ABSTRACT

Title of dissertation: GABOR FRAMES FOR QUASICRYSTALS AND K-THEORY

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Dissertation directed by: Professor Jonathan Rosenberg
Department of Mathematics

We study the connection between Gabor frames for quasicrystals, the topology of the hull Ω_Λ of a quasicrystal Λ and the K-theory of an associated twisted groupoid algebra. In particular, we construct a finitely generated projective module over this algebra, and any multiwindow Gabor frame for Λ can be used to construct a projection representing this module in K-theory. For the case of lattices, modules of this kind were first constructed over noncommutative tori in [31]. Luef developed connections with Gabor analysis and showed that the operator algebraic framework tied together many unique aspects of lattice Gabor frames [25], [26]. Our work adapts their results to the setting of quasicrystals.

Along the way, we prove a variety of compatibility conditions between the topology of Ω_Λ and the associated Gabor frame operators. We give a version of Janssen's representation for the Gabor frame operator when Λ is a model set. We also prove that certain quasicrystals can never be the support of a tight multiwindow Gabor frame when the window functions are in the modulation space $M^1(\mathbb{R}^d)$.

As an application to noncommutative topology, we are able to deduce results on the twisted version of Bellissard's gap labeling conjecture. We show that the twisted gap labeling group of Λ always contains the image of the trace map of an associated noncommutative torus and we identify modules in K-theory corresponding to these gap labels. For lattice subsets in dimension two we prove that this constitutes the entire gap labeling group. As a byproduct of our analysis, we also show that when Ω_Λ is viewed as a fiber bundle over a torus with projection p, the pullback map p^* is injective on K^0.

GABOR FRAMES FOR QUASICRYSTALS AND K-THEORY

by

Michael Kreisel

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Advisory Committee:
Professor Jonathan Rosenberg, Chair/Advisor
Professor Radu Balan
Professor Wojciech Czaja
Professor Kasso Okoudjou
Professor Victor Yakovenko

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Chapter 1

Introduction

The first examples of mathematical quasicrystals were studied by Meyer in [28]. Meyer thought of quasicrystals as generalizations of lattices which retained enough lattice-like structure to be useful for studying sampling problems in harmonic analysis. In another direction, the mathematical theory of quasicrystals began developing rapidly after real, physical quasicrystals were discovered by Shechtman et. al. [35]. This led to the study of the topological dynamics of the hull Ω_Λ of a quasicrystal Λ, which are directly related to a variety of questions and constructions in symbolic dynamics (see [1] and [32] for an introduction). Bellissard's gap labeling conjecture provides a clear connection between the mathematics and physics [6]. While Bellissard's work demonstrates the value of topology and dynamics in studying the physics of quasicrystals, little has been done to integrate Meyer's original vision into this picture. The goal of the present paper is to show one avenue by which these strands of research can be connected. Namely, we will show how Gabor frames for a quasicrystal can be made compatible with its topological dynamics, and we use this connection to prove a twisted version of Bellissard's gap labeling conjecture for two-dimensional quasicrystals.

To elaborate, we will begin by describing Bellissard's gap labeling conjecture in detail. Given a quasicrystal Λ, we can imagine a material with an electron at each point in Λ. In order to analyze electron interactions in Λ, one studies a Schrodinger operator of the form

$$H_\Lambda = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{A}\right)^2 + \sum_{y\in\Lambda} v(\cdot - y)$$

acting on $L^2(\mathbb{R}^d)$, where v is a suitable potential([4] Section 2.7). The vector potential $\vec{A}$ models the effect of a constant, uniform magnetic field. With appropriate boundary conditions, it is possible to restrict H_Λ to an operator $H_{\Lambda,R}$ on $L^2(C_R(0))$ where $C_R(0)$ is the closed cube of side length R. Then we can define the integrated density of states (IDOS)

$$\mathcal{N}(E) = \lim_{R\to\infty} \frac{1}{|C_R(0)|} |\{E' \in \mathrm{Sp}(H_{\Lambda,R}) \,|\, E' \leq E\}|$$

which is used to express thermodynamical properties such as the heat capacity.

The IDOS can also be expressed using the language of operator algebras. There are natural C^* and Von Neumann (VN) algebras related to H_Λ which are generated by the resolvent of H_Λ. Essentially, the C^*-algebra the same as the twisted groupoid algebra $\mathcal{A}_\theta = C^*(R_\Lambda, \theta)$ described in Section 2.4, where the cocycle θ is determined by the magnetic field and is the restriction of a cocycle on $\mathbb{R}^{2d}$ (see [4] and [6] for details). The algebra $\mathcal{A}_\theta$ is simple and has a unique normalized trace. The associated VN algebra (its weak closure) will be denoted by $\mathcal{A}_\theta''$. The spectral projection of H_Λ onto $(-\infty, E]$ is denoted by $\chi_E(H_\Lambda)$, and lies in $\mathcal{A}_\theta''$. This allows us to describe the IDOS using the trace on $\mathcal{A}_\theta''$ as

$$\mathcal{N}(E) = \mathrm{Tr}(\chi_E(H_\Lambda))$$

which is known as Shubin's formula ([4] Section 2.7). When E lies in a spectral gap,

$\chi_E(H_\Lambda)$ lies in the C^*-algebra $\mathcal{A}_\theta$. In this case, the value of the IDOS is constant over the gap and can be described using the trace on $\mathcal{A}_\theta$.

Thus there is physical interest in computing the image of the trace map

$$\mathrm{Tr}_* : K_0(\mathcal{A}_\theta) \to \mathbb{R}$$

which we will call the **gap labeling group**. Moreover, from physical considerations we would expect that the gap labels can be computed from only the structure of Λ and θ. A large part of the gap labeling group can be computed by looking at the structure of $\mathcal{A}_\theta$ as a groupoid algebra. Since the unit space of R_Λ is a Cantor set, any clopen set of the unit space gives a projection in $\mathcal{A}_\theta$. The trace of the corresponding projection is simply the measure of the clopen set, which is given by a patch frequency as described in Section 2.1. This leads to Bellissard's gap labeling conjecture:

Conjecture 1.1 ([6] Problem 1.15)**.** *When the magnetic field $\theta = 0$, the set of gap labels is given by*

$$\mathrm{Tr}_*(K_0(\mathcal{A}_{\theta=0})) = \int_{\Omega_{trans}} C(\Omega_{trans}, \mathbb{Z}),$$

which is precisely the group generated by the normalized patch frequencies of Λ.

There are many proofs of Conjecture 1.1 in low dimensions and other special cases, and there are at least three proofs of the conjecture in full generality [5], [7], [17]. However, all three of these papers depend upon the results of [11], which have been shown to be incorrect [12]. Thus the conjecture appears to remain open in its full generality, although all cases of physical interest have been settled.

Despite the success of Bellissard's gap labeling program when $\theta = 0$, nothing seems to be known when a magnetic field is present. Part of the difficulty lies in the simplicity of Conjecture 1.1. When $\theta = 0$, we can ignore all classes in $K_0(\mathcal{A}_{\theta=0})$ which do not come from projections on the unit space, at least for the purpose of gap labeling. However, once we twist by θ the other summands in $K_0(\mathcal{A}_\theta)$ may contribute to the gap labeling group. Additionally, the methods of [5], [7], and [17] all apply some version of transverse index theory. This allows them to prove Conjecture 1.1 without knowing how to construct all the classes in $K_0(\mathcal{A}_\theta)$. Thus one might ask:

Question 1.1. *How can we construct the classes in $K_0(\mathcal{A}_\theta)$ which do not come from projections on the unit space of R_Λ?*

If we could construct these elements directly then we would immediately be able to compute the gap labeling group.

Motivated by Question 1.1, we look at the simpler case when Λ is a lattice. In this case, the algebra $\mathcal{A}_\theta$ is a noncommutative torus. In [31], Rieffel describes a general procedure for constructing all modules over noncommutative tori. In [25] and [26] Luef shows how Rieffel's construction is related to Gabor analysis. In particular, he shows how Gabor frames for lattices can be used to construct projections in noncommutative tori which represent Rieffel's modules. In order to prove that his modules are finitely generated and projective, Rieffel uses arguments that rely heavily on the group structure of a lattice. The construction of Luef's projections is more flexible since Gabor frames can be defined not only for lattices, but for any point set. One roadblock to generalizing Luef's results to the setting of quasicrystals is that lattice Gabor frames are understood much better than non-uniform frames. The recent results of Gröchenig, Ortega-Cerda, and Romero in [15] have greatly increased our understanding of non-uniform frames and they comprise the main technical results that we need.

The main goal of this thesis is to adapt Rieffel's construction to the setting of quasicrystals. In Chapter 2 and we review the background on quasicrystals needed to understand our main results. In Chapter 3 we review the work of Gröchenig, Ortega-Cerda, and Romero in [15], which allows us to prove the following result:

Theorem 1.1. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a quasicrystal. Then there exist functions $g_1, \ldots, g_N$ in $M^1(\mathbb{R}^d)$ so that for any $T \in \Omega_\Lambda$, $\mathcal{G}(g_1, \ldots, g_N, T)$ is a frame for $L^2(\mathbb{R}^d)$ and an M^p-frame for all $1 \leq p \leq \infty$.*

We also show various ways in which the topology of the hull Ω_Λ is compatible with the frame operators for point sets contained in Ω_Λ.

In Chapter 4 we give a review of Rieffel's and Luef's work on noncommutative tori to motivate our constructions, then we construct an $\mathcal{A}_\theta$ module $\mathcal{H}_\Lambda$ which is a representation of $\mathcal{A}_\theta$ by time-frequency shifts. We then use the results from Chapter 3 to show that $\mathcal{H}_\Lambda$ is finitely generated and projective. In order to compute the dimension of $\mathcal{H}_\Lambda$, we apply a deep result from [3] on the frame measure for non-uniform Gabor frames. This yields the following theorem:

Theorem 1.2. *The module $\mathcal{H}_\Lambda$ is finitely generated and projective as an $\mathcal{A}_\theta$-module, and thus defines a class $[\mathcal{H}_\Lambda] \in K_0(\mathcal{A}_\theta)$. The dimension of $\mathcal{H}_\Lambda$ is given by*

$$\mathrm{Tr}([\mathcal{H}_\Lambda]) = \frac{1}{\mathrm{Dens}(\Lambda)},$$

so that $\frac{1}{\mathrm{Dens}(\Lambda)}$ lies in the gap labeling group of $\mathcal{A}_\theta$.

We also construct a projection in $M_N(\mathcal{A}_\theta)$ representing $[\mathcal{H}_\Lambda]$ in $K_0(\mathcal{A}_\theta)$ and use this to give $\mathcal{H}_\Lambda$ the structure of a Hilbert C^*-module. We describe some features of $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{H}_\Lambda)$ which indicate that a Janssen representation should exist for Gabor frames when Λ has pure discrete spectrum. We then prove a Janssen representation for model sets rigorously using a Poisson summation formula. This representation suggests that when Λ is a quasicrystal and $g_1, \ldots, g_N \in M^1(\mathbb{R}^d)$ then $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ can never be a tight frame. We are able to prove this under certain assumptions on the eigenvalues of Ω_Λ, and we indicate how some of these assumptions may be able to be relaxed.

Theorem 1.2 indicates that when the cocycle σ is nontrivial, the gap labeling group may be generated by more than just the patch frequencies. Note that while the density of Λ is intrinsic to the space Ω_Λ, it is not an isomorphism invariant of the groupoid R_Λ. For example, we can apply a linear map A with $\det(A) \neq 1$ to Λ. The groupoids R_Λ and $R_{A\Lambda}$ are isomorphic, however $\mathrm{Dens}(A\Lambda) = \frac{\mathrm{Dens}(\Lambda)}{|\det(A)|}$. If the cocycle σ is preserved by the isomorphism then $\mathcal{H}_\Lambda$ and $\mathcal{H}_{A\Lambda}$ are both modules over $\mathcal{A}_\sigma$, but represent different elements in $K_0(\mathcal{A}_\sigma)$. Thus by deforming Λ in a way which does not alter the groupoid R_Λ or the cocycle σ, we can construct many modules over $\mathcal{A}_\sigma$ using our methods.

In Chapter 5, we illustrate this point by computing the gap labeling group for any standard cocycle θ when $\Lambda \subset \mathbb{R}^2$ is contained in a lattice. This involves showing exactly how the modules $\mathcal{H}_{A\Lambda}$ fit into $K_0(\mathcal{A}_\theta)$, which can be computed easily using the Pimsner-Voiculescu exact sequence. Thus we have:

Theorem 1.3. *When $\Lambda \subset \mathbb{Z}^2$ is a quasicrystal and θ is a standard cocycle, the gap labeling group of $\mathcal{A}_\theta$ is*

$$\mathrm{Tr}_*(K_0(\mathcal{A}_\theta)) = \mu(C(\Omega_{trans}, \mathbb{Z})) + \frac{\theta}{\mathrm{Dens}(\Lambda)}\,\mathbb{Z}\,.$$

Here the cocycle θ is determined by a single real number, also denoted θ.

In higher dimensions $K_0(\mathcal{A}_\theta)$ is larger and the modules $\mathcal{H}_{A\Lambda}$ are not enough to generate the rest of $K_0(\mathcal{A}_\theta)$, although they do give us some information about the gap labeling group. We are able to prove the following theorem which is a partial generalization of our results to higher dimensions. Let $\Lambda \subset \mathbb{R}^d$ be a marked lattice with an aperiodic coloring satisfying the definition of a quasicrystal. Then Ω_Λ naturally has the structure of a fiber bundle

$$p : \Omega_\Lambda \to \mathbb{T}^d$$

over the torus $\mathbb{T}^d$ with the Cantor set as its fibers. The fiber bundle structure comes from viewing Ω_Λ as the suspension of the Cantor set Ω_{trans} by an action of $\mathbb{Z}^d$.

Theorem 1.4. *The induced map $p^* : K^0(\mathbb{T}^d) \to K^0(\Omega_\Lambda)$ is injective. Furthermore, we can compare the image of p^* with the image of r_*,*

$$r_* : K_0(C(\Omega_{trans})) \to K_0(C(\Omega_{trans}) \rtimes \mathbb{Z}^d) \cong K_0(C(\Omega_\Lambda) \rtimes \mathbb{R}^d) \cong K^0(\Omega_\Lambda)$$

where r_* is induced by the inclusion $r : C(\Omega_{trans}) \to C(\Omega_{trans}) \rtimes \mathbb{Z}^d$. The intersection of the images of p^* and r_* is generated by $[\mathbf{1}]$, the class of the trivial bundle.

By results of Sadun and Williams [33], given any quasicrystal Λ we can find a marked lattice L so that Ω_Λ is homeomorphic to Ω_L, and thus Theorem 1.4 holds in much greater generality than at first it may appear. The proof of Theorem 1.4 shows that it can be useful to study the twisted algebras $\mathcal{A}_\sigma$ even if one's primary goal is to understand the topology of Ω_Λ.

Appendices A and B collect various basic facts about C^*-algebras, K-theory, and Morita equivalence which will be used implicitly throughout the text.

Chapter 2

Quasicrystals

2.1 Topological Dynamics

The main objects of our investigation are quasicrystals, so we begin with a review of the topological and dynamical properties of a quasicrystal, as well as properties of the associated operator algebras. We will state the basic definitions and theorems for even dimensional quasicrystals since it will simplify notation later, however the same definitions and theorems apply in any dimension. We will always think of $\mathbb{R}^{2d} \cong \mathbb{R}^d \times \hat{\mathbb{R}}^d$ as time-frequency space, and elements $z \in \mathbb{R}^{2d}$ will be written as $z = (x, \omega)$ when it is necessary to emphasize this point of view.

Definition 2.1. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a discrete set.*

1. *The **hole** of Λ is defined to be*

$$\rho(\Lambda) := \sup_{z \in \mathbb{R}^{2d}} \inf_{\lambda \in \Lambda} |z - \lambda|$$

   *and Λ is called **relatively dense** if $\rho(\Lambda) < \infty$.*

2. *Λ is called **relatively separated** if*

$$\mathrm{rel}(\Lambda) := \sup\{\#(\Lambda \cap C_1(z)) : z \in \mathbb{R}^{2d}\} < \infty$$

   where $C_1(z)$ is the cube of side length 1 centered at z.

3. *Λ is called* ***uniformly discrete*** *if there is an open ball $B_r(0)$ s.t. $(\Lambda - \Lambda) \cap B_r(0) = \{0\}$.*

If Λ is both relatively dense and uniformly discrete then it is called a ***Delone set****. A Delone set Λ is called* ***aperiodic*** *if $\Lambda - z \neq \Lambda$ for any $z \in \mathbb{R}^{2d}$.*

In Gabor analysis, the goal is to recover a function from samples of its Short Time Fourier Transform on a discrete set (see Chapter 3). Often the sampling set is assumed to be a lattice; however there are now a variety of results available which treat sampling on non-uniform sets as well ([3], [15]). While these results are able to deal with sampling on arbitrary Delone sets, we will restrict our attention to quasicrystals, which are Delone sets with additional regularity properties.

Definition 2.2. *Let Λ be a Delone set. The sets $B_r(z) \cap \Lambda$ where $z \in \Lambda$ are called the* **r*-patches*** *of Λ.*

1. *If for any fixed r there are only finitely many r-patches up to translation, then Λ is said to be of* ***finite local complexity (FLC).***

2. *For an r-patch P and a set $A \subset \mathbb{R}^{2d}$ we define*

$$L_P(A) = \#\{z \in \mathbb{R}^{2d} \mid P - z \subset A \cap \Lambda\}.$$

Thus $L_P(A)$ counts the number of times P appears in A. For a sequence of balls $B_{r_k}(z)$ in $\mathbb{R}^{2d}$ such that r_k goes to ∞, we define the ***patch frequency*** *of P to be*

$$freq(P, \Lambda) = \lim_{k \to \infty} \frac{L_P(B_{r_k} - z)}{vol(B_{r_k})}$$

*if this limit exists uniformly in z and independent of the choice of balls B_{r_k}. If the patch frequencies exist for all patches $P \subset \Lambda$ then Λ is said to have **uniform cluster frequencies (UCF)**.*

*A Delone set is called a **quasicrystal** if it is FLC and has UCF.*

The study of quasicrystals has to a large extent been driven by the study of electron interactions in aperiodic solids. Given an electron in an aperiodic solid, we might assume that it will only interact with nearby electrons since the forces drop off rapidly as distances increase. Thus for the study of electron interactions, it is natural to treat two quasicrystals Λ and Λ' as the same if they contain precisely the same local patterns. One could formalize this by saying that any r-patch appearing in Λ also appears as an r-patch in Λ' and vice versa, and in this case we say Λ and Λ' are **locally isomorphic**. For example, any translate $\Lambda - z$ is clearly locally isomorphic to Λ. The collection of all quasicrystals which are locally isomorphic to Λ will be called the **hull** of Λ (denoted Ω_Λ), and this object is useful in studying the physics of aperiodic solids (see [6]).

Now we present another construction of the hull which demonstrates how Ω_Λ can be given the structure of a topological dynamical system. Given two Delone sets Λ, Λ', define

$$R(\Lambda, \Lambda') = \sup\{r \mid \exists z \in \mathbb{R}^{2d} \text{ with } ||z|| < \frac{1}{r}, B_r \cap (\Lambda - z) = \Lambda' \cap B_r\}.$$

We can define the distance between Λ and Λ' as

$$d(\Lambda, \Lambda') = \min\left\{1, \frac{1}{R(\Lambda, \Lambda')}\right\}.$$

Intuitively, two Delone sets are close if they agree in a large ball around the origin after a small translation. This defines a metric d on the space of all Delone subsets of $\mathbb{R}^{2d}$, and the resulting topology on point sets is known as the **local topology**.

Definition 2.3. *Given a Delone set Λ, the* ***orbit*** *of Λ is $O_\Lambda = \{\Lambda - z \mid z \in \mathbb{R}^{2d}\}$. The* ***hull*** *$\Omega_\Lambda$ is the closure of O_Λ in the metric d.*

The hull Ω_Λ comes with a natural action of $\mathbb{R}^{2d}$ by translation. The following proposition shows how regularity properties of Λ can be translated into properties of the dynamical system $(\Omega_\Lambda, \mathbb{R}^{2d})$:

Proposition 2.1 ([6], [23]). *Let Λ be an aperiodic Delone set.*

1. *Λ is FLC iff Ω_Λ is compact.*

2. *Λ has UCF iff the dynamical system $(\Omega_\Lambda, \mathbb{R}^{2d})$ is minimal and uniquely ergodic.*

Thus we see that for any quasicrystal Λ we have an associated dynamical system $(\Omega_\Lambda, \mathbb{R}^{2d})$ which is compact, minimal, and uniquely ergodic. In fact, we have an explicit description of the ergodic measure μ using patch frequencies. Given a patch P in Λ, and $V \subset \mathbb{R}^{2d}$ a precompact open set, define the **cylinder set**

$$\Omega_{P,V} = \{\Lambda' \in \Omega_\Lambda \mid P - z \subset \Lambda' \text{ for some } z \in V\}.$$

The cylinder sets form a basis for the topology on Ω_Λ, so it suffices to describe the ergodic measure for cylinder sets. Fix $\eta(\Lambda)$ so that any ball of radius $\eta(\Lambda)$ contains at most one point of Λ. If $\mathrm{diam}(V) < \eta(\Lambda)$, then the measure of $\Omega_{P,V}$ is given by

$$\mu(\Omega_{P,V}) = Vol(V) freq(P, \Gamma)$$

where Γ is an element of Ω_Λ. Since we can also describe the hull using local isomorphism classes, for any $\Lambda' \in \Omega_\Lambda$ the quantities $\mathrm{rel}(\Lambda')$ and $\rho(\Lambda')$ are equal to $\mathrm{rel}(\Lambda)$ and $\rho(\Lambda)$ respectively. Thus we may think of these quantities as associated to the hull itself, and not just to a particular point set contained in it. Furthermore, the patch frequencies are also independent of the choice of point set $\Lambda' \in \Omega_\Lambda$ so that the patch frequencies and density can be associated to the tiling space as a whole as well.

While the hull Ω_Λ appears naturally from physical considerations, we will consider now a different space which appears more naturally in the context of harmonic analysis. We would like to think of a quasicrystal Λ as a collection of shifts we can apply to a function. The shifts might simply be translations (see [27]), but in the case of Gabor analysis they will be time-frequency shifts. In this vein, we consider

$$O^\Lambda_{trans} := \{\Lambda - z \mid z \in \Lambda\},$$

the collection of Delone sets which are translates of Λ by points in Λ.

Definition 2.4. *We define the **canonical transversal** Ω_{trans} as the closure of O^Λ_{trans} in the metric d.*

Note that the canonical transversal can also be defined as

$$\Omega_{trans} = \{\Lambda' \in \Omega_\Lambda \,|\, 0 \in \Lambda'\},$$

and is a transversal to the action of $\mathbb{R}^{2d}$ on the hull.

Topologically Ω_{trans} is a Cantor set, and it comes with a measure which, by abuse of notation, we shall also call μ. Given a patch $P \subset \Lambda$, we can define

$$\Omega_P := \{\Lambda' \in \Omega_{trans} \,|\, P - z = B_r(0) \cap \Lambda' \text{ for some } r \in \mathbb{R}, z \in \mathbb{R}^{2d}\}.$$

The set Ω_P contains exactly the point sets in Ω_{trans} which have the pattern P centered at the origin. The sets Ω_P form a clopen basis for the topology on Ω_{trans}, and $\mu(\Omega_P) = freq(P, \Lambda)$. The hull Ω_Λ is locally the product of Ω_{trans} and $\mathbb{R}^{2d}$ as both a topological space and a measure space. By results of Sadun and Williams we can always realize Ω_Λ as a fiber bundle over a torus with Cantor set fibers [33], however it is not always the case that Ω_{trans} carries an action of $\mathbb{Z}^{2d}$ so that Ω_Λ is the suspension of Ω_{trans}. This will be an important point to keep in mind during Chapter 5.

2.2 Examples

Now we will present two classes of quasicrystals which comprise the most commonly studied examples: model sets and substitutions. To construct model sets in $\mathbb{R}^d$, we first embed $\mathbb{R}^d$ into a higher dimensional $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$, or more generally as part of a product $\mathbb{R}^d \times G$ where G is a locally compact abelian group. Denote by p_1 and p_2 the projections onto the factors $\mathbb{R}^d$ and G respectively. Then we choose a lattice $D \subset \mathbb{R}^d \times G$ so that p_1 is injective on D and $p_2(D)$ is dense in G. Instead of projecting all of D onto $\mathbb{R}^d$, we will project only a piece of D, ensuring that the resulting collection of points in $\mathbb{R}^d$ is a quasicrystal. This is summarized in the definition below:

Definition 2.5 (Model sets)**.** *Consider the space* $\mathbb{R}^d \times G$*, where* G *is a locally com-*

pact abelian group. Fix $D \subset \mathbb{R}^d \times G$ a discrete cocompact subgroup and $W \subset G$ a relatively compact subset whose boundary has Haar measure 0. Also assume that π_1 is injective on D and $\pi_2(D)$ is dense in W. The triple $(\mathbb{R}^d, G, D)$ is known as the ***cut and project scheme****. We define the* ***model set*** *or* ***cut and project set*** *Λ_W by*

$$\Lambda_W = \{\pi_1(d) \mid d \in D, \pi_2(d) \in W\}.$$

When $x \in \Lambda_W$, we define $x^\star := p_2(p_1^{-1}(x))$. The group G is known as the ***internal space*** *and $\mathbb{R}^d$ is known as the* ***physical space****.*

Any model set (except for a lattice) is aperiodic, FLC, and has UCF ([6], [23]). In fact, we can calculate the patch frequencies of patterns in Λ_W using the cut and project scheme:

Proposition 2.2 (Corollary 7.3 in [1])**.** *Let Λ be a model set for the cut and project scheme $(\mathbb{R}^d, G, D)$ and suppose the window W is compact. If $P \subset \Lambda_W$ is a finite subset, we can determine the relative frequency of P as*

$$\textit{rel freq}(P, \Lambda_W) = \frac{\mathrm{vol}\left(\bigcap_{x \in P}(W - x^\star)\right)}{\mathrm{vol}(W)}$$

and we have the equality

$$freq(P, \Lambda_W) = \mathrm{Dens}(\Lambda_W)\textit{rel freq}(P, \Lambda_W).$$

We can also compute the density of Λ_W as $\mathrm{Dens}(\Lambda_W) = \frac{\mathrm{vol}(W)}{\mathrm{vol}(D)}$.

Many interesting examples of model sets can be constructed using objects from number theory. Consider the number ring $\mathbb{Z}[\zeta_8] \subset \mathbb{C}$ where ζ_8 is a primitive eighth root of unity. We have an automorphism $x \to x^\star$ of $\mathbb{Z}[\zeta_8]$ given by extending the

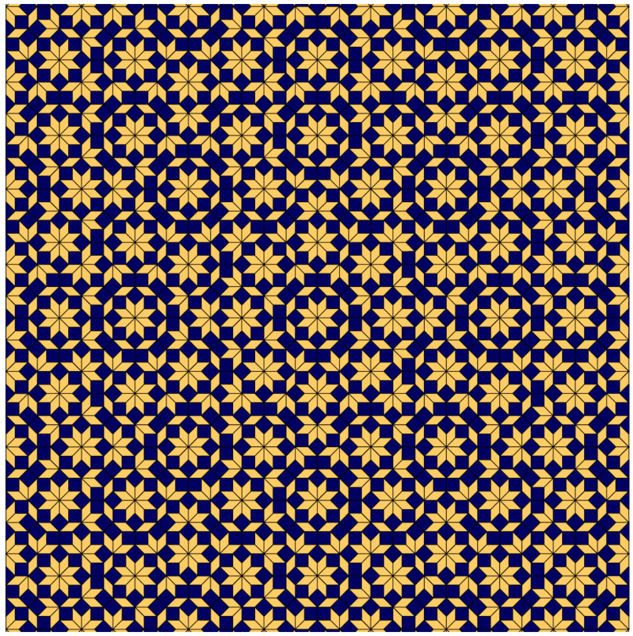

Figure 2.1: A patch of the Ammann-Beenker tiling.

map $\zeta_8 \to \zeta_8^3$. The map $x \to (x, x^\star)$ gives an embedding of $\mathbb{Z}[\zeta_8]$ as a lattice $\mathcal{L}_8$ in $\mathbb{C}^2 \cong \mathbb{R}^4$. We can write $\mathcal{L}_8 = \sqrt{2}R_8\,\mathbb{Z}^4$ where R_8 is the rotation matrix

$$R_8 = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \\ \sqrt{2} & -1 & 0 & 1 \\ 0 & 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

The projections p_1 and p_2 are given by projecting onto the first and second copies of $\mathbb{C} \cong \mathbb{R}^2$ respectively. Collectively, we have described a cut and project scheme $(\mathbb{R}^2, \mathbb{R}^2, \mathcal{L}_8)$. The window W will be a regular octagon in $\mathbb{R}^2$ centered at the origin with side length one, and oriented so that all edges are perpendicular to some eighth root of unity. The resulting model set Λ_W is known as the **Ammann-Beenker**

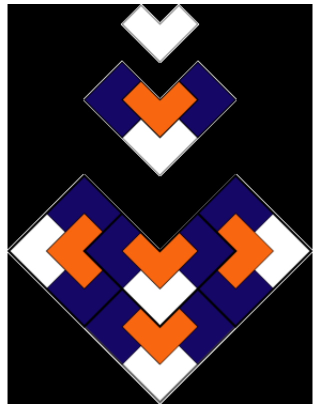

Figure 2.2: The substitution rule for the chair tiling.

point set and can be seen as the vertices of the tiles in Figure 2.1[1].

Now we will describe another way to build aperiodic tilings of the plane known as the substitution method. We begin with a finite collection of tiles $\mathcal{T} = \{T_i\}$ which are polygonal subsets of $\mathbb{R}^d$. We also have a linear map A and a rule for each i which allows us decompose AT_i into a union of tiles from $\mathcal{T}$. This is called the **substitution rule**. By iterating this procedure, we can build a tiling of the plane, and the vertices of the tiling will be a quasicrystal.

This procedure is better understood by looking at a particular example. Figure 2.2 shows the substitution rule for the **chair tiling**. Our collection of tiles contains "L" shaped blocks in a variety of orientations. Recall that two tiles are considered to be of the same tile type if they are the same after a translation, but *not* after a rotation. Thus in the second picture, the orange and white blocks are of the same tile type, but each blue block is its own tile type. Here the linear map A is simply multiplication by two. After iterating this substitution rule a few times, we will

[1]All pictures in this section were obtained from the Tilings Encyclopedia project.

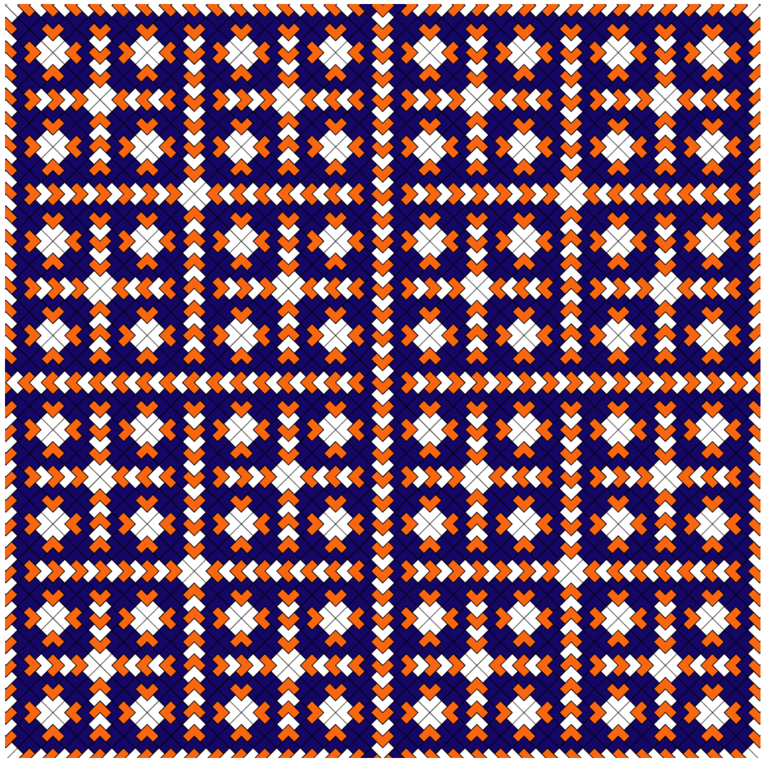

Figure 2.3: A patch of the chair tiling.

have encountered all the different tile types present in the full tiling. It is clear that we can tile one quarter of the plane by repeatedly iterating the substitution rule. To get a tiling of the full plane, we simply rotate this picture to tile each quarter plane. The vertices of the tiling form an aperiodic subset of $\mathbb{Z}^2$.

Although the descriptions of model sets and substitution tilings are quite different, these classes often overlap. In particular, the Ammann-Beenker tiling and the chair tiling can be realized as both model sets and substitutions. The chair tiling can be viewed as a union of lattices of increasing volume, which allow it to be written as a model set where the internal space is the group of 2-adic numbers.

2.3 Poisson Summation Formulas

In studying Gabor analysis on model sets we shall sometimes need to make use of a generalized Poisson summation formula. Recall that for $f \in \mathcal{S}(\mathbb{R}^d)$ and a lattice $L \subset \mathbb{R}^d$ the classical Poisson summation states

$$\sum_{z \in L} f(z) = \sum_{l \in L^*} \hat{f}(l)$$

where L^* denotes the **dual lattice**

$$L^* := \{l \in \mathbb{R}^d \mid l \cdot z \in \mathbb{Z} \text{ for all } z \in L\}.$$

We can consider the translation bounded measure $\delta_L := \sum_{z \in L} \delta_z$ where δ_z is the Dirac delta function at z. Then we can rephrase the Poisson summation formula as an equality of measures

$$\widehat{\delta_L} = \delta_{L^*}.$$

Thus if we want an analogous formula for a quasicrystal Λ, we can begin by considering the measure

$$\delta_\Lambda := \sum_{z \in \Lambda} \delta_z$$

and try to compute $\widehat{\delta_\Lambda}$. Unfortunately, this computation is complicated by the fact that although δ_Λ is a measure, $\widehat{\delta_\Lambda}$ will *not* be a measure in general. In fact there are rather stringent restrictions on when this can occur.

Theorem 2.1 ([21] Theorem 3.7)**.** *Suppose that Λ and Λ' are discrete sets in $\mathbb{R}^d$, that $\{c(l) \,|\, l \in \Lambda'\}$ are positive numbers, and that the two distributions*

$$f_1 = \sum_{z \in \Lambda} \delta_z, \; f_2 = \sum_{l \in \Lambda'} c(l)\delta_l$$

are tempered distributions. If $f_2 = \hat{f}_1$, then Λ is a full rank lattice in $\mathbb{R}^d, \Lambda'$ is the dual lattice, and all $c(l) = \frac{1}{\mathrm{vol}(\Lambda)}$.

Thus we must be very careful while computing $\widehat{\delta_\Lambda}$.

For the remainder of the section we will assume that $\Lambda \subset \mathbb{R}^d$ is a model set with cut and project scheme $(\mathbb{R}^d, \mathbb{R}^n, D)$. We have assumed that the internal space is $\mathbb{R}^n$ largely for convenience, and all results carry over to the more general case ([34]). We define the **dual cut and project scheme** to be $(\mathbb{R}^d, \mathbb{R}^n, D^*)$ and for convenience we denote p_1 and p_2 as the projections for both the original cut and project scheme and its dual. We fix a relatively compact window W and consider the model set Λ_W. For a point $k \in \mathbb{R}^d$, we define the **Fourier-Bohr coefficient at k** to be

$$c_k := \lim_{R \to \infty} \frac{\mathrm{Dens}(\Lambda_W)}{|\Lambda_W \cap B_R|} \sum_{z \in \Lambda_W \cap B_R} e^{-2\pi i k z}.$$

For model sets this limit exists independently of the center of the ball B_R ([1] Prop 9.9). When $k \notin p_1(D^*)$ we have $c_k = 0$. It is possible to have $c_k = 0$ even though $k \in p_1(D^*)$, however for model sets such k form a discrete subset of $p_1(D^*)$. One can see this by computing the Fourier-Bohr coefficients using the Fourier transform of the characteristic function of the window, as in [1].

A cursory computation of $\widehat{\delta_{\Lambda_W}}$ shows that ([16] Section 5)

$$\widehat{\delta_{\Lambda_W}} = \sum_{k \in p_1(D^*)} c_k \delta_k.$$

However, since the RHS of this expression is not locally absolutely summable, we need a more precise way to interpret this sum. To do this, we consider a sequence of smooth functions φ_ϵ on W such that $\lim_{\epsilon \to 0} \varphi_\epsilon = \chi_W$, the characteristic function of W. We define a corresponding collection of measures

$$\delta_{\Lambda_W,\epsilon} := \sum_{z\in\Lambda_W} \varphi_\epsilon(z^\star)\delta_z.$$

It can be shown that

$$\widehat{\delta_{\Lambda_W,\epsilon}} = \sum_{k\in p_1(D^*)} c_k^\epsilon \delta_k$$

where the RHS is locally absolutely summable. Additionally, we have $\lim_{\epsilon\to 0} c_k^\epsilon = c_k$. We summarize this by the following theorem, whose proof can be found in [8] Section 2:

Theorem 2.2. *We have*

$$\lim_{\epsilon\to 0}\widehat{\delta_{\Lambda_W,\epsilon}} = \widehat{\delta_{\Lambda_W}}$$

in the sense of tempered distributions. Interpreted in this sense, for any $f \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\sum_{z\in\Lambda_W} \hat{f}(z) = \sum_{k\in p_1(D^*)} c_k f(k)$$

where

$$c_k = \lim_{R\to\infty} \frac{\mathrm{Dens}(\Lambda_W)}{|\Lambda_W \cap B_R|} \sum_{z\in\Lambda_W\cap B_R} e^{-2\pi ikz}.$$

To better motivate the definition of the Fourier-Bohr coefficients, we can give them a dynamical interpretation. For any point $k \in \mathbb{R}^d$ we say that k is an **eigenvalue** of the dynamical system $(\Omega_\Lambda, \mathbb{R}^d)$ with **eigenfunction** φ_k if φ_k is a measurable complex valued function on Ω_Λ satisfying

$$\varphi_k(T - z) = e^{2\pi ikz}\varphi_k(T)$$

for all $T \in \Omega_\Lambda$ and $z \in \mathbb{R}^d$. If an eigenfunction exists for an eigenvalue k then it is unique up to scaling by a unit complex number. If the eigenfunction is continuous, we call k a **continuous eigenvalue**. The continuous eigenvalues make up the discrete part of the dynamical spectrum. By this, we mean that we can decompose the joint spectrum of the translation operators on Ω_Λ, and that the continuous eigenfunctions make up the discrete part in this decomposition. When the continuous eigenfunctions make up the entire spectrum, we say that Ω_Λ has **pure discrete spectrum**. This is equivalent to saying that the linear span of the continuous eigenfunctions is dense in $L^2(\Omega_\Lambda)$. For the classes of quasicrystals we have described (namely model sets and substitutions) all eigenfunctions are continuous, and thus we do not have to draw the distinction between measurable and continuous eigenfunctions. However there are other classes where this distinction becomes important [20].

Now we can describe the connection between Fourier-Bohr coefficients and eigenvalues. For a model set Λ, Ω_Λ always has pure discrete spectrum and the collection of eigenvalues is exactly $p_1(D^*)$. In this case, we see that the Fourier-Bohr coefficient c_k is simply the integral of the eigenfunction φ_k over Ω_{trans} after applying Birkhoff's ergodic theorem. We will see the eigenfunctions appear again, along with the Fourier-Bohr coefficients, when we investigate Hilbert C^*-module structures and the Janssen representation in Section 4.4.

2.4 The Groupoid C^*-algebra $\mathcal{A}_\sigma$

We are now ready to describe our main object of study: the groupoid C^*-algebra $\mathcal{A}_\sigma$ associated to Ω_{trans}. We consider the equivalence relation

$$R_\Lambda = \{(T, T') \in \Omega_{trans} \times \Omega_{trans} \,|\, T \text{ is a translate of } T'\}$$

and an element of R_Λ will be written as $(T, T - z)$ where $z \in \mathbb{R}^{2d}$. We give R_Λ a topology by declaring that a sequence $(T_k, T_k - z_k) \to (T, T - z)$ iff $T_k \to T$ in Ω_{trans} and $|z_k - z| \to 0$. With this topology, R_Λ has the structure of a locally compact, principal, r-discrete groupoid (see [30]). The unit space of R_Λ is given by elements of the form (T, T). We can compose two elements $(T, T - z), (T', T' - w)$ only if $T' = T - z$, and in this case

$$(T, T - z) * (T - z, T - z - w) = (T, T - z - w).$$

This groupoid captures the idea of shifting by exactly the points in Λ. To see this, note that $(\Lambda, \Lambda - z) \in R_\Lambda$ iff $z \in \Lambda$. Thus the orbit of Λ in R_Λ is in correspondence with the points of Λ, and the element $(\Lambda, \Lambda - z)$ can be thought of as a shift by z. This will be made more explicit in Section 4.3, where we will construct a projective representation of R_Λ using time-frequency shifts. Anticipating this, we will describe the cocycle on R_Λ which will be involved in this projective representation. First, let θ be a 2-cocycle on $\mathbb{R}^{2d}$. We can use θ to construct a 2-cocycle on R_Λ, denoted θ_Λ, using the formula

$$\theta_\Lambda\left((T, T - z), (T', T' - w)\right) = \theta(z, w).$$

Cocycles of this form will be called **standard cocycles**, and when it is clear we will drop the subscript from θ_Λ and refer to both cocycles as θ. We will be particularly concerned with the symplectic cocycle σ on $\mathbb{R}^{2d}$ given by

$$\sigma(z, w) = e^{-2\pi i x \omega'}$$

where $z = (x, \omega)$ and $w = (x', \omega')$.

Following [30] and [6], we construct a C^*-algebra from R_Λ and a 2-cocycle θ. To construct the C^*-algebra $\mathcal{A}_\theta = C^*(R_\Lambda, \theta)$, we begin with $C_c(R_\Lambda)$, with the product

$$f * g(T, T - z) := \sum_{w \in T} f(T, T - w) g(T - w, T - z) \theta\left((T, T - w), (T - w, T - z)\right)$$

and involution defined by

$$f^*(T, T - z) := \overline{f(T - z, T)} \theta\left((T, T - z), (T - z, T)\right)$$

where $z = (x, \omega)$. For the symplectic cocycle σ, the multiplication and involution can be written as

$$f * g(T, T - z) := \sum_{w = (x', \omega') \in T} f(T, T - w) g(T - w, T - z) e^{2\pi i x'(\omega' - \omega)}$$

and

$$f^*(T, T - z) := \overline{f(T - z, T)} e^{2\pi i x \omega}$$

respectively. We can define a norm on $C_c(R_\Lambda)$ by taking the sup over all the norms coming from the bounded representations of $C_c(R_\Lambda)$ (see [30] Chapter 2 for details, or [6] Section 4.1 for a description specific to quasicrystals). After completing $C_c(R_\Lambda)$ in this norm, we obtain the C^*-algebra $\mathcal{A}_\theta$.

For a standard cocycle θ, we can also construct a cocycle on the action groupoid $C(\Omega_\Lambda) \rtimes \mathbb{R}^{2d}$, and in this case $\mathcal{A}_\theta$ is Morita equivalent to the twisted crossed product $C^*(C(\Omega_\Lambda) \rtimes \mathbb{R}^{2d}, \theta)$ [29]. Since the action of $\mathbb{R}^{2d}$ on Ω_Λ is minimal and uniquely ergodic, both algebras are simple and have a unique normalized trace given by integrating over the unit space of their respective groupoids. For a function $f \in C_c(R_\Lambda)$, the trace is given by

$$Tr(f) = \int_{\Omega_{trans}} f(T,T)dT,$$

and after applying Birkhoff's ergodic theorem we can write

$$Tr(f) = \lim_{k\to\infty} \frac{1}{|\Lambda \cap C_k|} \sum_{z\in(\Lambda\cap C_k)} f(T-z, T-z)$$

so that the trace is expressed as an average over the values of f on the orbit of Λ.

For a standard cocycle θ we can compute the K-theory of $\mathcal{A}_\theta$ by appealing to the following theorem of Gillaspy [13]:

Theorem 2.3 ([13] Thm. 5.1)**.** *Let G be a second countable locally compact Hausdorff group acting on a second countable locally compact Hausdorff space X such that G satisfies the Baum-Connes conjecture with coefficients, and let ω_t be a homotopy of continuous 2-cocycles on the transformation group $X \rtimes G$. For any $t \in [0,1]$, the $*$-homomorphism*

$$q_t : C_r^*(G \ltimes X \times [0,1], \omega) \to C_r^*(G \ltimes X, \omega_t),$$

given on $C_c(G \ltimes X \times [0,1])$ by evaluation at $t \in [0,1]$, induces an isomorphism

$$K_*(C_r^*(G \ltimes X \times [0,1], \omega)) \cong K_*(C_r^*(G \ltimes X, \omega_t)).$$

Theorem 2.3, combined with the Connes-Thom isomorphism and the Morita equivalence between $\mathcal{A}_\theta$ and $C^*(C(\Omega_\Lambda) \rtimes \mathbb{R}^{2d}, \theta)$, gives

$$K_*(\mathcal{A}_\theta) \cong K_*(C^*(C(\Omega_\Lambda) \rtimes \mathbb{R}^{2d}, \theta)) \cong K_*(C(\Omega_\Lambda) \rtimes \mathbb{R}^{2d}) \cong K_*(C(\Omega_\Lambda)) \cong K^*(\Omega_\Lambda).$$

Theorem 2.3 applies since any cocycle on $\mathbb{R}^{2d}$ is homotopic to the trivial cocycle, essentially by the straight line homotopy. Unfortunately, the K-theory of Ω_Λ can be quite complicated. In many cases $K^0(\Omega_\Lambda)$ will not be finitely generated, and there are examples where it has torsion [12]. Because of these complexities, it is in general difficult to see how our module $\mathcal{H}_\Lambda$ fits into $K_0(\mathcal{A}_\sigma)$. In Section 5, we will show that when $\Lambda \subset \mathbb{R}^2$ is a subset of a lattice these difficulties can be overcome, and an understanding of how $\mathcal{H}_\Lambda$ fits into $K_0(\mathcal{A}_\sigma)$ is enough to compute $Tr_*(K_0(\mathcal{A}_\sigma))$.

Chapter 3

Gabor Analysis

3.1 Time-Frequency Analysis

Now we will review some basic concepts from time-frequency analysis. For a point $z = (x, \omega) \in \mathbb{R}^{2d}$ we denote by $\pi(z)$ the **time-frequency shift** by z, which operates on $L^2(\mathbb{R}^d)$ by

$$\pi(z)f(t) = M_\omega T_x f(t) = e^{2\pi i \omega t} f(t - x).$$

Here M_ω denotes the **modulation operator**

$$M_\omega f(t) = e^{2\pi i \omega t} f(t)$$

and T_x denotes the **translation operator**

$$T_x f(t) = f(t - x).$$

Fix $g \neq 0 \in L^2(\mathbb{R}^d)$ which we will call the window function. Then the **Short Time Fourier Transform (STFT)** of $f \in L^2(\mathbb{R}^d)$ with respect to the window g is

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\omega} dt \text{ for } (x, \omega) \in \mathbb{R}^{2d}.$$

The STFT of a function f with respect to the window g is an attempt to decompose f into time-frequency shifts of g. If g is supported on a small set around the origin then we can view $V_g f$ as an attempt to measure the "local frequencies" present in

f. Similar to the Fourier transform, the STFT has the following continuous reconstruction formula:

Proposition 3.1 ([14])**.** *Fix $g, \gamma \in L^2(\mathbb{R}^d)$ s.t. $\langle g, \gamma \rangle \neq 0$. Then for all $f \in L^2(\mathbb{R}^d)$,*

$$f = \frac{1}{\langle g, \gamma \rangle} \int \int_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x \gamma \, d\omega dx.$$

A central goal in Gabor analysis is to look for discrete versions of this reconstruction formula. This idea is expressed through the language of frames.

Definition 3.1. *A sequence $(e_j)_{j \in J}$ in a separable Hilbert space $\mathcal{W}$ is called a **frame** if there exist constants $A, B > 0$ s.t. for all $f \in \mathcal{W}$*

$$A||f||^2 \leq \sum_{j \in J} |\langle f, e_j \rangle|^2 \leq B||f||^2.$$

*If $A = B$ then (e_j) is called a **tight frame**, and if $A = B = 1$ then (e_j) is called a **Parseval tight frame.***

Any frame (e_j) has an associated frame operator S given by

$$Sf = \sum_{j \in J} \langle f, e_j \rangle e_j,$$

which is the composition of the **analysis** and **synthesis operators**

$$(Cf)_j = \langle f, e_j \rangle$$

$$D(\{a_j\}_{j \in J}) = \sum_{j \in J} a_j e_j.$$

We have a (non-unique, non-orthogonal) expansion of f given by

$$f = \sum_{j \in J} \langle f, S^{-1} e_j \rangle e_j$$

where the elements $S^{-1}e_j$ are known as the **dual frame**. We also have an associated Parseval tight frame given by $(S^{-1/2}e_j)_{j\in J}$.

If we wish to discretize the STFT, we can choose a subset $\Lambda \subset \mathbb{R}^{2d}$ and a window g and ask whether the set

$$\mathcal{G}(g,\Lambda) =: \{\pi(z)g \,|\, z \in \Lambda\}$$

forms a frame for $L^2(\mathbb{R}^d)$. Such frames are called **Gabor frames** for Λ. More generally, we can choose finitely many functions $g_1, \ldots, g_N$ and look for **multiwindow Gabor frames** of the form

$$\mathcal{G}(g_1,\ldots,g_N,\Lambda) := \{\pi(z)g_i \,|\, i = 1\ldots,N, z \in \Lambda\}.$$

In this case elements of the dual frame will be denoted by $\tilde{g}_{iz} = S^{-1}(\pi(z)g_i)$. When Λ is a lattice, the dual frame will also have the structure of a Gabor frame given by $\mathcal{G}(\tilde{g_1},\ldots,\tilde{g_N},\Lambda)$ where $\tilde{g_i} = S^{-1}g_i$.

With this background in place, it is natural to ask:

Question 3.1. *Given a quasicrystal Λ, can we find functions $g_1,\ldots,g_N$ so that $\mathcal{G}(g_1,\ldots,g_N,\Lambda)$ is a Gabor frame for Λ?*

Much of the work in Gabor analysis has focused on the case where Λ is a lattice. However, recent results in [15] took a large step towards answering this question not just for quasicrystals, but for any discrete set Λ. In order to explain their results, it will be necessary to introduce the modulation spaces $M^p(\mathbb{R}^d)$.

Definition 3.2. *Fix a non-zero $g \in \mathcal{S}(\mathbb{R}^d)$. For $1 \le p \le \infty$ we define the* ***modula-***

tion spaces

$$M^p(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \,|\, V_g f \in L^p(\mathbb{R}^{2d})\}$$

with the norm $||f||_{M^p} = ||V_g f||_p$.

Different choices for g give rise to equivalent norms on $M^p(\mathbb{R}^d)$. The modulation space $M^1(\mathbb{R}^d)$ consists of good windows for Gabor analysis. When $g \in M^1(\mathbb{R}^d)$ the analysis and synthesis operators for a Gabor system $\mathcal{G}(g, \Lambda)$ are bounded between $M^p(\mathbb{R}^d)$ and $l^p(\Lambda)$:

$$||C_g^\Lambda f||_{l^p} \leq \mathrm{rel}(\Lambda)||g||_{M^1}||f||_{M^p}$$

$$||D_g^\Lambda c||_{M^p} \leq \mathrm{rel}(\Lambda)||g||_{M^1}||c||_{l^p}.$$

A Gabor system $\mathcal{G}(g, \Lambda)$ with $g \in M^1(\mathbb{R}^d)$ will be called an $\boldsymbol{M^p}$**-frame** if C_g^Λ is bounded below on $M^p(\mathbb{R}^d)$. This is equivalent to having constants A, B so that for all $f \in M^p(\mathbb{R}^d)$

$$\sqrt{A}||f||_{M^p} \leq ||S_g^\Lambda f||_{M^p} \leq \sqrt{B}||f||_{M^p}.$$

In this case the frame operator S_g^Λ is invertible on $M^p(\mathbb{R}^d)$.

Now we are ready to state the result from [15] which gives sufficient conditions for answering Question 3.1. For $g \in M^1(\mathbb{R}^d)$ and $\delta > 0$, we can define the M^1 modulus of continuity of g as

$$\omega_\delta(g) = \sup_{|z-w|\leq\delta} ||\pi(z)g - \pi(w)g||_{M^1}$$

It is clear that $\omega_\delta \to 0$ as $\delta \to 0$ since the representation π is strongly continuous in $B(M^1(\mathbb{R}^d))$.

Theorem 3.1 ([15]). *For $g \in M^1(\mathbb{R}^d)$ with $||g||_2 = 1$ choose $\delta > 0$ so that $\omega_\delta(g) < 1$. If $\Lambda \subset \mathbb{R}^{2d}$ is relatively separated and $\rho(\Lambda) < \delta$, then $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$.*

From this result, we can see that when $\rho(\Lambda)$ is small enough there will be many windows g for which $\mathcal{G}(g, \Lambda)$ is a Gabor frame. Furthermore, when g is one of these admissible windows, $\mathcal{G}(g, \Lambda')$ will also form a Gabor frame for any $\Lambda' \in \Omega_\Lambda$ since $\rho(\Lambda') = \rho(\Lambda)$. However, when $\rho(\Lambda)$ is large we cannot expect $\mathcal{G}(g, \Lambda)$ to form a Gabor frame for any g. In fact, the Balian-Low theorem for non-uniform frames proven in [15] shows that if $\mathcal{G}(g, \Lambda)$ is a frame then $\mathrm{Dens}(\Lambda) > 1$. In this case, we can only expect a multiwindow Gabor frame to exist.

Finally, we will need to introduce one more function space needed for the proofs in Section 3.2. The **Wiener amalgam space** $W(L^\infty, L^1)(\mathbb{R}^d)$ consists of all functions $f \in L^\infty(\mathbb{R}^d)$ such that

$$||f||_{W(L^\infty,L^1)} := \sum_{k\in\mathbb{Z}^d} ||f||_{L^\infty([0,1]^d+k)} < \infty.$$

It is a standard result (see [14] Proposition 12.1.11) that when $g \in M^1(\mathbb{R}^d)$ then for any $f \in M^1(\mathbb{R}^d), V_g f \in W(L^\infty, L^1)(\mathbb{R}^{2d})$ and $||V_g f||_{W(L^\infty,L^1)} \leq C||f||_{M^1}||g||_{M^1}$. Also note that if $f \in W(L^\infty, L^1)(\mathbb{R}^d)$ and $T \subset \mathbb{R}^d$ is a Delone set then we have the inequality

$$\sum_{t\in T} |f(t)| \leq \mathrm{rel}(T)||f||_{W(L^\infty,L^1)}. \tag{3.1}$$

If $T \in \Omega_\Lambda$ then the bound in this inequality is independent of T since $\mathrm{rel}(T) = \mathrm{rel}(\Lambda)$.

3.2 Gabor Frames for Quasicrystals

3.2.1 Existence of Multiwindow Gabor Frames

Our first goal will be to prove Theorem 1.1. Given a quasicrystal Λ, Theorem 3.1 gives sufficient conditions for a single window Gabor frame to exist for Λ based on the size of $\rho(\Lambda)$. To show that multiwindow frames exist, we first need the following lemma:

Lemma 3.1. *Suppose $\Lambda \subset \mathbb{R}^{2d}$ is FLC. Fix $\epsilon > 0$. We can find finitely many disjoint translates $\{\Lambda_i\}_{i=1}^N$ so that $\bar{\Lambda} = \bigcup_{i=1}^N \Lambda_i$ has $\rho(\bar{\Lambda}) < \epsilon$.*

Proof. First we let $R = \rho(\lambda) + \delta$ for some small δ. Since Λ is FLC, there are only finitely many patterns of the form $B_R(z) \cap \Lambda$ up to translation. These patterns contain all the possible types of holes in Λ, some of which have size larger than ϵ. Note that if we have a finite sequence z_n then $\bigcup_{n=1}^N \Lambda + z_n$ is also FLC. Thus we can systematically shrink these holes one by one by taking unions of translates of Λ. It will suffice to take a single pattern P which contains a hole of size larger than ϵ and show how we can shrink that hole by a factor of 2. Repeating the procedure will shrink the hole below a size of ϵ.

Choose a point $z \in P$ and let c denote the center of the largest hole in P. Then the set $\Lambda \cup (\Lambda - z + c)$ will no longer contain the patch P. Instead, all occurrences of the patch P in Λ will now have a point in the center of the largest hole of P, so that the largest hole will have been reduced in size by a factor of 2.

This method does not ensure that the sets Λ and $\Lambda - z + c$ will be disjoint, since the vector $z-c$ may lie in $\Lambda-\Lambda$. To fix this, note that we do not need to place a point exactly in the center of the hole, but only very close to the center, in order to reduce the hole by a significant amount. Thus if $z-c \in \Lambda-\Lambda$, we instead choose a point c' close enough to c so that $z-c' \notin \Lambda-\Lambda$ and the hole in P is reduced by a factor of $2-\eta$ for some small η. We can find such a point c' since Λ FLC implies that $\Lambda-\Lambda$ is discrete. $\square$

Proposition 3.2. *Given a Delone set $\Lambda \subset \mathbb{R}^{2d}$ with FLC and $g \in M^1(\mathbb{R}^d)$, we can find a multiwindow Gabor frame for Λ where the windows consist of time frequency translates of g. Furthermore, this multiwindow Gabor frame will be an M^p-frame for all p.*

Proof. Choose $\delta > 0$ so that $\omega_\delta(g) < 1$. Applying Lemma 3.1, we can find $\Lambda' = \bigcup_{i=1}^N (\Lambda + z_i)$ so that $\rho(\Lambda') < \delta$. Then by Theorem 1.1,

$$\mathcal{G}(g, \Lambda') = \bigcup_{i=1}^N \{\pi(z+z_i)g \mid z \in \Lambda\}$$

is a Gabor frame and by Theorem 5.1 of [15] it is an M^p-frame for all p. This is almost equal to the multiwindow Gabor system given by

$$\bigcup_{i=1}^N \mathcal{G}(\pi(z_i)g, \Lambda) = \bigcup_{i=1}^N \{\pi(z)\pi(z_i)g \mid z \in \Lambda\} = \bigcup_{i=1}^N \{e^{-2\pi i x \omega_i}\pi(z+z_i)g \mid z \in \Lambda\}$$

where $z = (x, \omega)$ and $z_i = (x_i, \omega_i)$. The functions in the two Gabor systems differ only by phase factors, so $\bigcup_{i=1}^N \mathcal{G}(\pi(z_i)g, \Lambda)$ will satisfy the same frame inequalities as $\mathcal{G}(g, \Lambda')$ and thus $\bigcup_{i=1}^N \mathcal{G}(\pi(z_i)g, \Lambda)$ is a multiwindow Gabor frame with the same frame bounds (and M^p-frame bounds) as $\mathcal{G}(g, \Lambda')$. $\square$

Corollary 3.1. *If $g \in M^1(\mathbb{R}^d)$ and $\bigcup_{i=1}^N \mathcal{G}(\pi(z_i)g, \Lambda)$ is a multiwindow Gabor frame as constructed above, then so is $\bigcup_{i=1}^N \mathcal{G}(\pi(z_i)g, \Lambda')$ for any $\Lambda' \in \Omega_\Lambda$.*

Proof. Since $\Lambda' \in \Omega_\Lambda$, it contains exactly the same patches as Λ. Thus the procedure in Lemma 3.1 also works to fill in the holes of Λ', so that the sets $\bigcup_{i=1}^N \Lambda + z_i$ and $\bigcup_{i=1}^N \Lambda' + z_i$ have the same sized hole. Then the argument from Proposition 3.2 applies in exactly the same way to Λ', showing that $\bigcup_{i=1}^N \mathcal{G}(\pi(z_i)g, \Lambda')$ is a multiwindow Gabor frame. □

Taken together, Proposition 3.2 and Corollary 3.1 immediately imply Theorem 1.1.

3.2.2 Continuity and Covariance Properties of the Frame Operator

Now we will investigate various continuity and covariance properties of the frame operator. When $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ is a multiwindow Gabor frame, we will denote the associated frame operator by $S^{\Lambda}_{\{g_i\}}$. We will often omit the subscripts and superscripts when they are clear from context. We would like to understand the relationship between the frame operators S^T and $S^{T'}$ when $T, T' \in \Omega_\Lambda$. First we shall show that when $T' = T - z$ then there is a covariance condition relating S^T and $S^{T'}$.

Proposition 3.3. *If $\mathcal{G}(g_1, \ldots, g_N, T)$ and $\mathcal{G}(g_1, \ldots, g_N, T - w)$ are multiwindow Gabor systems for T and $T - w$ respectively, then the frame operators S^T and S^{T-w} satisfy*

$$S^T \pi(w) = \pi(w) S^{T-w}.$$

Proof. Fix $f \in L^2(\mathbb{R}^d)$. On the one hand we have

$$S^T \pi(w) f = \sum_{i=1}^{N} \sum_{z \in T} \langle \pi(w) f, \pi(z) g_i \rangle \pi(z) g_i = \sum_{i=1}^{N} \sum_{z \in T} e^{2\pi i x' \omega} \langle f, \pi(z-w) g_i \rangle \pi(z) g_i.$$

where $z = (x, \omega)$ and $w = (x', \omega')$. On the other hand we have

$$\begin{aligned} \pi(w) S^{T-w} f &= \sum_{i=1}^{N} \sum_{z \in T} \langle f, \pi(z-w) g_i \rangle \pi(w) \pi(z-w) g_i \\ &= \sum_{i=1}^{N} \sum_{z \in T} e^{2\pi i x' \omega} \langle f, \pi(z-w) g_i \rangle \pi(z) g_i \end{aligned}$$

and so the two expressions are equal. $\square$

We would also like to know something about the continuity of the frame operators over Ω_Λ. If $T_k \to T$ in Ω_Λ, we cannot expect $S^{T_k} \to S^T$ in the operator norm. However, we do have that $S^{T_k} \to S^T$ in the strong operator topology.

Proposition 3.4. *Suppose $T_k \to T$ in Ω_Λ and the window functions $g_1, \ldots, g_N$ lie in $M^1(\mathbb{R}^d)$. Then $S^{T_k} \to S^T$ in the strong operator topology on $B(M^1(\mathbb{R}^d))$.*

Proof. Fix $f \in M^1(\mathbb{R}^d)$. Let $A = \max\{||g_i||_{M^1}\}$. Fix $\epsilon > 0$ and choose a large cube C so that for all i

$$\sum_{a \in \mathbb{Z}^n \setminus C} ||V_{g_i} f||_{L^\infty([0,1]^n + a)} < \frac{\epsilon}{4AN\mathrm{rel}(\Lambda)}$$

where N is the number of windows in the multiwindow frame. Since $T_k \to T$, we can choose K so that for all $k \geq K, T_k$ agrees with T on the cube C up to a small translation, so that

$$\left\| \sum_{i=1}^{N} \sum_{z \in T \cap C} \langle f, \pi(z) g_i \rangle \pi(z) g_i - \sum_{i=1}^{N} \sum_{z \in T_k \cap C} \langle f, \pi(z) g_i \rangle \pi(z) g_i \right\|_{M^1} < \frac{\epsilon}{2}.$$

Then for all $k \geq K$ we have

$$\begin{aligned}
||S^T f - S^{T_k} f||_{M^1} &\leq \left\| \sum_{i=1}^{N} \sum_{z \in T \setminus C} \langle f, \pi(z) g_i \rangle \pi(z) g_i - \sum_{i=1}^{N} \sum_{z \in T_k \setminus C} \langle f, \pi(z) g_i \rangle \pi(z) g_i \right\|_{M^1} + \frac{\epsilon}{2} \\
&\leq A \left(\sum_{i=1}^{N} \sum_{z \in T \setminus C} |\langle f, \pi(z) g_i \rangle| + \sum_{i=1}^{N} \sum_{z \in T_k \setminus C} |\langle f, \pi(z) g_i \rangle| \right) + \frac{\epsilon}{2} \\
&= A \left(\sum_{i=1}^{N} \sum_{z \in T \setminus C} |V_{g_i} f(z)| + \sum_{i=1}^{N} \sum_{z \in T_k \setminus C} |V_{g_i} f(z)| \right) + \frac{\epsilon}{2} \\
&\leq 2A \mathrm{rel}(\Lambda) \left(\sum_{i=1}^{N} \sum_{a \in \mathbb{Z}^n \setminus C} ||V_{g_i} f||_{L^\infty([0,1]^n + a)} \right) + \frac{\epsilon}{2} \\
&< 2AN \mathrm{rel}(\Lambda) \left(\frac{\epsilon}{4AN \mathrm{rel}(\Lambda)} \right) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

In the fourth inequality it is important to note that the inequality (3.1) holds not only for the norms, but also for the partial sums. The main reason this proof works is that $\mathrm{rel}(T)$ is constant on Ω_Λ. By applying inequality (3.1), this implies that we can find a cube C so that the sum $S^T f$ is arbitrarily small outside of C independent of $T \in \Omega_\Lambda$. $\square$

Even though the mapping $T \to S^T$ will not be continuous when $B(M^1(\mathbb{R}^d))$ is given the norm topology, we can still show that all the frames $\mathcal{G}(g_1, \ldots, g_N, T)$ have the same optimal frame bounds.

Proposition 3.5. *Suppose $\mathcal{G}(g_1, \ldots, g_N, T)$ is a frame for each $T \in \Omega_\Lambda$ and each $g_i \in M^1(\mathbb{R}^d)$. For any $T \in \Omega_\Lambda$ the optimal upper and lower frame bounds for $\mathcal{G}(g_1, \ldots, g_N, T)$ are the same as those for $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$. As a result, $||S^T||_{M^1} = ||S^\Lambda||_{M^1}$ and $||(S^T)^{-1}||_{M^1} = ||(S^\Lambda)^{-1}||_{M^1}$ where $||\cdot||_{M^1}$ denotes the operator norm on $B(M^1(\mathbb{R}^d))$.*

Proof. Let A and B denote the optimal lower and upper frame bounds for $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ so that for all $f \in M^1(\mathbb{R}^d)$

$$\sqrt{A}||f||_{M^1} \leq ||S^\Lambda f||_{M^1} \leq \sqrt{B}||f||_{M^1}.$$

Since the translates of Λ are dense in Ω_Λ, we can find a sequence of translates $\Lambda - z_k \to T$. Note that by Proposition 3.3 the frame bounds are constant on the orbit of Λ. Since $\Lambda - z_k \to T$, $S^{\Lambda - z_k} \to S^T$ in the strong topology by Proposition 3.4.

Now fix $f \in M^1(\mathbb{R}^d)$. We have $||S^{\Lambda - z_k} f||_{M^1} \to ||S^T f||_{M^1}$. Since $\sqrt{A}||f||_{M^1} \leq ||S^{\Lambda - z_k} f||_{M^1} \leq \sqrt{B}||f||_{M^1}$ for all k, we have $\sqrt{A}||f||_{M^1} \leq ||S^T f||_{M^1} \leq \sqrt{B}||f||_{M^1}$. By reversing the roles of T and Λ in this argument, we see that the upper and lower frame bounds for T and Λ must be equal. The last remark follows since the lower and upper frame bounds are equal to $||(S^T)^{-1}||_{M^1}$ and $||S^T||_{M^1}$ respectively. □

Corollary 3.2. *Suppose $g_1, \ldots, g_N \in M^1(\mathbb{R}^d)$ and $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ is an M^1-frame. Then for any $T \in \Omega_\Lambda, \mathcal{G}(g_1, \ldots, g_N, T)$ is also an M^1-frame.*

Proof. By examining the basic frame inequalities in Definition 3.1, we can see that if $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ is a frame then so is $\mathcal{G}(g_1, \ldots, g_N, \Lambda - z)$ for any $z \in \mathbb{R}^{2d}$. Given $T \in \Omega_\Lambda$, we can find a sequence of translates $\Lambda - z_k$ converging to T. By Proposition 3.4 we have $S^{\Lambda - z_k} \to S^T$ in the strong topology on $B(M^1(\mathbb{R}^d))$. The frame bounds for the frames $\mathcal{G}(g_1, \ldots, g_N, \Lambda - z_k)$ are all equal by Proposition 3.3, so S^T also satisfies those same frame bounds. In particular S^T is bounded below on $M^1(\mathbb{R}^d)$, and thus $\mathcal{G}(g_1, \ldots, g_N, T)$ is an M^1-frame. □

Note the difference between Corollary 3.1 and Corollary 3.2. Corollary 3.1 says that there exist windows $\{g_i\}_{i=1}^N \subset M^1(\mathbb{R}^d)$ so that $\mathcal{G}(g_1, \ldots, g_N, T)$ is a Gabor frame for any $T \in \Omega_\Lambda$. Corollary 3.2 says that when $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ is a multiwindow Gabor frame and each $g_i \in M^1(\mathbb{R}^d)$, then $\mathcal{G}(g_1, \ldots, g_N, T)$ is *automatically* also a Gabor frame for any $T \in \Omega_\Lambda$. The similarity between the Delone sets in Ω_Λ is the key to Proposition 3.4 which drives all of our results.

3.3 Comparison of Convergence Properties

It is interesting to compare our Proposition 3.4 to the results in [15]. They define a notion of convergence for point sets which is seemingly much stronger than the local topology defined in Section 2.1. For $\Lambda \subset \mathbb{R}^{2d}$ a Delone set, they consider a sequence of Delone sets $\{\Lambda_n \,|\, n \geq 1\}$ produced as follows. For each $n \geq 1$ let $\tau_n : \Lambda \to \mathbb{R}^{2d}$ be a map and define $\Lambda_n := \tau_n(\Lambda) = \{\tau_n(\lambda) \,|\, \lambda \in \Lambda\}$. We assume $\tau_n(\lambda) \to \lambda$ as $n \to \infty$. The sequence of sets Λ_n together with the maps τ_n is called a **deformation** of Λ. We will say that a sequence of sets Λ_n is a deformation of Λ with the understanding that the maps τ_n are also given.

Definition 3.3. *A deformation of Λ is called **Lipschitz**, denoted by $\Lambda_n \overset{\text{Lip}}{\to} \Lambda$, if:*

1. *Given $R > 0$,*

$$\sup_{\substack{\lambda, \lambda' \in \Lambda \\ |\lambda - \lambda'| \leq R}} |(\tau_n(\lambda) - \tau_n(\lambda')) - (\lambda - \lambda')| \to 0 \text{ as } n \to \infty.$$

2. *Given $R > 0$ there exists $R' > 0$ and $n_0 \in \mathbb{N}$ such that if $|\tau_n(\lambda) - \tau_n(\lambda)'| \leq R$ for some $n \geq n_0$ and some $\lambda, \lambda' \in \Lambda$ then $|\lambda - \lambda'| \leq R'$.*

The goal of Lipschitz convergence is to ensure that the separation, relative density, and hole of Λ_n are close to the corresponding quantities for Λ when n is large. This level of control allows them to prove the following result:

Theorem 3.2. *Let $g \in M^1(\mathbb{R}^d), \Lambda \subset \mathbb{R}^{2d}$ and assume that $\mathcal{G}(g, \Lambda)$ is a frame. If Λ_n is Lipschitz convergent to Λ then $\mathcal{G}(g, \Lambda_n)$ is a frame for sufficiently large n.*

The proof depends on a general characterization of Gabor frames which does not involve inequalities. Their result does not imply that $S^{\Lambda_n} \to S^{\Lambda}$ in any of the standard operator topologies, and does not yield estimates for the frame bounds of S^{Λ_n} when n is sufficiently large. This illustrates how much simpler the situation becomes when we restrict our attention to quasicrystals.

We would like to compare Lipschitz convergence with the local topology. In particular, if $\Lambda_n \overset{\text{Lip}}{\to} \Lambda$ and all $\Lambda_n \in \Omega_\Lambda$, does $\Lambda_n \to \Lambda$ in Ω_Λ? After reflecting upon this, it seems quite difficult for any deformation Λ_n to satisfy condition one in Definition 3.3 if all $\Lambda_n \in \Omega_\Lambda$. The sets in Ω_Λ were chosen based on their local structure, but condition one implies a kind of global convergence. We are led to conjecture:

Conjecture 3.1. *Suppose $\Lambda_n \in \Omega_\Lambda$ and $\Lambda_n \overset{\text{Lip}}{\to} \Lambda$. Then for any $\epsilon > 0$ we can find N_ϵ such that for all $n \geq N_\epsilon, \Lambda_n = \Lambda + v_n$ where $v_n \in \mathbb{R}^{2d}$ is a vector with $||v_n|| \leq \epsilon$.*

Conjecture 3.1 would imply that Lipschitz convergence implies convergence in the local topology, but is actually far stronger as any Lipschitz convergent sequence would be somewhat trivial. We can illustrate this by stating a slightly weaker conjecture:

Conjecture 3.2. *Suppose* $\Lambda_n \in \Omega_{trans}$ *and* $\Lambda_n \stackrel{\text{Lip}}{\to} \Lambda$. *Then the sequence* Λ_n *is eventually constant and equal to* Λ.

These conjectures seem reasonable in light of the fact that a local isomorphism between two quasicrystals need not imply global similarity. However, there is one class of point sets where local similarity *does* imply a kind of global similarity. In [2] the authors show that model sets which are close in the local topology are statistically similar in the following sense. Let Λ be a model set and suppose $\Lambda' \in \Omega_\Lambda$ and Λ' agrees with Λ on a large ball around the origin so that $d(\Lambda, \Lambda') < \epsilon$. Then there is a constant C independent of Λ, Λ' so that $\overline{\text{dens}}(\Lambda \,\Delta\, \Lambda') < C\epsilon$. Here $\overline{\text{dens}}(\Lambda)$ denotes the upper density of a point set Λ and Δ denotes the symmetric difference. Furthermore, they show that this property characterizes model sets among quasicrystals. It is unclear whether this statistical similarity is enough to construct a counterexample to either conjecture. It would be interesting to see whether these conjectures have different answers depending on the class of quasicrystals (e.g. model set or substitution) under consideration.

Chapter 4

Constructing $\mathcal{A}_\sigma$-modules

4.1 Lattice Gabor Frames and Modules over Noncommutative Tori

To motivate our construction of modules over $\mathcal{A}_\sigma$, we will review Rieffel's results in [31] on constructing modules over noncommutative tori and relate them to Gabor analysis as in [25], [26].

Definition 4.1. *Let $L \subset \mathbb{R}^{2d}$ be a lattice. The C^*-algebra A_L generated by the time-frequency shifts $\{\pi(z) \mid z \in L\}$ is called a* ***noncommutative torus****.*

We can also define noncommutative tori as twisted convolution algebras. We take $l^1(L)$ with twisted convolution

$$a * b(l) := \sum_{\mu \in L} a(\mu) b(l-\mu) \sigma(\mu, l-\mu)$$

where σ is the symplectic cocycle on $\mathbb{R}^{2d}$. This is equivalent to taking the twisted group algebra $A_\theta = C^*_r(\mathbb{Z}^{2d}, \theta)$ where $\theta = \sigma|_L$. The group algebra is generated by unitaries $U_{\vec{n}}$ which correspond to the Dirac δ-functions at the elements of $\mathbb{Z}^{2d}$. Any cocycle on $\mathbb{Z}^{2d}$ is given by a skew symmetric matrix Θ which describes the commutation relations between the $U_{\vec{n}}$:

$$U_{\vec{n}} U_{\vec{m}} = e^{2\pi i \vec{n}^t \Theta \vec{m}} U_{\vec{m}} U_{\vec{n}}.$$

When the off diagonal entries of this matrix are all irrational and rationally independent, we call the cocycle **totally irrational**. The standard trace on A_θ is given by

$$\mathrm{Tr}_{A_\theta}\left(\sum_{\vec{n}\in\mathbb{Z}^{2d}} a_{\vec{n}} U_{\vec{n}}\right) = a_0.$$

When θ is totally irrational the map $\mathrm{Tr}_{A_\theta *} : K_0(A_\theta) \to \mathbb{R}$ is injective, although in general it will not be [9].

Each of these definitions of the noncommutative torus comes with its own advantages. By viewing a noncommutative torus as a twisted group algebra A_θ we can easily compute its K-theory. Any skew symmetric matrix Θ is homotopic to the zero matrix by the straight line homotopy, so Theorem 2.3 applies[1] and shows $K_*(A_\theta) \cong K^*(\mathbb{T}^{2d})$. On the other hand, when we have a lattice L such that $\sigma|_L = \theta$, the algebra $A_L \cong A_\theta$ and describes A_θ in a specific representation. Rieffel's insight was that different lattices can produce different representations of A_θ, and that these representations exhaust the classes in $K_0(A_\theta)$.

More precisely, we define the **smooth noncommutative torus**

$$A_L^\infty := \left\{ \sum_{z\in L} a_z \pi(z) \in A_L \,|\, a_z \text{ decays faster than any polynomial} \right\},$$

and the analogous smooth subalgebra of A_θ is defined similarly. The algebra A_L^∞ is a spectrally invariant subalgebra of A_L. There is a canonical action of A_L^∞ on $\mathcal{S}(\mathbb{R}^d)$ by time-frequency shifts, and we denote this A_L^∞-module by V_L. We have

$$\mathrm{Tr}_{A_L *}([V_L]) = \mathrm{vol}(L) = \frac{1}{\mathrm{Dens}(L)},$$

and this last equality already suggests how the dimension of this module will generalize to quasicrystals. We identify a lattice L with a linear map A such that

[1]There are many ways to compute the K-theory of noncommutative tori, but we use Theorem 2.3 since we will need this specific isomorphism later.

$A\,\mathbb{Z}^{2d} = L$. If we fix a cocycle θ then $\sigma|_L = \theta$ exactly when $A^*\sigma = \theta$.

Theorem 4.1 (Rieffel [31])**.** *Fix a cocycle θ on $\mathbb{Z}^{2d}$. Any invertible linear map A such that $A^*\sigma = \theta$ gives rise to an A_θ^∞-module $V_{A\,\mathbb{Z}^{2d}}$. These modules are finitely generated and projective, and any class in $K_0(\mathcal{A}_\theta^\infty)$ can be represented as $[V_{A\,\mathbb{Z}^{2d}}]$ for some A.*

In order to promote V_L from an A_L^∞-module to an A_L-module we first endow it with the structure of a Hilbert C^*-module. For $f, g \in \mathcal{S}(\mathbb{R}^d)$, we define an A_L^∞ valued inner product by

$$ {}_L\langle f, g\rangle := \sum_{z\in L} \langle f, \pi(z)g\rangle \pi(z). $$

To prove that this inner product makes V_L into a Hilbert A_L^∞-module, we must show (among other, easier identities) that the inner product ${}_L\langle f, f\rangle$ always yields a positive element of A_L^∞. It suffices to show that for any $g \in \mathcal{S}(\mathbb{R}^d)$ we have

$$ \langle {}_L\langle f, f\rangle g, g\rangle \geq 0. $$

Simplifying the right hand side, we have

$$ \langle {}_L\langle f, f\rangle g, g\rangle = \sum_{z\in L} \langle f, \pi(z)f\rangle \langle \pi(z)g, g\rangle = \frac{1}{\mathrm{vol(L)}} \sum_{l\in L^\circ} \langle f, \pi(l)g\rangle \langle \pi(l)g, f\rangle \geq 0 $$

where

$$ L^\circ := \begin{pmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} L^* $$

is the **adjoint lattice**. Here the second equality follows from an application of the Poisson summation formula.

One of the advantages of giving V_L a Hilbert module structure is that we can investigate the dual structure $\mathrm{End}^0_{A_L^\infty}(V_L)$. Note that when $l \in L^\circ, \pi(l)$ commutes with $\pi(z)$ whenever $z \in L$. Thus we can consider the natural right action of $A_{L^\circ}^\infty$ on $\mathcal{S}(\mathbb{R}^d)$ and this action commutes with the left action of A_L^∞. We can define an $A_{L^\circ}^\infty$-valued inner product on $\mathcal{S}(\mathbb{R}^d)$ by

$$\{f, g\}_{L^\circ} := \frac{1}{\mathrm{vol(L)}} \sum_{l \in L^\circ} \pi(l)^* \langle \pi(l) g, f \rangle = \frac{1}{\mathrm{vol(L)}} \sum_{l \in L^\circ} \pi(l) \langle g, \pi(l) f \rangle.$$

We would hope that $\mathrm{End}^0_{A_L^\infty}(V_L) = A_{L^\circ}^\infty$, and we can show this by verifying the identity ${}_L\langle f, g\rangle h = f\{g, h\}_{L^\circ}$. It is enough to show that for all $k \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle {}_L\langle f, g\rangle h, k\rangle = \langle f\{g, h\}_{L^\circ}, k\rangle.$$

Simplifying this identity, we can see that it amounts to claiming

$$\sum_{z \in L} \langle f, \pi(z) g\rangle \langle \pi(z) h, k\rangle = \frac{1}{\mathrm{vol(L)}} \sum_{l \in L^\circ} \langle f, \pi(l) k\rangle \langle \pi(l) h, g\rangle$$

which follows again from an application of the Poisson summation formula. Thus V_L actually has the structure of a Hilbert A_L^∞-$A_{L^\circ}^\infty$ bimodule, demonstrating that A_L^∞ and $A_{L^\circ}^\infty$ are Morita equivalent. We can equip V_L with the norm

$$||f||_{A_L} := ||{}_L\langle f, f\rangle||_{A_L}^{1/2}.$$

After completing V_L in this norm, we end up with an A_L-A_{L° equivalence bimodule, and Theorem 4.1 holds in this case as well.

The reader may already have noticed some similarities between the arguments above and the analysis of lattice Gabor frames. In particular, the applications of the Poisson summation formula above are known in time-frequency analysis as the

Fundamental Identity of Gabor Analysis (FIGA). To draw out the comparison further, let's fix functions $f, g \in \mathcal{S}(\mathbb{R}^d)$ and consider the frame operator $S^L_{f,g}$ given by

$$S^L_{f,g} = \sum_{z \in L} \langle \,\cdot\,, \pi(z)f \rangle \pi(z) g.$$

This is precisely equal to the rank one operator ${}_L\langle \cdot, f \rangle g$, so the frame operator $S^L_{f,g}$ lies in $\mathrm{End}^0_{A^\infty_L}(V_L)$. However, we know from above that $\mathrm{End}^0_{A^\infty_L}(V_L) = A^\infty_{L^\circ}$, so we should be able to find an expression for $S^L_{f,g}$ in terms of time frequency shifts from L°. Indeed, an application of the FIGA shows that

$$S^L_{f,g} = \sum_{z \in L} \langle \,\cdot\,, \pi(z)f \rangle \pi(z) g = \frac{1}{\mathrm{vol}(L)} \sum_{l \in L^\circ} \langle g, \pi(l) f \rangle \pi(l) = \{f, g\}_{L^\circ}.$$

This is known as the **Janssen representation** of the Gabor frame operator.

Thus one could predict the existence of the Janssen representation from the Hilbert module structure of V_L. Indeed, one can actually deduce the coefficients in the representation by applying the trace on $A^\infty_{L^\circ}$. To this end, we can define a (non-normalized) trace $\mathrm{Tr}'_{A_{L^\circ}}$ on rank one operators in $A^\infty_{L^\circ}$ using the formula

$$\mathrm{Tr}'_{A_{L^\circ}}(\langle f, g \rangle_{L^\circ}) := \mathrm{Tr}_{A_L}({}_L\langle g, f \rangle).$$

This extends to all of $A^\infty_{L^\circ}$ and agrees with the standard trace up to a constant:

$$\mathrm{Tr}'_{A_{L^\circ}} = \mathrm{vol}(L) \mathrm{Tr}_{A_{L^\circ}}.$$

Thus we can compute

$$\mathrm{Tr}_{A_{L^\circ}}(\{f, g\}_{L^\circ}) = \frac{1}{\mathrm{vol}(L)} \mathrm{Tr}'_{A_{L^\circ}}(\{f, g\}_{L^\circ}) = \frac{1}{\mathrm{vol}(L)} \mathrm{Tr}_{A_L}({}_L\langle g, f \rangle) = \frac{1}{\mathrm{vol}(L)} \langle g, f \rangle.$$

Now to deduce the Janssen representation, we notice that for $l \in L^\circ$ the coefficient of $\pi(l)$ in the expansion of $S^L_{f,g}$ is given by $\mathrm{Tr}_{A_{L^\circ}}(S^L_{f,g}\pi(l)^*)$. However, $S^L_{f,g}\pi(l)^*$ is equal to $\{f, \pi(l)^* g\}_{L^\circ}$, so

$$\mathrm{Tr}_{A_{L^\circ}}(S^L_{f,g}\pi(l)^*) = \mathrm{Tr}_{A_{L^\circ}}(\{f, \pi(l)^* g\}_{L^\circ}) = \frac{1}{\mathrm{vol}(L)}\langle \pi(l)^* g, f\rangle = \frac{1}{\mathrm{vol}(L)}\langle g, \pi(l) f\rangle.$$

This is precisely the coefficient of $\pi(l)$ in the Janssen representation.

4.2 Projections in Noncommutative Tori

Whenever $\mathcal{G}(g, L)$ is a Gabor frame and $g \in \mathcal{S}(\mathbb{R}^d)$, we can use it to construct a projection in A^∞_L. Denote by $\tilde{g}_z$ the function $S_g^{-1}\pi(z)g = \pi(z)S_g^{-1}g$ which is an element of the canonical dual frame. We will denote by C the **noncommutative analysis operator** acting on $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$Cf = \sum_{z \in L} \langle f, \tilde{g}_z\rangle \pi(z) \in A^\infty_L.$$

Denote by D the **noncommutative synthesis operator** which takes an element $a \in A^\infty_L$ to $ag \in \mathcal{S}(\mathbb{R}^d)$. Since g generates a Gabor frame, we have that DC is the identity on $\mathcal{S}(\mathbb{R}^d)$, showing that V_L is finitely generated (by g) and projective as an A^∞_L module. Composing the operators in the other direction gives us a projection in A^∞_L representing the module V_L, which can be written as $P_g = \sum_{z\in L}\langle g, \tilde{g}_z\rangle \pi(z)$.

When g generates a Parseval tight Gabor frame, $\tilde{g}_z = \pi(z)g$ and the projection P_g is precisely the operator ${}_L\langle g, g\rangle$. This is no coincidence. We can identify the module V_L with the module $A^\infty_L \cdot P_g$ where A^∞_L acts by multiplication on the left. The isomorphism between V_L and A^∞_L is given by the analysis and synthesis operators.

After doing this, we can consider the canonical Hilbert module structure on $A_L^\infty \cdot P_g$ which is given by

$$\langle a, b \rangle = aP_g b^*.$$

We can use the analysis and synthesis operators to translate this inner product into an inner product on V_L, and this defines a new C^*-inner product given by

$${}_L^g\langle f, h \rangle = \sum_{z \in L} \langle f, S_g^{-1} \pi(z) h \rangle \pi(z).$$

Using this inner product, we have again that $P_g = {}_L^g\langle g, g \rangle$. We can see that this inner product is equivalent to our original one since

$${}_L^g\langle f, h \rangle = \sum_{z \in L} \langle f, S_g^{-1} \pi(z) h \rangle \pi(z) = \sum_{z \in L} \langle S_g^{-\frac{1}{2}} f, S_g^{-\frac{1}{2}} \pi(z) h \rangle \pi(z) =_L \langle S_g^{-\frac{1}{2}} f, S_g^{-\frac{1}{2}} h \rangle.$$

Thus we see that ${}_L^g\langle g, g \rangle$ is a projection iff ${}_L\langle S_g^{-\frac{1}{2}} g, S_g^{-\frac{1}{2}} g \rangle$ is a projection iff $S_g^{-\frac{1}{2}} g$ generates a Parseval tight frame. Here the last equivalence comes from Theorem 3.3 in [26].

We can see from the previous discussion that when f and g generate Gabor frames for L, the inner products ${}_L^f\langle \cdot\,, \cdot \rangle$ and ${}_L^g\langle \cdot\,, \cdot \rangle$ are equal iff $S_f = S_g$, though they are always isomorphic as Hilbert module structures. We can define an equivalence relation on functions $f, g \in \mathcal{S}(\mathbb{R}^d)$ by saying $f \sim g$ iff $S_f^L = S_g^L$. We call such functions **L frame equivalent.**

Question 4.1. *Can we classify functions up to L frame equivalence?*

This question is posed as an attempt to understand what types of frame operators are possible for a lattice L. Note that if f and g generate Parseval tight Gabor frames for L then they are L frame equivalent. We have a characterization of such functions called the **Wexler-Raz orthogonality relations** which say that $\mathcal{G}(g, L)$ is a Parseval tight frame iff $\langle g, \pi(l)g\rangle = \frac{1}{\mathrm{vol}(L)}\delta_{L^\circ,0}$. Thus we can already see the level of complexity inherent in Question 4.1 by examining Parseval tight frames. Given the connection between frame equivalence and equality of the Hilbert inner products ${}^{f}_{L}\langle \cdot\,, \cdot\rangle$ and ${}^{g}_{L}\langle \cdot\,, \cdot\rangle$, it would be interesting to see whether operator algebraic techniques could be used to tackle Question 4.1.

In the previous discussion we have used Gabor frames to construct projections, and then projections to construct Hilbert module structures. However, it can be advantageous to work in the opposite direction as well. For example, when $\mathrm{vol}(L) \geq 1$, we can never construct a single window Gabor frame for L. Regardless, we will always have a Hilbert bimodule structure on V_L. As in the proof of Proposition 3.3 in [31], we can always find a finite collection $\{g_i\}_{i=1}^N \subset M^1(\mathbb{R}^d)$ so that $\sum_{i=1}^N \{g_i, g_i\}_{L^\circ} = \mathbf{1}_{A_L^\circ}$. After unpacking the definitions, we see that this is precisely the condition that $\mathcal{G}(g_1, \ldots, g_N, L)$ is a Parseval tight multiwindow Gabor frame. Thus we have proven the existence of Parseval tight multiwindow Gabor frames for L using purely operator algebraic machinery! This result was first proven in [25] using these methods, but now has purely analytic proofs. Nonetheless, it still shows the benefits of using operator algebras to study Gabor systems.

4.3 Constructing $\mathcal{H}_\Lambda$

Now we are ready to define the module $\mathcal{H}_\Lambda$ described in Chapter 1. Let $\Lambda \subset \mathbb{R}^{2d}$ be a quasicrystal. Recall that σ is the standard symplectic cocycle on $\mathbb{R}^{2d}$. We construct a projective σ-representation of R_Λ by time-frequency shifts. Consider the (trivial) bundle of Hilbert spaces given by $\Omega_{trans} \times L^2(\mathbb{R}^d)$. Denote the fiber over a quasicrystal $T \in \Omega_{trans}$ by H_T. An element $(T, T-z) \in R_\Lambda$ acts as a map from $H_{T-z} \to H_T$ by

$$(T, T-z)f = \pi(z)f.$$

We could construct a module over $\mathcal{A}_\sigma$ by integrating this representation, however it would not have the correct topology to give a finitely generated projective module.

Instead, we define a module over $\mathcal{A}_\sigma^{L^1}$ which we will later complete to a module over $\mathcal{A}_\sigma$. Here $\mathcal{A}_\sigma^{L^1}$ denotes the continuous functions in $L^1_\sigma(R_\Lambda)$. We begin with $C(\Omega_{trans}, M^1(\mathbb{R}^d))$, the continuous functions on the transversal with values in $M^1(\mathbb{R}^d)$. Given $f \in \mathcal{A}_\sigma^{L^1}$ and $\Psi \in C(\Omega_{trans}, M^1(\mathbb{R}^d))$ we define an action I of $\mathcal{A}_\sigma^{L^1}$ by

$$I(f)\Psi(T) = \sum_{z \in T} f(T, T-z)\pi(z)\Psi(T-z).$$

Since Ω_{trans} is compact, $||\Psi(T)||_{M^1} \leq C$ for a constant C which is independent of T. Thus the series converges in $M^1(\mathbb{R}^d)$. This representation is faithful since $\mathcal{A}_\sigma^{L^1}$ is simple. We denote this $\mathcal{A}_\sigma^{L^1}$-module by $\mathcal{H}_\Lambda$. We will denote by $\mathcal{C}_\Lambda$ the linear subspace of $\mathcal{H}_\Lambda$ of **transversally constant functions** which can be naturally identified with $M^1(\mathbb{R}^d)$. For $g \in M^1(\mathbb{R}^d)$ we denote by $\Psi_g \in \mathcal{C}_\Lambda$ the function defined by $\Psi_g(T) = g$. When $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ is a multiwindow Gabor frame we will show that $\Psi_{g_1}, \ldots, \Psi_{g_N}$

generate $\mathcal{H}_\Lambda$ as an $\mathcal{A}_\sigma^{L^1}$-module and construct an associated projection in $\mathcal{A}_\sigma^{L^1}$.

To begin, fix $g_1, \ldots, g_N \in M^1(\mathbb{R}^d)$ so that for any $T \in \Omega_\Lambda, \mathcal{G}(g_1, \ldots, g_N, T)$ is an M^1-frame. By Theorem 1.1, it is always possible to find functions satisfying this requirement. Now we can define two maps, which are generalizations of the analysis and synthesis maps for frames. We define the **noncommutative synthesis operator**

$$D : (\mathcal{A}_\sigma^{L^1})^N \to \mathcal{H}_\Lambda$$

by

$$D(\mathbf{1}_i) = \Psi_{g_i}$$

where $\mathbf{1}_i$ denotes the element of $(\mathcal{A}_\sigma^{L^1})^N$ which is 0 except in the ith entry where it is equal to the identity element of $\mathcal{A}_\sigma^{L^1}$. We extend this map to all of $(\mathcal{A}_\sigma^{L^1})^N$ so that it is a continuous map of $\mathcal{A}_\sigma^{L^1}$-modules, effectively by letting an element in $\mathcal{A}_\sigma^{L^1}$ act on each Ψ_{g_i} and then summing over i.

Denote by $\tilde{g}_{iz}^T := (S^T)^{-1}\pi(z)g_i$ the ith dual frame element corresponding to $z \in T$. We now define the **noncommutative analysis operator** $C : \mathcal{H}_\Lambda \to (\mathcal{A}_\sigma^{L^1})^N$ which sends a function $f \in \mathcal{H}_\Lambda$ to

$$C(f) = (G_1, \ldots, G_N) \in (\mathcal{A}_\sigma^{L^1})^N$$

where

$$G_i(T, T - z) := \langle f(T), \tilde{g}_{iz}^T \rangle.$$

To see that $G_i \in \mathcal{A}_\sigma^{L^1}$, we compute

$$\int_{R_\Lambda} |G_i| = \int_{\Omega_{trans}} \sum_{z \in T} \left| \langle f(T), \tilde{g}_{i\,z}^{\,T} \rangle \right| dT$$
$$= \int_{\Omega_{trans}} \sum_{z \in T} \left| \langle (S^T)^{-1} f(T), \pi(z) g_i \rangle \right| dT$$

which holds since S^T is self-adjoint. For convenience we denote by F_T the function $(S^T)^{-1} f(T)$. Since S^T is invertible in $B(M^1(\mathbb{R}^d))$ we have $F_T \in M^1(\mathbb{R}^d)$. Now we have

$$\begin{aligned}
\int_{\Omega_{trans}} \sum_{z \in T} |\langle F_T, \pi(z) g_i \rangle| \, dT &= \int_{\Omega_{trans}} \sum_{z \in T} |V_{g_i} F_T(z)| dT \\
&\leq \mathrm{rel}(\Lambda) \int_{\Omega_{trans}} ||V_{g_i} F_T||_{W(L^\infty, L^1)} dT \\
&\leq C \, \mathrm{rel}(\Lambda) ||g_i||_{M^1} \int_{\Omega_{trans}} ||F_T||_{M^1} dT \\
&\leq C \, \mathrm{rel}(\Lambda) \, ||g_i||_{M^1} \int_{\Omega_{trans}} ||(S^T)^{-1}||_{M^1} ||f(T)||_{M^1} dT < \infty
\end{aligned}$$

The inequality in the third line comes from Proposition 12.1.11 in [14], and the constant C is independent of T. We see the integral is finite because the continuity of f implies $||f(T)||_{M^1}$ is bounded on Ω_{trans}, and because Proposition 3.5 shows that $||(S^T)^{-1}|| = ||(S^\Lambda)^{-1}||$ for all T.

Proposition 4.1. *The map C is a map of $\mathcal{A}_\sigma^{L^1}$-modules.*

Proof. First note that the transversally constant functions $\mathcal{C}_\Lambda$ are cyclic in $\mathcal{H}_\Lambda$ under the action of $\mathcal{A}_\sigma^{L^1}$. For example, we can get all transversally locally constant functions by applying characteristic functions of the unit space of R_Λ, and locally constant functions are dense in $C(\Omega_{trans}, M^1(\mathbb{R}^d))$. Thus it will suffice to prove that C is an $\mathcal{A}_\sigma^{L^1}$-module map when $\mathcal{A}_\sigma^{L^1}$ acts on $\mathcal{C}_\Lambda$.

So assume that $\Psi_f \in \mathcal{C}_\Lambda$ and that $a \in \mathcal{A}_\sigma^{L^1}$. On the one hand we have

$$\begin{aligned} C(I(a)\Psi_f)_i(T, T-z) &= \left\langle \sum_{w \in T} a(T, T-w)\pi(w)f,\ \tilde{g}_{i_z}^T \right\rangle \\ &= \sum_{w \in T} a(T, T-w)\langle \pi(w)f, \tilde{g}_{i_z}^T \rangle \\ &= \sum_{w \in T} a(T, T-w)\langle f, T_{-x'}M_{-\omega'}\tilde{g}_{i_z}^T \rangle. \end{aligned}$$

where $w = (x', \omega')$. On the other hand we have

$$a * C(\Psi_f)_i(T, T-z) = \sum_{w \in T} a(T, T-w)\langle f, e^{2\pi i x'(\omega'-\omega)}\tilde{g}_{i_{z-w}}^{T-w} \rangle.$$

where $z = (x, \omega)$. We will show that

$$T_{-x'}M_{-\omega'}\tilde{g}_{i_{(x,\omega)}}^T = e^{2\pi i x'(\omega'-\omega)}\tilde{g}_{i_{z-w}}^{T-w}.$$

Unpacking the definitions, we see that this is equivalent to showing

$$T_{-x'}M_{-\omega'}(S^T)^{-1}\pi(z)g_i = e^{2\pi i x'(\omega'-\omega)}(S^{T-w})^{-1}\pi(z-w)g_i$$

which is equivalent to

$$T_{-x'}M_{-\omega'}(S^T)^{-1}\pi(z)g_i = (S^{T-w})^{-1}T_{-x'}M_{-\omega'}\pi(z)g_i$$

after commuting $T_{-x'}$ past $M_{(\omega-\omega')}$ on the RHS. We can cancel the $\pi(z)$ from both sides and simply show the operator equality

$$T_{-x'}M_{-\omega'}(S^T)^{-1} = (S^{T-w})^{-1}T_{-x'}M_{-\omega'}.$$

By inverting both sides we see this is equivalent to showing

$$S^T\pi(w) = \pi(w)S^{T-w}$$

which follows from Proposition 3.3. $\square$

Now we can see that the maps D and C are well defined maps of $\mathcal{A}_\sigma^{L^1}$-modules. Composing these maps, we get that

$$DCf(T) = \sum_{i=1}^{N} \sum_{z \in T} \langle f(T), \tilde{g}_{i\,z}^{\,T} \rangle g_i = f(T)$$

where the last equality holds since this is exactly the reconstruction formula for $f(T)$ using the Gabor frame $\mathcal{G}(g_1, \ldots, g_N, T)$. Thus C splits the map D, showing that $\mathcal{H}_\Lambda$ is finitely generated (by the functions Ψ_{g_i}) and projective. Thus we have:

Theorem 4.2. *$\mathcal{H}_\Lambda$ is finitely generated and projective as a $\mathcal{A}_\sigma^{L^1}$-module.*

If we compose these maps in the opposite order, we can construct a projection matrix $P \in M_N(\mathcal{A}_\sigma^{L^1})$ which represents $\mathcal{H}_\Lambda$ in $K_0(\mathcal{A}_\sigma)$. We can write the elements of P explicitly as functions in $\mathcal{A}_\sigma^{L^1}$ as

$$P_{ij}(T, T - z) = \langle g_i, \tilde{g}_{j\,z}^{\,T} \rangle.$$

To compute the trace of this projection (and thus the dimension of the module $\mathcal{H}_\Lambda$) we apply the normalized trace on $M_N(\mathcal{A}_\sigma)$ to get

$$\mathrm{Tr}(P) = \frac{1}{N} \sum_{i=1}^{N} \int_{\Omega_{trans}} \langle g_i, \tilde{g}_{i\,0}^{\,T} \rangle dT.$$

By applying Birkhoff's Ergodic Theorem, the integrals can be replaced by averages over the orbits of Λ. Thus we get

$$\mathrm{Tr}(P) = \lim_{k \to \infty} \frac{1}{N|\Lambda \cap C_k|} \sum_{i=1}^{N} \sum_{z \in (\Lambda \cap C_k)} \langle g_i, \tilde{g}_{i\,(0,0)}^{\,\Lambda - z} \rangle$$

where C_k is the cube centered at the origin with side length k. We would like to rewrite this sum so that it involves only the dual frame for $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$. We can use Proposition 3.3 to rewrite $\tilde{g}_{i(0,0)}^{\Lambda-z}$ as

$$\tilde{g}_{i(0,0)}^{\Lambda-z} = (S^{\Lambda-z})^{-1}g_i = (S^{\Lambda-z})^{-1}T_{-x}M_{-\omega}M_\omega T_x g_i = T_{-x}M_{-\omega}(S^{\Lambda})^{-1}\pi(z)g_i.$$

Now we can rewrite the sum as

$$\begin{aligned}\mathrm{Tr}(P) &= \lim_{k\to\infty}\frac{1}{N|\Lambda\cap C_k|}\sum_{i=1}^{N}\sum_{z\in(\Lambda\cap C_k)}\langle g_i, T_{-x}M_{-\omega}(S^{\Lambda})^{-1}\pi(z)g_i\rangle \\ &= \lim_{k\to\infty}\frac{1}{N|\Lambda\cap C_k|}\sum_{i=1}^{N}\sum_{z\in(\Lambda\cap C_k)}\langle \pi(z)g_i, \tilde{g}_{i z}^{\Lambda}\rangle\end{aligned}$$

which involves only the Gabor frame $\mathcal{G}(g_1,\ldots,g_N,\Lambda)$ and its dual. These averages coincide precisely with the **frame measure** introduced in [3]. In Theorem 4.2 (b) they show that for a single window frame, the averages above are equal to $\frac{1}{\mathrm{Dens}(\Lambda)}$. Their results are easily generalized to show that this also holds for multiwindow frames, so we get the following result:

Corollary 4.1. *The dimension of $\mathcal{H}_\Lambda$ is equal to $\frac{1}{\mathrm{Dens}(\Lambda)}$.*

Thus we have completed the proof of Theorem 1.2. Note that the realization of the frame measure as the dimension of a projective module gives a structural reason why it should be independent of the choice of windows for the frame.

4.4 Hilbert C^*-module Structure

In Section 4.1 we saw that it was advantageous to give V_L the structure of a Morita equivalence bimodule. Understanding the structure of $\mathrm{End}^0_{A_L^\infty} V_L$ was particularly useful as it could be used to derive the Janssen representation of the frame operator. Motivated by this example, we would like to give $\mathcal{H}_\Lambda$ the structure of a Hilbert C^*-module and study the endomorphism algebra $\mathrm{End}^0_{\mathcal{A}_\sigma^{L^1}}\mathcal{H}_\Lambda$, which for brevity we will denote by $\mathcal{B}_\sigma^{L^1}$. We denote its completion to a C^*-algebra by $\mathcal{B}_\sigma$.

In the previous section we showed that $\mathcal{H}_\Lambda$ is a finitely generated projective module. When $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ is a multiwindow Gabor frame, we constructed an associated projection[2] $P \in M_N(\mathcal{A}_\sigma^{L^1})$ which represents $[\mathcal{H}_\Lambda]$ in $K_0(\mathcal{A}_\sigma)$. In principal, the projection P can be used to give $\mathcal{H}_\Lambda$ an $\mathcal{A}_\sigma^{L^1}$-valued inner product. We can also use P to describe $\mathcal{B}_\sigma^{L^1}$, though this description does not immediately identify $\mathcal{B}_\sigma^{L^1}$ as a familiar algebra (e.g. it is unclear whether $\mathcal{B}_\sigma^{L^1}$ is a twisted groupoid algebra associated to a quasicrystal Λ').

We now outline our strategy in greater detail. We can identify the module $\mathcal{H}_\Lambda$ with $(\mathcal{A}_\sigma^{L^1})^N P$ where $\mathcal{A}_\sigma^{L^1}$ acts by multiplication on the left. The isomorphism between $\mathcal{H}_\Lambda$ and $(\mathcal{A}_\sigma^{L^1})^N P$ is given by the noncommutative analysis and synthesis maps. For elements $a, b \in (\mathcal{A}_\sigma^{L^1})^N P$ the natural Hilbert C^*-inner product is given by

$$_{\mathcal{A}_\sigma^{L^1}}\langle a, b\rangle = aPb^*.$$

Using the noncommutative analysis and synthesis operators we can translate this inner product structure to $\mathcal{H}_\Lambda$. After doing this, we can identify the endomorphism algebra $\mathcal{B}_\sigma^{L^1}$ with $P(\mathcal{A}_\sigma^{L^1})^N P$ acting on $(\mathcal{A}_\sigma^{L^1})^N P$ by multiplication on the right. By utilizing the noncommutative analysis and synthesis maps we can take an element of the form PaP where $a \in (\mathcal{A}_\sigma^{L^1})^N$ and get an explicit formula for how it acts on an element in $\mathcal{H}_\Lambda$.

[2]Note that the projection P does depend on the choice of window functions $g_1, \ldots, g_N$, but the class of P in $K_0(\mathcal{A}_\sigma)$ is independent of this choice.

To carry out this strategy, we first make a simplifying assumption. We assume that there is a single window Gabor frame $\mathcal{G}(g, \Lambda)$ so that $P_g \in \mathcal{A}_\sigma^{L^1}$ is given by

$$P_g(T, T - z) = \langle g, \tilde{g}_z^T \rangle = \langle g, (S_g^T)^{-1} \pi(z) g \rangle.$$

Although it will not always be possible to construct such a frame, it serves to simplify the resulting formulas and does not change the nature of any of the results. Now suppose $\Psi, \eta \in \mathcal{H}_\Lambda$. Then we apply the noncommutative analysis operator C_g and compute

$${}^{g}_{\mathcal{A}_\sigma^{L^1}}\langle \Psi, \eta \rangle(T, T - z) = (C_g \Psi) * P * (C_g \eta)^*(T, T - z) = \langle \Psi(T), (S_g^T)^{-1} \pi(z) \eta(T - z) \rangle.$$

The final equality requires repeated use of the reconstruction formula for the Gabor frame $\mathcal{G}(g, \Lambda)$. This is quite similar to the inner product defined on V_L in the case of noncommutative tori. However, we might have guessed that

$${}_{\mathcal{A}_\sigma^{L^1}}\langle \Psi, \eta \rangle(T, T - z) = \langle \Psi(T), \pi(z) \eta(T - z) \rangle$$

would also define an inner product for $\mathcal{H}_\Lambda$. This would immediately be true if we could choose $\mathcal{G}(g, \Lambda)$ to be a tight frame, however it is not clear that this is possible.

Instead, we introduce the **global frame operator** S_g as an operator on $\mathcal{H}_\Lambda$ defined by

$$(S_g f)(T) = S_g^T f(T).$$

By Proposition 3.4 we see S_g is well defined and invertible. Now we compute

$$\begin{aligned}
{}^{g}_{\mathcal{A}_\sigma^{L^1}}\langle \Psi, \eta \rangle(T, T - z) &= \langle \Psi(T), (S_g^T)^{-1} \pi(z) \eta(T - z) \rangle \\
&= \langle (S_g^T)^{-\frac{1}{2}} \Psi(T), \pi(z) (S_g^{T-z})^{-\frac{1}{2}} \eta(T - z) \rangle \\
&= {}_{\mathcal{A}_\sigma^{L^1}}\langle (S_g)^{-\frac{1}{2}} \Psi, (S_g)^{-\frac{1}{2}} \eta \rangle(T, T - z).
\end{aligned}$$

Thus the operator $S_g^{-\frac{1}{2}}$ implements an isomorphism between ${}^{g}_{\mathcal{A}_\sigma^{L^1}}\langle\,\cdot\,,\,\cdot\,\rangle$ and ${}_{\mathcal{A}_\sigma^{L^1}}\langle\,\cdot\,,\,\cdot\,\rangle$, which shows that ${}_{\mathcal{A}_\sigma^{L^1}}\langle\,\cdot\,,\,\cdot\,\rangle$ is a well defined inner product.

In the case of noncommutative tori, the existence of an inner product like ${}_{\mathcal{A}_\sigma^{L^1}}\langle\,\cdot\,,\,\cdot\,\rangle$ immediately implied the existence of tight multiwindow Gabor frames. Unfortunately, we cannot conclude the same for the case of quasicrystals. To illustrate this, suppose for simplicity that we have an element $\Psi \in \mathcal{H}_\Lambda$ so that ${}_{\mathcal{A}_\sigma^{L^1}}\langle \Psi, \Psi\rangle$ is the identity operator. For noncommutative tori, this would immediately imply that Ψ generates a Parseval tight Gabor frame. This would also be true in our case if Ψ were transversally constant. Of course, we cannot guarantee that Ψ will be transversally constant. For example, let $\mathcal{G}(g, \Lambda)$ be a Gabor frame and consider Ψ_g, which we know generates $\mathcal{H}_\Lambda$ as an $\mathcal{A}_\sigma^{L^1}$-module. Then $(S_g)^{-\frac{1}{2}}\Psi_g$ satisfies ${}_{\mathcal{A}_\sigma^{L^1}}\langle (S_g)^{-\frac{1}{2}}\Psi_g, (S_g)^{-\frac{1}{2}}\Psi_g\rangle = \mathrm{Id}$, but is not transversally constant. This phenomenon is explained by the fact that the collection $\{(S_g^T)^{-\frac{1}{2}}\pi(z)g\}_{z\in T} = \{\pi(z)(S_g^{T-z})^{-\frac{1}{2}}g\}_{z\in T}$ is a Parseval tight frame, but not necessarily a Gabor frame.

It may seem disappointing that our method was unable to prove the existence of Parseval tight multiwindow frames for quasicrystals. After all, this was the first contribution of operator algebraic methods to understanding lattice Gabor frames. However, we will soon have evidence that it is *impossible* to construct Parseval multiwindow frames for certain quasicrystals. Thus we should not have expected the methods here to prove their existence. We might instead view the complications above in a positive light. If we were to determine a Janssen-type representation using the structure of $\mathcal{B}_\sigma$ then we should expect a similar representation to hold for frames like $\{(S_g^T)^{-\frac{1}{2}}\pi(z)g\}_{z\in T}$ even though they are not Gabor frames.

Next we would like to investigate some operators lying in $\mathcal{B}_\sigma^{L^1}$ which are of particular interest. In order to do this, it will help to have a succinct list of generators for $\mathcal{A}_\sigma^{L^1}$. To this end we introduce the sets $E_z = \{(T, T-z) \in R_\Lambda \mid z \in T\}$ and the operators $\chi_z := \chi_{E_z}$. The operator $I(\chi_z)$ acts on $\Psi \in \mathcal{H}_\Lambda$ by

$$I(\chi_z)\Psi(T) = \begin{cases} \pi(z)\Psi(T-z) & \text{if } z \in T \\ 0 & \text{otherwise} \end{cases}$$

These operators generate $\mathcal{A}_\sigma^{L^1}$ as a C^*-algebra and demonstrate how $\mathcal{A}_\sigma^{L^1}$ acts by time frequency shifts. The operator $I(\chi_z)$ does a time frequency shift by z if $z \in T$, and is the zero operator otherwise. In order to construct operators in $\mathcal{B}_\sigma^{L^1}$, we must first find operators which commute with all the $I(\chi_z)$. An operator O will commute with all the $I(\chi_z)$ iff it satisfies the following two properties. First, if $\Psi \in \mathcal{H}_\Lambda$ then $O\Psi(T)$ must depend only on $\Psi(T)$. This corresponds to commuting with projections on the unit space of R_Λ, which can be written as combinations of operators of the form $I(\chi_z\chi_z^*)$. Thus we can think of O as a continuous family of operators $O(T)$ each acting on $M^1(\mathbb{R}^{2d})$. Finally, the family of operators $O(T)$ must satisfy the covariance condition

$$O(T)\pi(z) = \pi(z)O(T-z).$$

From the conditions above and Proposition 3.3, we immediately see that the global frame operator S_g commutes with $\mathcal{A}_\sigma^{L^1}$. This mirrors the results of Section 4.1, where we found that the frame operator for a lattice Gabor frame was an element of $\mathrm{End}^0_{A_L^\infty}$. In looking for a Janssen representation, we would like to decompose S_g into a linear combination of time-frequency shifts. The first step would be to show that $\mathcal{B}_\sigma^{L^1}$ contains operators which look something like time-frequency shifts. We define operators τ_k on $\mathcal{H}_\Lambda$ by

$$(\tau_k \Psi)(T) = \varphi_k(T)\pi(\check{k})\Psi(T)$$

where φ_k is a continuous eigenfunction for Ω_Λ with eigenvalue k, and

$$\check{k} = \begin{pmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} k$$

is a symplectic transformation of the eigenvalue k. A simple computation shows that all τ_k commute with $\mathcal{A}_\sigma^{L^1}$. The operators τ_k play the same role as time-frequency shifts from the adjoint lattice did in the case of noncommutative tori.

Up until now we have only shown that S_g and the τ_k commute with $\mathcal{A}_\sigma^{L^1}$, but we have not actually shown that they are in $\mathcal{B}_\sigma^{L^1}$. This is a somewhat subtle point, but it becomes important if we consider operators τ_k when k is a measurable, but not continuous, eigenvalue. In this case, the operator τ_k is well defined as an operator in $B(L^2(\Omega_{trans} \times \mathbb{R}^d))$, but will not lie in $\mathcal{B}_\sigma^{L^1}$. It will, however, lie in the commutant of the von Neumann algebra generated by $\mathcal{A}_\sigma$. It is as yet unclear whether such an operator should contribute to something like a Janssen representation.

Nevertheless, we can show that S_g and all τ_k (for continuous eigenvalues k) lie in $\mathcal{B}_\sigma^{L^1}$. Showing $S_g \in \mathcal{B}_\sigma^{L^1}$ is simple, as S_g is exactly the rank one operator ${}_{\mathcal{A}_\sigma^{L^1}}\langle \cdot , \Psi_g\rangle \Psi_g$. Note that since all the C^*-inner products above are isomorphic, the corresponding endomorphism algebras are equal. Thus we can use any of the inner products to show that an operator lies in $\mathcal{B}_\sigma^{L^1}$, and we shall use this technique below. To show that the τ_k lie in $\mathcal{B}_\sigma^{L^1}$, we will identify $\mathcal{B}_\sigma^{L^1}$ with $P_g \mathcal{A}_\sigma^{L^1} P_g$. For a function

$a(T, T-z) \in \mathcal{A}_\sigma^{L^1}$, the element $P_g a P_g$ acts on $\Psi \in \mathcal{H}_\Lambda$ by

$$\Psi(P_g a P_g)(T) = \sum_{z \in T} \langle \Psi(T), \tilde{g}_z^T \rangle \sum_{w \in T} e^{2\pi i x(\omega - \omega')} a(T-z, T-w) \pi(w) g$$

where $z = (z, \omega), w = (x', \omega')$. When $a(T)$ is a function on the unit space of R_Λ, this formula simplifies to

$$\sum_{z \in T} \langle \Psi(T), \tilde{g}_z^T \rangle a(T-z) \pi(z) g.$$

We can interpret these operators as altering the reconstruction procedure of the frame $\mathcal{G}(g, T)$. If we define

$$\tilde{\tau}_k(T-z, T-w) = e^{-2\pi i x(\omega-\omega')} \varphi_k(T) \langle \pi(\check{k}) \pi(z) g, \tilde{g}_w^T \rangle$$

then we see that the function $\tilde{\tau}_k$ acts on $\mathcal{H}_\Lambda$ in exactly the same way as the operator τ_k. This demonstrates that τ_k lies in $\mathcal{B}_\sigma^{L^1}$. To compare with our earlier remarks, note that $\tilde{\tau}_k$ is a continuous function iff φ_k is a continuous eigenfunction.

Remark 4.1. *For a model set with internal space equal to $\mathbb{R}^n$, the operators τ_k generate a noncommutative torus inside $\mathcal{B}_\sigma$. Because $\mathcal{H}_\Lambda$ is finitely generated and projective, we can complete it to get a type II_1 representation of the von Neumann algebra generated by $\mathcal{A}_\sigma$. This implies that the representation of the rotation algebra generated by the τ_k (which lies in the VN completion of $\mathcal{B}_\sigma$) completes to a von Neumann algebra with a faithful, finite trace. This is curious, considering that the τ_k act something like time-frequency shifts from a dense subgroup of $\mathbb{R}^{2d}$. The standard representation of these time frequency shifts on $L^2(\mathbb{R}^d)$ completes to $B(L^2(\mathbb{R}^d))$ which has no such trace. This complication was one large obstruction to generalizing Linnell's results in [24]. By considering the noncommutative torus generated by the τ_k, it may be possible to sidestep this issue and generalize his results.*

The next step in reproducing the results from Section 4.1 is to describe the trace on $\mathcal{B}_\sigma^{L^1}$. There are two possible ways we might approach this. First, we can use the Hilbert module structure to describe the trace on rank one operators in $\mathcal{B}_\sigma^{L^1}$. For $\Psi, \eta \in \mathcal{H}_\Lambda$ we define the $\mathcal{B}_\sigma^{L^1}$-valued inner product $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{B}_\sigma^{L^1}}$ by

$$\{\Psi, \eta\}_{\mathcal{B}_\sigma^{L^1}} = {}_{\mathcal{A}_\sigma^{L^1}}\langle\,\cdot\,, \Psi\rangle\eta$$

so that $\{\Psi, \eta\}_{\mathcal{B}_\sigma^{L^1}}$ is the rank one operator corresponding to Ψ and η. In this case we can define a (non-normalized) trace $\mathrm{Tr}'_{\mathcal{B}_\sigma}$ by

$$\mathrm{Tr}'_{\mathcal{B}_\sigma}(\{\Psi, \eta\}_{\mathcal{B}_\sigma^{L^1}}) := \mathrm{Tr}_{\mathcal{A}_\sigma}\left({}_{\mathcal{A}_\sigma^{L^1}}\langle\eta, \Psi\rangle\right) = \int_{\Omega_{trans}} \langle\eta(T), \Psi(T)\rangle dT.$$

We know that $\mathrm{Tr}'_{\mathcal{B}_\sigma}(\mathrm{Id})$ is the dimension of $\mathcal{H}_\Lambda$, so the unique normalized trace on $\mathcal{B}_\sigma$ is given by

$$\mathrm{Tr}_{\mathcal{B}_\sigma} = \mathrm{Dens}(\Lambda)\mathrm{Tr}'_{\mathcal{B}_\sigma}.$$

We can also define traces $\mathrm{Tr}^g_{\mathcal{B}_\sigma}$ by identifying $\mathcal{B}_\sigma^{L^1}$ with $P_g\mathcal{A}_\sigma^{L^1}P_g$ and using the trace on $\mathcal{A}_\sigma$. For a rank one operator we have

$$\mathrm{Tr}^g_{\mathcal{B}_\sigma}(\{\Psi, \eta\}_{\mathcal{B}_\sigma^{L^1}}) = \int_{\Omega_{trans}} \sum_{z\in T} \langle g, (S^T)^{-1}\pi(z)\Psi(T-z)\rangle\langle\pi(z)\eta(T-z), (S^T)^{-1}g\rangle dT.$$

We know that all traces on $\mathcal{B}_\sigma^{L^1}$ are equal up to a constant multiple, and by evaluating on the projection P_g we see that

$$\mathrm{Tr}^g_{\mathcal{B}_\sigma} = \mathrm{Tr}'_{\mathcal{B}_\sigma}.$$

After some suitable choices, we can deduce the following identity whenever $\Psi, \eta \in \mathcal{H}_\Lambda$ and $\mathcal{G}(g, \Lambda)$ is a Gabor frame:

$$\int_{\Omega_{trans}} \sum_{z\in T} \langle g, \pi(z)\Psi(T-z)\rangle\langle\pi(z)\eta(T-z), g\rangle dT = \int_{\Omega_{trans}} \langle\eta(T), \Psi(T)\rangle dT.$$

This is of particular interest when $\Psi = \Psi_f, \eta = \Psi_h$ are transversally constant. In this case we get

$$\int_{\Omega_{trans}} \sum_{z \in T} \langle g, \pi(z) f \rangle \langle \pi(z) h, g \rangle dT = \langle h, f \rangle$$

whenever $f, h \in M^1(\mathbb{R}^{2d})$. These identities are suspiciously reminiscent of the FIGA. It seems as if some very general version of the Poisson summation formula must be at play here, but for now we see no way to prove these identities without the abstract machinery of operator algebras.

Now that we have described the trace on $\mathcal{B}_\sigma^{L^1}$, we can use it to suggest a decomposition of S_g into a linear combination of the operators τ_k. The coefficient of τ_k in the expansion of S_g should be equal to $\mathrm{Tr}_{\mathcal{B}_\sigma}(S_g \tau_k^*)$. Since S_g is the rank one operator $\{\Psi_g, \Psi_g\}_{\mathcal{B}_\sigma^{L^1}}$,

$$S_g \tau_k^* = \{\Psi_g, \tau_k^* \Psi_g\}_{\mathcal{B}_\sigma^{L^1}} = \{\tau_k \Psi_g, \Psi_g\}_{\mathcal{B}_\sigma^{L^1}}.$$

Computing the trace, we have

$$\begin{aligned} \mathrm{Tr}_{\mathcal{B}_\sigma}(\{\tau_k \Psi_g, \Psi_g\}_{\mathcal{B}_\sigma^{L^1}}) &= \mathrm{Dens}(\Lambda) \int_{\Omega_{trans}} \langle g, \varphi_k(T) \pi(\check{k}) g \rangle dT \\ &= \lim_{R \to \infty} \frac{\mathrm{Dens}(\Lambda) \langle g, \pi(\check{k}) g \rangle}{|\Lambda \cap B_R|} \sum_{z \in \Lambda \cap B_R} e^{-2\pi i k z} \\ &= c_k \langle g, \pi(\check{k}) g \rangle. \end{aligned}$$

This suggests that the coefficient of $\pi(\check{k})$ in the Janssen representation for the frame operator S_g^Λ is equal to $c_k \langle g, \pi(\check{k}) g \rangle$. We are careful to note that this computation does not give a proof of any type of Janssen representation. We know that the τ_k will almost never generate all of $\mathcal{B}_\sigma^{L^1}$ since the K-theory of Ω_Λ is rarely the same as the K-theory of a torus. To turn this into a proof, we would need more knowledge about the structure of $\mathcal{B}_\sigma^{L^1}$. Furthermore, we would only expect a Janssen represention of this form when Λ has pure discrete spectrum. Otherwise the distribution $\widehat{\delta_\Lambda}$ will have a continuous part, so we would expect the Janssen representation to reflect this.

Despite these difficulties in the case of general quasicrystals, the following results give a Janssen representation for $\mathcal{G}(g,\Lambda)$ when Λ is a model set:

Proposition 4.2 (FIGA for Model Sets)**.** *Let $\Lambda \subset \mathbb{R}^{2d}$ be a model set and fix $f, g, h, u \in \mathcal{S}(\mathbb{R}^d)$. Denote by $\operatorname{Eig}(\Lambda)$ the collection of continuous eigenvalues for Ω_Λ. Then*

$$\sum_{z\in\Lambda} \langle f, \pi(z)g\rangle\langle \pi(z)h, u\rangle = \sum_{k\in \operatorname{Eig}(\Lambda)} c_k \langle \pi(\check{k})f, u\rangle\langle h, \pi(\check{k})g\rangle$$

where c_k is the Fourier-Bohr coefficient for k. Here the RHS must be interpreted as in Theorem 2.2.

Proof. First define the function

$$F(k) = \langle \pi(\check{k})f, u\rangle\langle h, \pi(\check{k})g\rangle$$

which lies in $\mathcal{S}(\mathbb{R}^{2d})$ and has Fourier transform

$$\hat{F}(z) = \langle f, \pi(z)g\rangle\langle \pi(z)h, u\rangle.$$

Then applying Theorem 2.2 we have

$$\begin{aligned}\sum_{z\in\Lambda} \langle f, \pi(z)g\rangle\langle \pi(z)h, u\rangle &= \sum_{z\in\Lambda} \hat{F}(z)\\ &= \sum_{k\in\operatorname{Eig}(\Lambda)} c_k F(k)\\ &= \sum_{k\in\operatorname{Eig}(\Lambda)} c_k \langle \pi(\check{k})f, u\rangle\langle h, \pi(\check{k})g\rangle\end{aligned}$$

□

Theorem 4.3 (Janssen Representation for Model Sets)**.** *Let Λ be a model set and $S^{\Lambda}_{f,g} = \sum_{z\in\Lambda}\langle \cdot\,, \pi(z)f\rangle\pi(z)g$ be a generalized Gabor frame operator with $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then we have*

$$S^{\Lambda}_{f,g} = \sum_{k\in\ \mathrm{Eig}(\Lambda)} c_k\langle g, \pi(\check{k})f\rangle\pi(\check{k})$$

where the convergence on the RHS is interpreted as in Theorem 2.2.

Proof. Let $h, u \in \mathcal{S}(\mathbb{R}^d)$ be arbitrary. It suffices to show that

$$\langle S^{\Lambda}_{f,g}h, u\rangle = \left\langle \sum_{k\in\ \mathrm{Eig}(\Lambda)} c_k\langle g, \pi(\check{k}f)\rangle\pi(\check{k})h, u \right\rangle.$$

To see this, we compute

$$\begin{aligned}\langle S^{\Lambda}_{f,g}h, u\rangle &= \sum_{z\in\Lambda}\langle h, \pi(z)f\rangle\langle\pi(z)g, u\rangle \\ &= \sum_{k\in\ \mathrm{Eig}(\Lambda)} c_k\langle g, \pi(\check{k})f\rangle\langle\pi(\check{k})h, u\rangle \\ &= \left\langle \sum_{k\in\ \mathrm{Eig}(\Lambda)} c_k\langle g, \pi(\check{k})f\rangle\pi(\check{k})h, u \right\rangle\end{aligned}$$

where the second equality follows from Proposition 4.2. □

It is easily seen that Theorem 4.3 also holds when $f, g \in M^1(\mathbb{R}^d)$, since in this case $V_g f \in W(L^\infty, L^1)$. Additionally, the convergence on the RHS in Theorem 4.3 (properly interpreted) occurs in $\mathcal{S}(\mathbb{R}^d)$ since the approximating measures $\widehat{\delta_{\Lambda,\epsilon}}$ can be expressed using sums which are locally absolutely convergent.

As in the case of lattice Gabor frames, one would like to use the Janssen representation to characterize windows g for which $\mathcal{G}(g, \Lambda)$ is a (Parseval) tight frame. Since the representation in Theorem 4.3 involves a sum over a dense collection of time-frequency shifts, it is not immediately clear how to deduce something like the Wexler-Raz orthogonality relations. However, by using operator algebraic arguments we are able to prove the following theorem:

Theorem 4.4. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a quasicrystal. Assume that the group of continuous eigenvalues $\mathrm{Eig}(\Lambda)$ is dense in $\mathbb{R}^{2d}$, and that the Fourier-Bohr coefficients c_k are non-zero in a neighborhood of 0. Also assume that the operators τ_k generate a totally irrational noncommutative torus. Then it is not possible to find a tight multiwindow frame $\mathcal{G}(g_1, \ldots, g_N, \Lambda)$ where all $g_i \in M^1(\mathbb{R}^d)$.*

Proof. Since the noncommutative torus generated by the τ_k is totally irrational, it has a unique normalized trace. This implies that $\mathrm{Tr}_{\mathcal{B}_\sigma}(\tau_k) = \delta_{k,0}$. Now suppose $\{g_i\}_{i=1}^N \subset M^1(\mathbb{R}^d)$ generates a tight multiwindow Gabor frame for Λ. Then $S^\Lambda_{\{g_i\}} = A \cdot \mathrm{Id}$ where A denotes the upper (and lower) frame bound. By Proposition 3.5, $S^T_{\{g_i\}} = A \cdot \mathrm{Id}$ for all $T \in \Omega_\Lambda$. Thus the global frame operator $S_{\{g_i\}} = A \cdot \mathrm{Id}$ as an operator on $\mathcal{H}_\Lambda$. This implies that

$$\mathrm{Tr}_{\mathcal{B}_\sigma}(S_{\{g_i\}}\tau_k^*) = A \cdot \mathrm{Tr}_{\mathcal{B}_\sigma}(\tau_k^*) = A \cdot \delta_{k,0}.$$

However, we can compute $\mathrm{Tr}_{\mathcal{B}_\sigma}(S_{\{g_i\}}\tau_k^*)$ directly, and this implies

$$A \cdot \delta_{k,0} = \mathrm{Tr}_{\mathcal{B}_\sigma}(S_{\{g_i\}}\tau_k^*) = c_k \sum_{i=1}^N \langle g_i, \pi(\check{k}) g_i \rangle.$$

Since $\mathrm{Eig}(\Lambda)$ is dense in $\mathbb{R}^{2d}$, we can choose a sequence of eigenvalues $k_n \neq 0$ s.t. $k_n \to 0$ as $n \to \infty$. Then $\sum_{i=1}^N \langle g_i, \pi(\check{k_n}) g_i \rangle$ clearly converges to $\sum_{i=1}^N \langle g_i, g_i \rangle$. However, by our computation above, we have $\sum_{i=1}^N \langle g_i, \pi(\check{k_n}) g_i \rangle = 0$ for all k_n, and $\sum_{i=1}^N \langle g_i, g_i \rangle = \frac{A}{c_0}$, which is a contradiction. $\square$

Note that the operator algebraic approach helped us in two ways. First, it allowed us to avoid the messy convergence issues we ran into in Theorem 4.3 if we tried to prove the result using analytic methods. It also allowed us to generalize the result considerably, since we did not need to assume that Λ is a model set.

In Theorem 4.4, the assumption that the τ_k generate a totally irrational noncommutative torus may seem suspect. The following lemma shows that there are examples where this assumption holds:

Lemma 4.1. *Let $x_1 \ldots x_n \in \mathbb{R}^{2d}$. Then we can construct a cut and project scheme $(\mathbb{R}^{2d}, \mathbb{R}^k, D)$ and a window W so that $x_1, \ldots, x_n \in \Lambda_W$.*

Proof. (Sketch) Assume $n > 2d$, as the result is trivial otherwise. We consider $\mathbb{R}^{2d}$ as the first factor in $\mathbb{R}^{2d} \times \mathbb{R}^k$ where $2d + k = n$. To each point x_i, we have a k-dimensional affine subspace $V_i \subset \mathbb{R}^n$ so that the projection of V_i onto $\mathbb{R}^{2d}$ is exactly x_i. To ensure that each x_i will lie in our model set, it suffices to find a lattice D which contains at least one point from each V_i. The span of n vectors in $\mathbb{R}^n$ will quite generically form a lattice, so we can choose a lattice D which contains at least one point from each V_i. Now all x_i lie in $p_1(D)$, so we simply need to choose a window W large enough so that they all lie in Λ_W. $\square$

By using this lemma we can construct model sets with any specified (finitely generated) group of eigenvalues. This shows that the assumption in Theorem 4.4 actually occurs in a rather generic case for model sets.

Finally, we would be able to drop the irrationality assumption on $\mathrm{Eig}(\Lambda)$ if we knew more about the restriction of $\mathrm{Tr}_{\mathcal{B}_\sigma}$ to the algebra generated by the τ_k. We would need to show that $\mathrm{Tr}_{\mathcal{B}_\sigma}(\tau_k) = \delta_{k,0}$. This amounts to computing

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{B}_\sigma}(\tau_k) &= \int_{\Omega_{trans}} \varphi_k(T)\langle \pi(\check{k})(S_g^T)^{-\frac{1}{2}}g, (S_g^T)^{-\frac{1}{2}}g\rangle dT \\
&= \lim_{R\to\infty} \frac{1}{|\Lambda \cap B_r|} \sum_{z\in\Lambda\cap B_R} e^{2\pi i k z}\langle \pi(\check{k})(S_g^{\Lambda-z})^{-\frac{1}{2}}g, (S_g^{\Lambda-z})^{-\frac{1}{2}}g\rangle \\
&= \lim_{R\to\infty} \frac{1}{|\Lambda \cap B_r|} \sum_{z\in\Lambda\cap B_R} e^{2\pi i k z}\langle \pi(z)\pi(\check{k})(S_g^{\Lambda-z})^{-\frac{1}{2}}g, \pi(z)(S_g^{\Lambda-z})^{-\frac{1}{2}}g\rangle \\
&= \lim_{R\to\infty} \frac{1}{|\Lambda \cap B_r|} \sum_{z\in\Lambda\cap B_R} \langle \pi(\check{k})(S_g^{\Lambda})^{-\frac{1}{2}}\pi(z)g, (S_g^{\Lambda})^{-\frac{1}{2}}\pi(z)g\rangle \\
&\stackrel{?}{=} \frac{1}{\mathrm{Dens}(\Lambda)}\delta_{k,0}
\end{aligned}$$

whenever $g \in M^1(\mathbb{R}^d)$ generates a Gabor frame for Λ. If $\mathrm{Dens}(\Lambda) < 1$ then no such frame will exist, but a similar computation for multiwindow frames would suffice.

To put this in context, when $\mathcal{G}(g, L)$ is a Gabor frame for a lattice L, $(S_g^L)^{-\frac{1}{2}}g$ generates a Parseval tight multiwindow frame. Consequently, when $z \in L$, the functions $(S_g^L)^{-\frac{1}{2}}\pi(z)g = \pi(z)(S_g^L)^{-\frac{1}{2}}g$ all also generate Parseval tight frames. Thus we have

$$\langle \pi(l)(S_g^L)^{-\frac{1}{2}}\pi(z)g, (S_g^L)^{-\frac{1}{2}}\pi(z)g\rangle = \mathrm{vol}(L)\delta_{l,0}$$

whenever $z \in L$ and $l \in L^\circ$ by the Wexler-Raz orthogonality relations. So for lattices, the computation above clearly holds. We also know that the computation holds for $\check{k} = 0$; this was precisely the computation of the frame measure from [3]. It is possible that an extension of their methods could complete the computation in full, allowing as to remove the irrationality assumption in Theorem 4.4.

Remark 4.2. *To get the equality*

$$\mathrm{Tr}_{\mathcal{B}_\sigma}(\tau_k) = \int_{\Omega_{trans}} \varphi_k(T)\langle \pi(\check{k})(S^T)^{-\frac{1}{2}}g, (S^T)^{-\frac{1}{2}}g\rangle dT$$

we need to slightly tweak the discussion above. We identify $\mathcal{B}_\sigma^{L^1}$ *with* $P_g(\mathcal{A}_\sigma^{L^1})^N P_g$ *acting on* $(\mathcal{A}_\sigma^{L^1})^N P_g$ *by multiplication on the right. We have a formula for how an element* $P_g a P_g$ *acts on* $\Psi \in \mathcal{H}_\Lambda$*, but this formula depends on the choice of the noncommutative analysis and synthesis operators. In the original formula given for* $\tilde{\tau}_k$*, we used*

$$C_g\Psi(T, T-z) = \langle \Psi(T), \tilde{g}_z^T \rangle$$

$$D_g a = I(a)\Psi_g.$$

However to get the expression for the trace above, one instead should choose

$$C\Psi(T, T-z) = \langle \Psi(T), (S_g^T)^{-\frac{1}{2}}\pi(z)g \rangle$$

$$Da = I(a)(S_g)^{-\frac{1}{2}}g$$

where S_g *again denotes the global frame operator. This amounts to using the Parseval tight frame* $\{(S_g^\Lambda)^{-\frac{1}{2}}\pi(z)g \,|\, z \in \Lambda\}$ *rather than the Gabor frame* $\mathcal{G}(g, \Lambda)$*. In this case, the formula for* $\tilde{\tau}_k$ *is*

$$\tilde{\tau}_k(T-z, T-w) = e^{-2\pi i x(\omega-\omega')}\varphi_k(T)\langle \pi(\check{k})(S_g^T)^{-\frac{1}{2}}\pi(z)g, (S_g^T)^{-\frac{1}{2}}\pi(w)g \rangle$$

and the formula for the trace above is simply the integral of this function over the unit space of R_Λ*.*

Chapter 5

Twisted Gap Labeling

5.1 Gap Labeling for 2-D Lattice Subsets

Now we will look at the simpler case when Λ is a marked lattice and investigate the way that $\mathcal{H}_\Lambda$ fits into $K_0(\mathcal{A}_\sigma)$. A **marked lattice** is a lattice $L \subset \mathbb{R}^d$ where each point $l \in L$ is also assigned a color. For simplicity we shall always assume $L = \mathbb{Z}^d$, and the arguments given can be easily adapted to apply when L is a general lattice. We can construct the hull Ω_Λ in exactly the same way when Λ is a marked lattice. As point sets, all elements of Ω_Λ will be a translate of the integer lattice, however the sets are only considered equal when their colorings are also the same. We will always assume that a marked lattice has an aperiodic coloring with FLC and UCF.

Example 5.1. *Consider the chair tiling of Section 2.2 where the vertices of the tiles are contained in $\mathbb{Z}^2$. We denote the set of vertices of the tiling by V. We can take $\mathbb{Z}^2$ and color the points in V red and all other points blue. This is an example of a marked lattice whose hull has the same properties as the hull of a quasicrystal.*

When Λ is a marked lattice, the hull Ω_Λ has the structure of a fiber bundle $\Omega_{trans} \to \Omega_\Lambda \to \mathbb{T}^d$. It is the suspension of Ω_{trans} by an action of $\mathbb{Z}^d$. We can understand the second map using the associated C^*-algebras. Denote by $\mathcal{A} := C^*(R_\Lambda)$ the untwisted groupoid C^*-algebra of R_Λ which in this case is isomorphic to the crossed product $C(\Omega_{trans}) \rtimes \mathbb{Z}^d$. Then we have a map

$$i : C(\mathbb{T}^d) \cong \mathbb{C}_r^*(\mathbb{Z}^d) \to \mathcal{A}$$

where i takes a function $f \in C_0(\mathbb{Z}^d)$ and extends it to a function F on R_Λ by making it constant in the direction of Ω_{trans}, i.e. $F(T, T-z) = f(z)$. In other words, the image of i is generated by the functions on R_Λ which do not depend on the colorings of the points. The map i is the discrete analog of the map $C(\mathbb{T}^d) \to C(\Omega_\Lambda)$ induced by the fibration. Similarly when we twist by a standard cocycle θ we have an induced map

$$j : A_\theta \to \mathcal{A}_\theta$$

from a noncommutative torus into $\mathcal{A}_\theta$. Both i and j preserve the trace on $C_r^*(\mathbb{Z}^d)$ and A_θ respectively.

Our goal is to prove that the induced maps i_* and j_* are injective on K_0. When θ is totally irrational, the trace on A_θ is injective. Since j preserves the trace this immediately implies that j_* will be injective. We will show that the maps i_* and j_* are compatible, so that the injectivity of j_* for a totally irrational cocycle implies the injectivity of i_*.

Proposition 5.1. *Let $\Lambda = \mathbb{Z}^d$ be a marked lattice. Fix a cocycle θ_1 on $\mathbb{Z}^d$, and let the maps i and j be defined as above. Also fix a homotopy θ between $\theta_1 = \theta(1)$ and the trivial cocycle $\theta(0)$. Then we have a commutative diagram*

$$\begin{array}{ccc}
K_0(C_r^*(\mathbb{Z}^d)) & \xrightarrow{i_*} & K_0(\mathcal{A}) \\
\cong\big\downarrow & & \big\downarrow\cong \\
K_0(C_r^*(\mathbb{Z}^d \times [0,1], \theta)) & \xrightarrow{k_*} & K_0(C_r^*(\mathbb{Z}^d \rtimes \Omega_{trans} \times [0,1], \theta)) \\
\cong\big\downarrow & & \big\downarrow\cong \\
K_0(A_{\theta_1}) & \xrightarrow[j_*]{} & K_0(\mathcal{A}_{\theta_1})
\end{array}$$

where the vertical arrows come from the isomorphisms in Theorem 2.3 and the second horizontal map is induced by the map $k : C_r^*(\mathbb{Z}^d \times [0,1], \theta) \to C_r^*(\mathbb{Z}^d \rtimes \Omega_{trans} \times [0,1], \theta)$ *given by* i *on the fiber at* 0 *and the map* $j_t : A_{\theta(t)} \to \mathcal{A}_{\theta(t)}$ *on the fiber at* $0 < t \leq 1$.

Proof. We will prove only the commutativity of the upper square; commutativity of the lower square follows by a similar argument. Choose a projection $P \in M_N(C_r^*(\mathbb{Z}^d))$. We can lift this to a path of projections P_t, yielding an element of $K_0(C_r^*(\mathbb{Z}^d \times [0,1], \theta))$. When we map this via k_*, we simply extend the projection on each fiber by making it constant in the Ω_{trans} direction. Following the maps the other way around, we can take P and extend it to be constant in the Ω_{trans} direction, then lift it to a path of projections. It is clear that $k_*(P_t)$ is one such possible lift, so we are done. □

Theorem 5.1. *Let* $\Lambda = \mathbb{Z}^d$ *be a marked lattice and fix any cocycle* θ *on* $\mathbb{Z}^d$. *Then the maps* i_* *and* j_* *are injective. We can compare their images with the image of the canonical map* $r_* : K_0(C(\Omega_{trans})) \to K_0(\mathcal{A}_\theta)$ *and we find that the intersection is generated by* $[\mathbf{1}]$, *the class of the rank* 1 *trivial module.*

Remark 5.1. *Note that this immediately implies Theorem 1.4, since the map* i_* *is just the noncommutative version of the map* p^*. *By a result of Sadun and Williams*

[33], given any quasicrystal Λ we can find a marked lattice Λ' so that Ω_Λ and $\Omega_{\Lambda'}$ are homeomorphic. Thus for an arbitrary quasicrystal we can view Ω_Λ as a fiber bundle over a torus, and Theorem 1.4 holds in this case as well.

Proof. First note that when θ is totally irrational, the map $\mathrm{Tr}_* \circ j_*$ is injective, so j_* is injective as well. Thus by Proposition 5.1, we see that i_* must also be injective. Now let θ be any cocycle. Since i_* is injective, by Proposition 5.1 we see that j_* must be as well.

Now we compare the images of i_* and j_* with the image of r_*. First suppose θ is a totally irrational cocycle, and that the intersection of the groups $\mathrm{Tr}_{A_\theta *}(K_0(A_\theta))$ and $\mathrm{Tr}_*(K_0(\mathcal{A}))$ is equal to $\mathbb{Z} \subset \mathbb{R}$. This is possible since $\mathrm{Tr}_*(K_0(\mathcal{A}))$ is countable, so we can simply choose the entries in the matrix for θ to be rationally independent from $\mathrm{Tr}_*(K_0(\mathcal{A}))$. Now it is clear that the image of j_* is disjoint from the projections in $C(\Omega_{trans})$ (except for multiples of the identity) since this is true after applying the trace. Now note that projections in $C(\Omega_{trans})$ are preserved by the vertical isomorphisms on the RHS of the diagram in Proposition 5.1, so the same must be true for i_*. Finally, using the diagram from Proposition 5.1, the theorem holds when θ is an arbitrary cocycle. □

We can interpret the results above in terms of the modules $\mathcal{H}_\Lambda$. When Λ is a marked lattice, a Gabor frame for Λ is simply a lattice Gabor frame and does not depend at all on the colorings of the points in Λ. Furthermore, when $\Lambda = \mathbb{Z}^{2d}$ as a point set then the standard symplectic cocycle $\sigma|_\Lambda$ is the trivial cocycle. In this case, we can use the construction of V_Λ in Section 4.1 to get a module over

$C_r^*(\mathbb{Z}^{2d})$, and $i_*([V_\Lambda]) = [\mathcal{H}_\Lambda]$. To construct modules over $\mathcal{A}_\theta$ for general θ, we follow Rieffel's construction and apply a linear map A to Λ with $A^*\sigma = \theta$. Then we get a module V_Λ over the noncommutative torus $A_{A\Lambda}$ and $j_*([V_{A\Lambda}]) = [\mathcal{H}_{A\Lambda}]$. Thus our modules precisely describe the images of i_* and j_* for even dimensional Λ, and we can conclude that the twisted gap labeling group for a marked lattice always contains the image of the trace map on an associated noncommutative torus. With a little more work, it seems likely that Rieffel's more general method can be adapted to construct modules when Λ is odd dimensional as well.

Now we will describe these results in dimension two, where they allow us to determine the entire gap labeling group. Note that any cocycle θ on $\mathbb{Z}^2$ is determined by a single real number (also denoted θ), which is the only non-zero entry in the associated skew symmetric matrix. When $\Lambda = \mathbb{Z}^2$ is a marked lattice, we can compute its K-theory by applying the Pimsner-Voiculescu exact sequence twice, or by applying the associated Kasparov spectral sequence [18], [36]. In this case we have

$$K_0(C(\Omega_{trans}) \rtimes \mathbb{Z}^2) = C(\Omega_{trans}, \mathbb{Z})_{\mathbb{Z}^2} \oplus \mathbb{Z}$$

where $C(\Omega_{trans}, \mathbb{Z})_{\mathbb{Z}^2}$ denotes the group of coinvariants of the action of $\mathbb{Z}^2$ on Ω_{trans}. Here the extra copy of $\mathbb{Z}$ comes from the inclusion

$$K^0(\mathbb{T}^2) \cong K_0(C_r^*(\mathbb{Z}^2)) \to K_0(C(\Omega_{trans}) \rtimes \mathbb{Z}^2)$$

of the group algebra of $\mathbb{Z}^2$ into $C(\Omega_{trans}) \rtimes \mathbb{Z}^2$, and the summand $C(\Omega_{trans}, \mathbb{Z})_{\mathbb{Z}^2}$ comes from the inclusion

$$K_0(C(\Omega_{trans})) \to K_0(C(\Omega_{trans}) \rtimes \mathbb{Z}^2).$$

The extra generator is precisely the image of the Bott vector bundle in $K^0(\mathbb{T}^2)$. Thus from our results above, we immediately have

Proposition 5.2. *When $\Lambda = \mathbb{Z}^2$ is a marked lattice, the gap labeling group of $\mathcal{A}_\theta$ is*

$$\mathrm{Tr}_*(K_0(\mathcal{A}_\theta)) = \mu(C(\Omega_{trans}, \mathbb{Z})) + \theta\,\mathbb{Z}\,.$$

We can also determine the gap labeling group when we have a quasicrystal $\Lambda \subset \mathbb{Z}^2$. In this case, we can construct a marked lattice $\Gamma = \mathbb{Z}^2$ by coloring the points of Λ red and the remaining points blue. Then Ω^Λ_{trans} sits as a clopen set in Ω^Γ_{trans} with measure equal to $\mathrm{Dens}(\Lambda)$. This shows that the gap labeling group of $\mathcal{A}^\Lambda_\theta$ is equal to $\frac{1}{\mathrm{Dens}(\Lambda)}\ \mathrm{Tr}_*(K_0(\mathcal{A}^\Gamma_\theta))$, which is in turn equal to $\mu(C(\Omega^\Lambda_{trans}), \mathbb{Z}) + \frac{\theta}{\mathrm{Dens}(\Lambda)}\,\mathbb{Z}\,.$ Thus we have:

Theorem 5.2. *When $\Lambda \subset \mathbb{Z}^2$ is a quasicrystal, the gap labeling group of $\mathcal{A}_\theta$ is*

$$\mathrm{Tr}_*(K_0(\mathcal{A}_\theta)) = \mu(C(\Omega_{trans}, \mathbb{Z})) + \frac{\theta}{\mathrm{Dens}(\Lambda)}\,\mathbb{Z}\,.$$

Note that in dimension two a matrix A satisfies $A^*\sigma = \theta$ exactly when $\det(A) = \theta$. Thus the module $\mathcal{H}_{A\Lambda}$ has trace $\frac{1}{\mathrm{Dens}(A\Lambda)} = \frac{\theta}{\mathrm{Dens}(\Lambda)}$ and represents the extra generator in $K_0(\mathcal{A}_\theta)$.

5.2 Connections with Deformation Theory

In the previous section we were able to determine the twisted gap labeling group for lattice subsets in dimension two. There are two ways we might want to generalize this result. First, we might want to compute the twisted gap labeling group for any 2-D quasicrystal twisted by any cocycle, not merely lattice subsets twisted by standard cocycles. Bellissard's original (untwisted) gap labeling conjecture could be reduced to the case of marked lattices, essentially by the results of Sadun and Williams [33]. They show that a quasicrystal Λ can be deformed so that it is a marked lattice Λ', and that this deformation gives rise to a homeomorphism between Ω_Λ and $\Omega_{\Lambda'}$. Thus we might be tempted to say that we can reduce the twisted gap labeling conjecture so that it falls within the scope of our results. Unfortunately, our results hold only for standard cocycles (i.e. restrictions of cocycles on $\mathbb{R}^{2d}$) and in the process of deforming from Λ to Λ' we are likely to take a standard cocycle on Λ to a *nonstandard* cocycle on Λ'. Thus a general computation of the gap labeling group in dimension two still seems out of reach.

In another direction, our results do not give a full computation of the gap labeling group in higher dimensions, even for marked lattices with standard cocycles. We were able to apply linear maps to a marked lattice, and this allowed us to realize any standard cocycle as the restriction of the symplectic cocycle on $\mathbb{R}^{2d}$ in a number of different ways. For each linear map with $A^*\sigma = \theta$, we got a class in $K_0(\mathcal{A}_\theta)$. However, linear maps alone are not enough to fill out all the classes in K-theory. A linear deformation of a marked lattice ignores the colorings of points in the lattice, essentially avoiding the complexities that make it a quasicrystal.

To summarize, there are two difficulties preventing a more complete computation of twisted gap labels. First, there is the problem of passing from a general quasicrystal to a marked lattice, which really has to with extending our results from standard cocycles to all cocycles. The second problem has to do with finding more sophisticated deformations of a quasicrystal so that we can represent all classes in

$K_0(\mathcal{A}_\theta)$ when the dimension of Λ is greater than two. Actually, it seems reasonable to hope that a solution to the second problem will also resolve the first. Namely, we might expect that even given a nonstandard cocycle c we can find a deformation φ so that $\sigma|_{\varphi(\Lambda)} = c$. In fact this was already evident when we were describing the difficulties in dimension two.

Thus we present the following strategy for improving our results on the twisted gap labeling. First, we must get a sense of the types of cocycles by which we can twist. This involves computing the second cohomology group $H^2(R_\Lambda, S^1)$. Ideally $H^2(R_\Lambda, S^1)$ should be computable in terms of the cohomology of Ω_Λ. Next, we fix a cocycle c and give sufficient conditions for a deformation φ to yield the cocycle c on $\varphi(\Lambda)$. To expand upon this, we first assume that $\varphi(\Lambda)$ is a quasicrystal with $R_\Lambda \cong R_{\varphi(\Lambda)}$. Kellendonk has given sufficient conditions on a map $\varphi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ which will ensure this isomorphism of groupoids [19]. Furthermore, we need $\sigma|_{\varphi(\Lambda)} = c$. Once we have this, $\mathcal{H}_{\varphi(\Lambda)}$ gives a class in $K_0(\mathcal{A}_c)$ with trace equal to $\frac{1}{\mathrm{Dens}(\varphi(\Lambda))}$. Ideally, constructing enough maps of this form should exhaust the classes in $K_0(\mathcal{A}_c)$ and give us a full computation of the gap labeling group.

Appendices

Appendix A

K-theory for C^*-algebras

The discussion in these appendices is modeled largely on the explanations of K-theory in [10], [22]. We begin with some basic facts about C^*-algebras.

Definition A.1. *A* $\mathbf{C^*}$***-algebra*** *A is a $\mathbb{C}$-algebra equipped with a norm $||\cdot||$ and an involution $a \to a^*$ such that A is complete with respect to the norm, $||ab|| \leq ||a|| \cdot ||b||$, and $||a^*a|| = ||a||^2$ for all $a, b \in A$. Although a C^*-algebra need not have a multiplicative identity, all of our examples will. A homomorphism between C^*-algebras is a simply a homomorphism of $\mathbb{C}$-algebras which is continuous and preserves the involution.*

C^*-algebras and Banach algebras differ only by the final condition in the above definition. Small though it may seem, this additional axiom makes C^*-algebras especially amenable to topological techniques.

Example A.1. *$\mathbb{C}$ itself is a C^*-algebra, as is any matrix algebra $M_N(\mathbb{C})$. Here the norm is given by the operator norm, and the involution is by conjugate transpose. Similarly, for any C^*-algebra $A, M_N(A)$ is also a C^*-algebra. Finally, $B(H)$, the bounded linear operators on a Hilbert space, is a C^*-algebra. In fact it can be shown that any C^*-algebra embeds as a subalgebra of $B(H)$.*

Example A.2. *Let $L \subset \mathbb{R}^2$ be a lattice and consider the family of time-frequency* *shifts*

$$\{\pi(z) \mid z \in L\}.$$

The closure of this family of operators in the operator norm on $B(L^2(\mathbb{R}))$ is a C^*-algebra known as a ***noncommutative torus****.*

Example A.3. *Let X be any compact, Hausdorff topological space. The algebra $C(X)$ of complex valued functions on X forms a C^*-algebra, where we use the sup norm and the involution is given by complex conjugation.*

This last example is perhaps the most important to keep in mind, given the following theorem:

Theorem A.1 (Gelfand-Naimark)**.** *Suppose A is a commutative, unital C^*-algebra. Then there exists a compact Hausdorff space X such that $A \cong C(X)$.*

The Gelfand-Naimark theorem demonstrates the intrinsic link between C^*-algebras and topology. In light of this theorem, one is tempted to translate as much of topology as possible into C^*-algebraic language. The goal is to use topological techniques to study C^*-algebras which are not necessarily commutative (like the noncommutative torus above), and thus have no realization as an algebra of functions on some topological space. This is the goal of **noncommutative topology**. While it is possible to translate the ideas of homology and cohomology into C^*-algebraic language, it ends up being difficult. K-theory, however, admits a relatively simple translation into the world of C^*-algebras, and has been strikingly effective in a variety of programs which seek to classify C^*-algebras.

In preparation, we review some facts about K-theory for topological spaces.

Definition A.2. *Let X be a compact, Hausdorff topological space. A* ***vector bundle*** *over X is a topological space E equipped with a projection $\pi : E \to X$ such that for each $x \in X$, the fiber $\pi^{-1}(x)$ is a complex vector space of a fixed dimension n. Additionally, for each $x \in X$ there exists a neighborhood U_x such that $\pi^{-1}(U_x)$ is homeomorphic to $U_x \times \mathbb{C}^n$, where the homeomorphism preserves the fibers and is a linear isomorphism on each fiber.*

We can think of a vector bundle as a family of vector spaces parametrized by X. Two vector bundles E and F over X are isomorphic if there is a map from $E \to F$ which commutes with the corresponding projections to X and is a linear isomorphism on each fiber. We can take the direct sum of two vector bundles $E \oplus F$ by taking the direct sum of the fibers over each point in X. We denote by $\mathrm{Vect}(X)$ the abelian semigroup of isomorphism classes of vector bundles over X where the addition operation is direct sum. We can complete this semigroup to a group by formally adding inverses for each element, which gives us an abelian group known as $K^0(X)$. The association $X \to K^0(X)$ is a contravariant functor, which means that whenever $f : X \to Y$ is a continuous map, there is an induced map $f^* : K^0(Y) \to K^0(X)$. The group $K^0(X)$ is a homotopy invariant of a topological space, and carries with it much interesting topological information.

We shall now indicate how one can translate the concept of a vector bundle into the language of C^*-algebras. First we must make note of the following lemma:

Lemma A.1. *Given any vector bundle E over X, we can find a trivial bundle of the form $X \times \mathbb{C}^n$ such that E embeds as a subbundle of $X \times \mathbb{C}^n$.*

In other words, we can find a (potentially large) n so that each of the fibers of E exists as a subspace of $\mathbb{C}^n$ in a way which is continuous over X. We can think of this in a slightly different way. The bundle E, considered as a subbundle of $X \times \mathbb{C}^n$, specifies a projection matrix at each point of X. Namely, the projection at $x \in X$ is the projection from $\mathbb{C}^n$ to the fiber of E over x. Furthermore, this choice of projection matrix is continuous over X, so gives a continuous map from X into $M_N(\mathbb{C})$. However, $C(X, M_N(\mathbb{C})) \cong M_N(C(X))$, so we naturally get a projection in the C^*-algebra $M_N(C(X))$.

We can also consider the continuous sections $\Gamma(E)$ of the bundle E, which are maps $s : X \to E$ such that $\pi(s(x)) = x$. Given any section, we can always multiply it pointwise by a function in $C(X)$. Thus $\Gamma(E)$ becomes a $C(X)$-module, and we can check that it will always be finitely generated and projective. So now we have two ways to express the concept of a vector bundle in the language of C^*-algebras. First it gives us a projection in a matrix algebra over $C(X)$, and second it give us a finitely generated, projective $C(X)$-module. We shall see that these points of view are actually equivalent, and will allow us to extend K-theory to noncommutative C^*-algebras.

Motivated by K-theory for topological spaces, for a C^*-algebra A we denote by $\mathcal{P}_N(A)$ the collection of projections in $M_N(A)$ and let $\mathcal{P}_\infty(A) = \bigcup_{N=1}^{\infty} \mathcal{P}_N(A)$. We can take the direct sum of two projections $p, q \in \mathcal{P}_\infty(A)$ by

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Two projections $p, q \in \mathcal{P}_N(A)$, are called **Murray-Von Neumann equivalent,** denoted $p \sim q$, if there exists $v \in M_N(A)$ with $vv^* = p$ and $v^*v = q$. We can extend this equivalence relation to $\mathcal{P}_\infty(A)$ by defining $p \sim_0 q$ if there exist matrices of all zeros $0_n, 0_m$ of sizes n and m respectively so that $p \oplus 0_n$ and $q \oplus 0_m$ are the same size and $p \oplus 0_n \sim q \oplus 0_m$. This is a well defined equivalence relation on $\mathcal{P}_\infty(A)$ and respects the direct sum operation. We denote by $\mathcal{D}(A)$ the set of equivalence classes $\mathcal{P}_\infty(A)/ \sim_0$. This gives us an abelian semigroup $\mathcal{D}(A)$, and after completing $\mathcal{D}(A)$ to a group we arrive at $K_0(A)$, the C^*-algebraic K-theory group. The assignment $A \to K_0(A)$ is a covariant functor, so that whenever $\varphi : A \to B$ is a homomorphism there is a corresponding map $\varphi_* : K_0(A) \to K_0(B)$. To complete the analogy with the K-theory of topological spaces, we have:

Theorem A.2. *When X is a compact Hausdorff space, $K_0(C(X)) \cong K^0(X)$.*

Alternatively, we could have constructed the K_0 group by examining finitely generated, projective left (or right) A-modules. The isomorphism classes of such modules form an abelian semigroup under direct sum. After completing this to a group by adding formal inverses for the elements, we end up with a group isomorphic to $K_0(A)$. To see the connection between this construction and the one above, choose $p \in \mathcal{P}_n(A)$. Then $A^n p$ is a finitely generated, projective left A-module. This association $p \to A^n p$ gives a map between the two different versions of C^*-algebraic K-theory we have described. When A has a unique normalized trace, we can assign each finitely generated, projective A-module a **dimension**. We simply find a projection $p \in M_N(A)$ which represents the module in K_0 and apply the trace to p. The terminology comes from looking at projections in $M_N(\mathbb{C})$, where applying the usual matrix trace gives the dimension of the range of the projection. Thus to any trace we can define a homomorphism from $K_0(A)$ to $\mathbb{R}$ simply by applying the trace to classes of projections in $\mathcal{P}_\infty(A)$. For C^*-algebras coming from quasicrystals, the image of the trace map has a physical interpretation. Determining this image is the subject of Bellissard's gap labeling conjecture, described in Chapter 1.

Appendix B

Hilbert C^*-modules and Morita Equivalence

In Appendix A, we have seen how isomorphism classes of finitely generated, projective A-modules can be used to construct $K_0(A)$. In this section we will explore the concept of A-modules further. It is possible to endow all finitely generated, projective A-modules with the structure of a **Hilbert C*-module**, which is meant to emulate the inner product structure on a Hilbert space.

Definition B.1. *A (left)* ***pre-C*-module*** *over a C^*-algebra A is a complex vector space $\mathcal{E}$ which is also an A-module with a pairing ${}_A\langle \cdot , \cdot \rangle : \mathcal{E} \times \mathcal{E} \to A$ satisfying the following conditions for all $r, s, t \in \mathcal{E}$ and $a \in A$:*

1. ${}_A\langle r + s, t\rangle = {}_A\langle r, t\rangle + {}_A\langle s, t\rangle$
2. ${}_A\langle ar, s\rangle = a\,{}_A\langle r, s\rangle$
3. ${}_A\langle r, s\rangle = {}_A\langle s, r\rangle^*$
4. ${}_A\langle s, s\rangle > 0$ *when* $s \neq 0$

The last condition means that ${}_A\langle s, s\rangle$ gives a positive element of the C^*-algebra A.

We can use the inner product to define a norm on $\mathcal{E}$ by

$$||s||_{\mathcal{E}} := \sqrt{||\langle s, s\rangle_A||}.$$

We call the completion of $\mathcal{E}$ in this norm a Hilbert C^*-module. Denote by ${}_A\langle\mathcal{E},\mathcal{E}\rangle$ the linear span of all elements of the form ${}_A\langle r, s\rangle$. We call $\mathcal{E}$ a **full** Hilbert C^*-module if ${}_A\langle\mathcal{E},\mathcal{E}\rangle$ is dense in A. When A is unital, this implies ${}_A\langle\mathcal{E},\mathcal{E}\rangle = A$.

Example B.1. *A itself is a Hilbert A-module, where the inner product is given by*

$${}_A\langle a, b\rangle = ab^*.$$

Moreover, A^n is a Hilbert A-module with inner product

$${}_A\langle(a_1,\dots,a_n),(b_1,\dots,b_n)\rangle := \sum_{i=1}^{n} a_i b_i^*.$$

Example B.2. *Whenever $P \in \mathcal{P}_n(A)$ is a projection, A^nP becomes a Hilbert A-module with inner product*

$${}_A\langle\vec{a},\vec{b}\rangle := \vec{a}P\vec{b}^*.$$

This last example shows how $K_0(A)$ can be described using the (seemingly more complicated) concept of Hilbert A-modules. Any finitely generated, projective A-module is represented by a projection $P \in \mathcal{P}_n(A)$ for some n. Thus it is isomorphic to A^nP for some projection P, which comes with a natural Hilbert A-module structure.

The beauty of Hilbert A-modules is that they let us to construct new C^*-algebras which are related to A but not equal to A, allowing us to understand the structure of A using a variety of techniques.

Definition B.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert A-modules. A map $T : \mathcal{E} \to \mathcal{F}$ is* ***adjointable*** *if there exists a map $T^* : \mathcal{F} \to \mathcal{E}$, called the* ***adjoint*** *of T, such that*

$$ {}_A^{\mathcal{F}}\langle r, Ts\rangle = {}_A^{\mathcal{E}}\langle T^* r, s\rangle $$

whenever $r \in \mathcal{F}, s \in \mathcal{E}$.

We shall be primarily concerned with $\mathrm{End}_A(\mathcal{E})$, the space of adjointable operators from $\mathcal{E}$ to itself. An important class of adjointable operators is given by the **A-finite rank operators**, which are operators of the form ${}_A\langle \,\cdot\,, r\rangle s$ where $r, s \in \mathcal{E}$. The A-finite rank operators form a vector space denoted by $\mathrm{End}_A^{00}(\mathcal{E})$, and the closure of $\mathrm{End}_A^{00}(\mathcal{E})$ will be known as the collection of **A-compact operators**, denoted by End_A^0.

Example B.3. *If $P \in \mathcal{P}_n$ then $A^n P$ is a Hilbert A-module and $\mathrm{End}_A^0(A^n P) = PM_n(A)P$. See [10] Lemma 2.18 for details.*

We can think of A acting on $\mathcal{E}$ on the left while $B = \mathrm{End}_A^0$ acts on $\mathcal{E}$ on the right. Additionally, we can give $\mathcal{E}$ the structure of a right Hilbert B-module by defining the inner product

$$ \{r, s\}_B := {}_A\langle \,\cdot\,, r\rangle s. $$

This leads to the definition of a Hilbert A-B bimodule.

Definition B.3. *A* ***pre-C* A-B-bimodule*** *is a complex vector space $\mathcal{E}$ which is both a left Hilbert A-module and a right Hilbert B-module where the inner products satisfy the following compatibility condition:*

$$ {}_A\langle r, s\rangle t = r\{s, t\}_B \text{ for all } r, s, t \in \mathcal{E}. $$

*A pre-C^* A-B-bimodule is called **full** if ${}_A\langle\mathcal{E},\mathcal{E}\rangle$ and $\{\mathcal{E},\mathcal{E}\}_B$ are dense in A and B respectively.*

For bimodules, we could define a norm on $\mathcal{E}$ using either of the inner products. As it turns out, these norms are equivalent, and the completion of $\mathcal{E}$ with respect to either norm is known as a **Hilbert A-B-bimodule**. Two C^*-algebras A and B are called **Morita equivalent** if there exists a full Hilbert A-B-bimodule $\mathcal{E}$. In this case, $B = \mathrm{End}^0_A(\mathcal{E})$.

Morita equivalence is a sort of "homotopy equivalence" for noncommutative spaces. When A and B are Morita equivalent, we have $K_0(A) \cong K_0(B)$. Given a trace Tr_A on A, we can define a trace Tr_B on the finite rank operators in B by

$$\mathrm{Tr}_B(\{r,s\}_B) := \mathrm{Tr}_A(\langle s,r\rangle_A)$$

when $r, s \in \mathcal{E}$. This extends to a trace on B and defines a bijection between the traces on A and B. There is much more to be said about Morita equivalence, however these facts will be enough to understand the analysis in the text above.

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