ABSTRACT

Title of Document: APPROACHES TO PROOF IN GEOMETRY TEXTBOOKS: COMPARING TEXTS FROM THE 1980S AND 2000S

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Eight American textbooks were studied, four each from the 1980s and the 2000s, with the purpose of identifying differences between the two groups of textbooks in their approaches to teaching proof and proof writing. All of the exercises in each text were coded using parameters established by the author for proofs, types of proof, and other justification and reasoning tasks. Additionally, numbers of proofs in the exposition of each textbook were determined. Thematic analyses of attention to form, presentation of theorems, and introduction to proof and proof writing were also included in the research design. Results suggest both quantifiable and qualitative differences in students' opportunities to engage in and practice proof writing as found in the exercises. Other differences in the newer textbooks include a conjecture-based approach to theorems, greater attention to placing proof in the context of mathematical reasoning, and emphasis on alternatives to the two-column form.

APPROACHES TO PROOF IN GEOMETRY TEXTBOOKS: COMPARING TEXTS FROM THE 1980S AND 2000S

By

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Chapter 1: Introduction

In his 1931 article summarizing three decades of changes in geometry teaching, Shibli enthusiastically described a shift toward emphasizing student-written proofs and away from memorizing proofs of basal propositions. He called the change "nothing less than revolutionary" (p. 370), and he was probably right. Not surprisingly, it was during those decades of "revolutionary" change that the familiar arrangement of proofs in two columns, with numbered steps, became popular in American textbooks.

The two-column proof that emerged during the early 20th century was still dominating the course when I studied geometry as a high school freshman in Montgomery County, Maryland in the late 1980s. But when I returned in 1999 to the same public school district to teach, the course had changed – and it changed even more during the nine years that I was a classroom teacher. Proofs, and the two-column format in particular, were still a significant part of the curriculum, but they no longer dominated instruction or assessment at my school. My perception was that there were other changes as well, among them increased attention to alternate forms of proof and more time spent on real-world applications.

Research Questions

My research will explore the differences in approach to proof in geometry textbooks from the 1980s and the first decade of the 21st century. To get as complete a picture as possible, I will include a look at the narrower category of two-column proofs and the broader topic of mathematical reasoning. My chosen time period is of particular interest, because it includes the “pre-reform" textbooks published before NCTM’s *Curriculum and Evaluation Standards* (1989) as well as textbooks published when the reform movement was well underway. It also has personal significance to me, since it spans my years as both a geometry learner and teacher.

“How do geometry textbooks from the 1980s and the 2000s differ in their approach to proof?” is a broad question with a multi-faceted answer. My research will investigate several sub-questions: Is proof emphasized less in the newer textbooks, relative to the earlier books? And is there increased attention to inductive reasoning as well as explanations that do not constitute formal proof? How do the texts introduce proof and proof writing? What attention is given to the two-column form, other forms of proof, and form in general? Since proofs and theorems are closely linked, I also explore the question of how theorems are presented in the texts.

Historical Background

Proof in the Late 1800s and Early 1900s

When Geometry first emerged as an American high school course in the 1840s, students studied Euclidean geometry in order to understand, memorize, and reproduce geometrical relationships and the associated theorems (Herbst, 2002; Shibli, 1931). That began to change, as new textbooks written by Greenleaf in 1858 and Chauvenet in 1870 included exercises at the end of the text that gave students the opportunity to write original proofs for corollaries or theorems not already proven in the body of the text (Herbst, 2002, p. 290). It is important to note that these “originals” did not entirely resemble today’s exercises for students. They dealt with generalized geometric truths, rather than assertions about specific figures. Thus, they were intended to add to the student's body of knowledge (Herbst, 2002, p. 290) instead of existing primarily as practice for using previous propositions. And the originals were difficult, perhaps prohibitively so for most students. According to Shibli (1931), "There still existed a general feeling that they were formidable and far beyond the ability of pupils to be included in the course" (p. 369).

George Wentworth's geometry textbook series of the late 1800s was a landmark in the evolution of student proving, both because of its popularity and because of its attempt to make writing proofs accessible to students and teachers (Donoghue, 2003). Wentworth's textbooks were notable for their long sets of exercises (Donoghue, 2003; Herbst, 2002). In the preface to his first edition geometry, Wentworth (1878) assured readers that his original exercises are "not so difficult as to discourage the beginner" (p. iv). In his 1888 *A Textbook of Geometry*, he "invites particular attention to the systematic and progressive series of exercises in this edition" (p. vii), pointing out that there are more than 700 of them, "*carefully graded and specially adapted to beginners*[1]" (p. vii).

Wentworth (1878) introduced a specific format for the proofs in his textbooks. Notice the attention given to the smallest details (e.g. the typeface used) of format in the explanation below:

> In each proposition a concise statement of what is given is printed in one kind of type, of what is required in another, and the demonstration in still another. The reason for each step is indicated in small type between that step and the one following . . . The number of the section, however, on which the reason depends is placed at the side of the page. (p. iv)

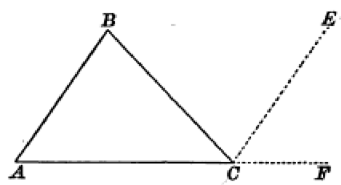
[1] Emphasis original.

Wentworth also took care to complete each proof on no more than one page, so that it was not necessary to turn the page in order to finish the argument (p. iv).

Wentworth's proofs, with their distinct steps – each consisting of a statement and clearly defined reason – are a sort of prototype of the two-column proofs[2] that would later become popular in American textbooks. (See Figure 1.)

PROPOSITION XXIII. THEOREM.

138. ***The sum of the three angles of a triangle is equal to two right angles.***



Let ABC ***be a triangle.***

To prove $\angle B + \angle BCA + \angle A = 2$ rt. $\angle$s.

Proof. Suppose CE drawn ∥ to AB, and prolong AC to F.

Then $\angle ECF + \angle ECB + \angle BCA = 2$ rt. $\angle$s, § 92

(the sum of all the $\angle$s *about a point on the same side of a straight line* $= 2$ *rt.* $\angle$s*).*

But $\angle A = \angle ECF$, § 106

(being ext.-int. $\angle$s *of* ∥ *lines).*

and $\angle B = \angle BCE$, § 104

(being alt.-int. $\angle$s *of* ∥ *lines).*

Substitute for $\angle ECF$ and $\angle BCE$ the equal $\angle$s A and B.

Then $\angle A + \angle B + \angle BCA = 2$ rt. $\angle$s.

Q. E. D.

Figure 1. Proof of the triangle sum theorem, Wentworth. From *A Textbook of Geometry* (Wentworth. 1892, p. 42, Ginn & Company, Retrieved from books.google.com).

But in Wentworth's careful attention to format, I wonder if we also see precedence for what would become the target of criticism 100 years later: the over-emphasis on form, to the detriment of meaning. Schoenfeld (1988), for example,

[2] While I frequently refer to "two-column proofs," I have not found evidence that the authors of the early 20th century employed that phrase. For example, Hart and Feldman (1911) give the description, "Argument and reasons are arranged in parallel form" (p. iv). Other authors use similar wording.

found in his study of "well-taught" geometry students that, "As a result of their instruction [students in his study] came to believe that it is the form of expression, as much as the substance of the mathematics, that is important" (p. 158).

The influential Committee of Ten convened in 1892, and its mathematics subcommittee made several recommendations pertaining to teaching proof in geometry. In its *Report on the Committee of Secondary School Studies*, the committee called for instruction in the axiomatic nature of demonstrative geometry (National Education Association, 1894, p. 112) and in elementary principles of logic (p. 113). The need for all students – not just the brightest – to learn to construct original proofs was emphasized, and the process of demonstration was hailed as a means for "quickening and developing creative talent" (p. 115)[3].

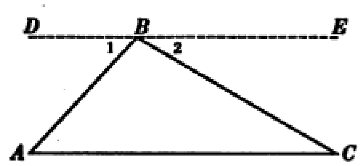
In keeping with the ideas put forward by the Committee of Ten, the textbooks of the early 20th century took a different approach to that of those first texts (Greenleaf and then Chauvenet) that gently encouraged students to write their own proofs (Herbst, 2002). Those exercises had been few in number and too difficult for most students (or teachers) to attempt (Shibli, 1931). Now, the teaching of proof was to be systematized. New editions of textbooks included instruction on specific proof-writing techniques (Herbst, 2002). Exercises were to be numerous, easier, and graded for difficulty. Moreover, the propositions to be proven did not usually add to the body of knowledge. The logic of the argument was emphasized; the result was often trivial. They were not theorems of any consequence; they were exercises in the truest sense

[3] It is interesting to note that the committee placed a high value on oral demonstrations of propositions. The authors lamented students' inability to present a coherent and elegant oral argument. However, they claimed that, "The remedy is obvious: abundance of oral recitation – for which there is no proper substitution – and the rejection of all proofs which are not formally perfect" (NEA, 1893, p. 115).

of the word. Thus, the process of proving began to be divorced from the substance of mathematical content. (Herbst, 2002).

Among the first textbooks to introduce proofs written in two columns divided by a vertical line was Hart and Feldman's *Plane Geometry*, published in 1911. The *argument* and *reasons* columns consisted of numbered steps that were aligned between columns. (See Figure 2.)

204. ***The sum of the angles of any triangle is two right angles.***



Given $\triangle ABC$.

To prove $\angle A + \angle ABC + \angle C = 2$ rt. $\measuredangle$.

Argument	Reasons
1. Through B draw $DE \parallel AC$.	1. Parallel line post. § 179.
2. $\angle 1 + \angle ABC + \angle 2 = 2$ rt. $\measuredangle$.	2. The sum of all the $\measuredangle$ about a point on one side of a str. line passing through that point $= 2$ rt. $\measuredangle$. § 66.
3. $\angle 1 = \angle A$.	3. Alt. int. $\measuredangle$ of $\parallel$ lines are equal. § 189.
4. $\angle 2 = \angle C$.	4. Same reason as 3.
5. $\therefore \angle A + \angle ABC + \angle C = 2$ rt. $\measuredangle$. Q.E.D.	5. Substituting for $\measuredangle$ 1 and 2 their equals, $\measuredangle$ A and C, respectively.

Figure 2. Proof of the triangle sum theorem, Hart and Feldman. From Plane Geometry (Hart & Feldman, 1892, p. 77, American Book Co., Retrieved from books.google.com).

Hart and Feldman (1911) enumerated the virtues of the two-column format in the textbook's preface, writing, "This arrangement gives a definite model for proving exercises, renders the careless omission of the reasons in a demonstration impossible, leads to accurate thinking, and greatly lightens the labor of reading papers" (p. v). The innovation of two parallel columns seems to be part of the answer to the demands of the Committee of Ten that all students be taught to write proofs and that the demonstrative nature of geometry be emphasized. According to Hart and Feldman in the quote above, the two-column format provided a model to replicate, ease of assessment for teacher and student, and a built-in focus on the need to justify one's assertions. But it was more than that as well. Herbst argued that the two-column format offered a stabilizing consistency between the business of proving substantive theorems (found in the body of the text) and the very different process of proving the kinds of trivial propositions found in student exercises (2002, p. 306). Herbst called the custom of two-column proving a "reduction of mathematical reasoning to its logical, formal dimensions" (2002, p. 285). We might call it reductive, but it was reductive by design.

Rumblings of Change in the 1980s

There is evidence that many students in the 1980s were failing to master proof-writing, even though much of the course was devoted to the study of proof. Senk's study (conducted in 1981 and published in 1985) of more than 1500 students across five states presented rather dismal results, finding that only 31 percent of geometry students in the sample were able to write three out of four proofs correctly (1985, p. 453). Many students could not even begin the arguments; some used the theorem to be proved as part of their argument (p. 455). Senk asserted a need for more effective instruction and – significantly – a reexamination of the value of a traditional proof-centered geometry curriculum (p. 455). The numbers from Senk's research look even worse when we consider that only 47% of high school graduates in 1982 had completed geometry (Snyder & Dillow, 2012), and that a significant number of geometry students were enrolled in a “basic” course that did not include proof-writing (Senk, 1985, p. 448).

Schoenfeld (1988) found that even when students did become competent proof writers, they often missed more important lessons about mathematics. The students in his study were successful at writing two-column proofs, but the process of proving did not help them solve problems or discover truth (p. 151). They made no connection between constructive and deductive geometry (p. 150). Additionally, the elevated status of the two-column format was causing students to focus on form at least as much as substance (p. 158).

Other critics saw problems with the way students were learning about proof: Hanna (1990) argued that formal proofs did not necessarily help students understand why a theorem is true, making a distinction between “proofs that prove and proofs that explain” (p. 9). Chazan (1993) found that some students did not understand the power of deductive proof to establish a general mathematical truth (p. 384).

Usiskin (1980) argued out that for most geometry students, proof was divorced from exploration, a stark contrast with the way true mathematicians develop proof (p. 420). He claimed that the geometry student was often misled about the nature of proof: “By spending so much time on the same kind of proof -- most often two-column proof based on properties of angles, lines, or congruent triangles -- we are misteaching our students into believing that this is what proof is” (p. 420).

Research related to the van Hiele theory of geometric thinking, and its implications for geometry instruction, burgeoned in the 1980s[4]. The van Hieles, two Dutch researchers, had identified five *levels* of reasoning about geometric

[4] For example: Usiskin (1982), Senk (1989), Burger & Shaughnessy (1986), Fuys et al. (1988).

relationships and shapes, which students must attain sequentially (Shaughnessy & Burger, 1985, p. 420). Shaughnessy and Burger's 1985 paper reported on an interview study that revolved around the van Hiele theory. Shaughnessy and Burger found that no secondary school student in their study was reasoning at the van Hiele level 3, which is the level required for writing proof (p. 425). They concluded that geometry students should not write formal proofs for at least half a year, though informal deductive reasoning should be encouraged (p. 426). Even more boldly, they recommended that most high school students should never be instructed in formal proof-writing (p. 426). According to Shaughnessy and Burger, "It would be a mistake to force all students . . . to take a formal geometry course. Our interviews suggest that such a step would be a disaster for both students and teachers" (p. 426).

But in fact, the percentage of American high school graduates completing geometry rose from 47% in 1982 to 63% in 1990 to 83% in 2005 (Snyder & Dillow, 2012). Locally, the state of Maryland made geometry a requirement for graduation in 2005 (E. Nolan, personal communication, Feb. 13, 2013, citing COMAR 13.03.02.04 A). It seems reasonable to consider the increase in enrollment as a possible factor contributing to currents of change in the curriculum.

The National Council of Teachers of Mathematics (NCTM) published its *Curriculum and Evaluation Standards* in 1989, which explicitly prescribed a de-emphasis on deductive reasoning and proof in geometry instruction and increased attention to inductive reasoning (p. 159). The document put heavy emphasis on applications in the sciences, the arts, and practical tasks (p. 157) and downplayed formal proof. The two-column format, in particular, was designated as a "topic to receive decreased attention" (p. 127), while "deductive arguments expressed orally and in sentence or paragraph form" (p. 126) were to receive increased attention.

Formal proof was placed in the context of meaningful discovery of truth: verifying properties of rectangles, squares, and rhombuses for example, after a period of class discussion and exploration (p. 159). For instance, a full paragraph is devoted to a method of using a tape measure to determine whether a figure is a rectangle, with a brief mention of proof at the end: "Some students, especially the college intending, should be expected to cite or prove theorems to justify these methods" (p. 158). NCTM identified "geometry from an algebraic perspective" (p. 161) and synthetic geometry as separate standards. In the algebraic strand, attention was given to transformations (called the "geometric counterpart of functions" (p. 161)) and vectors. Educators were advised that, "deducing properties of geometric figures using their coordinate representations is often easier for students than synthetic proofs are" (p. 162). Taken altogether, NCTM's description of geometry is a broad vision, where proof is in the landscape but is not the central feature. Significantly, the words used by NCTM to describe its approach to geometry are "more eclectic" and "based on informal explorations and short local axiomatic sequences" (p. 162).

In the addenda series to the *Standards*, Coxford (1991) expanded on the possibility of using sets of "local axioms" (p. 61). The general idea is to start with a few statements pertaining to one topic, triangle congruence for example, and derive theorems within that topic. One could assume a few postulates about triangle congruence and alternate interior angles of parallel lines, and then go on to prove theorems about properties of parallelograms (p. 61). This approach allows proof-writing to be confined to a few units of study (Wu, 1992, p. 228) freeing up time for more work with coordinate geometry, transformations, and real-world application, as Hirsch explained in the editor's preface (Coxford, 1991, p. v). Coxford distinguished between formal proof and mathematical justification, saying that a justification "may consist of a set of examples that seem to support the proposition, or it may be an intuitive argument" (p. 61).

We should pause here and consider this inclusive view of "justification" (so radically different from the vision of the Committee of Ten 100 years earlier), and its serious implications for how one conceives of mathematical knowing. Not everyone supported the shift away from deductive reasoning espoused by NCTM. In a 1996 article,which was based on a 1992 speech, Wu ardently defended a Euclidean proof-based curriculum (p. 227), though he recommended including some more advanced and more modern topics as well (p. 232). In contrast to Coxford's proposal to use "local axiomatics," Wu argued that there is value in a course where one can "start proving theorems from the ground up" (p. 227), and that geometry is the only course where that can be reasonably managed with high school students (p. 227-8). He made multiple references to the NCTM addenda series, criticizing the curriculum sharply:

> [It] fails to make the students see the most important feature of what constitutes Mathematical knowledge, namely, that once the truth of a small set of statements (axioms) is granted, then the truth of everything else within this body of knowledge –deduced as they are from the former by precise logical reasoning – will leave no room for doubt. (p. 225)

Wu was no radical defender of the status quo, however. He joined the reformers in calling for a departure from a full year of classical Euclidean geometry (p. 227), and while he fiercely defended the value of two-column proofs, he urged that they be abandoned in favor of paragraph style proofs after a few weeks, as one would abandon training wheels when learning to ride a bicycle (p. 227). Of course, his ideas are in sharp contrast with Shaughnessy and Burger's (1985) van Hiele-inspired recommendation that no student should encounter formal proof until the second half of the year, if at all.

Above, I have outlined several areas of criticism of the traditional proof-based curriculum: the low success rate among students in writing correct proofs (Senk 1985) and understanding their meaning and purpose (Hanna, 1990; Schoenfeld, 1988); the need for more time spent on inductive reasoning and exploration (NCTM, 1989) and developing informal justifications (Coxford, 1992); and NCTM's recommended de-emphasis on proof, with a de-emphasis on the two-column form in particular, in favor of paragraph proofs and oral arguments (NCTM, 1989).

In light of these points, is there evidence that, compared with textbooks from the 1980s, post-*Standards* textbooks are substantively different in their approach to proof? Is there indeed a de-emphasis on proof in the newer textbooks, relative to the earlier books and increased emphasis on informal justifications and other explanations that fall short of proof? Does the newer group of textbooks give more attention to inductive reasoning and exploration than does the earlier group? Does the two-column form receive less attention in the newer textbooks, and are paragraph proofs (and other forms) better represented? How do the newer textbooks treat form itself? Do the ways in which the newer texts introduce proof, proof writing, and theorems address some of the concerns of the critics?

In the next chapter, I will review the literature relevant to my research and describe my research methods. My research has two parts. The first is a counting scheme that involves coding the exercises and elements of the exposition, and the second consists of thematic observations. I report the results of my research in the third and fourth chapters, with the third chapter focusing mainly on the counting results and the fourth presenting my thematic analysis. The fifth and final chapter examines how the results of my research answer the research questions posed in this chapter.

Chapter 2: Methodology

With the goal of investigating the differences in approach to proof between textbooks from the 1980s and textbooks from the 2000s, I chose four textbooks from each decade to analyze. I initially formulated a scheme for analysis that included counting the total number of proofs in the exposition and exercises of each textbook, classifying the proof exercises into several subcategories, and counting the total number of exercises in each textbook requiring justifications or explanations that did not constitute proof. Realizing that these numbers alone did not capture a complete picture of the textbooks' differing approaches, I added a qualitative component to the analysis. This component had two branches and consisted of examining the ways in which textbooks introduced proof and proof writing as well as how theorems were presented in the texts.

Analytical Approaches of Prior Relevant Research

There is precedent in the literature for classifying exercises, according to well-defined criteria, from a sample of textbooks and analyzing the results. Thompson, Senk, and Johnson (2009) studied students' opportunities for mathematical reasoning in twenty contemporary Algebra 1, Algebra 2, and Precalculus textbooks. Using selected topics rather than coding entire texts, they calculated the percentage of student exercises that are proof-related. They identified twelve categories of proof-related reasoning in the exercises, distinguishing between conjectures and arguments, and between general and specific cases (p. 262). The code takes into account the wide variety of activities that may fall under the broad umbrella of proof-related exercises.

For example, an exercise might require a student to determine the error in an invalid argument. That particular type of problem is coded as CG or CS, depending on whether the argument is about a general or specific case (p. 262). The authors also coded reasoning and proof in the narrative of the text. Mathematical statements were either given no justification, or they were justified in one of three ways: a proof (or general argument), an argument based on a specific case, or directions for the student to justify the property (p. 261).

Vincent and Stacey (2008) built on the findings of a 1999 TIMSS Video Study of Australian eighth grade math lessons and designed a textbook study revolving around the “shallow teaching syndrome” described in the Video Study (p. 82). In order to investigate how closely the textbooks in their study aligned with the lessons in the Video Study, the authors used the same definitions and classifications that were used in the Video Study (p. 82, 83). They classified textbook problems from selected topics according to five criteria, one of which was whether or not the problems involved proof, verification or derivation (PVD) (p. 90). The code required that a PVD problem be a general argument rather than a specific case. Additionally, PVD problems had to use deductive reasoning and be non-numeric (p. 90). In contrast with the Thompson, Senk, and Johnson’s study, there was no distinguishing between different varieties of proof-related reasoning. The percentage of problems in the selected geometry topics found to be PVD problems was calculated (p. 96).

In a more recent study, Otten, Gilbertson, Males, and Clark (2011) sampled 44% of sections from six stand-alone geometry textbooks, coding both the expositions and exercises (p. 349). They calculated the percentage of exercises related to reasoning-and-proving, and they classified reasoning-and-proving exercises according to the type of activity expected from the student. For example, *developing a mathematical proof* was distinguished from *developing a rationale*, i.e. explaining a statement in a way that does not necessarily constitute a proof. Other subcategories included *making a conjecture*, and *finding a counterexample.* The authors found that exercises requiring students to develop a proof comprised a small percentage (3-7%) of the total (p. 351). Writing a proof from an outline and filling in blanks of a proof were not classified as *developing a mathematical proof*, but were grouped together in an "other" category. Providing a proof outline was also designated as "other" (p. 351).

Otten et al. (2011) also distinguished between mathematical statements of a general nature and of a particular nature, in both the exposition and the exercises. Finally, the authors made a count of the numbers of statements and exercises that were *about* reasoning-and-proving (as opposed to involved in the activity of reasoning-and-proving), finding that number to be low (pp. 352-3).

There are some missing pieces in the background literature relevant to this study. Notably, I have not found any studies in the literature that analyze and compare proofs in texts from two different eras, as I am doing. Also missing are studies related to the qualitative component of my research. Though there are many papers that make recommendations on improving teaching of proof, I have not found any that look specifically at how proof writing is introduced in textbooks (geometry of otherwise) or how theorems are presented in textbooks. I did find a study by Fujita (2001) analyzing the order of theorems in early 20th century geometry textbooks. I deemed the topic of that study to be only tangentially related to my research, and I do not include it in this literature review.

Textbook Selection

My intention was not to choose books based on a particular philosophy or movement, but to analyze the most popular geometry textbooks from each decade, avoiding those that appear to be geared toward students who are not college-bound (e.g. Jurgensen's *Basic Geometry*). This turned out to be difficult, as I did not find a reliable way to determine which textbooks were most widely used during the decades in question. Information about textbooks in the 1980s was especially scarce. Adding to the difficulty, almost none of the books were available in libraries or local used bookstores; they had to be purchased sight unseen via the internet.

The 2000s

I have better information about popular textbooks in the 2000s than I have for the 1980s. According to Dossey, Halvorson, and McCrone (2008), the three leading publishers of secondary school mathematics textbooks in 2005 were Glencoe-McGraw Hill, Houghton Mifflin-McDougal Littell-Heath, and Pearson-Prentice Hall-Addison Wesley-Scott Foresman (p. 20). With that in mind, I have selected the following textbooks:

- Boyd, C. J., Cummins, J., Malloy, C., Carter, J., & Flores, A. (2005). *Geometry.* New York, NY: Glencoe/McGraw-Hill.
- Larson, R., Boswell, L., Kanold, T.D., & Stiff, L. (2007). *Geometry.* Evanston, IL: McDougal Littell.

- Bass, L.E., Hall, B.R., Johnson, A., & Wood, D.F. (2001). *Geometry.* Needham, MA: Prentice Hall.

In addition, I include the 2004 edition of the textbook used by Montgomery County Public Schools county-wide (though their website cites a publishing date of 2005):

- Schultz, J.E., Hollowell, K.A., Ellis Jr., W., & Kennedy, P.A. (2004). *Geometry.* Austin, TX: Holt, Rinehart, & Winston.

While a textbook's reform agenda was not a factor in its selection, it is worth noting that James E. Schultz, senior series author for the Holt 2004 textbook, was a co-author of the 1989 *NCTM Curriculum and Evaluation Standards for School Mathematics.* Another author of that same textbook, Paul A. Kennedy, is described in the authors' page as "a leader in mathematics education reform" (p. iii). In addition, one of the authors of the McDougal Littell 2007 textbook is Lee Stiff, past president of NCTM for the years 2000-2002 (p. v).

The 1980s

Not knowing the leading publishers from the 1980s, the selection process for this group was more difficult. However, there was one textbook that stood out as a clear choice: the textbook by Jurgensen, Brown, and Jurgensen (1988).

- Jurgensen, R.C., Brown, R.G., & Jurgensen, J.W. (1988). *Geometry.* Boston, MA: Houghton Mifflin.

Ray Jurgensen was one of the mathematicians involved in the National Science Foundation funded School Mathematics Study Group (SMSG) in the late 1950s and early 1960s (Donoghue, 2003, p. 371). In 1965, he coauthored a geometry textbook with Donnelly and Dolciani, based on the approach developed by SMSG (p. 374). He published geometry textbooks, with various coauthors, through the end of the 20th century. His books were published by Houghton Mifflin, which eventually merged with McDougal Littell. Though I have no hard data on textbook popularity, judging from the abundant supply and numerous editions available through Amazon and other online booksellers, his texts were widely used. A later book in the Jurgensen series was used during the first year that I taught geometry at Magruder High School, and at least two magnet programs in Montgomery County Public Schools still use a Jurgensen text (personal communication, L. Loomis, Jan. 22, 2013).

My criteria for choosing the other three textbooks from the 1980s are somewhat less clear. Here are the other selections:

- Moise, E., & Downs Jr., F. (1982). *Geometry*. Menlo Park, CA: Addison-Wesley Publishing Co.

Moise and Downs produced a geometry textbook series that began in the 1960s. The latest edition that I could find was published in 1991. They both participated in SMSG, along with Jurgensen. In fact, the authors indicate in the preface that the text contains portions of the SMSG geometry text (p. vi), though it is not clear how extensively the SMSG text was used. According to Wikipedia, Addison-Wesley is an imprint of Pearson, which also owns Prentice Hall, the publisher of one of my selections from the 2000s. Copies of the Moise and Downs textbooks are not as plentiful the Jurgensen books, but they still appear to be well within the mainstream.

- Nichols, E. D., Edwards, M. L., Garland, E. H., Hoffman, S.A., Mamary, A., & Palmer, W.F. (1991). *Geometry*. Austin, TX: Holt Rinehart, & Winston, Inc.

This was a natural choice, because it is published by Holt, Rinehart & Winston, the same publisher as the 2004 *Geometry,* which I am also analyzing. Nichols's *Geometry* was published in several printings, and Amazon.com is well stocked with inexpensive copies.

I had hoped to find a Glencoe/McGraw-Hill textbook published in the 1980s, so that I could compare it with the one published in 2005. But none was available on Amazon or Alibris books, a good indication that if such a textbook did exist, it must have been relatively obscure.

After looking at the available options, I settled on Foster, Cummins, and Yunker's (1984) *Geometry* as the fourth textbook in the group.

- Foster, A.G., Cummins, J.J., & Yunker, L.E. (1984). *Geometry.* Columbus, Ohio: Charles E. Merrill Publishing Company.

The publisher is well known, and used copies are easy to find. This group of authors is unusual in that all three were high school teachers in Illinois and none was a college professor. Note also that this textbook shares one of its authors, Jerry Cummins, with the Glencoe 2005 text.

Throughout this paper, I will refer to textbooks by their publisher and date of publication (e.g. Glencoe 2005 or the Glencoe 2005 text). Given the large number of authors of some of the textbooks, this method will be more convenient than using the convention of author(s) and date.

Research Design, Part 1: Coding Analysis

My research design was informed by the overarching question, "How do geometry textbooks from the 1980s and the 2000s differ in their approach to proof?"

Since I was interested in emphasis on proof, I decided to make a total count for each textbook of the number of *proofs* in the exposition and the number of proofs elicited in the exercises. As did Otten et al. (2001), I distinguished between *unassisted proofs* (i.e. exercises that required students to develop proofs without help from the text), and proof writing exercises that provided some assistance. I identified two subcategories of assisted proofs and decided to determine a count for each: *fill-in proofs* (where the task was to fill in blanks of a proof provided by the text) and *planned proofs* (where students developed a proof from an outline). At the other end of the assistance-independence continuum, I established a subcategory of *unassisted proofs* I called *independent set-up proofs*. These were problems requiring students to write a proof with no "given" statement, "prove" statement, or diagram provided in the text. I was interested in what the relative number of *independent set-up* proofs might say about expectations for students regarding level of mastery of proof writing, and what those expectations, in turn, might say about the overall emphasis on proof in each textbook.

It soon became apparent that there were many exercises that targeted one or more element of proof writing but that did not qualify as proofs. Examples include the tasks of writing the first statement of an indirect proof and writing a plan for a proof. Given my goal of investigating a possible de-emphasis on proof in the newer textbooks, I wanted to account for these proof-related exercises. For such exercises, I established a *proof development* category. One of the questions that interested me was whether the newer textbooks gave more attention to justifications and explanations that fall short of proof. For this reason, I decided to count exercises that elicited *justification and explanation* but not necessarily proof. In my scheme, the following five categories are mutually exclusive: unassisted proof, fill-in proof, planned proof, proof development, and justification and explanation. In addition, independent set-up proofs are counted as a subcategory of unassisted proofs.

Finally, I addressed the question, “What attention is given to the two-column form, other forms of proof, and form in general?” To explore this theme of form, I counted the number of exercises in each textbook that specifically required a proof in one of three forms other than the two-column form. The three categories that I established are *flow proofs, paragraph proofs,* and *coordinate proofs.* The numbers for these categories do not fully capture the various approaches to form found in the textbooks, however. To further address this question, I made non-numerical observations about each textbook’s treatment of form and about the changing status of the two-column form in particular.

While the research design identifies only one element to be counted in the exposition (i.e. proofs), the scheme for coding the exercises is more complicated. The table below lists the categories and subcategories of exercises identified in the research design. (See Table 1.)

Table 1

Categories and Subcategories of Exercises

In Exercises	
<u>Types</u>	<u>Forms</u>
Unassisted Proofs	Flow Proofs
Subset: Independent Set-up Proofs	Paragraph Proofs
Assisted Proofs	Subset: Indirect Proofs
Subset: Fill-in Proofs	Subset: Direct Proofs
Subset: Planned Proofs	
Proof Development Items	Coordinate Proofs
Justification and Explanation	

Parameters for Each Category

Below, I describe in greater detail each category and subcategory, with examples for greater clarity.

In the exposition. The scope of my research included the entire text, including non-canonical sections such as enrichment, investigative, and extension sections. (I applied this rule when coding both exposition and exercises.) After comparing the organization schemes for different texts, I realized that if I excluded non-canonical sections, it would be difficult to treat the texts equitably. Moreover I would not get a complete picture of the differences between the two groups of texts, since the newer textbooks tended to have far more enrichment sections than did the older textbooks.

Proofs. For the purposes of this research, my definition of *proof* is, "a deductive argument consisting of a sequence of statements and reasons." A proof may take various forms, but I excluded outlines, plans, and informal explanations. I included proofs of theorems, corollaries, and any other assertion with mathematical content. Occasionally, I would find a proof of a statement with no mathematical content (e.g. Holt 1991 on p. 48 used a two-column format to prove the statement “I qualify to try out for J.V. volleyball”), and such proofs were excluded.

Though some textbooks provide detailed justifications of compass and straight-edge constructions, they are not identified as “proofs” in the exposition. As do formal proofs, such justifications draw on theorems and postulates to justify steps in the construction, but they argue the correctness of a mathematical process rather than a statement. Though one could argue that some justifications of constructions might qualify as proofs, I excluded the entire category from the proof count for the sake of simplicity.

Example of proof in exposition. Addison-Wesley 1982, p. 41

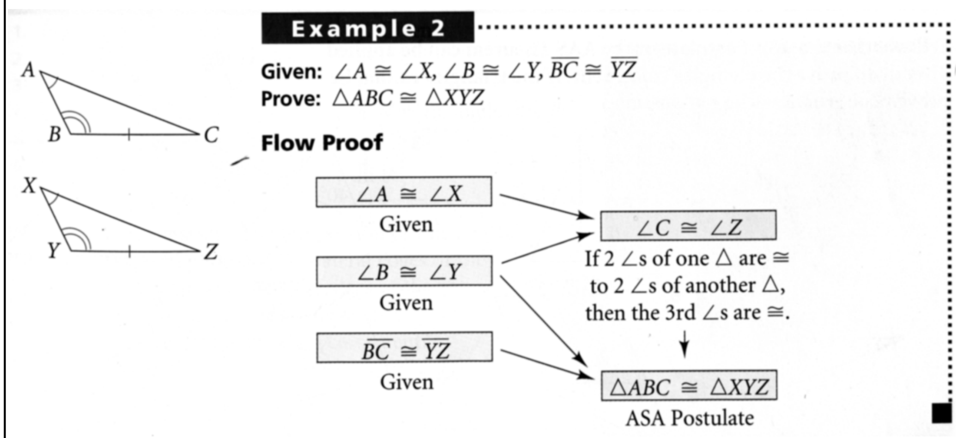
Theorem 2–5. The Point-Plotting Theorem

Let $\overrightarrow{AB}$ be a ray, and let x be a positive number. Then there is exactly one point P of $\overrightarrow{AB}$ such that $AP = x$.

Proof. By the Ruler Placement Postulate, we can choose a coordinate system for the line $\overleftrightarrow{AB}$ in such a way that the coordinate of A is 0 and the coordinate of B is a positive number r.

Let P be the point whose coordinate is the given number x. Then P lies on $\overrightarrow{AB}$, because $x > 0$; and, by the Ruler Postulate, $AP = |x - 0| = |x| = x$. (By the definition of absolute value, $|x| = x$ when $x > 0$.) Therefore (by the Ruler Postulate), only one point of the line corresponds to x. Only one point of the ray lies at a distance x from A.

Example of proof in exposition. Prentice Hall 2001, p. 415



In the exercises. As in the exposition, I considered the entire text, including non-canonical sections. Because I was trying to capture the potential student experience with homework exercises from the textbook, I attempted to include all potential homework problems as *exercises* to be coded. To this end, I excluded examples embedded in the exposition. In order to treat textbooks as equitably as possible, I included all review exercises, challenge problems, and extra practice problems, including those found at the very end of the textbook. This was necessary because of the differences among textbooks in how exercise sets were organized. For example, one textbook might have a separate “mixed review” set of problems covering material from previous chapters. Another textbook might make no such distinction but incorporate older material into the main exercise set instead. Thus, leaving out “mixed review” sections would unfairly exclude problems covering older material from the first textbook while including such problems in the other. I made the arbitrary decision to exclude college entrance exam preparation exercise sets.

Because of the nature of these exercises (i.e. mostly multiple choice and one-word answers), this decision had a minimal impact on my results.

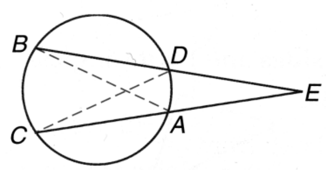
Counting the exercises was not always straightforward. I counted each proof or justification/explanation task as a separate item, regardless of how it was numbered. For example, if a problem was divided into parts (a) and (b), and each part required a separate proof or justification, I counted that problem as two items. On the other hand, the Holt 2004 textbook counted each blank in its fill-in proofs as a separate numbered exercise, but I counted the entire proof as one item (see the example for *fill-in proofs* in this section).

While I originally intended to report the number of proof exercises in each textbook both as a raw number and as a percentage of the total number of exercises, I eventually decided to report only the raw number. Because the textbooks used such different numbering styles (see Holt 2004, for example), I determined that it would not be feasible for me to determine the total number of exercises in each textbook. But there was another more compelling reason as well. The NCTM *Standards* had made it clear that the geometry curriculum was to include a rich set of real-world applications (1989, p. 157). Thus I expected that the exercise sets in the newer textbooks would contain many real-world problems and potentially introduce other kinds of problems that were unknown in the earlier texts. I reasoned that if the newer textbooks were indeed widening their scope to "focus on more than deductive reasoning and proof," (p. 160) as NCTM recommended, then a comparison of raw numbers would be the fairer comparison between the two groups of texts. Comparing percentages (i.e. number of proofs in exercises as percentage of total number of exercises) could potentially hide the fact that some newer books had large numbers of proofs in exercises, simply because they had added more real-world applications or other types of problems. Thus, because my analysis compared textbooks across decades, the raw number would be more meaningful than the percentage.

Unassisted proof. This category includes only proof-writing tasks that require the student to develop the proof without help from the text. Fill-in-the blank proofs or proofs where a plan or outline is provided are excluded (though a hint is permissible). The assertion to be proved may be any statement with mathematical content. The category includes coordinate algebra proofs, or even proofs of solutions to algebraic equations.

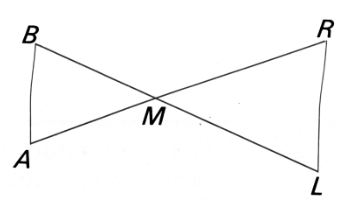
Example of unassisted proof. Glencoe 2005, p. 573

30. **PROOF** Write a two-column proof of Theorem 10.16.
Given: secants $\overline{EC}$ and $\overline{EB}$
Prove: $EA \cdot EC = ED \cdot EB$



Example of unassisted proof. Holt 1991, p. 317

12. Given: $\overline{AB} \parallel \overline{RL}$. Write a similarity statement. Prove triangle similarity. ***8.3***



Independent set-up proof. This is a subset of *unassisted proofs*. For an *independent set-up* exercise, the text does not provide the student with any of the following: the given statement, the prove statement, and the diagram. For example, the exercise above from Holt 1991, p. 317 does not count as an *independent set-up*, even though the student must determine what is to be proved, because the given statement and diagram are provided.

> Example of independent set-up proof. Houghton Mifflin 1988, p. 172
>
> 25. Prove: If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

> Example of independent set-up proof. McDougal Littell 2007, p. 301
>
> In Exercises 44 and 45, write a coordinate proof. . .
>
> 45. Any two congruent right isosceles triangles can be formed to form a single right isosceles triangle.

Fill-in proof. These are exercises that require students to fill in at least four blanks of a proof. Proofs with three or fewer blanks fall in the *proof development* category of exercises. As seen in the examples below, fill-in proofs need not be in two-column form.

Example of fill-in proof. Holt 2004, p. 248 (counts as one fill-in proof)

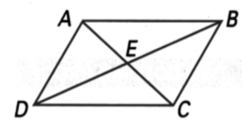
Complete the paragraph proof below of Theorem 4.5.5

Theorem

The diagonals of a parallelogram bisect each other. **4.5.5**

Given: parallelogram $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at point E

Prove: Point E is the midpoint of $\overline{AC}$ and $\overline{BD}$.



Proof:

$\overline{AB}$ and $\overline{CD}$ are parallel by **42.** ___?___, so $\angle BDC$ and $\angle DBA$ are congruent **43.** ___?___ angles. Also, $\angle ACD$ and $\angle CAB$ are congruent alternate interior angles, and $\overline{AB} \cong \overline{CD}$ because **44.** ___?___ sides of a parallelogram are **45.** ___?___. $\triangle ABE \cong \triangle CDE$ by **46.** ___?___. $\overline{BE} \cong \overline{DE}$ because **47.** ___?___, so point E is the midpoint of $\overline{BD}$ by definition. Also, $\overline{AE} \cong \overline{CE}$ because **48.** ___?___, so point E is the midpoint of $\overline{AC}$ by the definition of a midpoint.

Planned proof. Students are given a plan or outline, somewhere within the section. The plan may be in the exposition or in the exercise itself.

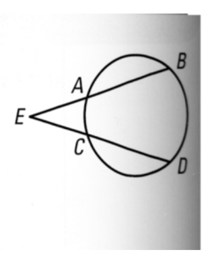
Example of planned proof. McDougal 2007, pg. 694

25. PROVING THEOREM 10.15 Use the plan to prove Theorem 10.15.

GIVEN ▶ $\overline{EB}$ and $\overline{ED}$ are secant segments.

PROVE ▶ $EA \cdot EB = EC \cdot ED$

Plan for Proof Draw $\overline{AD}$ and $\overline{BC}$. Show that $\triangle BCE$ and $\triangle DAE$ are similar. Use the fact that corresponding side lengths in similar triangles are proportional.



Proof development. An exercise in this category does not ask for a complete proof, but the task is to produce a component of proof. Here is a partial list of what the category includes: writing a plan for a proof, rewriting a proof in a different format, filling in three or fewer blanks, writing the beginning statement of an indirect proof, writing the "given" and the "prove" based on a figure provided.

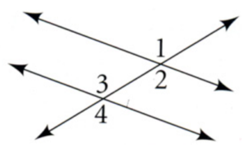
Example of proof development exercise. Merrill, 1984, p. 25

For each of the following theorems, name the given, the prove statement, and draw a diagram you would use in a formal proof.

19. If two planes intersect, then their intersection contains at least two points.

Example of proof development exercise. Prentice Hall, 2001, p. 197

10. Rewrite this paragraph proof as a two-column proof.
Given: $\angle 1 \cong \angle 4$
Prove: $\angle 2 \cong \angle 3$



$\angle 1$ and $\angle 2$ are vertical angles, as are $\angle 3$ and $\angle 4$. Vertical angles are congruent, so $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$. We are given that $\angle 1 \cong \angle 4$. By the Transitive Property of Congruence, $\angle 2 \cong \angle 4$ and $\angle 2 \cong \angle 3$.

Example of proof development exercise. Holt, 1991, p. 405

7. Discuss a plan for a proof that the median of a trapezoid is parallel to the base.

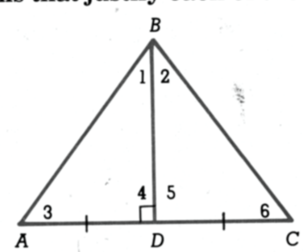
Justification and explanation, not proof. These are exercises that require students to develop a rationale for a statement or an answer that is not necessarily a proof. Responses to problems in this category include informal explanations, justifying an isolated statement using a theorem or postulate, providing a counterexample, and explaining why a given argument is incorrect.

Sometimes, instructions at the beginning of a true/false problem set included the direction to justify any “false” answers. In such cases, rather than determine the truth value of each statement, I counted all problems in the set as *justification and explanation* problems. Admittedly, my decision surely inflated the numbers for this category. Similar situations occurred elsewhere. For example, the student might be asked to draw the figure described, if possible and, in the case where it was not possible, to explain why.

Example of justification and explanation exercise. Merrill 1984, pg. 169

Name the postulates or theorems that justify each of the following statements.

11. The midpoint of $\overline{AC}$ is D.
12. $m\angle 5 = 90$
13. $\overline{BD} \cong \overline{BD}$
14. $\triangle ABD \cong \triangle CBD$
15. $AB = CB$
16. $\angle 1 \cong \angle 2$



Example of justification and explanation exercise.
Houghton Mifflin 1988, p. 34

39. Explain why the measure of a complement of an angle can never be exactly half the measure of a supplement of an angle.

Flow proofs. For an example of a flow proof, see the second example of a proof in exposition on p. 25. Flow proofs (also called flowchart proofs) use arrows to show the logical dependence of some statements on others. For an exercise to qualify for this category, the instructions had to specifically require a flow proof. For example, “Write a flow proof,” qualifies, but “Write a two-column proof, paragraph proof, or flow proof,” does not. A similar rule applies to paragraph proofs.

Paragraph proofs. These exercises fell into two subcategories: indirect proofs and direct proofs. Of course, in high school geometry textbooks, the direct approach is the default. Of the two, only the indirect approach is specified in the exercises. In every textbook, indirect proofs in the exposition were written primarily in paragraph form. Therefore, I assumed that when an indirect proof was asked for in the exercises, it would be written in paragraph form, and I counted it in the total number of paragraph proofs. However, of greater interest to me was the number of paragraph proofs in the exercises that were *not* indirect. I believe that this number tells us much more about how textbooks value and emphasize the paragraph form, and this is the number that I report in my results chapter.

Coordinate proofs. Also called “coordinate geometry proofs,” these are proofs that use coordinate algebra rather than synthetic methods. Some coordinate proofs qualified as *independent set-up* proofs, as in the second example below. Though textbooks often presented coordinate proofs in complete sentences and paragraphs (and almost never in two-column form), I did not include them in the category of paragraph proofs.

Example of coordinate proof, Addison-Wesley 1982, p. 447

8. Prove that the triangle whose vertices are $A(-3, 7)$, $B(2, -2)$, and $C(11, 3)$ is an isosceles right triangle.

Example of coordinate proof. Houghton Mifflin 1988, p. 508

13. Use coordinate geometry to prove that the median of a trapezoid is parallel to each base.

Limitations

There are several important limitations of the research methods outlined above, the most frustrating of these being the difficulty in establishing clear boundaries between categories. For example, though I established a working definition of *proof*, it was sometimes impossible to come to a definitive conclusion about whether the required response to a particular problem constituted a proof or a less formal explanation. The line between justification and explanation exercises, and exercises that did not fit into any of the relevant categories was also blurred.

Guiding principles that seemed straightforward in theory seemed to break down when faced with real examples. For instance, in the justification and explanation category, I included problems that required the student to explain their reasoning (e.g. "explain how you know. . ."), and I generally excluded problems that required students to explain a procedure (e.g. "explain how to find the circumference . . ."). But what about problems that end with the one-word imperative, "Explain."?[5] And where does explanation of a process end and explanation of reasoning begin? I made hundreds of decisions involving quandaries like these. As I did, my working parameters for each category continually evolved to accommodate new variations. Though I did strive for consistency, I eventually realized that perfect consistency would be impossible to achieve. This led me to view my results as rough counts rather than definitive numbers.

As my research got underway, I realized that the justification and explanation category was too broad. It was not effectively capturing the prevalence of the

[5] Answer: I usually included such problems in the justification and explanation category, assuming that they were asking students to explain their reasoning.

informal justifications recommended by Coxford (1991) and others. The category includes informal justifications but also short, formulaic responses such as citing a postulate or theorem in order to justify an isolated statement. While it might have been fruitful to divide this category into subcategories (perhaps drawing on the distinctions that Otten et al. (2011) made among types of reasoning tasks), looking at such a fine grain would have added significantly to the demands of the project. Looking at difficulty levels of the proof and justification exercises was also well outside the scope of this research.

Research Design, Part 2: Thematic Observations

In this section, I will discuss the qualitative thematic observation component of my research design. There are three themes that I follow in each of the textbooks: attention to form, presentation of theorems, and introduction to proof and proof writing.

Attention to Form

As I mentioned earlier in this chapter, I wanted a more complete picture of the treatment of form in the various texts (i.e. two-column form, other forms, and the concept of form itself) than numbers alone could provide. I added a thematic analysis component to my study of form, reporting observations on the treatment of form found in each textbook. I looked at options of form available to students, implied status of the two-column proof vs. other forms of proof, and conceptualization of form itself.

Presentation of Theorems

Since proofs and theorems are so closely linked, I wanted to include analysis of how individual theorems are introduced and discussed in each textbook, as well as a look at how the texts conceptualized *theorem*. I began with a holistic approach to gathering observations. This method yielded interesting insights, but I soon realized that I wanted to be able to use more concrete language than “sometimes” or “most of the time” to describe the patterns that were emerging. To this end, I chose a sample of five theorems for further study. So as not to bias the results, I chose the theorems as blindly as I could. That is, I chose them without first checking to see how they were discussed in the textbooks, though I had probably looked at most of them in at least some of the textbooks during my initial holistic perusal of the texts. My criteria for selection were (a) that the theorems had to be sufficiently well-known to be covered by each textbook, and (b) that they spanned several units of study, with representation from earlier and later chapters in the textbook. Below I have listed the five theorems that I chose. (I have arbitrarily chosen the Merrill 1984 text to provide the wording for each theorem listed here.)

- *Isosceles Triangle Theorem* If two sides of a triangle are congruent, then the angles opposite those sides are congruent (p. 129).
- *Angle-Angle-Side* If two angles and a nonincluded side of one triangle are congruent to the corresponding angles and nonincluded side of another triangle, then the triangles are congruent (p. 121).
- If a quadrilateral is a parallelogram, then its opposite sides are congruent (p. 217).

- If a trapezoid has an area of A square units, bases of b_1 units and b_2 units, and an altitude of h units, then $A = \frac{1}{2} h (b_1 + b_2)$ (p. 368).
- If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency (p. 324).

In retrospect, I can see that I had a third criterion in mind. Without being fully aware of what I was doing, I instinctively chose theorems that are fairly easy to grasp intuitively and are well-suited to visual explanations. This probably influenced my results; the patterns that became clear were likely stronger because of this characteristic of the theorems in the sample.

I noted the following aspects of the discussion surrounding each theorem: (a) the extent to which the theorem was placed in the context of investigation and conjecture, (b) any real-world applications for the theorem, and (c) if, where, and how the theorem was proven in the text. "Discussion" here includes canonical sections of the exposition, any investigation sections that preceded the theorem, and exercises immediately following the exposition (but not including chapter review exercises, etc.).

Though I looked for activities in the texts that would lead students to discover the content of the theorem, I did leave out one possible avenue for investigation and conjecture. From my experience teaching high school geometry, I know that textbooks on occasion formulate exercises designed to help students anticipate a theorem that will appear in a future section. I did not search for such exercises in the sections preceding each theorem. Had I done so, my results would certainly have been more complete.

Introduction to Proof and Proof Writing

To investigate how textbooks introduced students to proof writing, and to the concept of proof itself, I analyzed the material in the textbooks at and preceding the point where students were first expected to write their own proofs. I was particularly interested in extra-geometric material that was clearly designed to set the stage for proof by teaching reasoning skills (e.g. analyzing conditional statements, identifying inductive reasoning) or providing practice with components of proof writing (e.g. drawing and labeling a diagram to illustrate a "given" statement). Of course, one could argue that any section teaching material that will be used later to write proofs (e.g. definitions, postulates) is preparation for proof. But I was looking for specific instruction outside the domain of geometric concepts, terms, postulates, theorems, etc. I should mention here that it was sometimes difficult to pinpoint the moment in the text where proof writing was introduced. The phrase "introduction to proof" should be understood to be lacking in precision.

To get a sense of how textbooks conceptualized *proof*, I looked for definitions of "proof" in the expositions and in the glossaries of each textbook. I also looked at chapter and section titles for a clearer picture of how each group of textbooks framed the concepts of proof and reasoning. As my research progressed, I saw that I could not separate the textbooks' treatment of proof from their treatment of reasoning, so I incorporated the topic of reasoning into my discussion.

In the following two chapters, I present my analysis of the research, beginning with the counting results and continuing with the thematic analysis.

Chapter 3: Results, Part 1

This chapter analyzes the results of the coding analysis that comprises the first part of my research, reporting totals for the categories and subcategories that I established in the previous chapter. I begin with the number of *unassisted proofs* in the exercises for each textbook, followed by the subcategory of *independent set-up proofs*. Then I report the numbers of *fill-in proofs* and *planned proofs* in the exercises, both in the broader category of assisted proofs, followed by the numbers of *proof development exercises.* Next, I turn to the exposition, looking at the number of *complete proofs in the exposition* of each textbook. The next section presents the totals for the *justification and explanation* category of exercises. The next two sections look at the numbers of *coordinate proofs*, *flow proofs,* and *direct paragraph proofs* in the exercises of each textbook. The final section of the chapter is a thematic analysis of the attention to form in each textbook.

Results of the Coding Analysis

Unassisted Proofs in Exercises

First, I consider the number of *unassisted proofs* in the exercises¸ with the goal of gaining insight into the relative emphasis placed on proof in the two groups of texts. As described in Chapter 2, these are exercises that require students to develop proofs without help from the text in the form of an outline, plan, or fill-in-the-blank format (See p. 27). The results show that in the newer textbooks, the number of exercises that require students to write unassisted proofs is generally lower than in the earlier textbooks. This finding seems to support the hypothesis that the later textbooks deemphasize proof in that it appears that in the earlier textbooks there is a greater expectation for students to be able to produce proofs without assistance. (See Figure 3.)

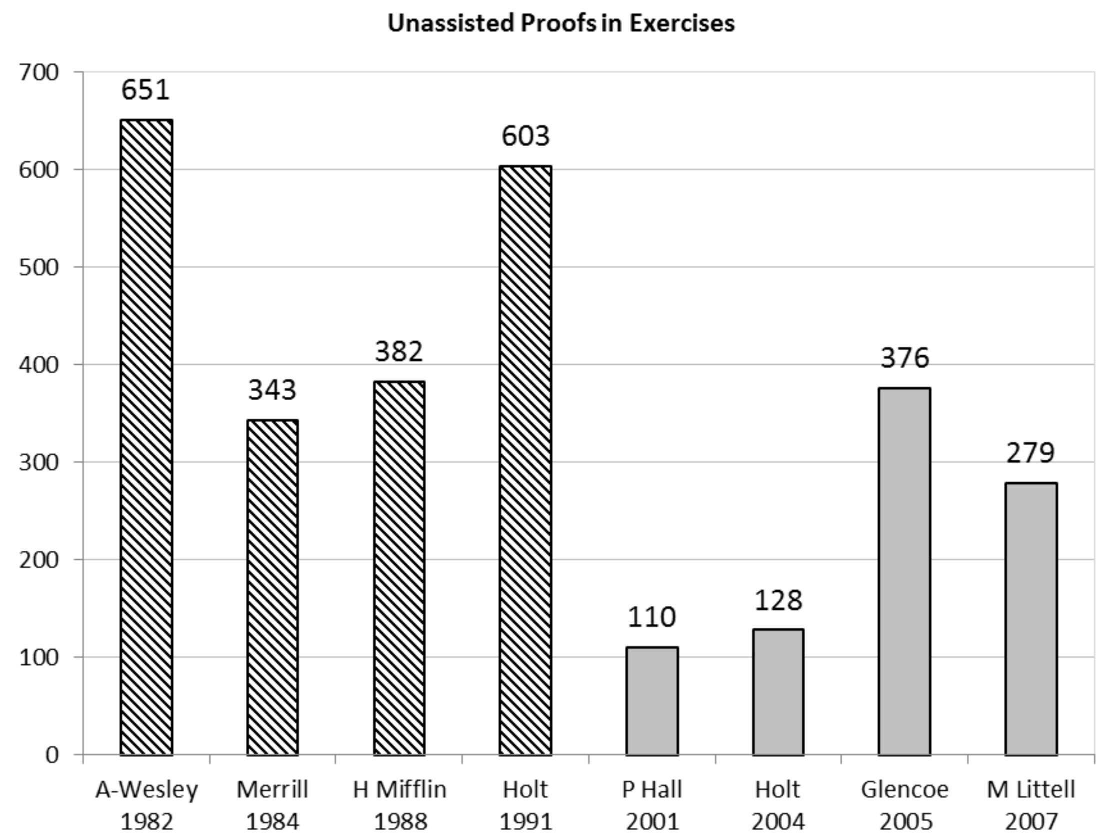


Figure 3. Unassisted proofs in exercises. The hashed bars are for textbooks from the older group, while the newer group uses solid bars. The number that appears over each bar gives the total count for that textbook.

Since the number of textbooks in each set is so small, the variation within the group should temper the importance we give to the mean. That said, the mean for the earlier four texts is 495, while the mean for the newer texts is less than half of that, at 223. Only one of the newer textbooks ranked higher in this category than any of the earlier textbooks. These results are strengthened by the fact that the newer textbooks are generally longer. (The earlier textbooks have an average of 655 pages, while the newer textbooks have an average of more than 885 pages.) However, we must not ignore the considerable variation among the textbooks within each decade. For example, among the newer texts, Glencoe 2005 stands out. With 376 unassisted proofs in the exercises, it is comparable to the Merrill 1984 and Houghton-Mifflin 1988 texts.

It is also interesting to compare the two textbooks published by Holt, Rinehart and Winston. The difference there is striking. In the 1991 textbook, there are nearly five times as many unassisted proofs in the exercises compared with the 2004 textbook (603 vs. 128). This gap in the number of unassisted proofs found in textbooks out of the same publishing house seems to tell a dramatic story of shift in emphasis. As we will see in the next several sections, that shift is manifest in other areas besides the count of unassisted proofs.

There are some oddities buried in these numbers, with one worth mentioning in the Prentice Hall 2001 textbook. The total count of unassisted proofs in the exercises was only 110, the lowest of all books in the study. But further, 31 of those proofs appear in a problem set titled “Challenge Problems”, found at the very end of the textbook. In addition, more than half of the Independent Set-up exercises (11 out of 19) appear in the Challenge Problems at the end of the book. These numbers make it clear that the Prentice Hall textbook demonstrates the lightest emphasis on proof in the exercises of all the textbooks that I studied.

The category of exercises analyzed in this section, unassisted proofs, is particularly important to my research. It is significant that, in their analysis framework, Otten et al. (2011) included *only* what I call "unassisted proofs" in the category of exercises that involved students in "developing a mathematical proof" (p. 351). Indeed, only unassisted proofs, not fill-in or planned proofs, require students to conceive of the argument's deductive structure.

Though the Prentice Hall 2001 text is an extreme example, it is apparent that the newer textbooks, on the whole, provide fewer opportunities in the exercises for students to write unassisted proofs. If we make the plausible assumption that the majority of practice writing proofs is done in the exercises, and if we use Otten et al.'s (2011) name for this category[6], we can make a stronger statement: On the whole, the newer textbooks provide fewer opportunities for students to develop a mathematical proof. This could have implications for how well students learn to write proofs. At the very least, the relative scarcity of unassisted proofs in the Prentice Hall 2001 and Holt 2004 texts should give us pause. Can they be as effective in helping students master proof writing as the Addison-Wesley 1982 and Holt 1991 texts?

Independent Set-up Proofs in Exercises

In this section, I compare the number of exercises that require students to generate the "given" and "prove" statements and the diagram, in addition to writing the proof without assistance. (See p. 28 for examples.) The number of *independent set-up proofs* in the earlier textbooks averaged 119 while the newer textbooks averaged 27, and all of the newer textbooks ranked higher than all of the earlier textbooks in this category. This result presents one of the clearest differences between

[6] Otten et al.'s parameters most likely differ in some respects from mine. My category of *unassisted proofs* and their category of exercises that require students to *develop a mathematical proof* are essentially the same, but probably not identical.

the two groups of textbooks. Notice as well the stark contrast between the two Holt textbooks. (See Figure 4.)

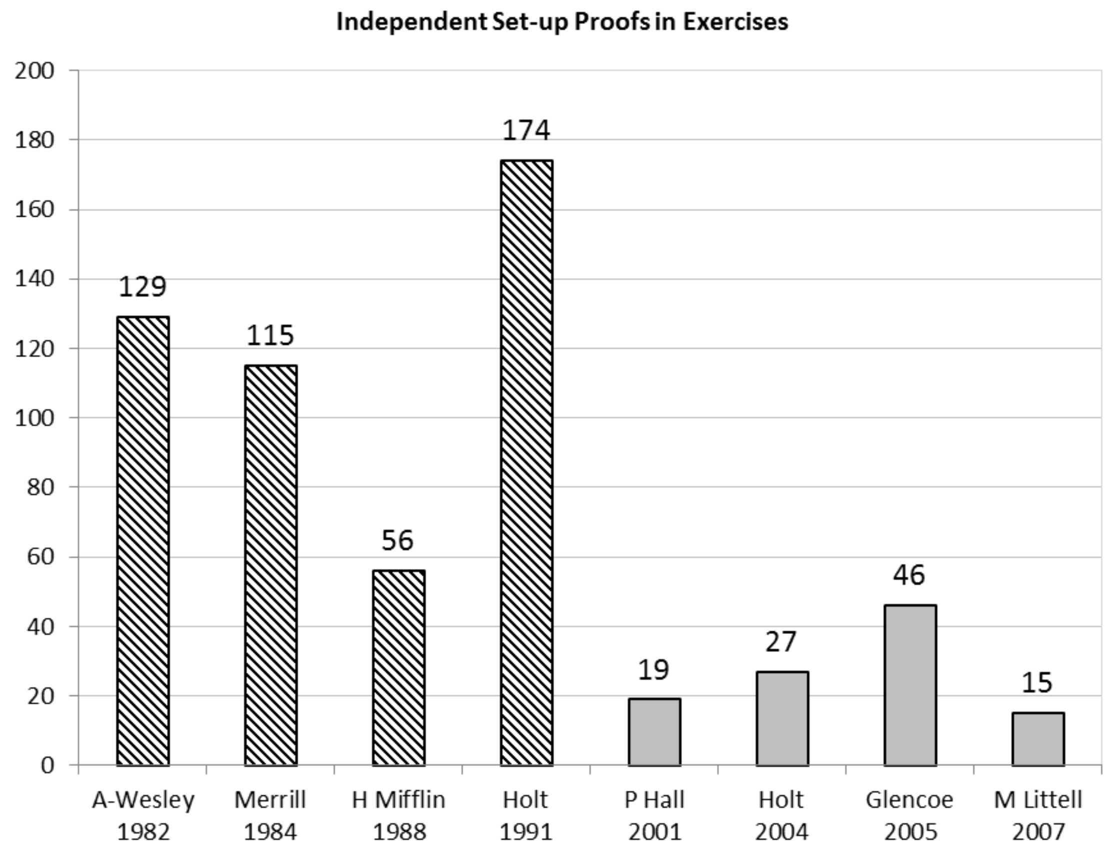


Figure 4. Independent set-up proofs in exercises.

Independent set-up proofs require more than unassisted proofs do: The student must read a mathematical assertion, extract the given information, draw a diagram, and formulate what is to be proved for the specific diagram. The older textbooks seem to establish a stronger expectation that students develop this set of skills. Arguably, a successfully completed independent set-up proof exercise demonstrates a greater level of sophistication than one in which the "given," "prove," and diagram are provided. Note that a mathematical theorem is almost always stated as a general conditional statement, not as an assertion about a particular diagram, and not as separate "given" and "conclusion" statements. We should consider the possibility that fewer opportunities to practice writing independent set-up proofs might negatively affect students' ability to see their own proof-writing activity as a means for establishing a generalizable mathematical truth.

Assisted Proofs in Exercises

Recall from Chapter 2 that I have established two categories of exercises requiring students to write proofs with assistance provided by the text: *fill-in proofs*, where students complete several blanks in a proof that has been partially completed in the text, and *planned proofs*, where students write a proof with a plan or outline provided. (See p. 28.)

Comparing the numbers of assisted proofs in the exercises of each textbook does not yield any obvious patterns. (See Figure 5.) The results are not proportional to the total number of proofs in the exercises. Neither is there any generalization that I can make about newer textbooks vs. older textbooks. However, it is notable that in the two textbooks with the fewest proofs in the exercises (Prentice Hall 2001 and Holt 2004), the share of assisted proofs as a percentage of the total number of proofs (i.e. the sum of unassisted and assisted proofs) is the highest, at 23.1% and 32.7%, respectively. Perhaps this finding points to softer expectations for students' mastery of proof writing. It is also interesting to follow the story we began in a previous section of the two Holt textbooks. The Holt 2004 text, which has barely one fifth as many unassisted proofs as Holt 1991, actually has more assisted proofs than the earlier text. Looking more closely, we see that Holt 2004 favors fill-in proofs, while Holt 1991 has more planned proofs.

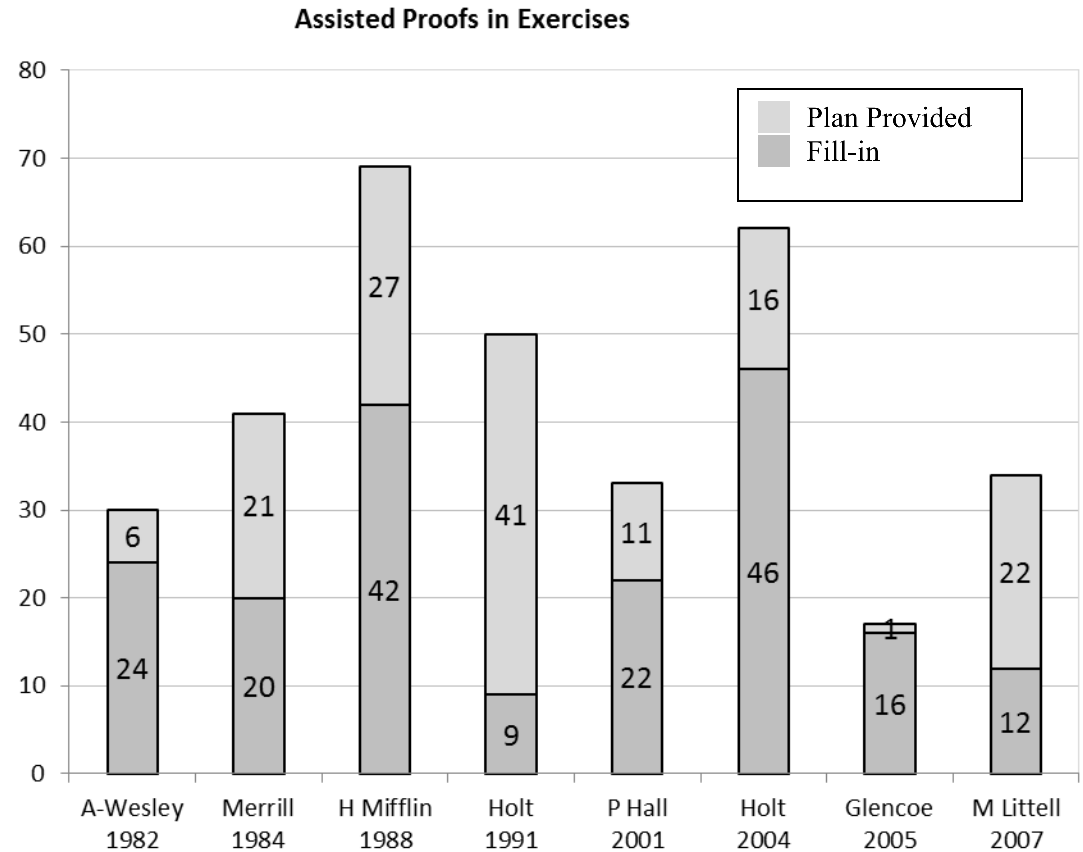


Figure 5. Assisted proofs in exercises. The upper portion of each bar, shown in lighter gray, indicates the number of planned proofs. The lower portion of the bar, shown in darker grey, indicates the number of fill-in proofs.

Proof Development Exercises

Proof development exercises is a broad category which encompasses various exercises that require students to write a plan for a proof or some component of a proof. (See p. 29.) There is no obvious pattern in the numbers of proof development items in the exercises. That is not entirely surprising, considering that this is a miscellaneous category that includes a wide variety of exercises, as I discuss in Chapter 2. Still, I speculated at the outset that the number of proof development exercises might correlate to an overall emphasis on proof in the exercises. Alternatively, I wondered if the newer textbooks might contain more proof development exercises in response to concerns that students were not developing proficiency at proof writing. Neither guess proved correct. (See Figure 6.)

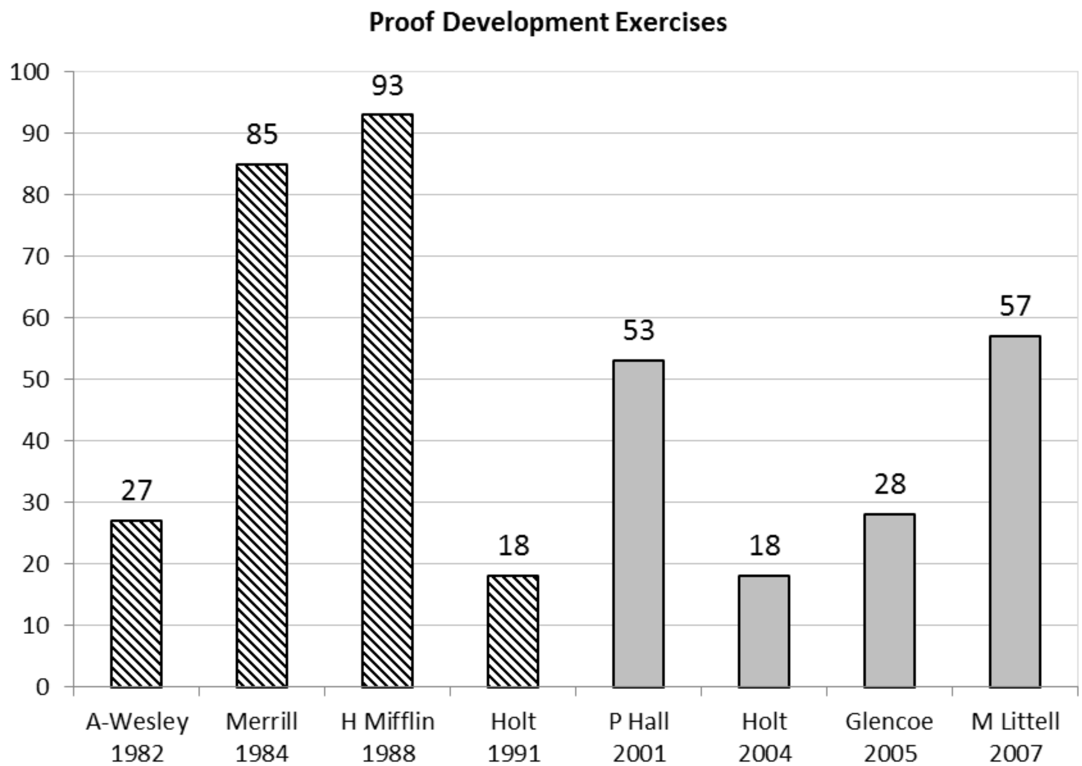


Figure 6. Proof development exercises.

The numbers of proof development exercises are probably most meaningful when added to the total number of proofs in the exercises for each textbook. A plausible interpretation of these numbers would be that the total sums reflect the overall emphasis in the exercises on the activity of writing proofs, or parts of proofs. Notice that the numbers are higher in the earlier books. (See Figure 7.)

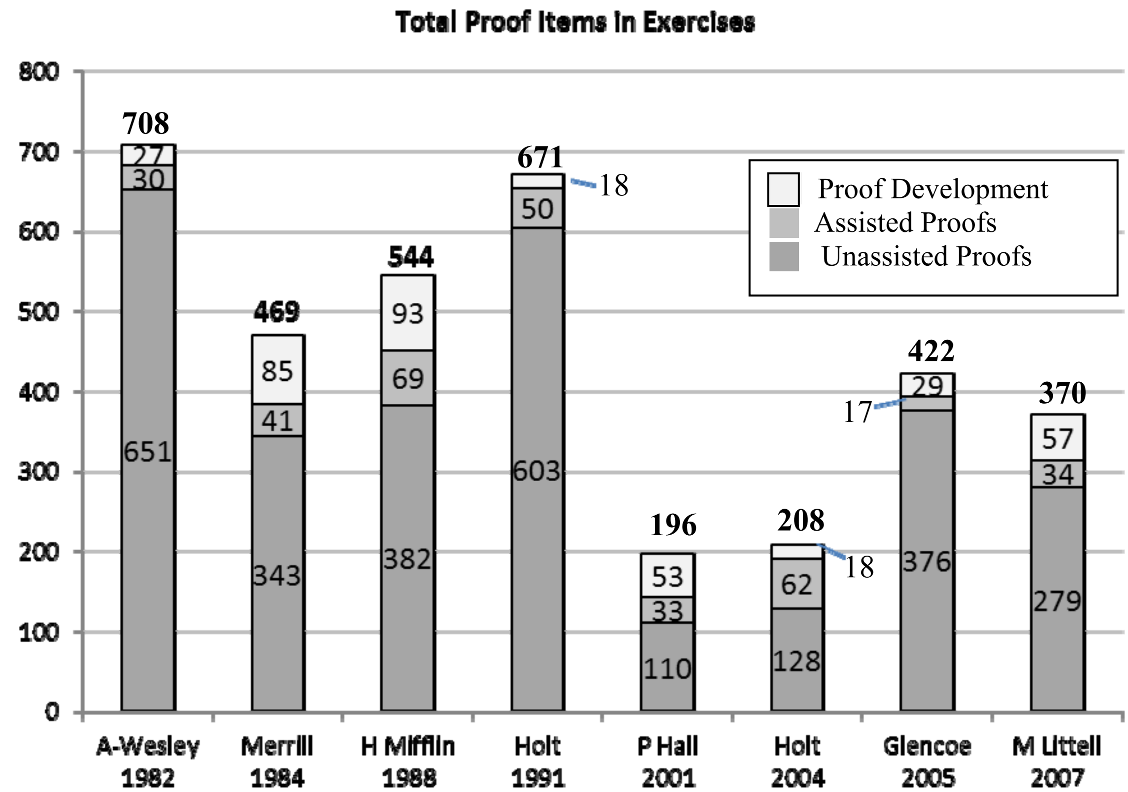


Figure 7. Total proof items in exercises. Each number above the bar is the sum of proof development items, assisted proofs, and unassisted proofs.

Complete Proofs in the Exposition

This section analyzes the presence of proof in the exposition, rather than in the exercises, reporting the numbers of complete proofs in the exposition. (See Figure 6.) Prior to counting proofs in the exposition, I had expected to find consistently and dramatically higher numbers in the older set of textbooks. The results did not meet my expectations; they look somewhat muddier than the results from the sections counting unassisted proofs and independent set-up proofs in the exercises. (Compare with Figures 3 and 4.) It is true that the Addison-Wesley 1982 and Holt 1991 texts have far more proofs in their expositions than do any of the other textbooks. But the other two textbooks in that group, Merrill 1984 and Houghton Mifflin 1988, have numbers comparable to the newer textbooks. Certainly, the numbers suggest that the Addison-Wesley 1982 and Holt 1991 texts, in harmony with their proof-heavy problem sets, place a higher emphasis on proof in the exposition than do the other texts.

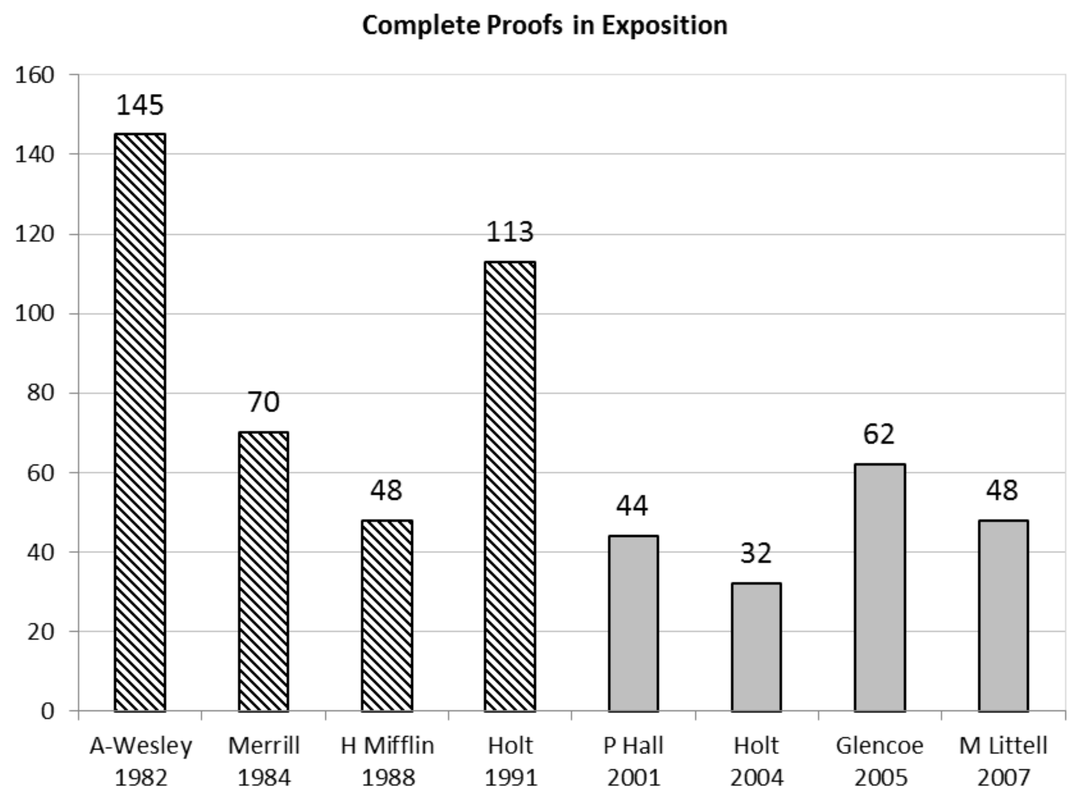


Figure 8. Complete proofs in exposition.

These numbers do not include plans for proofs or incomplete proofs in the exposition. Nor do they take into account the number of times students were presented with a theorem in the exposition (with or without a plan for proof), and then instructed to prove it in the exercises. It would be interesting to know how those exclusions might change the results. However, the present methods do not allow for this analysis.

Justification and Explanation Exercises

This section considers justification and explanation problems that do not require a formal proof. (See p. 30.) As I discuss in greater detail in Chapter 2, responses to these exercises include informal explanations as well as short, formulaic answers such as citing a theorem or postulate to justify a statement.

The data suggest that the newer textbooks place a somewhat greater emphasis on this type of problem than do the older textbooks, with the exception of Houghton Mifflin 1988. (See Figure 9.)

However, as I discussed in Chapter 2, the results for this section may not fully capture the difference in emphasis between the two groups of textbooks. The category is so broad that it includes not only informal arguments, but also exercises with short formulaic responses such as providing counterexamples or citing a theorem to justify an isolated statement. This helps explain the result for Houghton Mifflin 1988, a textbook in the earlier group. That textbook has 715 justification and explanation exercises, a number in line with the newer textbook group. But the number is potentially misleading, because the Houghton Mifflin 1988 textbook has a particularly large number of problems asking students to cite a theorem or postulate in order to justify an isolated statement. Those problems fall within my parameters for the justification and explanation category, but they do not represent the kind of informal argument that I had in mind when I formulated my research.

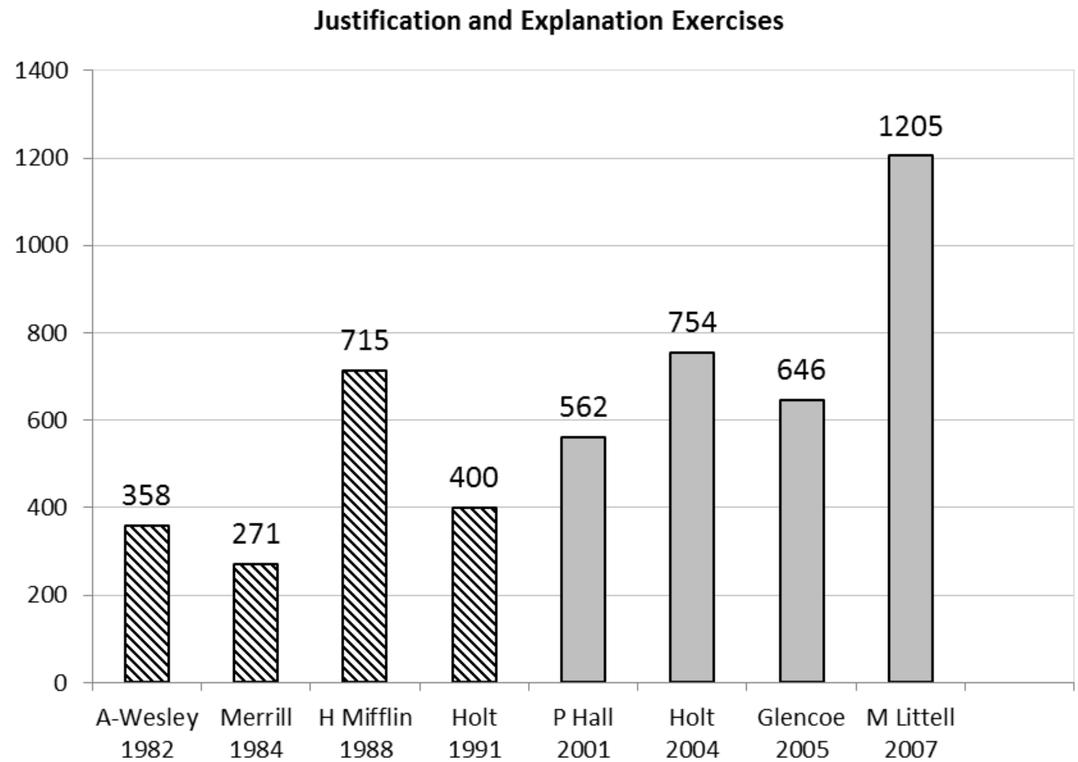


Figure 9. Justification and explanation exercises.

As I conducted the analysis for this section, my overall impression was that the newer textbooks had far more problems asking for informal justification than did the older textbooks. Though my methodology did not allow me to capture that insight with concrete data, it is an important piece of the shift in emphasis in the approach to proof. The relative value that a textbook places on formal proof vs. informal argumentation might have implications for students' attitudes and beliefs about justification.

Coordinate Proofs in Exercises

A *coordinate proof* uses algebraic methods to prove a statement, rather than synthetic geometry. (See p. 32.) NCTM's 1989 *Standards* give the topic of geometry from an algebraic perspective its own chapter, separate from synthetic geometry (p.

161). This Standard includes coordinate representations of geometric figures, though it encompasses the study of vectors and transformations as well. Because a separate Standard is devoted to geometry from an algebraic perspective, I initially expected to find a significantly higher number of coordinate proofs in the newer textbooks, as compared with the earlier textbooks. That did not turn out to be the case. (See Figure 10.)

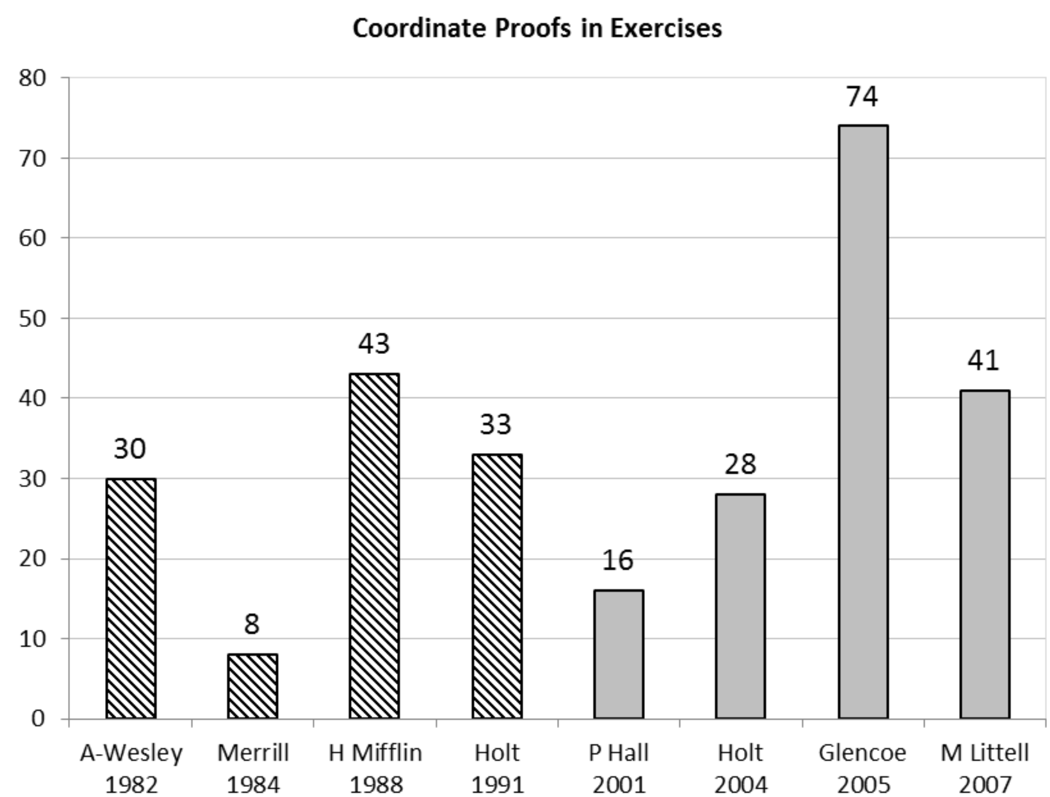


Figure 10. Coordinate proofs in exercises.

It was difficult to settle on a definition of “coordinate proof” that made sense in every situation, so it is possible that these results reflect some ambiguities in the research method. But I do not believe that an imperfect counting method can fully account for the finding that, except for Glencoe 2005, the newer group of textbooks showed no increase in the number of coordinate proofs. Excluding the Glencoe text, the data for the two groups are remarkably similar. In fact, their respective means are virtually the same, both rounding to 28.

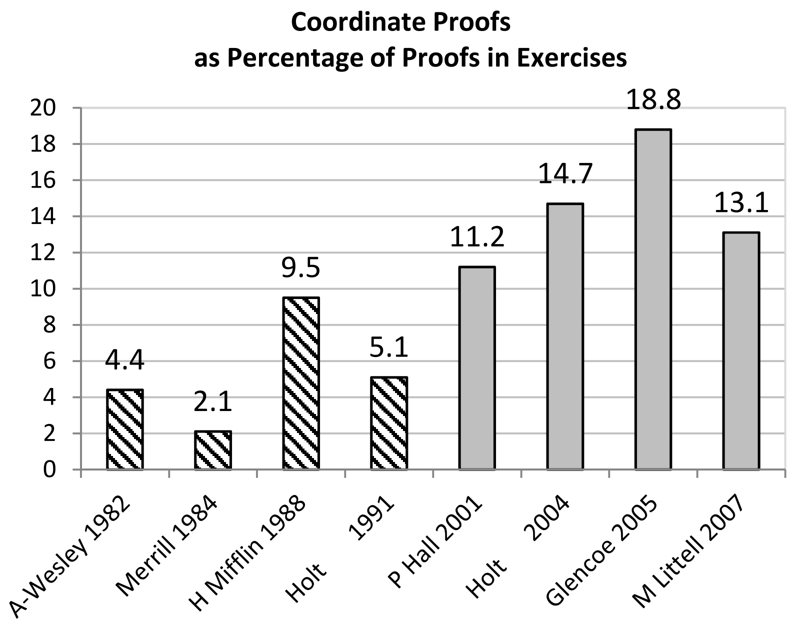


Figure 11. Coordinate proofs as percentage of proofs in exercises. Here, "proofs" is defined as the sum of unassisted and assisted proofs.

Though the raw numbers in this section do not indicate a significant difference between the two groups of textbooks, when we look at the number of coordinate proofs in the exercises as a percent of the number of proofs (the sum of assisted proofs and unassisted proofs) in the exercises, we see that the newer textbooks do devote a larger share of proof exercises to coordinate proofs. (See Figure 11.) Still, the increase is not as large as I had predicted at the outset. It should be noted that the measurement in this section is a narrow one: the number of coordinate geometry proofs in the exercises. It would be interesting to take a broader look at coordinate geometry and compare the treatment and emphasis of that branch of geometry in each group of textbooks. However, I will not undertake analysis of that sort here.

Flow and (Direct) Paragraph Proofs in Exercises

Here I look at the number of *flow proofs* and the number of direct *paragraph proofs* (p. 31) specifically elicited in the exercises of each textbook. (See Figure 12.)

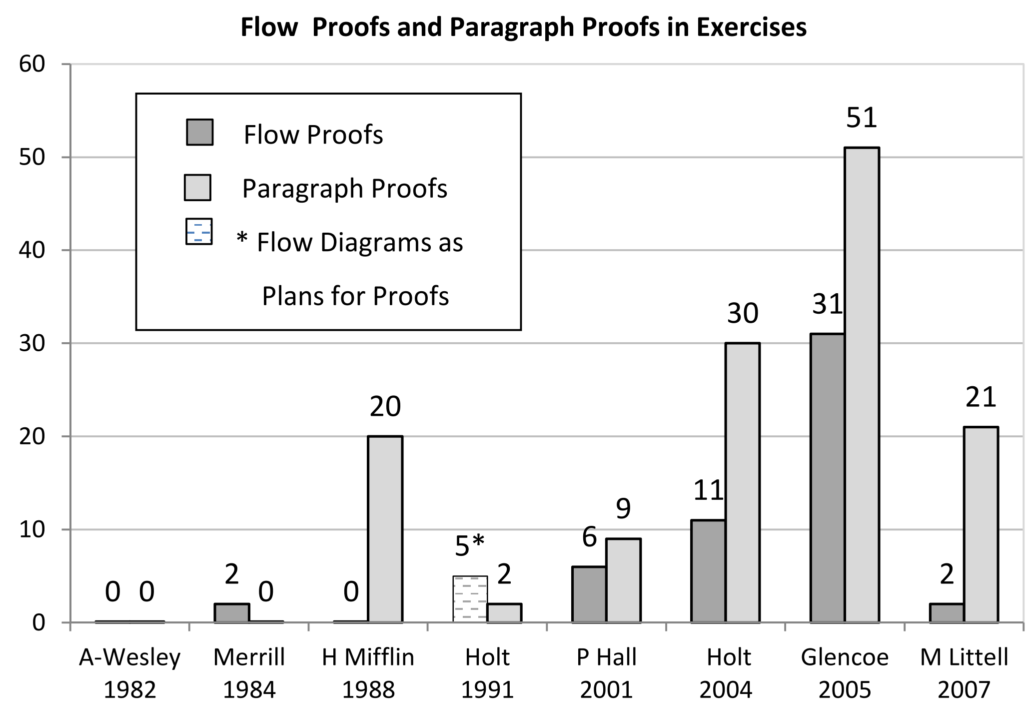


Figure 12. Flow proofs and paragraph proofs in exercises. The paragraph proofs here are direct proofs (as opposed to indirect proofs). The patterned shading of the first bar for Holt 1991 indicates that the flow diagrams in that textbook are plans for proofs, not proofs themselves.

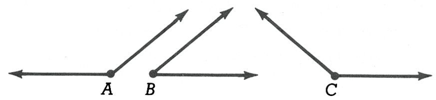
Despite wide variation of the data, the newer group clearly outstrips the earlier group in both flow proofs and direct paragraph proofs in the exercises. (For the remainder of this section, "paragraph proof" will be understood to exclude indirect proofs.) In fact, two of the four older textbooks (Addison-Wesley 1982 and Merrill

1984) do not include any specific requirements for paragraph proofs in the exercises, and two (Addison-Wesley 1982 and Houghton Mifflin 1988) include no flow proofs. It is important to note, however, that in the Addison-Wesley 1982 text, many of the proofs in the exposition are in paragraph form, as will be discussed later in the chapter.

Though flow proofs are found in the literature as early as the 1970s (for example, see R. McMurray 1978 and D. Basinger 1979, both in *The Mathematics Teacher*), they are scarcely represented in the 1980s textbooks, and they are not mentioned at all in two of the four textbooks from that group. Holt 1991 does contain a few “flow diagrams,” as they are called in that textbook. They are not presented as proofs themselves, but rather as plans for proofs or organizational aids, as demonstrated in this sentence from the text: “In more complex proofs . . . it is sometimes helpful to draw a flow diagram before actually writing out the steps” (p. 185). Merrill 1984 relegated flow proofs to one enrichment section, separate from the main sequence of topics. I found it interesting that the flow proofs in the Merrill text have a different look than those in the newer textbooks; they are more compact and less visually robust. (See Figure 13.)

Flow proofs are found in all four of the newer geometry textbooks, but to widely varying degrees. Still, there seems to be an expectation in each textbook from the 2000s that students should become familiar with the form and know how to use it to write proofs. In all four newer textbooks, flow proofs appear in both the exposition and the exercises.

Given: $\angle A$ is a supplement of $\angle B$.
$\angle C$ is a supplement of $\angle B$.
Prove: $\angle A \cong \angle C$



Flow Proof:

$$\left.\begin{array}{l}\angle A \text{ is supplement of } \angle B. \xrightarrow{1} m\angle A + m\angle B = 180 \\ \angle C \text{ is supplement of } \angle B. \xrightarrow{1} m\angle C + m\angle B = 180\end{array}\right\} \xrightarrow{2} m\angle A + m\angle B = m\angle C + m\angle B \xrightarrow{3}$$

$$\xrightarrow{3} m\angle A = m\angle C \xrightarrow{4} \angle A \cong \angle C$$

1. Definition of Supplementary Angles
2. Substitution
3. Subtraction Property of Equality
4. Definition of Congruent Angles

Figure 13. Flow proof from Merrill 1984 (p. 80). Notice that the reasons are printed below the flow chart. For an example of a flow proof in a newer book, see p. 25 of this paper.

It is evident that the newer textbooks are somewhat more inclined to specifically ask for paragraph proofs in the exercises, though there is significant overlap in the results for the two groups. However, since I reported only the paragraph proofs elicited in the exercises, these results do not show the large number of paragraph proofs in the exposition of the Addison-Wesley 1982 text. In the next section I will explore this thread further as I look more closely at the attention given in the various texts to proof format or style.

Thematic Analysis of Attention to Form

As I undertook the counting phase of my research, I had a growing sense that the numbers alone were not capturing differences that I saw in the way textbooks treated the concept of form in general, and the two-column form I particular. Since the two-column form has such an important role in the history of high school geometry, I wanted to examine more closely the evolution of its role in the 21st century. With these ideas in mind, I added a thematic analysis component to my study of form, reporting observations on the treatment of form found in each textbook. I looked at options of form available to students, implied status of the two-column proof vs. other forms of proof, and conceptualization of form itself.

In the Addison-Wesley 1982 textbook the two-column form is introduced as a pedagogical aide for writing proofs, not as a form that is inherently preferable or more correct than the paragraph form. The text advises, “Using the two-column form for the proof makes it easier to organize your work when you are first learning” (p. 110). In the sections of text where proof writing was introduced and developed, the two-column form is found in the exposition. Later, paragraph proofs are more frequently used. But nowhere does the exposition draw attention to the paragraph form as an object of study (and I do not believe that the phrase “paragraph proof” is ever used). It seems to be taken for granted that the two-column form will naturally give way to paragraphs as students become more familiar with proof and proving.

The two-column format is the dominant form in the Merrill 1984 and Holt 1991 texts. Except for coordinate proofs, the Merrill text does not present any alternatives to the two-column format. Barring coordinate proofs, nearly all of the proofs in the exposition are in two-column form. All of the proofs in the answers to odd problems given at the end of the textbook are also in two-column from.

The Holt 1991 text does mention paragraph proofs briefly, but they are given little attention. Throughout the text, most proofs in the exposition are written in two columns. Notably, there are several instances in the exposition where flow diagrams are used, but each is designated as a “plan” for a proof, not a proof itself. As in the Addison-Wesley textbook, there is not much discussion of form itself in Holt 1991 and Merrill 1984.

In contrast with Addison-Wesley 1982 (where the two-column form's pedagogical function is made clear, with the paragraph form's virtues left as an implicit assumption), and Holt 1991 and Merrill 1984 (where the two-column form is virtually the only option), the newer textbooks make the point that a proof can take various forms, all worthy of study and equally valid.

In Prentice Hall 2001, the numbers of flow proofs and paragraph proofs specifically elicited in the exercises are low (six and nine, respectively), but those numbers do not tell the whole story. The various forms that proofs might take are heavily emphasized. In the section introducing proof writing, the discussion of geometry proofs begins with the sentence, "Proofs in geometry can take several forms" (p. 194). In that introductory section, there are examples of both two-column proofs and paragraph proofs, and both forms are represented in the exercises. After flow proofs are established in Chapter 7, most proof writing exercises give students a choice of form: "Write a paragraph proof, a two-column proof, or a flow proof" (p. 509, for example). This instruction is repeated over and over again in the exercises of the later chapters, as if to ensure that students do not forget that various forms are possible. The wording is telling. It draws attention to form itself, it establishes flow proof, paragraph proof, and two-column proof as separate non-overlapping entities, and it treats them as equally valid options. Contrast that with the three earliest textbooks, where form is rarely mentioned, and, at least in the case of Merrill 1984 and Holt 1991, the two-column proof is expected throughout the book.

Holt 2004 introduces proof writing using what it calls "free-form proofs" in the first lesson – informal arguments about concepts that are generally outside the main course content (e.g. proving a shortcut for squaring numbers ending in 5). Then, in the same section, it lists five “styles for proofs” to be learned later: two-column proofs, paragraph proofs, flowchart proofs, coordinate proofs, table proofs (p. 82). As in Prentice Hall 2001, the message is clear that the two-column proof is just one style among several.

In the Glencoe 2005 the first section teaching proof writing focuses on paragraph proofs, and two-column proofs are not introduced until the following section. As do the other textbooks of its decade, it puts the two-column form in a one-of-several-forms context. In the exercises, paragraph proofs and flow proofs are heavily represented, more so than in any of the other textbooks. The Glencoe text’s definition of paragraph proof is an example of terminology that is specific to the textbook and does not concur with standard usage in the field of mathematics. In the text, paragraph proofs are specifically designated as informal proofs: “One type of proof is called a paragraph proof or informal proof (p. 90).” The use of the word “or” suggests that “paragraph proof” and “informal proof” are synonyms. Using similar wording, two-column proofs are defined as “formal proofs” (p. 95). These choices made by the authors are a deviation from the way mathematicians think about “formal” and “informal.”

Prentice Hall 2001, Holt 2004, and Glencoe 2005 all emphasize that the two-column format is one of several valid styles of proof. The McDougal Littell 2007 text departs from its group; it does not emphasize this message to the extent that the others do. It introduces paragraph proofs only after two-column proofs have been established, with almost no discussion of either the paragraph form or the concept of form itself. It does not give as much attention to flow proofs as do the others of its group, though it does elicit a moderate number of paragraph proofs.

My previous discussion of the early textbooks left out the Houghton Mifflin 1988 text, because of its similarity to the McDougal Littell 2007 in its emphasis on other formats (minus any mention of flow proofs). It departs from its group by setting a clear expectation that students learn to write paragraph proofs, with 20 paragraph proofs elicited in the exercises (compared with 21 in the McDougal Littell text). It introduces paragraph proofs more than 100 pages after the two-column form is established, with the statement, “A proof in mathematics can take many different forms” (p. 143). It is interesting that in Houghton Mifflin 1988, the description of paragraph proofs includes an element not explicitly addressed in the other textbooks: that “justifications that are expected to be clear to the reader are often omitted” (p. 143). Perhaps the practice of omitting some of the obvious justifications is the reason behind the Glencoe 2005 text’s definition that equates paragraph proofs with informal proofs.

In this section, I have analyzed each textbook’s attention to form. Even taking into account the idiosyncrasies of the textbooks and the variation within each group, we can identify distinguishing characteristics of each group of textbooks. Compared with the earlier books, the newer textbooks place a greater emphasis on alternatives to the two-column form, and they emphasize the role of the two-column form as one of several valid forms. In the next chapter, I will discuss the differences between the two groups in their approaches to presenting theorems and introducing proof writing.

Chapter 4: Results, Part 2

To get a clearer picture of the approach to proof taken by the textbooks in my study, I widened the scope of my research to include not only coding and counting of proofs and justifications, but also more qualitative analysis. My discussion of the qualitative portion of my research began in the previous chapter, with an analysis of the texts' attention to forms of proof. In this chapter, I discuss two additional themes: presentation of theorems and introduction to proof and proof writing. The first section of this chapter analyzes how the textbooks present individual theorems and provides insights on how theorems are conceptualized in the two groups of textbooks. The second section of this chapter looks at how each textbook introduces proof and proof writing. Both sections address themes that are essential to understanding differences in how the two groups of textbooks teach and talk about proof.

Presentation of Theorems

In comparing the way textbooks from the 1980s and the 2000s presented individual theorems, I used a sample of five theorems that were present in all eight textbooks to analyze, as detailed in Chapter 2. I also looked at how each textbook introduced the concept of "theorem."

At first glance, the definition of "theorem" in each textbook looks very similar: something like, "A theorem is a statement that is proven to be true." But a closer look reveals that two of the four textbooks from the newer group (and none of the earlier books) use the word "conjecture" in the definition. (See Figure 14, Definition of Theorem.) Prentice Hall 2001 gives the definition, "A conjecture that is

		Addison-Wesley, 1982	Merrill, 1984	Houghton-Mifflin, 1988	Holt, 1991	Prentice Hall, 2001	Holt, 2004	Glencoe, 2005	McDougal-Littell, 2007
Definition of Theorem		The statements that we prove will be called theorems. p. 2	Theorems are statements that must be proven before accepted. p. 19	Statements that are proved are called theorems. p. 23	A statement that has been proved true, not just taken for granted, is called a theorem. p. 27	A conjecture that is proven is a theorem. p. 49 and glossary	A statement that has been proven to be true deductively. Glossary, also on p. 109 in sentence form	A statement or conjecture that can be proven true by undefined terms, definitions, and postulates. Glossary	A theorem is a statement that can be proven. p. 113 A true statement that follows as a result of other true statements. Glossary
Discovery and Conjecture	Isosceles Triangle	-----	-----	-----	-----	Cutting paper	Draw, fold much earlier in text	Draw, fold	-----
	Angle-Angle-Side	-----	-----	-----	-----	-----	-----	Draw	-----
	Opp. Sides of a ▱ Are Congruent	-----	-----	-----	-----	Cutting paper	Draw, measure	Draw	Computer/ graphing calculator
	Area of a Trapezoid	-----	-----	-----	-----	Cutting paper	Cutting paper	-----	Draw, cut
	Tangent Line is Perp. to Radius	-----	-----	-----	-----	Drawing, paper/pencil	Draw w/ paper or software	Software investigation	-----
Proof	Isosceles Triangle	**Paragraph and Two-Column**	**Two-column**	Plan given in exp., proof as exercise	**Two-column**	Paragraph, given 200+ pages later	Plan given in exp., flow prf. as exercise	**Two-column**	**Two-column**
	Angle-Angle-Side	**Paragraph**	**Two-column**	Plan given	Short expln. given, proof as exercise	**Flowchart**	As exercise	**Two-column**	**Flowchart**
	Opp. Sides of a ▱ Are Congruent	-----	As exercise	Plan given in exp., proof as exercise	In exercises	**Flowchart**	As exercise (fill-in)	As exercise, two-column	**Two-column**
	Area of a Trapezoid	**Paragraph, some details omitted**	Explanation, not called a "proof"	Outline of proof given	**Two-column**	Studts derive frmla. in activity	Studts derive formula in activity	Explanation, formula not a "theorem"	In exercises, "write a plan for proof"
	Tangent Line is Perp. to Radius	**Paragraph, indirect**	Explanation, not called a "proof"	**Paragraph, indirect**	Proof as exercise, hint: indirect proof	**Paragraph, indirect**	As exercise, planned indirect proof	-----	As exercise, planned indirect proof

Figure 14. Theorems: Definition, discovery and conjecture, and proof.

		Addison-Wesley, 1982	Merrill, 1984	Houghton-Mifflin, 1988	Holt, 1991	Prentice Hall, 2001	Holt, 2004	Glencoe, 2005	McDougal-Littell, 2007
Real-world Context	Isosceles Triangle	-----	-----	Building photo, example of isos. triangles	-----	Photo in exposition, appl. in exercises	Applications of other theorems in the section	Photo of isos. triangles in art, design	Converse appl. in exposition, thm. in exercises
	Angle-Angle-Side	-----	-----	Photo of tower with triangles	-----	-----	In exercises	Appl. of other postulate	In exposition and exercises
	Opp. Sides of a ▱ Are Congruent	-----	Photograph of window with ▱	-----	-----	Photo, direct application of theorem	In exercises	Examples of ▱s, but not of this theorem	In exposition and exercises
	Area of a Trapezoid	In exercises	-----	-----	Photo of building with trap. shapes	In exposition & exercises	In exposition	In exercises	In exercises
	Tangent Line is Perp. to Radius	-----	-----	-----	-----	In exposition & exercises	In exercises	In exercises, tangent appl. in exposition	In exercises

Figure 15. Theorems: Real-world context.

proven is a theorem" (p. 49). Glencoe 2005 defines it this way: “A statement or conjecture that can be proven true by undefined terms, definitions, and postulates” (p. R28). This difference in wording indicates a significant change in how theorems are introduced and discussed in the newer textbooks. Giving students opportunities to make conjectures is a specific recommendation of NCTM’s Mathematics as Reasoning standard (1989, p. 143), and it is certainly a recommendation that the newer textbooks follow. Though only two of the textbooks in the newer group use “conjecture” in the definition of theorem, all four textbooks in the newer group, far more than in the earlier group, position theorems in the context of conjecture and discovery.

That difference in the approach to presenting theorems is evident in the structure of the exposition surrounding each theorem. In the earlier textbooks, theorems are usually placed at or near the beginning of their respective sections, sometimes preceded by definitions of terms or brief introductory remarks. In the newer books, the theorem in many cases follows an investigative activity, designed to lead students to a conjecture. Later, the conjecture is proven as a theorem. (For examples, see the appendix on p. 82.) In the newer group of textbooks, of the 20 cases that I looked at (five theorems, four textbooks), there are 14 in which the theorem is preceded by a discovery or conjecture activity. In the earlier group of textbooks, that number is zero out of 20. (See Figure 14, Discovery and Conjecture.) Those 14 investigative activities are all hands-on and inductive, rather than deductive. That is, students are led through activities such as cutting out paper triangles or using geometric software, rather than through a logical deductive process involving theorems or proof. (One example is in the appendix, on p. 82.) Most stated theorems in the newer books are still proven, at least somewhere in the textbook. Often the proof is left to students as an exercise, but this is true for both newer and earlier textbooks. (See Figure 14, Proof.)

The process of moving from conjecture to proof is reflective of the attention to inductive reasoning in the 1989 NCTM synthetic geometry standard. The *Standards* recommended that, “the organization of geometric facts from a deductive perspective should receive less emphasis, whereas the interplay between inductive and deductive experiences should be strengthened” (p. 159). Indeed, I found more than just a shift in emphasis in the newer textbooks. The newer textbooks make the point that theorems and their proofs work with, and flow from, inductive reasoning. (In the Holt 2004 text, there is even a section titled "Conjectures that Lead to Theorems.") In this spirit, three of the four newer textbooks introduce the term inductive reasoning before student proof writing begins. In contrast, in three of the four earlier books, inductive reasoning as a meta-concept is either taught after proof writing is already underway or, as is the case for one of the texts, not explicitly taught at all. It is noteworthy that Addison-Wesley 1982, the only one of its group that explicitly discusses inductive reasoning before proof, does not use the words “inductive” and “conjecture.” But it makes the same points that the newer books do about inductive reasoning and conjecture using words like *patterns*, *generalizations*, and *guessing* (p. 8).

The emphasis on inductive reasoning and conjecture found in the newer textbooks’ presentation of theorems has significant implications for how students understand the mathematical process. In the new approach, rather than beginning their study of a theorem with an assertion and then a proof, students begin with exploration and conjecture. An important question is whether this approach produces a more authentic understanding of the process mathematicians use to conceive of and establish mathematical truth.

The 1989 *Standards* recommended that high school geometry emphasize the "wide applicability in human activity" (NCTM, p. 157) of shapes and their properties. I found that, in keeping with NCTM's vision, the newer textbooks are far more likely to put the theorems in the context of real-world applications than are the older textbooks. This is no surprise: the illustrations of real-world situations in the newer books are obvious as soon as one opens them.

In 16 of the 20 cases that I examined in the newer textbooks, there is at least one real-world application directly related to the theorem, in the exposition or the exercises of the section in which it is presented. In stark contrast, out of all 20 cases in the earlier textbooks, there is only one real-world application (a trapezoidal area problem in the Addison-Wesley textbook). (See Figure 15.) Moreover, in the newer textbooks, in all 20 cases, there is some sort of real-world context for a portion of the material presented in the section, even if it does not directly relate to the specific theorem in my study sample. For example, in the section of the Holt 2004 textbook containing the Isosceles Triangle Theorem, there are two real-world applications of another theorem in the same section, the Converse of the Isosceles Triangle Theorem: an example in the exposition involving the distance across a canyon and an exercise involving the distance across a river (pp. 238-9). However, there are no direct real-world applications of the Isosceles Triangle Theorem itself.

Positioning the theorem in the context of real-world applications was clearly a priority for the authors of the newer textbooks. It is interesting to note that the proofs themselves still exist outside the "real world." Theorems are still proven, not using concrete examples from construction or science, but using abstract statements: postulates, theorems, and definitions.

Introducing Proof and Proof Writing

As I examined the various ways that proof and proof writing were introduced and defined in the texts, I found an element of metacognition in the approach taken by the textbooks from the 2000s, which is much stronger than that of the earlier textbooks. Of the three textbooks from the 1980s that have glossaries, none has a glossary entry for "proof." The one textbook that does not have a glossary, Addison-Wesley 1982, has no entry for proof in its index. In contrast, all of the newer textbooks define proof in the glossary (and elsewhere in the text). The fact that the newer textbooks have longer glossaries overall (about 34% longer on average, measured by number of entries) could be a factor in the difference between the two groups. But it is also true that, among the earlier textbooks, only in the Holt 1991 was I was able find a statement roughly following the form, "A ________ is a proof," or, "A proof is _______," anywhere in the exposition. Though the earlier textbooks outstrip the newer textbooks in their emphasis on the task of writing proofs, the newer textbooks, more than the earlier books, pay attention to the task of thinking about what a proof is.

This theme of metacognition is demonstrated in the material that introduces and precedes student proof writing. In most of the newer textbooks, there is an explicit discussion of "reasoning" before students begin writing proofs that is not usually found in the earlier textbooks. For example, in the 2007 McDougal Littell, we have sections 2.1: Use Inductive Reasoning, 2.2: Analyze Conditional Statements, 2.3: Apply Deductive Reasoning (discussing the Laws of Detachment and Syllogism), and section 2.4: Use Postulates and Diagrams. In section 2.5: Reason Using Properties from Algebra, students get their first taste of justifying steps in logical arguments, though they are not "proofs" yet. Finally, in section 2.6, "proof" is defined, and students begin to write proofs.

In contrast, consider the way that the Houghton Mifflin 1988 text introduces proof writing. In section 1.4: Introduction to Proof, students practice writing fill-in and independent proofs, justifying statements with properties from algebra and postulates about angles and segments. The preceding three sections cover strictly geometric content, walking through basic terminology and a few postulates. There is a discussion of inductive reasoning in this textbook, but it does not come until the end of Chapter 3.

These two textbooks represent the extremes of both approaches: introducing proof toward the end of a chapter-long discussion of reasoning vs. introducing proof right in the middle of the geometric content, with no preparatory discussion of reasoning or logic. When we look at the other six textbooks we see that most do not follow either extreme, and that the idiosyncrasies of each text make it impossible to describe a strict formula for either group. Yet there are similarities in each group that are significant.

Below, I list the sections in each textbook that precede student proof writing and that discuss logic and reasoning and/or explicitly prepare students for writing proofs.

Textbooks from the 2000s:

- Students using the Glencoe 2005 text begin writing proofs in section 2.5: Postulates and Paragraph Proofs. Immediately preceding that section are four sections titled Inductive Reasoning and Conjecture, Logic, Conditional Statements, and Deductive reasoning.
- The Holt 2004 text begins instruction on proof in section 2.1 with what it calls "free-form" proofs. These are informal logical arguments about topics that are not part of the core geometry course (e.g. using a diagram to explain why $(x + a)(x + b) = x^2 + ax + bx + ab$.) This is followed by 2.2: An Introduction to Logic, where students prove statements using logical chains such as, "All United States Postal workers are federal employees. John is a United States Postal worker (p. 94)." After Section 2.3 discussing definitions, section 2.4: Building a System of Geometry Knowledge introduces two-column and paragraph proofs.
- In Prentice Hall 2001, the instruction on proof and reasoning is oddly scattered. Section 1.1 is titled Using Patterns and Inductive Reasoning. Later in the chapter, students are briefly introduced to proof writing in section 1.7: Using Deductive Reasoning. No further proofs are found in the exposition or

exercises until chapter 4, which begins with section 4.1: Using Logical Reasoning, covering conditional statements. Finally, section 4.3: Preparing for Proof reintroduces proof writing.

Textbooks from the 1980s:

- Section 1-2: Two Kinds of Problems discusses the value of guessing and experimenting, and the importance of looking for patterns that lead to generalizations. Without using the word "inductive," this section introduces inductive reasoning. In Section 4.5: Equivalence Relations of the Addison-Wesley 1982 text, students are asked to give a reason for each statement in a "dependent sequence of steps (p.99)." Section 4.6: Some Theorems about Angles teaches the two-column form, and students begin writing proofs. Section 4.9: Writing Up Proofs gives suggestions on the proof writing process, such as drawing the diagram neatly.
- The Merrill 1984 text builds proof-writing skills slowly. Proofs are introduced in section 1.5: Theorems, but instead of writing complete proofs in this section, students practice skills like stating the hypothesis for a given theorem and drawing a diagram to illustrate a given statement. In section 1.6: Writing Proofs, again the exercises are not complete proofs, but short sequences of two statements and their reasons, with students expected to fill in a blank – either a statement or a reason. Section 1.7: Inverses and Contrapositives follows, giving students practice writing inverses, contrapositives, and negations of conditional statements. Section 2.3: Segments prepares further for proof by having students list definitions, postulates, or theorems that could

be used to prove statements about segments and midpoints. In section 2.4: Properties of the Real Number System, students finally see exercises that are complete proofs, albeit fill-in proofs. Notice that though there are several sections that build the components of proof writing, they are not explicit discussions of reasoning.

- Holt 1991's Section 2.2: Introduction to Proof is preceded by sections 1.7: Logic: Conjunction and Disjunction (where students determine whether "and" and "or" statements are true), 1.8: Graphing Conjunctions and Disjunctions, and 2.1: Drawing Conclusions. In the latter section, students practice drawing conclusions from given statements and diagrams.

In their approaches to introducing proof and proof writing, it is clear that there is wide variation among individual texts and some overlap between the two groups. Still, the generalization can be made that the newer textbooks give much greater attention to situating proof within the context of mathematical reasoning.

It is significant that, among the newer textbooks, the words "reason" and "reasoning" are found in the titles of four chapters (one in each book) and nine sections. Among the earlier textbooks, those numbers are zero and two, respectively. (To be fair, the section titles of the earlier textbooks occasionally use related words such as "logic" or "deductive.") That the newer textbooks more frequently use the word "reasoning" in section and chapter titles is not trivial. It points to differences in how the authors frame the discussion of proof, and of geometry writ large. It is not that the newer textbooks necessarily engage the students in more mathematical reasoning than do the earlier textbooks. The difference is in the stronger expectation that students think about their reasoning processes, and about writing proofs as a kind of reasoning.

In this chapter, I have analyzed the themes of how textbooks from each group treat the presentation of theorems and introduction to proof writing. Further discussion of these results and potential implications drawn will be addressed in the concluding chapter that follows.

Chapter 5: Discussion and Conclusions

My research has given me a framework for answering the question, "How do geometry textbooks from the 1980s and the 2000s differ in their approach to proof?" Below, I reiterate the sub-questions that have helped me piece together the answer to the overarching question.

1. Is proof emphasized less in the newer textbooks, relative to the earlier books?
2. In the newer textbooks, is there increased attention to inductive reasoning and to justifications that do not constitute formal proof?
3. How do the texts introduce proof and proof writing?
4. How are theorems presented in the texts?
5. What attention is given to the two-column form, other forms of proof, and form in general?

Informed by these questions, we can make some observations about the changes made in the textual curriculum between the 1980s and the 2000s.

Comparing the Textbooks

Research Question 1: Emphasis on Proof

Is proof emphasized less in the newer textbooks, relative to the earlier books?

Some measures seem to suggest a relative de-emphasis on proof in the newer textbooks, but the question warrants more than a simple yes or no. On the whole, the newer textbooks provide fewer opportunities for students to write proofs without assistance, as measured in number of exercises, though there was some overlap between the two groups. The newer textbooks also had smaller numbers of total proof items (i.e. the sum of unassisted proofs*,* fill-in proofs, paragraph proofs, and proof development items). But muddying the picture of emphasis in each group are the numbers of proofs in the exposition. The Addison-Wesley 1982 and Holt 1991 texts have greater numbers of proofs in the exposition than do the newer textbooks, but the other textbooks from the 1980s are comparable to the newer books. Still, we can make the generalization that, as an activity to be practiced, proof tends to be emphasized less in the newer textbooks than in the earlier textbooks.

But the wide variation within each group should not be ignored, and it has significant implications for textbook selection committees. Textbooks written even within a few years of each other may vary greatly in their numbers of opportunities for proof writing. Selection committees need to think carefully about their goals for the students in their districts. If proficiency in proof writing is a priority the Prentice Hall 2001 textbook would probably be a poor choice compared with the Glencoe 2005, for example. Of course, the assumption underlying the previous statement is that the number of opportunities to practice writing proofs is a key factor in developing proficiency. While this is a plausible assumption, and one that I accept, it has not, to my knowledge, been established in the literature.

The sharp decrease in *independent set-up* exercises in the newer books indicates that students using the newer textbooks have fewer opportunities to write a proof without any scaffolding provided. In an independent set-up proof, the responsibilities of providing a relevant diagram and formulating what is given and what is to be proved are shifted to the student. This lack of scaffolding is the norm in some advanced college mathematics textbooks (Munkres's *Topology* for example, published in 2000), and the de-emphasis on independent set-up proofs may have implications for those students who go on to take advanced mathematics courses. It is well worth investigating whether fewer opportunities in geometry textbooks to write proofs, and independent set-up proofs in particular, contribute to poor preparation for proof centered college mathematics courses has declined over the time period covered in this study. While there are anecdotal complaints from university professors that student preparation for proof has declined, I have not found evidence in the literature to support this claim.

The relative scarcity of independent set-up proofs raises another concern as well. In Chapter 3, I speculated about the possibility that fewer opportunities to practice independent set-up proofs might influence students' views of what their own proof-writing activity accomplishes. There is evidence that students may not realize that the conclusion of a proof is true for the entire class of figures satisfying the hypothesis. In a 1979 study, Williams (as cited by Chazan, 1993, p. 362) found that some students did not know that a given proof was generalizable for all triangles. Balacheff (as cited by Chazan, 1993, p. 362) argued that students may not understand that the figure accompanying a proof is "generic" in nature, meant to be representative of the characteristics of its class.

These findings beg for more research on independent set-up proofs, where students themselves provide the diagram, decide on the name of the figure (e.g. Δ *ABC*) and label its parts, in order to prove a (usually general) conditional statement. One might hypothesize that students would be more likely to understand of the generalizability of a proof and the generic aspect of the accompanying diagram, having drawn the diagram themselves.

Research Question 2: Inductive Reasoning and Justifications

In the newer textbooks, is there increased attention to inductive reasoning and to justifications that do not constitute formal proof?

There is a clear shift toward emphasizing inductive reasoning in the newer textbooks. As mentioned in Chapter 3, all of the newer textbooks explicitly discuss inductive reasoning, while only two of the earlier texts do. Further evidence of this increased emphasis is found in the way newer textbooks conceptualize theorems and proofs in the context of inductive reasoning and conjecture. The newer textbooks are much more likely to introduce theorems with discovery activities that draw on inductive reasoning. As discussed in Chapter 4, the newer books also take a stronger metacognitive approach to reasoning itself, both inductive and deductive.

There is some evidence that the newer textbooks devote more exercises that require some sort of justification, but not necessarily a formal proof. With the exception of Houghton Mifflin 1988, all of the earlier textbooks had fewer exercises that fell into the category of *justification and explanation* exercises, than did any of the newer texts.

While the newer textbooks generally provide fewer opportunities for students to practice writing proofs, they give students more opportunities to explain their reasoning in other ways. The newer textbooks also provide a framework for students to reflect on reasoning itself, much more so than the earlier textbooks. This is a decided shift in emphasis with significant implications for learning and teaching.

How the increased attention to reasoning, as an object of study, has influenced students' ability to reason mathematically would be an important question for further study.

Research Question 3: Introducing Proof

How do texts introduce proof and proof writing?

The textbooks from the 2000s take greater care to place proof within the context of reasoning than do the texts from the 1980s. They tend to prepare students for proof writing with explicit discussions of reasoning and logic, far more than the earlier texts do. By using the word "reasoning" to frame the discussion of proof, the newer textbooks emphasize the cognitive nature of proof writing and situate it in the same realm as other mental activities – looking for patterns, for example – that lead to mathematical truth.

One could argue that earlier textbooks also place a high priority on developing reasoning, with the large numbers of proofs in the exercises as evidence. But the earlier textbooks are less emphatic about identifying proof as a product of reasoning than the newer textbooks, which use chapter and section organization and the discursive itself to categorize proof writing as a reasoning process. And though this was not a specific piece of my coding scheme, I also noticed that the earlier textbooks are less likely to identify exercise tasks with the word "reasoning," compared with the newer textbooks.

The newer textbooks give more attention to the meaning of the term proof than do the earlier books, most of which do not explicitly define "proof" anywhere in the text. This omission may be further evidence of a weaker metacognitive component in the earlier group's approach to proof. The question remains whether the newer textbooks' stronger metacognitive elements and explicit conceptualization of proof writing as a reasoning process have had any substantial effect on how students themselves conceptualize proof.

Research Question 4: Presenting Theorems

How are theorems presented?

Here, I look at two differences in the presentation of theorems between the textbooks from the 2000s and those from the 1980s. First, the newer textbooks provide more real-world applications of theorems than do the earlier textbooks. This is reflective of NCTM's 1989 *Standards*, which focus on the "wide applicability in human activity" of shapes and their properties (p. 157). The emphasis on real-world applications says something about the role of theorems in the geometry curriculum. Why do students learn theorems in geometry? Is it because of their role in building a deductive system, or is it because of their role in helping us understand the world we live in? Each of the textbooks in the study would certainly answer that question, "A mixture of both," but the newer textbooks emphasize the latter role more than the earlier texts do.

Second, in the newer textbooks, the relationship between theorems and conjecture is emphasized, far more than in the earlier texts. Before stating a theorem, the newer textbooks are more likely to encourage investigation and conjecture leading to the theorem. By making clear the connection between theorems and conjectures, the newer textbooks repeatedly reinforce the value of inductive reasoning. Proving theorems is not abandoned in the newer textbooks, but the proof itself is no longer the sole means of persuasion or explanation, as it so often is in the earlier texts.

What do the newer textbooks accomplish by bringing investigation and conjecture to the forefront? The possibilities are rich. One outcome might be that students develop a better intuitive understanding of the theorems, as they explore shapes and investigate their properties. Another possible benefit might be that the students develop a more authentic understanding of mathematical proof and reasoning. In a landmark 1980 article for *The Mathematics Teacher*, Usiskin argued for greater authenticity in the way geometry students were taught proof.

Concerned with the way geometry students were learning about proof, Usiskin (1980) posited that for mathematicians, "proof is done only after some amount of exploration. The real learning often as not comes from this exploration" (p. 420). Geometry students, he claimed, were getting the opposite impression. They "seldom explore and are almost always told what they should prove" (p. 420). With a superficial reading, the newer textbooks' approach to presenting theorems seems to be a remedy for the problem that Usiskin identified. But a more careful examination leaves me unsure. The investigations that precede theorems in the newer textbooks are usually short, controlled activities, contrived to lead to a particular conjecture. Arguably, they are not the same kinds of open-ended, sustained explorations that mathematicians engage in.

The potential of the newer textbooks' discovery-conjecture-theorem-proof approach for helping students develop a stronger understanding of "the interplay between inductive and deductive experiences" (NCTM, p. 159) is certainly exciting.

However, I am only cautiously optimistic about the degree to which this approach fulfills that potential. This is a question for further research: What, if anything, does the literature say about how a conjecture based approach to theorem strengthens students' understanding of proof and reasoning?

Research Question 5: Attention to Form

What attention is given to the two-column form, other forms of proof, and form in general?

In the newer textbooks, the two-column form is presented as one of several format options, all valid. In some of the newer textbooks, this point is made when proof is introduced, and then it is made over and over again, through specific elicitations of alternative forms in the exercises. The treatment of form is very different in the earlier textbooks. With the exception of Houghton Mifflin 1988, which spends time teaching paragraph proofs, the earlier textbooks say very little about alternative forms. Certainly, none of them emphasize the variety of forms to the extent that the newer textbooks do. The newer textbooks seem to be responding to NCTM's recommendation that the two-column proof be used as a teaching tool, with the expectation that students eventually learn to write paragraph proofs (p. 145). We are also reminded of Schoenfeld's (1988) concern that the elevated status of the two-column format was causing students to focus on form at least as much as substance (p. 158).

We also see that flow proofs, given scant attention in the earlier textbooks, have established a significant role in the newer textbooks, although they never outnumber two-column proofs. Flow proofs do more than simply provide an example of an alternative to the two-column form. By visually demonstrating the logical flow of the proof, they function as a resource to make proof more accessible to students. They have a reflective function as well, forcing students to think about the relationships between statements.

The relative presence (or absence) of flow proofs is one of the starkest differences between the two groups of geometry textbooks, so it was surprising to me how little I found on the subject in the literature. All of the other distinguishing characteristics of the newer textbooks (e.g. more paragraph proofs, more inductive reasoning and conjecture, fewer formal proofs) correspond to recommendations made by the *Standards* and by researchers in the 1980s and 1990s. I have found nothing on flow proofs aside from two articles in *The Mathematics Teacher* from the late 1970s, both written by classroom teachers.

Flow proofs raise historical and pedagogical questions that intrigue me: What influences led to the rise of flow proofs as an alternative format in the newer textbooks? And how effective are they in helping students to understand the logic and deductive structure of proofs? The latter question is particularly interesting in light of Senk's (1985) finding that most geometry students were not mastering proof, and Shaughnessy and Burger's (1985) argument that most high school students were not intellectually mature enough to learn proof. Anecdotal evidence from my colleagues suggests that flow proofs may indeed help weaker students to successfully construct proofs – simple ones, at least – and understand their structure.

Further Implications and Questions

Though textbooks are an important component of curriculum, there is much that textual analysis does not tell us: How much of the textbook does the teacher actually cover? What supplementary materials are used? Which of the textbook exercises are assigned, and which are not assigned? Extra-textual influences such as teacher beliefs, student preparation, school district requirements, and standardized assessments are all part of the picture of classroom instruction. Remillard and Bryans (2004), in a study of teachers implementing an Investigations curriculum that was new to them, found that each teacher's unique set of beliefs and attitudes mediated the way the curriculum was implemented in the classroom (p. 364). Students' practices also shaped curricular instruction (p. 364).

External pressures play a significant role in shaping curriculum as well. As a teacher in Montgomery County Public Schools, my colleagues and I paid as much attention to the school district's curriculum guide and final exam review as we did to the textbook. We deemphasized some textbook sections and left out others altogether. My experience is probably typical of teachers in the county who, my anecdotal evidence suggests, teach with the final exam in mind.

There is evidence that teacher beliefs may often be in conflict with reform efforts, and this has implications for how reform is implemented (Handal & Herrington, 2003). On the other hand, according to Remillard and Bryans (2004), it is also true that teachers' beliefs could be "challenged and altered when they examined unfamiliar mathematical tasks and interpreted students' work on them while teaching" (p. 355). One fruitful field of study might be work on how teachers' beliefs about proof have interacted with changes to the intended geometry curriculum, and what that means for the curriculum that is delivered to the student.

We have seen significant changes in how textbooks approach proof: a decrease in the number of proofs in the exercises, an increased emphasis on inductive reasoning, an increase in attention to forms other than the two-column proof. There are more nuanced changes as well in the textual discussion surrounding proof, theorems, and reasoning. My research provides a background for further inquiry into the ways that students learn about proof in geometry. This textual analysis addresses only the beginning of the path from textbook, to classroom, to student. More work needs to be done exploring how the textbooks' changing approach to proof influences classroom teaching, and how both textbook and teacher ultimately shape students' learning.

Appendix: Examples of Presentation of Theorems in Exposition

Figure A1. Exposition preceding and including the Isosceles Triangle Theorem in Prentice Hall 2001 (p.188). The exposition is continued in Figure A2.

What You'll Learn

- Using and applying properties of isosceles triangles

...And Why

To understand a geometric figure used in the designs of many buildings and bridges

What You'll Need

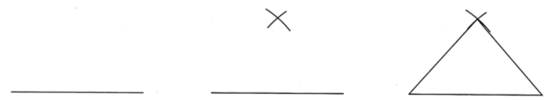
- straightedge
- compass
- scissors

4-2 Connections Landscaping . . . and more

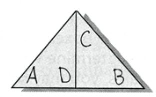
Isosceles Triangles

WORK TOGETHER

Have each member of your group construct a different isosceles triangle and then cut it out. Be sure to include acute and obtuse triangles.



- Label the triangle $\triangle ABC$, with A and B opposite the congruent sides.
- Bisect $\angle C$ by folding the triangle so that the congruent sides overlap. Label the intersection of the fold line and $\overline{AB}$ as point D.

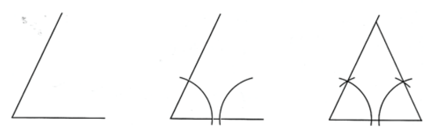


TECHNOLOGY HINT

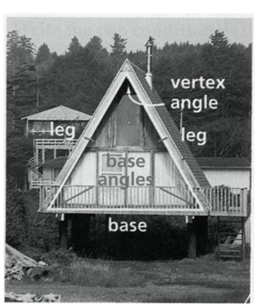
The Work Together could be done using geometry software.

1. What do you notice about $\angle A$ and $\angle B$? Compare results within your group.
2. a. What types of angles do $\angle CDA$ and $\angle CDB$ appear to be?
   b. What do you notice about $\overline{AD}$ and $\overline{DB}$?
   c. Use your answers to parts (a) and (b) to complete the statement: $\overline{CD}$ is the __?__ of $\overline{AB}$.

- Construct a new triangle that has two congruent angles. Cut out the triangle and label it $\triangle EFG$, where $\angle E \cong \angle F$.



3. Fold the triangle so that the congruent angles overlap.
   a. What do you notice about $\overline{EG}$ and $\overline{FG}$? Compare results within your group.
   b. What type of triangle is $\triangle EFG$?



THINK AND DISCUSS

Isosceles triangles are common in the real world. You can find them in structures such as bridges and buildings. The congruent sides of an isosceles triangle are the **legs.** The third side is the **base.** The two congruent sides form the **vertex angle.** The other two angles are the **base angles.**

188 **Chapter 4** Triangle Relationships

Figure A2. Exposition preceding and including the Isosceles Triangle Theorem in Prentice Hall 2001 (p.188). This is a continuation of Figure A1.

Your observations from the Work Together suggest the following theorems. The proofs of these theorems involve properties of congruent triangles that you will study in Chapter 8.

Theorem 4-1
Isosceles Triangle Theorem

If two sides of a triangle are congruent, then the angles opposite those sides are also congruent.

If $\overline{AC} \cong \overline{BC}$, then $\angle A \cong \angle B$.

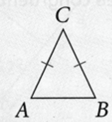


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Figure A3. Exposition preceding and including the Isosceles Triangle Theorem in Merrill (p.80).

4–7 Isosceles Triangles

Some proofs include geometric figures that are *not* given in the theorem. Often, these figures are lines or parts of lines. They are included on the diagram in color. The proof of the following theorem uses an **auxiliary** segment.

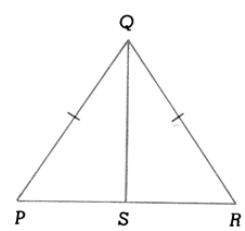
If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

Theorem 4–6
Isosceles Triangle Theorem

example

1 Prove Theorem 4–6.

Given: $\triangle PQR$
$\overline{PQ} \cong \overline{RQ}$
Prove: $\angle P \cong \angle R$



Proof:

STATEMENTS	REASONS
1. Call S the midpoint of $\overline{PR}$, and draw $\overline{QS}$.	**1.** Theorem 2–1: If a segment is given, then it has exactly one midpoint, and Postulate 1–1: Through any two points, there is exactly one line.
2. $\overline{PS} \cong \overline{RS}$	**2.** Theorem 2–5: Midpoint Theorem
3. $\overline{PQ} \cong \overline{RQ}$	**3.** Given
4. $\overline{QS} \cong \overline{QS}$	**4.** Theorem 2–2: Congruence of segments is reflexive.
5. $\triangle PQS \cong \triangle RQS$	**5.** Postulate 4–1: SSS *Use steps 2, 3, and 4.*
6. $\angle P \cong \angle R$	**6.** Definition of Congruent Triangles *CPCTC*

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