

On Number Of Partitions Of An Integer Into A Fixed Number of Positive Integers

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Abstract

This paper focuses on the number of partitions of a positive integer n into k positive summands, where k is an integer between 1 and n . Recently some upper bounds were reported for this number in [Merca14]. Here, it is shown that these bounds are not as tight as an earlier upper bound proved in [Andrews76-1] for $k \leq 0.42n$. A new upper bound for the number of partitions of n into k summands is given, and shown to be tighter than the upper bound in [Merca14] when k is between $O(\frac{\sqrt{n}}{\ln n})$ and $n - O(\frac{\sqrt{n}}{\ln n})$. It is further shown that the new upper bound is also tighter than two other upper bounds previously reported in [Andrews76-1] and [Colman82]. A generalization of this upper bound to number of partitions of n into at most k summands is also presented.

1 Introduction

Partitions of an integer play an important role in the solutions of combinatorial problems and this article is motivated in part by such a problem that arises in counting multicast calls between n callers and n receivers in a switching network [Oruç15]. In particular, three types of partitions will be of interest in this paper as stated below and we refer the reader to [Andrews76-1] for basic concepts in partition theory.

1. A partition of n is an unordered sum of n that comprises up to n positive integers. The number of such sums is often denoted by $p(n)$. For example, $p(3) = 3$ as $3 = 3, 3 = 1 + 2$, and $3 = 1 + 1 + 1$.
2. A partition of n into exactly k parts is an unordered sum of n that uses exactly k positive integers. The number of such partitions will henceforth be denoted by $p(n, k)$. For example, $p(5, 3) = 2$ as $5 = 1 + 2 + 2$ and $5 = 1 + 1 + 3$ are the only two sums of 5 that can be formed using three positive integers.

3. A partition of n into at most k parts is an unordered sum of n that uses at most k positive integers. Following the asterisk notation in [Colman82], the number of such partitions will henceforth be denoted by $p^*(n, k)$. For example, $p^*(5, 3) = 5$ as $5 = 2 + 2 + 1$, $5 = 3 + 1 + 1$, $5 = 2 + 3$, $5 = 4 + 1$, and $5 = 5$ are the only sums of 5 that can be formed using one, two, or three integers.

No exact closed-form expressions are known to compute the values of $p(n)$, $p(n, k)$, and $p^*(n, k)$. For $p(n)$, Hardy-Ramanujan-Rademacher formula provides an asymptotic approximation to $p(n)$ [Andrews76-2]:

$$p(n) \approx \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}. \quad (1)$$

Using Remark 1 in [Kane06], it can be shown that

$$0.02556 \leq \lim_{n \rightarrow \infty} \frac{p(n)}{\frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}} \leq 37.6393, \quad (2)$$

while

$$\lim_{n \rightarrow \infty} \left| \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} - p(n) \right| = \infty. \quad (3)$$

In the sequel, we will also need Kane's inequality:

$$\frac{C_{1,1}^-}{n} e^{\pi\sqrt{\frac{2n}{3}}} \leq p(n) \leq \frac{C_{1,1}^+}{n} e^{\pi\sqrt{\frac{2n}{3}}}. \quad (4)$$

where $C_{1,1}^-$ is any number less than $\frac{5e^{-2-\frac{3\gamma}{2}}}{8\sqrt{3}\pi^{3/2}}$, $C_{1,1}^+$ is any number greater than $\frac{27}{4} \left(\frac{e}{\pi}\right)^{3/2}$, and $\gamma = 0.57721\dots$, is the Euler constant. We will let $C_{1,1}^- = 0.0036$ and $C_{1,1}^+ = 5.44$.

In the remainder of the paper, we analyze the previously reported upper bounds for $p(n, k)$ and present a sharper upper bound for the same. This new bound is used to obtain an upper bound on $p^*(n, k)$ as well. It is assumed that i, j, k, m, n, s are all positive integers unless otherwise stated.

2 Partitions With A Fixed Number Of Parts

The following lower and upper bounds for $p(n, k)$, $1 \leq k \leq n - 1$ are well-known [Andrews76-1]:

$$\frac{1}{k!} \binom{n-1}{k} \leq p(n, k) \leq U_a(n, k) = \frac{1}{k!} \left(n + \frac{k(k-1)}{2} - 1 \right). \quad (5)$$

Another upper bound, attributed to Rieger and stated below, was shown to be effective in [Colman82] in estimating $p(n, k)$ when $\sqrt{72}n > k^{5/2}$,

$$p(n, k) \leq U_r(n, k) = \frac{1}{k!(k-1)!} \left(n + \frac{k(k-3)}{4} \right)^{k-1}, \quad 4 \leq k \leq n. \quad (6)$$

Expanding the binomial term in $U_a(n, k)$, we find

$$U_a(n, k) = \frac{1}{k!(k-1)!} \prod_{j=1}^{k-1} \left(n + \frac{k(k-1)}{2} - j \right)! \quad (7)$$

$$\geq \frac{1}{k!(k-1)!} \left(n + \frac{k(k-1)}{2} - k + 1 \right)^{k-1} \quad (8)$$

$$\geq \frac{1}{k!(k-1)!} \left(n + \frac{(k-1)(k-2)}{2} \right)^{k-1}. \quad (9)$$

Comparing $U_r(n, k)$ in (6) with the last inequality shows that $U_r(n, k) \leq U_a(n, k)$ for all $k, 1 \leq k \leq n$.

More recently, a series of upper bounds was established in [Merca14] with increasing numbers of terms in the formulas, using a relation between multinomial and binomial coefficients. The simplest two of these upper bounds are¹

$$p(n, k) \leq \frac{1}{2} \binom{n-1}{k-1} + \frac{1}{2} \delta_{0, n \bmod k}, \quad (10)$$

$$p(n, k) \leq \frac{1}{3} \binom{n-1}{k-1} + \frac{2}{3} \delta_{0, n \bmod k} + \frac{1}{3} \delta_{2, k} \left\lfloor \frac{n-1}{2} \right\rfloor, \quad (11)$$

where $\delta_{u,v}$ is the Kronecker delta function and so the following the upper bounds obviously apply:

$$p(n, k) \leq U_{M1}(n, k) = \frac{1}{2} \binom{n-1}{k-1} + \frac{1}{2}, \quad (12)$$

$$p(n, k) \leq U_{M2}(n, k) = \frac{1}{3} \binom{n-1}{k-1} + \frac{2}{3} + \frac{1}{3} \left\lfloor \frac{n-1}{2} \right\rfloor. \quad (13)$$

¹A number of other upper bounds have been provided in [Merca14], all having a dominating term of $\binom{n-1}{k-1}$ that is multiplied by a constant $1/t, 2 \leq t \leq 10$. Even though $1/t$ decreases as t increases, it does not alter the order of complexity of the upper bound given that t is a constant. An upper bound was also given in Theorem 2 in the same paper for any positive integer $t > 1$, and it was shown that this upper bound converges to $p(n, k)$ as t tends to $k!$. However, this bound has a sum term whose value was not settled in the paper.

What is not so obvious is whether $U_{M1}(n, k) \leq U_a(n, k)$ or $U_{M2}(n, k) \leq U_a(n, k)$ except when $k = 1$ and $k = 2$. For $k = 1$ and $k = 2$, $U_{M1}(n, k)$ coincides with $U_a(n, k)$. For $k = 1$ and $k = 2$, and $n > 2$, $U_a(n, k) \leq U_{M2}(n, k)$. To settle this question for $k, 3 \leq k \leq n - 1$, we note that $U_a(n, k)$ can be expressed as

$$U_a(n, k) = \frac{\binom{n-1}{k-1}}{k!} \prod_{j=0}^{s-1} \frac{n-1+s-j}{n-1+s-j-(k-1)}, \quad (14)$$

where $s = \frac{k(k-1)}{2}$. Let

$$f(s, k, n) = \frac{1}{k!} \prod_{j=0}^{s-1} \frac{1}{1 - \frac{(k-1)}{n-1+s-j}}. \quad (15)$$

Then $U_a(n, k) = \binom{n-1}{k-1} f(s, k, n)$ and

Remark 1. if $f(s, k, n) \leq \frac{1}{2}$, then $U_a(n, k) \leq U_{M1}(n, k)$. □

Similarly²,

Remark 2. if $f(s, k, n) \leq \frac{1}{3}$, then $U_a(n, k) \leq U_{M2}(n, k)$. □

To proceed further, we will invoke the Geometric-Harmonic mean inequality:

$$\left(\prod_{j=0}^{s-1} u_j \right)^{\frac{1}{s}} \geq \frac{s}{\sum_{j=0}^{s-1} \frac{1}{u_j}}, \quad (16)$$

where we let $u_j = 1 - \frac{(k-1)}{k-1+s-j}$. Thus,

$$\prod_{j=0}^{s-1} u_j \geq \left(\frac{s}{\sum_{j=0}^{s-1} \frac{1}{u_j}} \right)^s. \quad (17)$$

Now, let $w(s, k, n) = \prod_{j=0}^{s-1} u_j$ and $z(s, k, n) = \sum_{j=0}^{s-1} \frac{1}{u_j}$. Then

$$f(s, k, n) = \frac{1}{k! w(s, k, n)}, \quad (18)$$

²As it will be shown later, it suffices to focus on the first inequality.

$$w(s, k, n) \geq \left(\frac{s}{z(s, k, n)} \right)^s, \quad (19)$$

and

$$z(s, k, n) = \sum_{j=0}^{s-1} \frac{1}{u_j} \quad (20)$$

$$= \sum_{j=0}^{s-1} \frac{1}{1 - \frac{k-1}{n-1+s-j}} \quad (21)$$

$$= \sum_{j=0}^{s-1} \frac{n+s-j-1}{n+s-j-k}. \quad (22)$$

Replacing the sum by the integral $\int_{x=0}^s \frac{n+s-x-1}{n+s-x-k} dx$ in the last expression results in the inequality

$$z(s, k, n) \leq s + 2(k-1) \operatorname{arctanh} \left(\frac{s}{s-2k+2n} \right) \quad (23)$$

$$\leq s + (k-1) \left[\ln \left(1 + \frac{s}{s-2k+2n} \right) - \ln \left(1 - \frac{s}{s-2k+2n} \right) \right] \quad (24)$$

$$\leq s + (k-1) \ln \left(\frac{2s-2k+2n}{2n-2k} \right) \quad (25)$$

$$\leq s + (k-1) \ln \left(1 + \frac{s}{n-k} \right). \quad (26)$$

Combining the last inequality with (19), we find

$$w(s, k, n) \geq \left(\frac{s}{s + (k-1) \ln \left(1 + \frac{s}{n-k} \right)} \right)^s \quad (27)$$

$$\geq \frac{1}{\left(1 + \frac{k-1}{s} \ln \left(1 + \frac{s}{n-k} \right) \right)^s} \quad (28)$$

and combining the last inequality with (18)

$$f(s, k, n) \leq \frac{\left(1 + \frac{k-1}{s} \ln \left(1 + \frac{s}{n-k} \right) \right)^s}{k!}. \quad (29)$$

Hence, by Remark 1, $U_a(n, k) \leq U_{M1}(n, k)$ if

$$\frac{\left(1 + \frac{k-1}{s} \ln \left(1 + \frac{s}{n-k}\right)\right)^s}{k!} \leq \frac{1}{2}. \quad (30)$$

Recalling that $s = k(k-1)/2$, this inequality can be simplified as follows:

$$\begin{aligned} \frac{\left(1 + \frac{2}{k} \ln \left(1 + \frac{k(k-1)}{2n-2k}\right)\right)^{k(k-1)/2}}{k!} &\leq \frac{1}{2} \\ \ln \left(1 + \frac{k(k-1)}{2n-2k}\right) &\leq \frac{k}{2} \left(\left(\frac{k!}{2}\right)^{\frac{2}{k(k-1)}} - 1 \right) \\ \frac{k(k-1)}{2n-2k} &\leq \exp \left(\frac{k}{2} \left(\left(\frac{k!}{2}\right)^{\frac{2}{k(k-1)}} - 1 \right) \right) - 1 \\ k + \frac{k(k-1)}{2 \exp \left(\frac{k}{2} \left(\left(\frac{k!}{2}\right)^{\frac{2}{k(k-1)}} - 1 \right) \right) - 2} &\leq n. \end{aligned}$$

It follows that if

$$n \geq k + \frac{k(k-1)}{2 \exp \left(\frac{k}{2} \left(\left(\frac{k!}{2}\right)^{\frac{2}{k(k-1)}} - 1 \right) \right) - 2} \quad (31)$$

then $U_a(n, k) \leq U_{M1}(n, k)$. Let $g(k) = \frac{(k-1)}{\exp \left(\frac{k}{2} \left(\left(\frac{k!}{2}\right)^{\frac{2}{k(k-1)}} - 1 \right) \right) - 1}$. Then

$U_a(n, k) \leq U_{M1}(n, k)$ if

$$n \geq k \left(1 + \frac{g(k)}{2}\right). \quad (32)$$

It can be shown that $g(k)$ has a minimum when k is near 2π and $1 \leq g(k) \leq e$ if $k \geq 3$. Thus, the inequality in (32) is satisfied if $n \geq k(1 + \frac{e}{2})$, and therefore, $U_a(n, k) \leq U_{M1}(n, k)$, $3 \leq k \leq \frac{n}{1 + \frac{e}{2}} = \lfloor 0.42n \rfloor$.

It can further be shown that the same limit holds when the $k!/2$ term is replaced by $k!/3$ in $g(k)$ and that the exact same upper bound on k , i.e., $k \leq \lfloor 0.42n \rfloor$ also holds if $U_{M2}(n, k)$ is used in the comparison. It should also be noted that $U_r(n, k) \leq U_{M1}(n, k)$ and $U_r(n, k) \leq U_{M2}(n, k)$ over the same domain of values of k , given that it has already been established in the preceding section that $U_r(n, k) \leq U_a(n, k)$, $1 \leq k \leq n$.

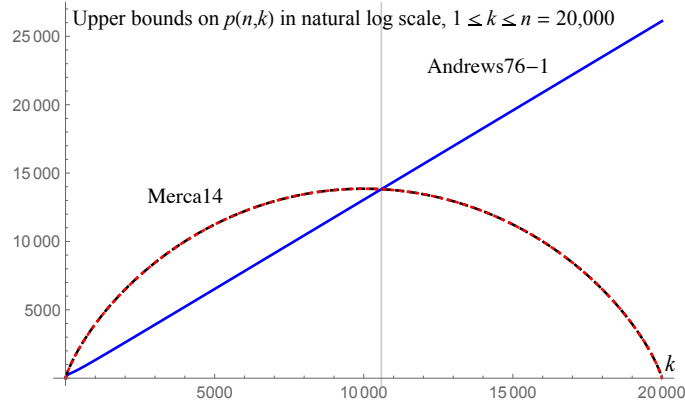


Figure 1: Comparison of the upper bounds in [Merca14] and [Andrews76-1] in natural logarithmic scale, $1 \leq k \leq n = 20,000$. (The solid curve represents the upper bound in (5). The dashed and dotted curves are the upper bounds in (10) and (11) and they nearly coincide.)

Remark 3. The upper limit on k is not tight since we used an upper bound for $f(s, k, n)$ to determine it. The numerical comparison of $U_a(n, k)$ with $U_{M1}(n, k)$ reveals that $U_a(20,000, k) \leq U_{M1}(20,000, k)$ for all $k \leq 10,590$. This suggests that the upper limit tends to $\lfloor 0.53n \rfloor$ as n gets large, but proving this requires a more precise estimation of $f(s, k, n)$. \square

3 A Tighter Upper Bound

The value of $p(n, k)$ can be predicted much more accurately for certain values of k and n using the following theorem.

Theorem 1. $p(n, k) = p(n - k)$, $k \leq n \leq 2k$.

Proof. The proof is based on an “ n urns and k balls distribution” argument, where each summand in a partition with exactly k summands represent the number of balls in each of the k urns. Each urn must clearly have at least one ball. With one ball in each urn, the remaining $n - k$ balls can be placed in $p(n - k)$ ways, where each such placement corresponds to a partition of $n - k$, as long as $n - k \leq k$ or $n \leq 2k$. Given that the urns are indistinguishable, the statement follows. \square

Combining this result with Hardy-Ramanujan-Rademacher asymptotic formula yields the asymptotic formula

$$p(n, k) \approx \frac{1}{4\sqrt{3}(n - k)} e^{\pi\sqrt{\frac{2(n-k)}{3}}}, k \leq n \leq 2k. \quad (33)$$

Furthermore, the following upper bound applies to $p(n, k)$.

Corollary 1.

$$p(n, k) \leq \frac{5.44}{(n-k)} e^{\pi \sqrt{\frac{2(n-k)}{3}}}, 1 \leq k \leq n-1. \quad (34)$$

Proof. Using the same "urns and balls" analogy in Theorem 1, in any partition of n into exactly k parts, each urn must contain at least one ball. The remaining $n-k$ balls cannot be distributed to the k urns in more than $p(n-k)$ ways. Therefore, $p(n, k) \leq p(n-k)$ and the statement follows from Kane's upper bound in inequality (4). \square

Let

$$U_{\text{new}}(n, k) = \frac{5.44}{n-k} e^{\pi \sqrt{\frac{2(n-k)}{3}}}. \quad (35)$$

The next three results compare the new upper bound with the previous upper bounds.

Corollary 2. For $n \geq 171$, and $\left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil + 1 \leq k \leq n - \left\lfloor \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rfloor$,

$$U_{\text{new}}(n, k) \leq U_{\text{M1}}(n, k). \quad (36)$$

Proof. For the specified values of k and n , we need to show

$$\frac{5.44}{n-k} e^{\pi \sqrt{\frac{2(n-k)}{3}}} \leq \frac{1}{2} \binom{n-1}{k-1} \quad (37)$$

or, given that $k \leq n - \left\lfloor \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rfloor$, we have $n-k \geq \left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil \geq 13$

for $n \geq 171$.

Thus, it is sufficient to prove

$$\frac{10.88e^{\pi \sqrt{\frac{2(n-1)}{3}}}}{13\sqrt{3}} \leq \binom{n-1}{k-1}. \quad (38)$$

Now, $\binom{n-1}{m-1} = \binom{n-1}{n-m}$, $1 \leq m \leq n$, and $\binom{n-1}{m-1} \leq \binom{n-1}{k-1}$, $m \leq k \leq n-m+1$.

In particular, if $\left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil + 1 \leq k \leq n - \left\lfloor \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rfloor$ then

$$\binom{n-1}{\left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil} \leq \binom{n-1}{k-1}. \quad (39)$$

Thus, (38) holds if

$$\frac{10.88e^{\pi \sqrt{\frac{2(n-1)}{3}}}}{13} \leq \binom{n-1}{\left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil} \quad (40)$$

holds, or using the inequality, $(x/y)^y \leq \binom{x}{y}$, it suffices to prove the inequality

$$\frac{10.88e^{\pi\sqrt{\frac{2(n-1)}{3}}}}{13} \leq \left(\frac{n-1}{\left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil} \right)^{\left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil}. \quad (41)$$

It is not difficult to see that this inequality is satisfied if the following inequality holds:

$$\frac{10.88e^{\pi\sqrt{\frac{2(n-1)}{3}}}}{13} \leq \left(\frac{n-1}{\frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)}} \right)^{\frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)}}. \quad (42)$$

Hence, taking the logarithm of both sides of the inequality, we need to prove

$$\pi\sqrt{\frac{2(n-1)}{3}} + \ln \frac{10.88}{13} \leq \frac{2\pi\sqrt{\frac{2(n-1)}{3}}}{\ln(n-1)} \ln \left(\frac{(n-1)\ln(n-1)}{2\pi\sqrt{2(n-1)/3}} \right). \quad (43)$$

Applying the logarithm to the terms inside the expression on the right gives

$$\pi\sqrt{\frac{2(n-1)}{3}} - 0.179 \leq \pi\sqrt{\frac{2(n-1)}{3}} + \frac{2\pi\sqrt{\frac{2(n-1)}{3}}}{\ln(n-1)} \ln \left(\frac{\ln(n-1)}{2\pi\sqrt{2/3}} \right). \quad (44)$$

The inequality clearly holds if $\ln(n-1) \geq 2\pi\sqrt{\frac{2}{3}}$ or $n \geq 171$ and the statement follows. \square

We state the following analogous result without a proof.

Corollary 3. For $n \geq 171$, and $\left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil + 1 \leq k \leq n - \left\lceil \frac{2\pi\sqrt{2(n-1)/3}}{\ln(n-1)} \right\rceil$,

$$U_{\text{new}}(n, k) \leq U_{M_2}(n, k). \quad \square$$

These results establish that the new upper bound is tighter than the upper bounds in [Merca14] if k is between $O(\sqrt{n}/\ln n)$ and $n - O(\sqrt{n}/\ln n)$. The bounds on k can be sharpened using a tighter lower bound on $\binom{n}{k}$ for the same lower bound on n given in Corollary 2. The lower bound for k in Corollary 2 implies that $k \geq 14$ when $n = 171$ for the new upper bound to be less than $U_{M_1}(n, k)$ and $U_{M_2}(n, k)$. If we use the original inequality in (37) then the new upper bound drops below $U_{M_1}(n, k)$ and $U_{M_2}(n, k)$ at $k = 8$ when $n \geq 171$.

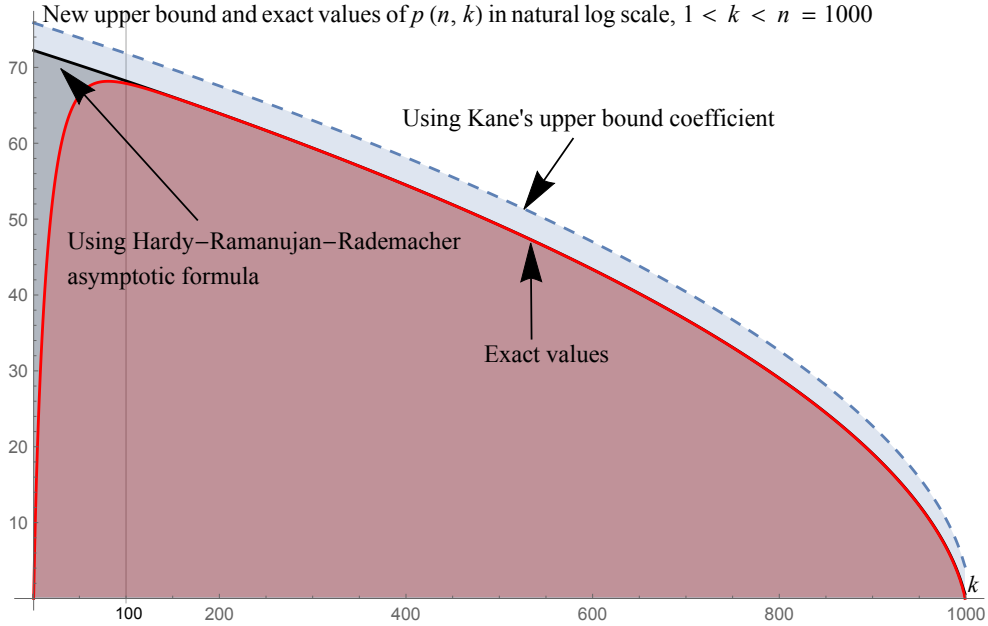


Figure 2: The new upper bound and exact values of $p(n, k)$.

Remark 4.

The new upper bound (the dashed curve) is plotted in natural log scale against the values of $p(n, k)$ for $n = 1000$ and $1 \leq k \leq n - 1$ in Figure 2. The Hardy-Ramanujan-Rademacher asymptotic formula for $p(n, k)$ in (33) is also plotted in the figure over the same interval (solid dark curve). Both bounds closely track the values of $p(n, k)$, with Hardy-Ramanujan-Rademacher formula nearly coinciding with $p(n, k)$ when k is between 100 and 999. This is expected as $p(n, k)$ approaches $p(n - k)$ as k approaches $n/2$ in which case Hardy-Ramanujan-Rademacher formula provides a tight approximation to $p(n, k)$. However, it is only an asymptotic approximation, not an upper bound. It should also be noted that the new upper bound differs from the Hardy-Ramanujan-Rademacher formula in (33) only by a constant factor of Kane's upper bound coefficient $\frac{27}{4} \left(\frac{e}{\pi}\right)^{3/2}$ multiplied by $4\sqrt{3}$, that is, 37.6393. \square

We now establish that $U_{\text{new}}(n, k)$ is also tighter than $U_a(n, k)$ when k is between $O(\sqrt{n})$ and $n - 1$.

Corollary 4. If $1.77(6.28\sqrt{0.56n - 0.92 \ln n + 9.87} + 0.92 \ln n - 19.74) \leq k \leq n$ then

$$U_{\text{new}}(n, k) \leq U_a(n, k). \tag{45}$$

Proof. We first note that the inequalities, $\binom{r}{q} > (r/q)^q$ and $r! < \sqrt{2\pi r} \left(\frac{r}{e}\right)^r e^{\frac{1}{12r}}$ imply

$$U_a(n, k) = \frac{1}{k!} \binom{n + \frac{k(k-1)}{2} - 1}{k-1} \geq \frac{e^k}{\sqrt{2\pi k} e^{\frac{1}{12k}} k^k} \left(\frac{n + \frac{k(k-1)}{2} - 1}{k-1} \right)^{k-1}. \quad (46)$$

Rearranging the terms,

$$U_a(n, k) \geq \frac{e^k}{\sqrt{2\pi k} e^{\frac{1}{12k}} k} \left(\frac{n-1}{k(k-1)} + \frac{1}{2} \right)^{k-1}. \quad (47)$$

Thus, it is sufficient to prove the following inequality in the specified interval for k .

$$U_{\text{new}}(n, k) = \frac{5.44}{(n-k)} e^{\pi \sqrt{\frac{2(n-k)}{3}}} \leq \frac{e^k}{\sqrt{2\pi k} e^{\frac{1}{12k}} k} \left(\frac{n-1}{k(k-1)} + \frac{1}{2} \right)^{k-1}. \quad (48)$$

Taking the logarithm of both sides and rearranging the terms, we find

$$\pi \sqrt{\frac{2(n-k)}{3}} \leq k - \frac{1}{12k} + \ln \frac{n-k}{k} - \frac{1}{2} \ln 2\pi k + (k-1) \ln \left(\frac{n-1}{k(k-1)} + \frac{1}{2} \right) - \ln 5.44. \quad (49)$$

Noting that $0 \leq \frac{n-1}{k(k-1)}$ and $-\frac{1}{24} \leq -\frac{1}{12k}$ for $k \geq 2$, to prove (48), it suffices to prove the inequality:

$$\pi \sqrt{\frac{2(n-k)}{3}} \leq k - \frac{1}{24} + \ln \frac{n-k}{k} - \frac{1}{2} \ln 2\pi k + (k-1) \ln \frac{1}{2} - \ln 5.44 \quad (50)$$

in the specified interval for k . Now, suppose that $k \leq \frac{n}{2}$. Then $0 \leq \ln \frac{n-k}{k}$ and $-\frac{1}{2} \ln \pi n \leq -\frac{1}{2} \ln 2\pi k$. Therefore, if $k \leq \frac{n}{2}$ and if

$$\pi \sqrt{\frac{2(n-k)}{3}} \leq k - \frac{1}{24} - \frac{1}{2} \ln \pi n + (k-1) \ln \frac{1}{2} - \ln 5.44 \quad (51)$$

holds or equivalently

$$\pi \sqrt{\frac{2(n-k)}{3}} \leq k(1 - \ln 2) - \frac{1}{2} \ln n - \ln 2.77\sqrt{\pi} - \frac{1}{24} \quad (52)$$

$$\leq 0.3069k - \frac{1}{2} \ln n - 1.6147 \quad (53)$$

holds then (50) holds as well. Solving for k gives

$$(1.629 \ln n - 29.667) + 3.909 \sqrt{4.572n - 7.449 \ln n + 55.793} \leq k \leq \frac{n}{2} \quad (54)$$

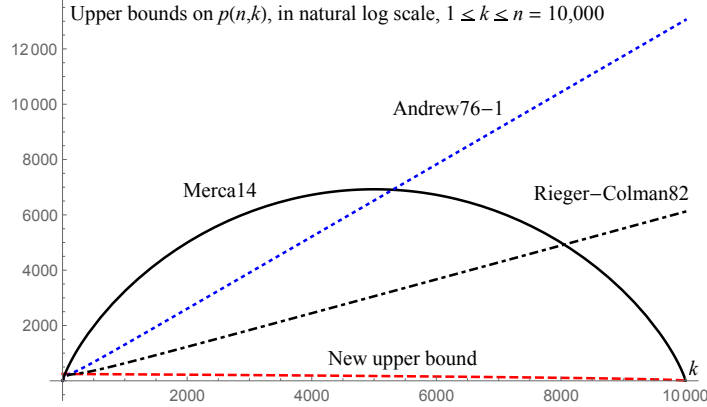


Figure 3: The new upper bound (red dashed curve closest to the horizontal axis) versus the previous upper bounds, where $n = 10,000$ and $1 \leq k \leq n-1$.

On the other hand, if $\frac{n}{2} < k < n$, then $-\ln n \leq \ln \frac{n-k}{k}$, $-\frac{1}{2} \ln 2\pi n \leq -\frac{1}{2} \ln 2\pi k$, and $n-k \leq \frac{n}{2}$. Therefore if $\frac{n}{2} < k < n$ and

$$\pi \sqrt{\frac{n}{3}} \leq k - \frac{1}{24} - \ln n - \frac{1}{2} \ln 2\pi n + (k-1) \ln \frac{1}{2} - \ln 5.44 \quad (55)$$

then (50) also holds. Isolating k , we find

$$\frac{\pi \sqrt{\frac{n}{3}} + \frac{3}{2} \ln n + \frac{1}{24} - \ln 2.77\sqrt{2\pi}}{1 - \ln 2} \leq k < n. \quad (56)$$

This proves that (50) also holds if $n/2 < k \leq n-1$ and the statement follows. \square

Remark 5. The new upper bound, $U_{\text{new}}(n, k)$ (dashed curve) is plotted against $U_a(n, k)$ (dotted curve), $U_{M1}(n, k)$ (solid curve), and $U_r(n, k)$ (dotted and dashed curve), for $n = 10,000$ and $1 \leq k \leq n-1$ in natural log scale in Figure 3. It is seen that the new upper bound remains negligible as compared to the previous upper bounds throughout the interval of interest. It is also seen that $U_r(n, k)$ is tighter than $U_a(n, k)$ and this agrees with the result that was established in Section 2. \square

Remark 6. That $U_{\text{new}}(n, k)$ is also tighter than $U_r(n, k)$ can be proved using a similar approach. The upper bound, $U_r(n, k)$ in inequality in (6) can be put into the same form as the inequality in (47) using the factorial

inequality $r! < \sqrt{2\pi r} \left(\frac{r}{e}\right)^r e^{\frac{1}{12r}}$ as shown below

$$U_r(n, k) \geq \frac{e^{2k-1}}{\sqrt{4\pi^2 k(k-1)} e^{\frac{1}{12k} + \frac{1}{12(k-1)}} k} \left(\frac{n}{k(k-1)} + \frac{k-3}{4(k-1)} \right)^{k-1}. \quad (57)$$

It can then similarly be shown that the inequality

$$\frac{5.44e^{\pi\sqrt{\frac{2(n-k)}{3}}}}{n-k} \leq \frac{e^{2k-1}}{\sqrt{4\pi^2 k(k-1)} e^{\frac{1}{12k} + \frac{1}{12(k-1)}} k} \left(\frac{n}{k(k-1)} + \frac{k-3}{4(k-1)} \right)^{k-1}, \quad (58)$$

where the term on the left is $U_{\text{new}}(n, k)$, is satisfied if

$$21.7501\sqrt{0.3208n - 0.7705\ln n + 7.8957} + 2.4015\ln n - 62.5556 \leq k < n. \quad (59)$$

Thus, in this case, $U_{\text{new}}(n, k) \leq U_r(n, k)$ when k is between $O(\sqrt{n})$ and $O(n)$. \square

4 Partitions With At Most k Parts

The Hardy-Ramanujan-Rademacher type bound for $p(n, k)$ can be extended to $p^*(n, k)$ by combining Corollary 1 with the following obvious identity [Szekeres51]³

$$p^*(n, k) = \sum_{j=1}^k p(n, j). \quad (60)$$

Thus, the following inequality must hold:

$$p^*(n, k) \leq 5.44 \sum_{j=1}^k \frac{e^{\pi\sqrt{\frac{2(n-j)}{3}}}}{n-j}. \quad (61)$$

Replacing the sum by an integral and computing it gives

$$p^*(n, k) \leq 5.44 \int_{x=0}^k \frac{e^{\pi\sqrt{\frac{2(n-x)}{3}}}}{n-x} dx \quad (62)$$

$$\leq 10.88 \left(\text{Ei} \left(\pi\sqrt{\frac{2n}{3}} \right) - \text{Ei} \left(\pi\sqrt{\frac{2(n-k)}{3}} \right) \right). \quad (63)$$

³In [Szekeres51] $p^*(n, k)$ is denoted by $P(n, k)$.

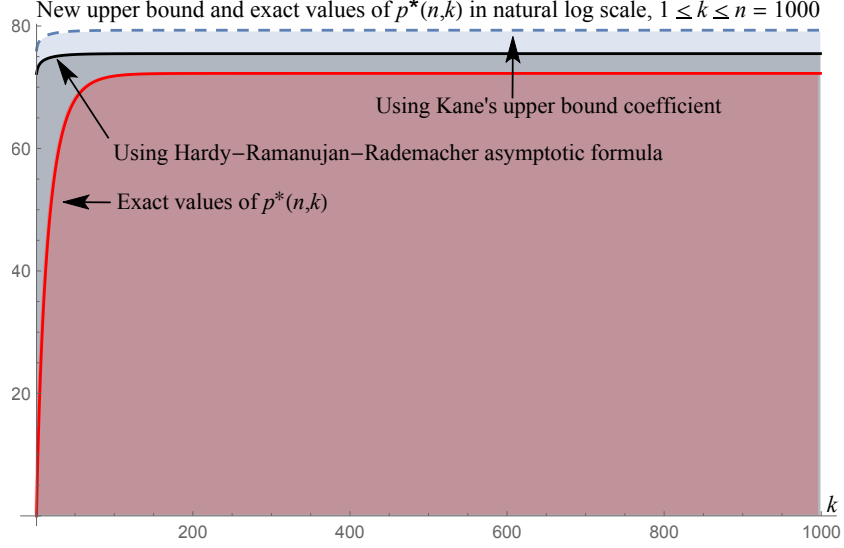


Figure 4: The new upper bound and exact values of $p^*(n, k)$.

Now, using the following two-sided inequality⁴,

$$-e^x \ln \left(1 - \frac{1}{x} \right) \leq \text{Ei}(x) \leq -\frac{1}{2} e^x \ln \left(1 - \frac{2}{x} \right), \quad x > 2, \quad (64)$$

we obtain the following upper bound for $p^*(n, k)$,

$$p^*(n, k) \leq 10.88 \left(e^{\pi \sqrt{\frac{2(n-k)}{3}}} \ln \left(1 - \frac{1}{\pi \sqrt{\frac{2(n-k)}{3}}} \right) - \frac{1}{2} e^{\pi \sqrt{\frac{2n}{3}}} \ln \left(1 - \frac{2}{\pi \sqrt{\frac{2n}{3}}} \right) \right). \quad (65)$$

Furthermore, replacing 10.88 by $\frac{1}{2\sqrt{3}}$ gives an asymptotic formula for $p^*(n, k)$, $1 \leq k \leq n - 1$. Both the upper bound and asymptotic formula are plotted against $p^*(n, k)$ for $n = 1000$ and $1 \leq k \leq n - 1$ in Figure 4. It is seen that the upper bound and asymptotic values for $p^*(n, k)$ are not as tight as those for $p(n, k)$ even though they track the exact values as curves. This is expected as $p^*(n, k)$ is obtained by replacing the sum $\sum_{j=1}^k p(n, j)$ with the sum $\sum_{j=1}^k p(n - j)$ and the error term $p(n - j) - p(n, j)$ accumulates as k tends to $n - 1$. Thus, a tighter upper bound for $p^*(n, k)$ will likely involve a different method of computation.

⁴We combined identity 5.1.20 for the exponential integral $E_1(x)$ on p. 229 in [Abromowitz-Stegun72] with the relation $\text{Ei}(x) = -E_1(-x)$ to derive this inequality.

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