ABSTRACT

Title of Thesis: ON THE GENERALIZED TOWER OF HANOI PROBLEM: AN INTRODUCTION TO CLUSTER SPACES.

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In this thesis, we examine the Tower of Hanoi puzzle with p posts ($p \geq 3$) and n disks ($n \in \mathbb{N}$). We examine the puzzle in the context of a *cluster space*: a hierarchical partitioning of the space of all possible disk configurations. This thesis includes two theorems that address the topic of minimal paths connecting disk configurations.

ON THE GENERALIZED TOWER OF HANOI PROBLEM: AN INTRODUCTION TO CLUSTER SPACES

by

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TABLE OF CONTENTS

List of Figures iv

Introduction 1

Chapter 1: Path Trees and Cluster Spaces 3

- Definition 1.1 5
- Lemma 1.1 6
- Definition 1.2 6
- Lemma 1.2 6
- Definition 1.3 7
- Definition 1.4 7
- Lemma 1.3 8

Chapter 2: On Minimality within Cluster Spaces 10

- Definition 2.1 10
- Definition 2.2 10
- Definition 2.3 11
- Definition 2.4 11
- Definition 2.5 12
- Definition 2.6 13
- Lemma 2.1 14

Theorem 2.1 14

Theorem 2.2 16

Conclusion 20

References 21

LIST OF FIGURES

Lucas' classic Tower of Hanoi puzzle 1

Example of a configuration sequence 4

Example of a path tree 4

Introduction

In 1883, the French number theorist Édouard Lucas introduced the now-classic Tower of Hanoi puzzle to the mathematical community as a *récréation mathématique*. In his original puzzle, three posts are attached to a single base, and eight annular disks with mutually distinct diameters are stacked in descending order on one of the posts (i.e., with the largest disk on the bottom of the stack). The goal of the puzzle is to transfer the initial "tower" of disks to a tower on a different post, according to the following rules:

i. each turn consists of transferring at most one disk from the top of one stack to the top of another (possibly empty) stack;

ii. the stacking of any disk on top of a smaller disk is prohibited.

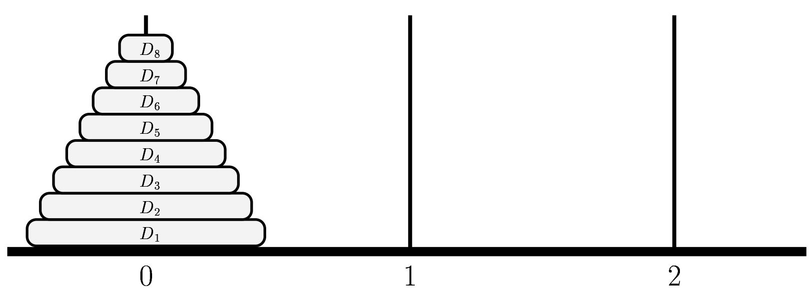

Figure 1: Lucas' classic Tower of Hanoi puzzle

Numerous variants of the original problem have been considered over the past century; some examples include introducing an arbitrary number of disks or extending the number of posts beyond the original three (see Stockmeyer [8] for a listing of over two hundred references). There is, however, a lack of understanding of the underlying mathematical structure of the problem (this issue is aptly summarized by Hinz in [4]). For example, one can find over ten papers in Stockmeyer's list that contribute nothing more than solutions that are equivalent to the well-known solutions of Frame [2] and Stewart [7]. Moreover, one can find arguments based on "intuitive," and sometimes incorrect, assumptions (e.g., see Lemmas 4 and 5 in [5]), or arguments about minimality in the three-post problem (e.g., see either the textbook by Graham, Knuth, and Patashnik [3] or the article by Er [1]). In this thesis, we examine the combinatorial underpinnings of the puzzle with p posts ($p \geq 3$) and n disks ($n \in \mathbb{N}$), and we obtain results concerning minimality within this structure that apply to the discussions in several of the articles alluded to above.

To date, the notation used in analyzing and discussing the problem has not been standardized. The notation used to represent the puzzle in this thesis is as follows: Each of the p posts is represented by an element of the set $\{0, \ldots, p-1\}$. The n disks are denoted by $D_1, \ldots, D_n$, where D_j is the j-th largest disk for every $j \in \{1, \ldots, n\}$. A configuration of the disks is represented by a string $a_1 \ldots a_n$ of elements of $\{0, \ldots, p-1\}$, which indicates that D_j occupies post a_j for every $j \in \{1, \ldots, n\}$. Lastly, the domain of all variables, unless otherwise stated, is the set of positive integers.

Chapter 1: Path Trees and Cluster Spaces

Graphical structures have served as the basis of analysis for the classical Tower of Hanoi problem since the 1940s. The *state graph* $H_p^n = (S_p^n, E)$ of the three-post problem appears in Scorer, Grundy, and Smith [6], where the vertex set S_p^n is the space of valid disk configurations of n disks on p posts, and the edge set E consists of all pairs of vertices corresponding to pairs of disk configurations that differ by a valid disk transfer under the rules of the puzzle. Several authors have investigated the mathematical properties of the state graphs and have demonstrated natural relationships between them and several well-known mathematical entities such as the Arithmetic Triangle (a.k.a. Pascal's Triangle) and Sierpinski's Gasket. According to Hinz [4], graphical structures could provide further insight into the problem with more than three posts.

In this section, we consider a new graphical structure in the analysis of the Tower of Hanoi problem. A *path tree* represents a valid sequence of disk configurations in the puzzle, and each branch of the tree represents a configuration in the sequence. Specifically,

1. the path tree is of height n;

2. for every $j \leq n$, the nodes of the tree at height j designate the positions of D_j in the various configurations in the sequence;

3. a pair of adjacent siblings at height j corresponds to a transfer of D_j from one post to another.

An example of a valid configuration sequence in the three-disk, three-post puzzle is shown in Figure 2. This sequence is represented by the path tree illustrated in Figure 3.

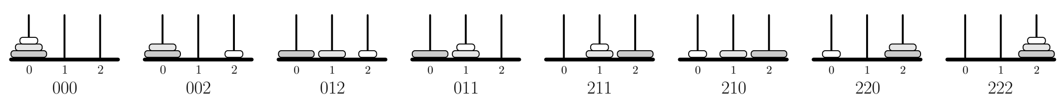

Figure 2: Example of a configuration sequence

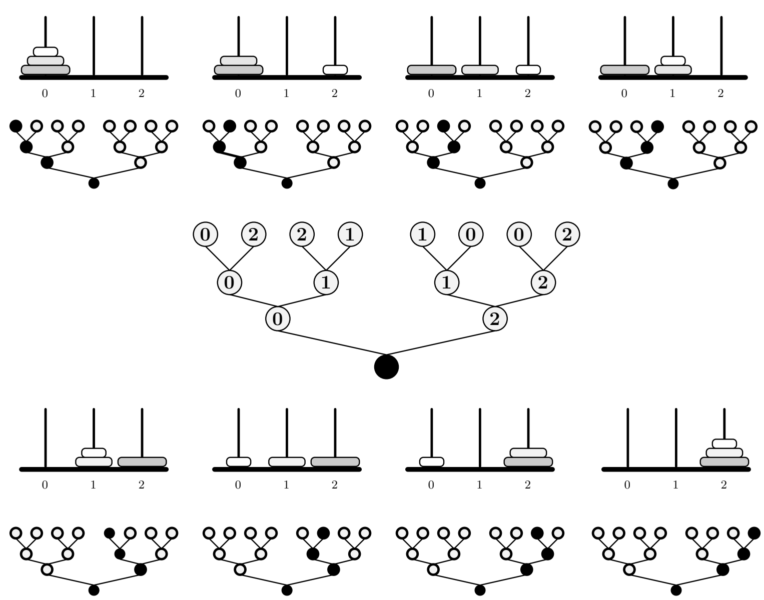

Figure 3: Example of a path tree

We derive the combinatorial structure of the Tower of Hanoi problem with p posts and n disks as follows: Assuming $n \geq 1$, we begin by partitioning the set S_p^n into subsets according to the configurations of the disks, beginning with the largest disk and proceeding in order to the smallest. For example, if $n \geq 2$, there are p disjoint subsets of the form $\{a_1 x \mid x \in S_p^{n-1}\}$ within the entire set of disk configurations, and each subset in the partition is distinguished by the position of D_1 (we will define $S_p^1 = \{0, \ldots, p-1\}$). Furthermore, if $n \geq 3$, each of these p subsets can be partitioned into p distinct subsets of the form $\{a_1 a_2 x \mid x \in S_p^{n-2}\}$. This partition generates p^2 disjoint subsets of disk configurations, and each subset naturally corresponds to one of the p^2 possible configurations of both D_1 and D_2. This recursive method of partitioning the set S_p^n can be extended in a straightforward manner, and leads to our first definition.

Definition 1.1. A cluster of index j is a complete set of valid disk configurations in which $D_1, \ldots, D_j$ are in a fixed configuration. We will denote a cluster of index j by C^j. If D_i occupies post a_i in C^j for every $i \leq j$, then the radix notation for C^j is $\mathbf{a_1} \ldots \mathbf{a_j}$. If $j = 1$, the notation $\mathbf{a_1} \ldots \mathbf{a_{j-1}}$ will represent the empty cluster.

Lemma 1.1. Let $j \leq k \leq n$, and let $C^j = \mathbf{a_1} \ldots \mathbf{a_j}$. Then there exist exactly p^{k-j} clusters $C^k = \mathbf{a_1} \ldots \mathbf{a_j a_{j+1}} \ldots \mathbf{a_k}$ of index k with the property that $C^k \subseteq C^j$.

Proof: The proof is by the combinatorial counting principle. When $k > j$, there are exactly p valid positions of D_k. $\square$

For all pairs j, k with $j \leq k \leq n$, we will denote the set $\{C^k | C^k \subseteq C^j\}$ by $S^k(C^j)$. Also, for every $j \leq n$, we will write S^j to designate the set of all clusters of index j.

Definition 1.2. Let C_a^j and C_b^j be clusters of index j, and let $i \leq j$. Then C_a^j and C_b^j are D_i-adjacent if there exist $C_a^n \in S^n(C_a^j)$ and $C_b^n \in S^n(C_b^j)$ whose corresponding disk configurations differ by exactly one valid transfer of D_i. If this condition holds, then C_a^j and C_b^j are said to form a D_i-adjacent cluster pair, which is written as $(C_a^j, C_b^j)^i$.

Lemma 1.2. Let $i \leq j \leq k$. For every D_i-adjacent pair $(C_a^j, C_b^j)^i$, there exist exactly $(p-2)^{k-j}$ D_i-adjacent pairs $(C_a^k, C_b^k)^i$ with the property that $C_a^k \subseteq C_a^j$ and $C_b^k \subseteq C_b^j$.

Proof: Since C_a^j and C_b^j differ by the transfer of D_i, it follows that all disks smaller than D_i must occupy the $p-2$ "intermediate" posts (i.e., posts other than the source and destination posts occupied by D_i). From this observation, the statement in the lemma follows by the combinatorial counting principle.

□

The following definition introduces the notion of a stack of contiguous disks occupying the same post.

Definition 1.3. Let $j \leq n$, and let $\mathbf{a_1} \ldots \mathbf{a_j}$ be a cluster of index j. We will denote the cluster $\mathbf{a_1} \ldots \mathbf{a_j} \underbrace{\mathbf{a_j} \ldots \mathbf{a_j}}_{n-j}$ of index n by ${}_{(n)}\mathbf{a_1} \ldots \mathbf{a_j}$.

Now, we will define two cluster mappings that will be useful in the proofs of the theorems in the next section.

Definition 1.4. Let $j \leq n$, let $\mathbf{a_1} \ldots \mathbf{a_j}$ be a cluster of index j, and let $\mathbf{a_1} \ldots \mathbf{a_n} \in S^n(\mathbf{a_1} \ldots \mathbf{a_j})$. Then

i. for every cluster $\mathbf{b_1} \ldots \mathbf{b_j}$ of index j, $\Psi^j(\mathbf{a_1} \ldots \mathbf{a_n})$ denotes the cluster of index n defined by

$$\Psi^j(\mathbf{a_1} \ldots \mathbf{a_j} \mathbf{a_{j+1}} \ldots \mathbf{a_n}) \;=\; \mathbf{b_1} \ldots \mathbf{b_j} \mathbf{a_{j+1}} \ldots \mathbf{a_n}$$

under the mapping $\Psi^j \colon \mathbf{a_1} \ldots \mathbf{a_j} \rightarrow \mathbf{b_1} \ldots \mathbf{b_j}$;

ii. for every pair $q_1, q_2 \in \{0, \ldots, p-1\}$, $\Phi^j(\mathbf{a_1} \ldots \mathbf{a_n})$ denotes the cluster of index n defined by

$$\Phi^j(\mathbf{a_1} \ldots \mathbf{a_{j-1}} \mathbf{a_j} \ldots \mathbf{a_n}) = \mathbf{a_1} \ldots \mathbf{a_{j-1}} \mathbf{a_j}^{(\phi^\mathbf{j})} \ldots \mathbf{a_n}^{(\phi^\mathbf{j})}$$

under the mapping $\Phi^j\colon q_1 \leftrightarrow q_2$, where for every $k \geq j$:

a. if $a_k = q_1$, then $a_k^{(\phi^j)} = q_2$;

b. if $a_k = q_2$, then $a_k^{(\phi^j)} = q_1$;

c. if $a_k \neq q_1$ and $a_k \neq q_2$, then $a_k^{(\phi^j)} = a_k$.

As an example, let $p = 4$ and $n = 3$. Then the cluster $\Psi^2(\mathbf{123})$ equals $\mathbf{303}$ under the mapping $\Psi^2\colon \mathbf{12} \to \mathbf{30}$. Also, the cluster $\Phi^1(\mathbf{123})$ equals $\mathbf{213}$ under the mapping $\Phi^1\colon 1 \leftrightarrow 2$.

Lemma 1.3. Let $k \leq n$, and let C_1^n, C_2^n be clusters of index n with the property that $(C_1^n, C_2^n)^k$. Then

i. $k > 1 \implies$ for every $j < k$: if C_a^j is the cluster of index j with the property that $C_1^n, C_2^n \in S^n(C_a^j)$, then for every cluster C_b^j of index j the following hold under the mapping $\Psi^j\colon C_a^j \to C_b^j$:

   a. $\Psi^j(C_1^n), \Psi^j(C_2^n) \in S^n(C_b^j)$;

   b. $(\Psi^j(C_1^n), \Psi^j(C_2^n))^k$;

ii. for every $j \leq k$ and every pair $q_1, q_2 \in \{0, \ldots, p-1\}$, we have $(\Phi^j(C_1^n), \Phi^j(C_2^n))^k$ under the mapping $\Phi^j\colon q_1 \leftrightarrow q_2$.

Proof: The statements in (i) follow from the rules of the puzzle.

For (ii), let

$$C_1^n = \mathbf{a_1 \ldots a_{k-1} a_k a_{k+1} \ldots a_n}$$

and

$$C_2^n = \mathbf{a_1 \ldots a_{k-1} b_k a_{k+1} \ldots a_n}.$$

From the definition of $(C_1^n, C_2^n)^k$, it follows that

1. $a_k \neq b_k$;

2. for every $l > k$, we have that $a_k \neq a_l$ and $b_k \neq a_l$.

Thus from the definition of the mapping $\Phi^j \colon q_1 \leftrightarrow q_2$ and the fact that $j \leq k$, we also have

1. $a_k^{(\phi^j)} \neq b_k^{(\phi^j)}$;

2. for every $l > k$, we have that $a_k^{(\phi_j)} \neq a_l^{(\phi_j)}$ and $b_k^{(\phi^j)} \neq a_l^{(\phi^j)}$. $\square$

Chapter 2: On Minimality within Cluster Spaces

In this section, we obtain elementary results concerning minimality within cluster spaces. We begin with the following definitions.

Definition 2.1. Let $t \geq 1$ and $j \leq n$, and let C_1 and C_t be clusters of index j. A cluster *sequence* from C_1 to C_t, which we will designate by either $\sigma = \{C_1, \ldots, C_t\}$ or $\sigma(C_1, C_t)$, is a sequence of clusters of index j (beginning with C_1 and ending in C_t) with the property that if $t > 1$, then for every $s < t$, exactly one of the following holds:

i. $C_s = C_{s+1}$;

ii. $(C_s, C_{s+1})^i$ for some $i \leq j$.

A cluster *path* $\sigma = \{\alpha_1, \ldots, \alpha_t\}$ is a cluster sequence of index n (i.e., a cluster sequence that naturally corresponds to a valid sequence of disk configurations under the rules of the puzzle).

Definition 2.2. Let $j \leq n$, let C_a^j and C_b^j be clusters of index j, and let $\sigma(C_a^j, C_b^j)$ be a cluster sequence from C_a^j to C_b^j. We will write $|\sigma|$ to denote the length of σ. Moreover, we will say that σ is *minimal* if there is no sequence $\sigma'(C_a^j, C_b^j)$ with $|\sigma'| < |\sigma|$.

Note that each *occurrence* of every cluster in a sequence is counted in determining its length (e.g., the sequence $\{\mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}\}$ is of length 4).

We will now define some basic properties of cluster sequences.

Definition 2.3. Let $s < t$ and $j \leq n$, and let $\sigma_1 = \{C_1, \ldots, C_s\}$ and $\sigma_2 = \{C_{s+1}, \ldots, C_t\}$ be cluster sequences of index j with the property that either $C_s = C_{s+1}$ or $(C_s, C_{s+1})^i$ for some $i \leq j$. Then

i. the sum of σ_1 and σ_2 is $\sigma_1 + \sigma_2 = \{C_1, \ldots, C_s, C_{s+1}, \ldots, C_t\}$, which is also a cluster sequence of index j, and $|\sigma_1 + \sigma_2| = |\sigma_1| + |\sigma_2|$;

ii. the cluster sequence $\sigma = \{C_1, \ldots, C_t\}$ can be decomposed into the sum $\sigma_1(C_1, C_s) + \sigma_2(C_{s+1}, C_t)$.

We will also use the delimited summation notation $\sum_{l=1}^{m} \sigma_l$ to represent the sum $\sigma_1 + \cdots + \sigma_m$.

Definition 2.4. Let $t \geq 1$, and let $\sigma = \{C_1, \ldots, C_t\}$ be a cluster sequence with $|\sigma| = t$. Then

i. $t > 1 \Longrightarrow$ for every $s < t$: if $C_s = C_{s+1}$, then σ can be reduced to the cluster sequence

$$\{C_1, \ldots, C_{s-1}, C_{s+1}, \ldots, C_t\} = \{C_1, \ldots, C_s, C_{s+2}, \ldots, C_t\}$$

(the latter being just $\{C_1, \ldots, C_s\}$ if $s = t - 1$);

ii. if $t = 1$, or if $t > 1$ and $C_s \neq C_{s+1}$ for every $s < t$, then σ is said to be irreducible;

iii. σ can be reduced in a finite number of steps to an irreducible cluster sequence $\langle\sigma\rangle = \{C_{h(1)}, \ldots, C_{h(m)}\}$ with the property that $C_{h(1)} = C_1$, $C_{h(m)} = C_t$, and exactly one of the following holds:

a. $m = 1$;

b. $1 = h(1) < \cdots < h(m) = t$, and $C_{h(l)} \neq C_{h(l+1)}$ for every $l < m$.

Note that $|\langle\sigma\rangle| \leq |\sigma|$ in either case.

We will say that the cluster sequence $\{C_r, \ldots, C_s\}$ is empty when $r > s$.

Definition 2.5. Let $i \leq j \leq n$, and let $\sigma = \{C_1^j, \ldots, C_t^j\}$ be a cluster sequence of index j. The $[\sigma]^i$-decomposition of σ,

$$[\sigma]^i = \sum_{r=1}^{s} \sigma_r^i,$$

is the unique decomposition of σ with the property that for some irreducible cluster sequence $\sigma' = \{C_1^i, \ldots, C_s^i\}$ of index i, there exist $h(1), \ldots, h(s)$ with the following properties:

i. $1 \leq h(1) \leq \ldots \leq h(s) = t$;

ii. $\sigma_1^i = \{C_1^j, \ldots, C_{h(1)}^j\} \subseteq S^j(C_1^i)$;

iii. if $s > 1$, then $h(1) < \cdots < h(s)$, and for every $r < s$:

a. $\sigma_{r+1}^i = \{C_{h(r)+1}^j, \ldots, C_{h(r+1)}^j\} \subseteq S^j(C_{r+1}^i)$;

b. $(C^j_{h(r)}, C^j_{h(r)+1})^l$ for some $l \leq i$.

Under these conditions, we say that σ'contains σ (and that σ traverses σ').

For example, the cluster sequence $\sigma^3 = \{\mathbf{000}, \mathbf{001}, \mathbf{021}, \mathbf{022}\}$ traverses the cluster sequences $\sigma^2 = \{\mathbf{00}, \mathbf{02}\}$ and $\sigma^1 = \{\mathbf{0}\}$, while σ^1 contains both σ^2 and σ^3.

Definition 2.6. Let $j \leq n$, and let $\sigma = \{\alpha_1, \ldots, \alpha_t\}$ be a cluster path. Then

i. if $[\sigma]^j = \sum_{l=1}^m \sigma_l^j$ and the cluster sequence of index j traversed by σ is $\{C_1^j, \ldots, C_m^j\}$, then for every cluster C_a^j of index j, the trace of σ in C_a^j is the sequence

$$\begin{aligned}\Psi_a^j(\sigma) &= \sum_{l=1}^m \Psi_l^j(\sigma_l) \\ &= \sum_{l=1}^m \Psi_l^j(\{\alpha_{h(l-1)+1}, \ldots, \alpha_{h(l)}\}) \\ &= \sum_{l=1}^m \{\Psi_l^j(\alpha_{h(l-1)+1}), \ldots, \Psi_l^j(\alpha_{h(l)})\},\end{aligned}$$

where h is the function which indexes the clusters in σ that belong to the subpaths $\sigma_1, \ldots, \sigma_m$ ($h(0)$ is defined to be 0) and, for every $l \leq m$, Ψ_l^j denotes the mapping $\Psi_l^j \colon C_l^j \to C_a^j$;

ii. for every pair $q_1, q_2 \in \{0, \ldots, p-1\}$, $\Phi^j(\sigma)$ denotes the sequence $\{\Phi^j(\alpha_1), \ldots, \Phi^j(\alpha_t)\}$ under the mapping $\Phi^j \colon q_1 \leftrightarrow q_2$.

Lemma 2.1. The sequences $\Psi_a^j(\sigma)$ and $\Phi^j(\sigma)$ defined in items (i) and (ii) of Definition 2.6 are cluster paths. Moreover, $\Psi_a^j(\sigma) \subseteq S^n(C_a^j)$.

Proof: This is a direct consequence of Lemma 1.3. □

Now, we will consider the first of two theorems regarding minimality. The first theorem illustrates the property of acyclicality of minimal paths.

Theorem 2.1. Let $j \leq n$, let C_a^j be a cluster of index j, and let $C_1^n, C_2^n \in S^n(C_a^j)$. Then every minimal path $\mu(C_1^n, C_2^n)$ has the property that $\mu \subseteq S^n(C_a^j)$.

Proof: It can be shown inductively that the space of disk configurations is connected, hence that paths exist between any pair of clusters of index n in the space. We will show that for any path $\sigma(C_1^n, C_2^n) \not\subseteq S^n(C_a^j)$, there is a path $\sigma'(C_1^n, C_2^n) \subseteq S^n(C_a^j)$ with $|\sigma'| < |\sigma|$, from which it follows that every minimal path $\mu(C_1^n, C_2^n)$ has the property that $\mu \subseteq S^n(C_a^j)$.

Let $\sigma(C_1^n, C_2^n)$ be any path from C_1^n to C_2^n with $\sigma \not\subseteq S^n(C_a^j)$, and let $\{C_a^j, C_1^j, \ldots, C_m^j, C_a^j\}$ be the cluster sequence of index j that contains σ. Without loss of generality, we can assume that σ is irreducible. We will construct a path $\sigma'(C_1^n, C_2^n) \subseteq S^n(C_a^j)$ with the property that $|\sigma'| = |\sigma| - (m + 1)$.

If $m = 1$, consider the decomposition

$$\begin{aligned}[\sigma]^j &= \sigma_{a,1} + \sigma_1 + \sigma_{a,2} \\ &= \{\alpha_1, \ldots, \alpha_r\} + \{\alpha_{r+1}, \ldots, \alpha_s\} + \{\alpha_{s+1}, \ldots, \alpha_t\},\end{aligned}$$

where $\sigma_{a,1}, \sigma_{a,2} \subseteq S^n(C_a^j)$ and $\sigma_1 \subseteq S^n(C_1^j)$. Now $\Psi^j(\sigma_1) \subseteq S^n(C_a^j)$ under the mapping $\Psi^j \colon C_1^j \to C_a^j$; in addition, $\Psi^j(\alpha_{r+1}) = \alpha_r$ and $\Psi^j(\alpha_s) = \alpha_{s+1}$.

Therefore, the path

$$\begin{aligned}\sigma' &= \langle \Psi_a^j(\sigma) \rangle \\ &= (\sigma_{a,1})' + \Psi^j(\sigma_1) + (\sigma_{a,2})' \\ &= \{\alpha_1, \ldots, \alpha_{r-1}\} + \{\Psi^j(\alpha_{r+1}), \ldots, \Psi^j(\alpha_s)\} + \{\alpha_{s+2}, \ldots, \alpha_t\}\end{aligned}$$

has the property that $\sigma' \subseteq S^n(C_a^j)$ and $|\sigma'| = |\sigma| - 2$.

If $m > 1$, consider the decomposition

$$\begin{aligned}[\sigma]^j &= \sigma_{a,1} + \sum_{l=1}^{m} \sigma_l + \sigma_{a,2} \\ &= \{\alpha_{a,1}, \ldots, \alpha_{a,r}\} + \sum_{l=1}^{m} \{\alpha_{l,1}, \ldots, \alpha_{l,h(l)}\} + \{\alpha_{a,r+1}, \ldots, \alpha_{a,r+s}\},\end{aligned}$$

where $\sigma_{a,1}, \sigma_{a,2} \subseteq S^n(C_a^j)$ and $\sigma_l \subseteq S^n(C_l^j)$ for every $l \leq m$. Now for every $l \leq m$, we have $\Psi_l^j(\sigma_l) \subseteq S^n(C_a^j)$ under the mapping $\Psi_l^j \colon C_l^j \to C_a^j$; moreover,

i. $\Psi_1^j(\alpha_{1,1}) = \alpha_{a,r}$;

ii. $\Psi_m^j(\alpha_{m,h(m)}) = \alpha_{a,r+1}$;

iii. for every $l < m$, we have that $\Psi_l^j(\alpha_{l,h(l)}) = \Psi_{l+1}^j(\alpha_{l+1,1})$.

Therefore, the path

$$\begin{aligned}
\sigma' &= \langle \Psi_a^j(\sigma) \rangle \\
&= (\sigma_{a,1})' + \Psi_1^j(\sigma_1) + \sum_{l=2}^{m} (\Psi_l^j(\sigma_l))' + (\sigma_{a,2})' \\
&= \{\alpha_{a,1}, \dots, \alpha_{a,r-1}\} + \{\Psi_1^j(\alpha_{1,1}), \dots, \Psi_1^j(\alpha_{1,h(1)})\} \\
&\quad + \sum_{l=2}^{m} \{\Psi_l^j(\alpha_{l,2}), \dots, \Psi_l^j(\alpha_{l,h(l)})\} \\
&\quad + \{\alpha_{a,r+2}, \dots, \alpha_{a,r+s}\}
\end{aligned}$$

has the property that $\sigma' \subseteq S^n(C_a^j)$ and $|\sigma'| = |\sigma| - (m+1)$. □

Theorem 2.1 establishes the implicit assumptions made in Lemmas 4 and 5 in [5].

The second theorem extends minimal path containment from individual clusters to cluster pairs. This theorem, based on the Reflection Principle, illustrates the combinatorial analog of the triangle inequality in cluster spaces.

Theorem 2.2. Let $j \le n$, let $C_a^j = \mathbf{a_1} \dots, \mathbf{a_{j-1}} \mathbf{a_j}$ and $C_b^j = \mathbf{a_1} \dots, \mathbf{a_{j-1}} \mathbf{b_j}$ be clusters of index j with the property that $(C_a^j, C_b^j)^j$, and let $C_b^n \in S^n(C_b^j)$. Then every minimal path $\mu({}_{(n)}C_a^j, C_b^n)$ is contained in the sequence $\{C_a^j, C_b^j\}$.

Proof: We will show that for every cluster path $\sigma({}_{(n)}C_a^j, C_b^n)$ that is not contained in the sequence $\{C_a^j, C_b^j\}$, there is a path $\sigma'({}_{(n)}C_a^j, C_b^n)$ with the property that σ' is contained in $\{C_a^j, C_b^j\}$ and $|\sigma'| < |\sigma|$, from which it follows that every minimal path $\mu({}_{(n)}C_a^j, C_b^n)$ is contained in $\{C_a^j, C_b^j\}$.

Let $\sigma({}_{(n)}C_a^j, C_b^n)$ be a path connecting ${}_{(n)}C_a^j$ and C_b^n that is not contained in the sequence $\{C_a^j, C_b^j\}$, and let $\{C_a^j, C_1^j, \ldots, C_m^j, C_b^j\}$ be the cluster sequence of index j that contains σ. We can assume that σ is irreducible. Moreover, by Theorem 2.1 and the fact that $C_a^j, C_b^j \in S^j(\mathbf{a_1} \ldots \mathbf{a_{j-1}})$, we can assume that

i. $\{C_1^j, \ldots, C_m^j\} \subseteq S^j(\mathbf{a_1} \ldots \mathbf{a_{j-1}})$;

ii. no two of the clusters $C_a^j, C_1^j, \ldots, C_m^j, C_b^j$ are identical.

We will prove that there exists a path $\sigma'({}_{(n)}C_a^j, C_b^n) = \sigma'_a + \sigma'_b$ with the property that $\sigma'_a \subseteq S^n(C_a^j)$, $\sigma'_b \subseteq S^n(C_b^j)$, and $|\sigma'| = |\sigma| - m$. The proof is by induction on m.

For the case where $m = 1$, consider the decomposition

$$
\begin{aligned}
[\sigma]^j &= \sigma_a + \sigma_1 + \sigma_b \\
&= \{\alpha_1, \ldots, \alpha_r\} + \{\alpha_{r+1}, \ldots, \alpha_s\} + \{\alpha_{s+1}, \ldots, \alpha_t\},
\end{aligned}
$$

where $\sigma_a \subseteq S^n(C_a^j)$, $\sigma_1 \subseteq S^n(C_1^j)$, and $\sigma_b \subseteq S^n(C_b^j)$.

If d_j is the element of $\{0, \ldots, p-1\}$ with the property that $C_1^j = \mathbf{a_1} \ldots \mathbf{a_{j-1}} \mathbf{d_j}$, then $d_j \neq a_j$ and $d_j \neq b_j$. Under the mapping $\Phi^j : b_j \leftrightarrow d_j$ we have

i. $\Phi^j(\sigma_a) \subseteq S^n(C_a^j)$ and $\Phi^j(\sigma_1) \subseteq S^n(C_b^j)$;

ii. $\Phi^j(\alpha_1) = \alpha_1$ and $\Phi^j(\alpha_s) = \alpha_{s+1}$.

Therefore,

$$\sigma' = \Phi^j(\sigma_a + \sigma_1) + (\sigma_b)'$$
$$= \Phi^j(\{\alpha_1, \ldots, \alpha_s\}) + \{\alpha_{s+2}, \ldots, \alpha_t\}$$

is a path of length $|\sigma| - 1$ that connects ${}_{(n)}C_a^j$ to C_b^n and can be decomposed as $\sigma' = \sigma'_a + \sigma'_b$, where $\sigma'_a = \Phi^j(\sigma_a) \subseteq S^n(C_a^j)$ and $\sigma'_b = \Phi^j(\sigma_1) + (\sigma_b)' \subseteq\ S^n(C_b^j)$.

The case of $m > 1$ is resolved as follows: Consider the decomposition

$$[\sigma]^j = \sigma_a + \sum_{l=1}^{m} \sigma_l + \sigma_b$$
$$= \{\alpha_{a,1}, \ldots, \alpha_{a,r}\} + \sum_{l=1}^{m} \{\alpha_{l,1}, \ldots, \alpha_{l,h(l)}\} + \{\alpha_{b,1}, \ldots, \alpha_{b,s}\},$$

where $\sigma_a \subseteq S^n(C_a^j)$, $\sigma_b \subseteq S^n(C_b^j)$, and $\sigma_l \subseteq S^n(C_l^j)$ for every $l \leq m$. Applying the result of the base case to the path $\sigma_a + \sigma_1 + \sigma_2$, we get a path $\tau(\alpha_{a,1}, \alpha_{2,h(2)}) = \sigma''_a + \sigma''_2$ with the property that $\sigma''_a \subseteq S^n(C_a^j)$, $\sigma''_2 \subseteq S^n(C_2^j)$, and $|\tau| = |\sigma_a + \sigma_1 + \sigma_2| - 1$. Thus

$$\sigma'' = \tau(\alpha_{a,1}, \alpha_{2,h(2)}) + \textstyle\sum_{l=3}^{m} \sigma_l + \sigma_b$$

is a path of length $|\sigma| - 1$ that connects ${}_{(n)}C_a^j$ to C_b^n and is contained in a sequence of index j with only $m-1$ intermediate clusters, so the result follows from the induction hypothesis. $\square$

As an example of the process used in the proof of Theorem 2.2, let $p = 4$ and $n = 2$, and let

$$\sigma(\mathbf{11}, \mathbf{00}) = \{\mathbf{11}, \mathbf{12}, \mathbf{32}, \mathbf{31}, \mathbf{21}, \mathbf{01}, \mathbf{00}\}.$$

Note that $[\sigma]^1 = \sigma_{\mathbf{1}} + \sigma_{\mathbf{3}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{0}}$.

After applying the mapping $\Phi^1 \colon 3 \leftrightarrow 2$ to $\sigma_{\mathbf{1}} + \sigma_{\mathbf{3}}$ and reducing the resulting path, we obtain

$$\sigma' = \{\mathbf{11}, \mathbf{13}, \mathbf{23}, \mathbf{21}, \mathbf{01}, \mathbf{00}\},$$

which is of length $|\sigma| - 1$ and has the decomposition $[\sigma']^1 = (\sigma')_{\mathbf{1}} + (\sigma')_{\mathbf{2}} + (\sigma')_{\mathbf{0}}$.

After applying the mapping $\Phi^1 \colon 2 \leftrightarrow 0$ to $(\sigma')_{\mathbf{1}} + (\sigma')_{\mathbf{2}}$ and reducing the resulting path, we obtain

$$\sigma'' = \{\mathbf{11}, \mathbf{13}, \mathbf{03}, \mathbf{01}, \mathbf{00}\},$$

which is of length $|\sigma| - 2$ and is contained in $\{\mathbf{1}, \mathbf{0}\}$.

One important consequence of Theorem 2.2 is that for every $p \geq 3$ and every pair of distinct clusters C_a^1, C_b^1 in S^1, the largest disk transfers only once in any minimal path $\mu({}_{(n)}C_a^1, {}_{(n)}C_b^1)$ (see Wood [9]). In fact, we are now able to deduce that if D_1 occupies post a in C_a^1 and post b in C_b^1, then if $n > 1$ the $[\mu]^2$-decomposition of μ is

$$[\mu({}_{(n)}C_a^1, {}_{(n)}C_b^1)]^2 = \mu_{\mathbf{aa}} + \mu_{\mathbf{ac}} + \mu_{\mathbf{bc}} + \mu_{\mathbf{bb}}$$

for some intermediate post c.

Conclusion

The concept of a cluster space provides a rigorous mathematical basis for analyzing the Tower of Hanoi puzzle, a basis which is lacking throughout a century of literature on the problem. As noted by Hinz [4], the structure of the problem needs to be firmly established before one can make any claims regarding minimality of any proposed solution between towers of disks in the puzzle. As it currently stands, the minimality of the Frame–Stewart solution ([2], [7]) has yet to be demonstrated. In a following article, the author intends to demonstrate the minimality of this well-known solution by using the concept of cluster spaces and the results derived in this thesis.

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