ABSTRACT

Title of thesis: ERGODIC PROPERTIES OF
GIBBS MEASURES FOR EXPANDING MAPS

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Gibbs measure which are also called Sinai-Ruelle-Bowen Measure describe asymptotic behavior and statistical properties of typical trajectories in many physical systems. In this work we review several methods of studying Gibbs measures by Ya.G. Sinai, D. Ruelle, R. Bowen [4], and P. Walters [18]. First, using symbolic dynamics we show for subshifts of finite type that the invariant measure obtained in the Ruelle-Perron-Frobenius (R-P-F)Theorem is an ergodic Gibbs measure. Second, the proof of the R-P-F theorem is given following Walters approach, where he considers maps with infinitely many branches. In both cases, the idea is to find a fixed point $\rho \in \mathcal{C}(X)$ of the transfer operator which will allow us to define the measure $\mu = \rho \cdot m$ where m is the Lebesgue measure. Ergodic properties of μ are studied. In particular results are valid for expanding maps. These ideas are illustrated in the example of an expanding map with two branches where we show explicitly the existence of an invariant measure as well as we prove ergodicity, exactness, and the Rochlin Entropy formula.

ERGODIC PROPERTIES OF GIBBS MEASURES FOR EXPANDING MAPS

by

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Chapter 1: Preliminaries

Let $T : X \to X$ be a continuous map of a compact metric space and let $M(X)$ be the set of all Borel probability measures on X where $\mathcal{B}$ is the family of the Borel sets. One of the main important topics in Dynamical systems is to study the behavior of the orbits $\{T^n : n \in \mathbb{Z}\}$. The existence of an absolutely continuous invariant measures gives an important information about the system. In this chapter, we will introduce some concepts and properties of Ergodic Theory and the well known transfer operator which is also called Ruelle-Perron-Frobenius operator.

Definition 1.1. *Let μ be a measure in $M(X)$. We say that μ is invariant under T or T- invariant if for every Borel set B,*

$$T_*\mu(B) = \mu(B).$$

where $T_*\mu(B) = \mu(T^{-1}B)$.

Definition 1.2. *Let (T, μ) be measure preserving . We say that μ is ergodic if for every Borel sets $B \in \mathcal{B}$ such that $T^{-1}B = B$, $\mu(B) = 0$ or $\mu(B) = 1$.*

Let $\mathcal{C}(X)$ be the set of all continuous function on the set X. Recall that $\mathcal{C}(X)$ is a Banach space with the supreme norm $||\cdot||$.

Definition 1.3. *Let $\varphi \in C(X)$, $\bar{K} \in \mathbb{R}$, and let $T : X \to X$ be a continuous map such that cardinality of the set $\{T^{-1}x\}$ does not exceed $\bar{K}$, for each $x \in X$. The transfer operator $\mathcal{L}_\varphi$ is defined formally on functions $f : X \to \mathbb{C}$ by*

$$\mathcal{L}_\varphi f(x) = \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y).$$

Proposition 1.1. *Let X be a compact metric space and $T : X \to X$ be as in the previous definition. The transfer operator has the following properties.*

1. *$\mathcal{L}_\varphi : \mathcal{C}(X) \to \mathcal{C}(X)$ is linear and bounded.*

2. *If $f \in \mathcal{C}(X)$ is a positive function, then $\mathcal{L}_\varphi f$ is also positive.*

3. *For all $f, g \in \mathcal{C}(X)$,*

$$(\mathcal{L}_\varphi f) \cdot g = \mathcal{L}_\varphi(f \cdot (g \circ T)). \tag{1.1}$$

4. *If n is a positive integer number,*

$$\mathcal{L}^n_\varphi f(x) = \sum_{y \in T^{-n}x} e^{S_n\varphi(y)} f(y) \tag{1.2}$$

where $S_n\varphi(y) = \sum_{k=0}^{n-1} \varphi(T^k y)$.

Proof. 1. It is clear that $\mathcal{L}_\varphi$ is linear. To prove that $\mathcal{L}_\varphi$ is bounded we must show there is a constant $K > 0$ such that $||\mathcal{L}_\varphi f|| \leq K||f||$ for all $f \in C(X)$. Thus,

$$\begin{aligned} |\mathcal{L}_\varphi f(x)| &= \left| \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) \right| \\ &\leq \sum_{y \in T^{-1}x} |e^{\varphi(y)}|\, |f(y)| \\ &\leq ||f|| \sum_{y \in T^{-1}x} |e^{\varphi(y)}| \end{aligned}$$

$$\leq \quad (\bar{K}e^{||\varphi||})||f||.$$

Define $K = \bar{K}e^{||\varphi||}$, then

$$||\mathcal{L}_\varphi f|| \leq K||f||.$$

2. If f is positive, then it is immediately that $\mathcal{L}_\varphi f(x) = \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) > 0$

3. Let $f, g \in \mathcal{C}(X)$,

$$\begin{aligned}
((\mathcal{L}_\varphi f) \cdot g)(x) &= \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) \cdot g(x) \\
&= \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) \cdot g(T(y)) \\
&= \mathcal{L}_\varphi (f \cdot (g \circ T))(x).
\end{aligned}$$

4. We see that (1.2) holds for $n = 1$. Suppose (1.2) holds for n, then we want to prove that it also holds for $n + 1$. In fact,

$$\begin{aligned}
\mathcal{L}_\varphi^{n+1} f(x) &= \mathcal{L}_\varphi (\mathcal{L}_\varphi^n f)(x) \\
&= \sum_{y \in T^{-1}x} e^{\varphi(y)} (\mathcal{L}_\varphi^n f)(y) \\
&= \sum_{y \in T^{-1}x} e^{\varphi(y)} \left[\sum_{z \in T^{-n}(y)} e^{S_n\varphi(z)} f(z) \right] \\
&= \sum_{z \in T^{-n}(y)} e^{\varphi(T^{n+1}(z))} e^{S_n\varphi(z)} f(z) \\
&= \sum_{z \in T^{-(n+1)}(y)} e^{S_{n+1}\varphi(z)} f(z).
\end{aligned}$$

$\square$

Example 1.1. *Let $T : \mathbb{C} \to \mathbb{C}$ be defined by $T(z) = z^2$. We can note that every $w \in \mathbb{C}$, $w \neq 0$ has two pre-images z_1 and z_2, i.e $z_i^2 = w$ where $i = 1, 2$. Let $\varphi : \mathbb{C} \to \mathbb{R}$ define by $\varphi(z) = \ln(|z|^2 + 1)$. Then,*

$$
\begin{aligned}
\mathcal{L}_\varphi f(w) &= (|z_1|^2 + 1)f(z_1) + (|z_2|^2 + 1)f(z_2) \\
&= (|w| + 1)f(z_1) + (|w| + 1)f(z_2) \\
&= (|w| + 1)(f(z_1) + f(z_2))
\end{aligned}
$$

Note that if $w = 0$, then $\mathcal{L}_\varphi f(0) = 2f(0)$.

Definition 1.4. *The dual B^* of a Banach space B is the set of continuous linear functionals $\mu : B \to \mathbb{C}$ endowed with the weak* topology.*

For every $\mu \in B^*$, let us also define the dual $T^* : B^* \to B^*$ of a linear operator $T : B \to B$ by

$$T^*\mu(f) = \mu(T(f)) \tag{1.3}$$

for every $f \in B$.

Remark 1.1. *A sequence μ_n in the space B^* converges to $\mu \in B^*$ if and only if $\mu_n(f)$ converge to $\mu(f)$ for each $f \in B$.*

The following Theorem from Functional analysis identifies the space of probability measures with the dual space of continuous function on X.

Theorem 1.1 (Riesz Representation Theorem)**.** *For each $\mu \in M(X)$ define $\alpha_\mu \in \mathcal{C}(X)^*$ by*

$$\alpha_\mu(f) = \int f d\mu.$$

Then there is a bijection between the space of Borel probability measures, $M(X)$ and the set

$$\{\alpha \in \mathcal{C}(X)^* : \alpha(1) = 1 \text{ and } \alpha(f) \geq 0\}$$

A proof of this theorem can be found in [13]. The importance of this theorem is that we can identify the functional α_μ with the measure μ. Note that if μ is a probability measure $\alpha(1) = \int_X d\mu = \mu(X) = 1$. On the other hand, if $\alpha(1) = 1$, then $\mu(X) = \displaystyle\int_X d\mu = \alpha(1) = 1$.

Proposition 1.2. *As a consequence of the identification given in the Riesz Representation Theorem we can see that $\mu \in M(X)$ is invariant if and only if $\mu(f) = \mu(f \circ T)$ for all $f \in \mathcal{C}(X)$.*

Proof.

$$\begin{aligned}
\mu = T_*\mu &\iff \mu(A) = \mu(T^{-1}A) \\
&\iff \int f d\mu = \int f \, dT_*\mu \qquad \text{for all } f \in \mathcal{C}(X) \\
&\iff \int f d\mu = \int f \circ T \, d\mu \\
&\iff \mu(f) = \mu(f \circ T)
\end{aligned}$$

□

In Measure theory we define the absolute continuity of two measures. Let μ and ν be in $M(X)$, we say that ν is absolutely continuous with respect to μ, and it is denoted by $\nu \ll \mu$, if $\nu(B) = 0$ for every set $B \in \mathcal{B}$ such that $\mu(B) = 0$.

The major result that characterized the absolute continuity is the Radon Nikodyn Theorem which was proved for a special case by Johann Radon in 1913 and then generalized by Otto Nikodym in 1930.

Theorem 1.2 (Radon Nikodyn Theorem)**.** *Let m and μ be two probability measures on $M(X)$. Then, μ is absolutely continuous with respect to m if and only if there exist $f \in L(m)$, $f \geq 0$ and $\int f dm = 1$, such that $\mu(A) = \int_A f dm$ for all Borel set A. The function f is unique almost everywhere.*

The details of the proof can be found in [14].

The function f in the above theorem is called the Radon- Nikodym derivative of μ with respect to m, and it is denoted by $\dfrac{d\mu}{dm}$.

The Birkhoff Ergodic Theorem was proved by David Birkhoff in 1931 in his work: Proof of the ergodic theorem [2]. It is considered one of the most important theorems in Ergodic Theory. There are many different proof of this theorem, however we suggest to see Walters [18].

Theorem 1.3 (Birkhoff Ergodic Theorem for Measure Preserving Transformations)**.** *Let $(X, \mathcal{B}, \mu)$ be a finite measurable space. Let $T : X \to X$ be a measure preserving transformation. For any $f \in L^1(\mu)$, the limits*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$$

converges almost everywhere to a function $\bar{f} \in L^1(\mu)$. The function $\bar{f}$ satisfy that $\bar{f} \circ T = \bar{f}$ a.e, and $\int \bar{f}\, d\mu = \int f\, d\mu$.

The second version of this theorem gives a more explicit result for the physical average we want to study in the particular case when T is ergodic.

Theorem 1.4 (Birkhoff Ergodic Theorem for Ergodic transformations)**.** *Let* $(X, \mathcal{B}, \mu)$ *be a finite measurable space. Let* $T : X \to X$ *be an ergodic measure preserving transformation. For any* $f \in L^1(\mu)$*, the limits*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f \, d\mu$$

for μ *- almost every* $x \in X$*.*

Proof. Since T is ergodic, every function that is invariant almost everywhere is constant almost everywhere (See [18] pag.28). Suppose $\bar{f} = c$, where $c \in \mathbb{R}$, then

$$\int \bar{f} \, d\mu = c \cdot \mu(X) = c,$$

but $\int \bar{f} \, d\mu = \int f \, d\mu$, so

$$c = \int f \, d\mu.$$

Thus by Birkhoff ergodic theorem for measure preserving transformations.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = c = \int f \, d\mu$$

□

Chapter 2: Shift Spaces

2.1 R-P-F Theorem for a shift space

Symbolic dynamics study the structure of the orbits in a dynamical system using an infinite sequence of symbols. Usually it is used as an important tool to study dynamical systems by partitioning the space. The first person who introduce shift spaces, in 1898, was Hadamard [6] with the study of the geodesics on surfaces of negative curvature. In 1938, M. Morse and G. Hedlund presented the first systematic work named: Symbolic Dynamics [9]. Since then symbolic dynamics has been an important tool in different areas like Ergodic Theory, Topological Dynamics, Hyperbolic Dynamics, Information Theory, and Complex Dynamics.

In this section we will introduced the concept of one sided shift spaces and then we will state the famous Ruelle-Perron-Frobenius Theorem which will be crucial to obtain an absolutely continuous invariant measure.

The one-sided shift space is defined by

$$\sum\nolimits_A^+ = \{\underline{x} \in \prod_{i=0}^{\infty}\{1, \cdots, n\} \ : \ A_{x_i,x_{i+1}} = 1 \text{ for all } i \geq 0\}$$

where $A = (a_{ij})$ is a $n \times n$ matrix whose entries are zero and ones. Let

$$\mathcal{F}_A = \{\phi : \sum\nolimits_A^+ \to \mathbb{R} \text{ continuous } : \operatorname*{var}_k \phi \leq b \cdot \alpha^k \text{ some } b, \alpha \in (0,1), \text{ for all } k \geq 0\}$$

where $\mathrm{var}_k\,\phi = \sup\{|\phi(\underline{x}) - \phi(\underline{y})| : x_i = y_i \text{ for all } i \leq k\}$. Thus $\mathcal{F}_A$ is the space of continuous Hölder functions.

Example 2.1. *Define $\sigma : \sum_A^+ \to \sum_A^+$ by $\sigma(\underline{x})_i = x_{i+1}$. Note that σ is a surjective continuous map. Suppose $\varphi \in \sum_A^+ \cap \mathcal{F}_A$. Then, definition 1.3 can be written as:*

$$\mathcal{L}_\varphi f(\underline{x}) = \sum_{y \in \sigma^{-1} x} e^{\varphi(\underline{y})} f(\underline{y})$$

where for each $x \in \sum_A^+$, the set $\sigma^{-1}(x)$ has no more than n pre-images.

Definition 2.1. *We say that $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically mixing if for every U and V, non-empty open subsets of Σ_A^+, there exist N such that $\sigma^m U \cap V \neq \emptyset$ for all $m \geq N$.*

Theorem 2.1 (Ruelle-Perron-Frobenius)**.** *Let Σ_A be topologically mixing. Let $\varphi \in \mathcal{F}_A \cap C(\Sigma_A^+)$. There exist $\lambda > 0$, $h \in C(\Sigma_A^+)$ with $h > 0$ and $\nu \in M(\Sigma_A^+)$ such that*

1. $\mathcal{L}h = \lambda h$,
2. $\mathcal{L}^*\nu = \lambda \nu$,
3. $\nu(h) = 1$,
4. $\lim_{n\to\infty} ||\lambda^{-m}\mathcal{L}^m g - \nu(g)h|| = 0$ *for all* $g \in \mathcal{C}(\Sigma_A^+)$.

This theorem will be proved in Chapter 4 in a more general context. However, for this particular case of one-sided shift we refer to [4].

The R-P-F Theorem give us the existence of the measure ν, the eigenvalue λ, and h. Define $\mu = h \cdot \nu$ by

$$\mu(f) = \nu(hf).$$

Using Theorem 1.1, we can see that μ is a probability measure. In fact, $\mu(1) = \nu(h) = 1$ and $\mu(f) = \nu(hf) \geq 0$ since $h > 0$ by R-P-F Theorem.

Proposition 2.1. *The measure μ is σ-invariant on Σ_A^+.*

Proof. By Proposition 1.2 we are going to prove that $\mu(f) = \mu(f \circ \sigma)$ for all $f \in \mathcal{C}(\Sigma_A^+)$ In fact,

$$\begin{aligned} \mu(f) &= \nu(hf) \\ &= \nu((\lambda^{-1}\mathcal{L}h) \cdot f) \\ &= \lambda^{-1}\nu(\mathcal{L}(h \cdot (f \circ \sigma))) \qquad \text{(by 1.1)} \\ &= \lambda^{-1}\lambda\nu(h \cdot (f \circ \sigma)) \\ &= \mu(f \circ \sigma) \end{aligned}$$

□

Definition 2.2. *Let μ be a measure in $M(\Sigma_A^+)$. We say that μ is mixing if*

$$\lim_{n\to\infty} \mu(E \cap \sigma^{-n}F) = \mu(E)\mu(F)$$

for all Borel sets E and F.

Proposition 2.2. *Let the measure μ be σ-invariant. If μ is mixing, then μ is ergodic.*

Proof. Let E be any Borel set and suppose the measure μ is mixing. Then by the definition above $\lim_{n\to\infty} \mu(E \cap \sigma^{-n}E) = \mu(E)\mu(E)$. Now suppose $T^{-1}(E) = E$, then $\lim_{n\to\infty} \mu(E \cap \sigma^{-n}E) = \mu(E)$ and so $\mu(E) = (\mu(E))^2$. Thus, $\mu(E) = 1$ or $\mu(E) = 0$. □

Lemma 2.1. *Let $f \in \mathcal{C}(\Sigma_A^+)$ be such that $\operatorname{var}_r f = 0$ and let h be as in R-P-F Theorem, then for $n \geq s$ there exist a constant $A > 0$ and $\beta \in (0,1)$ such that*

$$||\lambda^{-n}\mathcal{L}^n(fh) - \nu(fh)h|| \leq A\nu(fh)\beta^{n-s}. \tag{2.1}$$

The proof of this lemma can be found in [4].

Proposition 2.3. *Let μ be the measure obtained in the R-P-F Theorem. Then μ is mixing for $\sigma : \Sigma_A^+ \to \Sigma_A^+$.*

Proof. First note that

$$\begin{aligned}
(\mathcal{L}^m f \cdot g)(x) &= \sum_{y \in \varphi^{-m}x} e^{S_m\varphi(y)} f(y)g(x) \\
&= \sum_{y \in \varphi^{-m}x} e^{S_m\varphi(y)} f(y)g(\sigma^m y) \\
&= \mathcal{L}^m(f \cdot (g \circ \sigma^m))(x)
\end{aligned}$$

Now let $E = \{y \in \Sigma_A : y_i = a_i, r \leq i \leq s\}$ $F = \{y \in \Sigma_A : y_i = b_i, u \leq i \leq v\}$ but since μ is $\sigma-$invariant, we can assume $r = u = 0$. Then,

$$\begin{aligned}
\mu(E \cap \sigma^{-n}F) &= \mu(\chi_E \cdot \chi_{\sigma^{-n}F}) \\
&= \mu(\chi_E \cdot (\chi_F \circ \sigma^n)) \\
&= \nu(h\chi_E \cdot (\chi_F \circ \sigma^n)) \\
&= \lambda^{-n}\mathcal{L}^{*n}\nu(h\chi_E \cdot (\chi_F \circ \sigma^n)) \\
&= \nu(\lambda^{-n}\mathcal{L}^n(h\chi_E \cdot (\chi_F \circ \sigma^n))) \\
&= \nu(\lambda^{-n}(\mathcal{L}^n(h\chi_E)) \cdot \chi_F))
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\mu(E\cap\sigma^{-n}F)-\mu(E)\mu(F)| &= |\mu(E\cap\sigma^{-n}F)-\nu(h\chi_E)\nu(h\chi_F)| \\
&= |\nu(\lambda^{-n}\mathcal{L}^n(h\chi_E)\cdot\chi_F)-\nu(h\chi_E)\nu(h\chi_F)| \\
&= |\nu[\lambda^{-n}\mathcal{L}^n(h\chi_E)\cdot\chi_F-\nu(h\chi_E)h\chi_F]| \\
&= |\nu([\lambda^{-n}\mathcal{L}^n(h\chi_E)-\nu(h\chi_E)h]\chi_F)| \\
&= |\nu(\lambda^{-n}\mathcal{L}^n(h\chi_E)-\nu(h\chi_E)h)||\nu(\chi_F)|
\end{aligned}$$

Since $\chi_E\in\mathcal{C}(\Sigma_A^+)$ and $\mathrm{var}_s(\chi_E)=0$, applying Lemma 2.1 we have:

$$\begin{aligned}
|\nu(\lambda^{-n}\mathcal{L}^n(h\chi_E)-\nu(h\chi_E)h)||\nu(\chi_F)| &\leq A\mu(E)\beta^{n-s}\nu(F) \\
&\leq A\mu(E)\beta^{n-s}
\end{aligned}$$

where $\beta\in(0,1)$. As the last expression tend to zero as n approaches to infinity. We get,

$$\mu(E\cap\sigma^{-n}F)\to\mu(E)\mu(F)$$

□

2.2 Gibbs Measures

Lemma 2.2. *Let* $A=\displaystyle\sum_{k=0}^{\infty}\operatorname*{var}_k\varphi<\infty$. *If* $x,y\in\sum_A^+$ *with* $x_i=y_i$ *for* $i\in[0,m)$. *Then,*

$$|S_m\varphi(x)-S_m\varphi(y)|\leq A.$$

Proof. Let $\varphi \in \mathcal{C}(\Sigma_A^+)$,

$$\begin{aligned}
|S_m\varphi(x) - S_m\varphi(y)| &= |\sum_{k=0}^{m-1}\varphi(\sigma^k x) - \sum_{k=0}^{m-1}\varphi(\sigma^k y)| \\
&\leq \sum_{k=0}^{m-1}|\varphi(\sigma^k x) - \varphi(\sigma^k y)| \\
&\leq \operatorname*{var}_{m-1-k}\varphi \\
&\leq A.
\end{aligned}$$

□

In the following theorem we will use the following inequalities which are immediately consequences of the previous lemma;

$$S_m\varphi(y) - S_m\varphi(x) \leq |S_m\varphi(x) - S_m\varphi(y)| \leq A \tag{2.2}$$

$$S_m\varphi(y) \leq S_m\varphi(x) + A \tag{2.3}$$

Theorem 2.2. *Suppose $\sum_A^+$ is topologically mixing and $\varphi \in \mathcal{C}(\Sigma_A^+)$ is a Holder Function. Then, there exist $\mu \in \sum_A^+$ an invariant probability measure and a number P such that:*

$$c_1 \leq \frac{\mu\{y : y_i = x_i \textit{ for all } i \in [0,m)\}}{e^{-Pm}e^{S_m\varphi(x)}} \leq c_2 \tag{2.4}$$

for every $x \in \sum_A^+$, $m \geq 0$.

Proof. Define the set

$$E = \{y : \; y_i = x_i \text{ for all } i \in [0,m)\}. \tag{2.5}$$

For any $z \in \sum_A^+$ there exist only one $\bar{y} \in \sigma^{-m}z$ with $\bar{y} \in E$. For example, if $z = (z_0, z_1, \ldots)$ we can take $\bar{y} = (x_0, x_1, \cdots, x_{m-1}, z_0, z_1, \ldots)$. Now, by (2.3)

$$\begin{aligned}
\mathcal{L}^m(h\chi_E)(z) &= \sum_{y\in\sigma^{-m}z} e^{S_m\varphi(y)}h(y)\chi_E(y). \\
&= e^{S_m\varphi(\bar{y})}h(\bar{y}) \\
&\leq e^{S_m\varphi(\bar{y})}||h|| \\
&\leq e^{S_m\varphi(x)}e^A||h||,
\end{aligned}$$

thus

$$\mathcal{L}^m(h\chi_E)(z) \leq e^{S_m\varphi(x)}e^A||h||. \tag{2.6}$$

Using inequality (2.6),

$$\begin{aligned}
\mu(E) &= \mu(\chi_E) \\
&= \lambda^{-m}\nu(\mathcal{L}^m(h\chi_E)) \\
&\leq \lambda^{-m}\int \mathcal{L}^m(h\chi_E)d\nu \\
&\leq \lambda^{-m}e^{S_m\varphi(x)}e^A||h||
\end{aligned}$$

.

Taking $c_2 = e^A||h||$ we have,

$$\frac{\mu(E)}{\lambda^{-m}e^{S_m\varphi(x)}} \leq c_2.$$

Moreover, let $P = \log\lambda$, then

$$\frac{\mu(E)}{e^{-Pm}e^{S_m\varphi(x)}} \leq c_2. \tag{2.7}$$

On the other hand, let $M > 0$ which exist since $\sum_A^+$ is topologically mixing, then for every $z \in \sum_A^+$ there exist at least one $\bar{y} \in \sigma^{-m-M} z$ where $\bar{y} \in E$.

Then,

$$\begin{aligned} \mathcal{L}^{m+M}(h\chi_E)(z) &= \sum_{y \in \sigma^{-m-M} z} e^{S_{m+M}\varphi(y)} h(y)\chi_E(y) \\ &\geq e^{S_{m+M}\varphi(\bar{y})} h(\bar{y}), \end{aligned}$$

but,

$$S_{m+M}\varphi(\bar{y}) = S_m\varphi(\bar{y}) + \sum_{k=m}^{M+m-1} \varphi(\sigma^k(\bar{y})),$$

so

$$\begin{aligned} \left|\sum_{k=m}^{M+m-1} \varphi(\sigma^k(\bar{y}))\right| &\leq \sum_{k=m}^{M+m-1} |\varphi(\sigma^k(\bar{y}))| \\ &\leq (M-1)||\varphi|| \leq M||\varphi|| \end{aligned}$$

Hence,

$$-M||\varphi|| \leq \sum_{k=m}^{M+m-1} \varphi(\sigma^k(\bar{y})).$$

This together with inequality (2.3) gives:

$$\begin{aligned} e^{S_{m+M}\varphi(\bar{y})} h(\bar{\bar{y}}) &\geq e^{S_m\varphi(y)} e^{-M||\varphi||} h(\bar{y}) \\ &\geq e^{S_m\varphi(x)-A} e^{-M||\varphi||} \min h \end{aligned}$$

Note that $\min h$ is positive since $h > 0$.

Finally,

$$
\begin{aligned}
\mu(E) &= \lambda^{-m-M}\nu(\mathcal{L}^{m+M}(h\chi_E)) \\
&\geq \lambda^{-m-M}e^{-M||\varphi||-A}(\min h)e^{S_m\varphi(x)} \\
&= \lambda^{-M}(\min h)e^{-M||\varphi||-A}\lambda^{-m}e^{S_m\varphi(x)}.
\end{aligned}
$$

Taking $c_1 = \lambda^{-M}(\inf h)e^{-M||\varphi||-A}$,

$$
\frac{\mu(E)}{\lambda^{-m}e^{S_m\varphi(x)}} \geq c_1. \tag{2.8}
$$

This last result together with (2.7) give us:

$$
c_1 \leq \frac{\mu(E)}{e^{-Pm}e^{S_m\varphi(x)}} \leq c_2.
$$

where $P = \log\lambda$. □

A measure μ that satisfy (2.4) is called a Gibbs measure of φ and it is denoted by μ_φ.

Theorem 2.3. *Suppose $\sum_A^+$ is topologically mixing and $\varphi \in \mathcal{C}(\Sigma_A^+)$ is a Hölder function. The measure $\mu \in M(\sum_A^+)$ and the constant P obtained in the previous theorem are unique.*

Proof. Let E be the set defined in the previous theorem, equation (2.5). Suppose there exist $\bar{\mu} \in M(\sum_A^+)$, and real constant $\bar{c_1}, \bar{c_2}$ and P which satisfies 2.4, i.e,

$$
\bar{c_1} \leq \frac{\bar{\mu}(E)}{e^{-\bar{P}m}e^{S_m\varphi(x)}} \leq \bar{c_2}. \tag{2.9}
$$

Let x be in $\sum_A^+$, and define the set

$$E_m(x) = \{y \in \sum_A^+ \ : \ y_i = x_i \text{ for all } i \in [0, m)\}.$$

Let T_m be a subset of $\sum_A^+$ such that:

1. The cardinality of T_m is finite.

2. $\sum_A^+ = \bigsqcup_{x \in T_m} E_m(x)$.

Re-writing (2.9), we have

$$\bar{c_1} e^{-\bar{P}m} e^{S_m \varphi(x)} \leq \bar{\mu}(E_m) \leq \bar{c_2} e^{-\bar{P}m} e^{S_m \varphi(x)} \tag{2.10}$$

so,

$$\bar{c_1} e^{-\bar{P}m} \sum_{x \in T_m} e^{S_m \varphi(x)} \leq \sum_{x \in T_m} \bar{\mu}(E_m) = 1 \leq \bar{c_2} e^{-\bar{P}m} \sum_{x \in T_m} e^{S_m \varphi(x)}.$$

Then,

$$\frac{\log \bar{c_1}}{m} + \frac{-\bar{P}m}{m} + \frac{\log \sum_{x \in T_m} e^{S_m \varphi(x)}}{m} \leq 0.$$

Similarly,

$$0 \leq \frac{\log \bar{c_2}}{m} + \frac{-\bar{P}m}{m} + \frac{\log \sum_{x \in T_m} e^{S_m \varphi(x)}}{m}.$$

Taking limit when $m \to \infty$,

$$\bar{P} \leq \lim_{m \to \infty} \frac{1}{m} \log \sum_{x \in T_m} e^{S_m \varphi(x)}$$

and

$$\lim_{m \to \infty} \frac{1}{m} \log \sum_{x \in T_m} e^{S_m \varphi(x)} \leq \bar{P}$$

respectively.

Therefore,

$$\bar{P} = \lim_{m\to\infty} \frac{1}{m} \log \sum_{x\in T_m} e^{S_m\varphi(x)}.$$

Applying the same argument to the measure μ, we get $P = \bar{P}$. Now we will prove that μ is unique. In fact, from (2.10) we have

$$(\bar{c_2})^{-1}\bar{\mu}(E_m) \leq e^{-\bar{P}m}e^{S_m\varphi(x)}$$

and similarly for the measure μ,

$$e^{-Pm}e^{S_m\varphi(x)} \leq (c_1)^{-1}\mu(E_m)$$

but since $P = \bar{P}$,

$$\begin{aligned}(\bar{c_2})^{-1}\bar{\mu}(E_m) &\leq (c_1)^{-1}\mu(E_m)\\ \bar{\mu}(E_m) &\leq (c_1)^{-1}\bar{c_2}\mu(E_m)\end{aligned}$$

Let $K = (c_1)^{-1}\bar{c_2}$ then,

$$\bar{\mu}(E) \leq K\mu(E)$$

for all Borel sets E. Thus, $\bar{\mu}$ is absolutely continuous with respect to μ. By the Radon Nikodyn Theorem, theorem 1.2 there exist a function f which is μ- measurable such that $\bar{\mu} = f\mu$.

Then,

$$\bar{\mu} = \sigma_*\bar{\mu}$$

$$
\begin{aligned}
&= (f \circ \sigma)\sigma_* \mu \\
&= (f \circ \sigma)\mu \\
&= (f \circ \sigma)
\end{aligned}
$$

i.e $f = f \circ \sigma$ μ-almost everywhere. and so $f = c$.

Now since $1 = \bar{\mu}(\sigma_A^+) = \int c \, d\mu = c$, we conclude:

$$\bar{\mu} = \mu.$$

$\square$

Chapter 3: Expanding maps with finitely many branches

3.1 Existence of an invariant measure

In the previous Chapter, we introduced subshifts of finite type which are a powerful tool to study hyperbolic systems. Indeed, it is often used to prove ergodic properties without any advance knowledge of measure theory. In the 1970's, Sinai, Ruelle, and Bowen introduced the invariant measures which nowadays are called SRB measures. In their works, they constructed Markov partitions and they studied SRB measures for subshifts of finite type. However, the study of non-hyperbolic systems often requires countable Markov partitions. In the particular case of the quadratic family $f_\lambda(x) = \lambda x(1-x)$, which was studied by Jakobson [7], there exists a set of positive measure where the respective power maps have infinitely many expanding branches and satisfy conditions of the Folklore Theorem which gives the existence of an absolutely continuous invariant measure.

Theorem 3.1 (Folklore Theorem)**.** *Let $X = [0,1]$ and let $\{I_i\}$ be a countable disjoint collection of open subsets of [0,1] such that $\cup I_i$ has full measure in [0,1]. Suppose $f_i : I_i \to [0,1]$ are $\mathcal{C}^2$-expanding maps,suppose there is a constant K such that:*

$$\sup_{x\in I_i} \frac{|D^2 f_i(x)|}{|Df_i(x)|}|I_i| \leq K$$

for all i. Let T be a map defined a.e on [0,1] by $T \mid I_i = f_i$. Then T has a unique invariant probability measure which is equivalent to the Lebesgue Measure on [0,1].

The proof of the theorem, as stated above, can be found in the work of Jakobson [7]. An earlier proof of a similar result can be found in Adler [1]. Adler refers to earlier similar results by Renyi and Sinai, so it is traditionally called "Folklore Theorem". In the following chapter we will generalize the Folklore Theorem using Waters [18] approach, but first we would like to present a simple example which will connect the two ideas between Chapter 2 and Chapter 3.

In order to start getting familiar with our general problem of finding absolutely continuous invariant measures, we will consider the case of an expanding map with two branches.

Let us consider the following example. Let $T : [0,1] \to [0,1]$ be an $\mathcal{C}^2$-expanding map with two branches. We say that T is expanding if there exist $K_0 > 1$ such that

$$|DT| \geq K_0.$$

Let us consider $\varphi = \log\left(|DT(x)|\right)^{-1}$ according to notation in Chapter 1. We want to find an absolutely continuous invariant measure for T which is invariant with respect to the Lebesgue measure. We say that a measure is absolutely continuous when it is absolutely continuous with respect to the Lebesgue measure. Thus, for this particular φ, the transfer operator becomes:

$$\mathcal{L}_\varphi f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|DT(y)|}.$$

Note that the function DT is just the Jacobian of T with respect to the

Lebesgue measure and more explicitly, since we are considering two branches, if $\{x_1, x_2\}$ is the set of pre-images of $x \in [0, 1]$, we have:

$$\mathcal{L}_\varphi f(x) = \frac{f(x_1)}{|DT(x_1)|} + \frac{f(x_2)}{|DT(x_2)|}.$$

Let m be the Lebesgue measure on $[0, 1]$. Then by a simple change of variable we have that $\mathcal{L}$ satisfy the following equation:

$$\int (f \circ T) \cdot g \; dm = \int (\mathcal{L}_\varphi g) \cdot f \; dm \tag{3.1}$$

Thus, the Lebesgue measure m is a fixed point for the dual operator $\mathcal{L}_\varphi^*$. In fact, let f be in $\mathcal{C}([0, 1])$,

$$\begin{aligned} m(f) &= \int f \; dm \\ &= \int (1 \circ T) \cdot f \; dm \\ &= \int (\mathcal{L}_\varphi f) \cdot 1 \; dm \qquad \text{(by equation 3.1)} \\ &= \int (\mathcal{L}_\varphi f) \; dm \\ &= m(\mathcal{L}_\varphi f) \\ &= \mathcal{L}_\varphi^* m(f) \end{aligned}$$

Now, we will show the existence of a fixed point for the operator $\mathcal{L}_\varphi$. To do this we need to state the following famous theorem which will be a very important tool for the proof of existence, not only in this case but also in the more general case that we consider in Chapter 4.

Theorem 3.2 (Schauder-Tychonoff Fixed Point Theorem)**.** *Let K be a compact convex subset of a locally convex space V, and let $F : K \to F(K)$ be a continuous map of K into itself. Then there exists a point $k \in K$ such that $F(k) = k$.*

This Theorem is a generalization of the Schauder Fixed Point Theorem which was proved for Banach spaces by Juliusz Schauder in 1930. Four years latter, Tychonoff generalized the proof for a compact convex subset of a locally convex space. The proof can be found in Royden [14].

Proposition 3.1. *The operator $\mathcal{L}_\varphi$ has a fixed point.*

Proof. Let us fix some $K > 0$ and let

$$\Lambda = \{f : \frac{f(x_1)}{f(x_2)} \leq e^{Kd(x_1,x_2)} \text{ for all } x_1, x_2 \in [0,1], \text{ and } \int f dm = 1\}.$$

Note that the set Λ is convex and compact. Let f be in Λ we want to show that if K is sufficiently large, then $\mathcal{L}_\varphi f \in \Lambda$. Consider any $x, y \in [0,1]$ and let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be the set of pre-images under T for x and y respectively.

$$\mathcal{L}_\varphi f(x) = \frac{f(x_1)}{|DT(x_1)|} + \frac{f(x_2)}{|DT(x_2)|},$$
$$\mathcal{L}_\varphi f(y) = \frac{f(y_1)}{|DT(y_1)|} + \frac{f(y_2)}{|DT(y_2)|}.$$

Also for any $x, y \in [0,1]$ we have $d(x,y) \geq K_0 \cdot d(x_1, y_1)$. In fact,

$$\begin{aligned} d(x,y) &= \int_x^y DT(x)\, dx \\ &\geq \int_{x_1}^{y_1} DT(x)\, dx \\ &\geq K_0 \cdot d(x_1, y_1) \end{aligned}$$

so, since $f \in \Lambda$,

$$\frac{f(x_1)}{f(y_1)} \leq e^{Kd(x_1,y_1)} \leq e^{\frac{K}{K_0}d(x,y)}. \tag{3.2}$$

On the other hand, by Mean Value Theorem, there exist $\theta \in [0,1]$ such that if $g(x) = \log|DT(x)|$ we have:

$$\begin{aligned}
\log\left(\frac{DT(y_1)}{DT(x_1)}\right) &= \log DT(y_1) - \log DT(x_1) \\
&\leq g'(\theta)d(x_1,y_1) \\
&= \left|\frac{D^2T(\theta)}{DT(\theta)}\right| d(x_1,y_1).
\end{aligned}$$

Let $K_1 = \max\limits_{x\in[0,1]}\left|\frac{D^2T(\theta)}{DT(\theta)}\right|$. Since T is $\mathcal{C}^2$, K_1 exist and then

$$\frac{DT(y_1)}{DT(x_1)} \leq e^{K_1 d(x_1,y_1)} \leq e^{\frac{K_1\ d(x,y)}{K_0}} \tag{3.3}$$

Thus, by equations (3.2) and (3.3),

$$\begin{aligned}
\frac{\mathcal{L}f(x)}{\mathcal{L}f(y)} &\leq \max\{\frac{f(x_1)DT(y_1)}{f(y_1)DT(x_1)}, \frac{f(x_2)DT(y_2)}{f(y_2)DT(x_2)}\} \\
&\leq \exp(\frac{K\ d(x,y)}{K_0}) \cdot \exp(\frac{K_1\ d(x,y)}{K_0}) \\
&\leq \exp\left\{\left(\frac{K+K_1}{K_0}\right) d(x,y)\right\}
\end{aligned}$$

As K_1 is fixed and $K_0 > 1$ we get that for a sufficiently large K, $\frac{K+K_1}{K_0} < K$. Then,

$$\frac{\mathcal{L}f(x)}{\mathcal{L}f(y)} \leq \exp\{K \cdot d(x,y)\}$$

and we have proved that

$$\mathcal{L}_\varphi(\Lambda) \subset \Lambda.$$

Then, by Schauder-Tychonoff fixed point Theorem, Theorem 3.2, there exist a fixed point, ρ, for the operator $\mathcal{L}_\varphi$, i.e $\mathcal{L}_\varphi \rho = \rho$. $\square$

Define

$$\mu = \rho \cdot m$$

where ρ is given in the previous proposition. Note that from the definition of Λ it follows that there exists a constant C_0 such that $\rho \geq C_0 > 0$. Now, we can prove that μ is a T-invariant measure. In fact, using Proposition 1.2,

$$\begin{aligned}
\mu(f \circ T) &= \rho \cdot m(f \circ T) \\
&= \int (f \circ T) \cdot \rho \, dm \\
&= \int (\mathcal{L}_\varphi \rho) f \, dm, \qquad (\text{ by } 3.1) \\
&= \int \rho \cdot f \, dm, \qquad (\text{since } \rho \text{ is a fixed point of } \mathcal{L} \text{ }) \\
&= \int f \cdot \rho \, dm \\
&= \rho \cdot m(f) \\
&= \mu(f).
\end{aligned}$$

Hence, μ is T-invariant and it is clear that it is absolutely continuous with respect to the Lebesgue measure m.

The next goal is to prove Bounded Distortion for T^n. The idea is very similar to the one we did in Proposition 3.1. Since we are considering $T : [0,1] \to [0,1]$ an expanding map with with two branches. Let us define the partition $\xi_n = \xi_{w_0} \cap$

$T^{-1}\xi_{w_1} \cap \ldots \cap T^{-(n-1)}\xi_{w_{n-1}}$ where w_i are either 0 or 1 and its elements are denoted by $I_{w_0,\ldots,w_{n-1}}$. Note that the map T^n maps $I_{w_0,\ldots,w_{n-1}}$ onto $[0,1]$.

Proposition 3.2. *Let T be a $\mathcal{C}^2$ expanding map with two branches, then Bounded Distortion Property holds for T^n, i.e there exist $C > 0$ such that*

$$C^{-1} < \frac{(T^n)'(x)}{(T^n)'(y)} < C \tag{3.4}$$

whenever x, y lie in the same partition element $I_{w_0,\ldots,w_{k-1}}$.

Proof. Let $x, y \in [0,1]$. Note that if $x, y \in I_{w_0,\ldots,w_{k-1}}$, then $T^i x$ and $T^i y$ belong to the same partition element I_0 or I_1 for all $i = 0,\ldots,n-1$. Then by the Mean Value Theorem,

$$\begin{aligned}
\left|\log \frac{(T^n)'(x)}{(T^n)'(y)}\right| &= \left|\log \frac{\prod_{i=1}^{n-1} T'(T^i(x))}{\prod_{i=1}^{n-1} T'(T^i(y))}\right| \\
&\le \sum_{i=1}^{n-1} \left|\log(T'(T^i(x))) - \log(T'(T^i(y)))\right| \\
&\le \max_{\theta\in[0,1]} \left|\frac{T''(T^i(\theta))}{T'(T^i(\theta))}\right| \sum_{i=0}^{n-1} d(T^i(x), T^i(y))
\end{aligned}$$

but since $d(T^i x, T^i y) \le \frac{1}{K_0^{n-i}}$, $i = 1, 2, \ldots, n-1$, we have:

$$\begin{aligned}
\sum_{i=0}^{n-1} d(T^i(x), T^i(y)) &\le \lim_{n\to\infty} \sum_{i=0}^{n-1} d(T^i(x), T^i(y)) \\
&\le \lim_{n\to\infty} \sum_{i=1}^{n-1} \frac{1}{K_0^{n-i}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{K_0^n} - 1}{1 - K_0} \\
&= \frac{1}{K_0 - 1}.
\end{aligned}$$

Taking $C_1 = (\frac{1}{K_0 - 1}) \left(\max_{\theta \in [0,1]} \left| \frac{T''(T^i(\theta))}{T'(T^i(\theta))} \right| \right)$ we have,

$$\left| \log \left(\frac{(T^n)'(x)}{(T^n)'(y)} \right) \right| < C_1,$$

$$\leftrightarrow -C_1 < \log \left(\frac{(T^n)'(x)}{(T^n)'(y)} \right) < C_1$$

Let $C = e^{C_1}$

$$C^{-1} < \frac{(T^n)'(x)}{(T^n)'(y)} < C$$

□

3.2 Ergodic properties of T

In this section we will prove two important ergodic properties for our system (T, μ). First, we will start by showing that (T, μ) is ergodic and we will conclude with the proof of exactness.

Lemma 3.1. *Let T be a $\mathcal{C}^2$ expanding map with two branches, then*

$$\frac{m(T^n(A))}{m(T^n(B))} < C \frac{m(A)}{m(B)}$$

for any $A, B \subset [0,1]$ which belong to the same partition element, where C is the constant obtained from the Bounded Distortion Property.

Proof. Using Bounded distortion property, by 3.4, there exist $C > 0$ such that for any x, y in the same element of the partition,

$$(T^n)'(x) \leq C \cdot (T^n)'(y)$$

Let A, B be subsets which belong to the same partition element. Integrating the last expression with respect to the Lebesgue measure m over the set A, we get for y in that element,

$$\begin{aligned}\int_A (T^n)'(x) dm(x) &\leq C \int_A (T^n)'(y)\, dm(x) \\ &\leq C \cdot ((T^n)'(y)) \cdot m(A).\end{aligned}$$

Then integrating over the set B with respect to y we get

$$\left(\int_A (T^n)'(x)\, dm(x)\right) \cdot m(B) \;\leq\; C \cdot m(A) \left(\int_B (T^n)'(y)\, dm(y)\right)$$

Thus,

$$\frac{m(T^n)(A)}{m(T^n)(B)} = \frac{\int_A (T^n)'(x)\, dm(x)}{\int_B (T^n)'(y)\, dm(y)} \leq C \cdot \frac{m(A)}{m(B)}$$

Therefore we have proved:

$$\frac{m(T^n)(A)}{m(T^n)(B)} \leq C \cdot \frac{m(A)}{m(B)}$$

□

We recall the well known Lebesgue's Density Point Theorem, the proof can be found in [5].

Theorem 3.3 (Lebesgue's Density Point)**.** *Let A be a Lebesgue measurable subset of $[0, 1]$ and let $B_\epsilon(x)$ be a ϵ- neighborhood of a point $x \in \mathbb{R}$. Then for almost all $x \in A$ the limit*

$$\lim_{\epsilon \to 0} \frac{m(A \cap B_\epsilon(x))}{m(B_\epsilon(x))} \tag{3.5}$$

exist and equals 1.

The points for which 3.5 hold are called density points of the set A.

Theorem 3.4. *(T, μ) is ergodic.*

Proof. Suppose by contradiction that T is not ergodic. Let A be a Borel set such that $T^{-1}A = A$ and $0 < \mu(A) < 1$. Let x be a density point of A, then by the above result we know that for the Lebesgue measure m,

$$\lim_{\epsilon \to 0} \frac{m(A \cap B_\epsilon(x))}{m(B_\epsilon(x))} = 1.$$

Re-writing this,

$$\begin{aligned}
&\leftrightarrow \lim_{\epsilon \to 0} \frac{m(A \cap B_\epsilon(x))}{m(B_\epsilon(x))} = 1 \\
&\leftrightarrow \lim_{\epsilon \to 0} \frac{m(B_\epsilon(x))}{m(B_\epsilon(x))} - \frac{m(B_\epsilon(x) \setminus A)}{m(B_\epsilon(x))} = 1 \\
&\leftrightarrow \lim_{\epsilon \to 0} 1 - \frac{m(B_\epsilon(x) \setminus A)}{m(B_\epsilon(x))} = 1 \\
&\leftrightarrow \lim_{\epsilon \to 0} \frac{m(B_\epsilon(x) \setminus A)}{m(B_\epsilon(x))} = 0 \\
&\leftrightarrow \lim_{\epsilon \to 0} \frac{m(B_\epsilon(x) \cap A^c)}{m(B_\epsilon(x))} = 0
\end{aligned}$$

Thus, given δ there exist ϵ_0 such that

$$\frac{m(B_\epsilon(x) \cap A^c)}{m(B_\epsilon(x))} \leq \delta \tag{3.6}$$

for any $\epsilon < \epsilon_0$.

Now consider an interval of size ϵ around x which is the union of intervals $I_{w_0, w_1, \ldots, w_n}$ up to a set of Lebesgue measure zero. Then, 3.6 holds for at least one

$I_{w_0,w_1,\ldots,w_n}$, i,e

$$\frac{m(I_{w_0,\ldots,w_n} \cap A^c)}{m(I_{w_0,\ldots,w_n})} \leq \delta$$

On the other hand, note that since A is T invariant, then A^c is also T - invariant and since $T^n : I_{w_0,w_1,\ldots,w_n} \to [0,1]$ is one to one, we get that $T^n(I_{w_0,\ldots,w_n} \cap A^c) = A^c$ up to a set of Lebesgue measure zero.

Then from Theorem 3.1 we get

$$\begin{aligned} m(A^c) &= \frac{m(A^c)}{m([0,1])} \\ &= \frac{m(T^n(I_{w_0,\ldots,w_n} \cap A^c))}{m(T^n(I_{w_0,\ldots,w_n}))} \\ &\leq C \cdot \frac{m(I_{w_0,\ldots,w_n} \cap A^c)}{m(I_{w_0,\ldots,w_n})} \\ &\leq C \cdot \delta \end{aligned}$$

Thus if we choose δ small enough we get $m(A^c) = 0$, but we showed that μ is absolutely continuous with respect to the Lebesgue measure m, then $\mu(A^c) = 0$, and so $\mu(A) = 1$ which is a contradiction since we suppose $0 < \mu(A) < 1$. Hence, (T, μ) is ergodic. □

Our next goal is to prove that (T, μ) is an exact endomorphism. To do this we will start defining exactness which will be used again in the next Chapter where will prove exactness for a more general system.

Definition 3.1. *We say that T is and exact endomorphism if*

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N}$$

where $\mathcal{B}$ is the given $\sigma-$algebra and $\mathcal{N}$ is the $\sigma-$algebra of sets of measure 0 *or* 1.

In other words, (T, X) is exact if there is no set A such that $0 < \mu(A) < 1$ and for every n there exists a set $B_n \in \mathcal{B}$ which satisfies $A = T^{-n}B_n$.

Theorem 3.5. *(T, μ) is exact.*

Proof. Suppose by contradiction that there exist A such that $0 < \mu(A) < 1$ and for each n there exists $B_n \in \mathcal{B}$ such that $A = T^{-n}(B_n)$. Let x be a density point of A. Then by Lebesgue's Density theorem

$$\lim_{\epsilon \to 0} \frac{m(A \cap B_\epsilon(x))}{m(B_\epsilon(x))} = 1,$$

where B_ϵ is a ϵ-neighborhood of the point x. Then,

$$\lim_{\epsilon \to 0} \frac{m(A^c \cap B_\epsilon(x))}{m(B_\epsilon(x))} = 0$$

where A^c is the complement of the set A. Thus, for all $\delta > 0$ there exist $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$,

$$\frac{m(A^c \cap B_\epsilon(x))}{m(B_\epsilon(x))} \leq \delta. \tag{3.7}$$

Now consider an interval of size ϵ around x which is the union of intervals $I_{w_0,w_1,\ldots,w_n}$ up to a set of Lebesgue measure zero. Then, (3.7) holds for at least one $I_{w_0,w_1,\ldots,w_n}$, i.e

$$\frac{m(A^c \cap I_{w_0,\ldots,w_n})}{m(I_{w_0,\ldots,w_n})} \leq \delta.$$

Let D_n be the complement of B_n, then $A^c = T^{-n}D_n$. Considering this, the fact that $T^n : I_{w_0,\ldots,w_n} \to [0, 1]$ is injective, and lemma 3.1 we get:

$$\begin{aligned} m(D_n) &= \frac{m(D_n)}{m([0,1])} \\ &= \frac{m(T^n(A^c \cap I_{w_0,\ldots,w_n}))}{m(T^n(I_{w_0,\ldots,w_n}))} \\ &\leq C \cdot \frac{m(A^c \cap I_{w_0,\ldots,w_n})}{m(I_{w_0,\ldots,w_n})} \\ &\leq C \cdot \delta. \end{aligned}$$

Choosing δ small enough, we have that $\lim_{n\to\infty} m(D_n) = 0$, but μ is absolutely continuous with respect to the Lebesgue measure, then $\lim_{n\to\infty} \mu(D_n) = 0$. On the other hand, since T is μ-invariant and $A^c = T^{-n}D_n$ we have

$$\begin{aligned} \mu(D_n) &= \mu(T^{-n}(D_n)) \\ &= \mu(A^c) \end{aligned}$$

but we proved $\lim_{n\to\infty} \mu(D_n) = 0$, then $\mu(A^c) = 0$. Therefore, $\mu(A) = 1$ which contradicts our assumption that $0 < \mu(A) < 1$. Hence, (T, μ) is exact. $\square$

3.3 Rochlin Entropy Formula

Entropy theory was developed essentially by Rochlin, Sinai ,and Kolmogorov in the late 1950's. Here we will introduce the concept of the entropy of a measure preserving transformation.

Let $\xi = \{A_i\}_{i=1}^k$ be a finite partition of $(X, \mathcal{B}, \mu)$. Let

$$H(\xi) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

We define the Entropy with respect to the partition ξ by

$$h(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\xi)$$

and the Entropy with respect to the measure μ is define by

$$h_\mu(T) = \sup_\xi h(T,\xi).$$

A partition ξ of X is called a generator for a measure preserving transformation T if

$$\bigvee_0^\infty T^{-n}\xi = \mathcal{B},$$

in other words, every measurable set can be arbitrary well approximated by elements of $\bigvee_0^\infty T^{-n}\xi$.

Theorem 3.6 (Kolmogorov-Sinai Theorem)**.** *If ξ is a generator, then*

$$h_\mu(T) = h(T,\xi)$$

Kolmogorov proved this theorem for Bernoulli partitions and Sinai [16] generalized the proof in 1959. We now introduce Shannon-McMillan-Breiman Theorem which will be an essential tool to prove Rochlin Entropy formula.

Theorem 3.7 (Shannon-McMillan-Breiman)**.** *Let T be an ergodic measure preserving transformation of $(X,\mathcal{B},\mu)$. Let ξ be a finite partition of X and let $B_n(x)$ denote the member of the partition $\bigvee_{i=0}^{n-1} T^{-i}\xi$ to which x belongs. Then,*

$$\lim_{n\to\infty} \frac{-1}{n} \log \mu(B_n(x)) = h(T,\xi) \qquad a.e.$$

For details about this theorem we suggest Parry [11] and Walters [19].

Theorem 3.8. *Let T be a $\mathcal{C}^2$ expanding map with two branches. Then, the absolutely continuous invariant measure μ satisfies Rochlin entropy formula, i.e,*

$$h_\mu(T) = \int \log(\frac{dT}{dx})\, d\mu.$$

Proof. Consider the partition $\xi_n = \xi_0 \vee T^{-1}\xi_{w_0} \vee \dots T^{-(n-1)}\xi_{w_{n-1}}$ with elements $E_{w_0,\dots,w_{n-1}}$.

Note that by the Mean Value Theorem, there exist $\theta \in E_{w_0,\dots,w_{n-1}}$ such that

$$m(T^n(E_{w_0,\dots,w_{n-1}})) = DT^n(\theta) m(E_{w_0,\dots,w_{n-1}})$$

where m denotes the Lebesgue measure. As T^n maps $E_{w_0,\dots,w_{n-1}}$ onto $[0,1]$,

$$1 = DT^n(\theta)\; m(E_{w_0,\dots,w_{n-1}}). \tag{3.8}$$

and since $\mu = \rho \cdot m$ and $\rho \in \mathcal{C}([0,1])$ there exist K_1, K_2 positive constants such that

$$K_2 \cdot m(E_{w_0,\dots,w_n}) \leq \mu(E_{w_0,\dots,w_n}) = \int_{E_{w_0,\dots,w_n}} \rho\, d\mu \leq K_1 \cdot m(E_{w_0,\dots,w_n}).$$

So, by 3.8 we get that for every $x \in E_{w_0,w_1,\dots,w_{n-1}}$,

$$K_2(DT^n(x))^{-1} \leq \mu(E_{w_0,\dots,w_n}) \leq K_1(DT^n(x))^{-1}.$$

Then,

$$\begin{aligned} \log \mu(E_{w_0,\dots,w_n}) &\leq \log K_1 + \log(DT^n(x))^{-1} \\ -\frac{1}{n}\log \mu(E_{w_0,\dots,w_n}) &\geq -\frac{1}{n}\log K_1 - \frac{1}{n}\log(DT^n(x))^{-1}. \end{aligned}$$

Taking limits in this last expression when n tend to infinity we have that the left terms on the right side go to zero so:

$$\lim_{n\to\infty} -\frac{1}{n}\log(DT^n(x))^{-1} = \lim_{n\to\infty} -\frac{1}{n}\log\mu(E_{w_0,\ldots,w_n}).$$

But by Shannon-McMillan-Breiman Theorem (3.7),

$$\lim_{n\to\infty} \frac{-1}{n}\log\mu(E_{w_0,\ldots,w_{n-1}}(x)) = h_\mu(T)$$

for μ-almost all x, which implies that

$$\lim_{n\to\infty} -\frac{1}{n}\log(DT^n(x))^{-1} = h_\mu(T).$$

Now, let fix such an x, then

$$\begin{aligned}\lim_{n\to\infty} -\frac{1}{n}\log(DT^n(x))^{-1} &= \lim_{n\to\infty} \frac{-1}{n}\log\left(\frac{dT^n(x)}{dx}\right)^{-1} \\ &= \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{dT^n(x)}{dx}\right) \\ &= \lim_{n\to} \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{dT(T^i(x))}{dx}\right)\end{aligned}$$

so

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{dT(T^i(x))}{dx}\right) = h_\mu(T).$$

On the other hand by Birkhoff Ergodic Theorem, theorem 1.4, for μ- almost all x,

$$\lim_{n\to} \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{dT(T^i(x))}{dx}\right) = \int \log\frac{dT}{dx}\,d\mu.$$

Hence choosing the same x from Shannon-McMillan-Breiman Theorem and from Birkhoff Ergodic Theorem we get,

$$h_\mu(T) = \int \log \frac{dT}{dx}\, d\mu.$$

$\square$

Chapter 4: Expanding maps with infinitely many branches

4.1 Existence and ergodic properties of Gibbs measures

The problem of showing the existence of invariant measure has been approaching from different points of view in the previous chapters. Here we present a generalization of this problem following Walter's paper [18]. In particular, expanding maps are treated as a particular case. The proof of the Ruelle's Perron Frobenius Theorem will allow us to study the existence of absolutely invariant measures and their ergodic properties.

Let $\bar{X}$ be a compact metric space and let X and X_0 be open dense subsets of $\bar{X}$ such that $X_0 \subset X \subset \bar{X}$. Let us also consider $T : X_0 \to X$ a continuous map with the following conditions:

(i_T) There exist $\epsilon_0 > 0$ such that for every $x \in X$,

$$T^{-1}(B_{2\epsilon_0}(x) \cap X) = \bigsqcup_i A_i(x)$$

where the union can be countable. Here $A_i(x)$ are open subsets of X_0 and $T|A_i : A_i \to B_{2\epsilon_0}(x) \cap X$ is an homeomorphism non-decreasing distances, i.e, if two element belong to the same set $A_i(x)$, then their distance under T is grater or equal to the distance between them.

(ii_T) For every ϵ there exists $M > 0$ such that for each $x \in X$, $T^{-M}x$ is ϵ-dense in X.

Now, let φ be a continuous function on X_0, and let $\epsilon > 0$. Suppose there exists K such that:

$(i)_\varphi$ $\sum_{y \in T^{-1}x} e^{\varphi(y)} \leq K$ for all $x \in X$, and

$(ii)_\varphi$ if $d(x, x') < \epsilon_0$, then

$$C_\varphi(x, x') = \sup_{n>1} \sup_{y \in T^{-n}x} \sum_{i=0}^{n-1} [\varphi(T^i y) - \varphi(T^i y')]$$

exist and it is bounded from above by a constant C_φ.

Proposition 4.1. *Let $G(X_0) = \{g \in C(X) | \ \ g > 0$ and $\sum_{y \in T^{-1}x} g(y) = 1 \ \ \forall x \in X\}$. If $g \in G(X_0)$, then for $\varphi = \log g$ condition $(i)_\varphi$ is satisfied.*

Proof. Note that

$$\sum_{y \in T^{-1}x} e^{\varphi(y)} = \sum_{y \in T^{-1}x} g(y) = 1 \text{ for all } x \in X. \tag{4.1}$$

and so condition $(i)_\varphi$ is satisfied with $K = 1$. $\square$

Let $g \in G(X_0)$ and $\varphi = \log g$ as in the the above proposition. Note that if φ satisfies condition (ii_φ), then

$$\text{If } \ d(x, x') < \epsilon_0, \ \text{then } C_\varphi(x, x') = \sup_{n>1} \sup_{y \in T^{-n}x} \prod_{i=0}^{n-1} \frac{g(T^i y)}{g(T^i y')} \tag{4.2}$$

exist and it is bounded. In fact,

$$\sup_{n>1} \sup_{y \in T^{-n}x} \sum_{i=0}^{n-1} [\varphi(T^i y) - \varphi(T^i y')] \quad = \quad \sup_{n>1} \sup_{y \in T^{-n}x} \sum_{i=0}^{n-1} [\log g(T^i y) - \log g(T^i y')]$$

$$= \sup_{n>1} \sup_{y\in T^{-n}x} [\log \prod_{i=0}^{n-1} g(T^i y) - \log \prod_{i=0}^{n-1} g(T^i y')]$$
$$= \sup_{n>1} \sup_{y\in T^{-n}x} [\log \prod_{i=0}^{n-1} \frac{g(T^i y)}{g(T^i y')}].$$

Thus, condition (ii_φ) is equivalent to condition (4.2).

Now, we will state the following lemma which will be used in the next theorem where we prove the existence of a fixed point for the Dual operator $\mathcal{L}^*_{\log g}$

Lemma 4.1. *Let $T : X_0 \to X$ be as above and let φ satisfy conditions (i_φ) and (ii_φ). Then, for any $\epsilon > 0$ there exist a positive natural number N and a real constant a such that for any $x, w \in X$ there exist $y \in T^{-N}x \cap B_\epsilon(w)$ which satisfy*

$$\sum_{i=1}^{N-1} \varphi(T^i y) \geq a.$$

The proof can be found in [18].

Theorem 4.1. *Let T be as above and let $g \in G(X_0)$ satisfying (4.2). Then, there exists $\mu \in M(\bar{X})$ such that:*

1. *For all $f \in \mathcal{C}(\bar{X})$*

$$\lim_{n\to\infty} |\mathcal{L}^n_{\log g} f - \mu(f)| \to 0.$$

2. *μ is the only fixed point of $\mathcal{L}^*_{\log g}$.*

Proof. In this case our function $\varphi = \log g$. Let f be in $\mathcal{C}(\bar{X})$, we will show the existence of the measure μ by using Arzela Ascoli Theorem (See [12]). Define the set

$$L_n = \{\mathcal{L}^n f \ : \ n \geq 0\}.$$

We want to prove that L_n is equicontinuous. In fact, let x, x' be in X such that $d(x,x') < \epsilon < \epsilon_0$. Then,

$$\begin{aligned}
\left|\mathcal{L}_\varphi^n f(x) - \mathcal{L}_\varphi^n f(x')\right| &= \left|\sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y))f(y) - \sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y'))f(y')\right| \\
&= |\sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y))f(y) - \sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y))f(y') \\
&+ \sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y))f(y') - \sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y'))f(y')| \\
&\leq \left|\sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y))f(y) - \sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y))f(y')\right| \\
&+ \left|\sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y))f(y') - \sum_{y\in T^{-n}x}\prod_{i=1}^{n-1} g(T^{i-1}(y'))f(y')\right| \\
&\leq \sup\{f(u) - f(v) : d(u,v) < \epsilon\} \\
&+ ||f|| \sum_{y\in T^{-n}} \left(\prod_{i=1}^{n-1} g(T^{i-1}(y'))\right)\left(\frac{\prod_{i=1}^{n-1} g(T^{i-1}(y))}{\prod_{i=1}^{n-1} g(T^{i-1}(y'))} - 1\right) \\
&\leq \sup\{f(u) - f(v) : d(u,v) < \epsilon\} + ||f||C_\varphi(x,x')
\end{aligned}$$

where the last inequality is because g satisfy (4.2). Thus, we have proved that L_n is equicontinuous. Also, we can easily see that the set $\bar{L}_n$ is compact since $||\mathcal{L}_\varphi^n f|| \leq ||f||$ for all $f \in \mathcal{C}(\bar{X})$, so by Arzela-Ascoli Theorem there exist a sequence $\{n_i\}$ and a continuous function $\bar{f}$ such that $\mathcal{L}_\varphi^{n_i} f \to \bar{f}$.

On the other hand we have that:

$$\begin{aligned}
\min(f) &\leq \min(\mathcal{L}_\varphi f) \leq \ldots \leq \min(\bar{f}) \leq \max(\bar{f}) \leq \ldots \\
&\leq \max(\mathcal{L}_\varphi f) \leq \max f
\end{aligned}$$

Clearly

$$\min(\mathcal{L}_\varphi^k \bar{f}) = \min(\bar{f}) \text{ for all } k > 0. \tag{4.3}$$

That implies (see Walters [18]),

$$\min(\bar{f}) = \bar{f}. \tag{4.4}$$

Then $\bar{f}$ is constant and so we can define $\mu(f) = \bar{f}$. Thus, μ is a measure in $\bar{X}$, $\mu : \mathcal{C}(\bar{X}) \to \mathbb{R}$. Now we claim that $\mathcal{L}_\varphi^* \mu = \mu$. In fact,

$$\begin{aligned}
\mathcal{L}_\varphi^* \mu(f) &= \mu(\mathcal{L}_\varphi f) \\
&= \overline{(\mathcal{L}_\varphi f)} \qquad \text{by 4.4} \\
&= \min \overline{(\mathcal{L}_\varphi f)}, \qquad \text{by 4.3} \\
&= \min \mathcal{L}_\varphi(\bar{f}), \\
&= \min(\bar{f}) \\
&= \bar{f} \\
&= \mu(f)
\end{aligned}$$

Hence, we have proved that μ is a fixed point of $\mathcal{L}_\varphi^*$ and that

$$\mathcal{L}_\varphi^n f \to \mu(f). \tag{4.5}$$

Now to prove uniqueness, suppose there exist $m \in M(X)$ such that $\mathcal{L}_\varphi^* m = m$, then integrating 4.5 with respect to m we have:

$$\int \mathcal{L}_\varphi^n f \; dm \to \mu(f).$$

On the other hand,

$$\begin{aligned}
\int \mathcal{L}_\varphi^n f \; dm &= m(\mathcal{L}_\varphi^n f) \\
&= (\mathcal{L}_\varphi^n)^* m(f) \\
&= m(f).
\end{aligned}$$

This implies that $m(f) = \mu(f)$ for all $f \in \mathcal{C}(\bar{X})$. $\square$

Theorem 4.2 (Ruelle-Perron-Frobenius Theorem)**.** *Let $T : X_0 \to X$ be as before and let $\varphi \in \mathcal{C}(X_0)$ satisfy $(i)_\varphi$ and $(ii)_\varphi$. Then, there exist $\nu \in M(\bar{X})$, $h \in \mathcal{C}(\bar{X})$, $h > 0$ and a positive real number λ such that:*

1. $\mathcal{L}_\varphi^* \nu = \lambda \nu$.

2. $\mathcal{L}_\varphi h = \lambda h$

3. $\nu(h) = 1$

4. $\frac{1}{\lambda^n} \mathcal{L}_\varphi^n \to h \cdot \nu(f)$ *for all* $f \in \mathcal{C}(X)$

Proof. 1. First note that

$$\mathcal{L}_\varphi 1(x) = \sum_{y \in T^{-1}x} e^{\varphi(y)} > 0$$

Now consider the function

$$\begin{array}{rccc} F: & M(\bar{X}) & \to & M(\bar{X}) \\ & \nu & \mapsto & \dfrac{\mathcal{L}_\varphi^* \nu}{(\mathcal{L}_\varphi^* \nu)(1)} \end{array}$$

In order to use Theorem 3.2 , we need to prove that $F(\nu) \in M(\bar{X})$. However, by Theorem 1.1 we only need to prove that $F(\nu)$ belongs to the set

$$\{\alpha \in \mathcal{C}(X)^* : \alpha(1) = 1 \text{ and } \alpha(f) \geq 0\}.$$

In fact,

- $\dfrac{\mathcal{L}_\varphi^* \nu(1)}{(\mathcal{L}_\varphi^* \nu)(1)} = 1$, and

- $\dfrac{\mathcal{L}_\varphi^*\nu(f)}{(\mathcal{L}_\varphi^*\nu)(1)} > 0$ for all $f \in \mathcal{C}(X)$.

Thus by Shauder-Tychonoff Fixed Point Theorem, there exist $\nu \in M(\bar{X})$ such that $F(\nu) = \nu$, i.e

$$\mathcal{L}_\varphi^*\nu = \left[\mathcal{L}_\varphi^*\nu(1)\right]\nu.$$

Taking $\lambda = \mathcal{L}_\varphi^*\nu(1) > 0$, we have $\mathcal{L}_\varphi^*\nu = \lambda\nu$.

2. Consider the set

$$\Gamma = \{f \in \mathcal{C}(\bar{X}) : f > 0, \nu(f) = 1, f(x) < e^{C(x,x')} f(x') \text{ if } x, x' \in X \text{ with } d(x, x') < \epsilon_0\}.$$

Note that Γ is not empty. In fact we can check that $\lambda^{-1}\mathcal{L}1 \in \Gamma$ since:

- $\lambda^{-1}\mathcal{L}1 > 0$ since $\lambda > 0$ and $\mathcal{L}1 > 0$.

-
$$\begin{aligned}
\nu(\lambda^{-1}\mathcal{L}1) &= \lambda^{-1}\nu(\mathcal{L}1) \\
&= \lambda^{-1}\mathcal{L}^*\nu(1) \\
&= \lambda^{-1}\lambda\nu(1) \\
&= \nu(1) = 1.
\end{aligned}$$

- Let x and x' be in X such that $d(x, x') < \epsilon_0$, then

$$\mathcal{L}1(x) = \sum_{y^{Tx}} e^{\varphi(y)}$$

$$
\begin{aligned}
&\leq \sum_{y\in T^{-1}x} e^{\varphi(y)}(\sum_{y'\in T^{-1}x'} e^{\varphi(y')} - e^{\varphi(y')}) \\
&= \sum_{y\in T^{-1}x} e^{\varphi(y)-\varphi(y')} \sum_{y'\in T^{-1}x'} e^{\varphi(y')} \\
&\leq \exp\{\sup_{y\in T^{-1}x} \varphi(y)-\varphi(y')\} \sum_{y'\in T^{-1}x'} e^{\varphi(y')} \\
&\leq \exp\{C(x,x')\}\mathcal{L}1(x').
\end{aligned}
$$

Now we will prove Γ is bounded and equicontinuous. To prove Γ is bounded, let ϵ be such that $\epsilon < \epsilon_0$. Note that by Lemma 4.1, there exist N and a constant a. Let x, w be in X, then choose $y_0 \in T^{-N}x \cap B_\epsilon(w)$ such that

$$\sum_{i=1}^{N-1} \varphi(T^i y_0) \geq a.$$

Thus,

$$
\begin{aligned}
\mathcal{L}^N f(x) &= \sum_{y\in T^{-N}x} e^{S_N\varphi(y)} f(y) \\
&\geq e^{S_N\varphi(y_0)} f(y_0) \\
&\geq e^{a} f(y_0) \\
&\geq e^{a-C} f(w).
\end{aligned}
$$

Hence, $f(w) \leq e^{C-a}\mathcal{L}^N f(x)$ for all $w, x \in X$, so we have that $f(w) \leq e^{C-a}\nu(\mathcal{L}^N f) = e^{C-a}\lambda^N$. Let $K = e^{C-a}\lambda^N$, then $f(w) \leq K$ for all $w \in X$ and so Γ is bounded.

In order to prove that Γ is equicontinuous, let f be in Γ and let $x, x' \in X$ such that $d(x,x') < \epsilon_0$. Note that since $f \in \Gamma$, $f(x) < e^{C(x,x')} f(x')$, writing $C = C(x,x')$ we have,

$$|f(x) - f(x')| = \max(f(x) - f(x'), f(x') - f(x))$$

$$
\begin{aligned}
&\leq \max(e^C f(x') - f(x'), e^C f(x) - f(x)) \\
&= \max(f(x')(e^C - 1), f(x)(e^C - 1)) \\
&= K \max(e^C - 1, e^C - 1)
\end{aligned}
$$

Now, by condition $(ii)_\varphi$, if $d(x, x') < \epsilon$, C is bounded above by C_φ, then

$$|f(x) - f(x')| < \epsilon_1,$$

where, $\epsilon_1 = K \max(e^{C_\varphi} - 1, e^{C_\varphi} - 1)$, and so Γ is equicontinuous.

Since Γ is clearly convex and closed it only remain to prove that $\lambda^{-1}\mathcal{L}(\Gamma) \in \Gamma$. In fact, let f be in Γ. Clearly $\lambda^{-1}\mathcal{L}f > 0$ and by definition of the dual, definition 1.4,

$$\nu(\lambda^{-1}\mathcal{L}f) = \lambda^{-1}\mathcal{L}^*\nu(f) = \nu(f) = 1.$$

It is only left to prove the last condition in our set Γ. Let $x, x' \in X$ such that $d(x, x') < \epsilon_0$. Since $f \in \Gamma$, in particular satisfies that $f(x) \leq e^{C(x,x')} f(x')$. Then,

$$
\begin{aligned}
\lambda^{-1}\mathcal{L}f(x) &= \lambda^{-1} \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) \\
&\leq \lambda^{-1} \sum_{y \in T^{-1}x} e^{\varphi(y)} e^{C(x,x')} f(y') e^{\varphi(y') - \varphi(y')} \\
&= \lambda^{-1} \sum_{y \in T^{-1}x} e^{\varphi(y\prime)} f(y') [e^{\varphi(y) - \varphi(y')}] e^{C(x,x')} \\
&\leq \lambda^{-1}\mathcal{L}f(x') e^{C(x,x')}
\end{aligned}
$$

Applying Schauder-Tychonoff Theorem there exist a fixed point $h \in \Gamma$ such that $\lambda^{-1}\mathcal{L}h = h$, i.e

$$\mathcal{L}h = \lambda h.$$

Since $h \in \Gamma$, it follows that $\nu(h) = 1$ and $h > 0$ as we wanted. So we have proved (2) and (3).

Now, let $g = e^{\varphi}h/(\lambda h \circ T)$. Note that $g \in G(X_0)$ and moreover, g satisfies equation (4.2). In fact, let x and x' be in X and $d(x, x') < \epsilon_0$. If $y \in T^{-n}x$ then

$$\begin{aligned}\prod_{i=0}^{n-1} \frac{g(T^i y)}{g(T^i y')} &= \exp\left(S_n\varphi(y) - S_n\varphi(y')\right) \frac{h(y)}{h(y')} \frac{h(T^n y')}{h(T^n y)} \\ &= \exp\left(S_n\varphi(y) - S_n\varphi(y')\right) \frac{h(y)}{h(y')} \frac{h(x')}{h(x)}.\end{aligned}$$

Since

$$C_{\varphi}(y', y) = \sup_{n \leq 1} \sup_{y \in T^{-n}x} S_n\varphi(y') - S_n\varphi(y),$$

then,

$$\begin{aligned}\exp\left(S_n\varphi(y) - S_n\varphi(y') - C(y', y) - C(x, x')\right) &\leq \prod_{i=0}^{n-1} \frac{g(T^i y)}{g(T^i y')} \\ &\leq \exp\left(S_n\varphi(y) - S_n\varphi(y') + C(y', y) + C(x, x')\right)\end{aligned}$$

and so,

$$\exp\left(-C(x, x') - C(x', x)\right) \leq \prod_{i=0}^{n-1} \frac{g(T^i y)}{g(T^i y')} \leq \exp\left(C(x, x') + C(x', x)\right)$$

which proved that $(iii)_G$ is verified.

Now, by Theorem 4.1 there exist $\mu \in M(\bar{X})$ such that $\mathcal{L}^n_{\log g} f$ converges uniformly to $\mu(f)$ for all $f \in \mathcal{C}(\bar{X})$ where $\mu \in M(\bar{X})$ satisfy $\mathcal{L}^*_{\log g}\mu = \mu$.

Since $g = \frac{e^{\varphi}h}{\lambda \cdot h \circ T}$, then $\mathcal{L}_{\varphi}^{n} f(x) = \lambda^{n} h(x)(\mathcal{L}_{\log g} f/h)(x)$. Thus, $\mathcal{L}_{\varphi}^{n} f$ converges to $\lambda^{n} h\mu(f/h)$. Now we want to prove that

$$\mu(f/h) = \nu(f), \tag{4.6}$$

which will implies that $\mathcal{L}_{\varphi}^{n} f \to \lambda^{n} h\nu(f)$ as we want. In fact, define $m(f) = \nu(hf)$. Then,

$$\begin{aligned} m(\mathcal{L}_{\log g} f) &= \nu(h \cdot \mathcal{L}_{\log g} f) \\ &= \frac{1}{\lambda}\nu(\mathcal{L}_{\varphi}(f \cdot h)) \\ &= \nu(f \cdot h) \\ &= m(f), \end{aligned}$$

□

Note that proving 4.6 we actually prove that $\mu(f) = \nu(hf)$, i.e the measure that was obtained in Theorem 4.1 is equivalent to the measure ν. Thus $\mu = h\nu$ is the absolutely continuous invariant measure for T.

Corollary 4.1. *The measure μ and the scalar λ are uniquely determine by the conditions: $\lambda > 0$, $\nu \in M(\bar{X})$, and $\mathcal{L}_{\varphi}^{*}\nu = \lambda\nu$.*

Proof. We showed in the previous theorem that

$$\lim_{n\to\infty} \frac{1}{\lambda^{n}} {\mathcal{L}_{\varphi}}^{n}(f) = h\nu(f)$$

for all $f \in C\bar{X}$. In particular, for $f = 1$,

$$\lim_{n\to\infty} \frac{1}{\lambda^{n}} {\mathcal{L}_{\varphi}}^{n}(1) = h \cdot \nu(1)$$

$$= \nu(h \cdot 1)$$

$$= \nu(h) = 1$$

Applying logarithm,

$$\begin{aligned}
\lim_{n\to\infty} \log \frac{\mathcal{L}_{\varphi}^{n}(1)}{\lambda^{n}} &= 0 \\
\lim_{n\to\infty} [\log \mathcal{L}_{\varphi}{}^{n}(1) - n \log \lambda] &= 0 \\
\log \lambda &= \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{\varphi}^{n}(1)
\end{aligned}$$

On the other hand, by R-P-F Theorem we have that $\mathcal{L}_{\varphi}^{n} h = \lambda^{n} h$ which implies that we can write $h = \frac{1}{\lambda^{n}} \mathcal{L}_{\varphi}{}^{n} h$. Replacing this in part 4 of R-P-F Theorem, we get

$$\lim_{n\to\infty} \mathcal{L}_{\varphi}^{n} f = \frac{1}{\lambda^{n}} (\mathcal{L}_{\varphi}^{n} h) \nu(f).$$

Hence,

$$\nu(f) = \lim_{n\to\infty} \frac{\mathcal{L}_{\varphi}^{n} f}{\mathcal{L}_{\varphi}^{n} h}$$

□

Proposition 4.2. *Let $T : X_0 \to X$ satisfy $(i)_T$ and $(ii)_T$. Then T is a measure preserving transformation and $\mu(X_0) = \nu(X_0) = 1$.*

Proof. Let $f > 0$ be in $\mathcal{C}(X)$ with compact support inside $X \cap B_{2\epsilon_0}(x)$ for some ϵ_0. Then by Proposition1.2 we want to show:

$$\mu(f \circ T) = \mu(f).$$

Let $\{A_i\}_{i=1}^{\infty}$ be the component of $T^{-1}(X \cap B_{2\epsilon}(x))$ and define $T_i = T \mid A_i$.

Thus, $f \circ T_i$ has compact support inside A_i and we can extend it over the whole space by defining $f \circ T_i = 0$ on $\bar{X} \setminus A_i$. Then,

$$(f \circ T)(x) = \sum_i f(T_i(x))$$

and by Theorem 4.1.

$$\begin{aligned}
\mu(f \circ T_i) &= \mathcal{L}^*_{\log g}\mu(f \circ T_i) && (4.7)\\
&= \int \mathcal{L}_{\log g}(f \circ T_i) d\mu && (4.8)\\
&= \int_{X \cap B_{2\epsilon_0}(x)} g(T_i^{-1}(x)) f(x) d\mu(x) && (4.9)\\
& && (4.10)
\end{aligned}$$

Now summing this last expression, we get $\mu(f \circ T_i) = \mu(f)$.

To prove that μ is concentrate in X_0 see [18]. □

Definition 4.1. *A measure μ is said to have no atoms if for any measurable set A of positive measure there exist $B \subset A$ such that*

$$\mu(A) > \mu(B) > 0.$$

Corollary 4.2. *Let $T : X_0 \to X$ as before and let $\varphi \in \mathcal{C}(X_0)$ satisfy $(i)_\varphi$ and $(ii)_\varphi$. Consider $\lambda, \nu, h, \mu,$ and g as in Theorem 4.2. Then*

1. *μ is positive on non-empy open sets and has no atoms.*

2. *$\nu \circ T^{-n}$ converges to μ in $M(\bar{X})$.*

Proof. Let $\epsilon > 0$. By Lemma4.1 there exist $N > 0$ and $b > 0$ such that if $x, w \in X$ there exist $y \in T^{-N}x \cup B_\epsilon$ with

$$\prod_{i=0}^{N-1} g(T^i y) \geq b.$$

Thus using Theorem 4.1

$$\begin{aligned}
\mu(B_\epsilon) &= \mu(\chi_{B_\epsilon}(w)) \\
&= \mu(\mathcal{L}^N_{\log g}(\chi_{B_\epsilon}(w))) \\
&= \int \mathcal{L}^N \chi_{B_\epsilon}(w)(x) d\mu(x) \\
&= \int \sum_{y \in T^{-N}x \cap B_\epsilon} \prod_{i=0}^{N-1} g(T^i y) d\mu(x)
\end{aligned}$$

where the last expression is grater or equal than b. Therefore, μ is positive on nonempty open sets. Now let x_0 be a point with the largest mass among all atoms. Then,

$$\begin{aligned}
\mu(x_0) &= \mathcal{L}^*_{\log g} \mu(\chi_{x_0}) \\
&= \mu(\mathcal{L}_{\log g} \chi_{x_0}) \\
&= \int \mathcal{L}_{\log g} \chi_{x_0}(x) d\mu(x) \\
&= \int \sum_{y \in T^{-1}x} g(y) \chi_{x_0}(x) d\mu(x) \\
&= \int g(x_0) d\mu(T x_0) \\
&= g(x_0) \mu(T x_0)
\end{aligned}$$

but since $g(x_0) \leq 1$ and by the choice of x_0 we must have $g(x_0) = 1$ which contradicts that $g > 0$ and

$$\sum_{z \in T^{-1}(T(x_0))} g(z) = 1$$

To prove part 2, first note that

$$\begin{aligned}
\mathcal{L}_\varphi^n(f \circ T^n) &= \sum_{y \in T^{-n}x} e^{S_n\varphi(y)}(f \circ T^n)(y) \\
&= \sum_{y \in T^{-n}x} e^{S_n\varphi(y)} f(x)) \\
&= f \circ \mathcal{L}_\varphi^n 1
\end{aligned}$$

Thus by Theorem 4.2,

$$\begin{aligned}
\int f d(\nu \circ T^{-n}) &= \int f \circ T^n d\nu \\
&= \frac{1}{\lambda^n} \int \mathcal{L}_\varphi^n(f \circ T^n) d\nu \\
&= \frac{1}{\lambda^n} \int f \circ \mathcal{L}_\varphi^n 1 d\nu \\
&= \int f \cdot h d\nu \\
&= \nu(f \cdot h) \\
&= \mu(f)
\end{aligned}$$

Hence, $f \circ T^n$ converge to μ in $M(\bar{X})$ $\square$

Definition 4.2 (Conditional Expectation)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\mathcal{C}$ be a sub σ-algebra of $\mathcal{B}$. We define the conditional expectation operator*

$$E(\cdot \mid \mathcal{C}) : Ł^1(X, \mathcal{B}, \mu) \to L^1(X, \mathcal{C}, \mu).$$

Let f be in $L^1(X, \mathcal{B}, \mu)$, $f \geq 0$ and define

$$\mu_f(C) = k \int_C f d\mu,$$

where $k = (\int_X f d\mu)^{-1}$. Clearly, μ_f is a probability measure which is absolutely continuous with respect to μ. Then, by Radon- Nikodym Theorem there exist a function $E(f \mid \mathcal{C}) \geq 0$ in $L^1(X, \mathcal{C}, \mu)$ such that:

$$\int_C E(f \mid \mathcal{C})\, d\mu = \int_C f\, d\mu$$

If f is a real valued function on $L^1(X, \mathcal{B}, \mu)$, we define the conditional expectation linearly by considering the positive and negative parts of f.

Definition 4.3. *A transformation $T : \bar{X} \to \bar{X}$ is said to be an exact endomorphism if*

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \mathcal{N}$$

where $\mathcal{B}$ is the given $\sigma-$algebra and $\mathcal{N}$ is the $\sigma-$algebra of sets of measure 0 or 1.

To prove that (T, μ) is an exact endomorphism is the same as to prove

$$E(f \mid \bigcap_{n=0}^{\infty} T^{-n}\mathcal{B}) = \mu(f)$$

almost everywhere for all $f \in L^1(\mu)$.

Lemma 4.2. *Let T satisfy condition (i_φ), and let $g \in G(X)$ satisfy (4.2). Let μ be as in the Theorem 4.1. Then,*

$$E_\mu(f \mid T^{-1}\mathcal{B})(x) = \mathcal{L}f \circ T(x)$$

μ-almost everywhere. Note that $\mathcal{L}f \circ T(x) = \sum_{y \in T^{-1}(Tx)} g(y)f(y)$

Proof. Let f be in $\mathcal{C}(\bar{X})$.

$$\begin{aligned}
\int_{\bar{X}} \mathcal{L}_{\log g} f d\mu &= \int_X \mathcal{L}_{\log g} f d\mu \\
&= \int_X \mathcal{L}_{\log g} f d\mu \\
&= \int_X \mathcal{L}_{\log g} f dT_* \mu \\
&= \int_X \mathcal{L}_{\log g} f \circ T d\mu \\
&= \int_X \sum_{y \in T^{-1}x} g(y) f(y) \circ T d\mu \\
&= \int_X \sum_{y \in T^{-1}(Tx)} g(y) f(y) d\mu
\end{aligned}$$

Then, by definition 4.2,

$$E(f \mid T^{-1}\mathcal{B})(x) = \sum_{y \in T^{-1}(Tx)} g(y) f(y)$$

□

Applying Lemma 4.2 consecutively we get the following corollary.

Corollary 4.3. *Let T satisfy condition (i_φ), and let $g \in G(X)$ satisfy (4.2). Let μ be as in the Theorem 4.1. Then,*

$$E_\mu(f \mid T^{-N}\mathcal{B})(x) = \mathcal{L}^{-N} f \circ T^N(x).$$

μ-almost everywhere.

Theorem 4.3. *Let $T : X_0 \to X$ as before and let $\varphi \in \mathcal{C}(X_0)$ satisfy $(i)_\varphi$ and $(ii)_\varphi$. Consider $\lambda, \nu, h, \mu,$ and g as in Theorem 4.2. Then (T, μ) is an exact endomorphism.*

Proof. Let $\epsilon > 0$, we can choose $l \in \mathcal{C}(\bar{X})$ with $\int |f - l| < \frac{\epsilon}{3}$. This implies the following observation:

$$|\mu(f) - \mu(l)| \leq \int |f - l| d\mu \leq \frac{\epsilon}{3}. \tag{4.11}$$

Now, since by Theorem 4.1, $\mathcal{L}^n_{\log g} f \to \mu(f)$ for all $f \in \mathcal{C}(\bar{X})$:

$$\begin{aligned}
\int |\mathcal{L}^n_{\log g} f - \mu(f)| d\mu &= \int |\mathcal{L}^n_{\log g} f - \mathcal{L}^n_{\log g} l + \mathcal{L}^n_{\log g} l - \mu(l) + \mu(l) - \mu(f)| d\mu \\
&\leq \int |\mathcal{L}^n_{\log g} f - \mathcal{L}^n_{\log g} l| d\mu + \int |\mathcal{L}^n_{\log g} l - \mu(l)| d\mu + \int |\mu(l) - \mu(f)| d\mu \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Thus $\mathcal{L}^n_{\log g} f$ converge to $\mu(f)$ in $L^1(\mu)$. Next, we estimate:

$$\begin{aligned}
\int |E(f \mid \cap_{n=0}^{\infty} T^{-n}\mathcal{B}) - \mu(f)|\, d\mu &= \int |E(f \mid \cap_{n=0}^{\infty} T^{-n}\mathcal{B}) - E(f \mid T^{-N}\mathcal{B}) + E(f \mid T^{-N}\mathcal{B}) \\
&\quad - \mu(f)|\, d\mu \\
&\leq \int |E(f \mid \cap_{n=0}^{\infty} T^{-n}\mathcal{B}) - E(f \mid T^{-N}\mathcal{B})| + \\
&\quad \int |E(f \mid T^{-N}\mathcal{B}) - \mu(f)|\, d\mu
\end{aligned}$$

For large N the first term is small by the Martingale Theorem, and for the second term we use Corollary 4.3 and get

$$\begin{aligned}
\int |E(f \mid T^{-N}\mathcal{B}) - \mu(f)|\, d\mu &= \int |\mathcal{L}^N f \circ T^N - \mu(f)| d\mu \\
&= \int |\mathcal{L}^N f - \mu(f)| d\mu
\end{aligned}$$

which is small as we proved above. Thus the result follows when n tends to infinity.

$\square$

4.2 Expanding Maps

In this section we will apply the results obtained in Section 4.1 for the case when T is an expanding map where all conditions stated before are satisfied. Let X be a compact connected manifold, in this case, $X = \bar{X} = X_0$. Let ν be a smooth probability measure on X, we would like to find a T-invariant probability measure $\mu \in M(X)$ which is equivalent to ν. The result will be an immediate consequence from the results proved in the previous section.

Definition 4.4. *Let $T : X \to X$ be a C^1 map, $n \geq 0$. We say that T is expanding if there exist constants $\gamma > 1$ and $K > 0$ such that*

$$||DT^n v|| \geq K\gamma^n ||v||$$

for all tangent vectors v, where DT is the tangent map of T.

This constants depends on the choice of the Riemannian metric and then an appropriate metric can be chosen so that we can consider $K = 1$. Also, if d is the metric on X which is determined by the Riemannian metric, there exist $\delta > 0$ such that if $d(x, x') < \delta$, then

$$d(Tx, Tx') \geq \gamma d(x, x').$$

Lemma 4.3. *Let $T : X \to X$ be expanding, then T satisfies conditions (i_T) and $(ii_T.)$*

Since T is an expanding map, then T is a covering map. In fact, the set $T^{-1}B_{2\epsilon}(x) = \bigsqcup_{i=1}^{k} A_i(x)$ and $T : A_i \to B_{2\epsilon}(x)$ is a homeomorphism. A good reference for more details is the well known lectures notes from Viana [17]. To prove that (ii_T) holds, we refer to [18] and [15].

Lemma 4.4. *Let $T : X \to X$ be an expanding $\mathcal{C}^2$-map. If $\varphi(x) = -\log|T'(x)|$, then φ satisfies condition $(i)_\varphi$ and $(ii)_\varphi$.*

Proof. Since we showed that the set $\{T^{-1}x\}$ is bounded, there exist a constant $\bar{K} > 0$ such that

$$\sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) \leq \bar{K}.$$

To prove $(ii)_\varphi$, let $y \in T^{-n}(x)$, and $\epsilon > 0$. Suppose that $d(x, x') < \epsilon$ for $x, x' \in X$. Then, for $n > s$ we prove,

$$\begin{aligned} d(x, x') &= d(T(T^{-1}x), T(T^{-1}x')) \\ &\geq \gamma d(T^{-1}x, T^{-1}x'). \end{aligned}$$

Then inductively we get that

$$d(T^{-j}x, T^{-j}x') \leq \gamma^{-j} d(x, x') \qquad \text{for } j > 0.$$

Then, as $-\log|T'(x)|$ is a $\mathcal{C}^2$-map we get,

$$\begin{aligned} |\sum_{i=0}^{n-1} \varphi(T^i y) - \varphi(T^i y')| &\leq C \sum_{i=0}^{n-1} d(T^i y, T^i y') \\ &\leq C \sum_{i=0}^{n-1} d(T^{i-n}(T^n y), T^{i-n}(T^n y')) \end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{i=0}^{n-1} d(T^{i-n}(x), T^{i-n}(x')) \\
&\leq C\sum_{i=0}^{n-1} \gamma^{i-n} d(x, x') \\
&\leq C\frac{d(x, x')}{(1-\gamma)} \\
&\leq C\frac{\epsilon}{(1-\gamma)}
\end{aligned}$$

Therefore $C_\varphi(x, x') = \sup_{n>1} \sup_{y\in T^{-n}x} \sum_{i=0}^{n-1}[\varphi(T^i y) - \varphi(T^i y')]$ is bounded. □

Since we have verified the conditions over T and φ. We can apply the results obtained in the previous section which give the following important Theorem for expanding maps.

Theorem 4.4. *Let $T : X \to X$ be expanding and let $\nu \in M(X)$ be a smooth measure. Then, there exist μ a T-invariant measure which is equivalent to ν and $h \in \mathcal{C}(X)$, $h > 0$ such that:*

1. *$\mathcal{L}_\varphi^n f \to h \cdot \nu(f)$ for all $f \in \mathcal{C}(X)$. Note that in this case*

$$\mathcal{L}_\varphi^n f = \sum_{y\in T^{-n}x} \frac{f(y)}{|(T^{-n})'(y)|}.$$

2. *The measure $\mu = h \cdot \nu$ is T-invariant.*

3. *$\nu \circ T^{-n} \to \mu$ in $M(X)$.*

4. *(T, X) is an exact endomorphism.*

5. *The measure ν is T-invariant if and only if $\sum_{y\in T^{-1}(x)} \frac{1}{|T'(y)|} = 1$, for all $x \in X$.*

The proof is an immediately consequence of the Theorems 4.2, 4.3, and Corollary 4.2. To prove the last statement, we can se that ν is invariant if and only if $h \equiv 1$. In this case, $\mathcal{L}_\varphi 1 = 1$, so ν is T- invariant if and only if $\displaystyle\sum_{y \in T^{-1}x} \frac{1}{|T'(y)|} = 1$.

Example 4.1. *Consider $T : z \mapsto z^2$ on $|z| = 1$ or equivalently $T : x \mapsto 2x$ (mod 1). In this case, $T' \equiv 2$, and ν is the Lebesgue measure.*

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