ABSTRACT

Title of dissertation: ON CONFORMALLY FLAT CIRCLE BUNDLES OVER SURFACES

Son Lam Ho, Doctor of Philosophy, 2014

Dissertation directed by: Professor William M. Goldman
Department of Mathematics

We study surface groups Γ in $SO(4,1)$, which is the group of conformal automorphisms of S^3, and also the group of isometries of $\mathbb{H}^4$. We consider such Γ so that its limit set Λ_Γ is a quasi-circle in S^3, and so that the quotient $(S^3 - \Lambda_\Gamma)/\Gamma$ is a circle bundle over a surface. This circle bundle is said to be conformally flat, and our main goal is to discover how twisted such bundle may be by establishing a bound on its Euler number.

We have two results in this direction. First, given a surface group Γ which admits a nice fundamental domain with n sides, we show that $(S^3 - \Lambda_\Gamma)/\Gamma$ has Euler number bounded by n^2. Second, if Γ is purely loxodromic acting properly discontinuously on $\mathbb{H}^4$, and Γ satisfies a mild technical condition, then the disc bundle quotient $\mathbb{H}^4/\Gamma$ has Euler number bounded by $(4g-2)(36g-23)$ where g is the genus of the underlying surface. Both results are proven using a direct combinatorial approach. The above are not tight bounds, improvements are possible in future research.

ON CONFORMALLY FLAT CIRCLE BUNDLES OVER SURFACES

by

Son Lam Ho

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2014

Advisory Committee:
Professor William M. Goldman, Chair/Advisor
Professor Ted Jacobson
Professor John J. Millson
Professor Karin Melnick
Professor Richard Wentworth

Dedication

To Ba Tan and Me Tam.

Acknowledgments

First and foremost I'd like to thank my advisor, professor Bill Goldman, for introducing me to this mathematical area, and for countless number of enlightening conversations. I also thank him for his patient guidance, generosity and friendship during these many years.

I'm grateful to professor Feng Luo for his valuable suggestions which are important to the development of this work.

I thank professor Karin Melnick and the UMD writing workshop group for their helpful comments on an early draft.

I'd like to acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "NMS: Geometric Structures and Representation Varieties" (the GEAR Network).

Last but not least, I thank my family and friends for their moral support.

Contents

Chapter 1: Introduction

Let E be the total space of a circle bundle over a closed surface Σ_g of genus g with $g \geq 2$. If we fix g, the topological type of E is classified by its Euler number, denoted by $e(E)$. Gromov, Lawson, and Thurston [1], Kuiper [2], and Kapovich [3] constructed examples of flat conformal $(SO(4,1), S^3)$ structures on circle bundles E with non-zero Euler number. These examples are constructed using fundamental domains in S^3 that are bounded by a "necklace" of 2-spheres which are arranged along an unknotted embedding of a circle. It's interesting that the arrangement of these spheres determines the topology of E.

All constructed examples of flat conformal structures on E satisfy the inequality

$$|e(E)| \leq |\chi(\Sigma_g)| = 2g - 2. \tag{1.1}$$

It is conjectured in [1] that this inequality is a necessary condition for the existence of a flat conformal structure on E. We will refer to this as the GLT conjecture. If it is true, it would be an example of the general principle that existence of geometry on a manifold often restricts its topology.

The case when $g = 1$ was established by Goldman in [4]. In this article we consider the case $g \geq 2$, in particular the case where the conformally flat manifold E is a quotient of the domain of discontinuity of a surface group Γ. Partial results are obtained in some nice cases. These are presented in chapter 4.

Representations of fundamental groups of closed surfaces (surface groups) into a Lie group G is a well-studied subject. This is especially true when G is $\mathrm{Isom}(\mathbb{H}^2)$ and $\mathrm{Isom}(\mathbb{H}^3)$. Here we study the natural higher dimensional analogue: surface groups in $G = \mathrm{Isom}(\mathbb{H}^4)$. Just as in lower dimension cases, we want to study the space of "nice" surface groups called quasi-Fuchsian groups. Following [5] we define:

Definition 1.1. *A surface group in* $\mathrm{Isom}(\mathbb{H}^n)$ *(for* $n \geq 2$*) is* quasi-Fuchsian *if its limit set is a topological circle in* $\partial_\infty(\mathbb{H}^n) = S^{n-1}$.

Indeed, all known examples of conformally flat circle bundles are constructed as quotients of a domain in S^3 by a quasi-Fuchsian surface group which admits a finite sided fundamental domain. In non-trivial cases (where $e(E) \neq 0$,) the limit set of one of these group is a fractal topological circle.

The manuscript is organized as follows. In chapter 2, we present background material in flat conformal geometry. In section 3.1 we describe the combinatorial construction of a circle bundle, section 3.2 discusses examples and an algorithm to compute the Euler number using the fundamental domain. In chapter 4 we show two approaches to bound the Euler number of a conformally flat circle bundle in nice cases.

Chapter 2: Preliminaries on flat conformal geometry

Let us recall some basic concepts and notations. Let (X, g) be a Riemannian manifold. Isometries from X to itself are maps preserve the Riemannian metric g. These maps form a group which we call Isom(X), the group of isometries on X. The group of orientation preserving isometries is denoted Isom$^+(X)$.

Two Riemannian metrics g, h on X are conformally equivalent if there is a positive function λ on X such that $g_x(u, v) = \lambda(x)h_x(u, v)$ on each tangent space. A class of conformally equivalent metrics on X is called a *conformal structure*. Given two Riemannian manifolds (X, g) and (Y, h), a local diffeomorphism $X \to Y$ is called a *conformal map* if the pull back metric h^* on X is conformally equivalent to g.

For the sphere S^n, the conformal maps from S^n to itself form a group which we will name Möb(S^n).

We will now introduce the hyperbolic space and its conformal sphere boundary at infinity. More details can be found in [6]. For $m \geq 2$, we let $\mathbb{R}^{m,1}$ be $\mathbb{R}^{m+1}$ with a Lorentzian metric of signature $(m, 1)$. That is, in metric is given by the quadratic form $B(x) = -x_0^2 + x_1^2 + ... + x_m^2$. Let $SO(m, 1)$ be the group of $(m + 1) \times (m + 1)$ matrices of determinant 1 preserving this quadratic form.

The hyperbolic m space, denoted $\mathbb{H}^m$ is defined to be the level set

$$H := \{x \in \mathbb{R}^{m,1} | B(x) = -1\} \subset \mathbb{R}^{m,1}$$

with the inherited metric on each tangent space. This is the hyperboloid model of the hyperbolic space. The hyperboloid H is asymptotic to the light-cone

$$L := \{x \in \mathbb{R}^{m,1} | B(x) = -x_0^2 + \sum_{i=1}^{m} x_i^2 = 0\} \subset \mathbb{R}^{m,1}.$$

The projectivization of L identifies with $\{x \in \mathbb{R}^{m,1} | \sum_{i=1}^m x_i^2 = 1, x_0 = 1\} = S^{m-1}$. This is a sphere of dimension $m-1$ and it is naturally the boundary at infinity of hyperbolic space $\mathbb{H}^m$. Unless otherwise noted we give S^{m-1} the standard spherical metric inherited from $\mathbb{R}$.

It can be shown that $SO(m,1)$ acts on $\mathbb{H}^m$ by hyperbolic isometries, and also on S^{m-1} by conformal automorphisms. That is, Möb$(S^{m-1}) =$ Isom$(\mathbb{H}^m) = SO(m,1)$.

An element $A \in$ Möb$^+(S^{m-1})$ can be classified by the dynamics of its action on $\mathbb{H}^m \cup S^{m-1}$:

1. Loxodromic: this is when A has two fixed points (one attracting and one repelling) on S^{m-1}, and it leaves invariant a geodesic in $\mathbb{H}^m$ whose ideal end points are the fixed points of A. All loxodromic actions on $S^{m-1} = \mathbb{R}^{m-1} \cup \{\infty\}$ are conjugate to $x \mapsto R(\lambda x)$ for $x \in \mathbb{R}^{m-1}$, $1 \neq \lambda \in \mathbb{R}$ a scalar, and R a matrix in $SO(m-1)$ representing a rotation.
2. Parabolic: this is when A has exactly one fixed point on S^{m-1}. We can think of a parabolic element as a limit of loxodromic elements when the two fixed points come together. All parabolic actions on $S^{m-1} = \mathbb{R}^{m-1} \cup \{\infty\}$ are conjugate to

$x \mapsto Rx + t$ for $R \in SO(m-1)$ a rotation matrix, and $t \in \mathbb{R}^{m-1}$ a non-zero vector of translations.

3. Elliptic: this is when A has one or more fixed point(s) in $\mathbb{H}^m$. In this case, the differential of the action of A at the fixed point $p \in \mathbb{H}^m$ can be represented as a matrix in $SO(m)$.

(See [5], [7], and [6] for more details on this classification.) One can easily show that loxodromic elements form an open set in Möb$^+(S^{m-1})$ using Brouwer fixed point theorem.

The following are well known facts in lower dimensions. For $n = 1$, we have Möb$^+(S^1) =$ Isom$^+(\mathbb{H}^2) \cong PSL(2, \mathbb{R})$ the group of projectivized 2×2 real matrices of determinant 1. For $n = 2$, we have $Aut(\mathbb{C}P^1) =$ Möb$^+(S^2) =$ Isom$^+(\mathbb{H}^3) \cong PSL(2, \mathbb{C})$ the group of projectivized 2×2 complex matrices of determinant 1.

The basic objects in conformal geometry are subspheres inside S^{m-1}. These objects arise naturally as the boundaries of totally geodesic hyperbolic subspaces inside $\mathbb{H}^m$. Under actions of elements of Möb(S^{m-1}), one subsphere must be mapped to another of the same dimension. Also, given two subspheres of the same dimension in S^{m-1}, there always exist conformal transformations taking one to the other. Moreover, under a stereographic projection $(S^{m-1} - \infty) \to \mathbb{R}^{m-1}$ these subspheres are mapped to Euclidean subspheres and in $\mathbb{R}^{m-1}$. We now reserve the words "circle" and "sphere" only for these natural geometric objects in S^{m-1}. From this point we restrict our attention to the case $m = 4$. In all figures, S^3 will be presented as its stereographic projection model $\mathbb{R}^3 \cup \{\infty\}$.

Definition 2.1. *A loxodromic transformation* $A \in$ Möb$^+(S^3)$ *is said to be* non-rotating *if up to a conjugation there is a* $\lambda \in \mathbb{R}+$ *so that* $A(x) = \lambda x$ *for all* $x \in \mathbb{R}^3 = S^3 - \infty$.

Otherwise we say that A is rotating.

2.1 Further classifications of elliptic transformations

Let $A \in \text{Möb}^+(S^3)$ be an elliptic transformation. We say A is *regular elliptic* if it fixes exactly one point in $\mathbb{H}^4$, thus it has no fixed points in S^3. For example, if we let $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, then the transformation $(z, w) \mapsto (e^{i\theta}z, e^{i\psi}w)$ for $\theta, \psi \neq k2\pi$ would be a regular elliptic transformation.

If A is not regular elliptic, that is, it fixes two points in $\mathbb{H}^4$, then A fixes the geodesic connecting those two points. So the differential of A at a fixed point can be represented as $d_pA \sim \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ with $R \in SO(3)$. But any $SO(3)$ rotation has a fixed axis, so

$$d_pA \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T \end{pmatrix}$$

with $T \in SO(2)$. Thus if A is not regular elliptic, then A fixes a totally geodesic plane inside $\mathbb{H}^4$ along with its ideal boundary - a circle in S^3. In this case, we say that A is *singular-elliptic*.

This is an important class of Möbius transformations. The centralizer of almost every 1-parameter subgroup of $\text{Möb}^+(S^3)$ contains a singular-elliptic subgroup. (1-parameter subgroups generated by certain parabolic elements may have trivial centralizer.) Also, among all 1-parameter subgroups of $\text{Möb}^+(S^3)$, the singular-elliptic ones have the largest centralizers, which can be shown to be isomorphic to $SO(2) \times PSL(2, \mathbb{R})$.

2.2 Möbius Annulus

Let C be a circle in S^3. We denote Fix(C) the subgroup of Möb$^+(S^3)$ that fixes every point on C. Then Fix(C) contains only singular-elliptic elements, and it is isomorphic to $SO(2)$. Its action on S^3 is rotation around the axis C.

Consider Inv(C), the subgroup of Möb$^+(S^3)$ that leaves a circle C invariant. That is, elements of Inv(C) mapping C to itself. Clearly Fix(C) $\hookrightarrow$Inv(C) a normal subgroup. Let Conf$^+(C)$ be the group of conformal automorphisms of C. Then Inv(C) $\rightarrow$Conf$^+(C)$ $\cong$ Isom$^+(\mathbb{H}^2)$ $\cong$ $PSL(2,\mathbb{R})$ is a surjective map defined by restricting the elements of Inv(C) to act on C. So we have an exact sequence

$$0 \to \text{ Fix}(C) \hookrightarrow \text{ Inv}(C) \to \text{Conf}^+(C) \to 0.$$

In fact there is a splitting Conf$^+(C)$ $\hookrightarrow$ Inv(C) and the image commutes with Fix(C) . So we have Inv(C) $\cong$ Fix(C) $\times$ Conf$^+(C)$ $\cong SO(2) \times PSL(2,\mathbb{R})$.

Now let S be a 2-sphere in S^3. We would still have a similar exact sequence like above. And since Fix(S) is the trivial group, we get Inv(S) $\cong$ Conf$^+(S)$ $\cong PSL(2,\mathbb{C})$. We can then think of a 2-sphere in S^3 as a copy of $\mathbb{C}P^1$ with $PSL(2,\mathbb{C})$ acting on it.

Let H be a connected subset of a 2-sphere S with boundary being a circle, that is, H is a half sphere. Then Inv(H) is a subset of Inv(S) $\cong PSL(2,\mathbb{C})$ that fixes a half-sphere. So Inv(H) $\cong PSL(2,\mathbb{R})$ the isometry group of $\mathbb{H}^2$. We can then think of H as a copy of the hyperbolic plane.

An important object to study is defined below:

Definition 2.2. *A* Möbius annulus *is a 2-sphere minus two disjoint half-spheres.*

Let $\mathcal{A}$ be a Möbius annulus, and denote Inv$^*(\mathcal{A})$ the subgroup of Möb$^+(S^3)$ that leaves $\mathcal{A}$ and its two boundary components invariant. (Elements of this group will not swap the two boundary components of $\mathcal{A}$.) Let $\partial_1\mathcal{A}$ and $\partial_2\mathcal{A}$ be the two boundary components which are both circles. Suppose that half-sphere H contains $\mathcal{A}$ with $\partial H = \partial_1\mathcal{A}$. Then we have Inv$^*(\mathcal{A}) \subset$ Inv$(H) \cong PSL(2,\mathbb{R})$, and Inv$^*(\mathcal{A})$ preserve a disk in H. So

$$\text{Inv}^*(\mathcal{A}) \cong SO(2)$$

a subgroup of elliptic elements in $PSL(2,\mathbb{R})$.

Consider a Möbius annulus in the plane given by

$$\mathcal{A}_l = \{z \in \mathbb{C} \mid \frac{1}{e^l} < |z| < 1\} \quad \text{for } l > 0$$

which is considered to be a subset of the Poincare unit disk model of the hyperbolic plane. If $0 < l_1 < l_2 < 1$ then $\mathcal{A}_{l_1}$ is not equivalent to $\mathcal{A}_{l_2}$. This is because conformal automorphism of the unit disk cannot change hyperbolic area.

We now claim that every Möbius annulus in S^3 is conformally equivalent to $\mathcal{A}_l$ for some l. Given any Möbius annulus $\mathcal{A}$, it is contained in a halfsphere H such that $\partial H = \partial_1\mathcal{A}$. Since H can be mapped conformally to the unit disk, this map takes $\partial_1\mathcal{A}$ to $\{|z| = 1\}$ and takes the other boundary component $\partial_2\mathcal{A}$ to some circle inside the unit disk. A composition with some element of Inv(H) will take $\partial_2\mathcal{A}$ to a circle centered at 0 with radius $1/e^l$. So $\mathcal{A}$ is equivalent to $\mathcal{A}_l$ for some $l > 0$. Thus we have the following:

Remark 2.3. *There is a one parameter invariant given by $l > 0$ for Möbius annulus $\mathcal{A}$. We denote this by* mod$(\mathcal{A})$ *and we call it the modulus of A.*

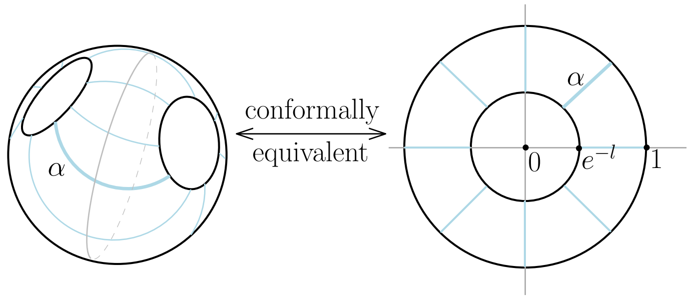

Fig. 2.1: Conformally equivalent Möbius annuli.

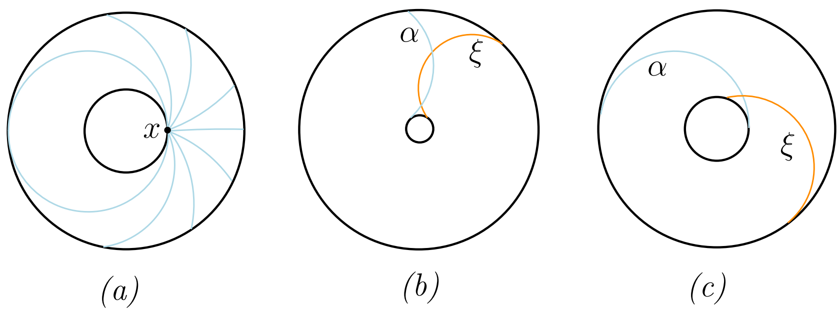

Fig. 2.2: (a) circle arcs in $\mathcal{A}$ connecting x to every point in the other boundary component, *(b)* circle arcs α, ξ with $i(\alpha, \xi) = 0$ and *(c)* with $i(\alpha, \xi) = 1$.

Definition 2.4. *Let $\mathcal{A}$ be a Möbius annulus and x and y be two points on the boundary components $\partial_1\mathcal{A}$ and $\partial_2\mathcal{A}$ respectively. We say x, y is a singular pair on the boundary of $\mathcal{A}$ if there is a non-contractible circle C subset of the closure $\mathrm{cl}(\mathcal{A})$ so that C contains both x and y.*

Lemma 2.5. *(See Figure 2.2.) We call a connected segment of a circle a* circle arc. *Let $\partial_1\mathcal{A}$ and $\partial_2\mathcal{A}$ be the boundary components of a Möbius annulus $\mathcal{A}$. Then:*

1. *Every point from $\partial_1\mathcal{A}$ can be connected to any point on $\partial_2\mathcal{A}$ by a circle arc on $\mathcal{A}$;*

2. *if x, y are a singular pair on $\partial\mathcal{A}$, then there are two non-homotopic circle arcs connecting x and y.*

3. *if x, y are on two different components of $\partial\mathcal{A}$ but are not a singular pair, then all circle arcs connecting x and y belong to the same homotopy class (rel. endpoints.)*

Proof. As suggested in figure 2.2 we can find a family of circle arcs in $cl(\mathcal{A})$ connecting a point on $\partial_1\mathcal{A}$ to any other point on $\partial_2\mathcal{A}$. Without loss of generality, we assume the annulus $\mathcal{A}$ is given by $\mathcal{A} = \{z \in \mathbb{C} \mid e^{-l} < |z| < 1\}$. Let α be an oriented circle arcs on $\mathcal{A}$ connecting the two boundary components. Using polar decomposition, let $r : \mathbb{C} \to (0, \infty)$ be the function giving the distance between 0 and a point on $\mathbb{C}$, and let $\theta : \mathbb{C} \to \mathbb{R}$ be the multivalued function representing the angle. We have $d\theta$ is a closed 1-form on $\mathbb{C} - \{0\}$. For any curve γ not passing thru 0, the winding number of γ is

$$w(\gamma) = \frac{1}{2\pi}\int_\gamma d\theta.$$

This allows us to define the winding number of a curve on a Möbius annulus $\mathcal{A}$ because Inv$^*(\mathcal{A}) \cong SO(2)$ in the planar picture are Euclidean rotations around 0. This is well-defined up to an orientation on the annulus. Since $\alpha : [0,1] \to cl(\mathcal{A})$ is a circle arc connecting two boundary components, α is part of a circle $C_\alpha \subset \mathbb{C}$ that's not centered at 0. So there is a point z_{min} on C_α that's closest to 0, and a point z_{max} on C_α that is furthest from 0. It's easy to see that z_{min}, z_{max} and 0 lie on a straight line in the plane, and z_{min}, z_{max} uniquely determine the circle C_α. So α is a subset of a half circle connecting z_{min} and z_{max}. We use the notation $\frac{1}{2}C_\alpha$ to refer to this half circle. We have the winding number $w(\frac{1}{2}C_\alpha) = \pm\frac{1}{2}$ or 0, and thus any subsegment $\alpha \subset \frac{1}{2}C_\alpha$ has winding number $|w(\alpha)| \leq 1/2$. We have equality if and only if $\alpha = \frac{1}{2}C_\alpha$ and the endpoints of α are z_{min}, z_{max}.

Now let α, β be two circle arcs with the same endpoints on $\partial\mathcal{A}$. Note than $\alpha\beta^{-1}$ is a closed loop with integer winding number, and $|w(\alpha\beta^{-1})| = |w(\alpha) - w(\beta)| \le \frac{1}{2} + \frac{1}{2} = 1$. We have α and β are not homotopic only when $|w(\alpha\beta^{-1})| = 1$. This happens only when α and β are two halves of the same circle $C \subset \mathcal{A}$, which means the end-points are a singular pair. Otherwise, $w(\alpha\beta^{-1}) = 0$ and α, β are homotopic rel. end points. □

Definition 2.6. *Given a Möbius annulus $\mathcal{A}$, we define a radial orientation on it to be an equivalence class of bijective conformal maps $\mathcal{A} \to \{z \in \mathbb{C} \mid e^{-\text{mod}(A)} < |z| < 1\}$ where two maps are equivalent if they are related by a composition with an $SO(2)$ rotation.*

The difference between two such orientation is an inversion on $\mathbb{C}$ that exchanges the two boundary components of $\{z \in \mathbb{C} \mid e^{-\text{mod}(A)} < |z| < 1\}$. Basically, a radial orientation of $\mathcal{A}$ defines which boundary component of $\mathcal{A}$ is the inner and which is the outer one.

Definition 2.7. *Given a Möbius annulus $\mathcal{A}$ which is conformally equivalent to $\{z \in \mathbb{C} \mid e^{-l} < |z| < 1\}$, the* natural S^1-fibration *of $\mathcal{A}$ is a continuous map $f : \mathcal{A} \to [e^{-l}, 1]$ such that the fibers are circles : $f^{-1}(x) = \{z \in \mathbb{C} : |z| = x\}$.*

Definition 2.8. *A* marking *on a radially oriented Möbius annulus $\mathcal{A}$ is a homotopy class $[\alpha]$ (rel. end points) of curves $\alpha : [0, 1] \to cl(\mathcal{A})$ such $\alpha(0)$ is in the inner boundary component and $\alpha(1)$ is in the outer one, and that the winding number $|w(\alpha)| \le 1/2$. The pair $(\mathcal{A}, [\alpha])$ is then called a marked annulus.*

Note that the definition of winding number is in the proof of lemma 2.5. The lemma implies that each marking has a circle arc representative (not unique). Moreover, a marking $[\alpha]$ on a radially oriented Möbius annulus is completely determined by one end-point of α and the winding number $w(\alpha)$, a real number in $[-1/2, 1/2]$.

Chapter 3: Conformal structures and fundamental domains

We will first define the *flat conformal structure* on a manifold. Following Thurston's more general notion of (G, X) structures as in [8] , we let $G = \text{Möb}(S^3)$ and $X = S^3$. Let M be a 3-manifold. Then a $(\text{Möb}(S^3), S^3)$ structure on M is an atlas with charts from open sets of M to S^3, and transition maps are restrictions of actions by $\text{Möb}(S^3)$ elements. We will call this a flat conformal structure on M.

This more rigid type of manifold structure allows us to extend charts along curves and define a developing map on the universal over dev : $\widetilde{M} \to S^3$ which is a local diffeomorphism. We also get a holonomy representation $\rho : \pi_1(M) \to \text{Möb}(S^3)$ which is equivariant with respect to dev. That is, for $A \in \pi_1(M)$ acting by deck transformation on $\widetilde{M}$, $\text{dev}(A.p) = \rho(A).\text{dev}(p)$. From an equivariant pair (dev, ρ) one can construct an atlas as in the definition. So a flat conformal structure can be seen as a pair (dev, ρ) satisfying the above conditions.

3.1 Circle bundles, combinatorial construction

The 3-manifolds in which we are interested are total space of oriented circle bundles with structure group $\text{Homeo}^+(S^1)$. Let E be the total space of such circle bundle. Homeomorphisms of the circle can be extended to the unit disc, so we have an associated disc bundle. The Euler number of E can be viewed as the self-intersection number of a section of the associated disc bundle. This is the point of view taken in [1] as they estimate the Euler number of conformally flat circle bundles coming from tesselations by regular polyhedrons.

There is another equivalent formulation of the Euler number coming from the fundamental group of circle bundle E which we will present below. Regard the surface Σ_g as a $4g$-gon D^2_A whose edges are paired and identified and a standard way. Its fundamental group can then be presented as

$$\pi_1(\Sigma_g, p) = \langle A_1, ..., A_{2g} \mid W(A_1, ..., A_{2g}) = 1 \rangle$$

where W is the word obtained by listing the oriented edges of the $4g$-gon in order, that is $W(A_1, ..., A_{2g}) = [A_1, A_2]...[A_{2g-1}, A_{2g}]$. Here A_i are (homotopy classes of) loops based at p in Σ_g. These loops are joined only at p and their concatenation $W(A_1, ..., A_{2g})$ is a deformation retract of $\Sigma_g - \{\text{a point}\}$.

Suppose D^2_B is a different polygon with n sides which are paired and indentified to get the same surface Σ_g. (n is an even number $\geq 4g$.) Let $W_B(B_1, ..., B_{n/2})$ be again the word obtained by listing the oriented edges in order. Here B_i are paths in Σ_g and not necessarily loops. Then $W(A_1, ..., A_{2g}) \simeq W_B(B_1, ..., B_{n/2})$ are homotopic loops, since they are both deformation retracts of $\Sigma_g - \{\text{a point}\}$.

Let C be the fiber loop over p and we pick a base point $\hat{p} \in C \hookrightarrow E$. Since E is oriented, the fiber bundle restricted to a loop A_i is a torus embedded in E. We can choose loops $\hat{A}_i$ in E based at $\hat{p}$, so $\hat{A}_i$ are (non-unique)“lifts” of A_i. So we have relations $[\hat{A}_i, C] = 1$ coming from the embbeded tori. There are $2g$ tori coming from the $2g$ generating loops on the surface. We cut along these tori and we are left with a circle bundle over a disc which must be trivial. We give this trivial bundle a natural boundary and get $P_A \cong cl(D_A^2) \times S^1$ a closed solid torus (however we remember the gluing identification $\sim$). Note that P_A has $4g$ boundary pieces, each a topological annulus, which are paired up and identified to recover the circle bundle E, that is $E = P_A/\sim$. Thus we call P_A a fundamental region for E. We have P_A is combinatorially equivalent to D_A^2. Under the quotient/identification map $P_A \to (P_A/\sim) = E$, the boundary $\partial P_A \cong T^2$ is mapped to the bundle restricted over the loop $W(A_1, ..., A_{2g})$. This is a non-injective piecewise immersion of ∂P.

From the above combinatorial construction we can compute the fundamental group of E using van Kampen theorem, so we have

$$\pi_1(E) = \left\langle \hat{A}_i, C \mid [\hat{A}_i, C] = 1, W(\hat{A}_1, ..., \hat{A}_{2g}) = C^k \right\rangle.$$

We define this number k to be the Euler number $e(E)$ of circle bundle E. Note that k does not depend on the choice of generators $\hat{A}_i$. If we choose a different lift $\hat{A}_i'$ over A_i, then $\hat{A}_i'$ is also in the torus over A_i, and $\hat{A}_i' \simeq \hat{A}_i C^{a_i}$ for some integer a_i. We have $W(\hat{A}_1, ..., \hat{A}_{2g})$ contains $\hat{A}_i$ and $\hat{A}_i^{-1}$ exactly once, and C commutes with everything, so replacing $\hat{A}_i$ with $\hat{A}_i'$ does not change the relation. Thus $e(E) = k$ is well-defined. This also shows that even though individual lifts $\hat{A}_i$ over A_i are not unique, the lift $W(\hat{A}_1, ..., \hat{A}_{2g})$ of $W(A_1, ..., A_{2g})$ is unique up to homotopy.

We can repeat the above construction with respect to the other fundamental region D_B^2 for Σ_g; thus we get $P_B \cong cl(D_B^2) \times S^1$ with n sides which are annuli, and the sides are paired and glued to recover the same circle bundle E. Similarly, P_B is combinatorially equivalent to an n-gon fundamental region for Σ_g. Again $\partial P_B \cong T^2$ is mapped to the bundle restricted over the loop $W_B(B_1, ..., B_{n/2})$. Similar to above, we choose lifts $\hat{B}_i$ over B_i and get $W_B(\hat{B}_1, ..., \hat{B}_{n/2})$ a lift of $W_B(B_1, ..., B_{n/2})$. We must also have $W_B(\hat{B}_1, ..., \hat{B}_{n/2}) \simeq C^k$.

Now would be a good place to remark on the Milnor-Wood inequality. In [9], it is shown that that for a circle bundle E over a closed surface Σ_g (with $g \geq 1$), the following are equivalent:

- $|e(E)| \leq |\chi(\Sigma_g)| = 2g - 2$.
- E is induced by a representation $\phi : \pi_1(\Sigma_g) \to \text{Homeo}^+(S^1)$.
- There is a foliation of the total space with leaves transverse to the fibers.

A foliation on E transverse to the fibers when restricted to the interior of the fundamental region int$(P) \cong D^2 \times S^1$ must be the product foliation. Each leaf is a closed disc whose boundary is a circle and this creates a foliation on the torus boundary of P. So an equivalent condition for the Milnor-Wood inequality for $E = P/\sim$ is: the boundary identifications ($\sim$) preserve a product foliation on the boundary of P (mapping leaf to leaf.)

3.2 Structures with fundamental domains

We will only consider conformal structures on E with holonomy representation $\rho : \pi_1(E) \to \text{Möb}(S^3)$ that factors into

$$\begin{array}{ccc} \pi_1(E) & \to & \text{Möb}(S^3) \\ \downarrow & \nearrow & \\ \pi_1(\Sigma_g) & & \end{array}$$

In other words, the fiber generator $C \in \pi_1(E)$ is mapped to $\rho(C) = 1$. Let $\Gamma = \rho(\pi_1(E)) \subset \text{Möb}(S^3)$ be the image of ρ, so Γ is isomorphic to a fundamental group of a surface, Γ is said to be a surface group in $\text{Möb}(S^3)$.

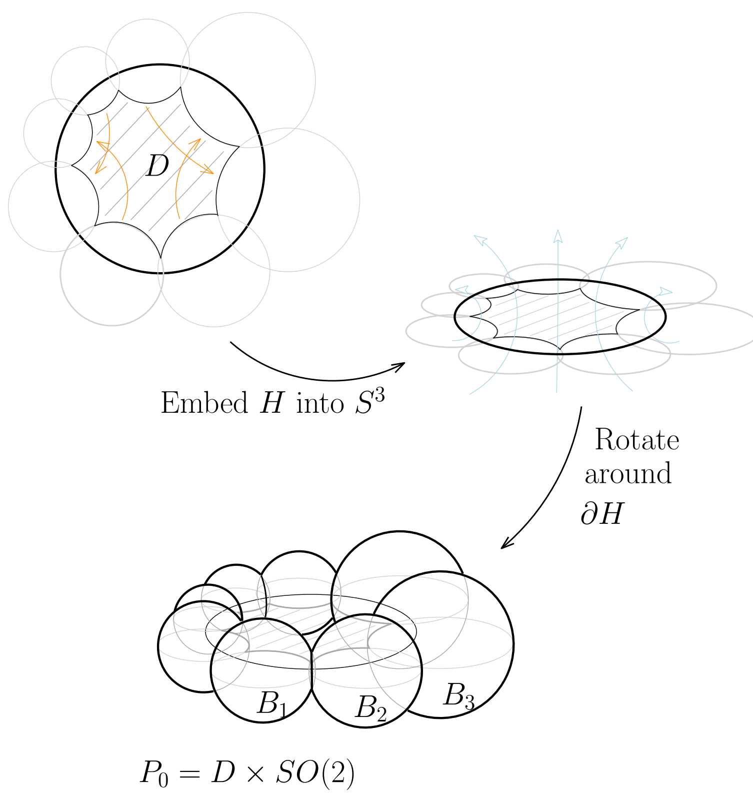

Fig. 3.1: Example 1, Fuchsian example illustration. In this picture, infinity is inside P_0.

Example 1. An easy example of a conformally flat manifold can be described as follows. Take a hyperbolic surface that has fundamental domain D with a *standard identification pattern*; embed the unit disk model of $\mathbb{H}^2$ into S^3 as a half-sphere H; then rotate H by the singular-elliptic group Fix(∂H). The domain D under Fix(∂H)-action will sweep out a polyhedron in S^3 which we denote by P_0, and we have $P_0 = S^3 - \cup_{i=1}^n B_i$ where B_i are open balls. So the faces of P_0 are all *aligned*, that is, the spheres $\partial B_1, \partial B_2, ..., \partial B_n$ are all orthogonal to the same circle ∂H. (Note that by open ball we mean the open connected region in S^3 that's bounded by a 2-sphere.)

This example is basically obtained from a totally geodesic embedding $\mathbb{H}^2 \hookrightarrow \mathbb{H}^4$ along with an embedding of Fuchsian group $\Gamma_0 \hookrightarrow PSL(2, \mathbb{R}) \hookrightarrow$ Möb$^+(S^3)$. P_0 is a fundamental domain for the action of Γ_0 on $S^3 = \partial_\infty \mathbb{H}^4$. Moreover, $(S^3 - \partial H)/\Gamma_0$ is conformally flat trivial bundle over a surface.

Example 2. The first examples of non-trivial circle bundles with flat conformal structures were constructed by Gromov-Lawson-Thurston [1], Kapovich [3] and Kuiper [2]. All these examples are constructed by fundamental domains and/or tesselations of $\mathbb{H}^4$.

Next, some definitions.

Definition 3.1. *A polyhedron in S^1 is either a circle arc bounded by 2 points or the whole circle. A* polyhedron *$P \subset S^m$ is a closed region with non-empty interior $int(P)$ such that:*

- *$cl(int(P)) = P$,*

- *the boundary ∂P is a union of polyhedra in S^{m-1} such that the intersection between two of them are either empty or polyhedra in S^{m-2}.*

The codimension 1 pieces of the boundary are called faces *and the codimension 2 pieces are called* edges.

Definition 3.2. *Let $P \subset S^m$ be a polyhedron of dimension m. We use $\{P^{m-k}\}$ to denote the set of codimension k polyhedra on the boundary of P. We can also call it the combinatorial $(m-k)$-skeleton of ∂P.*

Definition 3.3. *A polyhedron $P \subset S^m$ is* convex *if $P = S^m - \cup B_i$ where each B_i is a ball with boundary an $(m-1)$-sphere. That is P is the intersection of a collection of half spaces.*

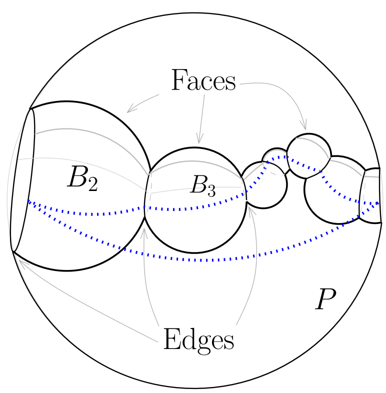

Fig. 3.2: A solid torus polyhedron P with $n = 8$ faces. In this picture, infinity is in $S^3 - P$.

Following [10] we define the following:

Definition 3.4. *A* cornerless *polyhedron is one whose boundary contains only codimension 1 and codimension 2 pieces.*

Let Q be a (2-dimensional) polygon with n edges and hence n vertices. Let $\sim_Q$ be an identification of the edges of Q such that $Q/\sim_Q$ is topologically a closed orientable surface. Let $\Sigma_Q = Q/\sim_Q$ be this topological surface. There is an equivalence relations on the set of edges $\{Q^1\}$ and on the set of vertices $\{Q^0\}$ of Q induced by the above identification; abusing notation we will call this equivalence relation $\sim_Q$ as well.

Let P be a (cornerless) solid torus polyhedron with n faces so that the faces are Möbius annuli. Let $\sim_P$ be an equivalence relation on the set of faces $\{P^2\}$ and the set of edges $\{P^1\}$ of P. We say that $(P, \sim_P)$ and $(Q, \sim_Q)$ are *combinatorially equivalent* if and only if there's a bijective map $f : \{P^2\} \to \{Q^1\}$, $f : \{P^1\} \to \{Q^0\}$ such that $P_i^k \sim_P P_j^k$ exactly when $f(P_i^k) \sim_Q f(P_j^k)$ for $k = 1, 2$ and $i, j \in \mathbb{Z}/n\mathbb{Z}$.

Note that even though $Q/\sim_Q$ is well defined, $P/\sim_P$ is not well-defined if we only have the combinatorial information: equivalence relations on the set of faces $\{P^2\}$ and on the set of edges $\{P^1\}$. We would need actual homeomorphisms between paired faces to define $P/\sim_P$ as a manifold using quotient topology.

Remark 3.5. *The natural S^1-fibration of ∂P is a continuous map $\partial P \to \partial Q$ that extends the natural S^1-fibration on each face of P. See definition 2.7.*

Definition 3.6. *The* domain of discontinuity *of $\Gamma \subset SO(4,1)$, denoted by Ω_Γ, is defined to be the largest open subset of S^3 on which Γ acts properly discontinuously. The* limit set *Λ_Γ of Γ is defined to be $S^3 - \Omega_\Gamma$.*

Theorem 3.7. *Let $P \subset S^3$ be a corneless solid torus polyhedron with n faces and suppose P is a fundamental domain for a surface group $\Gamma \subset$ Möb$^+(S^3)$ acting on $S^3 - \Lambda_\Gamma$. Additionally suppose that the face-pairing transformations in Γ define an equivalence* *relation $\sim_P$ on the set of faces and the set of edges of P combinatorially equivalent to $\sim_Q$ on the set of edges and the set of vertices of a polygon Q that results in an orientable closed surface. Conclusion: Then $E = (S^3 - \Lambda_\Gamma)/\Gamma$ is a conformally flat circle bundle over a closed surface, and there is a loop $\gamma \subset \partial P$ composed of n circular arcs such that $[\gamma] = e(E) \in \pi_1(P) \cong \mathbb{Z}$ (with appropriate orientation for the generator of $\pi_1(P)$).*

Proof. Following [1](section 5 and 7) we have Λ_Γ is a topological circle in S^3, and $S^3 - \Lambda_\Gamma$ is homeomorphic to a solid torus.

The fundamental domain condition implies that we have a tesselation of $S^3 - \Lambda_\Gamma$ by action of Γ on P. The side-pairing Möbius tranformations are unique, and they realize $\sim_P$, thus manifold $E = P/\sim_P$ can be defined. Moreover $E = P/\sim_P = (S^3 - \Lambda_\Gamma)/\Gamma$ a conformally flat manifold because P is a fundamental domain for Γ.

We can define a fibration $S^1 \to P \to Q$ extending the natural S^1-fibration $\partial P \to \partial Q$. Note that the side-pairing Möbius tranformations preserve the natural S^1-fibration of ∂P. In addition, $(P, \sim_P)$ is combinatorially equivalent to $(Q, \sim_Q)$, so we get a fibration $S^1 \to (P/\sim_P) \to (Q/\sim_Q)$. Therefore E is a circle bundle over the closed surface $\Sigma_Q = Q/\sim_Q$.

Now we will describe an algorithm to construct the loop γ. Let $\mathcal{A}_1 ... \mathcal{A}_n$ be the faces of P and $E_1, ..., E_n$ be the edges so that for $i \in \mathbb{Z}/n\mathbb{Z}$ we have $\mathcal{A}_i, \mathcal{A}_{i+1}$ are adjacent and share an edge: E_{i+1}, which means $\mathcal{A}_i$ contains edges E_i, E_{i+1} for $i \in \mathbb{Z}/n\mathbb{Z}$. (The faces and the vertices of P are cyclically ordered.) Suppose edges $E_{i_1}, ..., E_{i_m}$ are identified under face pairing Möbius transformations. We pick a point $p_{i_1} \in E_{i_1}$. The identification maps give us $p_{i_k} \in E_{i_k}$ for $k = 2, ..., m$. Note that these are lifts of the same point in E.

We can do the above for every other equivalence class of edges. Now we have a point p_i on each edge E_i for $i = 1, ..., n$. Face $\mathcal{A}_i$ contains edges E_i, E_{i+1} for $i \in \mathbb{Z}/n\mathbb{Z}$, so we connect p_i to p_{i+1} by a circle arc $\gamma_i : [0, 1] \to \mathcal{A}_i$. This is possible by lemma 2.5. Suppose A_i is identified with A_j by the Möbius transformation $B_{i,j}$, then we have $B_{i,j}(p_i) = p_{j+1}$ and $B_{i,j}(p_{i+1}) = p_j$. Then let $\gamma_j = B_{i,j}(\gamma_i^{-1})$ which is a circle arc connecting p_j to p_{j+1}. We can repeat this process until there is a circle arc on each face connecting $p_1, ..., p_n$, and therefore the concatenation $\gamma = \gamma_1\gamma_2...\gamma_n$ is a closed loop.

Consider the quotient map $p : P \to (P/\sim_P) = E$, and the induced $p^* : \pi_1(P) \to \pi_1(Q)$, and also the quotient map $q : Q \to Q/\sim_Q = \Sigma_Q$ and the induced $q^* : \pi_1(Q) \to \pi_1(\Sigma_Q)$. Let c be a loop in P so that $[c]$ generates $\pi_1(P)$. By construction, $p^*([c]) \in \pi_1(E)$ is the generator corresponding to the fiber of E. Let $a_1, ..., a_n$ be the edges of the polygon Q corresponding to $\mathcal{A}_1, ..., \mathcal{A}_j$ under the equivalence. If $\mathcal{A}_i, \mathcal{A}_j$ are identified faces then $p(\gamma_i) = p(\gamma_j^{-1})$ in E, and also $p(a_i) = p(a_j^{-1})$ in Σ_Q. We have $p(\gamma_i)$ is a lift of $q(a_i)$ for $i = 1, ..., n$. So following the discussion in section 3.1 we have $p^*([\gamma]) = [p(\gamma)] = [p(\gamma_1)...p(\gamma_n)] = p^*([c])^k = p^*([c]^k)$ for some integer k. Thus $[\gamma] = [c]^k \in \pi_1(P)$ since p^* is injective. Moreover $e(E) = k$ as defined, the theorem follows. $\square$

As a corollary, with the same hypothesis as in theorem 3.7, we get $|e(E)| < n^2$. The proof will be in chapter 4.

Example 3. First let us describe a new non-trivial example with Euler number computation. A new feature is that there will be side-pairing Möbius transformations that are loxodromic without rotation.

Let $S_1, S_2, ..., S_{25}$ be open Euclidean spheres in $\mathbb{R}^3 = S^3 - \{\infty\}$ centered at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix},$
$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix},$
$\begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix}$, respectively. We choose r_1 to be the radius for S_i for odd i between 1 and 25, and r_2 to be the radius for B_i for even i between 1 and 25. It is possible to choose r_1, r_2 so that only adjacent spheres intersect. For $i = 1, ..., 24$ we let $E_{i+1} = S_i \cap S_{i+1}$ which are all Euclidean circles. Notice that by construction, the radii of E_i are the same and its value depends on r_1, r_2. Let r be the radius of E_i, technically r is a function $r(r_1, r_2)$. Let E_1 be the image of E_2 under the reflection $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$. Let E_{26} be the image of E_{25} under the reflection $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 16 - x \\ y \\ z \end{pmatrix}$. Let $S_0 = S_{28}$ be the sphere centered at

$\begin{pmatrix} -d \\ 0 \\ 0 \end{pmatrix}$ that contains E_1, and let S_{26} be the ball centered at $\begin{pmatrix} 8+d \\ 0 \\ 0 \end{pmatrix}$ that contains E_{26}. Note that S_0 and S_{26} are symmetric from the mid point $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$. Let S_{27} be a sphere centered at $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ with radius R so that S_{27} only intersect S_{26} and S_0, and let $E_{27} = S_{26} \cap S_{27}$, $E_0 = E_{28} = S_{27} \cap S_0$. For $i \in \mathbb{Z}/28\mathbb{Z}$ let $\mathcal{A}_i$ be the Möbius annulus contained in S_i, bounded by E_i, E_{i+1}. If d is large enough, we can choose radius R such that $\text{mod}(\mathcal{A}_{25}) = \text{mod}(\mathcal{A}_{27})$, so R is completely determined by r_1, r_2, d. Let $P = P(r_1, r_2, d)$ be the conformal polyhedron bounded by $\mathcal{A}_1, ..., \mathcal{A}_{28}$. Let $\theta = \theta(r_1, r_2, d)$ be the sum of inner dihedral angles at each edge of P. We have $\theta(\frac{1}{2}, \frac{1}{2}, 8) = 0$ and $\theta(\frac{3}{4}, \frac{3}{4}, 8) > 2\pi$. So we can choose r_1, r_2, d such that $\theta = 2\pi$.

Let $p_1 \in E_1$ such that the z-coordinate of p_1 is r. That is, p_1 is the point in E_1 with maximum z-coordinate. Given $p_i \in E_i$, choose $p_{i+1} \in E_{i+1}$ such that the winding number $w(\gamma_i) = 0$ for any circle arc $\gamma_i \subset \mathcal{A}_i$ connecting p_i to p_{i+1}. So we can choose circle arc γ_i connecting p_i to p_{i+1} so that it is orthogonal to both E_i, E_{i+1}. We can check that $\gamma = \gamma_1 ... \gamma_{28}$ form a closed loop and $[\gamma]$ generates $\pi_1(P)$.

We now have marked annuli $(\mathcal{A}_i, \gamma_i)$ for $i \in \mathbb{Z}/28\mathbb{Z}$. There are unique Möbius transformations identifying these marked faces in standard identification pattern, that is,

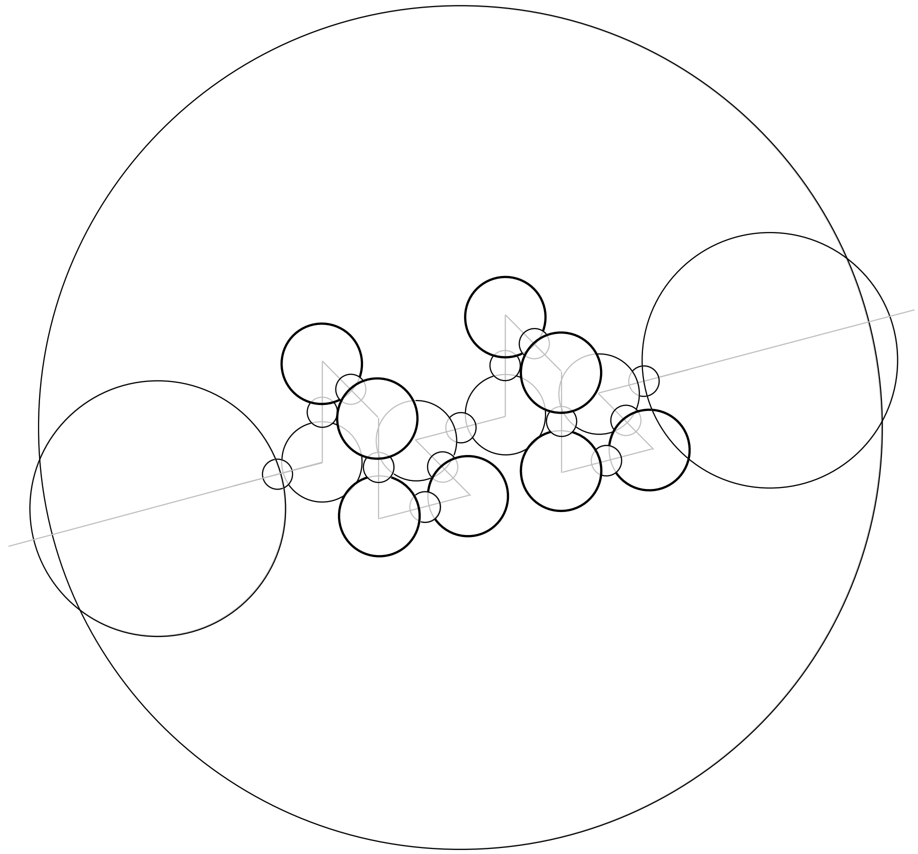

Fig. 3.3: Example 3 illustration.

$(\mathcal{A}_1 \xrightarrow{A_1} \mathcal{A}_3), (\mathcal{A}_4 \xrightarrow{B_1} \mathcal{A}_2), (\mathcal{A}_5 \xrightarrow{A_2} \mathcal{A}_7), (\mathcal{A}_8 \xrightarrow{B_2} \mathcal{A}_6)$, etc.

We can check that the edges $E_1 \xrightarrow{A_1} E_4 \xrightarrow{B_1} E_3 \xrightarrow{A_1^{-1}} E_2 \xrightarrow{B_1^{-1}} E_5 \xrightarrow{A_2} ... \xrightarrow{B_7^{-1}} E_1$ form a *geometric cycle of edges* as defined in [10], and by the main theorem in the same paper we have the face-pairing transformations generate a surface group Γ where P is a fundamental domain for Γ. (In order to check the *geometric cycle of edges* condition we can construct two other piecewise-circle-arc loops around ∂P compatible with side identification maps, this shows that the return map fixes 3 points on E_1 which must be singular-elliptic with rotation angle $\theta = 2\pi$ which must then be the identity.) Moreover, by theorem 3.7 we get $(S^3 - \Lambda_\Gamma)/\Gamma$ is a conformally flat circle bundle with

Euler number 1 over a surface of genus 7.

In this example we have $\Gamma = \langle A_1, B_1, ..., A_7, B_7 \mid \prod [A_i, B_i] = 1\rangle$ where A_i are loxodromics without rotation. This allows for more freedom of deformations as will be described below.

3.3 Deformations of surface groups

Let G be a Lie group. For any $A \in G$ we define $C_G(A)$ be the centralizer of A in G. Note that $C_G(A) \subset G$ is a Lie subgroup which contains 1-parameter subgroups through A (assuming A is in the identity component of G).

Let $\Gamma \subset G$ be a surface group with standard generators

$$\Gamma = \langle A_1, B_1, ..., A_g, B_g \mid W(A_1, B_1, ..., A_g, B_g) = \prod [A_i, B_i] = 1\rangle.$$

We will now describe algebraic deformations of surface group Γ which corresponds to earthquake/grafting in the level of representations.

Non-separating simple closed loops. Let $C \in C_G(A_1)$ not the identity, and let $B_1' = B_1C$. Then $A_1B_1'A_1^{-1}B_1'^{-1} = A_1B_1CA_1^{-1}C^{-1}B_1^{-1} = A_1B_1A_1^{-1}B_1^{-1}$. Then

$$\Gamma' = \langle A_1, B_1', A_2, B_2..., A_g, B_g \mid W(A_1, B_1', ..., A_g, B_g) = 1\rangle$$

is another surface group in G. In most cases, Γ' is not a conjugate of Γ.

Separating simple closed loops. Let k be an integer between 1 and $g-1$. (Think of k as the genus of a component of Σ_g with a separating simple closed loop removed.) Let $C \in C_G(\prod_{i=1}^{k}[A_i, B_i])$. For $i = 1, ..., k$, let $A_i' = CA_iC^{-1}$ and $B_i' = CB_iC^{-1}$. Let $A_i' = A_i$

and $B_i' = B_i$ for $i = k+1, ..., g$. We can check that $\prod_{i=1}^{k}[A_i', B_i']) = \prod_{i=1}^{k}[A_i, B_i])$ and thus

$$\Gamma' = \langle A_1', B_1', ..., A_g', B_g' \mid \prod [A_i', B_i'] = 1 \rangle$$

is another surface group.

Definition 3.8. *We call the above operation an* algebraic earthquake *on surface group representations.*

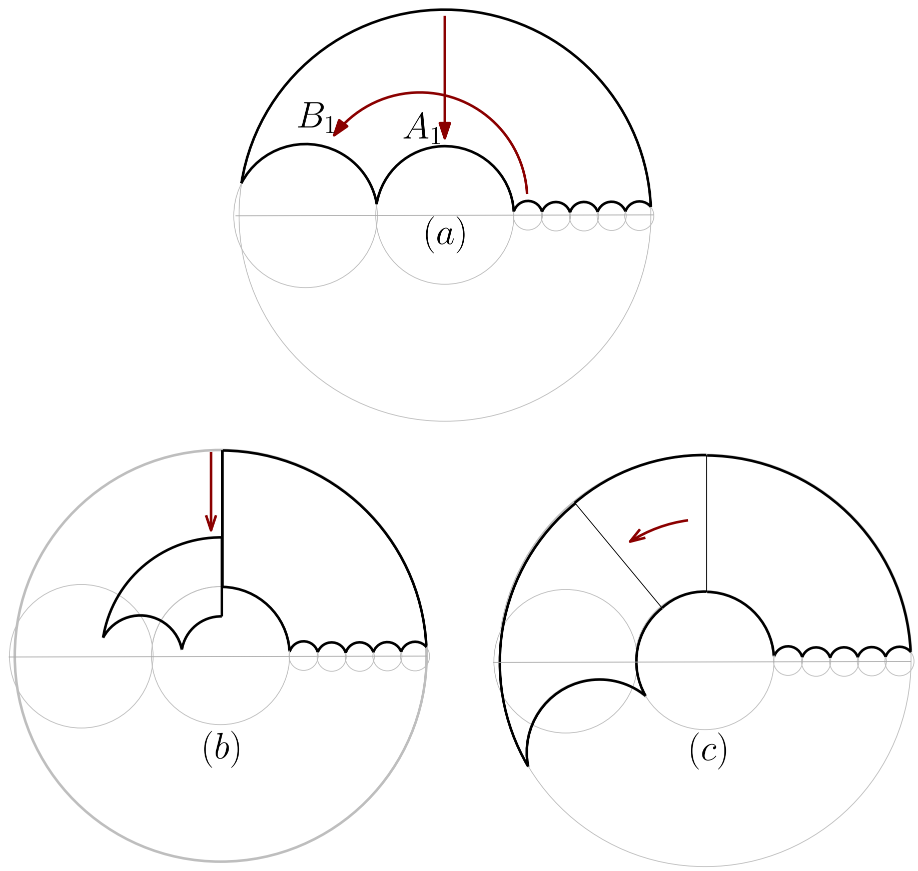

Fig. 3.4: Example 4 illustration. (a) Fundamental domain D of a Fuchsian surface group, (b) the domain after an earthquake $B_1' = B_1 C$ where C is in the $\mathbb{R}_+$ factor of $C_G(A_1)$, (c) the domain after a grafting $B_1' = B_1 C$ where C is in the $SO(2)$ factor of $C_G(A_1)$.

Example 4. Let $\rho : \pi_1(\Sigma_g) \to G = \text{Isom}^+(\mathbb{H}^3) \cong SO^+(3,1)$ be a Fuchsian representation and let $\Gamma = \rho(\pi_1(\Sigma_g))$ with standard generators. We have Γ is purely loxodromic preserving a totally geodesic $\mathbb{H}^2 \hookrightarrow \mathbb{H}^3$, so in particular A_1 is loxodromic without rotation. We have $C_G(A_1) \cong \mathbb{R}_+ \oplus SO(2)$ where the $\mathbb{R}_+$ factor corresponds to the 1-parameter group of non-rotating loxodromics containing A_1, and the $SO(2)$ factor correponds to the elliptic elements having the same fixed points (2 points in S^2) as A_1. Indeed deforming Γ using the $\mathbb{R}_+$ factor corresponds to an earthquake, and using the $SO(2)$ factor corresponds to a grafting along the free homotopy class of $\rho^{-1}(A_1)$ which is represented by a non-separating simple closed loop on the surface. The same analogy works for the case of earthquake/grafting along a separating simple closed loop $\rho^{-1}(\prod_{i=1}^{k}[A_i, B_i])$.

A question still remains: does algebraic earthquake completely generalize classical earthquake and grafting? More specifically, given the free homotopy class of a simple closed loop, do different choices of the standard generator set for Γ yield essentially different algebraic earthquake deformations?

Example 3 (continued). In the example constructed, A_i are non-rotating loxodromic generators. In particular, $\mathcal{A}_{25} \xrightarrow{A_7} \mathcal{A}_{27}$ is non-rotating loxodromic. So upto a conjugation of the whole surface group, A_7 acts as a scaling on $\mathbb{R}^3$: $A_7\vec{x} = \lambda\vec{x}$. Let P' be the image of P under this conjugation, so $\mathcal{A}'_{25}$ and $\mathcal{A}'_{27}$ are "concentric". So $C_G(A_i) \cong \mathbb{R}_+ \oplus SO(3)$. Consider an algebraic earthquake using the $SO(3)$ part of $C_G(A_7)$: we compose B_7 with such a rotation. This deformation of Γ can be realized as a deformation of P': rotating $\mathcal{A}'_{26}$. We can rotate $\mathcal{A}_{26}$ until it is tangent to another face, this represent a path in the space of representations to a group with accidental parabolic.

Chapter 4: Bounding the Euler number

The goal of this chapter is to bound the Euler number of a circle bundle which admits flat conformal structures. The first approach is by using a fundamental domain and applying theorem 3.7. The second approach relies on the formulation of the Euler number of a disc bundle over a surface as the self-intersection number of a section which may be constructed to be piecewise geodesic.

4.1 Fundamental domain approach

Definition 4.1. *Let X be a metric space. For $x \in X, \varepsilon > 0$, we denote $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$ which we call a* ball *of radius ε centered at x. For a subset $S \subset X$, we denote $N_\varepsilon(S) = \bigcup_{x \in S} B_\varepsilon(x)$ which is called the ε-neighborhood of S.*

We now prove the following:

Theorem 4.2. *With the same hypothesis as in theorem 3.7, we get $|e(E)| < n^2$ where n is the number of faces of the fundamental polyhedron.*

Proof. Let γ be the piecewise-circle-arc loop on ∂P as constructed in theorem 3.7. The strategy is to construct a nearby piecewise-circle-arc loop $\beta \in S^3 - P$, so that the Euler number $e(E)$ can be computed (up to a sign) as a linking number $lk(\gamma, \beta)$.

First, let's construct β. As before, let $\mathcal{A}_i, ..., \mathcal{A}_n$ be the faces of P, and let $E_1, ..., E_n$ be the edges of P such that $\mathcal{A}_i$ contains E_i, E_{i+1} for $i \in \mathbb{Z}/n\mathbb{Z}$. Let B_i be the bisecting 2-sphere between $\mathcal{A}_i$ and $\mathcal{A}_{i-1}$, note that B_i contains E_i.

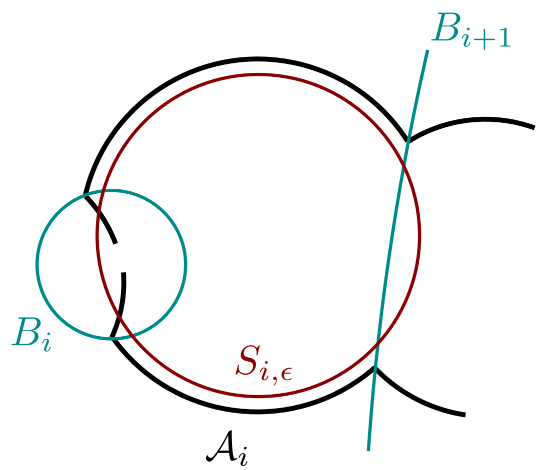

Fig. 4.1: Illustration of the thickened annulus $\mathbf{A_i}$

Let S_i be the 2-sphere containing the Möbius annulus $\mathcal{A}_i$. We know that each $S_i = \{x \in S^3 \mid d_S(x, x_i) = r_i\}$, a metric 2-sphere, where $x_i \in S^3, r_i \in \mathbb{R}_{>0}$ and d_S is the standard spherical metric. Also, $S_i = \exp(S(0, r_i))$ where $S(0, r_i)$ is a sphere of radius r_i centered at 0 in $T_{x_i}S^3$. If $p \in int(\mathcal{A}_i) \subset S_i$ such that $p = \exp(v)$ for $v \in T_{x_i}S^3$, we have for small $\epsilon > 0$, $\exp((1+\epsilon)v)$ is either in $int(P)$ or not in P. If $\exp((1+\epsilon)v) \in int(P)$, we let $S_{i,\epsilon} = \exp(S(0, (1-\epsilon)r_i) \subset T_{x_i}S^3)$ a sphere slightly "smaller" than S_i. Otherwise, we let $S_{i,\epsilon} = \exp(S(0, (1+\epsilon)r_i) \subset T_{x_i}S^3)$. We choose ϵ small enough so that $S_{i,\epsilon}$ intersects B_i and B_{i+1} transversely.

For each annulus $\mathcal{A}_i$, there is an associated thickened annulus $\mathbf{A_i}$ in $S^3 - P$. More specifically, $\mathbf{A_i}$ is a polyhedron bounded by $S_i, S_{i,\epsilon}, B_i, B_{i+1}$. Since P has finitely many faces, we can choose ϵ small enough so that the interior $int(\mathbf{A_i})$ is contained in $S^3 - P$ for all i. Let $F_{i,0}$ and $F_{i,1}$ be the boundary pieces of $\mathbf{A_i}$ which are contained the the spheres

B_i, B_{i+1}. These are closed annuli. Let $int(F_{i,0}), int(F_{i,1})$ be the interior of these annuli as subset of B_i, B_{i+1} respectively. The polyhedron $\mathbf{A_i}$ is a thickened Möbius annulus, it has the property that for any two points $q_i \in int(F_{i,0})$ and $q_{i+1} \in int(F_{i,1})$ can be connected by a circle arc lying completely in $int(\mathbf{A_i})$.

For $i \in \mathbb{Z}/n\mathbb{Z}$ we choose $q_i \in int(F_{i,0}) \cap int(F_{i-1,1})$. We can connect q_i, q_{i+1} by a circle arc β_i such that $\beta_i \subset int(\mathbf{A_i}) \subset S^3 - P$. Thus $\beta = \beta_1, ..., \beta_n$ is a loop composed of n circle arcs. Moreover, β generates $H_1(S^3 - P)$.

By theorem 3.7 we have $e(E) = [\gamma] \in \pi_1(P) \cong H_1(P)$. Consider $S^3 - \beta \supset P$, there is a deformation retract $S^3 - \beta$ to P, thus $\cong H_1(S^3 - \beta) \cong H_1(P)$ and we have $e(E) = \pm[\gamma] \in H_1(S^3 - \beta)$. Therefore $e(E) = \pm lk(\gamma, \beta)$.

Let us consider now the linking number computation by a planar link diagram which can be obtained by projecting γ, β to a generic plane in $\mathbb{R}^3 \subset S^3$. Each circle arc belongs to some circle; a pair of circles is projected to a pair of ellipses. Consider the crossings between these two ellipses, there are at most two $(+)$ crossings and two $(-)$ crossings. Both γ, β are composed of n-circle arcs, thus in the link diagram of γ, β, there are at most $2n^2$ crossings having the same sign. Therefore $e(E) \leq n^2$. Equality cannot be achieved since we can always find segments γ_i and β_j and a generic projection so that their projected images cross no more than once. Therefore we have $e(E) < n^2$. □

We will now provide comments on the possibility of a linear bound. Note that in the construction of β above, we can choose its vertices $q_1, ..., q_n$ arbitrarily close to the vertices $p_1, ..., p_n$ of γ. Consider γ, β as loops in $\mathbb{R}^3$. Replacing each circle arc segment of γ and β with a straight segment we obtain (Euclidean) polygonal unknots γ' and β' with the property that $|lk(\gamma', \beta') - lk(\gamma, \beta)| \leq 2n$. So in the interest of establishing a linear bound for $lk(\gamma, \beta)$, we can work with piecewise linear unknots instead. To show a linear bound on $e(E)$, it suffices to prove the following:

Conjecture 4.3. *Let γ be a piecewise linear unknot with vertices $p_i, ..., p_n$. Let $\varepsilon > 0$ be small enough such that $N_\varepsilon(\gamma)$ is a tubular neighborhood. Then there is a constant c such that for every choice of $q_i \in B_\varepsilon(p_i)$, the piecewise linear unknot β constructed by connecting $q_1, ..., q_n$ (in order) has the property: $lk(\gamma, \beta) < cn$.*

Proposition 4.4. *Let γ be piecewise linear unknot in $\mathbb{R}^3$ with vertices $p_1, ..., p_n$, and let β and β' be two piecewise linear unknots in $\mathbb{R}^3$ disjoint from γ with vertices $q_1, ..., q_n$ and $q'_1, ..., q'_n$ respectively. Let $\varepsilon > 0$ be small enough such that $N_\varepsilon(\gamma)$ is a tubular neighborhood of γ, and $B_\varepsilon(p_i)$ does not contain p_j for $j \neq i$. Suppose that $q_i, q'_i \in B_\varepsilon(p_i)$ for all i. Then*

$$|lk(\gamma, \beta) - lk(\gamma, \beta')| \leq 3n.$$

Proof. The goal is to homotope β to β' in n steps by moving each q_i to q'_i, and we estimate the number of times we cross γ along this homotopy. For notational purpose, let $\overline{ab}$ be the straight segment connecting two points $a, b \in \mathbb{R}^3$, let $\Delta(a, b, c)$ be the Euclidean triangle with vertices at $a, b, c \in \mathbb{R}^3$. By perturbing q_i, q'_i a small amount without changing $lk(\gamma, \beta), lk(\gamma, \beta')$ we can make sure that for all i, (q_i, q'_i, q_{i+1}) and (q_i, q'_i, q_{i-1}) are not colinear triples.

Consider the straight path $q_i(t) = (1-t)q_i + tq'_i$. let β^i_t be the corresponding piecewise linear loop obtained by replacing q_i with $q_i(t)$. So $\beta^i_0 = \beta$ and β^i_1 is a loop where we replace q_i by q'_i. We have $\Delta(q_i, q'_i, q_{i+1}) = (\bigcup_{x \in \overline{q_i q'_i}} \overline{x q_{i+1}})$ and $\Delta(q_i, q'_i, q_{i-1}) =$

$(\bigcup_{x\in\overline{q_i q_i'}} \overline{xq_{i-1}})$. Let $P_i = (\Delta(q_i, q_i', q_{i+1}) \cup \Delta(q_i, q_i', q_{i-1}))$ So each intersection point $\gamma \cap P_i$ corresponds to a place where β_t^i crosses γ as t varies, and each crossing corresponds to a jump $+1$ or -1 in the linking number $lk(\beta_t^i, \gamma)$ as t varies from 0 to 1. By tubular neighborhood condition

$$P_i \cap \gamma = P_i \cap (\overline{p_{i-2}p_{i-1}} \cup \overline{p_{i-1}p_i} \cup \overline{p_i p_{i+1}} \cup \overline{p_{i+1}p_{i+2}})$$

which contains at most 4 points. Each intersection point has a sign which depends on the orientation of P_i and γ. This sign corresponds to the $+1$ or -1 jump in linking number. Moreover, we can show that the total number of points of the same sign in $P_i \cap (\overline{p_{i-2}p_{i-1}} \cup \overline{p_{i-1}p_i} \cup \overline{p_i p_{i+1}} \cup \overline{p_{i+1}p_{i+2}})$ is at most 3. So we have

$$|lk(\beta_0^i, \gamma) - lk(\beta_1^i, \gamma)| \leq 3.$$

We homotope β to β' in n steps, thus

$$|lk(\gamma, \beta) - lk(\gamma, \beta')| \leq 3n.$$

□

The above is not a tight bound, but it is a linear bound which is sufficient for our purposes.

Definition 4.5. *Let γ be a piecewise linear knot with vertices $p_1, ..., p_n$. A* generic projection w.r.t. γ *is a parallel projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\pi(\gamma)$ is injective except at finitely many points, and $\pi(p_i) \neq \pi(p_j)$ for $i \neq j$.*

Definition 4.6. *Let γ be a piecewise linear knot (possibly trivial) in $\mathbb{R}^3$ and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a generic projection with respect to γ. We then denote $K(\gamma, \pi)$ to be the knot diagram of γ under projection π.*

Definition 4.7. *Let K be a knot diagram. The* Writhe number of K *is the sum of signed self-crossings of K, and it is denoted by $Wr(K)$.*

Proposition 4.8. *Let γ be piecewise linear unknot in $\mathbb{R}^3$ with vertices $p_1, ..., p_n$. Let K be the knot diagram for γ under a generic projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. Then for every $\varepsilon > 0$, every $i \in \mathbb{Z}/n\mathbb{Z}$, there is a choice of $q_i \in B_\varepsilon(p_i)$ so that the piecewise linear unknot β constructed by connecting $q_1, ..., q_n$ (in order) is such that $lk(\gamma, \beta) = Wr(K)$.*

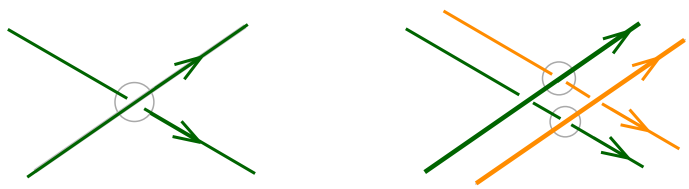

Fig. 4.2: (a) self-crossing of K and (b) corresponding 2 crossings between K and K'.

Proof. Since γ is piecewise linear, its knot diagram K is a projection of γ, so it is also piecewise linear. Here K contains not only the combinatorial information (the signed self-crossings), but also geometric information (vertices and straight segments) of γ. We have $\pi(p_1), ..., \pi(p_n)$ points in $\mathbb{R}^2$ are images of $p_1, ..., p_n$ under the projection. Let $\gamma_1, ..., \gamma_n$ be the segments of γ such that γ_i contains p_i, p_{i+1}. So $\pi(\gamma_1), ..., \pi(\gamma_n)$ are straight segments in $\mathbb{R}^2$ connecting $\pi(p_1), ..., \pi(p_n)$. Since π is generic we get $\pi(\gamma_1), ..., \pi(\gamma_n)$ are disjoint except at the self-crossings.

For each i, let s_i be the line through $\pi(p_i)$ that bisect the angle between $\pi(\gamma_{i-1})$ and $\pi(\gamma_i)^{-1}$. Pick a point $x_1 \in s_1$ of small distance $\delta > 0$ from $\pi(p_1)$. For $i = 1, ..., n-1$, let $x_{i+1} \in s_{i+1}$ such that the straight segment connecting x_i, x_{i+1} is parallel to $\pi(\gamma_i)$. For $i = 1, ..., n-1$, let b_i be the straight segment connecting x_i, x_{i+1}. By property of bisecting line, we have $d(b_i, \pi(\gamma_i))$ are the same and only depends on δ. Thus b_n, the straight segment connecting x_n to x_1, is also parallel to $\pi(\gamma_n)$. We say b_i intersects b_j non-trivially if b_i intersects b_j and $i \neq j \pm 1$.

We choose δ small enough such that $\delta < \varepsilon/3$, and $N_{2\delta}(\gamma)$ is a tubular neighborhood of γ, and such that b_i intersects b_j non-trivially only when $\pi(\gamma_i)$ intersects $\pi(\gamma_j)$ non-trivially (which correponds to a self-crossing of K). Thus by assigning the same sign to corresponding self-crossings, $b_1, ..., b_n$ form a knot diagram K' isomorphic to K.

By our choice of δ, we have $\pi^{-1}(x_i) \cap B_{2\delta}(p_i)$ is non-empty, and we pick a point q_i the intersection. Thus we can construct β as a piecewise linear loop connecting $q_1, ..., q_n$ in order. Indeed β is an unknot since it is contained in a tubular neighborhood of the unknot γ. Moreover, the knot diagram K' is in fact $K(\beta, \pi)$, the knot diagram of β under the same projection π. Thus we can compute $lk(\gamma, \beta)$ as half the sum of signed crossings between K' and K.

For every self-crossing of K, we have two crossings between K and K' of the same sign (see figure 4.2). Therefore $lk(\gamma, \beta) = Wr(K)$.

$\square$

The above two propositions imply that to prove conjecture 4.3, it's enough to establish an n-linear bound on

$$\min_{\text{projection } \pi} Wr(K(\gamma, \pi))$$

for arbitrary piecewise linear unknot γ with n vertices.

4.2 Self-intersection number approach

Suppose we have a disc bundle E over a closed surface Σ_g with a complete hyperbolic structure $E = \mathbb{H}^4/\Gamma$. The Euler number of this bundle, denoted by $e(E)$, is then the self-intersection number of a section $\Sigma_g \overset{s_0}{\hookrightarrow} E$. We can combinatorially construct a piecewise geodesic section as follows:

By the (dev, ρ) pair we can identify $\tilde{E} \cong \mathbb{H}^4$ and $\pi_1(\Sigma_g) \cong \tilde{\Gamma}$. We choose a standard generator set

$$\Gamma = \langle A_1, B_1, ..., A_g, B_g \mid \prod [A_i, B_i] = 1 \rangle.$$

Definition 4.9. *We define the* partial words *of* $\prod_{i=1}^{g} [A_i, B_i]$ *to be* $W_1 = A_1, W_2 = A_1 B_1, W_3 = A_1 B_1 A_1^{-1}, ..., W_{4g} = \prod [A_i, B_i] = 1$.

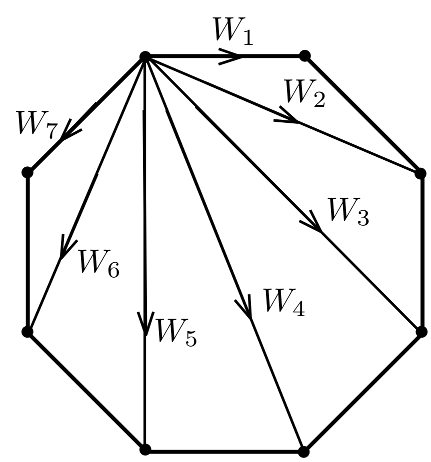

Fig. 4.3: Combinatorial picture of the partial words.

Pick a point $x_0 \in E$ and a representative $\tilde{x_0} \in \mathbb{H}^4$. Let $W_1, ..., W_{4g}$ be the partial words of $\prod [A_i, B_i]$, we can connect the points $W_1 \tilde{x_0}, W_2 \tilde{x_0}, ..., W_{4g} \tilde{x_0} = \tilde{x_0}$ by geodesics in $\mathbb{H}^4$ that form a triangulation of a $4g$-gon into $4g - 2$ triangles. Moreover there is a unique geodesic plane containing given 3 points in $\mathbb{H}^4$, so we have $4g-2$ geodesic triangles with vertices at $W_1\tilde{x_0}, W_2\tilde{x_0}, ..., W_{4g}\tilde{x_0}$. These triangles descend down to define a piecewise immersion $\Sigma_g \to E$ which is homotopic to a section.

Definition 4.10. *Let $s_{x_0} : \Sigma_g \to E$ be the piecewise immersion constructed above, it is well-defined (up to diffeomorphisms homotopic to identity) given a choice of vertex $x_0 \in E$. We call this a* standard $4g-2$ geodesic triangle immersion *of surface into E.*

Let $\tilde{s}_{\tilde{x}_0} : \widetilde{\Sigma}_g \to \mathbb{H}^4$ be the Γ-equivariant lift based at $\tilde{x_0}$.

A technical difficulty. Even though s_{x_0} is homotopic to a section, and it can be well approximated by a smooth immersion, we don't yet know for sure if s_{x_0} is regularly homotopic (or homotopic through immersions) to a section. If we know this then we can safely compute the Euler number of E as the self-intersection of s_{x_0}. One possible avenue to show this is to apply Hirsch's theorem in [11] which guarantees the existence of such a regular homotopy.

For now we will assume that s_{x_0} is regularly homotopic to a section. An approach to bound $e(E)$ suggested to the author by Feng Luo is as follows. The piecewise geodesic section described above is completely determined by the choice of an initial point $\tilde{x_0} \in \mathbb{H}^4$. We can move $\tilde{x_0}$ slightly and get a different piecewise section. We hope that the two sections only intersect transversely, and since each geodesic triangle transervely intersects another at no more than one point, we would have a rough bound $e(E) < (4g-2)(36g-23)$. This number comes from the $4g-2$ geodesic triangles in $\mathbb{H}^4$, and each triangle may possibly intersect $36g-23$ triangles around it if its vertices in $\mathbb{H}^4$ are perturbed.

We observe that there are 4 dimensions of freedom in choosing the base point which determines the piecewise geodesic section. However, the space of all geodesic planes intersecting a given plane non-transversely has dimension ≥ 4. So a non-trivial problem here is to show the existence of another piecewise geodesic section transverse to the original one.

Definition 4.11. *Let $A \in \mathrm{Isom}^+(\mathbb{H}^4) = \mathrm{Möb}^+(S^3)$ be a rotating loxodromic transformation with rotation angle not $2k\pi$ for some integer k. Then there is a circle C in S^3 containing both fixed points and C is invariant under A. We call C the* rotation axis *of A. Also, C bounds a complete totally geodesic 2-dimensional plane $H \subset \mathbb{H}^4$. Depending on the context, we also say H is the rotation axis of A.*

Lemma 4.12. *Let $A \in \mathrm{Isom}^+(\mathbb{H}^4) = \mathrm{Möb}^+(S^3)$ be a rotating loxodromic transformation with rotation angle not $k\pi$ for some integer k. Then there is a unique 3-dimensional totally geodesic subspace $\mathbb{H}^3 \hookrightarrow \mathbb{H}^4$ that is A-invariant.*

Proof. For all $x \in \mathbb{R}^3 = S^3 - \{\infty\}$, we have (up to a conjugation) $A(x) = \lambda R(x)$ for some $\lambda \in \mathbb{R}^+$ and $R \in SO(3) - \{I\}$. Any 3-dimensional subspace invariant under A corresponds to a 2-sphere in S^3 invariant under A. This invariant 2-sphere must contain the fixed points: $0, \infty$, so it must be a Euclidean plane through 0 in $\mathbb{R}^3$. There's only one such plane invariant under R which is the one orthogonal to its rotation axis. Thus there is a unique 3-dimensional totally geodesic subspace invariant under A. □

Note that the above is not true if the rotation angle of R is $k\pi$ for any interger k. If the rotation angle is π, then A leaves invariant any 2-sphere in S^3 that contains the rotation axis (which is a circle).

Definition 4.13. *Let $S_1, ..., S_k$ be subsets of $\mathbb{H}^n$. We define* $\text{span}(S_1, ..., S_k)$ *to be the smallest totally geodesic subspace $\mathbb{H}^m \hookrightarrow \mathbb{H}^n$ that contains $S_1, ..., S_k$.*

Lemma 4.14. *Let H_1, H_2 be two geodesic planes in $\mathbb{H}^4$. If span$(H_1, H_2) = \mathbb{H}^4$ then H_1, H_2 are either disjoint or intersecting transversely.*

Proof. If H_1, H_2 intersect non-transversely (along a plane or a geodesic), then $\text{span}(H_1, H_2)$ is at most 3-dimensional. □

Lemma 4.15. *Suppose we have a surface group $\Gamma \subset G = \text{Isom}^+(\mathbb{H}^4)$, and a choice of generators and partial words as in Definition 4.9 which are all loxodromic. Let A, B, P, Q be such loxodromic transformations. Then there is a point x_0' arbitrarily close to x_0 and x_1' arbitrarily close to x_1 so that* $\text{span}(x_0', Ax_0', Bx_0')$ *is either disjoint from* $\text{span}(x_1', Px_1', Qx_1')$ *or intersecting* $\text{span}(x_1', Px_1', Qx_1')$ *transversely.*

Proof. Let $\Delta(x_0, Ax_0, Bx_0)$ be the geodesic triangle with vertices x_0, Ax_0, Bx_0. The context is that this is a triangle in $\mathbb{H}^4$ which descends to one of the $4g - 2$ geodesic triangles immersed in E. Let $H = \text{span}(x_1, Px_1, Qx_1)$ and let $K = \text{span}(H, x_0, Ax_0, Bx_0)$. If K is 4-dimensional then we're done: $x_0' = x_0$. If K is 2-dimensional, then $H = \text{span}(x_0, Ax_0, Bx_0)$. Then we can move x_0 an arbitrarily small amount to x_0' outside of H and the problem is reduced to the case when K is 3-dimensional. This is when H and $\text{span}(x_0, Ax_0, Bx_0)$ intersect along a geodesic.

Case 1: Suppose K is not invariant under A. Then there is a vector $v \in T_{x_0}K$ so that $dA(v) \notin T_{Ax_0}K$. The condition $dA(v) \notin T_{Ax_0}K$ is an open condition, so we have the freedom to choose v such that $v \notin T_{x_0}H$. Then there is x_0' arbitrarily close to x_0 along the v direction where we have $x_0' \in K - H$ and $Ax_0' \notin K$. So $\text{span}(H, x_0') = K$, thus span$(H, x_0', Ax_0') = \mathbb{H}^4$. Therefore span$(x_0', Ax_0', Bx_0')$ and H are either disjoint or intersecting transversely.

A similar proof works for the case K is not invariant under either B or AB^{-1}.

Case 2: From now we assume that K is invariant under all three transformations A, B, and AB^{-1}. Since K is 3-dimensional, it divides $\mathbb{H}^4$ into two half-spaces.

Case 2a: If A is rotating with angle not $k\pi$ for an interger k, then by lemma there is a unique 3-dimensional geodesic subspace S_A invariant under A. We deform x_0 to x_0' outside of both K and S_A. Thus $K' =$ span(H, x_0', Ax_0', Bx_0') is either 4-dimesional (in which case we're done by lemma 4.14), or a 3-dimensional space distinct from K. In the latter case, A cannot preserve K' as well by lemma 4.12, so we are back in Case 1 which has already been resolved.

The above argument works if either A, B, or AB^{-1} is rotating with angle not $k\pi$.

Case 2b: Now suppose that A, B, AB^{-1} are all rotating with angle $k\pi$ for some integer k.

If A, B, AB^{-1} are all non-rotating, then their action on K is orientation preserving. Pick a vector $v \in T_{x_0}\mathbb{H}^4$ that is orthogonal to K. Its images under the differential maps are $dA(v) \in T_{Ax_0}\mathbb{H}^4$ and $dB(v) \in T_{Bx_0}\mathbb{H}^4$. So $v, dA(v), dB(v)$ must point into the same half space of $\mathbb{H}^4 - K$ because A, B are both orientation preserving. We can then find x_0' arbitrarily close to x_0 along the direction of v, and we have x_0', Ax_0', Bx_0' are all in the same half space, and also the distance from x_0', Ax_0', Bx_0' to K are the same. Thus span(x_0', Ax_0', Bx_0') is disjoint from K, in particular span(x_0', Ax_0', Bx_0') is disjoint from H.

The last case is when two out of the three transformations A, B, AB^{-1} are rotating with angle π, the other is non-rotating. Let C_1, C_2 be the rotation axes of these two loxodromic with π-rotation, these are circles in $\partial_\infty \mathbb{H}^4$. Since K is invariant under A, B, AB^{-1} (by Case 2 assumption), we have $\partial_\infty K \cong S^2$ contains the two rotation axes. We can move x_0 to x_0' outside of K. So we have $K' = \text{span}(H, x_0', Ax_0', Bx_0')$, and $K \cap K' = H$, thus $\partial_\infty K \cap \partial_\infty K' = \partial_\infty H$. If $\partial_\infty K'$ does not contain the two rotation axes then K' is not invariant under all three transformations A, B, AB^{-1} and we are reduced to Case 1. Otherwise, both $\partial_\infty K, \partial_\infty K'$ contain the two rotation axes, which means the two axes are the same circle and the same as $\partial_\infty H = \partial_\infty K \cap \partial_\infty K'$. Now recall that $H = \text{span}(x_1, Px_1, Qx_1)$. We can move x_1 an arbitrarily small amount to x_1' so that $H' = \text{span}(x_1', Px_1', Qx_1')$ is not the rotation axis of either A or B or AB^{-1}. Apply the same argument we get $\text{span}(x_0', Ax_0', Bx_0')$ is either disjoint from H' or intersecting H' transversely.

This concludes the proof of lemma 4.15

□

Theorem 4.16. *Let E be a disc bundle over a closed surface Σ_g with a complete uniformizable hyperbolic structure $E = \mathbb{H}^4/\Gamma$ where Γ is a surface group with a loxodromic standard generator set A_i, B_i and loxodromic partial words $W_1 = A_1, W_2 = A_1B_1, W_3 = A_1B_1A_1^{-1}, ..., W_{4g} = \prod[A_i, B_i] = 1$. Suppose also that all standard $4g-2$ geodesic triangle immersions $\Sigma_g \to E$ are regularly homotopic to a smooth section of E.*

Then

$$e(E) \leq (4g-2)(36g-23).$$

Proof. Fix a point $x_1 \in E$ and a lift $\tilde{x}_1 \in \mathbb{H}^4$. We can choose x_1 such that no lift $\tilde{x}_1$ is in the rotation axis of any loxodromic element of Γ. (We can do this since $\Gamma(\bigcup \text{all rotation axes})$ has measure 0 in $\mathbb{H}^4$.) As before we can construct $4g-2$ geodesic triangles in $\mathbb{H}^4$ with vertices at $W_1\tilde{x_1}, W_2\tilde{x_1}, ..., W_{4g}\tilde{x_1}$ which descends to a continuous map $s_{x_1} : \Sigma_g \to E$ which is homotopic to a section. The Euler number is the the self-intersection number of this section. The $4g-2$ geodesic triangle under Γ expand to a countable collection of geodesic triangles which we name $\{T_{x_1,i}\}_{i\in\mathbb{Z}^+}$.

We pick $\tilde{x}_0$ arbitraily close to $\tilde{x}_1$ in a way that $\tilde{x}_0$ is not in $T_{1,i}$ for any i. For every $\tilde{x}_0'$ in the ball $B(\tilde{x}_0, \epsilon) \subset \mathbb{H}^4$ we can once again construct $\{T_{\tilde{x}_0',i}\}_{i\in\mathbb{Z}^+}$ a countable collection of geodesic triangles from $\tilde{x}_0'$ and Γ. Let $s_{\tilde{x}_0'} : \Sigma_g \to E$ be the piecewise geodesic section based at $\tilde{x}_0'$. We have $\tilde{x}_0'$ is close to $\tilde{x}_0$ which is close to $\tilde{x}_1$, so the sections s_{x_1} and $s_{\tilde{x}_0'}$ are arbitrarily close. If we can make the two sections transverse (by choosing $\tilde{x}_0'$)), then $e(E)$ is the local intersection number between them.

Let $S_{i,j} \subset B(\tilde{x}_0, \epsilon)$ be the set of points $\tilde{x}_0'$ near $\tilde{x}_0$ such that $T_{\tilde{x}_0',j}$ intersects the fixed triangle $T_{x_1,i}$ non-transversely. By lemma 4.15, the set $S_{i,j}$ has empty interior. Moreover $S_{i,j}$ is a closed set since disjoint/transverse is an open condition. Therefore by Baire Category theorem,

$$S = \bigcup_{i,j\in\mathbb{Z}^+} S_{i,j}$$

has empty interior. So we can choose $\tilde{x}_0' \in B(\tilde{x}_0, \epsilon) - S$ and we have $T_{x_1,i}$ is disjoint or transverse to $T_{\tilde{x}_0',j}$ for any $i, j \in \mathbb{Z}^+$. Thus we have two arbitrarily close transverse sections s_{x_1} and $s_{\tilde{x}_0'}$ and therefore

$$e(E) = i(s_{x_1}, s_{\tilde{x}_0'}) \leq (4g-2)(36g-23).$$

□

Chapter 5: Further directions

The GLT conjecture is still open, and there is room for improvement on our results.

Deformations of flat conformal structures on such a circle bundle is a rich and unexplored area of research. All the possible degenerations of quasi-Fuchsian $(PSL(2,\mathbb{C}), \mathbb{C}P^1)$ surface groups can also happen in $(\text{Möb}(S^3), S^3)$ on the Euler number 0 component of discrete representations. Are there other kind of degenerations more specific to this geometry?

Fix a genus $g \geq 2$, we have the set of quasi-Fuchsian surface groups in $PSL(2,\mathbb{C})$ is connected, parametrized by $\mathcal{T}(\Sigma_g) \times \mathcal{T}(\Sigma_g)$. Going one dimension higher we find that the set of quasi-Fuchsian surface groups in $SO^+(4,1)$ is disconnected, because of different possible Euler numbers for the quotient $(S^3 - \Lambda_\Gamma)/\Gamma$. An interesting question is to what extend does the Euler number classify the components of this quasi-Fuchsian set?

Bibliography

[1] M. Gromov, H. Lawson, and W. Thurston. Hyperbolic 4-manifolds and conformally flat 3-manifolds. *Publ. Math. of IHES*, 68:27–45, 1988.

[2] N. Kuiper. Hyperbolic 4-manifolds and tesselations. *Math. Publ. of IHES*, 68:47–76, 1988.

[3] M. Kapovich. Flat conformal structures on three-dimensional manifolds: the existence problem. i. *Sibirsk. Mat. Zh.*, 30:60–73, 1989.

[4] W. Goldman. Conformally flat manifolds with nilpotent holonomy. *Trans. Amer. Math.Soc.*, pages 573–583, 1983.

[5] M. Kapovich. Kleinian groups in higher dimensions. *Progress in Mathematics*, 265:485–562, 2007.

[6] J. Ratcliffe. *Foundations of hyperbolic manifolds*. Springer, 1994.

[7] M. Kapovich. *Hyperbolic manifolds and discrete groups*. Birkhauser Boston Inc., Boston MA, 2001.

[8] W. Thurston. The geometry and topology of three-manifolds. Lecture notes, 1980.

[9] J. Wood. Bundles with totally disconnected structure group. *Comment. Math. Helv.*, 46:257–273, 1971.

[10] S. Anan'in and C. H. Grossi. Yet another poincare polyhedron theorem. *Proceedings of the Edinburgh Mathematical Society (Series 2)*, 54(02):297–308, 2011.

[11] M. W. Hirsch. Immersions of manifolds. *Transactions of the AMS*, 93:242–276, 1959.