ABSTRACT

Title of dissertation: REGULARITY OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR PIECEWISE EXPANDING UNIMODAL MAPS

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Doctor of Philosophy, 2014

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This dissertation consists of two parts. In the first part, we consider a piecewise expanding unimodal map (PEUM) $f : [0, 1] \to [0, 1]$ with $\mu = \rho dx$ the (unique) SRB measure associated to it and we show that ρ has a Taylor expansion in the Whitney sense. Moreover, we prove that the set of points where ρ is not differentiable is uncountable and has Hausdorff dimension equal to zero. In the second part, we consider a family $f_t : [0, 1] \to [0, 1]$ of PEUMs with μ_t the correspoding SRB measure and we present a new proof of [3] when considering the observables in $C^1[0, 1]$. That is, $\Gamma(t) = \int \phi d\mu_t$ is differentiable at $t = 0$, with $\phi \in C^1[0, 1]$, when assuming $J(c) = \sum_{k=0}^{\infty} \frac{v(f^k(c))}{Df^k(f(c))}$ is zero. Furthermore, we show that in fact $\Gamma(t)$ is never differentiable when $J(c)$ is not zero and we give the exact modulus of continuity.

REGULARITY OF MEASURES FOR PIECEWISE EXPANDING UNIMODAL MAPS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2014

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Chapter 1: Introduction

Sinai-Ruelle-Bowen (SRB) measures play important role in the study of statistical properties of dynamical systems. Let T be a map of a manifold M preserving a measure μ. A point x is called μ-regular if for every continuous function ϕ we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n x) \to \mu(\phi).$$

A measure μ is called SRB if the set of μ regular points has positive Lebesgue measure. In other words the SRB measure describes statistics of a Lebesgue positive measure set of initial conditions.

For example, if T preserves an absolutely continuous invariant measure which is ergodic then that measure is SRB.

Another case where SRB measures are known to exist is when some hyperbolicity is present. In particular, when the system is uniformly hyperbolic, for example, for topologically transitive Axiom A diffeomorphisms or for smooth expanding maps SRB measures exists are unique and have good statistical properties such as the Central Limit Theorem (CLT) [23]. The CLT states that if A is a Holder continuous function and x is chosen uniformly with respect to the Lebesgue measure then

$$\lim_{N\to\infty}\mathbb{P}\left(\frac{\sum_{n=0}^{N-1}\phi(T^n x)-n\mu(A)}{\sigma(A,T)\sqrt{N}}\le z\right)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$$

where the diffusion coefficient $\sigma(A,T)$ is defined by

$$\sigma^2(A,T) = \mu(A^2) + 2\sum_{n=1}^{\infty} \mu(A(A \circ T^n)).$$

In case A is smooth (rather than just Holder continuous) the normalizing constants

$$\mu_{SRB}(A,T) \text{ and } \sigma^2(A,T)$$

depend smoothly on T.

This smoothness plays important role in averaging theory including some problems of statistical mechanics [7, 9, 10, 17].

Uniformly hyperbolic systems appear rarely in applications. Much more common are systems which are either nonuniformly hyperbolic on the set of large measure (notable examples are quadratic family [11] and Henon family [6]) or are hyperbolic but have singularities (notable examples are Lorenz system [22] and Lorentz gas [8]).

While uniformly hyperbolic system provides us with a good understanding on what happens for more general chaotic maps, in the sense that many results first proven in the uniformly hyperbolic setting hold under much weaker conditions (see [23]) the families of uniformly hyperbolic maps are not good models for predicting what happens with more general families. Therefore our understanding of paramter dependence of invariant measures in weakly hyperbolic systems are quite poor. To remedy this situation David Ruelle suggested to look at families of piecewise expanding unimodal maps.

We call a map $f : [0,1] \to [0,1]$ piecewsie expanding unimodal map (PEUM) if there is a point c and two maps f_1 defined on $[0, c+\varepsilon]$ and f_2 defined on $[c-\varepsilon, 1]$

such that $f_1(c) = f_2(c)$ and there is a constant $\lambda > 1$ such that $|f_j'(x)| \geq \lambda$ for all x from the domain of f_j and

$$f(x) = \begin{cases} f_1(x) & \text{if } x \leq c \\ \\ f_2(x) & \text{if } x \geq c. \end{cases} \tag{1.1}$$

PEUMs have unique absolutely continuous invariant measure [15] which is ergodic (in fact it is mixing and even exponentially mixing [2, 23]) so it is the SRB measure for this system.

Several papers have been devoted to studying regularity of SRB absolutely continuous invariant measures in families of PEUMs. In particular some sufficient conditions for regular dependence of SRB measures on parameters have been found (those conditions however are exceptional in the sense that they do not hold for typical families).

The work also deals with families of PEUMs. We study regularity of both the density as a function of x and the regularity of $\mu_{SRB}(\phi, f)$ as a function of f. We have two (related) goals. First, we strive to provide explicit useful formulas for the change of quantities of interest. Secondly we would like to capture the exact modulus of continuity of these quantities. Before presenting our results let us review related papers.

1.1 Previous work

As described at the beginning, if $f_t : [a,b] \to [a,b]$ is a smooth one-parameter family of PEUMs with μ_t the (unique) SRB measure associated to each f_t, then we want to study the regularity of $\Gamma(t) = \int \phi \, d\mu_t$, with $\phi \in BV[0,1]$, as t approaches 0.

In [18] and [19], Ruelle considered the case $v = X \circ f$ and suggested a candidate for the derivative at $t = 0$, namely $\Psi(1)$ where

$$\Psi(t) = \sum_{n=0}^{\infty} \int z^n X(y) \partial_y (\phi \circ f^n)(y) d\mu_0(y).$$

In [1], Baladi studied properties of the complex function $\Psi(z)$, based on spectral perturbation theory for transfer operators. She found a different way to express $\Psi(1)$. The latter was used by Baladi and Smania in [3] to give sufficient conditions for the differentiability of $\Gamma(t)$ at $t = 0$ (the differentiability does not always hold as shown in [16], [1]). [3] also shows that the derivative equals $\Psi(1)$. Besides some routine differentiability and irreducibility assumptions on the family f_t, one important assumption is that the quantity

$$J(c,f) = \sum_{k=0}^{\infty} \frac{v(f^k(c))}{Df^k(f(c))}$$

equals zero. A different proof of differentiability is given in [4]. Their approach is based on arguments from thermodynamic formalism and it does not require any analysis based on the decomposition of ρ into its regular and saltus parts (definitions of these concepts will be explained later) used before in [3]. They also prove that if $J(c, f_t) = 0$ for all t and f and ϕ are sufficiently smooth then $\Gamma(t)$ has derivatives of higher order.

In this work we give yet another proof of the Baladi-Smania result. Our approach is more elementary. Also, it gives a more explicit expression for the derivative (in the style of [14]) comparing to Ruelle-Baladi-Smania approach. Our proof is presented in Section 4.

In the proof we keep a careful track of the places where the assumption $J(c, f) = 0$ is used. This allows to study later the regularity of Γ when $J(c, f)$ is not zero. The latter uses results from [20] and [21] which turn out to be very useful to conclude that $\Gamma(t)$ has modulus of continuity $t\sqrt{|\log(t)\log\log|\log(t)||}$ for almost all $t > 0$. This improves on Keller and Liverani's modulus of continuity, $t\log(t)$ obtained in [13].

Also, as we mentioned before, in [1], Baladi shows that there is a one-parameter family such that $J(c) \neq 0$ and concludes that $\Gamma(t)$ is not Lipschitz (and then it cannot be differentiable) for such a particular family. In our case, we fix a one-parameter family f_t and we show that $t \to \Gamma(t)$ is not Lipschitz for almost all t. This is discussed in Section 5.

Finally, Baladi [1] shows that $\rho' \in BV[0,1]$. We prove a stronger result which shows that ρ has a Taylor expansion in the sense of Whitney. Furthermore, the set of points where ρ is not differentiable is an uncountable set and has Hausdorff dimension equal to zero. These two facts are presented in Section 3.

We will start by formulating our results in Section 3. Some auxiliary facts which are of independent interest are presented in Section 4.

1.2 Results

1.2.1 Regularity of density

As in [3], we decompose the densities ρ_t of f_t as $\rho_t = \rho_{r,t} + \rho_{s,t}$, where $\rho_{r,t}$ and $\rho_{s,t}$ are continuous and discontinuous functions respectively (the definitions of these will be given in Section 5). Define $BV_1 = \{\phi \in BV \ : \ \phi' \in BV\}$. That is, ϕ' is equal almost everywhere to a BV function.

In all the results below, unless stronger conditions are explicitly imposed, we assume that f_1 and f_2 are C^2.

Proposition 1.2.1. *[1] The regular part of ρ is in BV_1.*

As a generalization of Proposition 1.2.1, we have the next two new results.

Theorem 1.2.2. *Assume that f_1 and f_2 are C^{k+2}. Then, there is a sequence of functions $\rho_0, \rho_1, \ldots, \rho_k \in BV$ such that $\rho_0 = \rho$ and for $j < k$, $\rho_j' = \rho_{j+1}$.*

Theorem 1.2.3. *The regular part of ρ is absolutely continuous. That is*

$$\rho_r(x_2) - \rho_r(x_1) = \int_{x_1}^{x_2} \rho'(x)dx.$$

Theorem 1.2.4. *(a) The set of points where ρ is non differentiable has Hausdorff dimension zero.*

(b) If the critical orbit is dense then the set of points where ρ is non differentiable is uncountable.

Theorem 1.2.5. *Let $k \geq 1$ and assume that f_1 and f_2 are of class C^{k+2}. Then, there is a set $\mathcal{N}_k$ such that $\mathcal{HD}(\mathcal{N}_k) = 0$ and ρ is k differentiable in the sense of Whitney on $[0,1] - \mathcal{N}_k$. That is, if $\bar{x} \notin \mathcal{N}_k$ then*

$$\rho(x) - \rho(\bar{x}) = \sum_{m=1}^{k} \frac{\rho_m(\bar{x})}{m!}(x-\bar{x})^m + o((x-\bar{x})^k).$$

Note that since $[0,1] - \mathcal{N}_k$ is not closed, ρ in general can not be extended to a smooth function on $[0,1]$.

The proofs of the above results will be given in Sections 2 and 3.

1.2.2 Regularity of the measure

Recall that expanding unimodal maps are defined by formula (1.1). Now we consider families of such maps. Namely, we assume that $f_{1,t}(x)$ is defined for $(t,x) \in [-\varepsilon, \varepsilon] \times [0, c+\varepsilon]$ and $f_{2,t}(x)$ is defined for $(t,x) \in [-\varepsilon, \varepsilon] \times [c-\varepsilon, 1]$ and that $f_{j,t}(x)$ are C^2 functions of their arguments. Then we let

$$f_t(x) = \begin{cases} f_{1,t}(x) & \text{if } x \leq c \\ f_{2,t}(x) & \text{if } x \geq c. \end{cases}$$

Thus we assume that c is a common critical point for all t. This assumption does not cause a loss of generality since it can always be achieved by t dependent change of variables.

Set

$$J(x,f) = \sum_{k=1}^{\infty} \frac{v(f^k x)}{Df^k(fx)}.$$

Note that J is two valued if x is precritical, that is $f^k x = c$ for some k, however it is well defined at all other points.

Let $u = \frac{v}{Df}$. Then, in Section 4, we prove the following theorem.

Theorem 1.2.6. *Let $\mu_t = \rho_t dx$ be the (unique) a.c.i.m of f_t and suppose (f_0, μ_0) is mixing. If $J(c, f_0) = 0$ then, for any $\phi \in C^1([0,1])$, the function $\Gamma(t) = \int \phi(x)\rho_t(x)\, dx$ is differentiable at $t = 0$. Moreover*

$$\lim_{t\to 0} \frac{\Gamma(t) - \Gamma(0)}{t} = \Delta_c + \Delta_d \tag{1.2}$$

where

$$\Delta_c = \sum_{k=1}^{\infty} \int \phi(f^k x)\left(u - u'\frac{\rho'}{\rho}\right)(x)\, d\mu_0,$$

$$\Delta_d = -\sum_{k=1}^{\infty} \phi(f^k c)\left[J_k(c)\rho(c)\left(\frac{1}{\lambda_L(c)} + \frac{1}{\lambda_R(c)}\right)\right]$$

if c is not periodic and

$$\Delta_d = -\sum_{k=1}^{p} \phi(f^k c)\left[\frac{J_k(c)\rho_L(c)}{\lambda_L(c)} + \frac{J_k(c)\rho_R(c)}{\lambda_R(c)}\right]$$

if c is periodic with minimal period p.

Note that $J(c, f)$ is in general multivalued if c is periodic so we discuss in Section 4.6 how the condition $J(c, f_0) = 0$ should be understood in the periodic case.

In the case $J(c, f)$ is not zero, we do not have differentiability for $\Gamma(t)$, as we state in the next theorem and prove in Section 5

Theorem 1.2.7. *Suppose that* $|J(c, f_t)| > \epsilon_1$ *for* $t \in (-\epsilon, \epsilon)$. *Then, for generic* $\phi \in C^1[0,1]$ *and for almost all* t *the following limit exists and is non-zero*

$$\limsup_{t \downarrow 0} \frac{\Gamma_t(\phi) - \Gamma_0(\phi)}{t\sqrt{|\log(t) \log\log|\log(t)||}}.$$

Chapter 2: Derivatives of the density.

2.1 Auxilary facts.

Lemma 2.1.1. *Let $g(s,r)$ be a function from $(\mathbb{Z}\cup\{0\})\times(\mathbb{Z}\cup\{0\})$ to $\mathbb{R}$. Suppose the series $\sum_{i=1}^{\infty}\sum_{j=0}^{i-1} g(i-j,j)$ converges absolutely. Then,*

$$\sum_{i=1}^{\infty}\sum_{j=0}^{i-1} g(i-j,j) = \sum_{c=1}^{\infty}\sum_{d=0}^{\infty} g(c,d)$$

Proof. Splitting the series $\sum_{i=1}^{\infty}\sum_{j=0}^{i-1} g(i-j,j)$, we have that

$$\sum_{i=1}^{\infty}\sum_{j=0}^{i-1} g(i-j,j) = g(1,0) + \Big(g(2,0)+g(1,1)\Big) + \Big(g(3,0)+g(2,1)+g(1,2)\Big) + \cdots$$

Using the series converges absolutely, after reordering the terms, it can be written as

$$
\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=0}^{i-1} g(i-j,j) &= g(1,0)+g(2,0)+g(3,0)+\cdots \\
&+ g(1,1)+g(2,1)+g(3,1)+\cdots \\
&+ g(1,2)+g(2,2)+g(3,2)+\cdots \\
&. \\
&. \\
&. \\
&= \sum_{c=1}^{\infty} g(c,0)+\sum_{c=1}^{\infty} g(c,1)+\sum_{c=1}^{\infty} g(c,2)+\cdots
\end{aligned}
$$

The last expression is just $\sum_{c=1}^{\infty}\sum_{d=0}^{\infty} g(c,d)$ proving the lemma. □

Denote by $\xi(z) = \frac{D^2 f(z)}{Df(z)}$. In the arguments of this section we will need to represent $D(|Df^m y|)$ as a sum. Namely we have

$$
D(|Df^m y|) = \frac{|Df^m y|}{Df^m y}\sum_{j=0}^{m-1} \xi(f^j y) Df^j y \text{ and } D(Df^m y) = \sum_{j=0}^{m-1} \xi(f^j y) Df^j y.
$$

Both formulas are easy consequnces of the chain rule.

Let $\widetilde{\mathcal{L}}(\psi)(x) = \sum_{f(y)=x} \frac{\psi(y)}{Df^n(y)|Df^n(y)|}$. More generally we shall use the following notation.

Definition 2.1.2. *For $\phi \in BV$, define the operator $\overset{m}{\widetilde{\mathcal{L}}}(\phi)$ by*

$$
\overset{m}{\widetilde{\mathcal{L}}}(\phi)(x) = \sum_{f(y)=x} \frac{\phi(y)}{(Df(y))^m |Df(y)|},
$$

where m is a nonnegative integer.

Definition 2.1.3. *Let k, $i_1, \dots, i_k$ and $m_1 > \cdots > m_k$ be positive integers. For functions $h_1, \dots, h_k$, define $\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}$ at $(h_1, \dots, h_k)$ by*

$$\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1, \dots, h_k) = \left(\overset{m_1}{\widetilde{\mathcal{L}}}\right)^{i_1}\left(h_1\left(\overset{m_2}{\widetilde{\mathcal{L}}}\right)^{i_2}\left(h_2 \cdots \left(\overset{m_k}{\widetilde{\mathcal{L}}}\right)^{i_k}(h_k)\right)\right).$$

We shall often use

Proposition 2.1.4.

$$\|\mathfrak{D}^{i_1,i_2,\dots,i_k}_{m_1,m_2,\dots,m_k}(h_1, \dots, h_k)\| \leq M(\lambda^{-i_1})^{m_1}(\lambda^{-i_2})^{m_2} \cdots (\lambda^{-i_k})^{m_k}$$

Proof. We use induction on k. For $k = 1$ we have

$$\left| \overset{m}{\widetilde{\mathcal{L}}}{}^i (h)(x)\right| = \left| \sum_{f^i y = x} \frac{h}{(Df^i(y))^m |Df^i(y)|} \right| \leq \sum_{f^i y = x} \left| \frac{|h|}{|Df^i(y)|^m |DF^i(y)|} \right|$$

$$\leq \frac{|h|}{\lambda^{im}} |\mathcal{L}^i(1)(x)| \leq \frac{C||h||}{\lambda^{im}}.$$

Now, supposing the result is true for $k-1$, we have

$$\begin{aligned}
\mathfrak{D}^{i,i_2,\dots,i_k}_{m,m_2,\dots,m_k}(h_1, \dots, h_k)(x)| &= \left(\overset{m}{\widetilde{\mathcal{L}}}\right)^i (h_1\, \mathfrak{D}^{i_2,\dots,i_s}_{m_2,\dots,m_s}(h_2, \dots, h_s)) \\
\leq \| \sum_{f^i y = x} \frac{h_1(y)\, \mathfrak{D}^{i_2,\dots,i_s}_{m_2,\dots,m_s}(h_2, \dots, h_s)(y)}{(Df^i(y))^m |Df^i(y)|} \| &\leq \lambda^{-im} \sum_{f^i y = x} \frac{\|h_1\, \mathfrak{D}^{i_2,\dots,i_s}_{m_2,\dots,m_s}(h_2, \dots, h_s)\|}{|Df^i(y)|} \\
\leq \lambda^{-im} \|h_1\, \mathfrak{D}^{i_2,\dots,i_s}_{m_2,\dots,m_s}(h_2, \dots, h_s)\| \|\mathcal{L}^i(1)\| &\leq \widetilde{M}(\lambda^{-i})^m \|\mathfrak{D}^{i_2,\dots,i_k}_{m_2,\dots,m_k}(h_2, \dots, h_k)\| \\
\leq \widetilde{K}(\lambda^{-i})^m \widetilde{M}(\lambda^{-i_2})^{m_2} \cdots (\lambda^{-i_s})^{m_s} &\leq M(\lambda^{-i})^m(\lambda^{-i_2})^{m_2} \cdots (\lambda^{-i_s})^{m_s}
\end{aligned}$$

Thus the claim holds for k and the Proposition is proven by induction. □

Lemma 2.1.5. *If* $\sum_{k=1}^{n} g_k \to g$ *in* BV *and* $\sum_{k=1}^{n} g'_k \to h$ *in* L_1- *then* $g' = h$ *a.e.*

Proof. Let $\epsilon > 0$. Then, there exists $N > 0$ such that, for all $n \geq N$,

$$\|g - \sum_{k=1}^{n} g_k\|_{BV} \leq \epsilon$$

Since $\|f'\|_{L_1} \leq \|f\|_{BV}$ for any function $f \in BV$, then

$$\|g' - \sum_{k=1}^{n} g'_k\|_{L_1} \leq \|g - \sum_{k=1}^{n} g_k\|_{BV} \leq \epsilon$$

Therefore, $\sum_{k=1}^{n} g'_k$ converges to g' in L_1, so then $g' = h$ as claimed. □

2.2 Explicit formulas for derivatives.

Define

$$\rho_1 = -\sum_{i=1}^{\infty} \widetilde{\mathcal{L}}^i(\xi \cdot \rho).$$

Lemma 2.2.1. *(a) Let* ρ *be the density of the invariant measure of* f*. Then,* $\rho' = \rho_1$ *almost everywhere.*

(b) $(\mathcal{L}^n 1)'(x)$ *converges to* $\rho_1(x)$ *uniformly for* x *which are not on the orbit of* c*.*

Proof. Since ρ is a fixed point of $\mathcal{L}$, then $\rho = \mathcal{L}^n(\rho)$ for all n. Because ρ is of bounded variation so is $\mathcal{L}^n(\rho)$, hence both are differentiable almost everywhere. In fact, differentiating both sides, we get $\rho' = (\mathcal{L}^n\rho)'$ almost every where. Next if $h \in BV$ then

$$(\mathcal{L}^n h)'(x) = \sum_{f^n y = x} \frac{h'(y)}{Df^n(y)|Df^n(y)|} - \sum_{f^n(y)=x} \frac{h(y)\cdot D(|Df^n(y)|)}{|Df^n(y)|^2} \qquad \text{a. e.}$$

In view of Proposition 2.1.4, $\displaystyle\sum_{f^n y = x} \frac{h'(y)}{Df^n(y)|Df^n(y)|} = \tilde{\mathcal{L}}^n(h')$ is bounded by λ^{-n} so it gets very small as n increases. Thus we focus on $\displaystyle\sum_{f^n(y)=x} \frac{h(y)\cdot D(|Df^n(y)|)}{|Df^n(y)|^2}$. Assuming that $y \notin \{c, f(c), \dots, f^{n-1}(c)\}$ for each y with $f^n y = x$ we have

$$\begin{aligned}
\sum_{f^n(y)=x} \frac{h(y)\cdot D(|Df^(y)|)}{|Df^n(y)|^2} &= \sum_{f^n(y)=x} \frac{h(y)}{|Df^n(x)|^2} D\left(\prod_{a=0}^{n-1} |Df(f^a y)|\right) \\
&= \sum_{f^n(y)=x} \frac{h(y)}{|Df^n(y)|^2} \frac{|Df^n(y)|}{Df^n(y)} \sum_{a=0}^{n-1} \xi(f^a(y)) Df^a(y) \\
&= \sum_{f^n(y)=x} \frac{h(y)}{|Df^n(y)|} \sum_{a=0}^{n-1} \frac{\xi(f^a(y))}{Df^{n-a}(f^a(y))} \\
&= \sum_{a=0}^{n-1} \sum_{f^{n-a}(z)=x} \frac{\xi(z)}{Df^{n-a}(z)} \sum_{f^a(y)=z} \frac{h(y)}{|Df^n(y)|}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{a=0}^{n-1} \sum_{f^{n-a}(z)=x} \frac{\xi(z)}{Df^{n-a}(z)|Df^{n-a}(z)|} \sum_{f^a(y)=z} \frac{h(y)}{|Df^a(y)|} \\
&= \sum_{a=0}^{n-1} \sum_{f^{n-a}(z)=x} \frac{\xi(z)}{Df^{n-a}(z)|Df^{n-a}(z)|} \mathcal{L}^a(h)(z) \\
&= \sum_{a=0}^{n-1} \widetilde{\mathcal{L}}^{n-a}(\xi(\mathcal{L}^a h))(x) \\
&= \sum_{i=1}^{n} \widetilde{\mathcal{L}}^{i}(\xi(\mathcal{L}^{n-i} h))(x)
\end{aligned}$$

Proposition 2.1.4 shows that we can take the limit $n \to \infty$ term-by-term. Since $\displaystyle\lim_{n\to\infty} (\mathcal{L}^{n-i}h)(x) = \left(\int_0^1 h(z)dz\right)\rho(x)$ both parts (a) and (b) follow. $\square$

Proposition 2.2.2. *The function ρ_1 is almost everywhere differentiable and*

$$\rho_1' = 3\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}(\xi\widetilde{\mathcal{L}}^{j}(\xi\rho)) + 2\sum_{i=1}^{\infty}\left(\widetilde{\widetilde{\mathcal{L}}}\right)^{i}\left(\xi^2\rho\right) - \sum_{i=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}(\xi'\rho) \tag{2.1}$$

In particular, there exists $\rho_2 \in BV$ such that $\rho_1' = \rho_2$ almost everywhere.

Proof. By Lemma 2.2.1 $\rho_1 = -\sum_{i=1}^{\infty}\widetilde{\mathcal{L}}^{i}(\xi\rho)$ almost everywhere. Therefore by Lemma 2.1.5

$$\rho_1'(x) = -\sum_{i=1}^{\infty}\left(\sum_{f^i y = x}\frac{\xi(y)\rho(y)}{Df^i(y)|Df^i(y)|}\right)' = -\sum_{i=1}^{\infty}\sum_{f^i y=x}\left(\frac{\xi(y)\rho(y)}{Df^i(y)|Df^i(y)|}\right)'$$

almost everywhere. Decompose

$$\left(\frac{\xi(y)\rho(y)}{Df^i(y)|Df^i(y)|}\right)' = \underbrace{\frac{(\xi(y)\rho(y))'}{Df^i(y)|Df^i(y)|}}_{(I)} - \underbrace{\frac{\xi(y)\rho(y)(Df^i(y)|Df^i(y)|)'}{(Df^i(y))^2|Df^i(y)|^2}}_{(II)}.$$

Let us first work on (I). We have

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{f^i y=x}(I) &= \sum_{i=1}^{\infty}\sum_{f^i y=x}\frac{\xi'(y)Dy\rho(y)+\xi(y)\rho'(y)Dy}{Df^i(y)|Df^i(y)|}\\
&= \sum_{i=1}^{\infty}\sum_{f^i y=x}\left(\frac{\xi'(y)\rho(y)}{Df^i(y)|Df^i(y)|}+\frac{\xi(y)\rho'(y)}{Df^i(y)|Df^i(y)|}\right)\\
&= \sum_{i=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}(\xi'\rho+\xi\rho')(x)
\end{aligned}$$

By Lemma 2.2.1 $-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}(\xi\rho') = \sum_{i=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}(\xi\widetilde{\mathcal{L}}^{j}(\xi\rho))$. Therefore

$$\sum_{i=1}^{\infty}\sum_{f^i y=x}(I) = \sum_{i=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}(\xi'\rho) - \sum_{i=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}(\xi\widetilde{\mathcal{L}}^{j}(\xi\rho)). \tag{2.2}$$

Now, let us analyze (II).

$$\sum_{i=1}^{\infty}\sum_{f^iy=x}(II)=\sum_{i=1}^{\infty}\sum_{f^iy=x}\frac{\xi(y)\rho(y)[D(Df^iy)|Df^iy|+D(|Df^iy|)(Df^i)(y)]}{(Df^iy)^2|Df^iy|^2}(Df^iy)^2|Df^iy|^2$$

$$=\sum_{i=1}^{\infty}\sum_{f^iy=x}\frac{\xi(y)\rho(y)\left[2|Df^iy|\sum_{j=0}^{i-1}\xi(f^jy)Df^jy\right]}{(Df^iy)^2|Df^iy|^2}=2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\sum_{f^iy=x}\frac{\xi(y)\rho(y)Df^jy\xi(f^jy)}{(Df^iy)^2|Df^iy|}$$

By making the change of variable $z=f^jy$, we obtain that

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{f^iy=x}(II) &= 2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\sum_{f^iy=x}\frac{\xi(y)\rho(y)(Df^j)(y)\xi(z)}{(Df^iy)^2|Df^iy|}\\ &= 2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\sum_{f^iy=x}\frac{\xi(y)\rho(y)(Df^j)(y)\xi(z)}{(Df^{i-j}z)^2(Df^jy)^2|Df^{i-j}z||Df^jy|}\\ &= 2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\sum_{f^iy=x}\frac{\xi(y)\rho(y)\xi(z)}{(Df^{i-j}z)^2Df^jy|Df^{i-j}z||Df^jy|}\\ &= 2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\sum_{f^{i-j}z=x}\frac{\xi(z)}{(Df^{i-j}z)^2|Df^{i-j}z|}\sum_{f^iy=z}\frac{\xi(y)\rho(y)}{Df^jy|Df^jy|}\end{aligned}$$

$$=2\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\left(\widetilde{\widetilde{\mathcal{L}}}\right)^{i-j}\left(\xi\widetilde{\mathcal{L}}^j(\xi\rho)\right).$$

By Lemma 2.1.1

$$\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\left(\widetilde{\widetilde{\mathcal{L}}}\right)^{i-j}\left(\xi\widetilde{\mathcal{L}}^j(\xi\rho)\right)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(\widetilde{\widetilde{\mathcal{L}}}\right)^{i-j}\left(\xi\widetilde{\mathcal{L}}^j(\xi\rho)\right).$$

Therefore

$$\sum_{i=1}^{\infty}\sum_{f^iy=x}(II)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(\widetilde{\widetilde{\mathcal{L}}}\right)^{i-j}\left(\xi\widetilde{\mathcal{L}}^j(\xi\rho)\right). \tag{2.3}$$

Combining (2.2) and (2.3), we finally obtain

$$\rho_1'=-\sum_{i=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^i(\xi'\rho)+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^i(\xi\widetilde{\mathcal{L}}^j(\xi\rho))+\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(\widetilde{\widetilde{\mathcal{L}}}\right)^j\left(\xi\widetilde{\mathcal{L}}^j(\xi\rho)\right)$$

$$= 3\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}\left(\xi\widetilde{\mathcal{L}}^{j}(\xi\rho)\right) + 2\sum_{i=1}^{\infty}\left(\widetilde{\widetilde{\mathcal{L}}}\right)^{i}\left(\xi^{2}\rho)\right) - \sum_{i=1}^{\infty}\widetilde{\widetilde{\mathcal{L}}}^{i}\left(\xi'\rho\right)$$

almost everywhere as claimed. □

2.3 Repeated derivatives of the density function

Lemma 2.2.1 shows that ρ' is in BV and so $\rho \in BV_1$. Then we saw in Proposition 2.2.2 that $\rho_1' = \rho_2 \in BV$. Here we show that these results can be extended to repeated differentiation of arbitrary order. We start with the following general result.

Proposition 2.3.1. *Let $k,\ i_1,\ldots,i_k$ and $m_1 > \cdots > m_k$ be positive integers with $i_1,\ldots,i_k \geq 1$. Let $h_1,\ldots,h_k$ be BV functions whose derivatives are in L^∞.*

(a) If $n \geq 1$, the derivative of $\displaystyle\sum_{k\leq i_1+\cdots+i_k\leq n}\mathfrak{D}^{i_1,\ldots,i_k}_{m,m_2,\ldots,m_k}(h_1,\ldots,h_k)$ is a finite sum of functions of the type $\displaystyle\sum_{\widetilde{k}\leq\widetilde{i}_1+\cdots+\widetilde{i}_{\widetilde{k}}\leq n}\mathfrak{D}^{\widetilde{i}_1\ldots,\widetilde{i}_{\widetilde{k}}}_{m+1,\widetilde{m}_2,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{k}})$, where $k \leq \widetilde{k} \leq k+1$,$\widetilde{i}_1,\ldots,\widetilde{i}_{\widetilde{k}} \geq 1$ and $\widetilde{m}_1 > \cdots > \widetilde{m}_{\widetilde{k}}$ are positive integers and $\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{k}} \in \{h_1,h_1^2,\ldots,h_k,h_k^2,h_1',\ldots,h_k',\xi,\xi'\}$.

(b) The derivative of $\displaystyle\sum_{i_1=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}\mathfrak{D}^{i_1,\ldots,j_k}_{m,m_2,\ldots,m_k}(h_1,\ldots,h_k)$ is a finite sum of functions of the type $\displaystyle\sum_{\widetilde{\widetilde{i}}_1}\cdots\sum_{\widetilde{\widetilde{i}}_{\widetilde{\widetilde{k}}}}\mathfrak{D}^{\widetilde{\widetilde{i}}_1\ldots,\widetilde{\widetilde{i}}_{\widetilde{\widetilde{k}}}}_{m+1,\widetilde{\widetilde{m}}_2,\ldots,\widetilde{\widetilde{m}}_{\widetilde{\widetilde{k}}}}(\widetilde{\widetilde{h}}_1,\ldots,\widetilde{\widetilde{h}}_{\widetilde{\widetilde{k}}})$, where $k \leq \widetilde{\widetilde{k}} \leq k+1$,$\widetilde{\widetilde{i}}_1,\ldots,\widetilde{\widetilde{i}}_{\widetilde{\widetilde{k}}} \geq 1$ and $\widetilde{\widetilde{m}}_1 > \cdots > \widetilde{\widetilde{m}}_{\widetilde{\widetilde{k}}}$ are positive integers and $\widetilde{\widetilde{h}}_1,\ldots,\widetilde{\widetilde{h}}_{\widetilde{\widetilde{k}}} \in \{h_1,h_1^2,\ldots,h_k,h_k^2,h_1',\ldots,h_k',\xi,\xi'\}$.

Proof. We will prove (a) by induction on m. For $m = 1$, we need to compute $\displaystyle\sum_{i=1}^{n}D\left(\widetilde{\mathcal{L}}^i(h)\right)$, so let us work on $D\left(\widetilde{\mathcal{L}}^i(h)\right)$. Then

$$D\left(\widetilde{\mathcal{L}}^i(h)\right)(x) = \sum_{f^i(y)=x}D\left(\frac{h(y)}{Df^iy|Df^iy|}\right) = \sum_{f^i(y)=x}\left[\frac{h'(y)}{Df^iy|Df^iy|} - \frac{h(y)D(Df^iy|Df^iy|)}{(Df^iy)^2|Df^iy|^2}\right]$$

$$
\begin{aligned}
&= \sum_{f^i(y)=x} \left[\frac{h'(y)Dy}{Df^iy|Df^iy|} - \frac{h(y)(D(Df^iy)|Df^iy| + Df^iyD(|Df^iy|))}{(Df^iy)^2|Df^iy|^2} \right] \\
&= \sum_{f^i(y)=x} \left[\frac{h'(y)}{(Df^iy)^2|Df^iy|} - \frac{h(y)(2|Df^iy|\sum_{j=0}^{i-1}\xi(f^j(x))Df^j(x))}{(Df^iy)^2|Df^iy|^2} \right] \\
&= \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{j=0}^{i-1}\sum_{f^i(y)=x} \frac{h(y)\xi(f^j(y))Df^j(y))}{(Df^iy)^2|Df^iy|}
\end{aligned}
$$

Let $z = f^j(y)$. Then

$$
\begin{aligned}
D\Big(\widetilde{\mathcal{L}}^i(h)\Big)(x) &= \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{j=0}^{i-1}\sum_{f^i(y)=x} \frac{h(y)\xi(f^j(y))Df^j(y))}{(Df^iy)^2|Df^iy|} \\
&= \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{j=0}^{i-1}\sum_{f^i(y)=x} \frac{h(y)\xi(z)}{(Df^{i-j}z)^2|Df^{i-j}z|(Df^jy)|Df^jy|} \\
&= \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{j=0}^{i-1}\sum_{f^{i-j}z=x} \frac{\xi(z)}{(Df^{i-j}z)^2|Df^{i-j}z|} \sum_{f^jy=z} \frac{h(y)}{(Df^jy)|Df^jy|} \\
&= \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{j=0}^{i-1} \widetilde{\widetilde{\mathcal{L}}}^{i-j}(\xi\widetilde{\mathcal{L}}^j(h))
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sum_{i=1}^{n} D\Big(\widetilde{\mathcal{L}}^i(h)\Big) &= \sum_{i=1}^{n} \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{i=1}^{n}\sum_{j=0}^{i-1} \widetilde{\widetilde{\mathcal{L}}}^{i-j}(\xi\widetilde{\mathcal{L}}^j(h)) \\
&= \sum_{i=1}^{n} \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{i=1}^{n} \widetilde{\widetilde{\mathcal{L}}}^i(\xi h) - 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} \widetilde{\widetilde{\mathcal{L}}}^{i-j}(\xi\widetilde{\mathcal{L}}^j(h)) \\
&= \sum_{1}^{n} \widetilde{\widetilde{\mathcal{L}}}^i(h')(x) - 2\sum_{i=1}^{n} \widetilde{\widetilde{\mathcal{L}}}^i(\xi h) - 2\sum_{\substack{2\le i+j\le n \\ 1\le i, 1\le j}} \widetilde{\widetilde{\mathcal{L}}}^i(\xi\widetilde{\mathcal{L}}^j(h))
\end{aligned}
$$

Therefore, the derivative is a finite sum of terms as described in the statement.

Assume the statement is true for $l < m$. Let us prove that it also holds for m. We are interested in the derivative of

$$\sum_{k+1\le i+i_1+\dots+i_k\le n} \mathfrak{D}^{i,i_2,\dots,i_k}_{m,m_1,\dots,m_k}(h,h_1,\dots,h_k) \tag{2.4}$$

with $i\ge 1, i_1\ge 1,\dots,i_k\ge 1$. For this, note that

$$\sum_{k+1\le i+i_1\cdots i_k\le n} \mathfrak{D}^{i,i_1,\dots,i_k}_{m,m_1,\dots,m_k}(h,h_1,\dots,h_k) = \sum_{i=1}^{n-k}\left(\overset{m}{\widetilde{L}}\right)^i\left(h\sum_{k\le i_1+\dots+i_k\le n-i}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)\right)$$

Thus, if we are interested in the derivative of (2.4), we can work on

$$\sum_{i=1}^{n-k} D\left[\left(\overset{m}{\widetilde{L}}\right)^i\left(h\sum_{k\le i_1+\dots+i_k\le n-i}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)\right)\right]$$

$$=\sum_{i=1}^{n-k}\sum_{f^iy=x} D\left[\frac{h(y)\ \sum_{k\le i_1+\dots+i_k\le n-i}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)}{(Df^iy)^m|Df^iy|}\right] = \sum_{i=1}^{n-k}\sum_{f^iy=x}(I)-(II)$$

where

$$(I) = \frac{D\left[h(y)\ \widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)\right]}{(Df^iy)^m|Df^iy|},$$

$$(II) = \frac{h(y)\ \widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)D\left[(Df^iy)^{2m}|Df^iy|^2\right]}{(Df^iy)^{2m}|Df^iy|^2}$$

and $\widehat{\sum}$ means $\sum_{k\le i_1+\dots+i_k\le n-i}$.

Let us first work on (II). Note that

$$D\left[(Df^iy)^m|Df^iy|\right] = m(Df^iy)^{m-1}D(Df^iy)|Df^iy| + (Df^iy)^mD(|Df^iy|)$$

$$
\begin{aligned}
&= m(Df^iy)^{m-1}|Df^iy|\sum_{j=0}^{i-1}\xi(f^jy)Df^jy + (Df^iy)^m\frac{|Df^iy|}{(Df^iy)}\sum_{j=0}^{i-1}\xi(f^jy)Df^jy \\
&= m(Df^iy)^{m-1}|Df^iy|\sum_{j=0}^{i-1}\xi(f^jy)Df^jy + (Df^iy)^{m-1}|Df^iy|\sum_{j=0}^{i-1}\xi(f^jy)Df^jy \\
&= (m+1)(Df^iy)^{m-1}|Df^iy|\sum_{j=0}^{i-1}\xi(f^jy)Df^jy
\end{aligned}
$$

Then $\sum_{i=1}^{n-k}\widehat{\sum}\sum_{f^iy=x}(II)$ equals

$$
\sum_{i=1}^{n-k}\widehat{\sum}\sum_{f^iy=x}\frac{(m+1)h(y)\ \mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,\ldots,h_k)(y)(Df^iy)^{m-1}|Df^iy|\sum_{j=0}^{i-1}\xi(f^jy)Df^jy}{(Df^iy)^{2m}|Df^iy|^2}
$$

$$
=\sum_{i=1}^{n-k}\sum_{j=0}^{i-1}\widehat{\sum}\sum_{f^iy=x}\frac{(m+1)h(y)\ \mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,\ldots,h_k)(y)\xi(f^jy)Df^jy}{(Df^iy)^{m+1}|Df^iy|}.
$$

Let $z = f^jy$. Then $\sum_{i=1}^{n-k}\widehat{\sum}\sum_{f^iy=x}(II)$ equals

$$
\begin{aligned}
&= \sum_{i=1}^{n-k}\sum_{j=0}^{i-1}\widehat{\sum}\sum_{f^iy=x}\frac{(m+1)h(y)\ \mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,\ldots,h_k)(y)\xi(f^jy)Df^jy}{(Df^iy)^{m+1}|Df^iy|} \\
&= \sum_{i=1}^{n-k}\sum_{j=0}^{i-1}\widehat{\sum}\sum_{f^iy=x}\frac{(m+1)h(y)\ \mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,\ldots,h_k)(y)\xi(z)Df^jy}{(Df^{i-j}z)^{m+1}|Df^{i-j}z|(Df^jy)^{m+1}|Df^jy|} \\
&= \sum_{i=1}^{n-k}\sum_{j=0}^{i-1}\widehat{\sum}\sum_{f^iy=x}\frac{(m+1)h(y)\ \mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,\ldots,h_k)(y)\xi(z)}{(Df^{i-j}z)^{m+1}|Df^{i-j}z|(Df^jy)^m|Df^jy|}
\end{aligned}
$$

$$= \sum_{i=1}^{n-k}\sum_{j=0}^{i-1}\widehat{\sum}\sum_{f^{i-j}z=x}\frac{\xi(z)}{(Df^{i-j}z)^{m+1}|Df^{i-j}z|}\sum_{f^jy=z}\frac{(m+1)h(y)\,\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)}{(Df^jy)^m|Df^jy|}$$

$$= (m+1)\sum_{i=1}^{n-k}\sum_{j=0}^{i-1}\widehat{\sum}\Big(\overset{m+1}{\widetilde{L}}\Big)^{i-j}\Big(\xi\,\Big(\overset{m}{\widetilde{L}}\Big)^{j}(h\,\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k))\Big)(x)$$

$$= (m+1)\sum_{i=1}^{n-k}\widehat{\sum}\Big(\overset{m+1}{\widetilde{L}}\Big)^{i}\Big(\xi\,\Big(h\,\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)\Big)(x)+$$

$$(m+1)\sum_{\substack{1\le i+j\le n-k\\1\le i,1\le j}}\widehat{\sum}\Big(\overset{m+1}{\widetilde{L}}\Big)^{i-j}\Big(\xi\,\Big(\overset{m}{\widetilde{L}}\Big)^{j}(h\,\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k))\Big)(x)=A+B$$

The last two terms can be rewritten as

$$A=(m+1)\sum_{1\le i+i_1+\dots+i_k\le n}\Big(\overset{m+1}{\widetilde{L}}\Big)^{i}\Big(\xi\,\Big(h\,\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)\Big)(x),$$

$$B=(m+1)\sum_{k+2\le i+j+i_1+\dots+i_k\le n}\Big(\overset{m+1}{\widetilde{L}}\Big)^{i}\Big(\xi\,\Big(\overset{m}{\widetilde{L}}\Big)^{j}(h\,\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k))\Big)(x)$$

Therefore $\sum_{i=1}^{n-k}\sum_{f^iy=x}(II)$ is a sum of terms described in the statement.

Now, let us analyze (I). Note that $\sum_{i=1}^{n-k}\sum_{f^iy=x}(I)$ equals to

$$\sum_{i=1}^{n-k}\sum_{f^iy=x}\frac{h'(y)Dy\widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)}{(Df^iy)^m|Df^iy|}-\frac{h(y)D\Big[\widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)\Big]}{(Df^iy)^m|Df^iy|}$$

$$= \sum_{i=1}^{n-k}\sum_{f^iy=x}\frac{h'(y)\widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)}{(Df^iy)^{m+1}|Df^iy|}-\frac{h(y)D\Big[\widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)\Big]}{(Df^iy)^m|Df^iy|}$$

$$= \sum_{i=1}^{n-k}\Big(\overset{m+1}{\widetilde{\mathcal{L}}}\Big)^{i}\Big(h'\,\widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)\Big)(y)$$

$$- \sum_{i=1}^{n-k}\sum_{f^iy=x}\frac{h(y)\widehat{\sum}D\Big[\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)(y)\Big]}{(Df^iy)^m|Df^iy|}$$

Using our inductive hypothesis, the derivative of $\widehat{\sum}\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k)$ is the finite sum of terms of the type $\sum_{\widetilde{k}\le\widetilde{i}_1+\cdots+\widetilde{i}_{\widetilde{k}}\le n-i}\mathfrak{D}^{\widetilde{i}_1,\widetilde{i}_2,\ldots,\widetilde{i}_{\widetilde{k}}}_{m_1+1,\widetilde{m}_2,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{k}})$. Hence, let us take one of these terms and analyze the expression

$$\sum_{i=1}^{n-k}\sum_{\widetilde{k}\le\widetilde{i}_1+\cdots+\widetilde{i}_{\widetilde{k}}\le n-i}\sum_{f^iy=x}\frac{h(y)\mathfrak{D}^{i_1+1,\widetilde{i}_2,\ldots,\widetilde{i}_{\widetilde{k}}}_{m_1+1,\widetilde{m}_2,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{k}})(y)\cdot Dy}{(Df^iy)^m|Df^iy|}$$

$$=\sum_{i=1}^{n-k}\sum_{\widetilde{k}\le\widetilde{i}_1+\cdots+\widetilde{i}_{\widetilde{k}}\le n-i}\sum_{f^iy=x}\frac{h(y)\mathfrak{D}^{i_1+1,\widetilde{i}_2,\ldots,\widetilde{i}_{\widetilde{k}}}_{m_1+1,\widetilde{m}_2,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{k}})(y)}{(Df^iy)^{m+1}|Df^iy|}$$

$$=\sum_{i=1}^{n-k}\sum_{\widetilde{k}\le\widetilde{i}_1+\cdots+\widetilde{i}_{\widetilde{k}}\le n-i}\left(\overset{m+1}{\widetilde{\mathcal{L}}}\right)^i\left(h\,\mathfrak{D}^{i_1+1,\widetilde{i}_2,\ldots,\widetilde{i}_{\widetilde{k}}}_{m_1+1,\widetilde{m}_2,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{k}})\right)$$

$$=\sum_{\widetilde{k}+1\le 1+\widetilde{i}_1+\cdots+\widetilde{i}_{\widetilde{k}}\le n}\left(\overset{m+1}{\widetilde{\mathcal{L}}}\right)^i\left(h\,\mathfrak{D}^{i_1+1,\widetilde{i}_2,\ldots,\widetilde{i}_{\widetilde{k}}}_{m_1+1,\widetilde{m}_2,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_{\widetilde{k}})\right)$$

Since we have a finite sums of terms as the one above, we obtained that our proposition also holds for m.Therefore, part (a) is established by induction.

(b) Lemma 2.1.5 and Proposition 2.1.4 allow us to take the limit $n\to\infty$. Then the condition $k\le i_1+\cdots+i_k\le n$ becomes $k\le i_1+\cdots+i_k\le\infty$ and using the condition $i_1\ge 1,\ldots,i_k\ge 1$ the sum $k\le i_1+\cdots+i_k\le n$ converges to

$$\sum_{i_1=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}.$$

Therefore part (b) follows from part (a). □

Proposition 2.3.1(b) allows us to derive Theorem 1.2.2.

Proof of Theorem 1.2.2. We have already defined ρ_1 in Lemma 2.2.1 and ρ_2 in Proposition 2.1. Continuing this procedure, we fix n, take $y\notin\{c,f(c),\ldots,f^{n-1}(c)\}$ and let $n\to\infty$ so that we can then define $\rho_k=\rho'_{k-1}$ for all $k\ge 1$. Then Proposition

2.3.1(b) can be used to show inductively that

$$\rho_k = \sum_{finite} \sum_{i_1,\ldots,i_s \geq 1} \mathfrak{D}^{i_1,\ldots,i_s}_{k,m_2,\ldots,m_s}(h_1,\ldots,h_{s-1},\rho)$$

where $s \geq 1,\ k-1 > m_2 > \cdots > m_s,$ and $h_1,\ldots,h_{s-1}$ are piecewise C^1 functions.

Now Proposition 2.1.4 implies that $\rho_k \in BV$ as claimed. □

Chapter 3: Differentiability set for the density.

3.1 Saltus part.

Any function of bounded variation ϕ can be decomposed as

$$\phi = \phi_r + \phi_s$$

where ϕ_r is a continuous function, called the regular part, and ϕ_s is constant except at discontinuities of ϕ. ϕ_s is called the saltus part, it is discontinuous on a countable set.

In fact, in the case of ρ, ρ_s can be explicitly written as ([1])

$$\rho_s = \sum_{j \geq 1} \alpha_j H_{c_j}$$

where $\alpha_j = \lim_{x \uparrow c_j} \rho(x) - \lim_{x \downarrow c_j} \rho(x)$ and H_{c_j} is defined as

$$H_{c_j}(x) = \begin{cases} 1 & \text{if } x < c_j \\ \frac{1}{2} & \text{if } x = c_j \\ 0 & \text{if } x > c_j \end{cases} \tag{3.1}$$

Lemma 3.1.1. *If c is not periodic then*

$$\alpha_j = \rho(c)\left[\frac{1}{|Df^j_+(c)|} + \frac{1}{|Df^j_-(c)|}\right].$$

Proof. We have

$$\alpha_j = \lim_{x\uparrow c}\rho(x) - \lim_{x\downarrow c}\rho(x).$$

Using the fact that ρ is a fixed point of $\mathcal{L}$ and $\mathcal{L}^j\rho(x) = \sum_{f^j y = x}\frac{\rho(y)}{Df^j(y)}$, we can see that ρ has a discontinuity at $x = c_j$. In fact, among all the $y's$ in the set $\{f^{-j}c_j\}$, the discontinuity comes from $y = c$, then

$$\alpha_j = \lim_{y\uparrow c}\frac{\rho(y)}{Df^j(y)} - \lim_{y\downarrow c}\frac{\rho(y)}{Df^j(y)}.$$

□

Proposition 3.1.2. *For $k \geq 0$, the element ρ_k of the sequence from Theorem 1.2.2 can be decomposed as $(\rho_k)_r + (\rho_k)_s$, where $(\rho_k)_r$ is a continuous function and $(\rho_k)_s = \sum_{m\geq 1}\alpha_{k,j}H_{c_j}$, with H_{c_j} defined in* (3.1) *and $\alpha_{k,j} = \lim_{x\uparrow c_j}\rho_k(x) - \lim_{x\downarrow c_j}\rho_k(x)$. Morever there exists $\theta < 1$ such that $|\alpha_{k,j}| \leq K\theta^j$*

Proof. The existence of decomposition follows from the fact that, due to Theorem 1.2.2, $\rho_k \in BV$-function. We need to show that all discontinuities of ρ_k lie on the critical orbit and bound the size of discontinuity.

Let z be a discontinuity point of ρ_k which is different from c_i for $i = 1\ldots j$. Let $\bar\rho = \mathcal{L}^j(1)$. In the proof of Proposition 1.2.2 we saw that

$$\rho_k = \sum_{finite}\sum_{i,i_2,\ldots,i_s\geq 1}\mathfrak{D}^{i,i_2\ldots,i_s}_{k,m_2,\ldots,m_k}(h_1,\ldots,h_{s-1},\rho)$$

$$= \sum_{finite} \sum_{i,i_2,\ldots,i_s \geq 1} \mathfrak{D}^{i,i_2\ldots,i_s}_{k,m_2,\ldots,m_k}(h_1,\ldots,h_{s-1},\bar{\rho}) + \sum_{finite} \sum_{i,i_2,\ldots,i_s \geq 1} \mathfrak{D}^{i,i_2\ldots,i_s}_{k,m_2,\ldots,m_k}(h_1,\ldots,h_{s-1},\rho-\bar{\rho}).$$

Denote $\Delta(h) = \lim_{x\uparrow z} h(x) - \lim_{x\downarrow z} h(x)$. Then

$$\Delta\left(\sum_{finite} \sum_{i,i_2,\ldots,i_s \geq 1} \mathfrak{D}^{i,i_2\ldots,i_s}_{k,m_2,\ldots,m_k}(h_1,\ldots,h_{s-1},\rho-\bar{\rho})\right) = O(\theta^j)l$$

in view of Proposition 2.1.4 and the fact that $\rho - \bar{\rho} = O(\theta^j)$.

Note that if $i, i_2, \ldots, i_s < j$ then $\left(\overset{k}{\widetilde{\mathcal{L}}}\right)^i$ and $\left(\overset{m_r}{\widetilde{\mathcal{L}}}\right)^{i_r}$ are continuous at z for $r = 2, \ldots, s$, so

$$\sum_{finite} \Delta\left(\sum_{i,i_2,\ldots,i_k<j} \mathfrak{D}^{i,i_2\ldots,i_s}_{k,m_2,\ldots,m_s}(h_1,\ldots,h_{s-1},\bar{\rho})\right) = 0.$$

Applying Proposition 2.1.4 again we see that

$$\sum_{finite} \sum_{\max(i,i_2,\ldots,i_s)>j} \mathfrak{D}^{i,i_2\ldots,i_s}_{k,m_2,\ldots,m_s}(h_1,\ldots,h_{s-1},\bar{\rho}) = O\left(\sum_{\max(i,i_2,\ldots,i_s)>j} \lambda^{-(i+i_2+\ldots i_s)}\right)$$

and since the expression in the right side is less than $Mj^s\lambda^{-j}$, we have that

$$\Delta\left(\sum_{finite} \sum_{\max(i,i_2,\ldots,i_s)>j} \mathfrak{D}^{i,i_2\ldots,i_s}_{k,m_2,\ldots,m_s}(h_1,\ldots,h_{s-1},\bar{\rho})\right) \leq 2Mj^s\lambda^{-j}.$$

In particular if z is not on the critical orbit then $\Delta\rho_k = 0$ and if $z = c_j$ then $\Delta\rho_k$ is exponentially small in j as claimed. □

3.2 Absolute continuity.

Proof of Theorem 1.2.3. Let $n \geq 1$ and let $x_2, x_1 \in [0,1]$. Then

$$(\mathcal{L}^n(1))(x_2) - (\mathcal{L}^n(1))(x_1) \tag{3.2}$$

$$= \int_{x_1}^{x_2} (\mathcal{L}^n(1))'(x)dx + \sum_{\substack{j \leq n \\ c_j \in [x_1, x_2]}} \triangle_j(\mathcal{L}^n(1)),$$

where $\triangle_j(\mathcal{L}^n(1)) = \lim_{x \uparrow c_j} \mathcal{L}^n(1)(x) - \lim_{x \downarrow c_j} \mathcal{L}^n(1)(x)$.

As $n \to \infty$, $(\mathcal{L}^n(1))(x) \to \rho(x)$. Hence, $\triangle_j(\mathcal{L}^n(1)) \to \triangle_j \rho$. By Lemma 2.2.1 $(\mathcal{L}^n(1))' \to \rho_1$ as $n \to \infty$. Thus letting $n \to \infty$ in (3.2) we get

$$\begin{aligned} \rho(x_2) - \rho(x_1) &= \int_{x_1}^{x_2} \rho_1(x)dx + \sum_{c_j \in [x_1, x_2]} \triangle_j \rho \\ &= \int_{x_1}^{x_2} \rho_1(x)dx + \rho_s(x_2) - \rho_s(x_1) \end{aligned}$$

Therefore $\rho_r(x_2) - \rho_r(x_1) = \int_{x_1}^{x_2} \rho_1(x)dx$.

□

Proposition 3.2.1. *Let $n \geq 1$ and $\epsilon > 0$. Suppose*

$$d(c_j, \bar{x}) > \epsilon, \tag{3.3}$$

for $j \leq n$.

If $d(x, \bar{x}) < \epsilon$, then there exist constants $K \geq 1$, $D \geq 1$ and $\varsigma < 1$ such that

$$|\rho(x) - \rho(\bar{x})| \leq K\epsilon + D\varsigma^n.$$

Proof. Decompose

$$\rho(x) - \rho(\bar{x}) = (\rho_r(x) - \rho_r(\bar{x})) + (\rho(x)_s - \rho_s(\bar{x})). \tag{3.4}$$

Combining Theorem 1.2.3 with Proposition 1.2.1 we get

$$|\rho_r(x) - \rho_r(\bar{x})| \leq K\epsilon. \tag{3.5}$$

Also, (3.3) implies

$$\rho_s(x) - \rho_s(\bar{x}) = \sum_{j\geq n} \alpha_j [H_{c_j}(x) - H_{c_j}(\bar{x})]. \tag{3.6}$$

By Lemma 3.1.1 $|\alpha_j| \leq \dfrac{2\|\rho\|_\infty}{\lambda^j}$. Hence, we can bound (3.6) as

$$\begin{aligned} |\rho_s(x) - \rho_s(\bar{x})| &\leq \sum_{j\geq n} |\alpha_j| \Big| H_{c_j}(x) - H_{c_j}(\bar{x}) \Big| \leq 2\|\rho\|_\infty \sum_{j\geq n} \frac{1}{\lambda^j} \\ &= 2\|\rho\|_\infty \frac{1}{\lambda^n} \sum_{j\geq 1} \frac{1}{\lambda^j} = 2\|\rho\|_\infty \left(\frac{\lambda}{\lambda - 1}\right) \frac{1}{\lambda^n} \end{aligned}$$

Taking $D = 2\|\rho\|_\infty \left(\frac{\lambda}{\lambda-1}\right)$, $\varsigma = \frac{1}{\lambda}$, we have

$$|\rho_s(x) - \rho_s(\bar{x})| \leq D\varsigma^n. \tag{3.7}$$

Combining (3.4), (3.5) and (3.7) we obtain the result. □

3.3 Differentiabilty points.

Recall that there is a constant $\theta < 1$ such that

$$\mathcal{L}^n h = \int h(z) dz \rho(x) + O\left(\theta^n ||h||_{BV}\right).$$

Theorem 3.3.1. *If $1 > \beta > \max(\theta, 1/\lambda)$ and if $\bar{x}$ is a point such that $d(\bar{x}, c_j) \geq \beta^j$ for all $j \geq j_0$ then ρ_k is differentiable at $\bar{x}$.*

Proof. Let $\epsilon > 0$ and let x such that $d(x, \bar{x}) = \epsilon$.

Let n be the maximal number such that $c_j \notin [x; \bar{x}]$ for all $j < n$. This implies, $\epsilon \geq \beta^n$, which implies

$$\epsilon \lambda^n \geq \beta^n \lambda^n$$

and

$$\frac{\epsilon}{\theta^n} \geq \frac{\beta^n}{\theta^n}.$$

By definition of β, $\beta\lambda > 1$ and $\frac{\beta}{\theta} > 1$, hence, $\beta^n \lambda^n \to \infty$ and $\frac{\beta^n}{\theta^n} \to \infty$ as $n \to \infty$. Therefore,

$$\epsilon \lambda^n \to \infty$$

and

$$\frac{\epsilon}{\theta^n} \to \infty.$$

as $n \to \infty$.

By Theorem 1.2.2

$$\rho_k(x) = \sum_{finite} \sum_{i_1,\ldots,i_k=1}^{\infty} \mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1, \ldots, h_k, \rho).$$

Let $\bar{\rho} = \mathcal{L}^n(1)$. Since $\rho = \bar{\rho} + O(\theta^n)$, Proposition 2.1.4 implies that we can write the above expression as

$$\rho_k(x) = \sum_{finite} \sum_{\substack{k\le i_1,\dots,i_k<n \\ 1\le i_1,\dots,1\le i_k}} \mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho) + O\left(\lambda^{-n}+\theta^n\right).$$

Therefore

$$\rho_k(x) - \rho_k(\bar x) = \tag{3.8}$$

$$\sum_{\substack{k\le i_1,\dots,i_k<n \\ 1\le i_1,\dots,1\le i_k}} \mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho)(x) - \mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho)(\bar x) + O\left(\lambda^{-n}+\theta^n\right).$$

Note that $\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho)$ is differentiable in $[x;\bar x]$. Thus

$$\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho)(x) - \mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho)(\bar x)$$
$$= \int_{\bar x}^{x} \left(\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho)\right)'(s)ds \tag{3.9}$$

By Proposition 2.3.1

$$\left(\sum_{\substack{k\le i_1,\dots,i_k<n \\ 1\le i_1,\dots,1\le i_k}} \mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1,\dots,h_k,\bar\rho)\right)' = \sum_{finite} \sum_{\widetilde{i}_1,\dots,\widetilde{i}_k} \mathfrak{D}^{\widetilde{i}_1,\dots,\widetilde{i}_k}_{m_1+1,\dots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\dots,\widetilde{h}_{n,k}\Upsilon_n),$$

where $\widetilde{h}_1,\dots,\widetilde{h}_k \in \{h_1,h_2,\dots,h_k,h_1',\dots,h_k',\xi,\xi'\}$ and $\Upsilon_n \in \{\bar\rho,\bar\rho'\}$. Hence

$$(3.9) = \int_{\bar x}^{x} \sum_{finite} \sum_{\widetilde{i}_1,\dots,\widetilde{i}_k} \mathfrak{D}^{\widetilde{i}_1,\dots,\widetilde{i}_k}_{m_1+1,\dots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\dots,\widetilde{h}_{n,k}\Upsilon_n)(s)ds$$

Decompose the last integral as

$$\int_{\bar x}^{x} \sum_{finite} \sum_{\widetilde{i}_1,\dots,\widetilde{i}_k} \mathfrak{D}^{\widetilde{i}_1,\dots,\widetilde{i}_k}_{m_1+1,\dots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\dots,\widetilde{h}_k\Upsilon_n)(s)ds = \sum_{finite} \sum_{\widetilde{i}_1,\dots,\widetilde{i}_k} \mathfrak{D}(\widetilde{h}_1,\dots,\widetilde{h}_k\Upsilon_n)(\bar x)(x-\bar x)+$$

$$+\int_{\bar{x}}^{x}\left[\mathfrak{D}^{\widetilde{i}_1,\ldots,\widetilde{i}_k}_{m_1+1,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_k\Upsilon_n)(s) \;-\; \mathfrak{D}^{\widetilde{i}_1,\ldots,\widetilde{i}_k}_{m_1+1,\ldots,\widetilde{m}_{\widetilde{k}}}(\widetilde{h}_1,\ldots,\widetilde{h}_k\Upsilon_n)(\bar{x})\right]ds$$

We now invoke Proposition 2.3.1 again to conclude that $\mathfrak{D}^{\widetilde{i}_1,\ldots,\widetilde{i}_k}_{m_1+1,\ldots,m_k}(\widetilde{h}_1,\ldots,\widetilde{h}_k,\Upsilon_n)$ is differentiable on $[x;\bar{x}]$, and moreover its derivative is bounded by a constant K. Hence the last integrand in the above formula is $O(\epsilon)$ and so the integral is $O(\epsilon^2)$. Accordingly

$$(3.9) = (x-\bar{x})\sum_{finite}\sum_{\widetilde{i}_1,\ldots,\widetilde{i}_k}\mathfrak{D}(\widetilde{h}_1,\ldots,\widetilde{h}_k\Upsilon_n)(\bar{x}) + O(\varepsilon^2).$$

Hence

$$\lim_{x\to\bar{x}}\frac{\rho_k(x)-\rho_k(\bar{x})}{x-\bar{x}} = \lim_{x\to\bar{x}}\sum_{finite}\sum_{1\le i_1,\ldots,i_k<n}\sum_{finite}\sum_{\widetilde{i}_1,\ldots,\widetilde{i}_k}\mathfrak{D}(\widetilde{h}_1,\ldots,\widetilde{h}_k\Upsilon_n)(\bar{x})+O\left(\epsilon+\frac{\lambda^{-n}+\theta^n}{\varepsilon}\right).$$

As x approaches $\bar{x}$, n goes to ∞, hence Υ_n converges to ρ or ρ_1. Thus,

$$\lim_{x\to\bar{x}}\frac{\rho_k(x)-\rho_k(\bar{x})}{x-\bar{x}} = \sum_{finite}\sum_{i_1,\ldots,i_k=1}^{\infty}\sum_{finite}\sum_{\widetilde{i}_1,\ldots,\widetilde{i}_k}\mathfrak{D}(\widetilde{h}_1,\ldots,\widetilde{h}_k\widetilde{\rho})(\bar{x}) = \rho_{k+1}(\bar{x}). \quad \square$$

Corollary 3.3.2. *If c is periodic of period p, then ρ differentiable except for a finite set of points.*

Proof. If $\bar{x}$ does not belong to the orbit of c (which is a finite set) then we can pick any $\beta > \max(\theta, 1/\lambda)$ and pick $j_0 \geq 1$ large enough so that $d(\bar{x},\bar{c}) \geq \beta^j$ for all $j \geq j_0$, where $\bar{c} = \max\{c_1, c_2, \ldots, c_p\}$. $\square$

3.4 Whitney smoothness

Proof of Theorem 1.2.5. Let us prove this by induction on k.

The case $k = 1$ follows from Theorem 2.3.1.

Now, assume

$$\rho_j(x) = \sum_{m=0}^{k} \frac{\rho_{j+m}(\bar{x})}{m!}(x-\bar{x})^m + O\left((x-\bar{x})^k\right). \tag{3.10}$$

Pick $1 > \beta > \max\{\lambda^{-\frac{n}{k}}, \theta^{\frac{n}{k}}\}$ and let $\bar{x} \in \mathcal{N}_\beta$. Let $\epsilon = |x - \bar{x}|$, define $\bar{n} = \max\{j : c_j \notin [x;\bar{x}]\}$ and let $n = \bar{n}+1$. Then, we have that $\epsilon^k > \lambda^{-n}$ and $\epsilon^k > \theta^n$.

Then, (3.10) implies

$$\int_{\bar{x}}^{x} \rho_j(x)dx = \sum_{m=0}^{k} \frac{\rho_{j+m}(\bar{x})}{(m+1)!}(x-\bar{x})^{m+1} + O\left(\epsilon^{k+1}\right) \tag{3.11}$$

On the other hand, letting $\bar{\rho} = \mathcal{L}^n(1)$, since $\rho = \bar{\rho} + O(\theta^n)$, Proposition 4.4 implies

$$\rho_{j-1}(x) = \sum_{finite} \widehat{\sum}_k^n \mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,h_2,\ldots,h_{s-1},\mathcal{L}^n(1))(x) + O\left(\lambda^{-n}+\theta^n\right), \tag{3.12}$$

where $\widehat{\sum}_k^n = \sum_{k\le i_1,\ldots,i_k<n \, 1\le i_1,\ldots,1\le i_k}$. This implies that $\rho_{j-1}(x) - \rho_{j-1}(\bar{x})$ equals

$$\sum_{finite} \widehat{\sum}_k^n \int_{\bar{x}}^{x} \left[\mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,h_2,\ldots,h_{s-1},\mathcal{L}^n(1))\right]'(y)dy + O\left(\left[\lambda^{-n}+\theta^n\right]\epsilon\right)$$

By Proposition (2.3.1), the derivative $\widehat{\sum}_k^n \left[\mathfrak{D}^{i_1,\ldots,i_k}_{m_1,\ldots,m_k}(h_1,h_2,\ldots,h_{s-1},\mathcal{L}^n(1))\right]'$ equals

$$\sum_{finite} \widehat{\sum}_k^n \mathfrak{D}^{\widetilde{i_1},\ldots,\widetilde{i_k}}_{\widetilde{m_1},\ldots,\widetilde{m_k}}\left(\widetilde{h_1},\widetilde{h_2},\ldots,\widetilde{h}_{s-1},\Gamma_n\right),$$

where $\Gamma_n = \rho$ or $\rho' + O\left(\theta^n\right)$

Therefore, by construction of ρ_j, we have that

$$\widehat{\sum_k}^{n} \left[\mathfrak{D}^{i_1,\dots,i_k}_{m_1,\dots,m_k}(h_1, h_2, \dots, h_{s-1}, \mathcal{L}^n(1))\right]' = \rho_j + O\left(\lambda^{-n} + \theta^n\right)$$

which implies

$$\begin{aligned}
\rho_{j-1}(x) - \rho_{j-1}(\bar{x}) &= \int_x^{\bar{x}} \rho_j(y)dy + O\left(\left[\lambda^{-n} + \theta^n\right]\right)\epsilon \\
&= \sum_{m=0}^{k} \frac{\rho_{j+m}(\bar{x})}{(m+1)!}\epsilon^{m+1} + O\left(\epsilon^{k+1}\right)
\end{aligned}$$

where the last equality holds since $\epsilon^k > \lambda^{-n}$ and $\epsilon > \theta^n$.

Therefore, by induction, the statement holds.

□

3.5 Nondifferentiability set.

Definition 3.5.1. *For $\beta < 1$, define*

$$\mathcal{N}_\beta = \{\bar{x} \ : \ d(c_n, \bar{x}) \leq \beta^n \textit{ for infinitely many } n's\}.$$

Proposition 3.5.2. *$\mathcal{HD}(\mathcal{N}_\beta) = 0$ where $\mathcal{HD}$ denotes the Hausdorff dimension.*

Proof. Define U_n as the ball centered at c_n of radius $\frac{\beta^n}{2}$. Let $\epsilon > 0$ and let $n_0 \leq 1$ such that $\frac{\beta^{n_0}}{2} \leq \epsilon$. Then, $\{U_n\}_{n \geq n_0}$ is an $\epsilon-$cover of $\mathcal{N}_\beta$.

Note that $|U_n| = \beta^n$. Hence, for any $s \geq 0$ we have that

$$\mathcal{H}^s_\epsilon(\mathcal{N}_\beta) \leq \sum_{n \geq n_0} |U_n|^s \leq \sum_{n \geq n_0} |U_n|^s = \frac{1}{1-\beta^s} < \infty.$$

Therefore $\mathcal{HD}(\mathcal{N}_\beta) = 0$. $\square$

Proposition 3.5.3. *If $\{c_n\}$ is dense in some interval $I \subset [0,1]$ then $\mathcal{N}_\beta$ is uncountable for all $\beta < 1$.*

Proof. Define $L_n = [c_n - \beta^n, c_n + \beta^n]$.

Since $\{c_n\}$ is dense, there exists c_{n_1} such that L_{n_1} is strictly contained in I. Set $M_1 = L_{n_1}$.

Now, again using the density of $\{c_n\}$, there exist $c_{n_{(1,1)}} \in (c_{n_1} - \beta^{n_1}, c_{n_1})$ and $c_{n_{(1,2)}} \in (c_{n_1}, c_{n_1} + \beta^{n_1})$ such that $L_{n_{(1,1)}}$ and $L_{n_{(1,2)}}$ are strictly contained in $(c_{n_1} - \beta^{n_1}, c_{n_1})$ and $(c_{n_1} - \beta^{n_1}, c_{n_1})$ respectively. Set $M_2 = L_{n_{(1,1)}} \cup L_{n_{(1,2)}}$.

Continuing this procedure we inductively define M_n and set $M = \bigcap_{n \geq 1} M_n$ which will be a Cantor set. Note that $M \subset \mathcal{N}_\beta$ and since M is uncountable, so is $\mathcal{N}_\beta$. $\square$

Lemma 3.5.4. *If $\beta \ll 1$ and $\bar{x} \in \mathcal{N}_\beta$ then ρ is non-differentiable at $\bar{x}$*

Proof. Suppose ρ is differentiable at $\bar{x}$. Let j be very large. Since $\bar{x} \in \mathcal{N}_\beta$, there exists c_{n_j} such that

$$d(\bar{x}, c_{n_j}) \leq \beta^{n_j}.$$

Without loss of generality, assume $\bar{x} < c_{n_j}$.

Let y_1 and y_2 be two points very close to c_{n_j} such that

$$\bar{x} < y_1 < c_{n_j} < y_2 < c_{n_j} + \beta^{n_j}.$$

Since ρ is assumed to be differentiable at $\bar{x}$, we have that

$$|\rho(y_i) - \rho(\bar{x})| \leq M\beta^{n_j} \text{ for } i = 1, 2.$$

and hence

$$|\rho(y_1) - \rho(y_2)| \leq 2M\beta^{n_j}.$$

Accordingly

$$\frac{c}{(\max|f'|)^{n_j}} \leq \alpha_{n_j} = \lim_{y_1 \uparrow c_{n_j}, y_2 \downarrow c_{n_j}} |\rho(y_2) - \rho(y_1)| \leq 2M\beta^{n_j}$$

where the first inequality follows from Lemma 3.1.1. If β is very small we get a contradiction. □

Proof of Theorem 1.2.4. Theorem 1.2.4 follows from Theorem 3.3.1, Proposition 3.5.3 and Lemma 3.5.4. □

Chapter 4: Differentiability of Invariant measure

4.1 Proof of Theorem 1.2.6

Proof. By uniform Lasota-Yorke estimates (see [2]) there exist $C \geq 1$ and $\delta \in (0,1)$ such that for all $n \geq 1$

$$\left| \int \phi(x)\rho_t(x)\, dx - \int \phi(f_t^n x)\, dx \right| \leq C\delta^n. \tag{4.1}$$

Hence

$$\Gamma(t) - \Gamma(0) = \int \phi(f_t^n x)\, dx - \int \phi(f^n x)\, dx + O(\delta^n). \tag{4.2}$$

Definition 4.1.1. *In* $[0,1]$*, we define the following sets*

$A_{t,n} = \{x \in [0,1]\ :\ $ *there exists* $y \in [0,1]$ *such that* $\omega_{t,n}(x) = \omega_n(y)$ *and* $f_t^n(x) = f^n(y)\}$,

$B_{t,n} = \{y \in [0,1]\ :\ $ *there exists* $x \in [0,1]$ *such that* $\omega_{t,n}(x) = \omega_n(y)$ *and* $f_t^n(x) = f^n(y)\}$.

The properties of these sets are described in the following lemma.

Lemma 4.1.2. *(a) The complement of* $B_{t,n}$ *equals*

$$[0,1]\backslash B_{t,n} = \bigcup_{k=0}^{n} f^{-k}\widetilde{I}_{n-k} \tag{4.3}$$

where

$$\widetilde{I}_k = \begin{cases} [c - t\frac{J_k(c)}{Df_L(c)}, c] + O(t^{1+\alpha}) & \text{if } J_k > 0 \\ [c, c + t\frac{J_k(c)}{Df_R(c)}] + O(t^{1+\alpha}) & \text{if} J_k \leq 0 \end{cases}$$

Moreover

$$Leb([0,1] \backslash B_{t,n}) = O(tn)$$

and if $J(c) = 0$ *then*

$$Leb([0,1] \backslash B_{t,n}) = O(t).$$

(b) the complement of $A_{t,n}$ *equals*

$$[0,1] \backslash A_{t,n} = \bigcup_{k=0}^{n} f_t^{-k} I_{n-k} \tag{4.4}$$

where

$$I_k = \begin{cases} [c + t\frac{J_k(c)}{Df_L(c)}, c] + O(t^{1+\alpha}) & \text{if } J_k \leq 0 \\ [c, c - t\frac{J_k(c)}{Df_R(c)}] + O(t^{1+\alpha}), & \text{if } J_k > 0. \end{cases}$$

Moreover

$$Leb([0,1] \backslash A_{t,n}) = O(tn)$$

and if $J(c) = 0$ *then*

$$Leb([0,1] \backslash A_{t,n}) = O(t).$$

From now on, we fix $t > 0$ and $n = K|\ln t|$ where K is a large constant.

Consider the following decomposition

$$\int \phi(f_t^n x) dx = \int_{A_{t,n}} \phi(f_t^n x) dx + \int_{[0,1] \backslash A_{t,n}} \phi(f_t^n x) dx. \tag{4.5}$$

We start with analyzing the first term in (4.5). By definition of $A_{t,n}$ we have

$$\int_{A_{t,n}} \phi(f_t^n x)dx = \int_{B_{t,n}} \phi(f^n y) \left(\frac{dx}{dy}\right) dy.$$

Lemma 4.1.3.

$$\left(\frac{dx}{dy}\right) = 1 - tR_n(y) + O(t^2)$$

where

$$R_n(y) = \sum_{k=0}^{n-1} \left\{ \frac{v'(f^k y)}{Df(f^k y)} - \frac{v(f^k y)}{Df(f^k y)} \sum_{j=0}^{k} \frac{\xi(f^j y)}{Df^{k-j}(f^j y)} \right\}$$

Accordingly

$$\begin{aligned}
\int_{A_{t,n}} \phi(f_t^n x)dx &= \int_{B_{t,n}} \phi(f^n y)dy - t\int_{B_{t,n}} \phi(f^n y)R_n(y)dy + O(t^2 n^2) \\
&= \int_{B_{t,n}} \phi(f^n y)dy - t\int_0^1 \phi(f^n y)R_n(y)dy + O(t^2 n^2)
\end{aligned}$$

where the last step uses Lemma 4.1.2(a). It follows that

$$\Gamma(t) - \Gamma(0) = -t\int_0^1 \phi(f^n y)R_n(y)dy + \int_{[0,1]\setminus A_{t,n}} \phi(f_t^n x)dx - \int_{[0,1]\setminus B_{t,n}} \phi(f^n y)dy + O(t^2 n^2).$$

Now Theorem 1.2.6 follows from the next two results

Lemma 4.1.4.

$$\lim_{n\to\infty} \int_0^1 \phi(f^n y)R_n(y)dy == -\Delta_c.$$

Lemma 4.1.5. *If* $J(c) = 0$ *then*

$$\lim_{t\to 0} \frac{1}{t}\left[\int_{[0,1]\setminus A_{t,n}} \phi(f_t^n x)dx - \int_{[0,1]\setminus B_{t,n}} \phi(f^n y)dy\right] = \Delta_d$$

4.2 Shadowable points.

Proof of Lemma 4.1.2. Let $x \in A_{t,n}$ and let $y_n(x)$ the corresponding y according to the definition of $A_{t,n}$. Then, $\omega_{t,n}(x) = \omega_n(y_n(x))$. lThe latter condition is the same as saying that, given $0 \le k \le n$, $f_t^k(x)$ and $f^k(y_n(x))$ are both in either $[0, c]$ or $[c, 1]$.

If $s \ge 1$ and $z \in A_{t,n}$, let us denote $y_s(z)$ by the point in $[0, 1]$ such that $f_t^s(z) = f^s(y_s(z))$.

Now, let $0 \le k \le n$ and let $x \in A_{t,n}$. By Chain Rule, given $s \ge 1$,

$$x = y_s(x) + t\frac{J_s(y_s(x))}{Df(y_s(x))} + o(t^2). \tag{4.6}$$

Note that

$$f^{n-k}(f^k(y_n(x))) = f^n(y_n(x)) = f_t^n(x) = f_t^{n-k}(f_t^k x)$$

Hence, $y_{n-k}(f_t^k(x)) = f^k(y_n(x))$.

Using the identity (4.6), we get

$$\begin{aligned} f_t^k x &= y_{n-k}(f_t^k x) + \frac{J_{n-k}(y_{n-k}(f_t^k x))}{Df(y_{n-k}(f_t^k x))} + o(t^2) \\ &= f^k(y_n(x)) + \frac{J_{n-k}(f^k(y_n(x))}{Df(f^k y_n(x))} + o(t^2) \end{aligned}$$

Since x and $y_n(x)$ have the same itinerary under f_t and f respectively up to the $n-$iteration, $f^k(y_n(x))$ must be sufficiently far away from c to assure that it is in the same side as $f_n^k(x)$. We claim that, in order to have that $f^k y_n(x)$ and $f_t^k x$ stay both in the same side, it is sufficient and necessary to require that

$$f^k(y_n(x)) \notin [c - t\frac{J_{n-k}(c)}{Df_L(c)} + o(t^2), c] \quad , \quad \text{if } J_{n-k}(c) > 0$$

or

$$f^k(y_n(x)) \notin [c, c - t\frac{J_{n-k}(c)}{Df_L(c)} + o(t^2)] \quad , \quad \text{if } J_{n-k}(c) < 0.$$

Indeed, suppose $J_{n-k}(c) > 0$ and $f^k y_n(x) \in [c - t\frac{J_{n-k}(c)}{Df_L(c)} + o(t^2), c]$.

Since $\frac{J_{n-k}(x)}{Df(x)}$ is left continuous and $f^k(y_n(x))$ is close to c (because t is assume to be small and $f^k(y_n(x)) \in [c - t\frac{J_{n-k}(c)}{Df_L(c)}, c]$), then, without loss of generality, the equality (4.7) can be replaced by

$$f_t^k x \quad = \quad f^k(y_n(x)) + t\frac{J_{n-k}(c)}{Df_L(c)} + o(t^2)$$

Now, we are assuming that $f^k(y_n(x))$ is in $[c-t\frac{J_{n-k}(c)}{Df_L(c)}+o(t^2), c]$, so in particular, $f^k(y_n(x)) > c - t\frac{J_{n-k}(c)}{Df_L(c)} + o(t^2)$. Hence, under the assumption that $J_{n-k}(c) > 0$, we have that

$$f_t^k(c) = f^k(y_n(x)) + t\frac{J_{n-k}(c)}{Df_L(c)} + o(t^2) > c - t\frac{J_{n-k}(c)}{Df_L(c)} + t\frac{J_{n-k}(c)}{Df_L(c)} + o(t^2) = c + o(t^2)$$

Thus, $f_t^k(c) > c$. Hence, $f_t^k(x)$ and $f^k(y_n(x))$ are in different sides. Therefore, if we want them to lie in the same side we must require the condition (4.7) as we claimed. The condition (4.7) is proved similarly.

Thus, we have seen that the range for such y's is the complement of the union $\bigcup_{k=0}^{n-1} f^{-k} I_k$. In other words,

$$B_{t,n} = [0,1] \backslash \bigcup_{k=0}^{n} f^{-k} \widetilde{I}_{n-k},$$

proving (4.3)

Similarly, the range of $x \in A_{t,n}$ is the complement of $\bigcup_{k=0}^{n-1} f_t^{-k} I_{n-k}$. In other words,

$$A_{t,n} = [0,1] \backslash \bigcup_{k=0}^{n} f_t^{-k} I_{n-k},$$

and then we also have (4.4)

Let us prove that if $J(c) = 0$ then the Lebesgue measure of $[0,1] \backslash \bigcup_{k=0}^{n} f^{-k} \widetilde{I}_k$ is of order $O(t)$. We have

$$Leb(\bigcup_{k=1}^{n-1} f^{-k} I_k) \leq \sum_{k=1}^{n-1} Leb(f^{-k} I_k) = \sum_{k=1}^{n-1} \int_{I_k} dx$$
$$= \sum_{k=1}^{n-1} \int \chi_{I_k}(f^k x) dx = \sum_{k=1}^{n-1} \int \mathcal{L}^k(1)(y) \chi_{I_k}(y) dy$$

$$= \sum_{k=1}^{n-1} \int (\rho(y) + O(\theta^k))\chi_{I_k}(y)dy = \sum_{k=1}^{n-1} \int \rho\chi_{I_k} dy + \int O(\theta^k)$$

$$\leq \sum_{k=1}^{n-1} tJ_{n-k}(c)(\|\rho\|_\infty + O(\theta^k)) + o(t^2) \leq t\sum_{k=1}^{n-1} J_{n-k}(c)(M + o(1))$$

Suppose $s \geq 1$. Then, using that $J(c) = 0$, we have that

$$|J_s(c)| = |J(c) - J_s(c)| = \left| \sum_{j=s}^{\infty} \frac{v(f^j(c))}{Df^j(c)} \right| \leq \|v\|_\infty \sum_{j=s}^{\infty} \frac{1}{\lambda^j} \leq \widetilde{C}\lambda^{-s}$$

Thus, $J_s(c) = O(\lambda^{-s})$. Hence, $J_{n-k}(c) = O(\lambda^{-(n-k)})$ and

$$\begin{aligned} Leb(\bigcup_{k=0}^{n} f^{-k}I_k) &\leq t\sum_{k=1}^{n-1} J_{n-k}(c)(M + o(1)) \\ &\leq t\sum_{k=0}^{n} O(\lambda^{n-k)})(M + o(1)) \\ &= t\sum_{j=1}^{n-1} O(\lambda^{-j)})(M + o(1)) \end{aligned}$$

Since the series $\sum_{j=1}^{\infty} \lambda^{-j}$ converges, the term $\sum_{j=1}^{n-1} O(\lambda^{-j})$ is bounded. Therefore, we conclude that

$$Leb(\bigcup_{k=0}^{n} f^{-k}I_k) = O(t) \tag{4.7}$$

□

4.3 Changing variables

Proof of Lemma 4.1.3. By definition of $A_{t,n}$, given $x \in A_{t,n}$, there is $y = y_n(x)$ such that $f_t^n(x) = f^n(y_n(x))$.

Since $\frac{\partial y_n}{\partial x} = \prod_{k=0}^{n-1} \frac{\partial y_{k+1}}{\partial y_k}$ (note that $y_0 = x$), we analyze the factors of this product. We have

$$\frac{\partial y_n}{\partial x} = \prod_{k=0}^{n-1}(1 + t\beta_k) = 1 + t\sum_{k=0}^{n-1}\beta_k + O(t^2n^2)$$

where $\beta_k(y) = \sum_{k=0}^{n-1}\left\{\frac{v'(f^k y)}{Df(f^k y)} - \frac{v(f^k y)}{Df(f^k y)}\sum_{j=0}^{k}\frac{\xi(f^j y)}{Df^{k-j}(f^j y)}\right\}$. Hence,

$$\left(\frac{\partial y_n}{\partial x}\right)^{-1} = 1 - t\sum_{k=0}^{n-1}\beta_k + O(n^2t^2) \quad \square$$

4.4 Contribution of shadowable points

We need to work with the integral

$$\int \phi(f^n y)R(y)\, dy \;=\; \int \phi(f^n y)\Bigg(\sum_{k=0}^{n-1}\Bigg\{\underbrace{\frac{v'(f^k y)}{Df(f^k y)}}_{(I)} - \underbrace{\frac{v(f^k y)}{Df(f^k y)}\sum_{j=0}^{k}\frac{\xi(f^j y)}{Df^{k-j}(f^j y)}}_{(II)}\Bigg\}\Bigg)\, dy.$$

Let us start by analyzing (I). Making the change of variables $z = f^k y$ and $w = f^k y$, we get

$$\begin{aligned}\sum_{k=0}^{n-1}\int \phi(f^n y)\frac{v'(f^k y)}{Df(f^k y)}\, dy &= \sum_{k=0}^{n-1}\int \phi(f^{n-k}z)l\frac{v'(z)}{Df(z)}\mathcal{L}^k(1)(z)\, dz \\ &= \sum_{k=0}^{n-1}\int \phi(w)\mathcal{L}^{n-k}\left(\frac{v'}{Df}\mathcal{L}^k(1)\right)(w)\, dw\end{aligned}$$

Let $0 < \theta < 1$ such that $\mathcal{L}^k(1) = \rho + O(\theta^k)$. Then,

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\int \phi(w)\mathcal{L}^{n-k}\Big(\frac{v'}{Df}\mathcal{L}^k(1)\Big)(w)\ dw \\
=\ &\sum_{k=0}^{n-1}\int \phi(w)\mathcal{L}^{n-k}\Big(\frac{v'}{Df}\Big[\rho+O(\theta^k)\Big]\Big)(w)\ dw \\
=\ &\sum_{k=0}^{n-1}\int \phi(w)\mathcal{L}^{n-k}\Big(\frac{v'}{Df}\rho\Big)(w)\ dw+\int \phi(w)\mathcal{L}^{n-k}\Big(O(\theta^k)\Big)(w)\ dw
\end{aligned}
$$

$$
\begin{aligned}
=\ &\sum_{k=0}^{n-1}\int \phi(w)\mathcal{L}^{n-k}\Big(\frac{v'}{Df}\rho\Big)(w)\ dw+\int \phi(w)O(\theta^k)\mathcal{L}^{n-k}(1)(w)\ dw \\
=\ &\sum_{k=0}^{n-1}\int \phi(w)\mathcal{L}^{n-k}\Big(\frac{v'}{Df}\rho\Big)(w)\ dw+O(\theta^k)\int \phi(w)\mathcal{L}^{n-k}(1)(w)\ dw \\
=\ &\sum_{j=1}^{n}\int \phi(f^jw)\frac{v'}{Df}\rho\ dw+\sum_{k=0}^{n-1}O(\theta^k)\int \phi(w)\mathcal{L}^{n-k}(1)(w)\ dw
\end{aligned}
$$

$$
\begin{aligned}
=\ &\sum_{j=1}^{n}\int \phi(f^jw)\frac{v'}{Df}\rho\ dw+\sum_{k=0}^{n-1}O(\theta^k)\int \phi(w)(\rho+O(\theta^{n-k}))dw \\
=\ &\sum_{j=1}^{n}\int \phi(f^jw)\frac{v'}{Df}\rho\ dw+\sum_{k=0}^{n-1}O(\theta^k)\Big(\int \phi\rho dw+\sum_{k=0}^{n-1}O(\theta^{n-k})\int \phi dw\Big)
\end{aligned}
$$

By assumption, $\int \phi\rho\ dx=0$ then the last expression equals

$$
\begin{aligned}
\sum_{j=1}^{n}\int \phi(f^jw)\frac{v'}{Df}\rho\ dw\quad +\quad &\sum_{k=0}^{n-1}O(\theta^k)\Big(\int \phi\rho dw+\sum_{k=0}^{n-1}O(\theta^{n-k})\int \phi dw\Big)= \\
&=\sum_{j=1}^{n}\int \phi(f^jw)\frac{v'}{Df}\rho\ dw+\int \phi dw\sum_{k=0}^{n-1}O(\theta^n) \\
&=\sum_{j=1}^{n}\int \phi(f^jw)\frac{v'}{Df}\rho\ dw+\ O(n\theta^n)
\end{aligned}
$$

Since $n \cdot \theta^n \to 0$ as $n \to \infty$

$$\sum_{k=0}^{\infty} \int \phi(w) \mathcal{L}^{n-k} \left(\frac{v'}{Df} \mathcal{L}^k(1) \right)(w)\, dw = \sum_{j=1}^{\infty} \int \phi(f^j w) \frac{v'}{Df} \rho \, dw.$$

Now, let us study (II). Making the change of variables $z = f^j y$ and $w = f^{k-j}(z)$, we obtain that

$$\int \sum_{k=0}^{n-1} \phi(f^n y) \frac{v(f^k y)}{Df(f^k y)} \sum_{j=0}^{k} \frac{\xi(f^j y)}{Df^{k-j}(f^j y)} = \sum_{k,j} \int \frac{\phi(f^n y) v(f^{k-j}(f^j y)) \xi(f^j y)}{Df(f^{k-j}(f^j y)) Df^{k-j}(f^j y)} dy$$

$$\begin{aligned}
= \quad & \sum_{k,j} \int \frac{\phi(f^{n-j} z) v(f^{k-j}(z)) \xi(z)}{Df(f^{k-j}(z)) Df^{k-j}(z)} \mathcal{L}^j(1)(z) dz \\
= \quad & \sum_{k,j} \int \frac{\phi(f^{n-k} w) v(w)}{Df(w)} \widetilde{\mathcal{L}}^{k-j} \Big(\xi \cdot \mathcal{L}^j(1) \Big)(w)\, dw \\
= \quad & \sum_{k=0}^{n-1} \sum_{j=0}^{k} \int \phi(q) \mathcal{L}^{n-k} \left(\frac{v}{Df} \widetilde{\mathcal{L}}^{k-j} \Big(\xi \cdot \mathcal{L}^j(1) \Big) \right) dq
\end{aligned}$$

where $\widetilde{\mathcal{L}}^{k-j} \Big(\xi \cdot \mathcal{L}^j(1) \Big)(w) = \displaystyle\sum_{f^{k-j}(z)=w} \frac{\xi(z) \mathcal{L}^j(1)(z)}{(Df^{k-j} z)^2}.$

We can rewrite the last expression as

$$\sum_{k=0}^{n-1} \sum_{j=0}^{k} \int \phi(q) \mathcal{L}^{n-k} \left(\frac{v}{Df} \widetilde{\mathcal{L}}^{k-j} \Big(\xi \cdot \mathcal{L}^j(1) \Big) \right) dq$$

$$= \sum_{l=1}^{n} \sum_{i=0}^{n-l} \int \phi(q) \mathcal{L}^l \left(\frac{v}{Df} \widetilde{\mathcal{L}}^i \Big(\xi \cdot \mathcal{L}^{n-l-i}(1) \Big) \right)(q) dq$$

Note that

$$\mathcal{L}^l(h)(x) = \rho(x) \int h(y) dy + O(\vartheta^l \|h\|) \tag{4.8}$$

and

$$\mathcal{L}^{n-l-i}(1) = \rho + \epsilon_{n,l,i}, \text{ where } \|\epsilon_{n,l,i}\| \leq \vartheta^{n-l-il}.$$

Letting $u = \frac{v}{Df}$, we have that

$$\begin{aligned}\int \phi(x)[(II)]\,dx &= \sum_{l=1}^{n}\sum_{i=0}^{n-l+1}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i(\mathcal{L}^{n-l-i}(1)\xi)\cdot\widetilde{u})(x)\,dx \\ &= \sum_{l=1}^{n}\sum_{i=0}^{n-l+1}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i([\rho+\epsilon_{n,l,i}]\xi)\cdot u)(x)\,dx\end{aligned}$$

and using the linearity of $\mathcal{L}^l$, we have

$$\int \phi(x)[(II)]\,dx = \sum_{l,i}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i(\rho\cdot\xi)\cdot u)(x)\,dx + \sum_{l,i}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i(\epsilon_{n,l,i}\cdot\xi)\cdot u)(x)\,dx.$$

We claim that $\sum_{l,i}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i(\epsilon_{n,l,i}\cdot\xi)\cdot u)(x)\,dx$ decrease to zero as n goes to infinity. Indeed, using that $\|\widetilde{\mathcal{L}}^i h\| \leq \frac{1}{\lambda^i}\|\mathcal{L}^i(h)\| \leq \frac{\|h\|}{\lambda^i}$. and

$$\int \phi(x)\mathcal{L}^l(h)(x)dx = \int h(y)dy\int \phi(x)\rho(x)dx + O(\vartheta^l\|h\|) = O(\vartheta^l\|h\|). \qquad (4.9)$$

(note that we use the assumption $\int \phi(x)\rho(x)dx = 0$ and (4.8)).

Hence,

$$\begin{aligned}\sum_{l,i}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i(\epsilon_{n,l,i}\cdot\xi)\cdot u)(x)dx &= \sum_{l,i} O(\vartheta^l\|\widetilde{\mathcal{L}}^i(\epsilon_{n,l,i}\cdot\xi)\cdot u\|)\\ &= \sum_{l,i} O(\vartheta^l(\frac{1}{\lambda})^i\|\epsilon_{n,l,i}\|)\end{aligned}$$

Set $\varrho = \max(\theta, \lambda^{-1})$. Then

$$\sum_{l,i}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i(\epsilon_{n,l,i}\cdot\xi)\cdot u)(x)dx = \sum_{l,i} O(\vartheta^l\|\widetilde{\mathcal{L}}^i(\epsilon_{n,l,i}\cdot\xi)\cdot u\|)$$

$$= \sum_{l,i} O(\varrho^l\varrho^i\|\epsilon_{n,l,i}\|) = \sum_{l,i} O(\varrho^l\varrho^i\varrho^{n-l-i}) = \sum_{l=1}^{n}\sum_{i=0}^{n-l+1} O(\varrho^n). = O(n\varrho^n)$$

where the last expression goes to zero as n approaches ∞ as we claimed.

Thus, we have that $\int \phi(x)[(II)]\,dx$ equals

$$\sum_{l\geq 1}\sum_{i\geq 0}\int \phi(x)\mathcal{L}^l(\widetilde{\mathcal{L}}^i(\rho\cdot\xi)\cdot u)(x)\,dx + O(n\varrho^n)$$

or equivalently,

$$\sum_{l\geq 1}\sum_{i\geq 0}\int \phi(f^l(x))\widetilde{\mathcal{L}}^i(\rho\cdot\xi)\cdot u(x)\,dx + O(n\varrho^n)$$

Now, recalling Lemma 2.2.1 we have

$$\begin{aligned}
&\sum_{l\geq 1}\sum_{i\geq 0}\int \phi(f^l x)\widetilde{\mathcal{L}}^i(\rho\cdot\xi)(x)\cdot u(x)\,dx\\
=&\sum_{l\geq 1}\Bigg[\int \phi(f^l x)\rho\cdot\xi(x)\cdot u(x)\,dx\\
=&\sum_{l\geq 1}\Bigg[\int \phi(f^l x)\rho\cdot\xi(x)\cdot u(x)\,dx+\sum_{i\geq 1}\int \phi(f^l x)\widetilde{\mathcal{L}}^i(\rho\cdot\xi)(x)\cdot u(x)\,dx\Bigg]+O(n\varrho^n)\\
=&\sum_{l\geq 1}\Bigg[\int \phi(f^l x)\rho\cdot\xi(x)\cdot u(x)\,dx+\int \phi(f^l x)\sum_{i\geq 1}\widetilde{\mathcal{L}}^i(\rho\cdot\xi)(x)\cdot u(x)\,dx\Bigg]+O(n\varrho^n)\\
=&\sum_{l\geq 1}\Bigg[\int \phi(f^l x)\rho\cdot\xi(x)\cdot u(x)\,dx+\int \phi(f^l x)\rho'(x)\cdot u(x)\,dx\Bigg]+O(n\varrho^n)
\end{aligned}$$

Thus, the integral $\int \phi(x)[(II)]\,dx$ equals

$$\sum_{l\geq 1}\Bigg[\int \phi(f^l x)\rho(x)\xi(x)\cdot u(x)\,dx+\int \phi(f^l x)\rho'(x)\cdot u(x)\,dx\Bigg]+O(n\varrho^n)$$

Combining the expressions we have for $\int \phi(x)[(I)]\,dx$ and $\int \phi(x)[(II)]\,dx$, we have that

$$\begin{aligned}
\int \phi(y)R(y)\,dy \;=\; -\Bigg[\sum_{j=1}^{\infty}\int \phi(f^j w)\frac{v'}{Df}\rho\,dw.-\sum_{j=1}^{\infty}\Bigg(\int \phi(f^j x)\rho(x)\xi(x)\cdot u(x)\,dx\\
+\int \phi(f^j x)\rho'(x)\cdot u(x)\,dx\Bigg)\Bigg]+O(n\varsigma^n),
\end{aligned}$$

where $\varsigma=\max\{\varrho,\vartheta\}$.

Noting that $\frac{v'}{Df}-\xi u=u'$, we finally have that

$$\int \phi(y)R(y)\,dy = -\Big[\sum_{j=1}^{\infty}\int \phi(f^j w)u(x)\rho\,dx - \int \phi(f^j x)\rho'(x)\cdot u(x)\,dx\Big]$$

$$= -\sum_{j=1}^{\infty}\int \phi(f^j w)\Big(u - u'\frac{\rho'}{\rho}\Big)\,d\mu + O(n\varsigma^n).$$

4.5 Contribution of almost precritical points. Nonperiodic case.

Here we prove Lemma 4.1.5 in case the critical point is non-periodic.

By Lemma 4.1.2 and because c is non-periodic, without loss of generality, we can work with the integral

$$\sum_{k=1}^{n-1}\int_{f^{-(n-k)t}I_k}\phi(f^n x)\,dx$$

by obtaining the asymptotics for fixed k. Thus

$$\sum_{k=1}^{n-1}\int \phi(f^n x)\chi_{I_k}(f^{n-k}x)\,dx = \sum_{k=1}^{n-1}\int \phi(y)\mathcal{L}^k(\chi_{I_k}\mathcal{L}^{n-k}1)(y)\,dy$$

$$= \sum_{k=1}^{n-1}\int_{I_k}\phi(f^k z)\mathcal{L}^{n-k}(z)\,dz = \sum_{k=1}^{n-1}\int_{I_k}\phi(f^k z)[\rho(z) + O(\theta^{n-k})]\,dz$$

Let $M \geq 1$ be the first time when c visits I_k. Then, the jumps of ρ inside I_k are bounded by $C\lambda^M$ for some constant C. Hence, if $z \in I_k$ we have that

$$\rho(x) = \rho(c) + O(t) + O(\lambda^{-M}),$$

where the term $O(t)$ is given since the regular part of ρ is absolutely continuous by Theorem 1.2.3 and $O(\lambda^{-M})$ is given because of the observation above.

Since $\phi(f^k z) = \phi(f^k c) + O(t)$, if we set $a_k = \int_{I_k} \phi(f^k z)[\rho(z) + O(\theta^{n-k})]\, dz$, we have that

$$
\begin{aligned}
\sum_{k=1}^{n-1} \frac{a_k}{t} &= \sum_{k=1}^{n-1} \frac{1}{t} \int_{I_k} \phi(f^k z)[\rho(z) + O(\theta^{n-k})]\, dz \\
&= \sum_{k=1}^{n-1} \frac{1}{t} \int_{I_k} \Big[\phi(f^k c) + O(t)\Big]\Big[\rho(c) + O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big] dz \\
&= \sum_{k=1}^{n-1} \frac{|I_k|}{t} \Big[\phi(f^k c) + O(t)\Big]\Big[\rho(c) + O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big] dz \\
&= \sum_{k=1}^{n-1} \frac{t J_k(c)}{t} \Big[\phi(f^k c)\rho(c) + \phi(f^k c)\Big(O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big) + \\
&\qquad + O(t)\Big(\rho(c) + O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big)\Big] \\
&= \sum_{k=1}^{n-1} \Big\{ J_k(c)\phi(f^k c)\rho(c) + J_k(c)\Big[\phi(f^k c)\Big(O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big)\Big] + \\
&\qquad + \ J_k(c) O(t)\Big(\rho(c) + O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big)\Big\}
\end{aligned}
$$

Since $J(c) = 0$ we have $J_k(c) = O\left(\lambda^{-k}\right)$. Thus the term

$$\sum_{k=1}^{n-1} J_k(c) O(t)\Big(\rho(c) + O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big)$$

will be small as $t \to 0$ because $\sum_{k=1}^{n-1} J_k(c) = \sum_{k=1}^{n-1} O\left(\lambda^{-k}\right)$ converges as $n \to \infty$. Then, we have to take care of

$$\sum_{k=1}^{n-1} J_k(c)\phi(f^k c)\rho(c) + J_k(c)\Big[\phi(f^k c)\Big(O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big)\Big]$$

The expression $\sum_{k=1}^{n-1} J_k(c)\Big[\phi(f^k c)\Big(O(t) + O(\lambda^{-M}) + O(\theta^{n-k})\Big)\Big]$ can be split as

$$\sum_{k=1}^{n-1} J_k(c)\phi(f^k c)O(t) + J_k(c)\phi(f^k c)O(\lambda^{-M}) + J_k(c)\phi(f^k c)O(\theta^{n-k}). \tag{4.10}$$

Since $t \to 0$, $\sum_{k=1}^{n-1} J_k(c)\phi(f^k c)O(t)$ gets smaller so that term is negligible. For $\sum_{k=1}^{n-1} J_k(c)\phi(f^k c)O(\theta^{n-k})$, note that this is of order $O(n\kappa^n)$, where $\kappa = \max\{\lambda^{-1}, \theta\} < 1$. Being n very large, $n\kappa^n$ gets very small, so then the term $\sum_{k=1}^{n-1} J_k(c)\phi(f^k c)O(\theta^{n-k})$ is negligible as well. Finally, for $\sum_{k=1}^{n-1} J_k(c)\phi(f^k c)O(\lambda^{-M})$, notice that, as $t \to 0$, $M(t) \to \infty$ and then λ^{-M} approaches zero; therefore, the whole expression (4.10) becomes very small.

Thus, $\sum_{k=1}^{n-1} \frac{a_k}{t}$ is essentially $\sum_{k=1}^{n-1} J_k(c)\phi(f^k c)\rho(c)$.

Now, since the complement of A is the union $\bigcup_{k=1}^{n-1} f^{-(n-k)}(I_k)$, then

$$\begin{aligned}
\frac{1}{t}\int_{[0,1]\setminus A} \phi(f^n x)\, dx &= \int_{\bigcup_{k=1}^{n-1} f^{-(n-k)}(I_k)} \phi(f^n x)\, dx \\
&= \frac{1}{t}\sum_{k=1}^{n-1} \int_{f^{-(n-k)}(I_k)} \phi(f^n x)\, dx \\
&= \sum_{k=1}^{n-1} J_k(c)\phi(f^k c)\rho(c)
\end{aligned}$$

Letting n goes to ∞, the last series converges.

4.6 Meaning of the condition $J(c) = 0$ in the periodic case.

Definition 4.6.1. *Let $n \geq 1$ and let $x \in [0,1]$. Define $\omega_{t,n}(x)$ as the itinerary of x under f_t up to the nth iteration. When $t = 0$, define $\omega_n(x) = \omega_{0,n}(x)$.*

Definition 4.6.2. *Let $p \geq 3$. If $k \geq 1$, we define k' as the smallest nonnegative integer number in the equivalence class of k modulo p.*

The condition $J(c) = 0$ when c is periodic of period p must be redefine since if $k \geq p$ the summand $\frac{v(c_k)}{Df^k(c_1)}$ is not well-defined. More precisely, the problem is in $Df^k(c_1)$ because $Df^k(c_1) = Df^{k'}(c_1)(Df^p(c_1))^{(k-k')/p}$ and $Df^p(c_1)$ does not take a single value. In fact, the term $Df^p(c_1)$ can be decomposed as

$$Df^p(c_1) = Df(c)Df^{p-1}(c_1),$$

so if we want this to make sense we can consider two expressions, namely, one with $Df_L(c)$ and another with $Df_R(c)$ instead of $Df_(c)$.

Thus, we shall define

$$J(c, f_L) = J_p(c) + \sum_{k=p}^{\infty} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}}$$

and

$$J(c, f_R) = J_p(c) + \sum_{k=p}^{\infty} \frac{v(c_k)}{Df^{k'}(c_1)(Df_R(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}},$$

where

$$J_p(c) = \sum_{k=0}^{p-1} \frac{v(c_k)}{Df^k(c_1)}$$

With this, we can even reduce these conditions to the following result.

Lemma 4.6.3. *Suppose c is periodic of period p. If $* \in \{L, R\}$, the condition $J(c, f_*) = 0$ implies*

$$J_p(c) = \sum_{k=0}^{p-1} \frac{v(c_k)}{Df^k(c_1)} = 0 \tag{4.11}$$

Also, if $s \geq 1$

$$J_s(c) = \frac{J_{s'}(c)}{(Df_R(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}} \tag{4.12}$$

In particular, if $l \geq 1$ then

$$J_{lp}(c) = 0 \tag{4.13}$$

Proof. Suppose $l \geq 1$ and let $1 \leq i \leq l$. Let us work assuming $J(c, f_L) = 0$ (the proof when $J(c, f_R) = 0$ is the same by replacing f_R whenever f_L appears).

Then,

$$\sum_{k=ip}^{ip+i'} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}}$$
$$= \frac{v(c)}{(Df_L(c))^i(Df^{p-1}(c_1))^i} + \frac{v(c_1)}{Df(c_1)(Df_L(c))^i(Df^{p-1}(c_1))^i} + \cdots$$
$$\cdots + \frac{v(c_{i'})}{Df^{i'}(c_1)(Df_L(c))^i(Df^{p-1}(c_1))^i} =$$
$$= \frac{1}{(Df_L(c))^i(Df^{p-1}(c_1))^i}\left[\sum_{k=0}^{i'} \frac{v(c_k)}{Df^k(c_1)}\right] = \frac{1}{(Df_L(c))^i(Df^{p-1}(c_1))^i} J_{i'+1}(c).$$

Therefore,

$$\sum_{k=ip}^{ip+i'} \frac{v(c_k)}{Df^k(c_1)} = \frac{J_{i'+1}(c)}{(Df_L(c))^i(Df^{p-1}(c_1))^i}. \tag{4.14}$$

Hence, if we set $\varpi = (Df_L(c))(Df^{p-1}(c_1)) \geq 1$ we have

$$J_{lp}(c) = \sum_{k=0}^{p-1} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}} +$$

$$\sum_{k=p}^{p+(p-1)} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}} + \cdots +$$

$$\sum_{k=(l-1)p}^{(l-1)p+(p-1)} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}}$$

$$= J_p(c) + \frac{1}{(Df_L(c))^p(Df^{p-1}(c_1))^p} J_p(c) + \cdots + \frac{1}{((Df_L(c))^p(Df^{p-1}(c_1))^p)^{l-1}} J_p(c)$$

$$= J_p(c) \sum_{i=0}^{l-1} \frac{1}{((Df_L(c))^p(Df^{p-1}(c_1))^p)} = J_p(c) \sum_{i=0}^{l-1} \frac{1}{(\varpi)^p}$$

Now, let $s \geq 1$. Then, s can be expressed as $s = lp + s'$, for some $l \geq 1$.

Then,

$$J_s(c) = \left(\sum_{k=0}^{lp-1} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}} \right)$$

$$+ \sum_{k=lp}^{s-1} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}}$$

$$= J_{lp}(c) + \sum_{k=lp}^{lp+(s'-1)} \frac{v(c_k)}{Df^{k'}(c_1)(Df_L(c))^{(k-k')/p}(Df^{p-1}(c_1))^{(k-k')/p}}$$

$$= J_{lp}(c) + \frac{J_{s'}(c)}{((Df_L(c))^p(Df^{p-1}(c_1))^p)^l} = J_p(c) \sum_{i=0}^{l-1} \frac{1}{\varpi^i} + \frac{J_{s'}(c)}{\varpi^l}$$

Now, if $s \to \infty$, so does l, then $\frac{1}{\varpi^l} \to 0$. Hence,

$$J(c) = \lim_{s\to\infty} J_s(c) = J_p(c) \left(\sum_{i=0}^{\infty} \frac{1}{\varpi^i} \right) = \left(\frac{1}{1 - \frac{1}{\varpi}} \right) J_p(c) = \left(\frac{\varpi}{\varpi - 1} \right) J_p(c).$$

Thus, since $J(c, f_L) = 0$, we have

$$J_p(c) = 0,$$

proving (4.11). Using the equality above and that $J_p(c) = 0$, we get

$$J_s(c) = J_p(c) \sum_{i=0}^{l-1} \frac{1}{\varpi^i} + \frac{J_{s'}(c)}{\varpi^l} = \frac{J_{s'}(c)}{\varpi^l}$$

and then (4.12) holds.

The equality (4.13) follows from (4.11) and (4.12). □

4.7 Analysis of the contribution of almost precritical points in the nonperiodic case.

Once again, our goal is to study the integral

$$\int_{\bigcup_{k=1}^{n} f_t^{-k} I_{n-k}} \phi(f_t^n x) dx. \tag{4.15}$$

Also, as before, let us make a change in the indices and write (4.15) as

$$\int_{\bigcup_{k=1}^{n} f_t^{-(n-k)} I_k} \phi(f_t^n x)\, dx. \tag{4.16}$$

Now, let us split the integral over certain sets to make it easier to analyze. Namely, let us write the union $\bigcup_{k=1}^{n} f_t^{-(n-k)} I_k$ as the disjoint union

$$\bigsqcup_{k=1}^{n} \left(f_t^{-(n-k)} I_k \setminus \bigcup_{j=1}^{k-1} f_t^{-(n-j)} I_j \right).$$

Thus, the integral (4.15) can be written as

$$\int_{\bigcup_{k=1}^{n} f_t^{-(n-k)} I_k} \phi(f_t^n x)\, dx = \sum_{k=1}^{n} \int_{f_t^{-(n-k)} I_k \setminus \bigcup_{j=1}^{k-1} f_t^{-(n-j)} I_j} \phi(f_t^n x)\, dx$$
$$= \sum_{k=1}^{n} \int_{f_t^{-(n-k)} I_k} \phi(f_t^n x) \chi_{\left(\bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j\right)^c}(x)\, dx$$

Making the change of variable $z = f_t^{n-k} x$, we have that

$$\sum_{k=1}^{n} \int_{f_t^{-(n-k)} I_k} \phi(f_t^n x) \chi_{\left(\bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j\right)^c}(x)\, dx \tag{4.17}$$
$$= \sum_{k=1}^{n} \int_{I_k} \phi(f_t^k z) \chi_{\left(\bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j\right)^c}(z) \mathcal{L}_t^{n-k} 1(z)\, dz$$

We claim that for $k > p$, the measure of $I_k \setminus \bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j$ is negligible.

Let $p < k \leq n$ and define $\lambda_{p,L} = \lim_{x\uparrow} Df^p(c)$ and $\lambda_{p,R} = \lim_{x\downarrow} Df^p(c)$.

To compute the measure of $I_k \setminus \bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j$, let us pick $j \equiv k \bmod(p)$. We claim that

$$Leb(I_k \setminus f_t^{-(k-j)} I_j) = o(t^2)$$

and then

$$Leb(I_k \setminus \bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j) = o(t^2)$$

Indeed, to determine explicitly $f_t^{-(k-j)}I_j$, we need to see how I_j depends on the sign of $J_j(c)$ to write how I_j is explicitly (according to its definition). It turns out that this can be easily see by analyzing two cases: when $f_t^{k-j}(c) > f^{k-j}(c)$ or $f_t^{k-j}(c) < f^{k-j}(c)$. We claim that if $f_t^{k-j}(c) > f^{k-j}(c)$ (resp. $f_t^{k-j}(c) < f^{k-j}(c)$), then $J_j(c) > 0$ (resp. $J_j(c) > 0$). Since the proof for both is basically the same, let us just work with

$$f_t^{k-j}(c) > f^{k-j}(c). \tag{4.18}$$

If x is close to c then, by chain rule,

$$f_t^{k-j}x - f^{k-j}x = t\frac{Df^{k-j}x}{Dfx}J_j(x) + o(t^2).$$

Then,

$$\begin{aligned}
0 &< f_t^{k-j}c - f^{k-j}c \\
&= \lim_{x\uparrow c} f_t^{k-j}x - f^{k-j}x \\
&= \lim_{x\uparrow c} t\frac{Df^{k-j}x}{Dfx}J_j(x) + o(t^2). \\
&= tJ_(c)\lim_{x\uparrow c}\frac{Df^{k-j}x}{Dfx}
\end{aligned}$$

Therefore,

$$tJ_j(c)\lim_{x\uparrow c}\frac{Df^{k-j}x}{Dfx} > 0. \tag{4.19}$$

Note that

$$\lim_{x\uparrow c}\frac{Df^{k-j}x}{Dfx}>0 \tag{4.20}$$

because if $x<c$ then, under the assumption (4.18), $Df^{k-j}x>0$, and since $Dfx>0$, (4.20) holds.

Thus, in order that (4.19) holds, we must require $J_j(c)>0$.

We can also analyze the same but with the limit

$$tJ_j(c)\lim_{x\downarrow c}\frac{Df^{k-j}x}{Dfx}>0 \tag{4.21}$$

Then, if $x>c$, $Df^{k-j}(x)<0$ (because of (4.18)) and $Df(x)<0$, therefore, (4.21) holds too, which allows us to conclude that $J_j(c)$ must be positive as well.

Therefore, in case, $f_t^{k-j}(c)>f^{k-j}(c)$ implies that $J_j(c)>0$.

Hence, by definition of I_j,

$$I_j=[c,c-t\frac{J_j(c)}{Df_R(c)}+o(t^2)]$$

Then,

$$f_t^{-(k-j)}I_j=[D(k,j,L),D(k,j,R)] \tag{4.22}$$

where $D(k,j,L)=f_{t,L}^{-(k-j)}(c-t\frac{J_j(c)}{Df_R(c)}+o(t^2))$ and $D(k,j,R)=f_{t,R}^{-(k-j)}(c-t\frac{J_j(c)}{Df_R(c)}+o(t^2))$.

Let us work first with the expression

$$D(k,j,L) = f_{t,L}^{-(k-j)}(c - t\frac{J_j(c)}{Df_R(c)} + o(t^2))$$

By Lemma (4.6.3), $J_j(c)$ can be written as

$$J_j(c) = \frac{J_{j'}(c)}{(Df^p(c_1))^{(j-j')/p}}$$

Then, $D(k,j,L)$ becomes

$$D(k,j,L) = f_{t,L}^{-(k-j)}(c - t\frac{J_{j'}(c)}{Df_R(c)(Df^p(c_1))^{(j-j')/p}} + o(t^2))$$

Using one more time chain rule, we have that

$$D(k,j,L) = f_L^{-(k-j)}(c) + t\left[\frac{J_{k-j}(f_L^{-(k-j)}(c))}{Df(f^{-(k-j)_L}(c))} - \frac{J_{j'}(c)}{Df_R(c)(Df^p(c_1))^{(j-j')/p}}\right] + O(t^2) \tag{4.23}$$

Since $k \equiv j \pmod p$, $f_L^{-(k-j)}(c) = c$ and then $J_{k-j}(f_L^{-(k-j)}(c)) = J_{k-j}(c)$, and, by Lemma (4.6.3), $J_{k-j}(c) = 0$. Also, $Df_L^{-(k-j)}(f_L^{-(k-j)}(c)) = (\lambda_{p,L}(c))^{-(k-j)/p}$. Hence,

$$D(k,j,L) = c - t\frac{J_{j'}(c)}{Df_R(c)(\lambda_{p,L}(c))^{(k-j)/p}(Df^p(c_1))^{(j-j')/p}}$$

However, note that, since Df^p is continuous at c_1

$$Df^p(c_1) = \lim_{x\uparrow c} Df^p(f(x)) = \lim_{x\uparrow c} Df^{1+(p-1)}(f(x)) = \lim_{x\uparrow c} Df(f^p(x))Df^{p-1}(f(x))$$

$$= \lim_{x \uparrow c} Df(x) Df^{p-1}(f(x)) = \lim_{x \uparrow c} Df^p(x) = \lambda_{p,L}(c),$$

where we used that, because of the periodicity of c, $\lim_{x \uparrow c} Df(f^p x) = \lim_{x \uparrow c} Df(x)$.

Then,

$$\begin{aligned} D(k,j,L) &= c - t \frac{J_{j'}(c)}{Df_R(c)(\lambda_{p,L}(c))^{(k-j)/p}(\lambda_{p,L}(c))^{(j-j')/p}} \\ &= c - t \frac{J_{j'}(c)}{Df_R(c)(\lambda_{p,L}(c))^{(k-j')/p}} \end{aligned}$$

Using a similar argument, we have that

$$D(k,j,R) = c - t \frac{J_{j'}(c)}{\lambda_R(c)(\lambda_{p,R}(c))^{(k-j')/p}}$$

Since $k \equiv j (\mathrm{mod}\ p)$, $j' = k'$. Therefore,

$$D(k,j,L) = c - t \frac{J_{k'}(c)}{Df_R(c) l (\lambda_{p,L}(c))^{(k-k')/p}}$$

and

$$D(k,j,R) = c - t \frac{J_{k'}(c)}{\lambda_R(c)(\lambda_{p,R}(c))^{(k-k')/p}}.$$

Comparing these two expressions with the extreme points of I_k, we conclude that $Leb\Big(I_k \backslash f_t^{-(k-j)} I_j \Big) = o(t^2)$ and thus

$$Leb\Big(I_k \backslash \bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j) \Big) = o(t^2).$$

Therefore, (4.17) is

$$\sum_{k=1}^{n} \int_{f_t^{-(n-k)} I_k} \phi(f_t^n x) \chi_{\left(\bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j\right)^c} \, dx \tag{4.24}$$

$$= \sum_{k=1}^{p} \int_{I_k} \phi(f_t^k z) \chi_{\left(\bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j\right)^c}(z) \mathcal{L}_t^{n-k} 1(z) \, dz.$$

Thus, we just have to deal with the sets I_1, $I_2 \backslash f_t^{-1} I_1, \ldots, I_p \backslash \bigcup_{j=1}^{p-1} f_t^{-(p-j)} I_j$.

Moreover, notice that if $1 \leq i \leq p$ and $1 \leq h \leq i-1$, then I_i and $f_t^{-(i-h)} I_h$ are disjoint. In fact, if x belongs to the intersection of these two sets, since t is assumed to be very small, x will be close to c, and since $f_t^{i-h}(c) \in I_h$ and I_h is a small segment around c. Thus c would be a periodic point of period $i - h < p$ which contradicts the assumption of the period being p. Therefore, we are just dealing with $I_1, \ I_2, \ldots, I_p$.

Hence, (4.25) is reduced to

$$\sum_{k=1}^{n} \int_{f_t^{-(n-k)} I_k} \phi(f_t^n x) \chi_{\left(\bigcup_{j=1}^{k-1} f_t^{-(k-j)} I_j\right)^c} \, dx = \sum_{k=1}^{p} \int_{I_k} \phi(f_t^k z) \mathcal{L}_t^{n-k} 1(z) \, dz \tag{4.25}$$

Letting $n \to \infty$, we get that $\mathcal{L}_t^{n-k} 1$ converges to ρ_L on $[0, c]$ and converges to ρ_R on $[c, 1]$, where ρ_L (resp. ρ_R) is the density ρ restricted to $[0, c]$(resp.$[c, 1]$).

$\square$

Chapter 5: Non differentiability of the invariant measure.

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In this section, we want to study the regularity of $\Gamma_t(\phi)$, with $\phi \in C^1[0,1]$, by assuming what happens when $J(c)$ is now nonzero.

First of all, to start with, we will need this proposition about recurrence of points in the orbit of c.

Proposition 5.0.1. *There exists $m > 1$ such that for almost all t in a small interval around 0*

$$|c_j(c) - c| > j^{-m} \tag{5.1}$$

if j is sufficiently large.

Proof. In Lemma (5.0.2) below, we will prove that

$$\left|\frac{\partial c_{n+1}}{\partial c_n}\right| \geq \lambda - \delta,$$

which implies that $c_n(t)$ is of bounded variation. Let C the constant that bound the quotient of derivatives with respect to t of $c_n(t)$.

Take k_0 such that $\lambda^{k_0} > 2C$.

Assume first that

$$c_k(t) \neq c, \tag{5.2}$$

for all $t \in I$ and all $k \leq k_0$. Define

$$w_n(t) = \{s \ : \ c_j(t) \text{ and } c_j(S) \text{ have the same itinerary for } j \leq n\},$$

$$W_n(t) = \{c_n(S)\}_{S \in w_n}, \text{ and}$$

$$\Gamma_n(t) = d\left(c_n(t), \partial W_n(t)\right)$$

Let us also define $Z_n = \sup_{0<\epsilon<1} \frac{mes(t:\Gamma_n(t)<\epsilon)}{\epsilon}$.

We claim that there exists $\widetilde{K}$ such that

$$Z_n < \widetilde{K}. \tag{5.3}$$

Assuming (5.3) take $\epsilon = n^{-m}$, for $n > 1$ and $m > 1$, then

$$\begin{aligned} mes(t : |C_n(t) - c| < n^{-m}) &\leq mes(t : |\Gamma_n(t)| < n^{-m}) \\ &\leq \widetilde{K} n^{-m} \end{aligned}$$

Then, $mes(t : |c_n(t) - c| < n^{-m}) \leq \widetilde{K} n^{-m}$, which implies that

$$\sum_{n=1}^{\infty} mes(t : |c_n(t) - c| < n^{-m}) < \infty$$

Therefore, we can by Borel-Cantelly lemma, there exist n_0 such that for all $n \geq n_0$

$mes(t : |c_n(t) - c| < n^{-m}) = 1$ as we want.

Hence, we need to prove (5.3).

In fact, we prove that there exist $n_0 > 1$, $\vartheta < 1$ and $M > 0$ such that

$$Z_{n+k} \leq Z_n \vartheta + M. \tag{5.4}$$

This will certainly imply (5.3). In order to prove (5.4), let us pick $\widetilde{\delta} << 1$. Then, we will analyze two cases

(1) The components of W_n that have measure less than $\widetilde{\delta}$.

(2) The components of W_n that have measure greater than $\widetilde{\delta}$.

Let us work in the first case and let V_n be a component of W_n such that $|V_n| < \widetilde{\delta}$ and let v_n the component of ω_n associated to V_n. Then, f^{k_0} maps V_n into, at most, two intervals contained in $W_{n+k_0} = \bigcup_t W_{n+k_0}(t)$. If V_n is split then V_n passes trough c at some point. Note that we cannot have more than two intervals because of (5.1). Suppose we have two intervals. Let us call them V'_{n+k_0} and V''_{n+k_0}.

Take $\epsilon > 0$ and note that by the expansivity of f^{k_0}, we have that $mes(\{t : c_{n+k_0}(t) \in V'_{n+k_0} \cup V''_{n+k_0}$ and $\Gamma_{n+k_0}(t) < \epsilon\})$ is less or equal than $mes(\{t : d(c_n(t), a) \leq \frac{\epsilon}{\lambda^{k_0}}$ or $\Gamma_n(t) \leq \frac{\epsilon}{\lambda^{k_0}}\})$, where a is the point where V_n reaches c at some point, this is, $f^j_{s_0}(a) = c$ for some $j \leq k_0$ and $s_0 \in V_n$. By bounded distortion, the measure of $\{t : d(c_n(t), a) \leq \frac{\epsilon}{\lambda^{k_0}}\}$ is comparable to $\{t \in v_n : \Gamma_n(t) \leq \frac{\epsilon}{\lambda^{k_0}}\}$, then

$$mes(\{t\ :\ c_{n+k_0}(t) \in V'_{n+k_0} \cup V''_{n+k_0} \text{ and } \Gamma_{n+k_0}(t) < \epsilon\}) \leq$$

$$\leq mes(\{t\ :\ d(c_n(t), a) \leq \frac{\epsilon}{\lambda^{k_0}} \text{ or } \Gamma_n(t) \leq \frac{\epsilon}{\lambda^{k_0}}\})$$

$$\leq 2Cmes(\{t\ :\ \Gamma_n(t) \leq \frac{\epsilon}{\lambda^{k_0}}\})$$

By summing over all components of W_n with measure less than $\widetilde{\delta}$, we have that

$$mes(\{t\ :\ c_{n+k_0}(t) \in V'_{n+k_0} \cup V''_{n+k_0} \text{ and } \Gamma_{n+k_0}(t) < \epsilon\}) \leq \frac{2C\epsilon}{\lambda^{k_0}} Z_n,$$

(note that we use the definition of Z_n as supremum). This suggest to take $\vartheta = \frac{2C\epsilon}{\lambda^{k_0}} < \epsilon$.

Now, let us analyze the case when the components have measure greater than $\widetilde{\delta}$. In fact, the idea is the same but we have that if $\widetilde{V}_n$ is component with measure greater or equal than delta, then $f^{k_0}(\widetilde{V}_n)$ will split in at most 2^{k_0} components inside W_{n+k_0}. Call $a_1, a_2, \ldots, a_{2^{k_0-1}}$ the points that visit c. Arguing as in the first case the first case we see that the measure of $\{t\ :\ d(c_n(t), a_i) \leq \frac{\epsilon}{\lambda^{k_0}}\}$ is comparable to

$$mes(\{t\ :\ \Gamma_n(t) \leq \frac{\epsilon}{\lambda^{k_0}}\}).$$

Therefore

$$mes(\{t\ :\ c_{n+k_0}(t) \in \widetilde{V}_{n+k_0,1} \cup \cdots \widetilde{V}_{n+k_0,2^{k_0}} \text{ and } \Gamma_{n+k_0}(t) < \epsilon\})$$

$$\leq \frac{2^{k_0}C\epsilon}{\lambda^{k_0}\delta} mes(\{t : c_n(t) \in \widetilde{V}_n\}.$$

Summing over components we get

$$mes(\{t\ :\ \Gamma_{n+k_0}(t) < \epsilon \text{ and } c_n(t) \text{ is in a long component}\} \leq M\varepsilon$$

where $M = \frac{2^{k_0} C\epsilon}{\lambda^{k_0}\delta}$.

Combining the two cases we get

$$Z_{n+k_0} \leq Z_n \vartheta + M$$

as claimed.

□

Let $m > 1$ as in the last proposition. Then, for almost all t in a small interval around 0

$$|c_j(t) - c| > j^{-m} \tag{5.5}$$

if j is sufficiently large.

Below we assume that t satisfies (5.5) and also satisfies Theorem 1.3 in [21] and Theorem 1.2 in [20].

Define n_1 such that there exists $s_1 \in [0, t]$ so that

$$f_{s_1}^{-n_1} \bar{I}_t \cap \bar{I}_t \neq \emptyset,$$

and

$$f_s^{-n} \bar{I}_t \cap \bar{I}_t = \emptyset,$$

for all $n < n_1$ and for all $s \in [0, t]$, where $\bar{I}_t = [c - tJ(c, f), c + tJ(c, f)]$.

Lemma 5.0.2. *For all $s_1, s_2 \in [0,t]$ and for all $n \leq n_1$*

$$\frac{1}{C} \leq \frac{|f^n_{s_1}(\bar{I}_t)|}{|f^n_{s_2}(\bar{I}_t)|} \leq C \tag{5.6}$$

and

$$\frac{1}{\widetilde{C}} \leq \frac{|f^n_{s_1}(\bar{I}_t)|}{|c_n(t) - c_n(0)|} \leq \widetilde{C} \tag{5.7}$$

Proof. Define $d_n = \sqrt{(s_1 - s_2)^2 + (c_n(s_1) - c_n(s_2))^2}$. By the Mean Value Theorem, there exists $\widetilde{s}$ between s_1 and s_2 such that

$$\frac{c_n(s_1) - c_n(s_2)}{s_1 - s_2} = \frac{\partial c_n}{\partial s}(\widetilde{s}).$$

By the Chain Rule,

$$\frac{\partial c_n}{\partial s}(\widetilde{s}) = J_n(c(\widetilde{s})) D f^n_{\widetilde{s}}(c_1(\widetilde{s}))$$

Since $D f^n_{\widetilde{s}}(c_1(\widetilde{s})) \geq \lambda^n$ and J_n converges as n goes to infinity

$$\left| \frac{\partial c_n}{\partial s}(\widetilde{s}) \right| \geq \lambda^n_{\widetilde{s}} C_{12}.$$

Hence $\left(\frac{\partial c_n}{\partial s}(\widetilde{s}) \right)^{-1} = O(\lambda^{-n})$. So

$$\begin{aligned} \sqrt{1 + \left(\frac{\partial c_n}{\partial s} \right)^2 (\widetilde{s})} &= \left| \frac{\partial c_n(\widetilde{s})}{\partial s} \right| \sqrt{1 + \left(\frac{\partial c_n}{\partial s} \right)^{-2} (\widetilde{s})} \\ &= \left| \frac{\partial c_n}{\partial s} \right| (\widetilde{s}) \left[1 + O(\lambda^{-2n}) \right]. \end{aligned}$$

Then,

$$\begin{aligned} d_n &= |s_1 - s_2| \sqrt{1 + \left(\frac{\partial c_n}{\partial s}\right)^2 (\widetilde{s})} \\ &= |s_1 - s_2| \left|\frac{\partial c_n}{\partial s}\right| (\widetilde{s}) \left[1 + O(\lambda^{-2n})\right] \\ &= |c_n(s_1) - c_n(s_2)| \left[1 + O(\lambda^{-2n})\right] \end{aligned}$$

Thus,

$$d_n = |c_n(s_1) - c_n(s_2)| \left[1 + O(\lambda^{-2n})\right] \tag{5.8}$$

We claim that

$$d_{n+1} \geq (\lambda - \delta) d_n \tag{5.9}$$

In fact, by (5.8), this is the same as proving

$$|c_{n+1}(s_1) - c_{n+1}(s_2)| \geq (\lambda - \delta)|c_n(s_1) - c_n(s_2)|. \tag{5.10}$$

Since

$$\frac{|c_{n+1}(s_1) - c_{n+1}(s_2)|}{|c_n(s_1) - c_n(s_2)|} = \left|\frac{\partial c_{n+1}}{\partial c_n}(\widetilde{\widetilde{s}})\right|$$

it suffices to show that

$$\left|\frac{\partial c_{n+1}}{\partial c_n}(\widetilde{\widetilde{s}})\right| \geq \lambda - \delta \tag{5.11}$$

so let us prove this last inequality.

Since $\left|\frac{\partial c_n}{\partial s}\right| \geq D\lambda^n$, in particular $\left|\frac{\partial c_n}{\partial s}\right| \neq 0$, so by the Implicit Function Theorem, $s = s(c_n)$ and

$$\left|\frac{\partial s}{\partial c_n}\right| = \left|\frac{1}{\frac{\partial c_n}{\partial s}}\right| \leq \frac{D}{\lambda^n}$$

Since $c_{n+1} = f_{s(c_n)}(c_n)$, by using Chain Rule,

$$|Df_s(c_n)| \leq \left|\frac{\partial c_{n+1}}{\partial c_n}\right| + \left|\frac{\partial c_n}{\partial s}(s)\frac{\partial s}{\partial c_n}\right|$$

This implies

$$\begin{aligned}\left|\frac{\partial c_{n+1}}{\partial c_n}\right| &\geq |Df_s(c_n)| - \left|\frac{\partial c_n}{\partial s}(s)\frac{\partial s}{\partial c_n}\right| \\ &\geq \lambda_s - \delta,\end{aligned}$$

where $\lambda^{-n}D < \delta \ll 1$.

Therefore, (5.11) holds, which, as discussed, implies

$$d_{n+1} \geq (\lambda - \delta)d_n.$$

With the above in mind, if D is the function on $[0,t]\times[0,1]$ defined by $D(s,x) = Df_s(x)$, then $\log\circ|D|$ is $P-$Lipschitz, for some constant P, hence

$$\begin{aligned}\log\frac{|Df^n_{s_1}(c)|}{|Df^n_{s_2}(c)|} &\leq \sum_{k=0}^{n-1}\log|Df_{s_1}(c_k(s_1))| - \log|Df_{s_2}(c_k(s_2))| \\ &\leq \sum_{k=0}^{n-1}Pd_k \leq P\sum_{k=0}^{n-1}\frac{d_n}{(\lambda-\epsilon)^{n-k}} \leq \widetilde{P}d_n \leq \hat{C},\end{aligned}$$

where $\widetilde{P} = \sum_{j=1}^{\infty} \frac{1}{(\lambda-\epsilon)^j}$ (note that d_n is bounded by 2). Therefore, $\frac{|Df_{s_1}^n(c)|}{|Df_{s_2}^n(c)|}$ is bounded above by some constant C_1 and since s_1 and s_2 are arbitrary then they are exchangeable so the expression $\frac{|Df_{s_1}^n(c)|}{|Df_{s_2}^n(c)|}$ is also bounded by below by the reciprocal of C_1.

Since

$$\frac{|Df_{s_1}^n(x)|}{|Df_{s_2}^n(y)|} = \frac{|Df_{s_1}^n(x)|}{|Df_{s_1}^n(c)|}\frac{|Df_{s_1}^n(c)|}{|Df_{s_2}^n(c)|}\frac{|Df_{s_2}^n(c)|}{|Df_{s_2}^n(y)|},$$

using that f_{s_1} and f_{s_2} are functions of bounded distortion, we have that

$$\frac{1}{C} \leq \frac{|Df_{s_1}^n(x)|}{|Df_{s_2}^n(y)|} \leq C, \tag{5.12}$$

where $C = C_1C_2C_3$ and C_2 and C_3 are the bounds for the distortion of f_{s_1} and f_{s_2} respectively.

Hence, if $x \in \bar{I}_t$, we have that

$$\frac{|\bar{I}_t|}{C} \leq \int_{\bar{I}_t} \frac{|Df_{s_2}^n(y)|}{|Df_{s_1}^n(x)|} dy \leq C|\bar{I}_t|.$$

Therefore

$$\frac{|f_{s_1}^n\bar{I}_t|}{|f_{s_2}^n\bar{I}_t|} = \frac{\int_{\bar{I}_t}|Df_{s_1}^n(x)|dx}{\int_{\bar{I}_t}|Df_{s_2}^n(y)|dy} = \int_{\bar{I}_t} \frac{1}{\int_{\bar{I}_t}\frac{|Df_{s_2}^n(y)|}{|Df_{s_1}^n(x)|}dy} dx \leq \int_{\bar{I}_t} \frac{1}{\frac{|\bar{I}_t|}{C}} dx = C.$$

Thus $\frac{|f_{s_1}^n\bar{I}_t|}{|f_{s_2}^n\bar{I}_t|} \leq C$. Similarly $\frac{1}{C} \leq \frac{|f_{s_1}^n\bar{I}_t|}{|f_{s_2}^n\bar{I}_t|}$, and so (5.6) holds.

To prove (5.7), note that $\frac{J(c(0))}{J_n(c(s))}$ is bounded above and below by some constant C_6 and $\frac{1}{C_6}$ respectively (because as t decreases so does s and n increases as well, so $J_n(c(s))$ converges to $J(c(0)))$. Then, using that $\frac{\partial c_n}{\partial s}(\widetilde{s}) = J_n(c(\widetilde{s}))Df^n_{\widetilde{s}}(c_1(\widetilde{s})$ (by the Chain Rule) we have

$$
\begin{aligned}
\frac{|f_{s_1}(\bar{I}_t)|}{|c_n(t) - c_n(s_1)|} &= \frac{1}{|t||\frac{\partial c_n}{\partial s}|}\int_{\bar{I}_t} |Df^n_{s_1}(x)|dx \\
&= \frac{1}{|t||J_n(c(\widetilde{s}))Df^n_{\widetilde{s}}(c_1(\widetilde{s})|}\int_{\bar{I}_t} |Df^n_{s_1}(x)|dx \\
&= \frac{1}{|t||J_n(c(\widetilde{s}))|}\int_{\bar{I}_t} \frac{|Df^n_{s_1}(x)|}{|Df^n_{\widetilde{s}}(c_1(\widetilde{s}))|}dx \\
&\leq \frac{C|\bar{I}_t|}{|t||J_n(c(\widetilde{s}))|} \\
&= 2C\frac{|J_n(c)|}{|J_n(c(\widetilde{s}))|} \\
&\leq \widetilde{C}
\end{aligned}
$$

where $\widetilde{C} = CC_6$ and we use (5.6) to bound $\frac{|Df^n_{s_1}(x)|}{|Df^n_{\widetilde{s}}(c_1(\widetilde{s}))|}$. Therefore

$$\frac{|f_{s_1}(\bar{I}_t)|}{|c_n(t) - c_n(s_1)|} \leq \widetilde{C}$$

Similarly, we can prove that

$$\frac{1}{\widetilde{C}} \leq \frac{|f_{s_1}(\bar{I}_t)|}{|c_n(t) - c_n(s_1)|}.$$

Thus we obtain (5.7). □

In order to make the analysis as simple as possible, let us set two useful lemmas. For simplicity, let us write $L_k = f_t^{-k}(I_t)$.

Lemma 5.0.3.

$$\int_{\bigcup_{k=1}^n L_k} \psi(x)\, dx - \sum_{k=1}^n \int_{L_k} \psi(x)\, dx \leq |\psi|_\infty \sum_{k_1,k_2} |L_{k_1} \cap L_{k_2}|$$

Proof. Define $\widetilde{L}_k = L_k - \bigcup_{j<k} L_j \cap L_k$. Note that $\bigcup_{k=1}^n L_k = \bigcup_{k=1}^n \widetilde{L}_k$ and that $\widetilde{L}_{k_1} \cap \widetilde{L}_{k_2} = \emptyset$, for all k_1, k_2, in particular,

$$\begin{aligned} \int_{\bigcup_{k=1}^n L_k} \psi(x)\, dx &= \int_{\bigcup_{k=1}^n \widetilde{L}_k} \psi(x)\, dx \\ &= \sum_{k=1}^n \int_{\widetilde{L}_k} \psi(x)\, dx. \end{aligned}$$

Now, we can work with $\int_{\widetilde{L}_k} \psi(x)\, dx$ and bound $\sum_{k=1}^n \int_{\widetilde{L}_k} \psi(x) - \sum_{k=1}^n \int_{L_k} \psi(x)\, dx$.

For this, since

$$\begin{aligned} \int_{\widetilde{L}_k} \psi(x)\, dx &= \int_{L_k} \psi(x)\, dx - \int_{L_k \setminus \widetilde{L}_k} \psi(x)\, dx \\ &= \int_{L_k} \psi(x)\, dx - \int_{\bigcup_{j<k} L_j \cap L_k} \psi(x)\, dx \end{aligned}$$

we have that

$$\begin{aligned}
\Big|\sum_{k=1}^{n}\int_{L_k}\psi(x)\,dx - \int_{\widetilde{L}_k}\psi(x)\,dx\Big| &= \sum_{k=1}^{n}\Big|\int_{\bigcup_{j<k} L_j\cap L_k}\psi(x)\,dx\Big| \\
&\leq \sum_{k=1}^{n}\Big|\bigcup_{j<k} L_j\cap L_k\Big|\|\psi\|_\infty \\
&\leq \sum_{k=1}^{n}\sum_{j<k}\Big|L_j\cap L_k\Big|\|\psi\|_\infty \\
&\leq \sum_{k_1,k_2=1}^{n}\Big|L_j\cap L_k\Big|\|\psi\|_\infty
\end{aligned}$$

Therefore,

$$\Big|\sum_{k=1}^{n}\int_{L_k}\psi(x)\,dx - \int_{\widetilde{L}_k}\psi(x)\,dx\Big| \leq \sum_{k_1,k_2=1}^{n}\Big|L_j\cap L_k\Big|\|\psi\|_\infty$$

□

Note that

$$n_1^{-m} \leq |c_{n_1}(\widetilde{t}) - c| \leq |c_{n_1}(\widetilde{t}) - c_{n_1}(s_1)| + |c_{n_1}(s_1) - c| \leq C_9|f_{s_1}^{n_1}(\bar{I}_t)| + C_{10}|f_{s_1}^{n_1}(\bar{I}_t)|,$$

where the first term is because of Lemma (5.0.2) and the second term is by definition of s_1. Therefore,

$$n_1^{-m} \leq C_{11}|f_{s_1}^{n_1}(\bar{I}_t)| \tag{5.13}$$

Since

$$\lambda^{n_1} t \leq |f_t^{n_1}(\bar{I}_t)| \leq 1,$$

we have that $n_1 \leq \tilde{C}_1 \log t$. Hence, the latter along with (5.13) gives

$$|f_t^{n_1}(\bar{I}_t)| \geq \tilde{C}_1 |\log t|^{-m} \tag{5.14}$$

On the other hand, if $\Lambda = \sup_{x,t} Df(x,t)$ then

$$(\tilde{C}_1 \log t)^{-m} \leq |f_t^{n_1}(\bar{I}_t)| \leq t\Lambda^{n_1}$$

and so

$$n_1 \quad \geq \quad \frac{|\log(t|\tilde{C}_1 \log t|^m)|}{\log \Lambda}$$

Now, since t is assume to be sufficiently small

$$\frac{|\log t| \log t|^m|}{\log \Lambda} \geq R|\log t|, \tag{5.15}$$

where $0 < R \ll 1$. Then,

$$n_1 \geq \frac{|\log(t|\tilde{C}_1 \log t|^m)|}{\log \Lambda} \geq \frac{|\log t| \log t|^m|}{\log \Lambda} - \tilde{C}_2 \geq R|\log t|,$$

where $\tilde{C}_2 = \frac{\tilde{C}_1}{\Lambda}$. Therefore,

$$n_1 \geq R|\log t| \tag{5.16}$$

This inequality will be used in the next lemma.

Lemma 5.0.4. *There exists $0 < \eta < 1$ such that*

$$\sum_{k_1,k_2=1}^{n} |L_{k_1} \cap L_{k_2}| \leq Cn^2 t^{1+\eta}$$

Proof. Let $1 \leq k_1, k_2 \leq n$. Without loss of generality, assume $k_1 \leq k_2$. Then, we can write $k_2 = k_1 + j$, for some $0 \leq j \leq n - k_1$. Then, using the fact that ρ is bounded below, we have

$$\begin{aligned} |L_{k_1} \cap L_{k_2}| &= |L_{k_1} \cap L_{k_1+j}| \\ &= \int \chi_{I_t}(f^{k_1}(x))\chi_{I_t}(f^{k_1+j}(x))dx \\ &\leq C_1 \int \rho(x)\chi_{I_t}(f^{k}(x))\chi_{I_t}(f^{k+j}(x))dx \end{aligned}$$

Since ρ is invariant, $\int \rho(x)\chi_{I_t}(f^k(x))\chi_{I_t}(f^{k+j}(x))dx = \int \rho(x)\chi_{I_t}(x)\chi_{I_t}(f^j(x))dx$. Now using that ρ is bounded from above, we have

$$\begin{aligned} |L_{k_1} \cap L_{k_2}| &= |L_{k_1} \cap L_{k_1+j}| \\ &\leq C_1 \int \rho(x)\chi_{I_t}(f(x))\chi_{I_t}(f^j(x))dx \\ &= C_2 \int \chi_{I_t}(f(x))\chi_{I_t}(f^j(x))dx \end{aligned}$$

If $j < n_1$ then $f_t^{-j} I_t \cap I_t = \emptyset$, and consequently

$$\int \chi_{I_t}(f(x))\chi_{I_t}(f^j(x))dx = 0.$$

If $j > n_1$ then, by (5.16), $\theta^j < \theta^{R|\log t|}$. Hence

$$\begin{aligned}
\int \chi_{I_t}(x)\chi_{I_t}(f_t^j(x))dx &= \int \chi_{I_t}(y)\mathcal{L}_t^j(\chi_{I_t})(y)dy \\
&= \int_{I_t} \mathcal{L}_t^j(\chi_{I_t})(y)dy \\
&= \int_{I_t} |I_t|\rho_t(y) + O(\theta^j)dy \\
&\leq \int_{I_t} |I_t|\rho_t(y)dy + \int_{I_t} O(\theta^j)dy
\end{aligned}$$

Since ρ_t is uniformly bounded on t and $|I_t| = O(t)$, the first integral is of order $O(t^2)$. For the second integral, since $\theta^j \leq \lambda^{R|\log(t)|}$, we have that $\theta^j \leq h^\eta$, where $\eta = R\log(\theta^{-1}) < 1$ since $R \ll 1$. Then, the second integral is of order $O(t^{1+\eta})$.

Therefore, $|L_{k_1} \cap L_{k_2}| \leq Ct^{1+\eta}$, for any $1 \leq k_1 < k_2 \leq n$ and then

$$\sum_{k_1<k_2} |L_{k_1} \cap L_{k_2}| \leq Ct^{1+\eta}n^2.$$

□

In the next theorem, as discussed at the beginning, we are interested in the limit

$$\limsup_{t\downarrow 0} \frac{\Gamma_t(\phi) - \Gamma_0(\phi)}{t\sqrt{|\log(t)\log\log|\log(t)||}}.$$

We have that

$$\Gamma_t(\phi)-\Gamma_0(\phi) = -t\int_0^1 \phi(f^n y)R_n(y)dy+\int_{[0,1]\setminus A_{t,n}} \phi(f_t^n x)dx-\int_{[0,1]\setminus B_{t,n}} \phi(f^n y)dy+O(t^2n^2).$$

Since $-t\int_0^1 \phi(f^n y)R_n(y)dy + O(t^2n^2)$ are negligible when dividing by $t\sqrt{|\log(t)\log\log|\log(t)||}$, we have to focus on the limit

$$\limsup_{t\downarrow 0} \frac{\int_{[0,1]\setminus A_{t,n}} \phi(f_t^n x)dx - \int_{[0,1]\setminus B_{t,n}} \phi(f^n y)dy}{t\sqrt{|\log(t)\log\log|\log(t)||}}. \tag{5.17}$$

We already know that

$$\int_{[0,1]\setminus A_{t,n}} \phi\circ f_t^n(x)dx = \int_{\bigcup_{k=1}^n f_t^{-k}I_{n-k}} \phi\circ f_t^n(x)dx. \tag{5.18}$$

Let $U_{n,k} = \bigcup_{k=1}^n f_t^{-k}I_{n-k}, V_{n,k} = \bigcup_{k=1}^n f_t^{-k}\bar{I}, P_{n,k} = \bigcup_{k=1}^n \left(f_t^{-k}I_{n-k}\setminus f_t^{-k}\bar{I}\right)$ and $Q_{n,k} = \bigcup_{k=1}^n \left(f_t^{-k}\bar{I}\setminus f_t^{-k}I_{n-k}\right)$. Then,

$$\begin{aligned}\left|\int_{U_{n,k}} \phi\circ f_t^n(x)dx - \int_{V_{n,k}} \phi\circ f_t^n(x)dx\right| = x\left|\int_{P_{n,k}} \phi\circ f_t^n(x)dx - \int_{Q_{n,k}} \phi\circ f_t^n(x)dx\right| \leq \\ \leq \int_{P_{n,k}} |\phi\circ f_t^n(x)|dx + \int_{Q_{n,k}} |\phi\circ f_t^n(x)dx| \leq \\ \leq \sum_{k=1}^n \int_{f_t^{-k}I_{n,k}\setminus f_t^{-k}\bar{I}} |\phi\circ f_t^n(x)|dx + \int_{f_t^{-k}\bar{I}\setminus f_t^{-k}I_{n,k}} |\phi\circ f_t^n(x)dx| = \\ = \sum_{k=1}^n \int_{I_{n,k}\setminus\bar{I}} |\phi\circ f_t^n(x)|\mathcal{L}_t^k dx + \int_{\bar{I}\setminus I_{n,k}} |\phi\circ f_t^n(x)\mathcal{L}_t^k dx| \leq \\ \leq \sum_{k=1}^n \breve{C}\Big(|I_{n-k}\setminus\bar{I}| + |\bar{I}\setminus I_{n-k}|\Big),\end{aligned}$$

where C comes from bounding ϕ and $\mathcal{L}_t^k$ (the latter is bounded since converges to ρ_t).

Now, note that $||I_{n-k}\setminus\bar{I}| = |\bar{I}\setminus I_{n-k}| = O(t\lambda^{-k}$. Therefore we have

$$\int_{\bigcup_{k=1}^n f_t^{-k}I_{n-k}} \phi\circ f_t^n(x)dx = \int_{\bigcup_{k=1}^n f_t^{-k}\bar{I}} \phi\circ f_t^n(x)dx + O(t). \tag{5.19}$$

Since $\frac{t}{t\sqrt{|\log(t)\log\log|\log(t)||}} \to 0$ as $t \downarrow 0$, we can work with

$$\int_{\bigcup_{k=1}^{n} f_t^{-k}\bar{I}} \phi \circ f_t^n(x)dx$$

instead of $\int_{\bigcup_{k=1}^{n} f_t^{-k}I_{n-k}} \phi \circ f_t^n(x)dx$.

By Lemmas (5.0.3) and (5.0.4), since $n^2t^{1+\eta}$ is negligible when dividing it by $t\sqrt{|\log(t)\log\log|\log(t)||}$, instead of working with $\int_{\bigcup_{k=1}^{n} f_t^{-k}\bar{I}}$, we can just focus on studying

$$\sum_{k=1}^{n} \frac{\int_{f_t^k\bar{I}} \phi \circ f_t^n(x)dx}{t\sqrt{|\log(t)\log\log|\log(t)||}}. \tag{5.20}$$

Similarly, instead of working with $\int_{\bigcup_{k=1}^{n} f^{-k}\bar{I}}$, we can just focus on studying

$$\sum_{k=1}^{n} \frac{\int_{f^k\bar{I}} \phi \circ f^n(x)dx}{t\sqrt{|\log(t)\log\log|\log(t)||}} \tag{5.21}$$

With this in mind, let us prove the following.

Theorem 5.0.5. *Suppose that $|J(c, f_t)| > \epsilon_1$ for $t \in (-\epsilon, \epsilon)$. Then, for generic $\phi \in C^1[0,1]$ and for almost all t the following limit exists and is non-zero*

$$\limsup_{t\downarrow 0} \frac{\Gamma_t(\phi) - \Gamma_0(\phi)}{t\sqrt{|\log(t)\log\log|\log(t)||}}.$$

Proof. According to what we saw, we have to study the limit

$$\Phi_1 = \limsup_{t\downarrow 0} \sum_{k=0}^{n} \frac{\int_{f_t^k\bar{I}} \phi \circ f_t^n(x)dx}{t\sqrt{|\log(t)\log\log|\log(t)||}} \tag{5.22}$$

and

$$\Phi_2 = \limsup_{t\downarrow 0} \sum_{k=0}^{n} \frac{\int_{f^k \bar{I}} \phi \circ f^n(x)dx}{t\sqrt{|\log(t)\log\log|\log(t)||}} \tag{5.23}$$

Let us start with (5.22).

We have that on $\bar{I}_t$, $\rho_t(x) = \rho_t(c) + O(t)$ so

$$\sum_{k=0}^{n} \int_{f_t^{-k}\bar{I}_t} \phi \circ f_t^n(x)dx = \sum_{k=0}^{n} \int_{\bar{I}_t} \phi \circ f_t^{n-k}(x)\mathcal{L}_t^k dx$$
$$= \sum_{k=0}^{n} \int_{\bar{I}_t} \phi \circ f_t^k(x)\mathcal{L}_t^{n-k} dx = \sum_{k=0}^{n} \int_{\bar{I}_t} \phi \circ f_t^k(x)\mathcal{L}_t^{n-k} dx$$
$$= \sum_{k=0}^{n} \int_{\bar{I}_t} \phi\circ f_t^k(x)\rho_t dx + \sum_{k=0}^{n} \int_{\bar{I}_t} \phi\circ f_t^k(x) O(\theta^k) dx \le \sum_{k=0}^{n} \int_{\bar{I}_t} \phi\circ f_t^k(x)\rho_t dx + |\bar{I}_t|\|\phi\| \sum_{k=0}^{\infty} O(\theta^k)$$
$$\le \sum_{k=0}^{n} \int_{\bar{I}_t} \phi \circ f_t^k(x)\rho_t dx + O(t) = \sum_{k=0}^{n} \left[\int_{\bar{I}_t} \phi \circ f_t^k(x)dx \right] \rho_t(c) + O(t).$$

Define $\widehat{I}_t = f^{n_1-1} I_t$. For $n \ge n_1$. Since $f_t^{n_1-1}$ is $1-1$ (by definition of n_1), we have

$$\begin{aligned}
\sum_{k=n_1+1}^{K|\log(t)|} \int_{\bar{I}_t} \phi(f_t^k)\, dx &\le \sum_{n=1}^{K|\log(t)|} \int_{\bar{I}_t} \phi(f_t^k)\, dx \\
&= \sum_{k=n_1+1}^{K|\log(t)|} \left[\int_{\widehat{I}_t} \phi(f_t^{k-n_1+1} y) \frac{Df^{n_1-1}(c)}{Df^{n_1-1}(f^{-(n_1-1)}(y))}\, dy \right] \frac{1}{Df^{n_1-1}(c)} \\
&= O\left(\frac{|\widehat{I}_t|}{Df^{n_1-1}(c)} |\log|\widehat{I}_t||\right),
\end{aligned}$$

where we used that $\frac{Df^{n_1-1}(c)}{Df^{n_1-1}(f^{-(n_1-1)}(y))}$ was bounded by (5.6).

Using bounded distortion, we have

$$\begin{aligned}\frac{|\widehat{I}_t|}{Df_t^{n_1-1}(c)} &= \frac{1}{Df_t^{n_1-1}(c)} \int_{\bar{I}_t} |Df_t^{n_1-1}(x)|\, dx \\ &= O(|\bar{I}_t|).\end{aligned}$$

By (5.14), $|\widehat{I}_t| \geq (\tilde{C}_1|\log(t)|)^{-m}$. Thus

$$\begin{aligned}\log|\widehat{I}_t| &\geq \log((\tilde{C}_1|\log(t)|)^{-m}) \\ &= -m\log(\tilde{C}_1|\log(t)|).\end{aligned}$$

This implies $|\log|\widehat{I}_t|| \leq m\log(\tilde{C}_1|\log(t)|)$

Thus, we finally obtain

$$\sum_{k=1}^{K\log(t)|} \int_{f_t^k(\bar{I})} \phi \circ f_t^n(x)\, dx = \left[\sum_{k=1}^{K|\log(t)|} \int_{\bar{I}_t} \phi(f_t^k x)\, dx\right]\rho_t(c) + O(t)$$

$$= \left[\sum_{k=1}^{n_1} \int_{\bar{I}_t} \phi(f_t^k x)\, dx\right]\rho_t(c) + \left[\sum_{k=n_1}^{K|\log(t)|} \int_{\bar{I}_t} \phi(f_t^k x)\, dx\right]\rho_t(c) + O(t)$$

$$= \left[\sum_{k=1}^{n_1} \int_{\bar{I}_t} \phi(f_t^k x)\, dx\right]\rho_t(c) + O(t\log(\log(t)))$$

Since ϕ is C^1, by the Mean Value Theorem, we have that

$$\begin{aligned}\phi(f_t^n(x)) &= \phi(f_t^n(c)) + O(|f_t^n\bar{I}_t|) \\ &= \phi(f_t^n(c)) + O(|\widehat{I}_n|\lambda^{-(n_1-n)}).\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{k=1}^{K|\log(t)|} \int_{f_t^k(\bar{I})} \phi \circ f_t^n(x)\, dx &= \left[\sum_{k=1}^{n_1} \int_{\bar{I}_t} \phi(f_t^k x)\, dx\right] \rho_t(c) + O(t\log(\log(t))) \\
&= \left[\rho_t(c)|\bar{I}_t| \sum_{k=1}^{n_1} \phi(f_t^k(c))\right] + O(t\log(\log(t))) \\
&= \left[2tJ(c,f)\rho_t(c) \sum_{k=1}^{n_1} \phi(f_t^k(c))\right] + O(t\log(\log(t))).
\end{aligned}$$

Therefore, we have that

$$\sum_{k=1}^{K|\log(t)|} \int_{f_t^k(\bar{I})} \phi \circ f_t^n(x)\, dx \tag{5.24}$$

$$= \left[2tJ(c,f)\rho_t(c) \sum_{k=1}^{n_1} \phi(f_t^k(c))\right] + O(t\log(\log(t))).$$

Now, define $n_2(t)$ as the smallest number such that

$$|Df_t^{n_2}(c)||\bar{I}_t| \geq 1. \tag{5.25}$$

We claim that $n_2 - n_1 \leq C \log|\log(t)|$. Indeed write

$$|Df_t^{n_2}(c)| = |Df^{n_2-1}(c_1)Df(c)|.$$

Using the definition of n_2 we have that $|Df_t^{n_2-1}(c_1)||\bar{I}_t| \leq 1$ so

$$|Df_t^{n_2}(c)||\bar{I}_t| \leq C_2, \tag{5.26}$$

where $C_2 = \max_x Df(x)$.

Note that

$$
\begin{aligned}
|Df_t^{n_2}(c)||\bar{I}_t| &= |Df_t^{n_2-n_1+1}(f_t^{-n_1+1}(c))||Df_t^{n_1-1}(c)||\bar{I}_t| \\
&\leq \lambda^{n_2-n_1+1}|Df_t^{n_1-1}(c)||\bar{I}_t|.
\end{aligned}
$$

As before, using bounded distortion, we have that

$$
|Df_t^{n_1-1}(c)||\bar{I}_t| \geq C_1|\widehat{I}_t|.
$$

Hence,

$$
|Df_t^{n_2}(c)||\bar{I}_t| \geq C_1\lambda^{n_2-n_1+1}|\widehat{I}_t| \tag{5.27}
$$

Using (5.26) and (5.27), we finally obtain

$$
C_1\lambda^{n_2-n_1+1}|\widehat{I}_t| \leq C_2, \tag{5.28}
$$

which implies

$$
\log(C_1) + (n_2 - n_1 + 1)\log(\lambda) + \log|\widehat{I}_t| \leq \log(C_2).
$$

Then,

$$
n_2 - n_1 + 1 = O(|\log|\widehat{I}_t||). \tag{5.29}
$$

Back to (5.24), we can decompose it as

$$\sum_{k=1}^{K|\log(t)|} \int_{f_t^k(\bar{I})} \phi \circ f_t^n(x)\, dx = \left[2tJ(c,f)\rho_t(c) \sum_{n=1}^{n_1} \phi(f_t^n(c))\right] + O(t\log(\log(t)))$$

$$= \left[2tJ(c,f)\rho_t(c) \sum_{n=1}^{n_2} \phi(f_t^n(c)) + \sum_{n=n_2+1}^{n_1} \phi(f_t^n(c))\right] + O(t\log(\log(t))).$$

Using (5.29), we have that

$$\sum_{k=1}^{K|\log(t)|} \int_{f_t^k(\bar{I})} \phi \circ f_t^n(x)\, dx = \left[2tJ(c,f)\rho_t(c) \sum_{n=1}^{n_2} \phi(f_t^n(c))\right] + O(t\log(\log(t)))$$

Finally, we have

$$\begin{aligned}
\Phi_1 &= \limsup_{t\downarrow 0} \frac{\sum_{k=1}^{K|\log(t)|} \int_{f_t^k(\bar{I})} \phi \circ f_t^n(x)\, dx}{t\sqrt{\log(t)\log\log|\log(t)|}} \\
&= 2\rho(c)J(c,f) \limsup_{\downarrow 0} \frac{\sum_{n=1}^{n_2} \phi(f_t^n(c))}{\sqrt{\log(t)\log\log|\log(t)|}} \\
&= 2\rho(c)J(c,f) \limsup_{\downarrow 0} \frac{\sum_{n=1}^{n_2} \phi(f_t^n(c))\sqrt{n_2\log\log n_2}}{t\sqrt{\log(t)\log\log|\log(t)|}\sqrt{n_2\log\log n_2}} \\
&= 2\rho(c)J(c,f) \limsup_{t\downarrow 0} \frac{\displaystyle\sum_{n=1}^{n_2} \phi(f_t^n(c))}{\sqrt{n_2\log\log n_2}} \limsup_{h\downarrow 0} \frac{\sqrt{n_2\log\log n_2}}{\sqrt{\log(t)\log\log|\log(t)|}}.
\end{aligned}$$

Note that, as $t \downarrow 0$, n_2 take all possible values and, in particular, goes to ∞, then

$$\frac{\displaystyle\sum_{n=1}^{n_2} \phi(f_t^n(c))}{\sqrt{n_2\log\log n_2}} = \frac{\displaystyle\sum_{n=1}^{n_2} \phi(f_t^n(c))}{\sqrt{n_2\log\log n_2}}$$

and by [21], the last sequence is convergent.

For the limit $\limsup_{t\downarrow 0} \frac{\sqrt{n_2 \log\log n_2}}{\sqrt{\log(t)\log\log|\log(t)|}}$, we will prove that $\limsup_{t\downarrow 0} \frac{n_2}{\log(t)}$ converges which implies the convergence of the limit we want. For this, note that

$$\log Df^{n_2}(c) = \sum_{j=0}^{n_2-1} Df(f^j(c))$$

Then,

$$\frac{\log Df^{n_2}(c)}{n_2} = \frac{\sum_{j=0}^{n_2-1} Df(f^j(c))}{n_2}.$$

By Theorem 1.2 in [20], the sequence $\frac{\sum_{j=0}^{n-1} Df(f^j(c))}{n}$ converges and so does its subsequence $\frac{\sum_{j=0}^{n_2-1} Df(f^j(c))}{n_2}$ so $\frac{\log Df^{n_2}(c)}{n_2}$ converges as $n_2 \to \infty$.

Also, as we already saw, $Df_t^{n_2}(c)|\bar{I}_t| = MtDf_t^{n_2}(c)$ is bounded by below (by 1 by definition) and by above for some constant C. Then

$$1 \leq |Df_t^{n_2}(c)||\bar{I}_t| \leq C,$$

which implies

$$\frac{|\log(t)|}{n_2} \leq \frac{\log|Df_t^{n_2}(c)|}{n_2} \leq \frac{|\log\frac{t}{C}|}{n_2},$$

and because $\frac{\log Df^{n_2}(c)}{n_2}$ converges so does $\frac{|\log\frac{t}{C}|}{n_2}$ as $t \to 0$.

Therefore, we finally conclude that the limit Φ_1 exists.

In order to analyze (5.21), we need to work on the limit Φ_2. For this, note that the previous argument for Φ_1 remains the same for Φ_2 up to the point where we obtained that

$$\Phi_1 = 2\rho(c)J(c,f)\limsup_{\downarrow 0} \frac{\sum_{n=1}^{n_2} \phi(f_t^n(c))}{\sqrt{\log(t)\log\log|\log(t)|}}.$$

In the case for (5.21), we will have that

$$\Phi_2 = 2\rho(c)J(c,f)\limsup_{\downarrow 0} \frac{\sum_{n=1}^{n_2} \phi(f^n(c))}{\sqrt{\log(t)\log\log|\log(t)|}},$$

where the limit on the right goes to zero.

Therefore, since

$$\limsup_{t\downarrow 0} \frac{\Gamma_t(\phi) - \Gamma_0(\phi)}{t\sqrt{\log(t)\log\log|\log(t)|}} = \Phi_1 - \Phi_2,$$

we have that such a limit exists for almost all t as claimed. □

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