

$$\mathcal{E} := \left\{ z \in \mathcal{H} : |C_\mu^{-\frac{1}{2}} z|_{\mathcal{H}} \leq \sqrt{2cr} \right\}$$

By Lemma 2.4.5, $z_n \in K^\mu_{\mathcal{E}, -N, 0}(r)$, for any $N \in \mathbb{N}$. Since $\mathcal{E}$ is compact in $\mathcal{H}$, in view of what we have seen above $K^\mu_{\mathcal{E}, -N, 0}(r) \subset C([-N, 0]; \mathcal{H})$ is compact, for each $N \in \mathbb{N}$. Then, by using a diagonalization procedure, we can find a subsequence of $\{z_n\}$ that converges uniformly to a limit $z^\mu \in C((-\infty, 0]; \mathcal{H})$, uniformly on $[-N, 0]$ for all N. This means that there exist controls ψ_N such that for $t \in [-N, 0]$,

$$z^\mu(t) = S_\mu(t + N) z^\mu(-N) + \int_{-N}^t S_\mu(t - s) B_\mu(z^\mu(s)) ds + \int_{-N}^t S_\mu(t - s) Q_\mu \psi_N(s) ds$$

and

$$\frac{1}{2} |\psi_N|^2_{L^2([-N, 0]; H)} \leq r.$$

All of these ψ_N coincide, because if $\varphi = \Pi_1 z^\mu$ satisfies the above equation,

$$\psi_N(t) = Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) - A\varphi(t) - B(\varphi(t))\right)$$

weakly. Therefore, we can let $\psi = \psi_N$ and notice that

$$\frac{1}{2}|\psi|^2_{L^2((-\infty,0);H)} \leq r$$

This implies that for each $N_0 \in \mathbb{N}$

$$z^\mu(t) = S_\mu(t+N_0)z^\mu(-N_0) + \int_{-N_0}^t S_\mu(t-s)B_\mu(z^\mu(s))ds + \int_{-N_0}^t S_\mu(t-s)Q_\mu\psi(s)ds.$$

Thus, by taking the limit as $N_0 \to +\infty$, we conclude that

$$z^\mu(t) = \int_{-\infty}^t S_\mu(t-s)B_\mu(z^\mu(s))ds + \int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)ds, \qquad t \leq 0.$$

Lastly, we need to show that

$$\lim_{t\to-\infty} |z^\mu(t)|_{\mathcal{H}} = 0.$$

By (2.66), each z_n has the property that

$$|z_n|_{L^2((-\infty,0);\mathcal{H})} \leq c\sqrt{r}.$$

Since $z_n \to z^\mu$ uniformly in $C((-N,0);\mathcal{H})$ for each N,

$$|z^\mu|_{L^2((-\infty,0);\mathcal{H})} = \lim_{N\to+\infty} |z^\mu|_{L^2((-N,0);\mathcal{H})} \leq c\sqrt{r}.$$

Next, by (2.56) and Hypothesis 4,

$$|z^\mu(t)|_{\mathcal{H}_1} = \left|\int_{-\infty}^t S_\mu(t-s)\left(B_\mu(z^\mu(s)) + Q\mu\psi(s)\right)ds\right|_{H_1} \leq c\,|z^\mu|_{L^2((-\infty,t);\mathcal{H})} + c\,|\psi|_{L^2((-\infty,t);H^{-2\beta})}.$$

Because $z^\mu \in L^2((-\infty,0);\mathcal{H})$, and $\psi \in L^2((-\infty,0);H)$,

$$\lim_{t\to-\infty} |z^\mu(t)|_{\mathcal{H}_1} = 0.$$

□

Corollary 2.5.2. *There exists $\mu_0 > 0$ such that for any $\psi \in L^2((-\infty, 0); H)$ and $\mu \leq \mu_0$ there exists $z_\psi^\mu \in C((-\infty, 0]; \mathcal{H})$ such that*

$$z_\psi^\mu(t) = \int_{-\infty}^t S_\mu(t-s)B_\mu(z_\psi^\mu(s))ds + \int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)ds, \qquad t \leq 0, \tag{2.74}$$

Moreover,

$$\lim_{t \to -\infty} |z_\psi^\mu(t)|_{\mathcal{H}} = 0.$$

Proof. A standard fixed point argument shows that for any $\mu > 0$ and $N \in \mathbb{N}$ there exists $z_N^\mu \in C([-N, 0]; \mathcal{H})$ satisfying

$$z_N^\mu(t) = \int_{-N}^t S_\mu(t-s)B_\mu(z_N^\mu(s))ds + \int_{-N}^t S_\mu(t-s)Q_\mu\psi(s)ds.$$

Each z_N^μ can be seen as an element of $C((-\infty, 0]; H)$, just by extending it to $z_N^\mu(t) = 0$, for all $t < -N$. According to Theorem 2.5.1, there exists a subsequence $\{z_{N_k}^\mu\}$ converging to some $z^\mu \in K_{-\infty}^\mu\left(\frac{1}{2}|\psi|^2_{L^2((-\infty,0);H)}\right)$, uniformly on compact sets. We notice that for any fixed $N_0 \in \mathbb{N}$ and $t \geq -N_0$

$$z_N^\mu(t) = S_\mu(t+N_0)z_N^\mu(-N_0) + \int_{-N_0}^t S_\mu(t-s)B_\mu(z_N^\mu(s))ds + \int_{-N_0}^t S_\mu(t-s)Q_\mu\psi(s)ds.$$

Therefore, by taking the limit as $N \to +\infty$, we obtain

$$z^\mu(t) = S_\mu(t+N_0)z^\mu(-N_0) + \int_{-N_0}^t S_\mu(t-s)B_\mu(z^\mu(s))ds + \int_{-N_0}^t S_\mu(t-s)Q_\mu\psi(s)ds.$$

Finally, if we let $N_0 \to +\infty$, we see that z^μ solves equation (2.74).

□

As $K_{-\infty}(r)$ is compact in $C((-\infty, 0]; H)$ with respect to the uniform convergence on bounded intervals, we have analogously that for any $\varphi \in L^2((-\infty, 0)$ there exists $\varphi_\psi \in C((-\infty, 0]; H)$ such that

$$\varphi_\psi(t) = \int_{-\infty}^t e^{(t-s)A} B(\varphi(s)) ds + \int_{-\infty}^t e^{(t-s)A} Q\psi(s) ds,$$

and

$$\lim_{t \to -\infty} |\varphi_\psi(t)|_H = 0.$$

In [6], it has been proved that the $V(x)$ can be characterized as

$$V(x) = \inf \left\{ I_{-\infty}(\varphi) : \lim_{t \to -\infty} \varphi(t) = 0, \varphi(0) = x \right\}.$$

Here, we want to prove that an analogous result holds for $V^\mu(x, y)$ and $V_\mu(x)$, at least for μ sufficiently small.

Theorem 2.5.3. *For small enough $\mu > 0$, we have the following representation for the quasipotentials $V^\mu(x, y)$*

$$V^\mu(x, y) = \min \left\{ I^\mu_{-\infty}(z) : \lim_{t \to -\infty} |z(t)|_{\mathcal{H}} = 0, z(0) = (x, y) \right\}, \tag{2.75}$$

and for $V_\mu(x)$

$$V_\mu(x) = \min \left\{ I^\mu_{-\infty}(z) : \lim_{t \to -\infty} |z(t)|_{\mathcal{H}} = 0, \Pi_1 z(0) = x \right\}, \tag{2.76}$$

whenever these quantities are finite.

Proof. From the definitions of $I^\mu_{t_1, t_2}$, it is clear that

$$V^\mu(x, y) = \inf \left\{ I^\mu_{t_1, 0}(z) : z(t_1) = 0, z(0) = (x, y), t_1 \leq 0 \right\}.$$

Now, if we define

$$M^\mu(x, y) = \inf \left\{ I^\mu_{-\infty}(\varphi) : \lim_{t \to -\infty} |z(t)|_{\mathcal{H}} = 0, z(0) = (x, y) \right\}, \tag{2.77}$$

it is immediate to check that $M^\mu(x,y) \le V^\mu(x,y)$, for any $(x,y) \in \mathcal{H}$. To see this, we observe that if $z \in C([t_1,0];\mathcal{H})$, with $z(t_1)=0$ and $z(0)=(x,y)$, then

$$\hat{z}(t) = \begin{cases} 0, & t \le t_1 \\ \\ z(t), & t_1 < t \le 0 \end{cases} \tag{2.78}$$

has the property that $\hat{z}(0)=(x,y)$, and $|\hat{z}(t)|_{\mathcal{H}} \to 0$, as $t \to -\infty$. Moreover,

$$I^\mu_{-\infty}(\hat{z}) = I^\mu_{t_1,0}(z).$$

Therefore, we need to show that $V^\mu(x,y) \le M^\mu(x,y)$, for all $(x,y) \in \mathcal{H}$.

If $M^\mu(x,y) = +\infty$ there is nothing to prove. So, assume that $M^\mu(x,y) < +\infty$. In view of Theorem 2.5.1, there is a minimizer $z^\mu \in C((-\infty,0];\mathcal{H}_{1+2\beta})$, with $z^\mu(0)=(x,y)$ such that

$$M^\mu(x,y) = I^\mu_{-\infty}(z^\mu).$$

Moreover, thanks to (2.54)

$$\lim_{t \to -\infty} |z^\mu(t)|_{\mathcal{H}_{1+2\beta}} = 0.$$

This means that for $\epsilon > 0$ fixed, there exists $t_\epsilon < 0$ such that

$$|z^\mu(t)|_{\mathcal{H}_{1+2\beta}} < \epsilon, \qquad t \le t_\epsilon.$$

Now, let us denote $z_\epsilon = z^\mu(t_\epsilon)$ and let us define

$$\psi_\epsilon = (L^\mu_{t_\epsilon - T_\mu, t_\epsilon})^{-1} z_\epsilon,$$

where $T_\mu > 0$ is the time introduced in Theorem 2.4.1. Then, by Theorem 2.4.1

$$|\psi_\epsilon|_{L^2((t_\epsilon - T_\mu, t_\epsilon);H)} \le c(\mu, T_\mu)|z_\epsilon|_{\mathcal{H}_{1+2\beta}} \le \epsilon\, c(\mu, T_\mu). \tag{2.79}$$

Next, for $t \in [t_\epsilon - T_\mu, t_\epsilon]$, we define

$$\zeta^\mu_\epsilon(t) = \int_{t_\epsilon - T_\mu}^t S_\mu(t-s) Q_\mu \psi_\epsilon(s) ds.$$

Clearly we have $\zeta^\mu_\epsilon(t_\epsilon - T_\mu) = 0$ and $\zeta^\mu_\epsilon(t_\epsilon) = z_\epsilon$. Moreover, thanks to (2.25), we have

$$|\zeta^\mu_\epsilon(t)|_{\mathcal{H}_{1+2\beta}} \leq \frac{M_\mu}{\mu} \int_{t_\epsilon - T_\mu}^t e^{-\omega_\mu(t-s)} |Q\psi_\epsilon(s)|_{H^{2\beta}}\, ds \leq \frac{cM_\mu}{\mu} \int_{t_\epsilon - T_\mu}^t e^{-\omega_\mu(t-s)} |\psi_\epsilon(s)|_H\, ds,$$

so that, due to (2.79)

$$\begin{aligned}
&\int_{t_\epsilon - T_\mu}^{t_\epsilon} |\zeta^\mu_\epsilon(t)|^2_{\mathcal{H}_{1+2\beta}}\, dt \leq \left(\frac{c\, M_\mu}{\mu}\right)^2 \int_{t_\epsilon - T_\mu}^{t_\epsilon} \left(\int_{t_\epsilon - T_\mu}^t M_\mu e^{-\omega_\mu(t-s)}\, |\psi_\epsilon(s)|_H\, ds\right)^2 dt \\
&\leq \left(\frac{M_\mu}{\omega_\mu \mu}\right)^2 |\psi_\epsilon|^2_{L^2((t_\epsilon - T_\mu, t_\epsilon); H)} \leq \left(\frac{M_\mu}{\omega_\mu \mu}\right)^2 c(\mu, T_\mu)^2 \epsilon^2.
\end{aligned} \tag{2.80}$$

Since

$$\zeta^\mu_\epsilon(t) = \int_{t_\epsilon - T_\mu}^t S_\mu(t-s) B_\mu(z^\mu_\epsilon(s))\, ds + \int_{t_\epsilon - T_\mu}^t S_\mu(t-s) Q_\mu \left(\psi_\epsilon(s) - Q^{-1} B(\Pi_1 z^\mu_\epsilon(s))\right)\, ds,$$

we have

$$I^\mu_{t_\epsilon - T_\mu, t_\epsilon}(\zeta^\mu_\epsilon) \leq 2\, |\psi_\epsilon|^2_{L^2((t_\epsilon - T_\mu, t_\epsilon); H)} + 2\, |Q^{-1} B(\Pi_1 z^\mu_\epsilon)|^2_{L^2((t_\epsilon - T_\mu, t_\epsilon); H)}.$$

Then, as due to Hypothesis 4

$$|Q^{-1} B(\Pi_1 \zeta^\mu_\epsilon(s))|_H \leq c\, |B(\Pi_1 \zeta^\mu_\epsilon(s))|_{H^{2\beta}} \leq c\, \gamma_{2\beta}\, |\Pi_1 \zeta^\mu_\epsilon(s)|_{H^{2\beta}} \leq c\, \gamma_{2\beta}\, |\zeta^\mu_\epsilon(s)|_{\mathcal{H}_{2\beta}},$$

thanks to (2.79) and (2.80), we can conclude

$$I^\mu_{t_\epsilon - T_\mu, t_\epsilon}(\zeta^\mu_\epsilon) \leq c_\mu\, \epsilon^2. \tag{2.81}$$

Finally, we define

$$\hat{\zeta}^\mu_\epsilon(t) = \begin{cases} \zeta^\mu_\epsilon(t), & t_\epsilon - T_\mu \leq t \leq t_\epsilon \\ \\ z^\mu(t), & t > t_\epsilon. \end{cases} \tag{2.82}$$

It is immediate to check that $\hat{\zeta}^\mu_\epsilon \in C([t_\epsilon - T_\mu, 0];\mathcal{H})$, $\hat{\zeta}^\mu_\epsilon = 0$ and $\hat{\zeta}^\mu_\epsilon(0) = (x,y)$. Moreover, thanks to (2.81)

$$I^\mu_{t_\epsilon - T_\mu, 0}(\hat{\zeta}^\mu_\epsilon) \leq I^\mu_{-\infty}(z^\mu) + I^\mu_{t_\epsilon - T_\mu, t_\epsilon}(\zeta^\mu_\epsilon) = M^\mu(x,y) + I^\mu_{t_\epsilon - T, t_\epsilon}(\zeta^\mu_\epsilon) \leq M^\mu(x,y) + c_\mu\,\epsilon^2. \tag{2.83}$$

Due to the arbitrariness of $\epsilon > 0$, this implies

$$V^\mu(x,y) \leq M^\mu(x,y),$$

and then (2.75) follows.

Finally, in order to prove (2.76), we just notice that there exists $\{y_n\} \subset H^{-1}$ such that

$$V_\mu(x) = \lim_{n\to\infty} V^\mu(x, y_n)$$

and

$$V^\mu(x, y_n) = I^\mu_{-\infty}(z_n),$$

for some $\{z_n\} \subset C((-\infty, 0];\mathcal{H})$ such that $z_n(0) = (x, y_n)$ and

$$\lim_{t\to-\infty} |z_n(t)|_{\mathcal{H}} = 0.$$

As

$$\sup_{n\in\mathbb{N}} I^\mu_{-\infty}(z_n) < \infty,$$

due to Theorem 2.5.1 we have that there exists a subsequence $\{z_{n_k}\}$ which is uniformly convergent on bounded sets to some $z \in C((-\infty, 0];\mathcal{H})$. In particular, $\Pi_1 z(0) = x$ and $|z(t)|_{\mathcal{H}} \to 0$, as $t \to -\infty$. Since $I^\mu_{-\infty}$ is lower semi-continuous, we have

$$I^\mu_{-\infty}(z) \leq \liminf_{k\to\infty} I^\mu_{-\infty}(z_{n_k}) = V_\mu(x),$$

and then $V_\mu(x) = I^\mu_{-\infty}(z)$, so that (2.76) holds true. $\square$

The characterization of $V^\mu(x,y)$ and $V_\mu(x)$ given in Theorem 2.5.3, implies that V^μ and V_μ have compact level sets.

Theorem 2.5.4. *For any $\mu > 0$ and $r \geq 0$ the level sets*

$$K^\mu(r) = \{(x,y) \in \mathcal{H} \,:\, V^\mu(x,y) \leq r\}$$

and

$$K_\mu(r) = \{x \in H \,:\, V_\mu(x) \leq r\}$$

are compact, in $\mathcal{H}$ and H, respectively.

Proof. We prove this result for V^μ and K^μ, as the proof for V_μ and K_μ is completely analogous. Let $\{(x_n,y_n)\}_{n\in\mathbb{N}} \subset K^\mu(r)$. In view of Theorem 2.5.3, for each $n \in \mathbb{N}$ there exists $z^n \in C((-\infty,0];\mathcal{H})$, with $z^n(0) = (x_n,y_n)$, and $|z^n(t)|_H \to 0$, as $t \downarrow -\infty$, such that $V^\mu(x_n,y_n) = I^\mu_{-\infty}(z^n)$. As $I^\mu_{-\infty}(z^n) \leq r$ and the level sets of $I^\mu_{-\infty}$ are compact in $C((-\infty,0];\mathcal{H})$, as shown in Theorem 2.5.1, there exists a subsequence $\{z^{n_k}\} \subseteq \{z^n\}$ converging to some $\hat{z} \in C((-\infty,0];\mathcal{H})$, with $I^\mu_{-\infty}(\hat{z}) \leq r$. Since

$$\lim_{k\to\infty}(x_{n_k},y_{n_k}) = \lim_{k\to\infty} z^{n_k}(0) = \hat{z}(0) =: (\hat{x},\hat{y}), \quad \text{in } \mathcal{H},$$

due to Theorem 2.5.3 we have

$$V^\mu(\hat{x},\hat{y}) \leq I^\mu_{-\infty}(\hat{z}) \leq r,$$

so that $(\hat{x},\hat{y}) \in K^\mu(r)$.

$\square$

2.6 Continuity of V^μ and V_μ

As a consequence of Theorem 2.5.4, the mappings $V^\mu : \mathcal{H} \to [0,+\infty]$ and $V_\mu : H \to [0,+\infty]$ are lower semicontinuous. Our purpose here is to prove that the mappings

$$V^\mu : \mathcal{H}_{1+2\beta} \to [0,+\infty), \quad V_\mu : H^{1+2\beta} \to [0,+\infty)$$

are well defined and continuous, uniformly in $0 < \mu < 1$.

Lemma 2.6.1. *Let us fix $(x,y) \in \mathcal{H}_{1+2\beta}$ and $\mu > 0$ and let $z(t) = S_\mu(-t)(x,-y)$, $t \leq 0$. Then, if we denote $\varphi(t) = \Pi_1 z(t)$, we have that φ is a weak solution to*

$$\begin{cases} \mu \dfrac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) + \dfrac{\partial \varphi}{\partial t}(t), \qquad t \leq 0 \\ \varphi(0) = x, \quad \dfrac{\partial \varphi}{\partial t}(0) = y. \end{cases} \tag{2.84}$$

and

$$\frac{1}{2}\int_{-\infty}^0 \left| Q^{-1}\left(\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t)\right)\right|_H^2 dt = \left|(-A)^{\frac{1}{2}} Q^{-1} x\right|_H^2 + \mu \left|Q^{-1} y\right|_H^2. \tag{2.85}$$

Moreover, $\varphi \in L^2((-\infty,0); H^{1+2\beta})$ and

$$\int_{-\infty}^0 |\varphi(t)|^2_{H^{1+2\beta}}\, dt \leq c\,(1+\mu+\mu^2)|(x,y)|^2_{\mathcal{H}_{1+2\beta}}. \tag{2.86}$$

Proof. The weak formulation (2.84) is clear because for $t < 0$

$$\frac{\partial z}{\partial t}(t) = -A_\mu S_\mu(-t)(x,-y) = (-\Pi_2 z(t), -\frac{1}{\mu}A\varphi(t) + \frac{1}{\mu}\Pi_2 z(t)),$$

so that

$$\mu \frac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) + \frac{\partial \varphi}{\partial t}(t).$$

Moreover,

$$\frac{\partial \varphi}{\partial t}(0) = -\Pi_2 z(0) = y.$$

Now, property (2.85) can be proven by noticing that

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{0}\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)+\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)\right)\right|_H^2 dt\\
&=\frac{1}{2}\int_{-\infty}^{0}\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)\right)\right|_H^2 dt\\
&+2\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t), Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-A\varphi(t)\right)\right\rangle_H dt\\
&=\left|Q^{-1}(-A)^{\frac{1}{2}}x\right|_H^2+\mu\left|Q^{-1}y\right|_H^2-\lim_{t\to-\infty}|C_\mu^{-1/2}z(t)|_H^2.
\end{aligned}$$

Then, (2.85) follows from (2.25), as

$$|C_\mu^{-1/2}z(t)|_H \leq |z(t)|_{\mathcal{H}_{1+2\beta}} \leq M_\mu\, e^{-\omega_\mu t}|(x,y)|_{\mathcal{H}_{1+2\beta}} \to 0, \quad \text{as } t\downarrow -\infty.$$

Finally, to obtain estimate (2.86), we notice that if

$$\varphi(t) = \Pi_1 S_\mu(-t)(x,-y)$$

then by (2.39),

$$|\varphi(t)|_{H^{1+2\beta}}^2 = \frac{1}{2}\frac{d}{dt}\left|\varphi(t)-\mu\frac{\partial\varphi}{\partial t}(t)\right|_{H^{2\beta}}^2+\frac{\mu}{2}\frac{d}{dt}\left|\varphi(t)\right|_{H^{1+2\beta}}^2.$$

Integrating, we obtain

$$\int_{-\infty}^{0}|\varphi(t)|_{H^{1+2\beta}}^2\,dt = \frac{1}{2}\left|x+\mu y\right|_{H^{2\beta}}^2+\frac{\mu}{2}\left|x\right|_{H^{1+2\beta}}^2,$$

which yields (2.86).

□

As a consequence of the previous lemma, we obtain the following bounds for $V^\mu(x,y)$ and $V_\mu(x)$.

Corollary 2.6.2. *For any $\mu > 0$ and $(x,y) \in \mathcal{H}_{1+2\beta}$, we have*

$$V^\mu(x,y) \leq c(1+\mu+\mu^2)|(x,y)|_{\mathcal{H}_{1+2\beta}} \tag{2.87}$$

and

$$V_\mu(x) \leq c(1+\mu)\,|x|^2_{H^{1+2\beta}} \tag{2.88}$$

Proof. The proof is based on the fact that

$$V^\mu(x,y) \leq I^\mu_{-\infty}(\Pi_1 S_\mu(-\cdot)(x,-y))$$

and

$$V_\mu(x) \leq I^\mu_{-\infty}(\Pi_1 S_\mu(-\cdot)(x,0)).$$

Now, if we set $z(t) = S_\mu(-t)(x,-y)$ and $\varphi(t) = \Pi_1 z(t)$, due to Hypothesis 4 we have

$$\begin{aligned}
I^\mu_{-\infty}(z) &= \frac{1}{2}\int_{-\infty}^0 \left| Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) - A\varphi(t) - B(\varphi(t))\right)\right|^2_H dt \\
&\leq \int_{-\infty}^0 \left| Q^{-1}\left(\frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) - A\varphi(t)\right)\right|^2_H dt + c\,\gamma^2_{2\beta}\int_{-\infty}^0 |\varphi(t)|^2_{H^{2\beta}}\,dt.
\end{aligned}$$

From (2.85) and (2.86), this give (2.87). Finally, (2.88) is a consequence of (2.87) and of the way $V_\mu(x)$ has been defined. □

Now, we can prove the continuity of V^μ and V_μ.

Theorem 2.6.3. *For each $\mu > 0$ the mappings $V^\mu : \mathcal{H}_{1+2\beta} \to [0,+\infty)$ and $V_\mu : H^{1+2\beta} \to [0,+\infty)$ are well defined and continuous. Moreover,*

$$\lim_{n\to\infty} |(x,y)-(x_n,y_n)|_{\mathcal{H}_{1+2\beta}} = 0 \Longrightarrow \lim_{n\to\infty}\sup_{0<\mu<1} |V^\mu(x,y) - V^\mu(x_n,y_n)| = 0. \tag{2.89}$$

and

$$\lim_{n\to\infty} |x-x_n|_{H^{1+2\beta}} = 0 \Longrightarrow \lim_{n\to\infty}\sup_{0<\mu<1} |V_\mu(x) - V_\mu(x_n)| = 0. \tag{2.90}$$

Proof. In view of Corollary 2.6.2, if $(x,y) \in \mathcal{H}_{1+2\beta}$, then $V^\mu(x,y) < +\infty$ and if $x \in H^{1+2\beta}$, then $V_\mu(x) < +\infty$. On the other hand, if $V^\mu(x,y) < +\infty$, thanks to Theorem 2.5.3 there exists $z^\mu \in C((-\infty,0];\mathcal{H})$ such that

$$V^\mu(x,y) = I^\mu_{-\infty}(z^\mu), \qquad z^\mu(0) = (x,y).$$

According to Lemma 2.4.3, this implies that $z^\mu \in C((-\infty,0];\mathcal{H}_{1+2\beta})$, so that $(x,y) = z^\mu(0) \in \mathcal{H}_{1+2\beta}$. Analogously, if $V_\mu(x) < +\infty$, we can prove that $x \in H^{1+2\beta}$, so that we can conclude that the mappings V^μ and V_μ are well defined in $\mathcal{H}_{1+2\beta}$ and $H^{1+2\beta}$, respectively.

Now, in order to prove (2.89), by using again Theorem 2.5.3, for each $n \in \mathbb{N}$ we can find $z^\mu_n \in C((-\infty,0];\mathcal{H})$ such that

$$V^\mu(x_n,y_n) = I^\mu_{-\infty}(z^\mu_n), \qquad z^\mu_n(0) = (x_n,y_n).$$

Then, if we define

$$\hat{z}^\mu_n(t) = S_\mu(-t)(x-x_n, y-y_n),$$

and

$$\varphi^\mu_n(t) = \Pi_1 z^\mu_n(t), \qquad \hat{\varphi}^\mu_n(t) = \Pi_1 \hat{z}^\mu_n(t), \qquad t \leq 0,$$

we have $\hat{z}^\mu_n(0) = (x-x_n, y-y_n)$ and for any $\epsilon > 0$

$$\begin{aligned}
&V^\mu(x,y) \leq I^\mu_{-\infty}(z^\mu_n + \hat{z}^\mu_n)\\
&\leq \frac{1}{2}\int_{-\infty}^0 \Bigg| Q^{-1}\left(\mu\frac{\partial^2\varphi^\mu_n}{\partial t^2}(t) + \frac{\partial\varphi^\mu_n}{\partial t}(t) - A\varphi^\mu_n(t) + B(\Pi_1\varphi^\mu_n(t))\right)\\
&+Q^{-1}\left(\mu\frac{\partial^2\hat{\varphi}^\mu_n}{\partial t^2}(t) + \frac{\partial\hat{\varphi}^\mu_n}{\partial t}(t) - A\hat{\varphi}^\mu_n(t)\right) + Q^{-1}\left(B(\varphi^\mu_n + \hat{\varphi}^\mu_n(t)) - B(\varphi^\mu_n(t))\right)\Bigg|^2_H dt\\
&\leq (1+\epsilon) I^\mu_{-\infty}(z^\mu_n) + (1+\frac{1}{\epsilon})\int_{-\infty}^0 \left| Q^{-1}\left(\frac{\partial^2\hat{\varphi}^\mu_n}{\partial t^2}(t) + \frac{\partial\hat{\varphi}^\mu_n}{\partial t}(t) - A\hat{\varphi}^\mu_n(t)\right)\right|^2_H dt\\
&+c\,(1+\frac{1}{\epsilon})\int_{-\infty}^0 |\hat{\varphi}^\mu_n(t)|^2_{H^{2\beta}}\,dt.
\end{aligned}$$

Now, by (2.85) and (2.86), we see that for $0 < \mu < 1$

$$
\begin{aligned}
&V_\mu(x,y) \leq (1+\epsilon) V_\mu(x_n,y_n) \\
&+c\,(1+\frac{1}{\epsilon})\,|(x-x_n,y-y_n)|^2_{\mathcal{H}_{1+2\beta}} + c^2(1+\frac{1}{\epsilon})\,|(x-x_n,y-y_n)|_{\mathcal{H}_{2\beta}}\,.
\end{aligned}
$$

If we follow the same procedure with z^μ as the minimizer of $V^\mu(x,y)$ and

$$\hat{z}^\mu_n(t) = S_\mu(-t)(x_n - x, y - y_n),$$

we see that for $0 < \mu < 1$

$$
\begin{aligned}
&V^\mu(x_n,y_n) \leq (1+\epsilon) V^\mu(x,y) \\
&+c\,(1+\epsilon^{-1})\,|(x-x_n,y-y_n)|^2_{\mathcal{H}_{1+2\beta}} + c\,(1+\epsilon^{-1})\,|(x-x_n,y-y_n)|_{\mathcal{H}_{2\beta}}\,.
\end{aligned}
$$

From these two estimates and Corollary 2.6.2, we see that

$$\sup_{0<\mu<1} |V^\mu(x,y) - V^\mu(x_n,y_n)| \leq c\epsilon\,|(x,y)|^2_{\mathcal{H}_{1+2\beta}} + c\,\left(1+\epsilon^{-1}\right)|(x-x_n,y-y_n)|^2_{\mathcal{H}_{1+2\beta}},$$

so that

$$\limsup_{n\to\infty} \sup_{0<\mu<1} |V^\mu(x,y) - V_\mu(x_n,y_n)| \leq c\,\epsilon\,|(x,y)|^2_{\mathcal{H}_{1+2\beta}}\,.$$

Due to the arbitrariness of $\epsilon > 0$, (2.89) follows. The proof of (2.90) is completely analogous to the proof of (2.89) and for this reason we omit it. □

2.7 Upper bound

In this section we show that for any closed set $N \subset H$

$$\limsup_{\mu\downarrow 0} \inf_{x\in N} V_\mu(x) \leq \inf_{x\in N} V(x). \tag{2.91}$$

First of all, we we notice that if $I_{-\infty}(\varphi) < \infty$, then

$$\varphi \in L^2((-\infty,0); H^{2(1+\beta)}), \quad \frac{\partial\varphi}{\partial t} \in L^2((-\infty,0); H^{2\beta}), \tag{2.92}$$

and

$$I_{-\infty}(\varphi) = \frac{1}{2}\int_{-\infty}^{0} \left| Q^{-1}\left(\frac{\partial \varphi}{\partial t}(t) - A\varphi(t) - B(\varphi(t))\right)\right|_H^2 dt. \tag{2.93}$$

Actually, if φ solves

$$\varphi(t) = \int_{-\infty}^{t} e^{(t-s)A} B(\varphi(s))ds + \int_{-\infty}^{t} e^{(t-s)A} Q\psi(s)ds$$

then we can check that (2.92) holds and

$$\psi(t) = Q^{-1}\left(\frac{\partial}{\partial t}\varphi(t) - A\varphi(t) - B(\varphi(t))\right),$$

so that (2.93) follows. Moreover, if

$$\varphi \in L^2((-\infty,0); H^{2(1+\beta)}), \qquad \frac{\partial \varphi}{\partial t}, \frac{\partial^2 \varphi}{\partial t^2} \in L^2((-\infty,0); H^{2\beta}),$$

then

$$I^{\mu}_{-\infty}(z) = \frac{1}{2}\int_{-\infty}^{0} \left| Q^{-1}\left(\mu\frac{\partial^2 \varphi}{\partial t^2}\varphi(t) + \frac{\partial}{\partial t}\varphi(t) - A\varphi(t) - B(\varphi(t))\right)\right|_H^2 dt,$$

where

$$z(t) = (\varphi(t), \frac{\partial \varphi}{\partial t}(t)).$$

Actually, if $I^{\mu}_{-\infty}(z) < \infty$, then z solves

$$z(t) = \int_{-\infty}^{t} S_\mu(t-s) B_\mu(z(s))ds + \int_{-\infty}^{t} S_\mu(t-s) Q_\mu \psi(s)ds$$

so that

$$\psi(t) = Q^{-1}\left(\mu\frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) - B(\varphi(t))\right)$$

weakly.

In particular, as in [10], where the finite dimensional case is studied, this means

$$
\begin{aligned}
I^{\mu}_{-\infty}(z) &= I_{-\infty}(\varphi) + \frac{\mu^2}{2}\int_{-\infty}^{0}\left|Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t)\right|_H^2 dt \\
&+\mu\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t), Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t) - A\varphi(t) - B(\varphi(t))\right)\right\rangle_H dt,
\end{aligned}
\tag{2.94}
$$

where $\varphi(t) = \Pi_1 z(t)$, as long as all of these terms are finite.

Now, for any $\mu > 0$ let us define

$$
\rho_\mu(t) = \frac{1}{\mu^\alpha}\,\rho\left(\frac{t}{\mu^\alpha}\right), \qquad t \in \mathbb{R}, \tag{2.95}
$$

for some $\alpha > 0$ to be chosen later, where $\rho \in C^\infty(\mathbb{R})$ is the usual mollifier function such that

$$
\operatorname{supp}(\rho) \subset\subset [0,2], \quad \int_{\mathbb{R}} \rho(s)ds = 1, \quad 0 \le \rho \le 1.
$$

This scaling ensures that

$$
\int_{\mathbb{R}} \rho_\mu(s)ds = 1.
$$

Next, we define φ_μ as the convolution

$$
\varphi_\mu(t) = \int_{-\infty}^{0} \rho_\mu(t-s)\varphi(s)ds. \tag{2.96}
$$

Lemma 2.7.1. *Assume that*

$$
\varphi \in L^2((-\infty,0); H^{2(1+\beta)}) \cap C((-\infty,0]; H^{1+2\beta}), \quad \frac{\partial\varphi}{\partial t} \in L^2((-\infty,0); H^{2\beta})
$$

with

$$
\varphi(0) = x \in H^{1+2\beta}, \quad \lim_{t\to-\infty} |\varphi(t)|_{H^{1+2\beta}} = 0.
$$

Then,

$$
\varphi_\mu \in L^2((-\infty,0); H^{2(1+\beta)}) \cap C((-\infty,0]; H^{1+2\beta}), \quad \frac{\partial\varphi_\mu}{\partial t} \in L^2((-\infty,0); H^{2\beta}), \tag{2.97}
$$

and

$$\lim_{t\to-\infty}\sup_{\mu>0}|\varphi_\mu(t)|_{H^{1+2\beta}}=0. \tag{2.98}$$

Moreover,

$$\frac{\partial^2\varphi_\mu}{\partial t^2}\in L^2((-\infty,0);H^{2\beta})$$

and for all $\mu>0$,

$$\left|\frac{\partial^2\varphi_\mu}{\partial t^2}\right|_{L^2((-\infty,0);H^{2\beta})}\leq\frac{c}{\mu^\alpha}\left|\frac{\partial\varphi_\mu}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})}. \tag{2.99}$$

Proof. Since we have

$$\varphi_\mu(t)=\int_{t-2\mu^\alpha}^t\rho_\mu(t-s)\varphi(s)ds,$$

it follows

$$\int_{-\infty}^0\left|\int_{t-2\mu^\alpha}^t\rho_\mu(t-s)\varphi(s)ds\right|^2_{H^{2(1+\beta)}}dt\leq\int_{-\infty}^0\left(\int_0^{2\mu^\alpha}\rho_\mu^2(s)ds\right)\left(\int_{t-2\mu^\alpha}^t|\varphi(s)|^2_{H^{2(1+\beta)}}ds\right)dt.$$

Therefore, as

$$\int_0^{2\mu^\alpha}\rho_\mu^2(s)ds\leq\frac{2}{\mu^\alpha},$$

we get

$$\begin{aligned}&|\varphi_\mu|^2_{L^2((-\infty,0);H^{2(1+\beta)})}dt\leq\frac{2}{\mu^\alpha}\int_{-\infty}^0\int_{t-2\mu^\alpha}^t|\varphi(s)|^2_{H^{2(1+\beta)}}dsdt\\&\leq\frac{2\mu^\alpha}{\mu^\alpha}\int_{-\infty}^0|\varphi(s)|^2_{H^{2(1+\beta)}}ds=2\,|\varphi|^2_{L^2((-\infty,0);H^{2(1+\beta)})}.\end{aligned} \tag{2.100}$$

Next, since

$$\lim_{t\to-\infty}|\varphi(t)|_{H^{1+2\beta}}=0,$$

we have that $\varphi:(-\infty,0]\to H^{1+2\beta}$ is uniformly continuous. Therefore, as

$$\begin{aligned}&\left|\int_{-\infty}^{t_1}\rho_\mu(t_1-s)\varphi(s)ds-\int_{-\infty}^{t_2}\rho_\mu(t_2-s)\varphi(s)ds\right|_{H^{1+2\beta}}\\&=\left|\int_0^\infty\rho_\mu(s)\left(\varphi(t_1-s)-\varphi(t_2-s)\right)ds\right|_{H^{1+2\beta}},\end{aligned}$$

we can conclude that φ_μ is uniformly continuous too, with values in $H^{1+2\beta}$.

Finally, since

$$\frac{\partial \varphi_\mu}{\partial t}(t) = \int_0^\infty \rho_\mu(s)\frac{\partial \varphi}{\partial t}(t-s)ds,$$

by proceeding as above we get

$$\frac{\partial \varphi_\mu}{\partial t} \in L^2((-\infty,0);H^{2\beta}),$$

so that, thanks to (2.100), we can conclude that (2.97) holds true.

Concerning (2.98), let us fix $\epsilon > 0$. Then there exists $T_\epsilon > 0$ such that

$$|\varphi(t)|_{H^{1+2\beta}} < \epsilon, \quad t < -T_\epsilon.$$

Then, for $t < -T_\epsilon$, we have

$$\left|\int_{-\infty}^t \rho_\mu(t-s)\varphi(s)ds\right|_{H^{1+2\beta}} \le \int_{-\infty}^t \rho_\mu(t-s)\,|\varphi(s)|_{H^{1+2\beta}}\,ds \le \epsilon\int_0^\infty \rho_\mu(s)ds = \epsilon.$$

and this yields (2.98).

Finally, let us prove (2.99). As

$$\frac{\partial \varphi_\mu}{\partial t}(t) = \int_{-\infty}^0 \rho_\mu(t-s)\frac{\partial \varphi}{\partial s}(s)ds,$$

we have

$$\frac{\partial^2 \varphi_\mu}{\partial t^2}(t) = \int_{-\infty}^0 \frac{d}{dt}\rho_\mu(t-s)\frac{\partial \varphi}{\partial s}(s)ds = \frac{1}{\mu^{2\alpha}}\int_{-\infty}^0 \rho'\left(\frac{t-s}{\mu^\alpha}\right)\frac{\partial \varphi}{\partial s}(s)ds.$$

This yields

$$\begin{aligned}
&\int_{-\infty}^0 \left|\frac{\partial^2 \varphi_\mu}{\partial t^2}(t)\right|_{H^{2\beta}}^2 dt = \frac{1}{\mu^{4\alpha}}\int_{-\infty}^0 \left|\int_{t-2\mu^\alpha}^t \rho'\left(\frac{t-s}{\mu^\alpha}\right)\frac{\partial}{\partial s}\varphi(s)ds\right|_{H^{2\beta}}^2 dt\\
&\le \frac{1}{\mu^{4\alpha}}\int_{-\infty}^0 \left(\int_{t-2\mu^\alpha}^t \left(\rho'\left(\frac{t-s}{\mu^\alpha}\right)\right)^2 ds\right)\left(\int_{t-2\mu^\alpha}^t \left|\frac{\partial \varphi}{\partial s}(s)ds\right|_{H^{2\beta}}^2 ds\right)dt\\
&\le \frac{2|\rho'|_\infty^2}{\mu^{3\alpha}}\int_{-\infty}^0\int_{t-2\mu^\alpha}^t \left|\frac{\partial \varphi}{\partial s}(s)\right|_{H^{2\beta}}^2 dsdt \le \frac{c}{\mu^{2\alpha}}\int_{-\infty}^0 \left|\frac{\partial \varphi}{\partial s}(s)\right|_{H^{2\beta}}^2 ds.
\end{aligned}$$

□

The following approximation results hold.

Lemma 2.7.2. *Under the same assumptions of Lemma 2.7.1, we have*

$$\lim_{\mu\to 0} |x - \varphi_\mu(0)|_{H^{1+2\beta}} = 0, \tag{2.101}$$

and

$$\lim_{\mu\to 0} \sup_{t\leq 0} |\varphi_\mu(t) - \varphi(t)|_{H^{1+2\beta}} = 0. \tag{2.102}$$

Moreover,

$$\lim_{\mu\to 0} |\varphi_\mu - \varphi|_{L^2((-\infty,0);H^{2(1+\beta)})} = 0, \tag{2.103}$$

and

$$\lim_{\mu\to 0} \left|\frac{\partial \varphi_\mu}{\partial t} - \frac{\partial \varphi}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})} = 0. \tag{2.104}$$

Proof. We have

$$\varphi_\mu(0) - x = \int_{-\infty}^0 \rho_\mu(-s)(\varphi(s) - \varphi(0))ds,$$

so that, by the continuity of φ in $H^{1+2\beta}$, (2.101) follows.

In order to prove (2.102), we have

$$|\varphi_\mu(t) - \varphi(t)|_{H^{1+2\beta}} \leq \int_{-\infty}^t \rho_\mu(t-s)\, |\varphi(s) - \varphi(t)|_{H^{1+2\beta}}\, ds.$$

Now, as $\varphi : (-\infty, 0] \to H^{1+2\beta}$ is uniformly continuous, for any fixed $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that. We use the uniform continuity of φ to find $\delta_\epsilon > 0$ such that

$$|t - s| < \delta_\epsilon \Longrightarrow |\varphi(s) - \varphi(t)|_{H^{1+2\beta}} < \frac{\epsilon}{2}.$$

Then if we pick μ small enough so that $\mu^\alpha < \delta_\epsilon/2$,

$$|\varphi_\mu(t) - \varphi(t)|_{H^{1+2\beta}} \leq \int_{-\infty}^t \rho_\mu(t-s)\, |\varphi(s) - \varphi(t)|_{H^{1+2\beta}}\, ds \leq \int_{t-2\mu^\alpha}^t \frac{1}{\mu^\alpha}\frac{\epsilon}{2} = \epsilon,$$

uniformly in t. This proves (2.102).

Limit (2.103) can be proved using the fact that

$$|\varphi_\mu - \varphi|_{L^2((-\infty,0);H^{2(1+\beta)})} = \sup_{|h|_{L^2((-\infty,0);H)}\leq 1} \int_{-\infty}^{0} \left\langle (-A)^{1+\beta} \left((\varphi_\mu)(t) - \varphi(t) \right), h(t) \right\rangle_H dt$$

$$= \sup_{|h|_{L^2((-\infty,0);H)}\leq 1} \int_{-\infty}^{0} \int_0^{2\mu^\alpha} \rho_\mu(s) \left\langle (-A)^{1+\beta} \left(\varphi(t-s) - \varphi(t) \right), h(t) \right\rangle_H dsdt$$

$$\leq \int_0^{2\mu^\alpha} \rho_\mu(s) \left| \varphi(\cdot - s) - \varphi(\cdot) \right|_{L^2((-\infty,0);H^{2(1+\beta)})} ds.$$

Because translation is continuous in L^2, this converges to 0 as $\mu \downarrow 0$. The same argument will show that (2.104) holds true.

□

Using these estimates we can prove the main result of this section.

Theorem 2.7.3. *For any $x \in H^{1+2\beta}$ we have*

$$\limsup_{\mu \downarrow 0} V_\mu(x) \leq V(x). \tag{2.105}$$

Proof. Let φ be the minimizer of $V(x)$. This means $\varphi(0) = x$, (2.93) holds and $I_{-\infty}(\varphi) = V(x)$. For each $\mu > 0$, let φ_μ be the convolution given by (2.96) and let $x_\mu = \varphi_\mu(0)$.

It is clear that

$$V_\mu(x_\mu) \leq I^\mu_{-\infty}(z_\mu), \tag{2.106}$$

where

$$z_\mu(t) = (\varphi_\mu(t), \frac{\partial \varphi_\mu}{\partial t}(t)), \quad t \leq 0.$$

According to Lemma 2.7.1, we can apply (2.94) and we have

$$\begin{aligned}
&I^{\mu}_{-\infty}(z_\mu) \leq \frac{\mu^2}{2}\int_{-\infty}^{0}\left|\frac{\partial^2\varphi_\mu}{\partial t^2}(t)\right|^2_{H^{2\beta}} dt + I_{-\infty}(\varphi_\mu)\\
&+\mu\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial^2\varphi_\mu}{\partial t^2}(t), Q^{-1}\left(\frac{\partial\varphi_\mu}{\partial t}(t) - A\varphi_\mu(t) - B(\varphi_\mu(t))\right)\right\rangle_H dt\\
&\leq \frac{\mu^2}{2}\int_{-\infty}^{0}\left|\frac{\partial^2\varphi_\mu}{\partial t^2}(t)\right|^2_{H^{2\beta}} + I_{-\infty}(\varphi_\mu) + \mu\left(\int_{-\infty}^{0}\left|\frac{\partial^2\varphi_\mu}{\partial t^2}(t)\right|^2_{H^{2\beta}} dt\right)^{1/2}(I_{-\infty}(\varphi_\mu))^{1/2}.
\end{aligned}$$

By (2.99), this implies

$$I^{\mu}_{-\infty}(z_\mu) \leq I_{-\infty}(\varphi_\mu) + c\mu^{2-2\alpha}\left|\frac{\partial\varphi}{\partial t}\right|^2_{L^2((-\infty,0);H^{2\beta})} + c\mu^{1-\alpha}\left|\frac{\partial\varphi}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})}(I_{-\infty}(\varphi_\mu))^{1/2},$$

and by (2.103) and (2.104)

$$\lim_{\mu\downarrow 0} I_{-\infty}(\varphi_\mu) = I_{-\infty}(\varphi) = V(x).$$

Therefore, if we pick $\alpha < 1$ in (2.95), we get

$$\limsup_{\mu\downarrow 0} V_\mu(x_\mu) \leq \limsup_{\mu\downarrow 0} I^{\mu}_{-\infty}(z_\mu) \leq V(x). \tag{2.107}$$

Since, in view of (2.101) and Theorem 2.6.3,

$$\limsup_{\mu\downarrow 0} V_\mu(x_\mu) = \limsup_{\mu\downarrow 0} V_\mu(x)$$

we can conclude that (2.105) holds. □

Corollary 2.7.4. *For any closed set $N \subset H$,*

$$\limsup_{\mu\to 0}\inf_{x\in N} V_\mu(x) \leq \inf_{x\in N} V(x) \tag{2.108}$$

Proof. If $\inf_{x\in N} V(x) = +\infty$ then the theorem is trivially true. So we assume that this is not the case. Then by the compactness of the level sets of V and the closedness of

N, there exists $x_0 \in N$ such that $V(x_0) = \inf_{x \in N} V(x)$. By (2.105), we can conclude, as

$$\limsup_{\mu \to 0} \inf_{x \in N} V_\mu(x) \leq \limsup_{\mu \downarrow 0} V_\mu(x_0) \leq V(x_0) = \inf_{x \in N} V(x).$$

□

2.8 Lower bound

Let $N \subset H$ be a closed set with $N \cap H^{1+2\beta} \neq \emptyset$. In particular, by Theorem 2.6.3 we have $\inf_{x \in N} V_\mu(x) < +\infty$. Due to (2.76) and Theorem 2.5.1, there exists $z^\mu \in C((-\infty, 0]; \mathcal{H})$ such that

$$x^\mu := \Pi_1 z^\mu(0) \in N, \qquad I^\mu_{-\infty}(z^\mu) = V_\mu(x^\mu) = \inf_{x \in N} V_\mu(x).$$

Now, let $\psi^\mu \in L^2((-\infty, 0); H)$ be the minimal control such that

$$z^\mu(t) = \int_{-\infty}^t S_\mu(t-s) B_\mu(z^\mu(s)) ds + \int_{-\infty}^t S_\mu(t-s) Q_\mu \psi^\mu(s) ds,$$

and

$$\inf_{x \in N} V_\mu(x) = V_\mu(x^\mu) = \frac{1}{2} |\psi^\mu|^2_{L^2((-\infty,0);H)} . \tag{2.109}$$

In what follows, we shall denote $y^\mu = \Pi_2 z^\mu(0)$. For any $\delta > 0$, we define the approximate control

$$\psi^{\mu,\delta}(t) = (I - \delta A)^{-\frac{1}{2}} \psi^\mu(t), \quad t \leq 0,$$

and in view of Corollary 2.5.2 we can define $z^{\mu,\delta}$ to be the solution to the corresponding control problem

$$z^{\mu,\delta}(t) = \int_{-\infty}^t S_\mu(t-s) B_\mu(z^{\mu,\delta}(s)) ds + \int_{-\infty}^t S_\mu(t-s) Q_\mu \psi^{\mu,\delta}(s) ds.$$

Notice that, according to (2.54),

$$\lim_{t\to-\infty}|z^{\mu,\delta}|_{\mathcal{H}_{1+2\beta}}=0.$$

Moreover, as $\psi^{\mu,\delta}\in L^2((-\infty,0);H^1)$, thanks to (2.54) we have

$$\lim_{t\to-\infty}|z^{\mu,\delta}|_{\mathcal{H}_{2(1+\beta)}}=0.$$

In what follows, we shall denote $(x^{\mu,\delta},y^{\mu,\delta})=z^{\mu,\delta}(0)$.

Lemma 2.8.1. *There exists $\mu_0>0$ such that*

$$\lim_{\delta\to 0}\sup_{\mu\leq\mu_0}\left|x^\mu-x^{\mu,\delta}\right|^2_{H^{2\beta}}=0. \tag{2.110}$$

Proof. By (2.67), there exists $\mu_0>0$ such that for $\mu<\mu_0$

$$|x^\mu-x^{\mu,\delta}|_{H^{2\beta}}\leq c\,|\psi^\mu-\psi^{\mu,\delta}|_{L^2((-\infty,0);H^{-1})}.$$

Now, since for any $h\in H$

$$\left|(-A)^{-\frac12}(I-\delta A)^{-\frac12}h-(-A)^{-\frac12}h\right|^2_H=\sum_{k=1}^\infty\frac{1}{\alpha_k}\left(1-\frac{1}{(1+\delta\alpha_k)^{\frac12}}\right)^2h_k^2,$$

and

$$\left(1-\frac{1}{(1+\delta\alpha_k)^{\frac12}}\right)^2\leq\alpha_k\delta,$$

we have

$$\left|(-A)^{-\frac12}(I-\delta A)^{-\frac12}h-(-A)^{-\frac12}h\right|^2_H\leq\delta|h|^2_H.$$

This implies

$$|x^\mu-x^{\mu,\delta}|^2_{H^{2\beta}}\leq c\,\delta\int_{-\infty}^0|\psi^\mu(s)|^2_H\,ds=c\,\delta\inf_{x\in N}V_\mu(x).$$

In Corollary 2.7.4 we have proved

$$\limsup_{\mu\downarrow 0}\inf_{x\in N} V_\mu(x) \le \inf_{x\in N} V(x),$$

and then we obtain

$$\sup_{\mu\le\mu_0} |x^\mu - x^{\mu,\delta}|_{H^{2\beta}} \le c\sqrt{\delta} \tag{2.111}$$

which implies (2.110). □

Now we can prove the main result of this section.

Theorem 2.8.2. *For any closed $N \subset H$, we have*

$$\inf_{x\in N} V(x) \le \liminf_{\mu\downarrow 0}\inf_{x\in N} V_\mu(x). \tag{2.112}$$

Proof. If the right hand side in (2.112) is infinite, the theorem is trivially true. Therefore, in what follows we can assume that

$$\liminf_{\mu\to 0}\inf_{x\in N} V_\mu(x) < +\infty. \tag{2.113}$$

We first observe that, if we define

$$\varphi^{\mu,\delta}(t) = \Pi_1 z^{\mu,\delta}(t), \quad t \le 0,$$

in view of (2.94)

$$\begin{aligned} V(x^{\mu,\delta}) \le I_{-\infty}(\varphi^{\mu,\delta}) = I^\mu_{-\infty}(z^{\mu,\delta}) - \frac{\mu^2}{2}\int_{-\infty}^0 \left| Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t)\right|_H^2 dt \\ -\mu\int_{-\infty}^0 \left\langle Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t), Q^{-1}\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t) - Q^{-1}A\varphi^{\mu,\delta}(t) - Q^{-1}B(\varphi^{\mu,\delta}(t))\right\rangle_H dt. \end{aligned} \tag{2.114}$$

Since

$$|\psi^{\mu,\delta}(t)|_H = |(I-\delta A)^{-1/2}\psi^\mu(t)|_H \le |\psi^\mu(t)|_H, \quad t \le 0, \tag{2.115}$$

we have

$$I^{\mu}_{-\infty}(z^{\mu,\delta}) \leq I^{\mu}_{-\infty}(z^{\mu}) = \inf_{x\in N} V_\mu(x),$$

so that

$$\begin{aligned}&V(x^{\mu,\delta}) \leq \inf_{x\in N} V_\mu(x)\\&-\mu\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t), Q^{-1}\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t) - Q^{-1}A\varphi^{\mu,\delta}(t) - Q^{-1}B(\varphi^{\mu,\delta}(t))\right\rangle_H dt.\end{aligned}$$

Thanks to (2.54) and Hypothesis 4, by integrating by parts

$$\begin{aligned}&V(x^{\mu,\delta}) \leq \inf_{x\in N} V_\mu(x)\\&-\frac{\mu}{2}|Q^{-1}y^{\mu,\delta}|^2_H - \mu\left\langle(-A)Q^{-1}x^{\mu,\delta}, Q^{-1}y^{\mu,\delta}\right\rangle_H + \mu\left\langle Q^{-1}B(x^{\mu,\delta}), Q^{-1}y^{\mu,\delta}\right\rangle_H\\&+c\,\mu\int_{-\infty}^{0}\left|\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)\right|^2_{H^{1+2\beta}}dt + c\,\gamma_{2\beta}\mu\int_{-\infty}^{0}\left|\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)\right|^2_{H^{2\beta}}dt = \inf_{x\in N} V_\mu(x) + \sum_{i=1}^{5} I_i^{\mu,\delta}.\end{aligned} \tag{2.116}$$

First, we note that

$$I_1^{\mu,\delta} \leq 0. \tag{2.117}$$

Next, by (2.66) see that

$$\begin{aligned}I_2^{\mu,\delta} + I_4^{\mu,\delta} &\leq c\sqrt{\mu}\left(|x^{\mu,\delta}|^2_{H^{2\beta+2}} + \mu|y^{\mu,\delta}|^2_{H^{2\beta+1}}\right) + c\mu\int_{-\infty}^{0}|z^{\mu,\delta}(t)|^2_{\mathcal{H}^{2+2\beta}}dt\\&\leq c(\mu+\sqrt{\mu})\int_{-\infty}^{0}|\psi^{\mu,\delta}(t)|^2_{H^1}dt.\end{aligned}$$

Since for any $h \in H$ we have $(I-\delta A)^{-\frac{1}{2}}h \in D(-A)^{\frac{1}{2}}$ and

$$\left|(-A)^{\frac{1}{2}}(I-\delta A)^{-\frac{1}{2}}h\right|_H \leq \delta^{-1/2}\,|h|_H\,,$$

we have

$$\left|\psi^{\mu,\delta}(t)\right|_{H^1} \leq \delta^{-1/2}\,\left|\psi^{\mu}(t)\right|_H\,, \quad t \leq 0.$$

Therefore, by (2.109),

$$I_2^{\mu,\delta} + I_4^{\mu,\delta} \leq c\,\delta^{-1/2}(\mu+\sqrt{\mu})\int_0^t|\psi^{\mu}(t)|^2_H = 2\,c\delta^{-1/2}(\mu+\sqrt{\mu})\inf_{x\in N} V_\mu(x). \tag{2.118}$$

By the same arguments, (2.66), and (2.115) give

$$I_3^{\mu,\delta} + I_5^{\mu,\delta} \leq c(\mu + \sqrt{\mu}) \inf_{x \in N} V_\mu(x). \tag{2.119}$$

Combining together (2.117), (2.118), and (2.119) with (2.116), we obtain,

$$V(x^{\mu,\delta}) \leq \inf_{x \in N} V_\mu(x) + c(\mu + \sqrt{\mu})(1 + \delta^{-1/2}) \inf_{x \in N} V_\mu(x). \tag{2.120}$$

From this, due to (2.113) we see that

$$\liminf_{\mu \to 0} V(x^{\mu,\sqrt{\mu}}) \leq \liminf_{\mu \to 0} \inf_{x \in N} V_\mu(x).$$

Since we are assuming (2.113), and, by [6, Proposition 5.1], the level sets of V are compact, there is a sequence $\mu_n \to 0$ and $x^0 \in H$ such that

$$\lim_{n \to \infty} |x^{\mu_n,\sqrt{\mu_n}} - x^0|_H = 0, \quad V(x^0) \leq \liminf_{\mu \to 0} V(x^{\mu,\sqrt{\mu}}).$$

By (2.110), we have that x^{μ_n} converges to x^0 in H, so that $x_0 \in N$. This means that we can conclude, as

$$\inf_{x \in N} V(x) \leq V(x^0) \leq \liminf_{\mu \to 0} V(x^{\mu,\sqrt{\mu}}) \leq \liminf_{\mu \to 0} \inf_{x \in N} V_\mu(x).$$

□

2.9 Application to the exit problem

In this section we study the problem of the exit of the solution u_ϵ^μ of equation (2.1) from a domain $G \subset H$, for any $\mu > 0$ fixed. Then we apply the limiting results proved in Theorems 2.7.3 and 2.8.2 to show that, when μ is small, the relevant quantities in the exit problem from G for the solution u^μ_ϵ of equation (2.1) can be approximated by the corresponding ones arising for equation (2.2).

First, let us give some assumptions on the set G.

Hypothesis 3. *The domain $G \subset H$ is an open, bounded, connected set, such that $0 \in G$. Moreover, for any $x \in \partial G \cap H^{1+2\beta}$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \bar{G}^c \cap H^{1+2\beta}$ such that*

$$\lim_{n\to+\infty} |x_n - x|_{\mathcal{H}_{1+2\beta}} = 0. \tag{2.121}$$

Example 2.9.1. *Assume now that G is an open, bounded and connected set such that, for any $x \in \partial G \cap H^{1+2\beta}$, there exists a $y \in \bar{G}^c \cap H^{1+2\beta}$ such that*

$$\{ty + (1-t)x : 0 < t \leq 1\} \subset \bar{G}^c. \tag{2.122}$$

Then it is immediate to check that (2.121) *is satisfied. Condition* (2.122) *is true, for example, if G is convex, because of the Hahn-Banach separation theorem and the density of $H^{1+2\beta}$ in H.*

Lemma 2.9.2. *Under Hypothesis 5*

$$V_\mu(\partial G) := \inf_{x\in\partial G} V_\mu(x) = V_\mu(x_{G,\mu}) < \infty, \tag{2.123}$$

for some $x_{G,\mu} \in \partial G \cap \mathcal{H}_{1+2\beta}$.

Proof. Since $\bar{G}^c$ is an open set, there exists $\tilde{x} \in \bar{G}^c \cap H^{1+2\beta}$. Because $0 \in G$, and the path $t \mapsto t\tilde{x}$ is continuous, there must exist $0 < t_0 < 1$ such that $t_0\tilde{x} \in \partial G$. Clearly, $t_0\tilde{x} \in H^{1+2\beta}$, so that, as $\partial G \cap H^{1+2\beta} \neq \emptyset$, according to Theorem 2.6.3

$$\inf_{x\in\partial G} V_\mu(x) < \infty.$$

Moreover, thanks to Theorem 2.5.4, the first equality in (2.123) implies that there exists $x_{G,\mu} \in \partial G \cap \mathcal{H}_{1+2\beta}$ such that

$$V_\mu(x_{G,\mu}) = V_\mu(\partial G). \tag{2.124}$$

$\square$

Now, if we denote by $z^\mu_{\epsilon,z_0} = (u^\mu_{\epsilon,z_0}, v^\mu_{\epsilon,z_0})$ the mild solution of (3.12), with initial position and velocity $z_0 = (u_0, v_0) \in \mathcal{H}$, we define the exit time

$$\tau^{\mu,\epsilon}_{z_0} = \inf\left\{t > 0 : u^\mu_{\epsilon,z_0}(t) \notin G\right\}. \tag{2.125}$$

The following theorem is the main result of this section.

Theorem 2.9.3. *There exists $\mu_0 > 0$ such that for $\mu < \mu_0$ the following conditions are verified. For any $z_0 = (u_0, v_0) \in \mathcal{H}$ such that $u_0 \in G$ and the unperturbed system has the property that $u^\mu_{0,z_0}(t) \in G$, for $t \geq 0$,*

1. *The exit time has the following asymptotic growth*

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left(\tau^{\mu,\epsilon}_{z_0}\right) = \inf_{x\in\partial G} V_\mu(x), \tag{2.126}$$

and for any $\eta > 0$,

$$\lim_{\epsilon\to 0} \epsilon \log\left(\tau^{\mu,\epsilon}_{z_0}\right) = \inf_{x\in\partial G} V_\mu(x) \text{ in probability.} \tag{2.127}$$

2. *For any closed $N \subset \partial G$ such that $\inf_{x\in N} V_\mu(x) > \inf_{x\in\partial G} V_\mu(x)$, it holds*

$$\lim_{\epsilon\to 0} \mathbb{P}\left(u^\mu_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0}) \in N\right) = 0. \tag{2.128}$$

Remark 2.9.4. The requirement that $u^{\mu}_{0,z_0}(t) \in G$ for all $t \geq 0$ is necessary because in Lemma 2.3.4 we showed that there exist $z_0 \in G \times H^{-1}$ such that u^{μ}_{0,z_0} leaves G in finite time. Of course, for these initial conditions, the stochastic processes u^{μ}_{ϵ,z_0} will also exit in finite time for small ϵ.

In [5] it has been proven that an analogous result to Theorem 2.9.3 holds for equation (2.22). If we denote by u_{ϵ,u_0} the mild solutions of equation (2.22), with initial condition $u_0 \in H$, we define the exit time

$$\tau^{\epsilon}_{u_0} = \inf \left\{ t > 0 : u_{\epsilon,u_0}(t) \notin G \right\}.$$

In [5] it has been proven that for any $u_0 \in G$ such that $u_{0,u_0}(t) \in G$, for any $t \geq 0$, it holds

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\left(\tau^{\epsilon}_{u_0}\right) = \inf_{x \in \partial G} V(x).$$

Similarly, as we would expect, it also holds that

$$\lim_{\epsilon \to 0} \epsilon \log \tau^{\epsilon}_{u_0} = \inf_{x \in \partial G} V(x), \quad \text{in probability,}$$

and if $N \subset \partial G$ is closed and $\inf_{x \in N} V(x) > \inf_{x \in \partial G} V(x)$,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(u^{\epsilon}_{u_0}(\tau^{\epsilon}_{u_0}) \in N\right) = 0.$$

The proof of these facts is analogous to the proof of Theorem 2.9.3.

In view of what we have proven in Sections 2.7 and 2.8 and of Theorem 2.9.3, this implies that the following Smoluchowski-Kramers approximations holds for the exit time.

Theorem 2.9.5. *For any initial conditions $z_0 = (u_0, v_0)$ such that the unperturbed system $u^{\mu}_{0,z_0}(t) \in G$ for all $t \geq 0$,* *1.*

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0}\epsilon\log\,\mathbb{E}\left(\tau^{\mu,\epsilon}_{z_0}\right)=\lim_{\epsilon\to 0}\epsilon\log\,\mathbb{E}\left(\tau^{\epsilon}_{u_0}\right)=\inf_{x\in\,\partial G}V(x). \tag{2.129}$$

2.

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0}\epsilon\log\left(\tau^{\mu,\epsilon}_{z_0}\right)=\lim_{\epsilon\to 0}\epsilon\log\left(\tau^{\mu,\epsilon}_{z_0}\right)=\inf_{x\in G}V(x)\ \textit{in probability.} \tag{2.130}$$

3. For any $N\subset\partial G$ such that $\inf_{x\in N}V(x)<\inf_{x\in\partial G}V(x)$, there exits $\mu_0>0$ such that for all $\mu<\mu_0$,

$$\lim_{\epsilon\to 0}\mathbb{P}\left(u^{\mu}_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0})\in N\right)=0. \tag{2.131}$$

4. If there exists a unique $\tilde{x}\in\partial G$ such that $V(\tilde{x})=\inf_{x\in\partial G}V(x)$, then

$$\lim_{\mu\to 0}\lim_{\epsilon\to 0}u^{\mu}_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0})=\lim_{\epsilon\to 0}u_{\epsilon,z_0}(\tau^{\epsilon}_{z_0})=\tilde{x}. \tag{2.132}$$

The above theorem demonstrates that the exponential divergence rates of the exit times and the exit place of the wave equation are accurately approximated by studying exit problems for the heat equation.

2.9.1 Proof to Theorem 2.9.3

In order to prove Theorem 2.9.3, we will need some preliminary lemmas, whose proofs are postponed to the next subsection.

Lemma 2.9.6. *For $\mu<(\alpha_1-\gamma_0)\gamma_0^{-2}$, there exists a constant $c(\mu)>0$ such that $z_1,z_2\in\mathcal{H}$*

$$\sup_{\psi\in\,L^2((0,+\infty);H)}\ \sup_{t\geq 0}\left|z^{\mu}_{\psi,z_1}(t)-z^{\mu}_{\psi,z_2}(t)\right|_{\mathcal{H}}\leq c(\mu)|z_1-z_2|_{\mathcal{H}}. \tag{2.133}$$

Lemma 2.9.7. *For any closed set $N \subset H$, and any $\nu < V_\mu(N)$, there exists $\rho_0 > 0$ such that if $z \in C((0,T);H)$, with $|z(0)|_{\mathcal{H}} < \rho_0$ and $I^\mu_{0,T}(z) < \nu$, then it holds*

$$\inf_{t \le T} \operatorname{dist}_H(\Pi_1 z(t), N) > |z(0)|_{\mathcal{H}}.$$

Lemma 2.9.8. *For any $\mu, \epsilon > 0$ and $z_0 \in \mathcal{H}$, let*

$$\tau^{\mu,\epsilon}_{z_0,\rho} := \inf\left\{ t > 0 : \Pi_1 z^\mu_{\epsilon,z_0}(t) \notin G \text{ or } \left|z^\mu_{\epsilon,z_0}(t)\right|_{\mathcal{H}} < \rho \right\},$$

where $\rho > 0$ is small enough so that $B_{\mathcal{H}}(\rho) \subset G \times H^{-1}$. Then

$$\lim_{t \to +\infty} \limsup_{\epsilon \to 0} \epsilon \log \left(\sup_{z_0 \in G \times H^{-1}} \mathbb{P}\left(\tau^{\mu,\epsilon}_{z_0,\rho} \ge t \right) \right) = -\infty. \tag{2.134}$$

Lemma 2.9.9. *Let $\tau^{\mu,\epsilon}_{z_0,\rho}$ be the exit time from Lemma 2.9.8 and let $N \subset \partial G$ be a closed set. Then*

$$\lim_{\rho \to 0} \limsup_{\epsilon \to 0} \epsilon \log \left(\sup_{z_0 \in B_{\mathcal{H}}((1+M_\mu)\rho)} \mathbb{P}\left(\Pi_1 z^\mu_{\epsilon,z_0}\left(\tau^{\mu,\epsilon}_{z_0,\rho}\right) \in N \right) \right) \le -V_\mu(N), \tag{2.135}$$

where $V_\mu(N) = \inf_{x \in N} V_\mu(x)$.

Lemma 2.9.10. *For fixed $\rho > 0$,*

$$\lim_{t \to 0} \limsup_{\epsilon \to 0} \epsilon \log \left(\sup_{z_0 \in B_{\mathcal{H}}(\rho)} \mathbb{P}\left(\sup_{s \le t} \left|z^\mu_{\epsilon,z_0}(s)\right|_{\mathcal{H}} \ge (1+M_\mu)\rho \right) \right) = -\infty. \tag{2.136}$$

Proof of Theorem 2.9.3. This proof is based on the approach by [13] but we have made important modifications to deal with both the infinite dimensionality of $\mathcal{H}$ as well as the unboundedness of the cylinder $G \times H^{-1}$. For completeness, the entire proof is included.

Upper Bound

As $G \subset H$ is a bounded set, there exists $R > 0$ such that $G \subset B_H(R-1)$ If $c(\mu, 1)$ is the constant from Lemma 2.3.4, for any $z_0 = (u_0, v_0) \in \mathcal{H}$ such that

$$u_0 \in G, \quad |v_0|_{H^{-1}} > R\, c(\mu, 1)^{-1} =: \kappa,$$

we have that $\Pi_1 z^\mu_{z_0}$ leaves B_R, (and therefore G) before time $t = 1$. Since for any $T > 0$

$$\lim_{\epsilon \to 0} \sup_{z_0 \in \mathcal{H}} \mathbb{E}\, |z^\mu_{\epsilon, z_0} - z^\mu_{z_0}|_{C([0,T];\mathcal{H})} = 0, \tag{2.137}$$

this yields

$$\lim_{\epsilon \to 0} \inf_{\substack{u_0 \in G \\ |v_0|_{H^{-1}} > \kappa}} \mathbb{P}\left(\tau^{\mu,\epsilon}_{z_0} < 1\right) \geq \lim_{\epsilon \to 0} \inf_{\substack{u_0 \in G \\ |v_0|_{H^{-1}} > \kappa}} \mathbb{P}\left(\left|z^\mu_{\epsilon, z_0} - z^\mu_{z_0}\right|_{C([0,T];\mathcal{H})} \leq 1\right) = 1. \tag{2.138}$$

Now, fix $\eta > 0$. According to (2.124), there exists $x_{G,\mu} \in \partial G \cap H^{1+2\beta}$ such that $V_\mu(x_{G,\mu}) = V_\mu(\partial G)$. Now, if $\{x_n\} \subset \bar{G}^c \cap H^{1+2\beta}$ is a sequence from (2.121) such that $x_n \to x_{G,\mu}$ in $H^{1+2\beta}$, as $n \to \infty$, due to Theorem 2.6.3 we have that $V_\mu(x_n) \to V_\mu(x_{G,\mu})$. This means that there exists $\bar{n}$ such that

$$V_\mu(x_{\bar{n}}) < V_\mu(x_{G,\mu}) + \frac{\eta}{4} = V_\mu(\partial G) + \frac{\eta}{4}.$$

In particular, there exists $T_1 > 0$ and $z^\mu_{\psi,0} \in C([0, T_1]; \mathcal{H})$ such that $z^\mu_{\psi,0}(0) = 0$ and $\Pi_1 z^\mu_{\psi,0}(T_1) = x_{\bar{n}} \in \bar{G}^c$ with

$$I^\mu_{0,T_1}(z^\mu_{\psi,0}) < V_\mu(x_{\bar{n}}) + \frac{\eta}{4} < V_\mu(\partial G) + \frac{\eta}{2}.$$

According to (2.133), the mapping $z_0 \in \mathcal{H} \mapsto z^\mu_{\psi,z_0} \in C([0, T_1]; \mathcal{H})$ is continuous, and therefore, we can find $\rho > 0$ such that

$$|z_0|_{\mathcal{H}} < \rho \Longrightarrow \text{dist}\left(z^\mu_{\psi,z_0}(T_1), (G \times H^{-1})\right) > \frac{1}{2}\text{dist}\left(z^\mu_{\psi,0}(T_1), (G \times H^{-1})\right) =: \alpha > 0.$$

In view of (2.70), we can see that there exists $\epsilon_1 > 0$ such that for all $\epsilon < \epsilon_1$, and all $|z_0|_{\mathcal{H}} < \rho$

$$\mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} < T_1\right) \geq \mathbb{P}\left(\left|z_{\epsilon,z_0}^{\mu} - z_{\psi,z_0}^{\mu}\right|_{C([0,T_1];\mathcal{H})} < \alpha\right) \geq e^{-\frac{1}{\epsilon}(V_\mu(G)+\eta)}. \tag{2.139}$$

Now, by Lemma 2.3.2 we can find $T_2 >$ such that

$$\sup_{\substack{u_0 \in G \\ |v_0|_{H^{-1}} \leq \kappa}} \left|z_{z_0}^{\mu}(T_2)\right|_{\mathcal{H}} < \frac{\rho}{2}.$$

Therefore, thanks to (2.137), there exists $0 < \epsilon_2 \leq \epsilon_1$ such that for $u_0 \in G$, and $|v_0|_{H^{-1}} \leq \kappa$,

$$\mathbb{P}\left(\left|z_{\epsilon,z_0}^{\mu}(T_2)\right|_{\mathcal{H}} < \rho\right) > \frac{1}{2}, \qquad \epsilon \leq \epsilon_2.$$

Thanks to (2.139), by the Markov property, this implies that for $u_0 \in G$ and $|v_0|_{H^{-1}} \leq \kappa$,

$$\mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} < T_1 + T_2\right) \geq \frac{1}{2} e^{-\frac{1}{\epsilon}(V_\mu(G)+\eta)}, \qquad \epsilon < \epsilon_2.$$

Hence, if we combine this with (2.138), we see that there exists $0 < \epsilon_0 \leq \epsilon_2$ such that for all $\epsilon < \epsilon_0$,

$$\inf_{z_0 \in G \times H^{-1}} \mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} < 1 + T_1 + T_2\right) \geq \frac{1}{2} e^{-\frac{1}{\epsilon}(V_\mu(G)+\eta)}. \tag{2.140}$$

By using again the Markov property, for any $k \in \mathbb{N}$ and $z_0 \in G \times H^{-1}$ this gives

$$\mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} \geq k(1 + T_1 + T_2)\right) \leq \left(\sup_{z_0 \in G \times H^{-1}} \mathbb{P}(\tau_{z_0}^{\mu,\epsilon} \geq (1 + T_1 + T_2))\right)^k \leq \left(1 - \frac{1}{2} e^{-\frac{1}{\epsilon}(V_\mu(G)+\eta)}\right)^k,$$

so that

$$\mathbb{E}\left(\tau_z^{\mu,\epsilon}\right) \leq (1 + T_1 + T_2) \sum_{k=0}^{\infty} \mathbb{P}\left(\tau_z^{\mu,\epsilon} \geq k(1 + T_1 + T_2)\right) \leq 2(1 + T_1 + T_2) e^{\frac{1}{\epsilon}(V_\mu(G)+\eta)}.$$

Thus, the upper bound of (2.126) follows as η was chosen arbitrarily small and the upper bound of (2.127), follows from this by using the Chebyshev inequality.

Lower Bound Let $\rho > 0$ to be chosen later. We define a sequence of stopping times.

$$\begin{aligned}
&\sigma_0 = 0, \\
&\tau_m = \inf\{t > \sigma_m : \Pi_1 z^\mu_{\epsilon,z_0}(t) \notin G \text{ or } |z^\mu_{\epsilon,z_0}(t)|_{\mathcal{H}} \leq \rho\}, \\
&\sigma_{m+1} = \inf\{t > \tau_m : |z^\mu_{\epsilon,z_0}(t)|_{\mathcal{H}} = (1+M_\mu)\rho\}.
\end{aligned} \tag{2.141}$$

Notice that these stopping times depend implicitly on μ, ϵ, and z_0 but we have suppressed those superscripts to simplify notation.

Fix $\eta > 0$. By Lemma 2.9.9, we can find $\rho > 0$ small enough so that

$$\limsup_{\epsilon\to 0} \epsilon \log \left(\sup_{z_0 \in B_{\mathcal{H}}((1+M_\mu)\rho)} \mathbb{P}\left(\Pi_1 z^\mu_{\epsilon,z_0}(\tau_0) \in \partial G\right) \right) < -V(\partial G) + \frac{\eta}{2}.$$

By the Markov property and the above formula, there exists $\epsilon_0 > 0$ so that for any $\epsilon < \epsilon_0$ and any $m \geq 1$

$$\sup_{z_0} \mathbb{P}\left(\tau^{\mu,\epsilon}_{z_0} = \tau_m\right) \leq \sup_{z_0 \in B_{\mathcal{H}}((1+M_\mu)\rho)} \mathbb{P}\left(\Pi_1 z^\mu_{\epsilon,z_0}(\tau_0) \in \partial G\right) < \exp\left(\frac{-V(\partial G) + \frac{\eta}{2}}{\epsilon}\right). \tag{2.142}$$

Furthermore, by Lemma 2.9.10, we can find $T_0 > 0$ small enough so that by possibly decreasing ϵ_0 it holds that for all $\epsilon < \epsilon_0$

$$\sup_{z_0} \mathbb{P}\left(\sigma_m - \tau_{m-1} \leq T_0\right) \leq \sup_{z_0 \in B_{\mathcal{H}}(\rho)} \mathbb{P}\left(\sup_{s \leq T_0} \left|z^\mu_{\epsilon,z_0}(s)\right|_{\mathcal{H}} \geq (1+M_\mu)\rho\right) \leq \exp\left(\frac{-V(\partial G)}{\epsilon}\right). \tag{2.143}$$

In [13], the authors observe that the event $\{\tau^{\mu,\epsilon}_{z_0} \leq kT_0\}$ implies that either $\{\tau^{\mu,\epsilon}_{z_0} = \tau_m\}$ for some $m \leq k$ or at least one of the excursions $\tau_{m+1} - \tau_m \leq T_0$. Then by (2.142) and (2.143) we can bound

$$\mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} \leq kT_0\right) \leq \mathbb{P}(\tau_{z_0}^{\mu,\epsilon} = \tau_0) \sum_{m=1}^{k} \left(\mathbb{P}(\tau_{z_0}^{\mu,\epsilon} = \tau_m) + \mathbb{P}\left(\sigma_m - \tau_{m-1}\right)\right)$$
$$\leq \mathbb{P}(\tau_{z_0}^{\mu,\epsilon} = \tau_0) + 2k \exp\left(\frac{-V(\partial G) + \frac{\eta}{2}}{\epsilon}\right).$$

We choose

$$k = \left[T_0^{-1} \exp\left(\frac{V(\partial G) - \eta}{\epsilon}\right)\right] + 1$$

and observe that for small $\epsilon > 0$,

$$\mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} \leq \exp\left(\frac{V(\partial G) - \eta}{\epsilon}\right)\right) \leq \mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} \leq kT_0\right) \leq \mathbb{P}\left(\tau_{z_0}^{\mu,\epsilon} = \tau_0\right) + \frac{4}{T_0} \exp\left(-\frac{\eta}{2\epsilon}\right).$$

We argue that this converges to zero. Recall that the initial condition was chosen so that the unperturbed equation z_{0,z_0}^{μ} never leaves G. Also that unperturbed system converges to any neighborhood 0 in finite time. In particular, we can find $T > 0$ such that $|z_{0,z_0}^{\mu}(T)|_{\mathcal{H}} \leq \frac{\rho}{3}$. By uniform convergence on finite time intervals, for small ϵ,

$$|z_{\epsilon,z_0}^{\mu}(T)|_{\mathcal{H}} \leq \frac{\rho}{2}$$

without first leaving G. Therefore $\mathbb{P}(\tau_{z_0}^{\mu,\epsilon} = \tau_0) \to 0$.

This concludes the proof of the lower bound of (2.127). The lower bound of (2.126) follows by Chebyshev inequality.

Exit place

Let $N \subset \partial G$ have the property that $V(N) > V(\partial G)$. Let $\eta > 0$ satisfy $\eta < \frac{1}{3}(V(\partial G) - V(N))$. As in the proof of the lower bound, we choose $\rho > 0$ small enough and ϵ_0 small enough so that by Lemma 2.9.9 for $\epsilon < e_0$,

$$\sup_{z_0 \in B_{\mathcal{H}}(\rho)} \mathbb{P}\left(\Pi_1 z_{\epsilon,z_0}^{\mu}(\tau_{z_0,\rho}^{\mu,\epsilon}) \in N\right) \leq \exp\left(\frac{-(V(N) - \eta)}{\epsilon}\right).$$

By possibly decreasing ϵ_0, we can use Lemma 2.9.10 to find T_0 such that for all $\epsilon < \epsilon_0$,

$$\sup_{z_0 \in B_{\mathcal{H}}(\rho)} \mathbb{P}\left(\sup_{s \le T_0} \left|z^{\mu}_{\epsilon,z_0}(s)\right|_{\mathcal{H}} \ge (1+M_\mu)\rho\right) \le \exp\left(-\frac{V(N)}{\epsilon}.\right)$$

Then for any integer l,

$$\sup_{z_0} \mathbb{P}\left(\tau_l \le lT_0\right) \le \sum_{m=1}^{l} \sup_{z_0 \in B_{\mathcal{H}}(\rho)} \mathbb{P}\left(\sup_{s \le T_0} \left|z^{\mu}_{\epsilon,z_0}(s)\right|_{\mathcal{H}} \ge (1+M_\mu)\rho\right) \le l \exp\left(-\frac{V(N)}{\epsilon}\right).$$

For $l \in \mathbb{N}$ to be chosen later

$$\begin{aligned}
&\mathbb{P}\left(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0} \in N)\right) \le \mathbb{P}\left(\tau^{\mu,\epsilon}_{z_0} > \tau_l\right) + \mathbb{P}\left(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau_0) \in N\right) \\
&+ \sum_{m=1}^{l} \mathbb{P}(\tau^{\mu,\epsilon}_{z_0} > \tau_{m-1}) \mathbb{P}(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau_m) \in N | \tau^{\mu,\epsilon}_{z_0} > \tau_m) \\
&\le \mathbb{P}(\tau^{\epsilon} > lT_0) + \mathbb{P}(\tau_l \le lT_0) + \mathbb{P}(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau_0) \in N) + l \sup_{z_0 \in B_{\mathcal{H}}(\rho)} \mathbb{P}\left(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0,\rho}) \in N\right) \\
&\le \mathbb{P}(\tau^{\epsilon} > lT_0) + \mathbb{P}(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau_0) \in N) + 2l \exp\left(-\frac{V(N)-\eta}{\epsilon}\right).
\end{aligned}$$

We choose $l = \left[\exp\left(\frac{V(\partial G)+2\eta}{\epsilon}\right)\right]$. By the upper bound of (2.129) and Chebyshev inequality we can guarantee that for some $T > 0$ and small enough $\epsilon > 0$

$$\mathbb{P}\left(\tau^{\epsilon} > lT_0\right) \le \frac{T}{lT_0} \exp\left(\frac{V(\partial G)+\eta}{\epsilon}\right).$$

This converges to 0 because $(V(N) - V(\partial G)) < 3\eta$. Finally,

$$\mathbb{P}(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau_0) \in N) \to 0$$

because z_0 was chosen in such a way that the deterministic path z^{μ}_{0,z_0} reaches $B_{\mathcal{H}}(\rho)$ before exiting G. From all of these estimates, (2.128) follows. $\square$

2.9.2 Proofs of Lemmas from 2.9.6 to 2.9.10

Proof of Lemma 2.9.6. If we let $\varphi(t) = \Pi_1 \left(z^\mu_{\psi,z_1}(t) - z^\mu_{\psi,z_2}(t) \right)$, then it is a weak solution to

$$\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) = A\varphi(t) + B(\Pi_1 z^\mu_{z_1,\psi}(t)) - B(\Pi_1 z^\mu_{z_2,\psi}(t)). \tag{2.144}$$

Therefore, we can conclude as in Lemma 2.3.1. $\square$

Proof of Lemma 2.9.7. Fix $\nu < V_\mu(N)$. Suppose by contradiction that there exist $\{z_n\} \subset \mathcal{H}$, $\{T_n\} \subset (0, +\infty)$ and $\{\psi_n\} \subset L^2((0, T_n); H)$ such that

$$\lim_{n\to\infty} |z_n|_{\mathcal{H}} = 0, \qquad \frac{1}{2}|\psi_n|^2_{L^2((0,T_n);H)} < \nu,$$

and

$$\mathrm{dist}_H(\Pi_1 z^\mu_{\psi_n,z_n}(T_n), N) \leq |z_n|_{\mathcal{H}}.$$

Now, if we set $x_n := \Pi_1 z^\mu_{\psi_n,0}(T_n)$, for any $n \in \mathbb{N}$ we have, by (2.133),

$$|x_n - \Pi_1 z^\mu_{\psi_n,z_n}(T_n)|_H \leq c(\mu)\, |z_n|_{\mathcal{H}},$$

so that

$$\mathrm{dist}_H(x_n, N) \leq c(\mu)|z_n|_{\mathcal{H}} + |z_n|_{\mathcal{H}}, \qquad n \in \mathbb{N}. \tag{2.145}$$

Recalling how V_μ is defined, we have

$$V_\mu(x_n) \leq \frac{1}{2}|\psi_n|^2_{L^2((0,T_n);H)} < \nu.$$

Now, as proven in Theorem 2.5.4, V_μ has compact level sets. Therefore, there is a sequence $\{x_{n_k}\}_k \subset H$ such that $x_{n_k} \to x$, so that $V_\mu(x) < \nu$. But, by (2.145), $x \in N$, and then $V_\mu(N) \leq V_\mu(x) < V_\mu(N)$, a contradiction. $\square$

Proof of Lemma 2.9.8. Fix $R > \sup_{x\in G}|x|_H + \rho + 1$ and, by Lemma 2.3.4, let us take $\kappa > 0$ such that if $|v_0|_{H^{-1}} \geq \kappa$ then $z^\mu_{z_0}$ leaves $B_R \times H^{-1}$ before time $t = 1$.

By Lemma 2.3.2, we can find $T_1 > 0$ such that

$$\sup_{\substack{u_0\in G\\ |v_0|_{H^{-1}}\leq\kappa}} \left|z^\mu_{z_0}(T_1)\right|_{\mathcal{H}} < \frac{\rho}{4},$$

and then for any $z_0 \in G \times H^{-1}$, $z^\mu_{z_0}(t)$ leaves $(G \times H^{-1}) \setminus B_{\mathcal{H}}(\rho/2)$ in less than time $T = T_1 + 1$. Let us set

$$a := \inf\left\{I^\mu_{0,T}(z) : z(t) \in (B_H(R-1) \times H^{-1}) \setminus B_{\mathcal{H}}(\rho/2) \text{ for } t \in [0,T]\right\}. \quad (2.146)$$

The set above contains no unperturbed trajectories. We would like to show that $a > 0$. Suppose that z^μ_{ψ,z_0} is a controlled trajectory with the property that $z^\mu_{\psi,z_0}(t) \in (B_H(R) \times H^{-1}) \setminus B_{\mathcal{H}}(\rho/2)$ for $t \in [0,T]$. Then because the unperturbed trajectory z^ψ_{0,z_0} either leaves $B_H(R) \times H^{-1}$ or enters $B_{\mathcal{H}}(\rho/4)$,

$$\frac{\rho}{4} \leq \left|z^\mu_{0,z_0} - z^\mu_{\psi,z_0}\right|_{C([0,T];\mathcal{H})}.$$

On the other hand, by arguments similar to those used in Lemma 2.4.5 we see that

$$\sup_{t\leq T} |\mathcal{I}_\mu(z^\mu_{0,z_0}(t) - z^\mu_{\psi,z_0}(t))|_{\mathcal{H}} \leq c|\psi|_{L^2((0,T);H)}.$$

Therefore, any such ψ has the property that

$$\frac{\rho}{4c} \leq |\psi|_{L^2((0,T);H)}$$

and

$$a \geq \frac{\rho^2}{32c^2} > 0.$$

By (2.71)

$$\limsup_{\epsilon\to 0} \epsilon \log\left(\sup_{z_0\in G\times H^{-1}} \mathbb{P}\left(\tau_0 \geq T\right)\right)$$
$$\leq \limsup_{\epsilon\to 0} \epsilon \log\left(\sup_{z_0\in G\times H^{-1}} \mathbb{P}\left(\mathrm{dist}_{C([0,T];\mathcal{H})}(z^{\mu}_{\epsilon,z_0}, K^{\mu}_{0,T}(a)) > \frac{\rho}{2}\right)\right) \leq -a.$$

By the Markov property, for any $k \in \mathbb{N}$,

$$\sup_{z_0\in G\times H^{-1}} \mathbb{P}(\tau_0 \geq kT) \leq \left(\sup_{z_0\in G\times H^{-1}} \mathbb{P}(\tau_0 \geq T)\right)^k$$

and therefore,

$$\lim_{\epsilon\to 0} \epsilon \log\left(\sup_{z_0\in G\times H^{-1}} \mathbb{P}\left(\tau_1 \geq Tk\right)\right) \leq -ka.$$

□

Proof of Lemma 2.9.9. Let $\Gamma_\rho := B_{\mathcal{H}}((1+M_\mu)\rho)$. For any $T > 0$ we have

$$\begin{aligned}&\sup_{z_0\in\Gamma_\rho} \mathbb{P}\left(\Pi_1 z^{\mu}_{\epsilon,z_0}(\tau^{\mu,\epsilon}_{z_0,\rho}) \in N\right)\\ &\leq \sup_{z_0\in\Gamma_\rho} \mathbb{P}(\tau^{\mu,\epsilon}_{z_0,\rho} > T) + \sup_{z_0\in\Gamma_\rho} \mathbb{P}(\Pi_1 z^{\mu}_{\epsilon,z_0}(t) \in N, \text{ for some } t \leq T).\end{aligned} \tag{2.147}$$

Next, thanks to Lemma 2.9.7, for any $\nu < V_\mu(N)$ fixed we can find $\rho_0 > 0$ such that for $\rho < \rho_0$ and any $T > 0$, the set

$$\left\{z : z(0) \in \Gamma_\rho, \, \mathrm{dist}_{C([0,T];\mathcal{H})}\left(z, K^{\mu}_{0,T}(\nu)\right) \leq (1+M_\mu)\rho\right\}$$

contains no trajectories that reach N by time T. Then by (2.71), for any $\eta > 0$, for small enough $\epsilon > 0$,

$$\sup_{z_0\in\Gamma_\rho} \mathbb{P}(\Pi_1 z^{\mu}_{\epsilon,z_0}(t) \in N \text{ for some } t \leq T)$$
$$\leq \sup_{z_0\in\Gamma_\rho} \mathbb{P}\left(\mathrm{dist}_{C([0,T];\mathcal{H})}(z^{\mu}_{\epsilon,z_0}, K^{\mu}_{0,T}(\nu)) > (1+M_\mu)\rho\right) \leq e^{-\frac{1}{\epsilon}(\nu-\eta)}.$$

Now, according to (2.134), we pick $T > 0$ so that, for small enough $\epsilon > 0$,

$$\sup_{z_0\in\Gamma_\rho} \mathbb{P}(\tau^{\mu,\epsilon}_{z_0,\rho} > T) \leq e^{-\frac{\nu}{\epsilon}}.$$

Due to (2.147), this implies our result, as $\nu < V_\mu(N)$ and $\eta > 0$ were arbitrary. $\square$

Proof of Lemma 2.9.10. If $z(t) = z^\mu_{\psi,z_0}(t)$, then

$$z(t) = S_\mu(t)z_0 + \int_0^t S_\mu(t-s)B_\mu(z(s))ds + \int_0^t S_\mu(t-s)Q_\mu\psi(s)ds,$$

so that, if $z_0 \in B_{\mathcal{H}}(\rho)$,

$$\sup_{s\leq t}|z(s)|_{\mathcal{H}} \leq M_\mu\rho + \frac{\gamma_0 t M_\mu}{\mu\sqrt{\alpha_1}}\sup_{s\leq t}|z(s)|_{\mathcal{H}} + \frac{M_\mu\|Q\|_{\mathcal{L}(H)}}{\mu\sqrt{\alpha_1}}\sqrt{t}|\psi|_{L^2((0,t);H)}.$$

Therefore, if $\sup\limits_{s\leq t}|z(s)| \geq (M_\mu + 1/2)\,\rho$, then we get

$$E_\mu(t) := \rho\left(\frac{1}{2} - \frac{\gamma_0 t M_\mu(M_\mu+\frac{1}{2})}{\sqrt{\alpha_1}\mu}\right)\frac{\mu\sqrt{\alpha_1}}{M_\mu\|Q\|_{\mathcal{L}(H)}\sqrt{t}} \leq |\psi|_{L^2((0,t);H)}.$$

This means that

$$\begin{aligned}&\limsup_{\epsilon\to 0}\epsilon\log\left(\sup_{z_0\in B_{\mathcal{H}}(\rho)}\mathbb{P}\left(\sup_{s\leq t}\left|z^\mu_{\epsilon,z_0}(s)\right|_{\mathcal{H}} \geq (1+M_\mu)\rho\right)\right)\\ &\leq \limsup_{\epsilon\to 0}\epsilon\log\left(\sup_{z_0\in B_{\mathcal{H}}(\rho)}\mathbb{P}\left(\mathrm{dist}_{C([0,t];\mathcal{H})}\left(z^\mu_{\epsilon,z_0}, K^\mu_{0,t}\left(\frac{1}{2}(E_\mu(t))^2\right)\right) > \frac{\rho}{2}\right)\right) \leq -\frac{(E_\mu(t))^2}{2},\end{aligned}$$

and our result follows as

$$\lim_{t\to 0}E_\mu(t) = +\infty.$$

$\square$

2.10 Gradient nonlinearities

We conclude this chapter by studying a special case of (2.1) and (2.2) where the nonlinearity B is of gradient type. That is, we assume that $B : H \to H$ is given by

$$B(x) = -Q^2DF(x) \tag{2.148}$$

where DF denotes the Frechet derivative of a function $F \in C^1(H;\mathbb{R})$ and $Q \in \mathcal{L}(H)$ is the covariance operator of the noise w^Q. We assume that $F(0) = 0$ and $F(x) \geq 0$. This means that $F(0)$ is necessarily a minimum of F and therefore $DF(0) = 0$. When B has this structure, we have explicit representations for V, V^μ, and V_μ. Moreover, V and V_μ coincide for all $x \in H$, $\mu > 0$.

First we introduce some examples of B that are of gradient type.

Example 2.10.1. *1. Assume $d = 1$ and $Q = I$ (white noise). Let $b : \mathbb{R} \to \mathbb{R}$ be a decreasing Lipschitz function with $b(0)$. Then the composition operator $B : H \to H$ defined by*

$$(B(x))(\xi) = b(x(\xi)), \xi \in D$$

is of gradient type. In this case we set

$$F(x) = -\int_D \int_0^{x(\xi)} b(\eta) d\eta d\xi, \quad x \in H.$$

Then $B(x) = -DF(x)$ and $F(0) = 0$. Furthermore, $F(x) \geq 0$ because we assumed that b is decreasing with $b(0) = 0$. Therefore, for any real number ζ

$$-\int_0^\zeta b(\eta) d\eta \geq 0.$$

This function also has the property that

$$\langle DF(x), x\rangle_H = -\int_D b(x(\xi))x(\xi)d\xi \geq 0.$$

2. Assume now $d \geq 1$ so that we cannot necessarily set $Q = I$. Let $b : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 with Lipschitz continuous first derivative such that

$b(0) = 0$ and $b(\eta) \geq 0$ for all $\eta \in \mathbb{R}$. For $x \in H$ set

$$F(x) = \int_D b(x(\xi))d\xi.$$

We can check that $F(0) = 0$ and $F(x) \geq 0$. Furthermore, for any $x \in H$

$$DF(x)(\xi) = b'(x(\xi)).$$

Therefore, the nonlinearity

$$B(x) = -Q^2 b'(x(\cdot)),\ x \in H$$

is of gradient type.

If $z \in C((-\infty, 0]; \mathcal{H})$ is such that $I^{\mu}_{-\infty,0}(z) < +\infty$, then we have

$$I^{\mu}_{-\infty}(z) = \frac{1}{2}\int_{-\infty}^{0} \left| Q^{-1}\left(\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t))\right)\right|_H^2 dt. \quad (2.149)$$

Actually, if $I^{\mu}_{-\infty,0}(z) < +\infty$, then there exists $\psi \in L^2((-\infty, 0); H)$ such that $\varphi = \Pi_1 z$ is a weak solution to

$$\mu \frac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) - \frac{\partial \varphi}{\partial t}(t) - Q^2 DF(\varphi(t)) + Q\psi.$$

This means that

$$\psi(t) = Q^{-1}\left(\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t))\right)$$

and (2.149) follows.

By the same argument, if $I_{-\infty,0}(\varphi) < +\infty$, then it follows that

$$I_{-\infty}(\varphi) = \int_{-\infty}^{0} \left| Q^{-1}\left(\frac{\partial \varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t))\right)\right|_H^2 dt \quad (2.150)$$

Theorem 2.10.2. *For any fixed $\mu > 0$ and $(x, y) \in D((-A)^{1/2}Q^{-1}) \times D(Q^{-1})$ it holds*

$$V^{\mu}(x,y) = \left|(-A)^{\frac{1}{2}}Q^{-1}x\right|_H^2 + 2F(x) + \mu\left|Q^{-1}y\right|_H^2. \tag{2.151}$$

Moreover

$$V(x) = \left|(-A)^{\frac{1}{2}}Q^{-1}x\right|_H^2 + 2F(x). \tag{2.152}$$

In particular, for any $\mu > 0$,

$$V_{\mu}(x) := \inf_{y \in H^{-1}} V^{\mu}(x,y) = V^{\mu}(x,0) = V(x).$$

Proof. First, we observe that if $\varphi(t) = \Pi_1 z(t)$, then

$$\begin{aligned} I^{\mu}_{-\infty}(z) = \frac{1}{2}\int_{-\infty}^{0} \left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t) - \frac{\partial\varphi}{\partial t}(t) - A\varphi(t) + Q^2 DF(\varphi(t))\right)\right|_H^2 dt \\ +2\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t), Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t) - A\varphi(t)\right) + QDF(\varphi(t))\right\rangle_H dt. \end{aligned} \tag{2.153}$$

Recall that in (2.42) $C_{\mu} : \mathcal{H} \to \mathcal{H}$ was defined as

$$C_{\mu}(u,v) = \left(Q^2(-A)^{-1}u, \frac{1}{\mu}Q^2(-A)^{-1}v\right), \qquad (u,v) \in \mathcal{H}.$$

Now, if

$$\lim_{t\to-\infty} |C_{\mu}^{-1/2} z(t)|_{\mathcal{H}} = 0,$$

then

$$\lim_{t\to-\infty} \left|(-A)^{\frac{1}{2}}Q^{-1}\varphi(t)\right|_H + \left|Q^{-1}\frac{\partial\varphi}{\partial t}(t)\right|_H = 0,$$

so that

$$2\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t), Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t) - A\varphi(t)\right) + QDF(\varphi(t))\right\rangle_H dt$$

$$= \left|(-A)^{\frac{1}{2}}Q^{-1}\varphi(0)\right|_H^2 + 2F(\varphi(0)) + \mu\left|Q^{-1}\frac{\partial\varphi}{\partial t}(0)\right|_H^2.$$

This yields

$$V^\mu(x,y) \geq \left|(-A)^{\frac{1}{2}} Q^{-1} x\right|_H^2 + 2F(x) + \mu \left|Q^{-1} y\right|_H^2.$$

Now, let $\tilde{z}(t)$ be a mild solution of the problem

$$\tilde{z}(t) = S_\mu(t)(x,-y) - \int_0^t S_\mu(t-s) Q_\mu Q DF(\tilde{z}(s)) ds,$$

and let $(x,y) \in D(C_\mu^{-1/2})$. Then $\tilde{\varphi}(t) = \Pi_1 \tilde{z}(t)$ is a weak solution of the problem

$$\mu \frac{\partial^2 \tilde{\varphi}}{\partial t^2}(t) = A\tilde{\varphi}(t) - \frac{\partial \tilde{\varphi}}{\partial t}(t) - Q^2 DF(\tilde{\varphi}(t)), \quad \tilde{\varphi}(0) = x, \quad \frac{\tilde{\varphi}}{\partial t}(0) = -y.$$

Moreover, as proven below in Lemma 2.10.3,

$$\lim_{t \to -\infty} \left|C_\mu^{-1/2} \tilde{z}(t)\right|_{\mathcal{H}} = 0.$$

Then, if we define $\hat{\varphi}(t) = \tilde{\varphi}(-t)$ for $t \leq 0$, we see that $\hat{\varphi}(t)$ solves

$$\mu \frac{\partial^2 \hat{\varphi}}{\partial t^2}(t) = A\hat{\varphi}(t) + \frac{\partial \hat{\varphi}}{\partial t}(t) - Q^2 DF(\hat{\varphi}(t)), \quad \hat{\varphi}(0) = x, \quad \frac{\partial \hat{\varphi}}{\partial t}(0) = y.$$

Thanks to (2.153) this yields

$$I_{-\infty}^\mu(\hat{\varphi}) = \left|(-A)^{\frac{1}{2}} Q^{-1} x\right|_H^2 + 2F(x) + \mu \left|Q^{-1} y\right|_H^2.$$

and then

$$V^\mu(x,y) = \left|(-A)^{\frac{1}{2}} Q^{-1} x\right|_H^2 + 2F(x) + \mu \left|Q^{-1} y\right|_H^2.$$

As known, an analogous result holds for $V(x)$. In what follows, for completeness, we give a proof. We have

$$\begin{aligned} I_{-\infty}(\varphi) &= \frac{1}{2} \int_{-\infty}^0 \left| Q^{-1} \left(\frac{\partial \varphi}{\partial t}(t) + A\varphi(t) - Q^2 DF(\varphi(t)) \right) \right|_H^2 dt \\ &+2 \int_{-\infty}^0 \left\langle Q^{-1} \frac{\partial \varphi}{\partial t}(t), Q^{-1} \left(-A\varphi(t) + Q^2 DF(\varphi(t)) \right) \right\rangle_H dt. \end{aligned} \tag{2.154}$$

From this we see that

$$V(x) \geq \left|(-A)^{\frac{1}{2}}Q^{-1}x\right|_H^2 + 2F(x).$$

Just as for the wave equation, for $x \in D((-A)^{\frac{1}{2}}Q^{-1})$, we define $\tilde{\varphi}$ to be the solution of

$$\tilde{\varphi}(t) = e^{tA}x - \int_0^t e^{(t-s)A}Q^2 DF(\tilde{\varphi}(s))ds.$$

We have

$$\lim_{t\to+\infty} |(-A)^{\frac{1}{2}}Q^{-1}\tilde{\varphi}(t)|_H = 0.$$

Then, if we define $\hat{\varphi}(t) = \tilde{\varphi}(-t)$ we get

$$\frac{\partial \hat{\varphi}}{\partial t}(t) = -A\hat{\varphi}(t) + Q^2 DF(\hat{\varphi}(t)),$$

so that

$$I_{-\infty}(\hat{\varphi}) = \left|(-A)^{\frac{1}{2}}Q^{-1}x\right|_H^2 + 2F(x)$$

and

$$V(x) = \left|(-A)^{\frac{1}{2}}Q^{-1}x\right|_H^2 + 2F(x).$$

$\square$

Now, in order to conclude the proof of Theorem 2.10.2, we have to prove the following result.

Lemma 2.10.3. *Let $(x,y) \in D((-A)^{\frac{1}{2}}Q^{-1}) \times D(Q^{-1})$ and let φ solve the problem*

$$\mu\frac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) - \frac{\partial \varphi}{\partial t}(t) - Q^2 DF(\varphi(t)), \qquad \varphi(0) = x, \quad \frac{\partial \varphi}{\partial t}(0) = y. \tag{2.155}$$

Then

$$z(t) = \left(\varphi(t), \frac{\partial \varphi}{\partial t}(t)\right) \in D((-A)^{\frac{1}{2}}) \times D(Q^{-1}), \qquad t \geq 0,$$

and

$$\lim_{t\to+\infty}\left|C_1^{-1/2}z(t)\right|_{\mathcal{H}}=0. \tag{2.156}$$

Proof. If in (2.155) we take the inner product with $2Q^{-2}\partial\varphi/\partial t(t)$, we have

$$2\left|Q^{-1}\frac{\partial\varphi}{\partial t}(t)\right|_H^2=-\frac{d}{dt}\left(\mu\left|Q^{-1}\varphi(t)\right|_H^2+\left|(-A)^{\frac{1}{2}}Q^{-1}\varphi(t)\right|_H^2+2F(\varphi(t))\right). \tag{2.157}$$

Therefore, if we define

$$\Phi_\mu(x,y)=\left|(-A)^{\frac{1}{2}}Q^{-1}x\right|_H^2+\mu\left|Q^{-1}y\right|_H^2+2F(x),$$

as a consequence of (2.157) we get

$$\Phi_\mu\left(z(t)\right)\leq\Phi_\mu(u,v). \tag{2.158}$$

Next, by (2.155) and the assumption that $\langle DF(x),x\rangle\geq 0$, we calculate that

$$\frac{d}{dt}\left|Q^{-1}\left(\mu\frac{\partial\varphi}{\partial t}(t)+\varphi(t)\right)\right|_H^2=2\left\langle Q^{-1}\left(\mu\frac{\partial\varphi}{\partial t}(t)+\varphi(t)\right),Q^{-1}A\varphi(t)-QDF(\varphi(t))\right\rangle_H$$
$$\leq-2\left|Q^{-1}(-A)^{\frac{1}{2}}\varphi(t)\right|_H^2-\mu\frac{d}{dt}\left|Q^{-1}(-A)^{\frac{1}{2}}\varphi(t)\right|_H^2-2\mu\frac{d}{dt}F(\varphi(t)).$$

A consequence of this is that

$$2\int_0^\infty\left|Q^{-1}(-A)^{\frac{1}{2}}\varphi(t)\right|_H^2dt\leq\left|\mu Q^{-1}y+Q^{-1}x\right|_H^2+\mu\left|Q^{-1}(-A)^{\frac{1}{2}}x\right|_H^2+2\mu F(x). \tag{2.159}$$

Now, if $z(t)=\left(\varphi(t),\frac{\partial\varphi}{\partial t}(t)\right)$, for any $t,T>0$ we have

$$z(T+t)=S_\mu(t)z(T)-\int_T^{T+t}S_\mu(T+t-s)Q_\mu QDF(\varphi(s))ds.$$

Because the semigroup S_μ is of negative type ($\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H})}\leq M_\mu e^{-\omega_\mu t}$) and DF is

Lipschitz continuous,

$$
\begin{aligned}
&\left|C_1^{-1/2}\int_T^{T+t} S_\mu(t+T-s)Q_\mu QDF(\varphi(s))ds\right|_{\mathcal{H}}\\
&\leq \int_T^{T+t}\left|S_\mu(t+T-s)Q_\mu(-A)^{\frac{1}{2}}DF(\varphi(s))\right|_{\mathcal{H}} ds\\
&\leq c\int_T^{T+t} e^{-\omega_\mu(t+T-s)}\left|(-A)^{\frac{1}{2}}QDF(\varphi(s))\right|_{H^{-1}} ds\\
&\leq c\int_T^{T+t} e^{-\omega_\mu(t+T-s)}\left|\varphi(s)\right|_H ds \leq c\left|\varphi\right|_{L^2((T,T+t);H)}.
\end{aligned}
$$

Therefore, by (2.159), for any $\epsilon > 0$ we can pick $T_\epsilon > 0$ large enough so that for all $t > 0$

$$
\left|C_1^{-1/2}\int_{T_\epsilon}^{T_\epsilon+t} S_\mu(t+T_\epsilon-s)Q_\mu QDF(\varphi(s))ds\right|_{\mathcal{H}} < \frac{\epsilon}{2}.
$$

We also note that C_1 commutes with S_μ. Because of this commutivity and the fact that S_μ is of negative type,

$$
\left|C_1^{-1/2}S_\mu(t)z(T)\right|_{\mathcal{H}} \leq M_\mu e^{-\omega_\mu t}\left|C_1^{-1/2}z(T)\right|_{\mathcal{H}}.
$$

Then, as

$$
|C_1^{-1/2}z|_{\mathcal{H}} \leq c\,\Phi_\mu(z), \quad z \in \mathcal{H},
$$

by (2.158) we can find a t_ϵ large enough so that for all $T > 0$ and $t > t_\epsilon$

$$
\left|C_1^{-1/2}S_\mu(t)z(T)\right|_{\mathcal{H}} < \frac{\epsilon}{2}.
$$

Then for $t > T_\epsilon + t_\epsilon$

$$
\left|C_1^{-1/2}z(t)\right|_{\mathcal{H}} < \epsilon
$$

which is what we were trying to prove. □

Chapter 3: Smoluchowski-Kramers approximation near a magnetic field

3.1 Introduction

We consider here the following two dimensional system of stochastic PDEs

$$
\begin{cases}
\mu \dfrac{\partial^2 u_\mu}{\partial t^2}(\xi,t) = \Delta u_\mu(\xi,t) + B(u_\mu(\cdot,t),t) + \vec{m} \times \dfrac{\partial u_\mu}{\partial t}(\xi,t) + G(u_\mu(\cdot,t),t)\,\dfrac{\partial w^Q}{\partial t}(\xi,t), \\
u_\mu(\xi,0) = u_0(\xi), \quad \dfrac{\partial u_\mu}{\partial t}(\xi,0) = v_0(\xi), \quad \xi \in D, \qquad u_\mu(\xi,t) = 0, \quad \xi \in \partial D,
\end{cases}
\tag{3.1}
$$

where D is a bounded regular domain in $\mathbb{R}^d$, with $d \geq 1$, B and G are suitable nonlinearities, $\vec{m} = (0,0,m)$ is a constant vector and $w^Q(t,\xi)$ is a cylindrical Wiener process, white in time and colored in space. In the case that the spatial dimension $d = 1$ we can take space-time white noise.

By Newton's law, the vector field $u_\mu : D \to \mathbb{R}^2$ models the displacement of a continuum of particles with constant density $\mu > 0$ in the region $D \subset \mathbb{R}^d$, in presence of a noisy perturbation and a constant magnetic field $\vec{m} = (0,0,m)$, which is orthogonal to the plane where the motion occurs (in what follows we shall assume just for simplicity of notations $m = 1$). For example, if $d = 1$ and $D = [0,1]$, this could model the displacement of a one-dimensional string, with fixed endpoints, that can move through two other spacial dimensions, where the Laplacian Δ models the forces neighboring particles exert on each other, B is some nonlinear forcing, and $\partial w^Q/\partial t$ is a Gaussian random forcing field, whose intensity G may depend on the state u_μ.

In [2] and [3], the authors prove the validity of the so-called Smoluchowski-Kramers approximation, in the case the magnetic field is replaced by a constant friction. Namely, it has been shown that, as μ tends to 0, the solutions of the second order system converge to the solution of the first order system which is obtained simply by taking $\mu = 0$. Moreover, in [8] and [7] we have studied the interplay between the Smoluchowski-Kramers approximation and the large deviation principle. In particular, we have shown how some relevant quantities associated with large deviations and exit problems from a basin of attraction for the second order problem can be approximated by the corresponding quantities for the first order problem, in terms of the small mass asymptotics described by the Smoluchowski-Kramers approximation.

One might hope that a similar result would be true in the case treated in the present paper. Namely, one would expect that for any $T > 0$ and $p \geq 1$

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} |u_\mu(t) - u(t)|^p_{L^2(D;\mathbb{R}^2)} = 0, \tag{3.2}$$

where $u(t)$ is the solution of the following system of stochastic PDEs

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi, t) = J_0^{-1}\Big[\Delta u(\xi, t) + B(u(\cdot, t), t) + G(u(\cdot, t), t)\dfrac{\partial w^Q}{\partial t}(\xi, t)\Big] \\ u(\xi, t) = 0, \quad \xi \in \partial D, \qquad u(\xi, 0) = u_0(\xi), \xi \in D, \end{cases} \tag{3.3}$$

where

$$J_0^{-1} = -\,J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Unfortunately, as shown in [4] such a limit is not valid, even for finite dimensional analogues of this problem. Actually, one can prove that if the stochastic term in (3.1) is replaced by a continuous function, then u_μ would converges uniformly in $[0, T]$ to the solution of (3.3). But if we have the stochastic noise term, this is not true anymore. An explanation of this lies in the fact that, while for any continuous function $\varphi(s)$ it holds

$$\lim_{\mu \to 0} \int_0^t \sin(s/\mu)\, \varphi(s)\, ds = 0,$$

but if we consider a stochastic integral and replace $\varphi(s)ds$ with $dB(s)$, we have

$$\lim_{\mu \to 0} \int_0^t \sin(s/\mu)\, dB(s) \neq 0,$$

since

$$\mathrm{Var}\left(\int_0^t \sin(s/\mu)\, dB(s) \right) = \int_0^t \sin^2(s/\mu)\, ds \to \frac{t}{2}, \qquad \text{as} \quad \mu \downarrow 0.$$

Nevertheless, this problem can be regularized in such a way that a counterpart of the Smoluchowski-Kramers approximation is still valid. One possible way consists in regularizing the noise (to this purpose, see [4] and [17] for the analysis of finite dimensional systems, both in the case of constant and in the case of state dependent magnetic field). Another possible way, which is the one we are using in the present paper, consists in introducing a small friction proportional to the velocity in equation

(3.1) and considering the regularized problem

$$\begin{cases} \mu \dfrac{\partial^2 u_\mu^\epsilon}{\partial t^2}(t) = \Delta u_\mu^\epsilon(t) + B(u_\mu^\epsilon(\cdot,t),t) + \vec{m} \times \dfrac{\partial u_\mu^\epsilon}{\partial t}(t) - \epsilon \dfrac{\partial u_\mu^\epsilon}{\partial t}(t) + G(u_\mu^\epsilon(\cdot,t),t)\dfrac{\partial w^Q}{\partial t}(t), \\ u_\mu^\epsilon(0) = u_0, \quad \dfrac{\partial u_\mu^\epsilon}{\partial t}(0) = v_0, \qquad u_\mu^\epsilon(\xi,t) = 0, \quad \xi \in \partial D, \end{cases} \tag{3.4}$$

which now depends on two small positive parameters ϵ and μ. Our purpose here is showing that, for any fixed $\epsilon > 0$, we can take the limit as μ goes to 0. Namely, we want to prove that for any $T > 0$ and $p \geq 1$

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} |u_\mu^\epsilon(t) - u_\epsilon(t)|^p_{L^2(D;\mathbb{R}^2)} = 0, \tag{3.5}$$

where $u_\epsilon(t)$ is the unique mild solution of the problem

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(\xi,t) = (J_0 + \epsilon\, I)^{-1} \Big[\Delta u_\epsilon(\xi,t) + B(u_\epsilon(\cdot,t),t) + G(u_\epsilon(\cdot,t),t)\dfrac{\partial w^Q}{\partial t}(\xi,t)\Big], \\ u_\epsilon(\xi,t) = 0, \quad \xi \in \partial D, \qquad u_\epsilon(\xi,0) = u_0(\xi), \xi \in D, \end{cases} \tag{3.6}$$

which is precisely what we get from (3.4) when we put $\mu = 0$.

The proof of (3.5) is not at all straightforward. First of all, it requires a thorough analysis of the linear semigroup $S_\mu^\epsilon(t)$ in the space $L^2(D) \times H^{-1}(D)$, associated with the differential operator

$$A_\mu^\epsilon(u,v) = \frac{1}{\mu}(\mu v, \Delta u - (J_0 + \epsilon I)v), \qquad (u,v) \in D(A_\mu^\epsilon) = H^1(D) \times L^2(D).$$

Suitable uniform bounds with respect to μ have to proven in order to prove the convergence in an appropriate sense of the semigroup $S_\mu^\epsilon(t)$ to the semigroup $T_\epsilon(t)$ associated with the linear differential operator $(J_0 + \epsilon I)^{-1}\Delta$ in equation (3.6).

Next, as the nonlinearities B and G are assumed to be Lipschitz-continuous, in order to obtain (3.5) the whole point is showing that the stochastic convolution associated with equation (3.4) converges to the stochastic convolution associated with equation (3.6). To this purpose, we have to distinguish the case of additive noise (G constant) and of multiplicative noise (G depending on the state u). As a matter of fact, while for additive noise the result is true in any space dimension, for multiplicative noise we are only able to treat the case of space dimension $d = 1$ (see also [3] for an analogous situation). In both cases, one of the key tools in the proof is the stochastic factorization formula combined with a-priori bounds.

Once we have obtained (3.5), we show that the regularized problems (3.4) and (3.6) provide a good approximation for the original problems (3.1) and (3.3), where the magnetic field is acting in absence of friction. Thus, we prove that for any fixed $\mu > 0$ and for any $\epsilon > 0$ small enough the solution u^ϵ_μ of the regularized system (3.4) is close to the solution of the original system (3.1). More precisely,

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}|u^\epsilon_\mu(t) - u_\mu(t)|^p_{L^2(D;\mathbb{R}^2)} = 0, \tag{3.7}$$

and

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\left|\frac{\partial u^\epsilon_\mu}{\partial t}(t) - \frac{\partial u_\mu}{\partial t}(t)\right|^p_{H^{-1}(D;\mathbb{R}^2)} = 0. \tag{3.8}$$

In the same way, we prove that

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}|u_\epsilon(t) - u(t)|^p_{L^2(D;\mathbb{R}^2)} = 0, \tag{3.9}$$

where $u(t)$ is the solution of system (3.3). To this purpose, we would like to stress that system (3.3) is not of parabolic type and the semigroup $T_0(t)$ associated with the differential operator

$$J_0^{-1}\Delta(u_1, u_2) = (-\Delta u_2, \Delta u_1)$$

is not analytic in $L^2(D;\mathbb{R}^2)$ (in fact, it is an isometry). In particular, equation (3.3) is not well posed in $L^2(D;\mathbb{R}^2)$ under the minimal regularity assumptions on the noise required for systems (3.1), (3.4) and (3.6) to be well posed and for limit (3.5) to hold. Actually, the noise in system (3.3) has to be assumed to be taking values in $L^2(D;\mathbb{R}^2)$ (which means that the covariance of the noise is a trace-class operator). Moreover, in spite of the fact that both system (3.1) and system (3.4) are well defined under weaker regularity conditions on the noise, limit (3.7) is true only if the covariance is trace-class.

3.2 Assumptions and notations

Let us assume that D is a bounded regular domain in $\mathbb{R}^d$, with $d \geq 1$. In what follows, we shall denote by H the Hilbert space $L^2(D,\mathbb{R}^2)$, endowed with the scalar product

$$\langle (x_1,y_1),(x_2,y_2)\rangle_H = \int_D x_1(\xi)x_2(\xi)\,d\xi + \int_D y_1(\xi)y_2(\xi)\,d\xi,$$

and the corresponding norm $|\cdot|_H$.

Now, let $\hat{A}$ denote the realization of the Laplace operator in $L^2(D;\mathbb{R})$, endowed with Dirichlet boundary conditions. Then there exists an orthonormal basis $\{\hat{e}_k\}$ for $L^2(D)$ and a positive sequence $\{\hat{\alpha}_k\}$ such that $\hat{A}\hat{e}_k = -\hat{\alpha}_k\hat{e}_k$, with $0 < \hat{\alpha}_1 \leq \hat{\alpha}_k \leq \hat{\alpha}_{k+1}$. Thus, if we define for any $k \in \mathbb{N}$,

$$e_{2k-1} = (\hat{e}_k, 0),\ \ \alpha_{2k} = \hat{\alpha_k},$$

$$e_{2k} = (0, \hat{e}_k),\ \ \alpha_{2k+1} = \hat{\alpha_k},$$

we have that $\{e_k\}_{k=1}^{\infty}$ is a complete orthonormal basis of H. Moreover, if we define

$$D(A) = D(\hat{A}) \times D(\hat{A}), \quad A(x,y) = (\hat{A}x, \hat{A}y), \quad (x,y) \in D(A),$$

we have that

$$Ae_k = -\alpha_k e_k, \qquad k \in \mathbb{N}.$$

Next, for any $\delta \in \mathbb{R}$, we define H^δ to be the completion of $C_0^\infty(D;\mathbb{R}^2)$ with respect to the norm

$$|u|_{H^\delta}^2 = \sum_{k=1}^{\infty} \alpha_k^\delta \langle u, e_k\rangle_H^2 .$$

Moreover, we define $\mathcal{H}_\delta := H^\delta \times H^{\delta-1}$, and in the case $\delta = 0$ we simply set $\mathcal{H} := \mathcal{H}_0$. Finally, for any $(x,y) \in \mathcal{H}_\delta$, we denote

$$\Pi_1(x,y) = x \in H^\delta, \qquad \Pi_2(x,y) = y \in H^{\delta-1}.$$

The cylindrical Wiener process $w^Q(t)$ is defined as the formal sum

$$w^Q(t) = \sum_{k=1}^{\infty} Qe_k\beta_k(t),$$

where $Q = (Q_1, Q_2) \in \mathcal{L}(H)$, $\{\beta_k\}$ is a sequence of identical, independently distributed one-dimensional, Brownian motions defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{e_k\}$ is the orthonormal basis of $\mathcal{H}$ introduced above.

Concerning the non-linearity B we assume the following conditions

Hypothesis 4. *The mapping $B : H \times [0,+\infty) \to H$ is measurable. Moreover, for any $T > 0$ there exists $\kappa_B(T) > 0$ such that*

$$|B(x,t) - B(y,t)|_H \leq \kappa_B(T)|x-y|_H, \qquad x,y \in H, \quad t \in [0,T],$$

and

$$\sup_{t\in[0,T]} |B(0,t)|_H \leq \kappa_B(T).$$

In the case there exists some measurable $b : \mathbb{R} \times D \times [0,+\infty) \to \mathbb{R}$ such that for any $x \in L^2(D)$ and $t \geq 0$

$$B(x,t)(\xi) = b(x(\xi),\xi,t), \quad \xi \in D,$$

then Hypothesis 4 is satisfied if $b(\cdot,\xi,t) : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and has linear growth, uniformly with respect to $\xi \in D$ and $t \in [0,T]$, for any $T > 0$.

Concerning the diffusion coefficient G, we assume the following

Hypothesis 5. *The mapping $G : H \times [0,+\infty) \to \mathcal{L}(L^\infty(D);H)$ is measurable and for any $T > 0$ there exists $\kappa_G(T) > 0$ such that*

$$|[G(x,t) - G(y,t)]\, z|_H \leq \kappa_G(T)|x-y|_H|z|_\infty, \quad x,y \in H, \;\; z \in L^\infty(D), \quad t \in [0,T],$$

and

$$\sup_{t\in[0,T]} |G(0,t)z|_H \leq \kappa_G(T)|z|_\infty, \quad z \in L^\infty(D), \quad t \in [0,T].$$

In particular, this implies that for any $x,y,z \in H$

$$|[G^\star(x,t) - G^\star(y,t)]z|_{(L^\infty(D))'} \leq \kappa_G(T)|x-y|_H\,|z|_H, \quad t \in [0,T]. \tag{3.10}$$

If for any $x \in L^2(D)$ and $z \in L^\infty(D)$ we define

$$[G(x,t)z](\xi) = g(x(\xi),\xi,t)z(\xi), \quad \xi \in D,$$

for some measurable $g : \mathbb{R}^2 \times D \times [0,+\infty] \to \mathcal{L}(\mathbb{R}^2)$, then Hypothesis 5 is satisfied if

$$\sup_{\xi\in D}\sup_{t\in[0,T]} |g(x,\xi,t) - g(y,\xi,t)|_{\mathcal{L}(\mathbb{R}^2)} \leq \kappa_T|x-y|_{\mathbb{R}^2}$$

and that it has linear growth

$$\sup_{\xi \in D} \sup_{t \in [0,T]} |g(x,\xi,t)|_{\mathcal{L}(\mathbb{R}^2)} \leq \kappa_T (1 + |x|_{\mathbb{R}^2}).$$

Actually, in this case

$$|(G(x_1,t) - G(x_2,t))y|_H^2 = \int_D |(g(x_2(\xi),\xi,t) - g(x_2(\xi),\xi,t))y(\xi)|_{\mathbb{R}^2}^2 d\xi$$
$$\leq \kappa_T \int_D |x_2(\xi) - x_1(\xi)|_{\mathbb{R}^2}^2 |y(\xi)|_{\mathbb{R}^2}^2 d\xi \leq |x_2 - x_1|_H^2 |y|_\infty^2,$$

and by the same reasoning

$$|G(x,t)y|_H \leq \kappa_T (1 + |x|_H)|y|_\infty. \tag{3.11}$$

Now, for any $\mu > 0$ and $\delta \in \mathbb{R}$, we define on $\mathcal{H}_\delta$ the unbounded linear operator

$$A_\mu(u,v) = \frac{1}{\mu}(\mu v, Au - J_0 v), \qquad (u,v) \in D(A_\mu) = \mathcal{H}_{\delta+1},$$

where J_0 is the skew symmetric 2×2 matrix

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It can be proven that A_μ is the generator of a strongly continuous group of bounded linear operators $\{S_\mu(t)\}_{t \geq 0}$ on each $\mathcal{H}_\delta$ (for a proof see [18, Section 7.4]).

Moreover, for any $\mu > 0$ we define

$$B_\mu : \mathcal{H} \times [0,+\infty) \to \mathcal{H}, \qquad (z,t) \in \mathcal{H} \times [0,+\infty) \mapsto \frac{1}{\mu}(0, B(\Pi_1 z, t)),$$

and

$$G_\mu : \mathcal{H} \times [0,+\infty) \to \mathcal{L}(L^\infty(D), \mathcal{H}), \qquad (z,t) \in \mathcal{H} \times [0,+\infty) \mapsto \frac{1}{\mu}(0, G(\Pi_1 z, t)).$$

With these notations, if we set

$$z_\mu(t) = \left(u_\mu(t), \frac{\partial u_\mu}{\partial t}(t)\right),$$

system (3.1) can be rewritten as the following stochastic equation in the Hilbert space $\mathcal{H}$

$$dz_\mu(t) = [A_\mu z_\mu(t) + B_\mu(z_\mu(t),t)]\, dt + G_\mu(z_\mu(t),t) dw^Q(t), \qquad z_\mu(0) = (u_0, v_0). \tag{3.12}$$

3.3 The approximating semigroup

In what follows we will consider (3.4). For any $\mu, \epsilon > 0$ and $\delta \in \mathbb{R}$, we define

$$A_\mu^\epsilon(u,v) = \frac{1}{\mu}(\mu\, v, Au - J_\epsilon v), \qquad (u,v) \in D(A_\mu^\epsilon) = \mathcal{H}_{\delta+1},$$

where

$$J_\epsilon = J_0 + \epsilon I = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix}, \qquad \epsilon > 0.$$

As we have seen in the previous section for A_μ, it is possible to prove that for any $\mu, \epsilon > 0$ the operator A_μ^ϵ generates a strongly continuous group of bounded linear operators $S_\mu^\epsilon(t)$, $t \geq 0$, on $\mathcal{H}_\delta$.

Lemma 3.3.1. *For any $(x,y) \in \mathcal{H}_\theta$, with $\theta \in \mathbb{R}$, and for any $\mu, \epsilon > 0$ let us define*

$$u_\mu^\epsilon(t) := \Pi_1 S_\mu^\epsilon(t)(x,y), \qquad v_\mu^\epsilon(t) := \Pi_2 S_\mu^\epsilon(t)(x,y).$$

Then

$$\mu \left|v_\mu^\epsilon(t)\right|^2_{H^{\theta-1}} + |u_\mu^\epsilon(t)|^2_{H^\theta} + 2\epsilon \int_0^t |v_\mu^\epsilon(s)|^2_{H^{\theta-1}}\, ds = \mu |y|^2_{H^{\theta-1}} + |x|_{H^\theta}, \tag{3.13}$$

and

$$\mu|u^\epsilon_\mu(t)|^2_{H^\theta} + \left|\mu v^\epsilon_\mu(t) + J_\epsilon u^\epsilon_\mu(t)\right|^2_{H^{\theta-1}} + 2\epsilon\int_0^t |u^\epsilon_\mu(s)|^2_{H^\theta}\,ds = \mu|x|^2_{H^\theta} + |\mu y + J_\epsilon x|^2_{H^{\theta-1}}. \tag{3.14}$$

Proof. Since

$$\frac{\partial u^\epsilon_\mu}{\partial t}(t) = v^\epsilon_\mu(t),$$

we have

$$\mu\frac{\partial v^\epsilon_\mu}{\partial t}(t) + J_\epsilon v^\epsilon_\mu(t) = A u^\epsilon_\mu(t). \tag{3.15}$$

Then, if we take the scalar product of both sides above with v^ϵ_μ in $H^{\theta-1}$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\mu\,\left|v^\epsilon_\mu(t)\right|^2_{H^{\theta-1}} + \left|u^\epsilon_\mu(t)\right|^2_{H^\theta}\right) = -\epsilon\left|v^\epsilon_\mu(t)\right|^2_{H^{\theta-1}},$$

which implies (3.13), as $u^\epsilon_\mu(0) = x$ and $v^\epsilon_\mu(0) = y$.

Next, by using again (3.15), we have

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left|\mu v^\epsilon_\mu(t) + J_\epsilon u^\epsilon_\mu(t)\right|^2_{H^{\theta-1}} = \left\langle \mu v^\epsilon_\mu(t) + J_\epsilon u^\epsilon_\mu(t), A u^\epsilon_\mu(t)\right\rangle_{H^{\theta-1}}\\
&= -\frac{\mu}{2}\frac{d}{dt}\left|u^\epsilon_\mu(t)\right|^2_{H^\theta} - \epsilon\left|u^\epsilon_\mu(t)\right|^2_{H^\theta},
\end{aligned}$$

and then, integrating with respect to time, we get (3.14). □

Notice that in particular this implies that for any $\mu, \epsilon > 0$ there exists $c_{\mu,\epsilon} > 0$ such that for any $(x, y) \in \mathcal{H}_\theta$

$$\int_0^\infty |S^\epsilon_\mu(t)(x,y)|^2_{\mathcal{H}_\theta}\,dt \le \frac{c_{\mu,\epsilon}}{2\epsilon}|(x,y)|^2_{\mathcal{H}_\theta}.$$

As a consequence of the Datko theorem (see [1] for a proof), we can conclude that there exist $M_{\mu,\epsilon}$, and $\omega_{\mu,\epsilon} > 0$ such that

$$\|S^\epsilon_\mu(t)\|_{\mathcal{L}(\mathcal{H}_\theta)} \le M_{\mu,\epsilon}\,e^{-\omega_{\mu,\epsilon}t}, \qquad t \ge 0.$$

Lemma 3.3.2. *For any $\mu, \epsilon > 0$, and for any $\theta \in \mathbb{R}$ and $\gamma \in [0,1]$ it holds*

$$\left|\Pi_1 S^\epsilon_\mu(t)(0,y)\right|_{H^\theta} \leq 2^\gamma \mu^{\frac{1+\gamma}{2}} |y|_{H^{\theta+\gamma-1}}, \quad t \geq 0, \quad y \in H^{\theta+\gamma-1}. \tag{3.16}$$

Proof. Let $u^\epsilon_\mu(t) := \Pi_1 S^\epsilon_\mu(t)(0,y)$. By (3.13),

$$\left|u^\epsilon_\mu(t)\right|^2_{H^{\theta+\gamma}} \leq \mu |y|^2_{H^{\theta-1}}.$$

Notice that for any $x \in \mathbb{R}^2$, $|J_\epsilon x|^2_{\mathbb{R}^2} = (1+\epsilon^2)|x|_{\mathbb{R}^2}$. Then by (3.14) and (3.13),

$$\begin{aligned}
&\left|u^\epsilon_\mu(t)\right|^2_{H^{\theta+\gamma-1}} \leq (1+\epsilon^2)\left|u^\epsilon_\mu(t)\right|^2_{H^{\theta+\gamma-1}} = \left|J_\epsilon u^\epsilon_\mu(t)\right|^2_{H^{\theta+\gamma-1}} \\
&\leq 2\left|\mu \frac{\partial u^\epsilon_\mu}{\partial t}(t) + J_\epsilon u^\epsilon_\mu(t)\right|^2_{H^{\theta+\gamma-1}} + 2\mu^2 \left|\frac{\partial u^\epsilon_\mu}{\partial t}(t)\right|^2_{H^{\theta+\gamma-1}} \leq 4\mu^2 |y|^2_{H^{\theta+\gamma-1}}.
\end{aligned}$$

Then, since for any $x \in H^{\theta+\gamma}$

$$|x|_{H^\theta} \leq |x|^\gamma_{H^{\theta+\gamma-1}} |x|^{1-\gamma}_{H^{\theta+\gamma}},$$

we conclude that

$$|u^\epsilon_\mu(t)|_{H^\theta} \leq |u^\epsilon_\mu(t)|^\gamma_{H^{\theta+\gamma-1}} |u^\epsilon_\mu(t)|^{1-\gamma}_{H^{\theta+\gamma}} \leq 2^\gamma \mu^{\frac{1+\gamma}{2}} |y|_{H^{\theta+\gamma-1}}.$$

$\square$

Now, for any $\mu > 0$ we define the bounded linear operator

$$Q_\mu : H \to \mathcal{H}, \quad x \in H \mapsto \frac{1}{\mu}(0, Qx) \in \mathcal{H}.$$

Lemma 3.3.3. *Assume that there exists a non-negative sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that*

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

Then, for any $0 < \delta < 1$ and $\epsilon > 0$ there exists a constant $c = c(\epsilon, \delta) > 0$ such that for any $k \in \mathbb{N}$ and $\theta > 0$

$$\sup_{\mu>0} \int_0^\infty s^{-\delta} \left|\Pi_1 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_H ds \leq c\, \frac{\lambda_k^2}{\alpha_k^{1-\delta}}, \tag{3.17}$$

and

$$\sup_{\mu>0} \mu^{1+\delta} \int_0^\infty s^{-\delta} \left|\Pi_2 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_{H^{\theta-1}} ds \leq c\,\frac{\lambda_k^2}{\alpha_k^{1-\theta}}. \tag{3.18}$$

Proof. We have

$$\int_0^\infty s^{-\delta} \left|\Pi_1 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_H ds = \int_0^{\alpha_k^{-1}} s^{-\delta} \left|\Pi_1 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_H ds + \int_{\alpha_k^{-1}}^\infty s^{-\delta} \left|\Pi_1 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_H ds.$$

Due to (3.16), with $\theta = 0$ and $\gamma = 1$, we have

$$\int_0^{\alpha_k^{-1}} s^{-\delta} \left|\Pi_1 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_H ds \leq 4\,|Qe_k|^2_H \int_0^{\alpha_k^{-1}} s^{-\delta}\, ds = \frac{4}{1-\delta}\frac{\lambda_k^2}{\alpha_k^{1-\delta}}. \tag{3.19}$$

Moreover, due to (3.14) we have

$$\int_{\alpha_k^{-1}}^\infty s^{-\delta} \left|\Pi_1 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_H ds \leq \frac{1}{2\epsilon}\alpha_k^{-\delta}|Qe_k|^2_{H^{-1}} = \frac{1}{2\epsilon}\frac{\lambda_k^2}{\alpha_k^{1-\delta}}.$$

Together with (3.19) this implies (3.17).

To establish (3.18), we write

$$\begin{aligned}
&\int_0^\infty s^{-\delta} \left|\Pi_2 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_{H^{\theta-1}} ds \\
&= \int_0^\mu s^{-\delta} \left|\Pi_2 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_{H^{\theta-1}} ds + \int_\mu^\infty s^{-\delta} \left|\Pi_2 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_{H^{\theta-1}} ds.
\end{aligned}$$

Thanks to (3.13) we have

$$\int_0^\mu s^{-\delta} \left|\Pi_2 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_{H^{\theta-1}} ds \leq \frac{2}{\mu^2}|Qe_k|^2_{H^{\theta-1}} \int_0^\mu s^{-\delta}\, ds = \frac{1}{\mu^{1+\delta}}\frac{2}{1-\delta}\frac{\lambda_k^2}{\alpha_k^{1-\theta}},$$

and

$$\int_\mu^\infty s^{-\delta} \left|\Pi_2 S^\epsilon_\mu(s) Q_\mu e_k\right|^2_{H^{\theta-1}} ds \leq \frac{1}{2\epsilon}\mu^{-\delta}\frac{1}{\mu}|Qe_k|^2_{H^{\theta-1}} = \frac{1}{2\epsilon}\frac{1}{\mu^{1+\delta}}\frac{\lambda_k^2}{\alpha_k^{1-\theta}},$$

and these two estimates together imply (3.18). $\square$

Now, for any $\epsilon > 0$ we define

$$A_\epsilon := J_\epsilon^{-1} A = \frac{1}{1+\epsilon^2}\begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix} \hat{A}, \tag{3.20}$$

and we denote by $T_\epsilon(t)$, $t \geq 0$, the strongly continuous semigroup generated by A_ϵ in H^θ, for any $\theta \in \mathbb{R}$. Moreover, we denote

$$Q_\epsilon = J_\epsilon^{-1} Q.$$

Lemma 3.3.4. *We have*

$$\|T_\epsilon(t)\|_{\mathcal{L}(H^\theta)} \leq e^{-\frac{\epsilon \alpha_1}{1+\epsilon^2} t}, \qquad t \geq 0. \tag{3.21}$$

Moreover, if there exists a non-negative sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ *such that*

$$Q e_k = \lambda_k e_k, \qquad k \in \mathbb{N},$$

then, for any $0 < \delta < 1$ *and* $\epsilon > 0$ *there exists a constant* $c = c(\delta, \epsilon)$ *such that for any* $k \in \mathbb{N}$

$$\int_0^\infty s^{-\delta} \left| T_\epsilon(s) Q_\epsilon e_k \right|_H^2 ds \leq c \frac{\lambda_k^2}{\alpha_k^{1-\delta}}. \tag{3.22}$$

Finally, for any $k \in \mathbb{N}$

$$\int_0^T s^{-\delta} \left| T_\epsilon(s) Q_\epsilon e_k \right|_H^2 ds \leq \frac{1}{1-\delta} T^{1-\delta} \lambda_k^2. \tag{3.23}$$

Proof. Let $x \in H^\theta$, and $u_\epsilon(t) = T_\epsilon(t)x$. This means

$$\frac{\partial u_\epsilon}{\partial t}(t) = A_\epsilon u_\epsilon(t) = \frac{1}{1+\epsilon^2} \begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix} \hat{A} u_\epsilon(t)$$

Then, if we take the scalar product in H^θ of the above equation by $u_\epsilon(t)$

$$\begin{aligned} &\frac{d}{dt}\left(|u_\epsilon(t)|_{H^\theta}^2 \right) = -\frac{2\epsilon}{1+\epsilon^2}\left(|u_\epsilon(t)|_{H^{1+\theta}}^2 \right) \\ &\leq -\frac{2\epsilon\alpha_1}{1+\epsilon^2}\left(|u_\epsilon(t)|_{H^\theta}^2 \right), \end{aligned}$$

and this implies (3.21).

In order to prove (3.22), we observe that if $u_\epsilon(t) = T_\epsilon(t)Q_\epsilon e_k$, then

$$\frac{\partial u_\epsilon}{\partial t}(t) = -\alpha_k J_\epsilon^{-1} u_\epsilon(t).$$

By the same arguments that we used above,

$$|u_\epsilon(t)|_H = e^{-\frac{\epsilon\alpha_k}{\epsilon^2+1}t}|Q_\epsilon e_k|_H = \frac{\lambda_k}{\sqrt{\epsilon^2+1}} e^{-\frac{\epsilon\alpha_k}{\epsilon^2+1}t},$$

and therefore,

$$\begin{aligned}
&\int_0^\infty s^{-\delta}\,|T_\epsilon(s)Q_\epsilon e_k|_H^2\,ds \leq \frac{\lambda_k^2}{(1+\epsilon^2)}\int_0^\infty s^{-\delta}e^{-\frac{2\epsilon\alpha_k}{1+\epsilon^2}s}ds \\
&\leq \frac{\lambda_k^2}{\epsilon^2+1}\left(\int_0^{\alpha_k^{-1}} s^{-\delta}ds + \alpha_k^\delta\int_{\alpha_k^{-1}}^\infty e^{-\frac{2\epsilon\alpha_1}{1+\epsilon^2}s}ds\right) \\
&\leq \frac{\lambda_k^2}{\epsilon^2+1}\left(\frac{1}{1-\delta}\alpha_k^{\delta-1} + \alpha_k^\delta\frac{1+\epsilon^2}{2\epsilon\alpha_k}\right) \leq c(\alpha,\epsilon)\,\frac{\lambda_k^2}{\alpha_k^{1-\delta}}.
\end{aligned}$$

Finally, in order to prove (3.23), we notice that

$$\int_0^T s^{-\delta}\,|T_\epsilon(s)Q_\epsilon e_k|_H^2\,ds \leq \lambda_k^2\int_0^T s^{-\delta}\,ds = \frac{T^{1-\delta}}{1-\delta}\,\lambda_k^2.$$

□

In view of the previous estimates for $S_\mu^\epsilon(t)$ and $T_\epsilon(t)$, we can prove the following convergence result.

Theorem 3.3.5. *For any $\epsilon > 0$, $0 < t_0 < T$, and $n \in \mathbb{N}$,*

$$\lim_{\mu\to 0}\sup_{t\leq T}\sup_{|x|_H\leq 1}\left|\Pi_1 S_\mu^\epsilon(t)(P_n x, 0) - T_\epsilon(t)P_n x\right|_H = 0, \tag{3.24}$$

and

$$\lim_{\mu\to 0}\sup_{0<t_0\leq t\leq T}\sup_{|y|_H\leq 1}\left|\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(t)(0, P_n y) - T_\epsilon(t)J_\epsilon^{-1}P_n y\right|_H = 0, \tag{3.25}$$

where P_n is the projection of H onto the n-dimensional subspace $H_n := \operatorname{span}\{e_1, \ldots, e_{2n}\}$.

Proof. Fix $k \in \mathbb{N}$, and let us consider the function $u^\epsilon_\mu(t) = \Pi_1 S^\epsilon_\mu(t)\left(x, \frac{y}{\mu}\right)$, with $x, y \in \text{span}\{e_{2k-1}, e_{2k}\}$. We have

$$\begin{cases} \mu \dfrac{\partial^2 u^\epsilon_\mu}{\partial t^2}(t) + J_\epsilon \dfrac{\partial u^\epsilon_\mu}{\partial t}(t) = -\alpha_{2k} u^\epsilon_\mu(t) \\ u^\epsilon_\mu(0) = x, \qquad \dfrac{\partial u^\epsilon_\mu}{\partial t}(0) = \dfrac{y}{\mu}, \end{cases}$$

so that

$$\frac{d}{dt}\left(e^{\frac{J_\epsilon}{\mu}t} \frac{\partial u^\epsilon_\mu}{\partial t}(t)\right) = -\frac{\alpha_{2k}}{\mu}\, e^{\frac{J_\epsilon}{\mu}t} u^\epsilon_\mu(t).$$

By integrating in time, we see that

$$\frac{\partial u^\epsilon_\mu}{\partial t}(t) = e^{-\frac{J_\epsilon}{\mu}t}\frac{y}{\mu} - \frac{\alpha_{2k}}{\mu}\int_0^t e^{-\frac{J_\epsilon}{\mu}(t-s)} u^\epsilon_\mu(s)ds.$$

Integrating once again, and exchanging the order of integration, we conclude that $\Pi_1 S^\epsilon_\mu(t)(x, y/\mu) = u^\epsilon_\mu(t)$ solves

$$u^\epsilon_\mu(t) = x + \left(I - e^{-\frac{J_\epsilon}{\mu}t}\right) J_\epsilon^{-1} y - \alpha_{2k}\int_0^t \left(I - e^{-\frac{J_\epsilon}{\mu}(t-s)}\right) J_\epsilon^{-1} u^\epsilon_\mu(s)ds. \tag{3.26}$$

Now, since $T_\epsilon(t)x$ solves the equation

$$T_\epsilon(t)x = x - \alpha_{2k}\int_0^t J_\epsilon^{-1} T_\epsilon(s)x ds,$$

from (3.26) and (3.13), we get

$$\begin{aligned} &\left|\Pi_1 S^\epsilon_\mu(t)(x,0) - T_\epsilon(t)x\right|_H \\ &\leq \alpha_{2k}\int_0^t \left|J_\epsilon^{-1}(\Pi_1 S^\epsilon_\mu(s)(x,0) - T_\epsilon(s)x)\right|_H ds + \alpha_{2k}\int_0^t |e^{-\frac{J_\epsilon}{\mu}(t-s)} J_\epsilon^{-1} S^\epsilon_\mu(s)(x,0)|_H ds. \\ &\leq \alpha_{2k}\int_0^t \left|\Pi_1 S^\epsilon_\mu(s)(x,0) - T_\epsilon(s)x\right|_H ds + \alpha_{2k}\int_0^t e^{-\frac{\epsilon}{\mu}(t-s)}\, ds |x|_H. \end{aligned}$$

From Grönwall's inequality we see that for $0 \leq t \leq T$,

$$\left|\Pi_1 S^\epsilon_\mu(t)(x,0) - T_\epsilon(t)x\right|_H \leq \frac{\mu}{\epsilon}\,\alpha_{2k}|x|_H e^{\alpha_{2k}T},$$

and this yields (3.24).

We prove (3.25) analogously, by taking $x = 0$ in (3.26). In this case, thanks to (3.16) and the fact that $\|J_\epsilon^{-1}\|_{\mathcal{L}(\mathbb{R}^2)} \leq 1$.

$$
\begin{aligned}
&\left|\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(t)(0,y) - T_\epsilon(t)J_\epsilon^{-1}y\right|_H \leq \left|e^{-\frac{J_\epsilon}{\mu}t}J_\epsilon^{-1}y\right|_H \\
&+\alpha_{2k}\int_0^t \left|J_\epsilon^{-1}\left(\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(s)(0,y) - T_\epsilon(s)J_\epsilon^{-1}y\right)\right|_H ds \\
&+\alpha_{2k}\int_0^t \left|e^{-\frac{J_\epsilon}{\mu}(t-s)}J_\epsilon^{-1}\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(s)(0,y)\right|_H ds \\
&\leq e^{-\frac{\epsilon}{\mu}t}|y|_H + \alpha_{2k}\int_0^t \left|\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(s)(0,y) - T_\epsilon(s)J_\epsilon^{-1}y\right|_H ds + \alpha_{2k}\int_0^t e^{-\frac{\epsilon}{\mu}(t-s)}\,ds|y|_H \\
&\leq \left(e^{-\frac{\epsilon}{\mu}t} + \frac{\alpha_{2k}\mu}{\epsilon}\right)|y|_H + \alpha_{2k}\int_0^t \left|\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(s)(0,y) - T_\epsilon(s)J_\epsilon^{-1}y\right|_H ds.
\end{aligned}
$$

By Grönwall's inequality,

$$
\left|\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(t)(0,y) - T_\epsilon(t)J_\epsilon^{-1}y\right|_H \leq \left(e^{-\frac{\epsilon t}{\mu}}\,|y|_H + \frac{\mu\alpha_{2k}}{\epsilon}|y|_H\right)e^{\alpha_{2k}T}, \qquad t \in [0,T],
$$

and this implies (3.25). $\square$

Corollary 3.3.6. *For any $\epsilon > 0$ and $T > 0$ and for any $(x,y) \in \mathcal{H}$,*

$$
\lim_{\mu\to 0}\sup_{t\leq T}|\Pi_1 S_\mu^\epsilon(t)(x,y) - T_\epsilon(t)x|_H = 0. \tag{3.27}
$$

Moreover, for any $y \in H$ and $0 < t_0 \leq T$,

$$
\lim_{\mu\to 0}\sup_{t_0\leq t\leq T}\left|\frac{1}{\mu}\Pi_1 S_\mu^\epsilon(t)\,(0,y) - T_\epsilon(t)J_\epsilon^{-1}y\right|_H = 0. \tag{3.28}
$$

Proof. We have

$$
\begin{aligned}
&|\Pi_1 S_\mu^\epsilon(t)(x,y) - T_\epsilon(t)x|_H \leq |\Pi_1 S_\mu^\epsilon(t)(0,y)|_H + |\Pi_1 S_\mu^\epsilon(t)(P_n x,0) - T_\epsilon(t)P_n x|_H \\
&+|\Pi_1 S_\mu^\epsilon(t)(x - P_n x,0)| + \sup_{t\leq T}|T_\epsilon(t)(x-P_n x)|_H := I_1(t) + \sum_{j=2}^4 I_{n,j}(t).
\end{aligned}
$$

By (3.13) and (3.21), for any $\eta > 0$ there exists $n_\eta \in \mathbb{N}$ such that

$$I_{n_\eta,3}(t) + I_{n_\eta,4}(t) \leq 3|x - P_{n_\eta} x|_H \leq \frac{\eta}{3}, \qquad t \geq 0.$$

Moreover, by (3.24), we can then find μ_1 such that for $\mu < \mu_1$

$$\sup_{t \in [0,T]} I_{n_\eta,2}(t) < \frac{\eta}{3},$$

and then since from (3.13)

$$\sup_{t \geq 0} I_1(t) \leq 2\mu |y|_H,$$

we can conclude that

$$\sup_{t \in [0,T]} |\Pi_1 S^\epsilon_\mu(t)(x,y) - T_\epsilon(t)x|_H \leq \eta, \quad \mu \leq \mu_0,$$

and (3.27) follows from the arbitrariness of $\eta > 0$.

In order to prove (3.28), we have

$$\left|\frac{1}{\mu} S^\epsilon_\mu(t)\,(0,y) - T_\epsilon(t) J_\epsilon^{-1} y\right|_H \leq \left|\frac{1}{\mu} S^\epsilon_\mu(t)\,(0, P_n y) - T_\epsilon(t) J_\epsilon^{-1} P_n y\right|_H$$
$$+ \left|\frac{1}{\mu}\Pi_1 S^\epsilon_\mu(t)\,(0, y - P_n y)\right|_H + \left|T_\epsilon(t) J_\epsilon^{-1}(y - P_n y)\right|_H := \sum_{j=1}^{3} I_{n,j}(t).$$

By Lemma 3.3.2 and (3.21), we have

$$I_{n,2}(t) + I_{n,3}(t) \leq c|y - P_n y|_H, \qquad t \geq 0.$$

Then, for any $\eta > 0$ we can fix $n_\eta \in \mathbb{N}$ such that

$$\sup_{t \geq 0} I_{n,2}(t) + I_{n,3}(t) \leq \frac{\eta}{2}.$$

Moreover, thanks to (3.25), we can find μ_0 such that for all $\mu < \mu_0$,

$$\sup_{t \in [t_0,T]} I_{n_\eta,1}(t) < \frac{\eta}{2}.$$

Because $\eta > 0$ was arbitrary, (3.28) follows. $\square$

Corollary 3.3.7. *For any $\epsilon > 0$, $T > 0$ and $p \geq 1$ and for any $\psi \in L^p(\Omega; L^p([0,T]; H))$,*

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} \left| \frac{1}{\mu} \int_0^t \Pi_1 S^\epsilon_\mu(t-s)(0,\psi(s))ds - \int_0^t T_\epsilon(t-s) J_\epsilon^{-1} \psi(s) ds \right|_H^p = 0. \quad (3.29)$$

Proof. For any $\psi \in L^p(\Omega; L^p([0,T]; H))$ and $n \in \mathbb{N}$, let us define

$$\psi_n(t) = I_{\{|\psi(t)|_H \leq n\}} P_n \psi(t), \qquad t \in [0,T].$$

We have clearly $\psi_n \in L^\infty(\Omega \times [0,T]; H_n)$ and by the dominated convergence theorem

$$\lim_{n \to \infty} \mathbb{E} |\psi_n - \psi|^p_{L^p(0,T;H)} = 0.$$

If for any $\mu, \epsilon > 0$ and $t \geq 0$ we define

$$\Phi^\mu_\epsilon(t) y = \frac{1}{\mu} \Pi_1 S^\epsilon_\mu(t)(0,y) - T_\epsilon(t) J_\epsilon^{-1} y, \qquad y \in H,$$

for any $0 < \delta < t$, we have

$$\begin{aligned}
&\frac{1}{\mu} \int_0^t \Pi_1 S^\epsilon_\mu(t-s)(0,\psi(s))ds - \int_0^t T_\epsilon(t-s) J_\epsilon^{-1} \psi(s) ds \\
&= \int_0^t \Phi^\mu_\epsilon(t-s)(\psi(s) - \psi_n(s))ds + \int_0^{t-\delta} \Phi^\mu_\epsilon(t-s)\psi_n(s)ds \\
&+ \int_{t-\delta}^t \Phi^\mu_\epsilon(t-s)\psi_n(s)ds := I_{n,1}(t) + I_{n,2}(\delta,t) + I_{n,3}(\delta,t).
\end{aligned}$$

As a consequence of Lemma 3.3.2 and (3.21), we have that for any fixed $\epsilon > 0$

$$\sup_{t \in [0,T]} \sup_{\mu > 0} \|\Phi^\mu_\epsilon(t)\|_{\mathcal{L}(H)} := M < +\infty,$$

so that

$$\mathbb{E} \sup_{t \in [0,T]} |I_{n,1}(t)|^p_H \leq T^{p-1} M^p \mathbb{E} |\psi - \psi_n|^p_{L^p([0,T];H)}.$$

Therefore, for any $\eta > 0$ there exists $n_\eta \in \mathbb{N}$ such that

$$\mathbb{E} \sup_{t \in [0,T]} |I_{n_\eta,1}(t)|^p_H < \frac{\eta}{3}. \quad (3.30)$$

Next, since $|\psi_{n_\eta}(s)|_H \leq n_\eta$, for every $s \in [0,T]$, we have

$$\delta_\eta(t) = \left(\frac{\eta}{3}\right)^{\frac{1}{p}} \frac{1}{n_\eta M} \wedge t \Longrightarrow \mathbb{E} \sup_{t\in[0,T]} |I_{n,3}(\delta_\eta(t),t)|_H^p \leq \frac{\eta}{3}. \tag{3.31}$$

Finally, by (3.25) we can find $\mu_0 > 0$ small enough so that for $\mu < \mu_0$

$$\sup_{t\in[\delta_\eta(t),T]} \sup_{|y|_H\leq n_\eta} |\Phi_\epsilon^\mu(t)P_{n_\eta}y|_H < \left(\frac{\eta}{3T}\right)^{\frac{1}{p}},$$

so that

$$\mathbb{E} \sup_{t\in[0,T]} |I_{n,2}(\delta_\eta(t),t)|_H^p \leq \frac{\eta}{3}.$$

Together with (3.30) and (3.31), this implies (3.29).

□

3.4 Approximation by small friction for additive noise

In this section, we assume that the noisy perturbation in system (3.1) is of additive type, that is $G(x,t) = I$, for any $x \in H$ and $t \geq 0$. Moreover, we assume that the covariance operator Q satisfies the following condition.

Hypothesis 6. *There exists a non-negative sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $Qe_k = \lambda_k e_k$, for any $k \in \mathbb{N}$. Moreover, there exists $\delta > 0$ such that*

$$\sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k^{1-\delta}} < \infty.$$

With the notations we have introduced in Sections 3.2 and 3.3, if we denote

$$z_\mu^\epsilon(t) = (u_\mu^\epsilon(t), \frac{\partial u_\mu^\epsilon}{\partial t}(t)), \quad t \geq 0,$$

the regularized system (3.4) can be rewritten as the abstract evolution equation

$$dz_\mu^\epsilon(t) = \left[A_\mu^\epsilon z_\mu^\epsilon(t) + B_\mu(z_\mu^\epsilon(t),t)\right] dt + Q_\mu dw(t), \qquad z_\mu^\epsilon(0) = (x,y) \tag{3.32}$$

in the Hilbert space $\mathcal{H}$.

Our purpose here is to prove that for any fixed $\epsilon > 0$ the process $u^\epsilon_\mu(t)$ converges to the solution $u_\epsilon(t)$ of the following system of stochastic PDEs

$$\left\{\begin{array}{l} \dfrac{\partial u_\epsilon}{\partial t}(t) = J_\epsilon^{-1}\Delta u_\epsilon(t) + B_\epsilon(u_\epsilon(t),t) + \dfrac{\partial w^{Q_\epsilon}}{\partial t} \\ u_\epsilon(0) = u_0, \quad\quad u_\epsilon(\xi,t) = 0, \quad \xi \in \partial D, \end{array}\right. \tag{3.33}$$

where for any $\epsilon > 0$ we have defined $Q_\epsilon = J_\epsilon^{-1} Q$ and

$$B_\epsilon(x,t) = J_\epsilon^{-1} B(x,t), \quad x \in H, \quad t \geq 0.$$

Notice that with these notations and A_ϵ defined by (3.20), system (3.33) can be rewritten as the abstract evolution equation

$$du_\epsilon(t) = [A_\epsilon u_\epsilon(t) + B_\epsilon(u_\epsilon(t),t)]\, dt + Q_\epsilon dw(t), \quad\quad u_\epsilon(0) = u_0, \tag{3.34}$$

in the Hilbert space H.

According to Lemma 3.3.3, due to Hypothesis 6 for any $t \geq 0$ we have

$$\int_0^t s^{-\delta} \sum_{k=1}^\infty |S^\epsilon_\mu(t-s) Q_\mu e_k|^2_{\mathcal{H}}\, ds \leq c\left(1+\mu^{-(1+\delta)}\right) \sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k^{1-\delta}}.$$

This implies that the stochastic convolution

$$\Gamma^\epsilon_\mu(t) := \int_0^t S^\epsilon_\mu(s) Q_\mu \, dw(s), \quad\quad t \geq 0,$$

takes values in $L^p(\Omega; C([0,T];\mathcal{H}))$, for any $T > 0$ and $p \geq 1$ (for a proof see [12]). Therefore, as the mapping $B_\mu(\cdot,t) : \mathcal{H} \to \mathcal{H}$ is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$, we have that there exists a unique process $z^\epsilon_\mu \in L^p(\Omega; C([0,T];\mathcal{H}))$ which solves equation (3.32) in the mild sense, that is

$$z^\epsilon_\mu(t) = S^\epsilon_\mu(t)(u_0,v_0) + \int_0^t S^\epsilon_\mu(t-s) B_\mu(z^\epsilon_\mu(s),s)\, ds + \Gamma^\epsilon_\mu(t).$$

In the same way, due to (3.22) we have that the stochastic convolution

$$\Gamma_\epsilon(t) := \int_0^t T_\epsilon(s) Q_\epsilon \, dw(s), \qquad t \geq 0,$$

takes values in $L^p(\Omega; C([0,T]; H))$, for any $T > 0$ and $p \geq 1$, so that, as the mapping $B_\epsilon(\cdot, t) : H \to H$ is Lipschitz-continuous, uniformly with respect to $t \in [0,T]$, we can conclude that there exists a unique process $u_\epsilon \in L^p(\Omega; C([0,T]; H))$ solving equation (3.34) in mild sense, that is

$$u_\epsilon(t) = T_\epsilon(t) u_0 + \int_0^t T_\epsilon(t-s) B_\epsilon(u_\epsilon(s), s) ds + \Gamma_\epsilon(t).$$

Theorem 3.4.1. *Under Hypotheses 4 and 6, for any $\epsilon > 0$, $T > 0$ and $p \geq 1$ and for any initial conditions $z_0 = (u_0, v_0) \in \mathcal{H}$, we have*

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \leq T} \left| u_\mu^\epsilon(t) - u_\epsilon(t) \right|_H^p = 0. \tag{3.35}$$

Proof. Due to Lemma 3.3.2 and the Lipschitz continuity of B, we have

$$\begin{aligned} &\left| u_\mu^\epsilon(t) - u_\epsilon(t) \right|_H \leq \left| \Pi_1 S_\mu^\epsilon(t) z_0 - T_\epsilon(t) u_0 \right|_H + c \int_0^t \left| u_\mu^\epsilon(s) - u_\epsilon(s) \right|_H ds \\ &+ \left| \int_0^t \left[\Pi_1 S_\mu^\epsilon(t-s) B_\mu((u_\epsilon(s), 0), s) - T_\epsilon(t-s) B_\epsilon(u_\epsilon(s), s) \right] ds \right|_H + |\Gamma_\mu^\epsilon(t) - \Gamma_\epsilon(t)|_H, \end{aligned}$$

and then, from Grönwall's Lemma, for any $p \geq 1$ we get

$$\begin{aligned} &\sup_{t \in [0,T]} \left| u_\mu^\epsilon(t) - u_\epsilon(t) \right|_H^p \leq c_p(T) \sup_{t \in [0,T]} \left| \Pi_1 S_\mu^\epsilon(t)(u_0, v_0) - T_\epsilon u_0 \right|_H^p \\ &+ c_p(T) \sup_{t \in [0,T]} \left| \int_0^t \left(\Pi_1 S_\mu^\epsilon(t-s) B_\mu(X_\epsilon(s), s) - T_\epsilon(t-s) B_\epsilon(X_\epsilon(s), s) \right) ds \right|_H^p \\ &+ c_p(T) \sup_{t \in [0,T]} \left| \Gamma_\mu^\epsilon(t) - \Gamma_\epsilon(t) \right|_H^p := c_p(T) \sum_{k=1}^3 \sup_{t \in [0,T]} |I_k(t)|_H^p. \end{aligned}$$

By (3.27)

$$\lim_{\mu \to 0} \sup_{t \in [0,T]} |I_1(t)|_H = 0.$$

Moreover, by (3.29), we know that

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |I_2(t)|_H^p = 0.$$

The analysis of $I_3(t)$ is more delicate. By using the factorization method (see [12, Chapter 5]) for any $\alpha \in (0,1)$ we have

$$\frac{\pi}{\sin(\pi\alpha)} I_3(t) = \int_0^t (t-\sigma)^{\alpha-1} \int_0^\sigma (\sigma-s)^{-\alpha} \left(\Pi_1 S_\mu^\epsilon(t-s)Q_\mu - T_\epsilon(t-s)Q_\epsilon\right) dw(s)d\sigma$$
$$= \int_0^t (t-\sigma)^{\alpha-1} T_\epsilon(t-\sigma) Y_{\mu,1}^\alpha(\sigma)d\sigma + \int_0^t (t-\sigma)^{\alpha-1}\Pi_1 S_\mu^\epsilon(t-\sigma)(0, Y_{\mu,2}^\alpha(\sigma))d\sigma$$
$$+ \int_0^t (t-\sigma)^{\alpha-1} \left[\Pi_1 S_\mu^\epsilon(t-\sigma)\Pi_1^\star - T_\epsilon(t-\sigma)\right] Y_{\mu,3}^\alpha(\sigma)d\sigma,$$

where

$$\begin{cases} Y_{\mu,1}^\alpha(\sigma) := \int_0^\sigma (\sigma-s)^{-\alpha} \left[\Pi_1 S_\mu^\epsilon(\sigma-s)Q_\mu - T_\epsilon(\sigma-s)Q_\epsilon\right] dw(s) \\ Y_{\mu,2}^\alpha(\sigma) := \int_0^\sigma (\sigma-s)^{-\alpha}\Pi_2 S_\mu^\epsilon(\sigma-s)Q_\mu dw(s), \\ Y_{\mu,3}^\alpha(\sigma) := \int_0^\sigma (\sigma-s)^{-\alpha}\Pi_1 S_\mu^\epsilon(\sigma-s)Q_\mu dw(s). \end{cases}$$

We have

$$\mathbb{E}\left|Y_{\mu,1}^\alpha(\sigma)\right|_H^2 \le \int_0^T s^{-2\alpha} \sum_{k=1}^\infty \left|\left[\Pi_1 S_\mu^\epsilon(s)Q_\mu - T_\epsilon(s)Q_\epsilon\right] e_k\right|_H^2 ds.$$

If we choose $\alpha = \frac{\delta}{4}$, then by (3.17), (3.22), (3.28), and Hypothesis 6, from the dominated convergence theorem we have

$$\lim_{\mu\to 0} \int_0^T s^{-\delta/2} \sum_{k=1}^\infty \left|\left[\Pi_1 S_\mu^\epsilon(s)Q_\mu - T_\epsilon(s)Q_\epsilon\right] e_k\right|_H^2 ds = 0.$$

Therefore, from the Gaussianity of $Y_{\mu,1}^{\delta/4}(\sigma)$, for any $p \ge 2$

$$\lim_{\mu\to 0} \sup_{\sigma\le T} \mathbb{E}|Y_{\mu,1}^{\delta/4}(\sigma)|_H^p = 0.$$

Thanks to (3.21), this implies that if we take p large enough so that

$p(\delta-4)/4(p-1) > -1$, we have by Hölder's inequality,

$$\begin{aligned}
&\lim_{\mu\to 0}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t (t-\sigma)^{\delta/4-1}T_\epsilon(t-\sigma)Y^{\delta/4}_{\mu,1}(\sigma)d\sigma\right|^p_H\\
&\leq \sup_{0\leq t\leq T}\|T_\epsilon(t)\|_{\mathcal{L}(H)}\left(\int_0^T \sigma^{\frac{p(\delta-4)}{4(p-1)}}d\sigma\right)^{p-1}\lim_{\mu\to 0}\int_0^T\mathbb{E}\left|Y^{\delta/4}_{\mu,1}(\sigma)\right|^p_H d\sigma=0.
\end{aligned}$$

Next, we remark that in view of (3.18) and Hypothesis 6,

$$\mathbb{E}|Y^{\delta/4}_{\mu,2}(\sigma)|^2_{H^{\delta-1}}\leq\ c\,\mu^{-(1+\frac{\delta}{2})}\sum_{k=1}^\infty\frac{\lambda_k^2}{\alpha_k^{1-\delta}}<\infty,$$

and by Lemma 3.3.2,

$$\left|\Pi_1 S^\epsilon_\mu(t)(0,Y^{\delta/4}_{\mu,2}(\sigma))\right|^2_H\leq 2^\delta\mu^{1+\delta}|Y^{\delta/4}_{\mu,2}(\sigma)|^2_{H^{\delta-1}}.$$

Therefore,

$$\sup_{\sigma\leq t\leq T}\mathbb{E}\left|\Pi_1 S^\epsilon_\mu(t-\sigma)(0,\Pi_2 Y^{\delta/4}_{\mu,2}(\sigma))\right|^p_{H^{\delta-1}}\leq c_p\,\mu^{\frac{p\delta}{4}}\left(\sum_{k=1}^\infty\frac{\lambda_k^2}{\alpha_k^{1-\delta}}\right)^{\frac{p}{2}}.$$

Therefore, if we pick again p large enough so that $p(\delta-4)/4(p-1)>-1$, we get

$$\begin{aligned}
&\lim_{\mu\to 0}\mathbb{E}\sup_{t\leq T}\left|\int_0^t (t-\sigma)^{\alpha-1}\Pi_1 S^\epsilon_\mu(t-\sigma)(0,Y^\alpha_{\mu,2}(\sigma))d\sigma\right|^p_H\\
&\leq T\left(\int_0^T \sigma^{\frac{(\delta-4)p}{4(p-1)}}d\sigma\right)^{p-1}\lim_{\mu\to 0}\sup_{\sigma\leq t\leq T}\mathbb{E}|\Pi_1 S^\epsilon_\mu(t-\sigma)Y^{\delta/4}_{\mu,2}(\sigma)|^p_{H^{\delta-1}}=0.
\end{aligned}$$

Finally, for any $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\int_0^t (t-\sigma)^{\frac{\delta-4}{4}}\left[\Pi_1 S^\epsilon_\mu(t-\sigma)\Pi_1^\star - T_\epsilon(t-\sigma)\right]Y^{\delta/4}_{\mu,3}(\sigma)d\sigma\\
&=\int_0^t (t-\sigma)^{\frac{\delta-4}{4}}\left[\Pi_1 S^\epsilon_\mu(t-\sigma)\Pi_1^\star - T_\epsilon(t-\sigma)\right]\left(P_nY^{\delta/4}_{\mu,3}(\sigma)+(Y^{\delta/4}_{\mu,3}(\sigma)-P_nY^{\delta/4}_{\mu,3}(\sigma))\right)d\sigma.
\end{aligned}$$

By (3.17),

$$\sup_{\mu>0,\,n\in\mathbb{N}}\mathbb{E}\left|P_nY^{\delta/4}_{\mu,3}(\sigma)\right|^2_H\leq\int_0^T s^{-\frac{\delta}{2}}\sum_{k=1}^\infty\left|\Pi_1 S^\epsilon_\mu(s)Q_\mu e_k\right|^2_{\mathcal{H}}ds\leq c\sum_{k=1}^\infty\frac{\lambda_k^2}{\alpha_k^{1-\frac{\delta}{2}}}.$$

and

$$\mathbb{E}\left|Y^{\delta/4}_{\mu,3}(\sigma)-P_nY^{\delta/4}_{\mu,3}(\sigma)\right|^2_H=\int_0^T s^{-\frac{\delta}{2}}\sum_{k=n+1}^\infty\left|\Pi_1 S^\epsilon_\mu(\sigma)Q_\mu e_k\right|^2_H ds\leq c\sum_{k=n+1}^\infty\frac{\lambda_k^2}{\alpha_k^{1-\frac{\delta}{2}}}.$$

This implies

$$\sup_{\mu>0,n\in\mathbb{N}}\ \sup_{\sigma\in[0,T]}\mathbb{E}\left|P_nY^{\delta/4}_{\mu,3}(\sigma)\right|^p_H<+\infty \tag{3.36}$$

and

$$\lim_{n\to+\infty}\sup_{\mu>0}\ \sup_{\sigma\in[0,T]}\mathbb{E}\left|Y^{\delta/4}_{\mu,3}(\sigma)-P_nY^{\delta/4}_{\mu,3}(\sigma)\right|^p_{\mathcal{H}}=0. \tag{3.37}$$

This implies that for any p such that $p(\delta-4)/4(p-1)>-1$

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t(t-\sigma)^{\frac{\delta-4}{4}}\left[\Pi_1S^\epsilon_\mu(t-\sigma)\Pi_1^\star-T_\epsilon(t-\sigma)\right]Y^{\delta/4}_{\mu,3}(\sigma)d\sigma\right|^p_H\\
&\leq\left(\int_0^T\sigma^{\frac{p(\delta-4)}{4(p-1)}}d\sigma\right)^{p-1}\left(\sup_{t\in[0,T]}\sup_{|x|_H\leq1}|\Pi_1S^\epsilon_\mu(t)(P_nx,0)-T_\epsilon(t)P_nx|^p_H\int_0^T\mathbb{E}\left|P_nY^{\delta/4}_{\mu,3}(\sigma)\right|^p_H d\sigma\right.\\
&\left.+\sup_{t\in[0,T]}\left(\|\Pi_1S^\epsilon_\mu(t)\Pi_1^\star\|_{\mathcal{L}(H)}+\|T_\epsilon(t)\|_{\mathcal{L}(H)}\right)\int_0^T\mathbb{E}\left|Y^{\delta/4}_{\mu,3}(\sigma)-P_nY^{\delta/4}_{\mu,3}(\sigma)\right|^p_{\mathcal{H}}d\sigma\right)
\end{aligned}$$

By choosing n large enough, (3.36) and (3.37) yield

$$\lim_{\mu\to0}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t(t-\sigma)^{\frac{\delta-4}{4}}\left[\Pi_1S^\epsilon_\mu(t-\sigma)\Pi_1^\star-T_\epsilon(t-\sigma)\right]Y^{\delta/4}_{\mu,3}(\sigma)d\sigma\right|^p_H=0.$$

□

3.5 Approximation by small friction for multiplicative noise

In this section we assume that the space dimension $d=1$ and D is a bounded interval, the diffusion coefficient G satisfies Hypothesis 5 and the covariance operator Q satisfies the following condition.

Hypothesis 7. *There exists a bounded non-negative sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that*

$$Qe_k=\lambda_ke_k,\qquad k\in\mathbb{N}.$$

We begin by studying the stochastic convolutions

$$\Gamma_\mu^\epsilon(z)(t) := \int_0^t S_\mu^\epsilon(t-s) G_\mu(z(s),s)\,dw^Q(s), \qquad z \in L^p(\Omega, C([0,T];\mathcal{H})),$$

and

$$\Gamma_\epsilon(u)(t) = \int_0^t T_\epsilon(t-s) G_\epsilon(u(s),s)\,dw^Q(s), u \in L^p(\Omega, C([0,T];H)).$$

With the notations introduced in Sections 3.16 and 3.19, the regularized system (3.4) can be rewritten as

$$dz_\mu^\epsilon(t) = \left[A_\mu^\epsilon z_\mu^\epsilon(t) + B_\mu(z_\mu^\epsilon(t),t)\right]\,dt + G_\mu(z_\mu^\epsilon(t),t)\,dw^Q(t), \quad z_\mu^\epsilon(0) = (u_0, v_0), \tag{3.38}$$

and the limiting problem (3.33) can be rewritten as

$$du_\epsilon(t) = \left[A_\epsilon u_\epsilon(t) + B_\epsilon(u_\epsilon(t),t)\right]\,dt + G_\epsilon(u_\epsilon(t),t)\,dw^Q(t), \qquad u_\epsilon(0) = u_0, \tag{3.39}$$

where

$$G_\epsilon(u,t) = J_\epsilon^{-1} G(u,t).$$

Lemma 3.5.1. *Under Hypotheses 5 and 7, for any $\mu, \epsilon > 0$, $T \geq 0$ and $p > 4$ we have*

$$z \in L^p(\Omega; C([0,T];\mathcal{H})) \Longrightarrow \Gamma_\mu^\epsilon(z) \in L^p(\Omega; C([0,T];\mathcal{H})).$$

Moreover, there exists a constant $c := c(\epsilon, \mu, p, T)$ such that

$$\mathbb{E}|\Gamma_\mu^\epsilon(z_1) - \Gamma_\mu^\epsilon(z_2)|^p_{C([0,T];\mathcal{H})} \leq c \int_0^T \mathbb{E}|\Pi_1 z_1 - \Pi_1 z_2|^p_{C([0,\sigma];H)}\,d\sigma. \tag{3.40}$$

Proof. It is sufficient to prove (3.40). By the factorization method, for any $\alpha \in (0,1/2)$ we have

$$\Gamma_\mu^\epsilon(z_1)(t) - \Gamma_\mu^\epsilon(z_2)(t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t (t-\sigma)^{\alpha-1} S_\mu^\epsilon(t-\sigma) Y^\mu(\sigma)\,d\sigma,$$

where

$$Y^\mu(\sigma) =: (Y_1^\mu(\sigma), Y_2^\mu(\sigma)),$$

with

$$Y_1^\mu(\sigma) = \int_0^\sigma (\sigma - s)^{-\alpha} \Pi_1 S_\mu^\epsilon(\sigma - s) \left[G_\mu(z_1(s), s) - G_\mu(z_2(s), s)\right] dw^Q(s)$$

and

$$Y_2^\mu(\sigma) = \int_0^\sigma (\sigma - s)^{-\alpha} \Pi_2 S_\mu^\epsilon(\sigma - s) \left[G_\mu(z_1(s), s) - G_\mu(z_2(s), s)\right] dw^Q(s).$$

Then, for any $p > 1/\alpha$ we have

$$\left|\Gamma_\mu^\epsilon(z_1)(t) - \Gamma_\mu^\epsilon(z_2)(t)\right|_{\mathcal{H}}^p \leq c_{\mu,\epsilon,p} \left(\int_0^T \sigma^{\frac{(\alpha-1)p}{p-1}} \, d\sigma\right)^{p-1} \int_0^t \left(|Y_1^\mu(\sigma)|_H^p + |Y_2^\mu(\sigma)|_{H^{-1}}^p\right) d\sigma. \tag{3.41}$$

By the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} &E|Y_1^\mu(\sigma)|_H^p \\ &\leq \frac{c_p}{\mu^p} \mathbb{E} \left(\int_0^\sigma (\sigma - s)^{-2\alpha} \sum_{k=1}^\infty \lambda_k^2 \left|\Pi_1 S_\mu^\epsilon(\sigma - s)(0, [G(\Pi_1 z_1(s), s) - G(\Pi_1 z_2(s), s)]e_k)\right|_H^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

Now, for any $v \in H^{-1}$ we have

$$\begin{aligned} &\Pi_1 S_\mu^\epsilon(t)(0, v) \\ &= \sum_{h=1}^\infty \left[\left\langle \Pi_1 S_\mu^\epsilon(t)(0, e_{2h-1}), e_{2h-1}\right\rangle_H \langle v, e_{2h-1}\rangle_H + \left\langle \Pi_1 S_\mu^\epsilon(t)(0, e_{2h}), e_{2h-1}\right\rangle_H \langle v, e_{2h}\rangle_H\right] e_{2h-1} \\ &+ \sum_{h=1}^\infty \left[\left\langle \Pi_1 S_\mu^\epsilon(t)(0, e_{2h-1}), e_{2h}\right\rangle_H \langle v, e_{2h-1}\rangle_H + \left\langle \Pi_1 S_\mu^\epsilon(t)(0, e_{2h}), e_{2h}\right\rangle_H \langle v, e_{2h}\rangle_H\right] e_{2h}. \end{aligned}$$

The above equation implies that

$$|\Pi_1 S_\mu^\epsilon(t)(0, v)|_H^2 \leq c \sum_{h=1}^\infty \left|\Pi_1 S_\mu^\epsilon(t)(0, e_h)\right|_H^2 |\langle v, e_h\rangle_H|^2,$$

so that

$$\sum_{k=1}^{\infty}\left|\Pi_1 S^\epsilon_\mu(\sigma-s)(0,[G(\Pi_1 z_1(s),s)-G(\Pi_1 z_2(s),s)]e_k)\right|_H^2$$
$$\leq c\sum_{k=1}^{\infty}\sum_{h=1}^{\infty}\left|\Pi_1 S^\epsilon_\mu(\sigma-s)(0,e_h)\right|_H^2|\langle[G(\Pi_1 z_1(s),s)-G(\Pi_1 z_2(s),s)]e_k,e_h\rangle_H|^2$$
$$=\sum_{h=1}^{\infty}\left|\Pi_1 S^\epsilon_\mu(\sigma-s)(0,e_h)\right|_H^2\sum_{k=1}^{\infty}|\langle[G^\star(\Pi_1 z_1(s),s)-G^\star(\Pi_1 z_2(s),s)]e_h,e_k\rangle_H|^2$$
$$=\sum_{h=1}^{\infty}\left|\Pi_1 S^\epsilon_\mu(\sigma-s)(0,e_h)\right|_H^2|[G^\star(\Pi_1 z_1(s),s)-G^\star(\Pi_1 z_2(s),s)]e_h|_H^2.$$

Therefore, thanks to (3.10) and (3.17), for any $\alpha<1/2$ we get

$$E|Y_1^\mu(\sigma)|_H^p\leq\frac{c_{p,T}}{\mu^p}\,\mathbb{E}\left(\int_0^\sigma(\sigma-s)^{-2\alpha}\sum_{h=1}^{\infty}\left|\Pi_1 S^\epsilon_\mu(\sigma-s)(0,e_h)\right|_H^2|\Pi_1 z_1(s)-\Pi_1 z_2(s)|_H^2 ds\right)^{\frac{p}{2}}$$
$$\leq\frac{c_{p,T}}{\mu^p}\mathbb{E}\,|\Pi_1 z_1-\Pi_1 z_2|^p_{C([0,T];H)}\left(\int_0^\sigma s^{-2\alpha}\sum_{h=1}^{\infty}\left|\Pi_1 S^\epsilon_\mu(s)(0,e_h)\right|_H^2 ds\right)^{\frac{p}{2}}$$
$$\leq c_{p,T}\,\mathbb{E}\,|\Pi_1 z_1-\Pi_1 z_2|^p_{C([0,T];H)}\left(\sum_{h=1}^{\infty}\frac{1}{\alpha_h^{1-2\alpha}}\right)^{\frac{p}{2}},$$

and if we take $\alpha<1/4$, we can conclude that

$$\mathbb{E}|Y_1^\mu(\sigma)|_H^p\leq c_{p,T}\,\mathbb{E}|\Pi_1 z_1-\Pi_1 z_2|^p_{C([0,\sigma];H)}. \tag{3.42}$$

In the above equation we recall that the eigenvalues grow as $\alpha_h\sim h^{\frac{2}{d}}$ and we assumed that $d=1$. By proceeding in the same way as for $Y_1^\mu(\sigma)$, for any $\theta<1/2$ we have

$$\mathbb{E}|Y_2^\mu(\sigma)|^p_{H^{\theta-1}}\leq\frac{c_{p,T}}{\mu^p}\mathbb{E}\,|\Pi_1 z_1-\Pi_1 z_2|^p_{C([0,\sigma];H)}\left(\int_0^\sigma s^{-2\alpha}\sum_{h=1}^{\infty}\left|\Pi_2 S^\epsilon_\mu(s)(0,e_h)\right|^2_{H^{\theta-1}}ds\right)^{\frac{p}{2}},$$

and then, thanks to (3.18)

$$\mathbb{E}|Y_2^\mu(\sigma)|^p_{H^{-1}}\leq c_{p,T}\,\mu^{-\frac{p(1+2\alpha)}{2}}\,\mathbb{E}|\Pi_1 z_1-\Pi_1 z_2|^p_{C([0,\sigma];H)}. \tag{3.43}$$

Therefore, according to (3.41), if $p>4$ we can find $\alpha_p\in(1/p,1/4)$ such that

$$\mathbb{E}\,\left|\Gamma^\epsilon_\mu(z_1)-\Gamma^\epsilon_\mu(z_2)\right|^p_{C([0,T];\mathcal{H})}\leq c_p\int_0^T\mathbb{E}\,|\Pi_1 z_1-\Pi_1 z_2|^p_{C([0,\sigma];H)}\,d\sigma,$$

for a constant c depending on $p,\ \mu,\ \epsilon$ and T. $\square$

Remark 3.5.2. From the proof of the Lemma above, we easily see that, as a consequence of estimates (3.13) and (3.16), for any $z_1, z_2 \in L^p(\Omega; C([0,T];\mathcal{H}))$

$$\sup_{\mu>0} \mathbb{E}|\Pi_1\Gamma^\epsilon_\mu(z_1) - \Pi_1\Gamma^\epsilon_\mu(z_2)|^p_{C([0,T];\mathcal{H})} \leq c_p \int_0^T \mathbb{E}|\Pi_1 z_1 - \Pi_1 z_2|^p_{C([0,\sigma];\mathcal{H})}\, d\sigma, \tag{3.44}$$

for a constant $c = c(\epsilon, p, T) > 0$.

In Lemma 3.5.1 we have proven that the mapping

$$z \in L^p(\Omega; C([0,T];\mathcal{H})) \mapsto \Gamma^\epsilon_\mu(z) \in L^p(\Omega; C([0,T];\mathcal{H})),$$

is Lipschitz continuous. Therefore, as the mapping $B_\mu(\cdot, t) : \mathcal{H} \to \mathcal{H}$, is Lipschitz continuous, uniformly for $t \in [0,T]$, we have that for any initial condition $z_0 = (u_0, v_0) \in \mathcal{H}$, system (3.38) admits a unique adapted mild solution $z^\epsilon_\mu \in L^p(\Omega; C([0,T];\mathcal{H}))$.

Lemma 3.5.3. *Under Hypotheses 5 and 7, for any $\epsilon, T \geq 0$ and any $p > 4$*

$$u \in L^p(\Omega; C([0,T]; H) \Longrightarrow \Gamma_\epsilon(u) \in L^p(\Omega; C([0,T]; H)).$$

Moreover, there exists a constant $c := c(\epsilon, p, T)$ such that for any $u, v \in L^p(\Omega; C([0,T]; H))$

$$\mathbb{E}\,|\Gamma_\epsilon(u) - \Gamma_\epsilon(v)|^p_{C([0,T];H)} \leq c \int_0^T \mathbb{E}|u - v|^p_{C([0,\sigma];H)}\, d\sigma. \tag{3.45}$$

If we assume that

$$\sum_{k=1}^\infty \lambda_k^2 < \infty,$$

then the constant c in (3.45) is independent of $\epsilon > 0$.

Proof. The proof is obtained from the same arguments used in the proof of Lemma 3.5.1, just by replacing the use of Lemmas 3.3.1, 3.3.2 and 3.3.3, with the use of Lemma 3.3.4. □

As a consequence of this lemma, since the mapping $B_\epsilon(\cdot, t) : H \to H$ is Lipschitz continuous, uniformly for $t \in [0, T]$, we have that for any initial condition $u_0 \in \mathcal{H}$, system (3.38) admits a unique adapted mild solution $u_\epsilon \in L^p(\Omega; C([0, T]; H))$.

Theorem 3.5.4. *For any fixed $\epsilon > 0$, $T > 0$ and $p \geq 1$, and any $u \in L^p(\Omega, C([0, T]; H))$*

$$\lim_{\mu \to 0} \mathbb{E} \left| \Pi_1 \Gamma^\epsilon_\mu((u, 0)) - \Gamma_\epsilon(u) \right|^p_{C([0,T];H)} = 0.$$

Proof. Once again, by the factorization method, we can write

$$\begin{aligned}
&\frac{\pi}{\sin(\pi\alpha)} \left[\Pi_1 \Gamma^\epsilon_\mu(u, 0)(t) - \Gamma_\epsilon(u)(t) \right] = \int_0^t (t - \sigma)^{\alpha - 1} T_\epsilon(t - \sigma) Y_1^\mu(\sigma) d\sigma \\
&+ \int_0^t (t - \sigma)^{\alpha - 1} [\Pi_1 S^\epsilon_\mu(t - \sigma)\Pi_1^\star - T_\epsilon(t - \sigma)] Y_2^\mu(\sigma) d\sigma \\
&+ \int_0^t (t - \sigma)^{\alpha - 1} \Pi_1 S^\epsilon_\mu(t - \sigma) \left(0, Y_3^\mu(\sigma)\right) d\sigma \\
&:= I_1^\mu(t) + I_2^\mu(t) + I_3^\mu(t).
\end{aligned}$$

where

$$\begin{aligned}
Y_1^\mu(\sigma) &= \int_0^\sigma (\sigma - s)^{-\alpha} [\Pi_1 S^\epsilon_\mu(\sigma - s) G_\mu((u(s), 0), s) - T_\epsilon(\sigma - s) G_\epsilon(u(s), s)] dw^Q(s) \\
Y_2^\mu(\sigma) &= \int_0^\sigma (\sigma - s)^{-\alpha} \Pi_1 S^\epsilon_\mu(\sigma - s) G_\mu((u(s), 0), s) dw^Q(s) \\
Y_3^\mu(\sigma) &= \int_0^\sigma (\sigma - s)^{-\alpha} \Pi_2 S^\epsilon_\mu(\sigma - s) G_\mu((u(s), 0), s) dw^Q(s)
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, for any $p \geq 2$

$$\begin{aligned}
&\mathbb{E}|Y_1^\mu(\sigma)|^p_H \\
&\leq c_p \, \mathbb{E} \left(\int_0^\sigma (\sigma - s)^{-2\alpha} \sum_{k=1}^\infty \left| [\Pi_1 S^\epsilon_\mu(\sigma - s) G_\mu((u(s), 0), s) - T_\epsilon(\sigma - s) G_\epsilon(u(s), s)] e_k \right|^2_H ds \right)^{\frac{p}{2}}.
\end{aligned}$$

Now, proceeding as in the proof of Lemma 3.5.1, we have

$$\sum_{k=1}^{\infty} \int_0^{\sigma} (\sigma - s)^{-2\alpha} \left|[\Pi_1 S^{\epsilon}_{\mu}(\sigma - s) G_{\mu}((u(s), 0), s) - T_{\epsilon}(\sigma - s) G_{\epsilon}(u(s), s)] e_k\right|^2_H$$
$$\leq c_{\alpha,T}\left(1 + |u|_{C([0,T];H)}\right) \sum_{h=1}^{\infty} \frac{1}{\alpha_h^{1-2\alpha}}.$$

Moreover, according to (3.28), for any fixed $0 \leq s < \sigma$ and $k \in \mathbb{N}$

$$\lim_{\mu \to 0} \left|[\Pi_1 S^{\epsilon}_{\mu}(\sigma - s) G_{\mu}((u(s), 0), s) - T_{\epsilon}(\sigma - s) G_{\epsilon}(u(s), s)] e_k\right|_H = 0, \quad \mathbb{P} - \text{a.s.}$$

Therefore, by the dominated convergence theorem for any $\sigma \leq T$,

$$\lim_{\mu \to 0} \mathbb{E}|Y_1^{\mu}(\sigma)|^p_H = 0,$$

so that, if $p > 4$ there exists $\alpha \in (1/p, 1/4)$ such that

$$|I_1^{\mu}|^p_{C([0,T];H)} \leq \left(\int_0^T \sigma^{\frac{p(\alpha-1)}{p-1}} d\sigma\right)^{p-1} \sup_{s \geq 0} \|T_{\epsilon}(s)\|^p_{\mathcal{L}(H)} \int_0^T |Y_1^{\mu}(\sigma)|^p_H d\sigma.$$

Due to the dominated convergence theorem, we can conclude that

$$\lim_{\mu \to 0} \mathbb{E}|I_1^{\mu}|^p_{C([0,T];H)} = 0. \tag{3.46}$$

For each $n \in \mathbb{N}$ we rewrite $I_2^{\mu}(t)$ as

$$I_2^{\mu}(t) = \int_0^t (t - \sigma)^{1-\alpha} [\Pi_1 S^{\epsilon}_{\mu}(t - \sigma) \Pi_1^{\star} - T_{\epsilon}(t - \sigma)](Y_2^{\mu}(\sigma) - P_n Y_2^{\mu}(\sigma)) d\sigma$$
$$+ \int_0^t (t - \sigma)^{1-\alpha} [\Pi_1 S^{\epsilon}_{\mu}(t - \sigma) \Pi_1^{\star} - T_{\epsilon}(t - \sigma)] P_n Y_2^{\mu}(\sigma) d\sigma.$$

Therefore, if $\alpha < 1/p$

$$|I_2^{\mu}|^p_{C([0,T];H)} \leq c_p \left(\int_0^T (T - \sigma)^{\frac{p(\alpha-1)}{p-1}} d\sigma\right)^{p-1}$$
$$\Bigg[\sup_{t \geq 0}\left(\|\Pi_1 S^{\epsilon}_{\mu}(t) \Pi_1^{\star}\|^p_{\mathcal{L}(H)} + \|T_{\epsilon}(t)\|^p_{\mathcal{L}(H)}\right) \int_0^T |Y_2^{\mu}(\sigma) - P_n Y_2^{\mu}(\sigma)|^p_H d\sigma$$
$$+ \sup_{0 \leq t \leq T} \sup_{|x|_H \leq 1} \left|S^{\epsilon}_{\mu}(s) \Pi_1^{\star} P_n x - T_{\epsilon}(s) P_n x\right|^p_H \int_0^T |P_n Y_2^{\mu}(\sigma)|^p_H d\sigma\Bigg] := I^{\mu}_{2,1}(n) + I^{\mu}_{2,2}(n).$$

Now, as we have seen above for $Y_1^\mu(t)$, for any $n \in \mathbb{N}$ we have

$$\mathbb{E}|P_n Y_2^\mu(\sigma)|_H^p \leq c_{p,T} \left(1 + \mathbb{E}\,|u|^p_{C([0,T];H)}\right). \tag{3.47}$$

Moreover, by proceeding as in the proof of Lemma 3.5.1, we have By the same arguments, we can show that

$$\begin{aligned}
&\mathbb{E}\,|Y_2^\mu(\sigma) - P_n Y_2^\mu(\sigma)|_H^p \\
&\leq \frac{c_{p,T}}{\mu^p} \left(1 + \mathbb{E}\,|u|^p_{C([0,T];H)}\right) \left(\int_0^\sigma s^{-2\alpha} \sum_{h=n+1}^\infty |\Pi_1 S_\mu^\epsilon(s)(0, e_h)|_H^2\, ds\right)^{\frac{p}{2}} \\
&\leq c_{p,T} \left(1 + \mathbb{E}\,|u|^p_{C([0,T];H)}\right) \left(\sum_{h=n+1}^\infty \frac{1}{\alpha_h^{1-2\alpha}}\right)^{\frac{p}{2}}.
\end{aligned}$$

Therefore, if $\alpha < 1/4$ we get

$$\lim_{n\to\infty} \mathbb{E}\,|Y_2^\mu(\sigma) - P_n Y_2^\mu(\sigma)|_H^p = 0. \tag{3.48}$$

Since by (3.13), $\|\Pi_1 S_\mu^\epsilon(s)\Pi_1^\star\|_{\mathcal{L}(H)}$ is uniformly bounded independently of s and μ, this means that for any $\eta > 0$there exists $n_\eta \in \mathbb{N}$ such that

$$\sup_{\mu>0} \mathbb{E} I_{2,1}^\mu(n_\eta) < \frac{\eta}{2}.$$

By (3.24) and (3.47) we can find $\mu_0 > 0$ small enough such that

$$\mathbb{E} I_{2,2}^\mu(n_\eta) < \frac{\eta}{2}, \qquad \mu \leq \mu_0,$$

and then, since η was arbitrary, we can conclude that

$$\lim_{\mu\to 0} \mathbb{E}|I_2^\mu|^p_{C([0,T];H)} = 0. \tag{3.49}$$

It remains to estimate $I_3^\mu(t)$. By proceeding as in the proof of Lemma 3.5.1, we have

$$|Y_3^\mu(\sigma)|_H^p \leq c_{p,T}\mu^{-\frac{p(1+2\alpha)}{2}} \mathbb{E}\left(1 + |u|^p_{C([0,T];H)}\right).$$

Then, for $\alpha < 1/p$ we have

$$|I_3^\mu|^p_{C([0,T];H)} \leq \left(\int_0^T \sigma^{\frac{2m(\alpha-1)}{2m-1}} d\sigma\right)^{p-1} \sup_{t\geq 0} \|\Pi_1 S_\mu^\epsilon(t)\Pi_2^\star\|^p_{\mathcal{L}(H)} \left(\int_0^T |Y_3^\mu(\sigma)|^p_H d\sigma\right).$$

From Lemma 3.3.2,

$$\|\Pi_1 S_\mu^\epsilon(s)\Pi_2^\star\|^p_{\mathcal{L}(H)} \leq c\mu^p.$$

Therefore,

$$\mathbb{E}|I_3^\mu|^p_{C([0,T];H)} \leq c\mu^{\frac{p(1-2\alpha)}{2}} \mathbb{E}(1 + |u|^p_{C([0,T];H)}),$$

and we can conclude that

$$\lim_{\mu\to 0} \mathbb{E}|I_3^\mu|^p_{C([0,T];H)} = 0.$$

This, together with (3.46) and (3.49) implies that for any $p > 4$

$$\lim_{\mu\to 0} \mathbb{E}|\Pi_1\Gamma_\mu^\epsilon(u,0) - \Gamma_\epsilon(u)|^p_{C([0,T];H)} = 0.$$

The case $p \geq 1$ is a consequence of the Hölder inequality. □

Theorem 3.5.5. *Let $z_\mu^\epsilon = (u_\mu^\epsilon, v_\mu^\epsilon)$ and u_ϵ be the mild solutions of problems* (3.38) *and* (3.39)*, with initial conditions $z_0 \in \mathcal{H}$ and $u_0 = \Pi_1 z_0 \in H$, respectively. Then, under Hypotheses 4, 5 and 7, for any $T > 0$, $\epsilon > 0$ and $p \geq 1$ we have*

$$\lim_{\mu\to 0} \mathbb{E}|u_\mu^\epsilon - u_\epsilon|^p_{C([0,T];H)} = 0.$$

Proof. We have

$$u_\mu^\epsilon(t) = \Pi_1 S_\mu^\epsilon(t)(u_0,v_0) + \Pi_1 \int_0^t S_\mu^\epsilon(t-s)B_\mu(z_\mu^\epsilon(s),s)ds + \Pi_1\Gamma_\mu^\epsilon(z_\mu^\epsilon)(t),$$

and

$$u_\epsilon(t) = T_\epsilon(t)u_0 + \int_0^t T_\epsilon(t-s)B_\epsilon(u_\epsilon(s),s) + \Gamma_\epsilon(u_\epsilon)(t).$$

Then

$$
\begin{aligned}
&\left|u_\mu^\epsilon(t) - u_\epsilon(t)\right|_H \le \left|\Pi_1 S_\mu^\epsilon(t)(u_0, v_0) - T_\epsilon(t)u_0\right|_H \\
&+ \left|\int_0^t \Pi_1 S_\mu^\epsilon(t-s)[B_\mu(z_\mu^\epsilon(s), s) - B_\mu((u_\epsilon(s), 0), s)]ds\right|_H \\
&+ \left|\frac{1}{\mu}\int_0^t \Pi_1 S_\mu^\epsilon(t-s)(0, B(u_\epsilon(s), s))ds - \int_0^t T_\epsilon(t-s)J_\epsilon^{-1}B(u_\epsilon(s), s)ds\right|_H \\
&+ \left|\Pi_1\left[\Gamma_\mu^\epsilon(z_\mu^\epsilon)(t) - \Gamma_\mu^\epsilon((u_\epsilon(t), 0))\right]\right|_H + \left|\Pi_1\Gamma_\mu^\epsilon(u_\epsilon(t), 0) - \Gamma_\epsilon(u_\epsilon)(t)\right|_H.
\end{aligned}
$$

By Lemma 3.3.2, and Hypothesis 4, there is a constant independent of μ and of $0 < s < t$, such that

$$
\left|\Pi_1 S_\mu^\epsilon(t-s)[B_\mu(z_\mu^\epsilon(s), s) - B_\mu((u_\epsilon(s), 0), s)]\right|_H \le c\,|u_\mu^\epsilon(s) - u_\epsilon(s)|_H,
$$

so that for any $p \ge 2$

$$
\left|\int_0^t \Pi_1 S_\mu^\epsilon(t-s)[B_\mu(z_\mu^\epsilon(s), s) - B_\mu((u_\epsilon(s), 0), s)]ds\right|_H^p \le c_p\, t^{p-1}\int_0^t |u_\mu^\epsilon - u_\epsilon|^p_{C([0,s];H)}ds.
$$

Thanks to (3.44), this implies

$$
\begin{aligned}
&\mathbb{E}\left|u_\mu^\epsilon - u_\epsilon\right|^p_{C([0,t];H)} \le c_p\, T^{p-1}\int_0^t \mathbb{E}\left|u_\mu^\epsilon - u_\epsilon\right|^p_{C([0,s];H)} ds \\
&+ c_p \sup_{s\le t}\left|\Pi_1 S_\mu^\epsilon(s)(u_0, v_0) - T_\epsilon(t)u_0\right|_H^p + c_p\mathbb{E}\left|\Pi_1\Gamma_\mu^\epsilon((u_\epsilon, 0)) - \Gamma_\epsilon(u_\epsilon)\right|^p_{C([0,t];H)} \\
&+ c_p\,\mathbb{E}\sup_{s\le t}\left|\frac{1}{\mu}\int_0^s \Pi_1 S_\mu^\epsilon(s-r)(0, B(u_\epsilon(r), r))dr - \int_0^s T_\epsilon(s-r)B_\epsilon(u_\epsilon(r), r)dr\right|_H^p,
\end{aligned}
$$

and the Grönwall's inequality yields

$$
\begin{aligned}
&\mathbb{E}\left|u_\mu^\epsilon - u_\epsilon\right|^p_{C([0,T];H)} \\
&\le c_p(T)\left(\sup_{s\le T}\left|\Pi_1 S_\mu^\epsilon(s)(u_0, v_0) - T_\epsilon(t)u_0\right|_H^p + \mathbb{E}\left|\Pi_1\Gamma_\mu^\epsilon((u_\epsilon, 0)) - \Gamma_\epsilon(u_\epsilon)\right|^p_{C([0,T];H)}\right) \\
&+ c_p(T)\,\mathbb{E}\sup_{s\le T}\left|\frac{1}{\mu}\int_0^s \Pi_1 S_\mu^\epsilon(s-r)(0, B(u_\epsilon(r), r))dr - \int_0^s T_\epsilon(s-r)B_\epsilon(u_\epsilon(r), r)dr\right|_H^p.
\end{aligned}
$$

Finally, the result follows because of (3.27), (3.29), and Theorem 3.5.4. □

3.6 The convergence for $\epsilon \downarrow 0$

In the previous sections, we have shown that under suitable conditions on the coefficients and the noise, for any fixed $\epsilon > 0$, $T > 0$ and $p \geq 1$

$$\lim_{\mu \to 0} \mathbb{E} \left| u_\mu^\epsilon - u_\epsilon \right|^p_{C([0,T];H)} = 0.$$

This limit is not uniform in $\epsilon > 0$, and the limit is not true for $\epsilon = 0$. In this section we want to show that

$$\lim_{\epsilon \to 0} \mathbb{E} \left| u_\epsilon - u \right|^p_{C([0,T];H)} = 0, \tag{3.50}$$

where u is the mild solution of the problem

$$du(t) = [A_0 u(t) + B_0(u(t),t)]\, dt + G_0(u(t),t)\, dw^Q(t), \qquad u(0) = u_0, \tag{3.51}$$

with

$$A_0 := J_0^{-1} A, \quad B_0 = J_0^{-1} B, \quad G_0 = J_0^{-1} G.$$

This statement is true if we strengthen Hypothesis 6. Actually, Hypothesis 6 is the weakest assumption on the regularity of the noise that implies Theorem 3.4.1 and Theorem 3.5.5, for $\epsilon > 0$. But in order to prove (3.50) we need to assume the following stronger condition on the covariance Q.

Hypothesis 8. *There exists a non-negative sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ such that $Qe_k = \lambda_k e_k$, for any $k \in \mathbb{N}$, and*

$$\sum_{k=1}^{\infty} \lambda_k^2 < +\infty.$$

In what follows, we shall denote by $T_0(t)$, $t \geq 0$, the semigroup generated by the differential operator A_0 in H, with $D(A_0) = D(A)$. The semigroup $T_0(t)$ is strongly continuous in H. Moreover, if we define $u(t) = T_0(t)x$, for $x \in D(A_0)$, we have

$$\begin{cases} \dfrac{\partial u_1}{\partial t}(t) = -\Delta u_2(t), \quad u_1(0) = x_1 \\ \dfrac{\partial u_2}{\partial t}(t) = \Delta u_1(t), \quad u_2(0) = x_2 \end{cases}$$

This means that if we take the scalar product in H^θ of the first equation by u_1 and of the second equation by u_2, we get

$$\frac{d}{dt}|u(t)|^2_{H^\theta} = 0,$$

so that

$$|T_0(t)x|_{H^\theta} = |x|_{H^\theta}, \quad t \geq 0, \tag{3.52}$$

for any $\theta \in \mathbb{R}$ and $x \in H$.

Now, let us consider the stochastic convolution associated with problem (3.51), in the simple case $G = I$

$$\Gamma(t) = \int_0^t T_0(t-s)Qdw(s), \quad t \geq 0.$$

As a consequence of (3.52), we have

$$\mathbb{E}\,|\Gamma(t)|^2_H = \int_0^t \sum_{k=1}^\infty |T_0(s)Qe_k|^2_H\,ds = \int_0^t \sum_{k=1}^\infty |Qe_k|^2_H\,ds = t\sum_{k=1}^\infty \lambda_k^2,$$

and this implies that Hypothesis 8 is necessary in order to have a solution in H for the limiting equation (3.51).

Lemma 3.6.1. *The matrix J_ϵ^{-1} converges to J_0^{-1} in $\mathcal{L}(\mathbb{R}^2)$. Furthermore, for any $T \geq 0$,*

$$\lim_{\epsilon\to 0} \sup_{t\in[-T,T]} \left\| e^{tJ_\epsilon^{-1}} - e^{tJ_0^{-1}} \right\|_{\mathcal{L}(\mathbb{R}^2)} = 0 \tag{3.53}$$

Proof. Recall that

$$J_\epsilon^{-1} = \frac{1}{1+\epsilon^2}\begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix} = \frac{\epsilon}{1+\epsilon^2} I + \frac{1}{1+\epsilon^2} J_0^{-1}.$$

Then,

$$J_\epsilon^{-1} - J_0^{-1} = \frac{\epsilon}{\epsilon^2+1} I - \frac{\epsilon^2}{1+\epsilon^2} J_0^{-1}$$

and this means that

$$\| J_\epsilon^{-1} - J_0^{-1} \|_{\mathcal{L}(\mathbb{R}^2)} \leq c\, \frac{\epsilon}{\epsilon^2+1}.$$

Moreover, we have

$$e^{tJ_\epsilon^{-1}} = e^{\frac{\epsilon t}{\epsilon^2+1}} \begin{pmatrix} \cos \frac{t}{\epsilon^2+1} & -\sin \frac{t}{\epsilon^2+1} \\ \sin \frac{t}{\epsilon^2+1} & \cos \frac{t}{\epsilon^2+1} \end{pmatrix}.$$

Therefore, limit (3.53) follows and is uniform with respect to $t \in [-T, T]$. □

Lemma 3.6.2. *For any $n \in \mathbb{N}$, and $T \geq 0$,*

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \sup_{|x|_H \leq 1} |T_\epsilon(t) P_n x - T_0(t) P_n x|_H = 0. \tag{3.54}$$

Proof. If $x \in \text{span}\{e_{2k-1}, e_{2k}\}$, then $T_\epsilon(t)x = e^{-\alpha_{2k} J_\epsilon^{-1} t} x$ and $T_0(t)x = e^{-\alpha_{2k} J_0^{-1} t} x$.

Therefore, by (3.53),

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \sup_{|x|_H \leq 1} |T_\epsilon(t) P_n x - T_0(t) P_n x|_H \leq \lim_{\epsilon \to 0} \sup_{t \in [-T\alpha_{2k}, 0]} \left\| e^{tJ_\epsilon^{-1}} - e^{tJ_0^{-1}} \right\|_{\mathcal{L}(\mathbb{R}^2)} = 0.$$

As we can extend this result to $\text{span}\{e_k\}_{k=1}^{2n}$, for any n, our result follows. □

Notice that, as in the proof of Theorem 3.4.1, this implies that for any $x \in H$

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} |T_\epsilon(t) x - T_0(t) x|_H = 0. \tag{3.55}$$

Now, as a consequence of (3.54), by proceeding as in the proof of Corollary 3.3.7, we obtain the following result.

Lemma 3.6.3. *For any* $\psi \in L^1(\Omega; L^1([0,T]; H))$

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \left(T_\epsilon(t-s) J_\epsilon^{-1} \psi(s) - T_0(t-s) J_0^{-1} \psi(s) \right) ds \right|_H = 0. \tag{3.56}$$

Now, u_ϵ is the unique mild solution in $L^p(\Omega; C([0,T]; H)$ of problem (3.34) (in the case of additive noise) or problem (3.39) (in the case of multiplicative noise), so that

$$u_\epsilon(t) = T_\epsilon(t) u_0 + \int_0^t T_\epsilon(t-s) B_\epsilon(u_\epsilon(s), s)\, ds + \Gamma_\epsilon(u_\epsilon)(t).$$

Moreover, $u(t)$ is the unique mild solution in $L^p(\Omega; C([0,T]; H))$ of the problem

$$du(t) = [A_0 u(t) + B_0(u(t), t)]\, dt + G_0(u(t), t)\, dw^Q(t), \qquad u(0) = u_0,$$

with $G_0 = J_0^{-1} I$ or $G_0 = J_0^{-1} G$, so that

$$u(t) = T_0(t) u_0 + \int_0^t T_0(t-s) B_0(u_\epsilon(s), s)\, ds + \Gamma_0(u_\epsilon)(t).$$

Then, in view the previous two lemmas, we have that the arguments used in the proof of Theorem 3.4.1 and Theorems 3.5.4 and 3.5.5 can be repeated and we have the following result.

Theorem 3.6.4. *Assume either* G *satisfies Hypothesis 5 or* $G(x,t) = I$. *Then, under Hypotheses 4 and 8, we have that for any* $T > 0$ *and* $p \geq 1$

$$\lim_{\epsilon \to 0} \mathbb{E}\, |u_\epsilon - u|^p_{C([0,T]; H)} = 0. \tag{3.57}$$

We conclude this section by showing that the convergence result proved above for $\epsilon \downarrow 0$ is also valid for the second order system, that is for every $\mu > 0$ fixed.

Theorem 3.6.5. *Assume either G satisfies Hypothesis 5 or $G(x,t) = I$. Then, under Hypotheses 4 and 8, we have that for any initial conditions (u_0, v_0) and $\mu > 0$*

$$\lim_{\epsilon \to 0} \mathbb{E}\, |z_\mu^\epsilon - z_\epsilon|^p_{C([0,T];\mathcal{H})},$$

for any $T > 0$ and $p \geq 1$.

As long as we can show that $S_\mu^\epsilon(t) P_n z \to S_\mu^0(t) P_n z$ for any fixed n, we can prove Theorem 3.6.5 by following the arguments of Theorems 3.4.1 and 3.5.5. Fortunately, we can prove something stronger. We show that $\sup_{t \geq 0} \|S_\mu^\epsilon(t) - S_\mu^0(t)\|_{\mathcal{L}(\mathcal{H})} = 0$.

To this purpose, we introduce an equivalent norm on $H \times H^{-1}$ by setting

$$|(x,y)|^2_{\mathcal{H}(\mu)} = |x|^2_H + \mu |y|^2_{H^{-1}}.$$

Because of (3.13), for any $\epsilon \geq 0$,

$$\sup_{t \geq 0} \|S_\mu^\epsilon(t)\|_{\mathcal{L}(\mathcal{H})} \leq 1. \tag{3.58}$$

Note that if $\epsilon = 0$, then, by (3.13), for any $z \in \mathcal{H}$ and $t \geq 0$,

$$|S_\mu^0(t) z|_{\mathcal{H}(\mu)} = |z|_{\mathcal{H}(\mu)}.$$

Lemma 3.6.6. *For fixed $\mu > 0$ and $T > 0$,*

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \|S_\mu^\epsilon(t) - S_\mu^0(t)\|_{\mathcal{L}(\mathcal{H}(\mu))} = 0. \tag{3.59}$$

Proof. If we fix $z \in \mathcal{H}$ and define $u_\mu^\epsilon(t) = \Pi_1 S_\mu^\epsilon(t) z$, then

$$\mu \frac{\partial^2 u_\mu^\epsilon}{\partial t^2}(t) + J_\epsilon \frac{\partial u_\mu^\epsilon}{\partial t}(t) = \Delta u_\mu^\epsilon(t).$$

Therefore, if we define $\gamma_\mu^\epsilon(t) = u_\mu^\epsilon(t) - u_\mu^0(t)$, we have

$$\mu \frac{\partial^2 \gamma_\mu^\epsilon}{\partial t^2}(t) + J_\epsilon \frac{\partial u_\mu^\epsilon}{\partial t}(t) - J_0 \frac{\partial u_\mu^0}{\partial t}(t) = \Delta \gamma_\mu^\epsilon(t).$$

Since $J_\epsilon - J_0 = \epsilon I$, we have

$$J_\epsilon \frac{\partial u_\mu^\epsilon}{\partial t}(t) - J_0 \frac{\partial u_\mu^0}{\partial t} = J_\epsilon \frac{\partial \gamma_\mu^\epsilon}{\partial t} + \epsilon \frac{\partial u_\mu^0}{\partial t},$$

so that

$$\mu \frac{\partial^2 \gamma_\mu^\epsilon}{\partial t^2}(t) + J_\epsilon \frac{\partial \gamma_\mu^\epsilon}{\partial t}(t) = \Delta \gamma_\mu^\epsilon(t) - \epsilon \frac{\partial u_\mu^0}{\partial t}(t), \quad \gamma_\mu^\epsilon(0) = \frac{\partial \gamma_\mu^\epsilon}{\partial t}(0) = 0.$$

This yields

$$\begin{aligned} &S_\mu^\epsilon(t)z - S_\mu^0(t)z = \left(\gamma_\mu^\epsilon(t), \frac{\partial \gamma_\mu^\epsilon}{\partial t}(t)\right) = -\frac{\epsilon}{\mu}\int_0^t S_\mu^\epsilon(t-s)\left(0, \frac{\partial u_\mu^0}{\partial s}(s)\right) ds \\ &= -\frac{\epsilon}{\mu}\int_0^t S_\mu^\epsilon(t-s)\left(0, \Pi_2 S_\mu^0(s)z\right) ds. \end{aligned} \tag{3.60}$$

Then, by (3.58), since $|(0, \Pi_2 z)|_{\mathcal{H}(\mu)} \leq |z|_{\mathcal{H}(\mu)}$, we conclude

$$\left|S_\mu^\epsilon(t)z - S_\mu^0(t)z\right|_{\mathcal{H}(\mu)} \leq \frac{\epsilon}{\mu}\int_0^t \left|(0, \Pi_2 S_\mu^0(s)z)\right|_{\mathcal{H}(\mu)} ds \leq \frac{\epsilon\, t}{\mu}|z|_{\mathcal{H}(\mu)},$$

and (3.59) follows. □

Bibliography

[1] A. Bensoussan, G. Da Prato, M.C. Delfour, S.K. Mitter, Representation and control of infinite-dimensional systems Vol. 1, Birkhäuser Boston, (1992).

[2] S. Cerrai, M. I. Freidlin, *On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom*, Probability Theory and Related Fields 135 (2006), pp. 363–394.

[3] S. Cerrai, M. I. Freidlin, *Smoluchowski-Kramers approximation for a general class of SPDE's*, Journal of Evolution Equations 6 (2006), pp. 657–689.

[4] S. Cerrai, F. Freidlin, *Small mass asymptotics for a charged particle in a magnetic field and long-time influence of small perturbations*, J Stat Phys 144 (2011), pp. 101-123.

[5] S. Cerrai, M. Freidlin, *Approximation of quasi-potentials and exit problems for multidimensional RDE's with noise*, Transactions of the AMS 363 (2011), pp. 3853-3892.

[6] S. Cerrai, M. Röckner, *Large deviations for invariant measures of stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term*, Annales de l'Institut Henri Poincaré (Probabilités et Statistiques) 41 (2005), pp. 69–105.

[7] S. Cerrai, M. Salins, *Smoluchowski-Kramers approximation and large deviations for infinite dimensional gradient systems*, Asymptotic Analysis 88 (2013), pp. 201–215.

[8] S. Cerrai, M. Salins, *Smoluchowski-Kramers approximation and large deviations for infinite dimensional non-gradient systems with applications to the exit problem*, to appear in Journal of Theoretical Probability (2015), arXiv:1403.5745.

[9] S. Cerrai, M. Salins, *On the Smoluchowski-Kramers approximation for a system with infinite degrees of freedom exposed to a magnetic field*, submitted to Electronic Journal of Probability (2014), arXiv:1409.0803.

[10] Z. Chen, M. Freidlin, *Smoluchowski-Kramers approximation and exit problems*, Stoch. Dyn. 5 (2005), pp. 569–585.

[11] F. Chenal, A. Millet, *Uniform large deviations for parabolic SPDE's and applications*, Stochastic Processes and their Applications 72 (1997), pp. 161-186.

[12] G. Da Prato, J. Zabczyk, STOCHASTIC EQUATIONS IN INFINITE DIMENSIONS, Cambridge University Press, Cambridge, 1992.

[13] A. Dembo, O. Zeitouni, LARGE DEVIATIONS TECHNIQUES AND APPLICATIONS, Springer-Verlag, 1998

[14] M. Freidlin, *Some remarks on the Smoluchowski-Kramers approximation*, J. Statist. Phys. 117 (2004), pp. 617–634.

[15] M. Freidlin, A. Wentzell RANDOM PERTURBATIONS OF DYNAMICAL SYSTEMS, Springer-Verlag, 1998.

[16] M. Fuhrman, *Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces,* Studia Mathematica 115 (1995), pp. 53-71.

[17] J. J. Lee, *Small mass asymptotics of a charged particle in a variable magnetic field*, Asymptotic Analysis 86 (2-14), pp. 99-121.

[18] A. Pazy, SEMIGROUPS OF LINEAR OPERATORS AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS, Springer-Verlag, New York (1983).